Finding a Formula for Fibonacci Numbers

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The Fibonacci sequence starts with 0 and 1, and each new term is calculated by adding the previous two terms together:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

We can define the sequence as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n \end{aligned}$$

Example: using this definition to calculate the third Fibonacci number:

$$\begin{aligned} &F_0 = 0 \\ &F_1 = 1 \\ &F_2 = F_1 + F_0 = 1 + 0 = 1 \\ &F_3 = F_2 + F_1 = 1 + 1 = 2 \end{aligned}$$

This process can be completed to calculate the nth Fibonacci number, but it can involve a lot of steps. We will use the relationship above to derive a formula that will calculate the nth Fibonacci number. To do this we will make use of generating functions and the formula for the sum of an infinite geometric series:

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + ax^4 + \cdots = \frac{a}{1-x} \quad (|x| < 1)$$

We will start by defining the generating function and finding a closed form for it. Then, we will take that closed form and express it as the sum of an infinite geometric series. These two expressions will be equal by construction, meaning that the coefficients of corresponding powers of x will be equal in both expressions. Importantly this applies to the general nth term, which will equate F_n to its closed form.

Finding the Generating Function

We will define our generating function as:

$$
\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= F_0 x^0 + F_1 x^1 + F_2 x^2 + F_3 x^3 + F_4 x^4 + F_5 x^5 + \cdots \\
&= 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots
\end{aligned}
$$

In our function, the variable x will be treated as a formal variable. It is there purely to assist with algebraic manipulation. Through our calculation we will find constraints on the values that x could take e.g. ($|x| < 1$). Our final goal doesn't require x to have any specific values so we can and will assume x has a value that satisfies those constraints.

Start by extracting the first two terms from the generating function:

$$
F(x) = 0 + x + \sum_{n=2}^{\infty} F_n x^n
$$

Next, we will reindex the sum:

$$
\begin{aligned}
\text{Change } n &\to n+2 \\
\text{Sum limit } n+2 &= 2 \\
n &= 0 \\
F(x) &= x + \sum_{n=0}^{\infty} F_{n+2} x^{n+2}
\end{aligned}
$$

Apply the recurrence relation to $Fn + 2$ and split into separate sums:

$$
\begin{aligned}
F(x) &= x + \sum_{n=0}^{\infty} F_{n+2} x^{n+2} \\
&= x + \sum_{n=0}^{\infty} (F_{n+1} + F_n) x^{n+2} \\
&= x + \sum_{n=0}^{\infty} F_{n+1} x^{n+2} + \sum_{n=0}^{\infty} F_n x^{n+2}
\end{aligned}
$$

Factorise x out of each sum so that the exponent for x matches the subscript for F:

$$
\begin{aligned}
F(x) &= x + \sum_{n=0}^{\infty} F_{n+1} x^{n+2} + \sum_{n=0}^{\infty} F_n x^{n+2} \\
&= x + x \sum_{n=0}^{\infty} F_{n+1} x^{n+1} + x^2 \sum_{n=0}^{\infty} F_n x^n
\end{aligned}
$$

Now reindex the middle sum so that the exponent of x and the subscript of F are both n, then add $F_0 = 0$ back into the middle sum:

$$\begin{aligned}
F(x) &= x + x\sum_{n=0}^{\infty} F_{n+1}x^{n+1} + x^2\sum_{n=0}^{\infty} F_n x^n \\
&= x + x\sum_{n=1}^{\infty} F_n x^n + x^2\sum_{n=0}^{\infty} F_n x^n && \text{(Reindex)} \\
&= x + x\sum_{n=0}^{\infty} F_n x^n + x^2\sum_{n=0}^{\infty} F_n x^n && \text{(Add } F_0 = 0\text{)}
\end{aligned}$$

Notice that the second and third terms contain the original generating function that we defined to be $F(x)$. We will make that substitution and then solve for $F(x)$:

$$\begin{aligned}
F(x) &= x + x\sum_{n=0}^{\infty} F_n x^n + x^2\sum_{n=0}^{\infty} F_n x^n \\
&= x + xF(x) + x^2F(x) \\
F(x) - xF(x) - x^2F(x) &= x \\
(1 - x - x^2)F(x) &= x \\
F(x) &= \frac{x}{1 - x - x^2}
\end{aligned}$$

Expressing as an Infinite Series

We will need to manipulate the generating function to get it into a form that can be directly converted to an infinite geometric series. To do this we will factorise the denominator into factors of an appropriate form, and then use partial fraction decomposition to break up the expression so that it can be expanded as an infinite geometric series.

We will first assume that this is possible, and use the variables a, b, A, B as placeholders for values we will need to calculate:

$$
\begin{aligned}
F(x) &= \frac{x}{1-x-x^2} \\
&= \frac{x}{(1-ax)(1-bx)} && \text{(factorise denominator)} \\
&= \frac{A}{1-ax} + \frac{B}{1-bx} && \text{(partial fraction decomposition)} \\
&= A\sum_{n=0}^{\infty}(ax)^n + B\sum_{n=0}^{\infty}(bx)^n && \text{(sum of infinite geometric series)} \\
&= \sum_{n=0}^{\infty} Aa^n x^n + \sum_{n=0}^{\infty} Bb^n x^n \\
&= \sum_{n=0}^{\infty}(Aa^n x^n + Bb^n x^n) \\
&= \sum_{n=0}^{\infty}(Aa^n + Bb^n)x^n \\
\sum_{n=0}^{\infty} F_n x^n &= \sum_{n=0}^{\infty}(Aa^n + Bb^n)x^n && \text{(definition of } F(x)\text{)} \\
\Rightarrow F_n &= Aa^n + Bb^n && \text{(extracting coefficients)}
\end{aligned}
$$

The first two values we will determine are a and b. From the step where we factorised the denominator we have the following:

$$
\begin{aligned}
(1-ax)(1-bx) &= 1-x-x^2 \\
1-(a+b)x+abx^2 &= 1-x-x^2
\end{aligned}
$$

Equating coefficients gives us the following system of equations to solve:

$$
\begin{cases}
a+b &= 1 \\
ab &= -1
\end{cases}
$$

Solving this system:

$$
\begin{aligned}
b &= 1 - a \\
ab = a(1-a) &= -1 \\
a - a^2 &= -1 \\
a^2 - a - 1 &= 0 \\
a &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot (1) \cdot (-1)}}{2 \cdot (1)} \quad \text{(quadratic formula)} \\
&= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

If $a = \dfrac{1+\sqrt{5}}{2}$:

$$
\begin{aligned}
b &= 1 - \frac{1+\sqrt{5}}{2} \\
&= \frac{2}{2} - \frac{1+\sqrt{5}}{2} \\
&= \frac{1-\sqrt{5}}{2}
\end{aligned}
$$

If $a = \dfrac{1-\sqrt{5}}{2}$:

$$
\begin{aligned}
b &= 1 - \frac{1-\sqrt{5}}{2} \\
&= \frac{2}{2} - \frac{1-\sqrt{5}}{2} \\
&= \frac{1+\sqrt{5}}{2}
\end{aligned}
$$

The system of equations is symmetric in a and b, meaning we could swap their values with no effect. Also, when a is assigned one of the roots, b has the value of the other. These two facts together indicate we have free choice in which root we assign to a and b, so we will assign the positive root to a.

$$
a = \frac{1+\sqrt{5}}{2} \quad b = \frac{1-\sqrt{5}}{2}
$$

This leaves A and B left to determine. From our partial fraction decomposition step we have:

$$
\begin{aligned}
\frac{x}{(1-ax)(1-bx)} &= \frac{A}{1-ax} + \frac{B}{1-bx} \\
x &= A(1-bx) + B(1-ax) \\
x &= (-Ab - Ba)x + (A + B)
\end{aligned}
$$

Equating coefficients of corresponding powers of x on both sides of the equation gives the following system of equations to be solved:

$$\begin{cases} -Ab - Ba &= 1 \\ A + B &= 0 \end{cases}$$

Solving this system:

$$\begin{aligned}
B &= -A \\
-Ab - (-A)a &= 1 \\
-Ab + Aa &= 1 \\
A(a - b) &= 1 \\
A &= \frac{1}{a - b} \\
&= \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} \\
&= \frac{1}{\sqrt{5}} \\
B &= -A \\
&= -\frac{1}{\sqrt{5}}
\end{aligned}$$

We were able to determine values for all four of our placeholder variables with no contradictions, indicating we do have a closed form. Now all that's required is to substitute them into our formula for F_n:

$$\begin{aligned}
F_n &= Aa^n + Bb^n \\
&= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)
\end{aligned}$$