Solutions to Project Euler Problem 6: Sum Square Difference

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https://projecteuler.net/problem=6

The problem reads:

> The sum of the squares of the first ten natural numbers is,
>
> $$1^2 + 2^2 + \cdots + 10^2 = 385$$
>
> The square of the sum of the first ten natural numbers is,
>
> $$(1 + 2 + \cdots + 10)^2 = 55^2 = 3025$$
>
> Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is $3025 - 385 = 2640$.
>
> Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.

Brute Force

This answer can be obtained by simply performing the artihmetic, although there are many calculations that each present an opportunity for a mistake. Doing this we get:

$$\begin{aligned}(1 + 2 + \cdots + 99 + 100)^2 &= 5050^2 = 25502500\\ 1^2 + 2^2 + \cdots + 99^2 + 100^2 &= 1 + 4 + \cdots + 9801 + 10000 = 338350\\ 25502500 - 338350 &= 25164150\end{aligned}$$

Python Implementation

```
def sum_numbers(n: int) -> int:
    return n * (n + 1) // 2

def sum_square_numbers(n: int) -> int:
    return n * (2 * n + 1) * (n + 1) // 6

def p6(n: int) -> int:
    return sum_numbers(n) ** 2 - sum_square_numbers(n)

print(p6(100))
```

Using Algebra

To reduce the potential for errors, we will derive a formula that can be used to calculate the result directly.

The triangle numbers are a well known sequence formed by adding the first n integers:

$$\begin{aligned} T_1 &= 1 \\ T_2 &= 1 + 2 = 3 \\ T_3 &= 1 + 2 + 3 = 6 \\ &\vdots \end{aligned}$$

The formula for a given triangle number is also well known:

$$T_n = \sum_{k=1}^{n} k = \frac{n}{2}(n+1)$$

Previously we examined the sum of squares sequence and determined that the formula for a given term in that sequence is:

$$S_n = \sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$$

Combining these formulae together will allow us to derive a new formula that we can use to calculate the desired result:

$$\begin{aligned}
\text{Let: } A_n &= (1+2+\cdots+n)^2 - (1^2+2^2+\cdots+n^2) \\
&= \left(\sum_{k=1}^{n} k\right)^2 - \left(\sum_{k=1}^{n} k^2\right) \\
&= \left(\frac{n}{2}(n+1)\right)^2 - \frac{1}{6}n(n+1)(2n+1) \\
&= \frac{n^2}{4}(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) \\
&= \frac{n}{12}\big(3n(n^2+2n+1) - 2(2n^2+3n+1)\big) \\
&= \frac{n}{12}\big(3n^3+2n^2-3n-2\big) \\
&= \frac{n}{12}(n-1)(n+1)(3n+2)
\end{aligned}$$

The question is asking for the 100th term. Plugging that in gives the desired result:

$$\begin{aligned}
A_{100} &= \frac{100}{12}(100-1)(100+1)(3\cdot 100+2) \\
&= \frac{100}{12}\cdot 99\cdot 101\cdot 302 \\
&= 25164150
\end{aligned}$$