Cyber Physical Systems - Discrete Models
Exercise Sheet 12 Solution

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Exercise 2: LTL Properties

(a)

$$\varphi_1 = a \land \bigcirc b : \tau = \{a\}\{b\}^\omega \vDash \varphi_1$$
$$\varphi_2 : \tau = \{a\}\{a\}\{a\}\{b\}^\omega$$
$$\varphi_3 : \tau = \{a\}\{a\}\{b\}\{a\}^\omega$$
$$\varphi_4 : \tau = \{b\}\{b\}\{c\}\{a\}^\omega$$
$$\varphi_5 : \tau = \{c\}\{c\}\{a\}^\omega$$
$$\varphi_6 : \tau = \{b\}\{b\}(\{a\}\{c\})^\omega$$

(b)

$$\neg\varphi_1 : \tau = \{a\}^\omega$$
$$\neg\varphi_2 : \tau = \{c\}^\omega$$
$$\neg\varphi_3 : \tau = \{a\}\{b\}^\omega$$
$$\neg\varphi_4 : \tau = \{c\}^\omega$$
$$\neg\varphi_5 : \tau = (\{b\}\{a\})^\omega$$
$$\neg\varphi_1 : \tau = \{c\}\{a\}^\omega$$

(c)

Let T be the Transition System

- $T \nvDash \varphi_1$. Counterexample: $\text{trace}(s_0 s_2 ...) = \{b\}\{a\}...$
- $T \vDash \varphi_2$. Because first trace is $\{b\}\{a\}...$ which immediately starts with b therefore satisfies and the second trace is $\{a,c\}\{a\}\{a,b\}...$ which also contains a until b.
- $T \nvDash \varphi_3$. Counterexample: $\text{trace}(s_1 s_2 s_3^\omega) = \{a,c\}\{a\}\{a,b\}^\omega$. Which satisfies $a \cup \Box b$ therefore violates φ_3.

- $T \nvDash \varphi_4$. Counterexample: $\text{trace}(s_0 s_2 s_3) = \{b\}\{a\}\{a,b\}^\omega$ doesn't contain a in the initial state and also there is no eventually c for the first state. Therefore it is not in $\text{Words}(\varphi_4)$.
- $T \vDash \varphi_5$. The infinite parts of each trace satisfies "always a". Therefore, all traces are in $\text{Words}(\varphi_5)$.
- $T \nvDash \varphi_6$. Counterexample: $\text{trace}(s_0 s_2 s_3^\omega) = \{b\}\{a\}\{a,b\}^\omega$ doesn't have c at all. Therefore, "eventually c" can't be satisfied.

(d)

$\text{Words}(\varphi_1) =$

$\left\{A_0 A_1 ... \in \left(2^{\text{AP}}\right)^\omega \mid a \in A_0 \land b \in A_1\right\}$

$\text{Words}(\varphi_2) =$

$\left\{A_0 A_1 ... \in \left(2^{\text{AP}}\right)^\omega \mid \exists i \in \mathbb{N}.\ \left(\forall j < i.\ a \in A_j\right) \land b \in A_i\right\}$

$\text{Words}(\varphi_3) =$

$\left\{A_0 A_1 ... \in \left(2^{\text{AP}}\right)^\omega \mid \forall i \in \mathbb{N}.\ \left(\exists j < i.\ a \notin A_j\right) \lor \left(\exists j \geq i.\ b \notin A_j\right)\right\}$

$\text{Words}(\varphi_4) =$

$\left\{A_0 A_1 ... \in \left(2^{\text{AP}}\right)^\omega \mid \exists i \in \mathbb{N}.\ (\forall j < i.\ (\exists k \geq j.\ c \in A_k)) \land \left(\forall j \geq i.\ a \in A_j\right)\right\}$

$\text{Words}(\varphi_5) =$

$\left\{A_0 A_1 ... \in \left(2^{\text{AP}}\right)^\omega \mid \exists i \in \mathbb{N}.\ \left(\forall j \geq i.\ a \in A_j\right)\right\}$

$\text{Words}(\varphi_6) =$

$\left\{A_0 A_1 ... \in \left(2^{\text{AP}}\right)^\omega \mid \forall i \in \mathbb{N}.\ \exists j \geq i.\ c \in A_j\right\}$

Exercise 3: Stating properties in LTL

$\varphi_a = \neg\ \text{Peter.use} \lor \neg\ \text{Betsy.use}$

$\varphi_b = (\Diamond\Box\neg\ \text{Peter.use}) \land (\Diamond\Box\neg\ \text{Betsy.use})$

$\varphi_c = (\text{Peter.request} \rightarrow \Diamond\ \text{Peter.use}) \land (\text{Betsy.request} \rightarrow \Diamond\ \text{Betsy.use})$

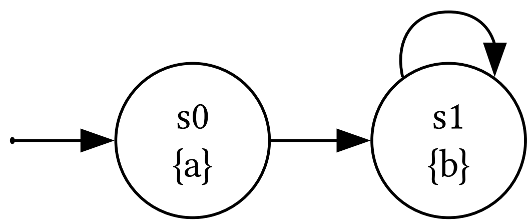
$\varphi_d = (\Box\Diamond\ \text{Peter.request} \rightarrow \Box\Diamond\ \text{Peter.use}) \land (\Box\Diamond\ \text{Betsy.request} \rightarrow \Box\Diamond\ \text{Betsy.use})$

$\varphi_e = (\text{Peter.use} \rightarrow (\neg\ \text{Peter.use}) \cup \text{Betsy.use}) \land (\text{Betsy.use} \rightarrow (\neg\ \text{Betsy.use}) \cup \text{Peter.use})$

Exercise 4: Equivalence of LTL formulas

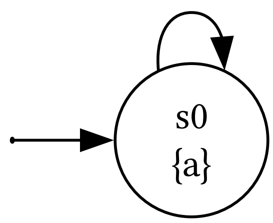
Note: Atomic propositions of the Transition System are notated under the state name.

- $\square a \land \bigcirc \Diamond a \stackrel{?}{\equiv} \square a = \text{true}$
- $\Diamond a \land \bigcirc \square a \stackrel{?}{\equiv} \Diamond a = \text{false}$. Counter example TS:



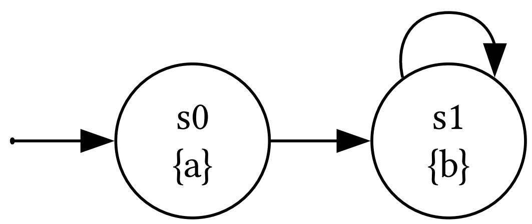
satisfies $\Diamond a$ but not for $\bigcirc \square a$

- $\square a \to \Diamond b \stackrel{?}{\equiv} a \cup (b \lor \neg a) = \text{true}$.
- $a \cup \text{false} \stackrel{?}{\equiv} \square a = \text{false}$. Counter example TS:



satisfies $\square a$ but not for $a \cup \text{false}$.

- $\square \bigcirc b \stackrel{?}{\equiv} \square b = \text{false}$. Counter example:



satisfies $\square \bigcirc b$ but not for $\square b$.

Proofs

Proof 1: $\square a \land \bigcirc \Diamond a \equiv \square a$
Assuming $\text{Words}(\square a) \subseteq \text{Words}(\bigcirc \square a)$, $\square a \land \bigcirc \Diamond a \equiv \square a$ because intersection with a subset results with the subset.

Proving $\text{Words}(\square a) \subseteq \text{Words}(\bigcirc \Diamond a)$:

$$\mathrm{Words}(\Box a) = \left\{A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i \in \mathbb{N}.\ a \in A_i\right\}$$

$$\mathrm{Words}(\bigcirc \Diamond a) = \left\{A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i > 0.\ \exists j \geq i.\ a \in A_i\right\}$$

Let $\sigma \in \mathrm{Words}(\Box a)$. $\sigma \in \mathrm{Words}(\bigcirc \Diamond a)$ because for any σ, we can take $i = 1$ and $j = 1$ which contains a and therefore $\sigma \vDash \bigcirc \Diamond a$.

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Proof 2: $\Box a \to \Diamond b \equiv a \cup (b \vee \neg a)$
$a \cup (b \vee \neg a) \equiv (\mathrm{true} \cup (b \vee \neg a))$, because a must necessarily hold until $b \vee \neg a$ occurs otherwise $b \vee \neg a$ would hold earlier. Also $\mathrm{true} \cup (b \vee \neg a) \equiv \Diamond(b \vee \neg a)$ from the definition of $\Diamond$ operator.

For $\Box a \to \Diamond b$:

$$\begin{aligned}
\Box a \to \Diamond b &\equiv \neg \Box a \vee \Diamond b \\
&\equiv \Diamond \neg a \vee \Diamond b \\
&\equiv \Diamond(\neg a \vee b)
\end{aligned}$$

Since both equations are equivalent for another LTL formula they are equivalent to each other as well.

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