Cyber Physical Systems - Discrete Models
Exercise Sheet 10 Solution

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Exercise 1: Invariants

A

Proposition: Let $E \subseteq \left(2^{\mathrm{AP}}\right)^{\omega}$ be an LT property. E is not an invariant if and only if there exists a trace $\sigma = A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^{\omega}$ such that $\sigma \notin E$, but for every $i \in \mathbb{N}$, the set A_i also occurs some other trace $\pi_i \in E$.

Proof: Assume that E is not an invariant and there exists a trace $\sigma = A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^{\omega}$ such that $\sigma \notin E$, but for every $i \in \mathbb{N}$, the set A_i also occurs some other trace $\pi_i \in E$.

Let Φ be the invariant condition of E then, by the definition of invariant we can conclude $\forall i \in \mathbb{N} \cdot \forall \sigma' \in \pi_i \cdot \sigma' \vDash \Phi$. Because $\sigma \notin E$, it means that $\exists i \in \mathbb{N} \cdot A_i \nvDash \Phi$. And there exists a corresponding $A_i \in \pi_i \wedge \pi_i \in E$. This means that $A_i \vDash \Phi$ which is a contradiction. So E must be an invariant given the properties it exhibits ■

B

E_1

$$E_1 = \left\{A_0A_1... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i \in \mathbb{N} \cdot \left(a \in A_i \to b \in A_{i+1}\right)\right\}$$

Let $w = A_0A_1... = a^\omega$ and $p_1 = \mathrm{ab}^\omega$. Clearly $w \notin E_1$ but $p_1 \in E_1$. Also $\forall i \in \mathbb{N} \cdot A_i = a$ and $a \in p_1$, hence E_1 is not an invariant.

E_2

$$E_2 = \left\{A_0A_1... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i, j \in \mathbb{N} \cdot A_i = A_j\right\}$$

$E = \{a^\omega, b^\omega\}$. Let $p_2 = A_0A_1... = a(b^\omega)$, $p_2 \notin E$. Since $A_0 \in a^\omega$ and $\forall i > 0 \cdot A_j = b \land A_j \in b^\omega$, all sets either contained in a^ω or b^ω. Therefore E_2 is not an invariant.

E_3

$$E_3 = \left\{A_0A_1... \in \left(2^{\mathrm{AP}}\right)^\omega \mid |\{\forall i \in \mathbb{N} \mid a \in A_i\}| \geq 2\right\}$$

Let $\pi_3 = aa(b^\omega) \in E_3$ and $\sigma = aaa(b^\omega) \notin E_3$. Both a and b is in π_3 and they also in σ. Therefore, E_3 is not an invariant.

Exercise 2: LT Properties

P_1

Part A

$$P_1 = \left\{A_0A_1... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i \in \mathbb{N} \cdot a \in A_i \lor b \in A_i\right\}$$

Part B

It's an invariant with invariant condition $\Phi = a \lor b$.

Part C

Since every invariant is a safety property, this is also a safety property. Set of bad prefixes can be denoted as
$\mathrm{BadPref} = \left\{A_0A_1...A_n \in \left(2^{\mathrm{AP}}\right)^+ \mid \exists i \in 0..n \cdot \emptyset = A_i\right\}$

Part D

It's not a liveness property, because P_1 contains prefixes that can't be extended to satisfy the language. For example $\sigma = \{a\}\emptyset\{a\}$ can't be extended so that it would satisfy the langage.

P_2

Part A

$$P_2 = \left\{A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^\omega \mid (|\{i \in \mathbb{N} \cdot a \in A_i\}| = 1) \lor (\forall i \in \mathbb{N} \cdot b \notin A_i)\right\}$$

Part B

P_2 is not an invariant since there is no such Φ that we can check for individual states.

Part C

P_2 is a safety property because once the condition is violated in a prefix, it can't be extended to satisfy it. It has the bad prefixes

$$\mathrm{BadPref} = \left\{A_0 A_1 ... A_n \in \left(2^{\mathrm{AP}}\right)^+ \mid (|\{i \in 0..n \cdot a \in A_i\}| > 1) \land (\exists i \in 0..n \cdot b \in A_i)\right\}$$

Part D

P_2 is not a liveness property because it contains prefixes that can't be added into language by appending some trace. For example: $\sigma = \{a\}\{a, b\}$.

P_3

The wording "b will never hold in the next step" is ambiguous. It's not clear if b doesn't hold in the next (subsequent) step or for all later steps. I am assuming that it's only the next step.

Part A

$$P_3 = \left\{A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i \in \mathbb{N} \cdot (a \in A_i \to b \notin A_{i+1})\right\}$$

Part B

P_3 is not an invariant because the constraint involves subsequent steps. Therefore, it's not possible to write a propositional logic formula Φ that would be evaluated for each step.

Part C

P_3 is a safety property because it has bad prefixes. Set of bad prefixes are:

$$\mathrm{BadPref} = \left\{A_0 A_1 ... A_n \in \left(2^{\mathrm{AP}}\right)^+ \mid \exists i \in 1..n \cdot a \in A_{i-1} \land b \in A_i\right\}$$

Part D

P_3 is not a liveness property because there are bad prefixes for this language. Those bad prefixes can't be extended to satisfy the language, so the language doesn't satisfy of the condition of liveness properties having $\left(2^{\mathrm{AP}}\right)^{+}$ as the prefix set.

P_4

Part A

$$P_4 = \left\{A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^{\omega} \mid \forall i \in \mathbb{N} \cdot \left(a \in A_i \rightarrow \left(\exists j \geq i \cdot b \in A_j\right)\right)\right\}$$

Part B

P_4 is not an invariant because the language constraint involves multiple steps to check. Therefore, it's not possible to write a propositional logic formula Φ that would be evaluated for each step.

Part C

P_4 is not a safety property, because for any prefix $\sigma \in \left(2^{\mathrm{AP}}\right)^{+}$ we can append $w = A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^{\omega} \cdot \left(\forall i \in \mathbb{N} \cdot \left(a \in A_i \rightarrow b \in A_{i+1}\right)\right)$ which means $\sigma w \in P_4$. Hence $\mathrm{BadPref} = \emptyset$.

Part D

P_4 is a liveness property because as explained in Part C we can extend any finite prefix $\sigma \in \left(2^{\mathrm{AP}}\right)^{+}$ with a trace $w \in \left(2^{\mathrm{AP}}\right)^{\omega}$ so that $\sigma w \in P_4$.

P_5

Part A

$$P_5 = \left\{A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^{\omega} \mid \forall i \in \mathbb{N} \cdot \{a, b\} \neq A_i\right\}$$

Part B

P_5 is an invariant with the invariant condition $\Phi = \neg(a \wedge b)$.

Part C

Since P_5 is an invariant, it's automatically a safety property. The set of bad prefixes are:

$$\mathrm{BadPref} = \left\{A_0 A_1 \ldots A_n \in \left(2^{\mathrm{AP}}\right)^{+} \mid \exists i \in 0..n \cdot \{a, b\} = A_i\right\}$$

Part D

Since P_5 is a safety property, it can't be a liveness property. A counter example is prefix $\sigma = \{a, b\}$. Because for any $\forall w \in \left(2^{\mathrm{AP}}\right)^{\omega} \cdot \sigma w \notin P_5$.

P_6

Part A

$$P_6 = \left\{A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^\omega \mid \left(\overset{\infty}{\exists} i \in \mathbb{N} \cdot a \in A_i\right) \to \left(\overset{\infty}{\exists} i \in \mathbb{N} \cdot b \in A_i\right)\right\}$$

Part B

P_6 is not an invariant because condition requires checking multiple steps at the same time. Therefore there is no boolean proposition formula Φ to check for a single step.

Part C

P_6 is not a safety property, because for any bad prefix $\sigma \in \left(2^{\mathrm{AP}}\right)^+$ we can append $w = A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^\omega \mid \overset{\infty}{\exists} i \in \mathbb{N} \cdot b \in A_i$ which means $\sigma w \in P_6$. Hence $\mathrm{BadPref} = \emptyset$.

Part D

P_6 is a liveness property because as explained in Part C we can extend any finite prefix $\sigma \in \left(2^{\mathrm{AP}}\right)^+$ with a trace $w \in \left(2^{\mathrm{AP}}\right)^\omega$ so that $\sigma w \in P_6$.

P_7

Part A

$$P_7 = \left\{A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^\omega \mid \exists i \in \mathbb{N} \cdot \forall j > i \cdot a \notin A_j\right\}$$

Part B

P_7 is not an invariant because the condition involves checking multiple steps at the same time. Therefore there is no boolean proposition formula Φ to check for a single step.

Part C

P_7 is not a safety property, because for any bad prefix $\sigma \in \left(2^{\mathrm{AP}}\right)^+$ we can append $w = A_0 A_1 \ldots \in \left(2^{\mathrm{AP}}\right)^\omega \mid \forall i \in \mathbb{N} \cdot a \notin A_i$ which means $\sigma w \in P_7$. Hence $\mathrm{BadPref} = \emptyset$.

Part D

P_6 is a liveness property because as explained in Part C we can extend any finite prefix $\sigma \in \left(2^{\mathrm{AP}}\right)^+$ with a trace $w \in \left(2^{\mathrm{AP}}\right)^\omega$ so that $\sigma w \in P_7$.

P_8

Part A

$$P_8 = \left\{A_0 A_1 ... \in \left(2^{\mathrm{AP}}\right)^\omega \mid \mathrm{true}\right\}$$

Part B

P_8 is an invariant with the invariant condition $\Phi = \mathrm{true}$.

Part C

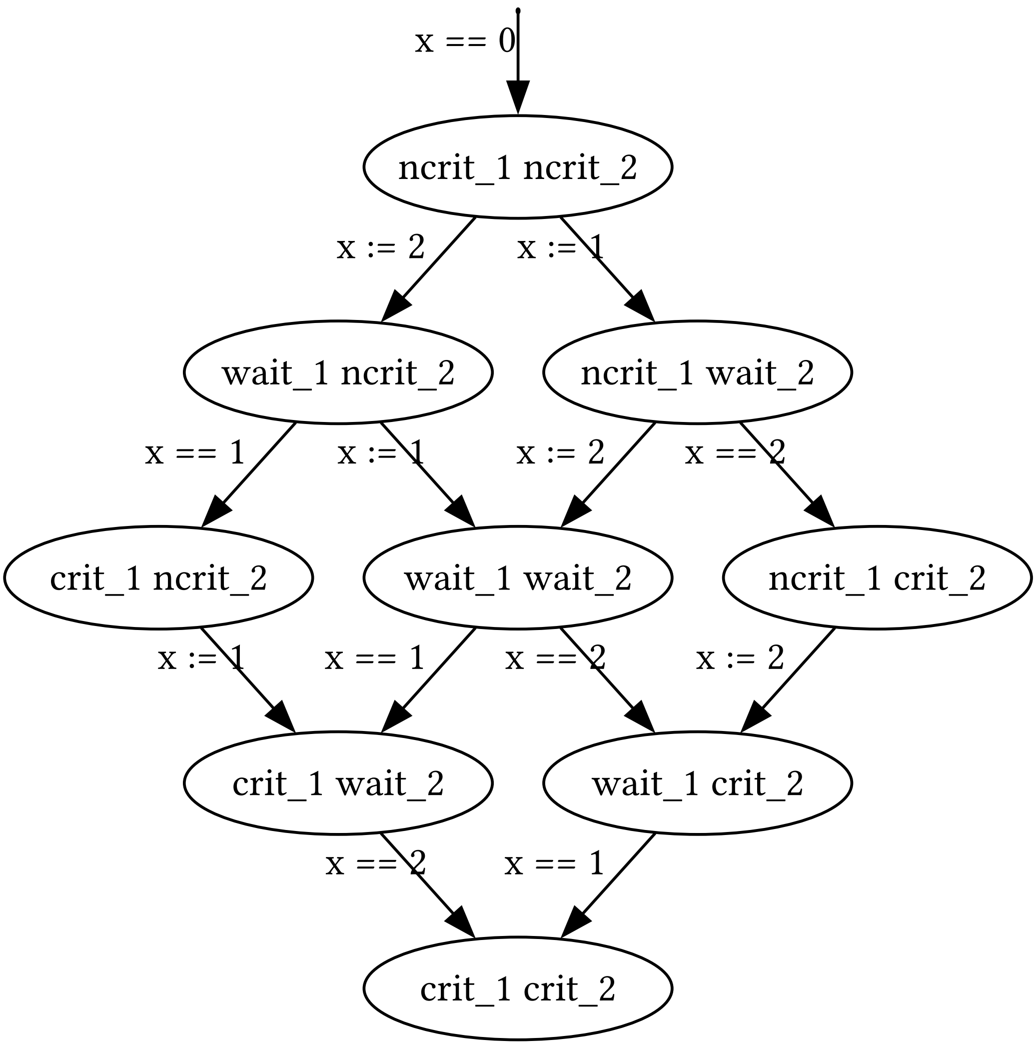
P_8 is a safety property, because even if it doesn't have any bad prefixes it doesn't have any traces that is not in the language either. So it doesn't need to have any bad prefixes.

Part D

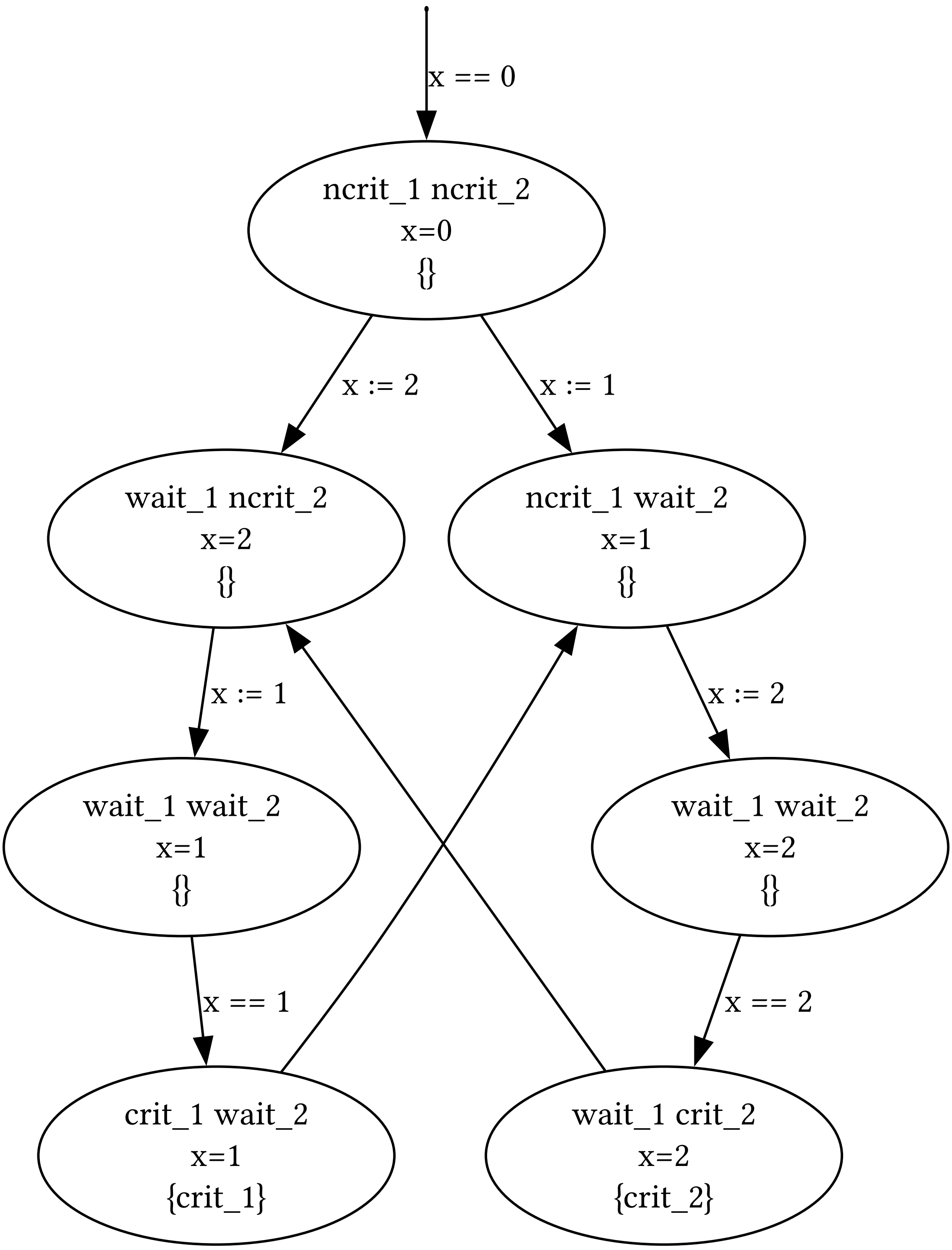
P_8 is a liveness property, because $\mathrm{pref}(P_8) = \left(2^{\mathrm{AP}}\right)^+$.

Exercise 3: Mutual Exclusion

Part A



Part B



Part C

Yes because in all states, invariatn $\Phi = \neg \text{crit}_1 \vee \neg \text{crit}_2$ is satisfied.

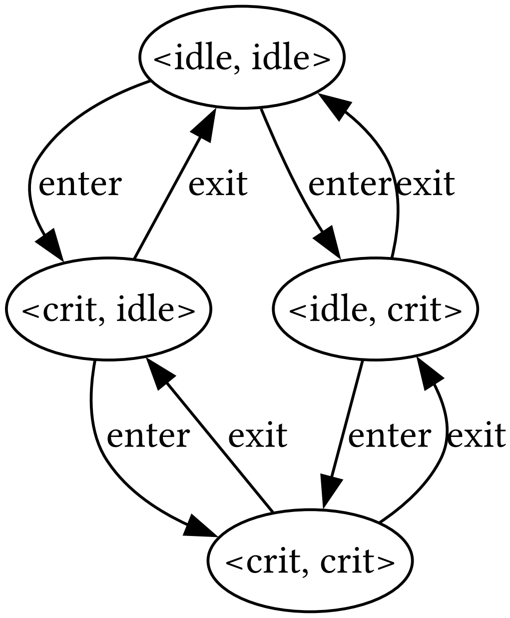
Part D

Yes, because both mutual exclusion and fairness is satisfied in this TS. Fairness is satisfied because the system forces alternating sequences of critical sequence entrance for both programs.

Exercise 4: Mutual Exclusion without Request

Part A

$\mathrm{TS}_1 \ ||| \ \mathrm{TS}_2$



Transitions

$$\langle \text{idle}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{idle} \rangle : \frac{\text{idle} \xrightarrow{\text{enter}}_1 \text{crit}}{\langle \text{idle}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{idle} \rangle} : \text{SOS}_1$$

$$\langle \text{idle}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{idle}, \text{crit} \rangle : \frac{\text{idle} \xrightarrow{\text{enter}}_2 \text{crit}}{\langle \text{idle}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{idle}, \text{crit} \rangle} : \text{SOS}_2$$

$$\langle \text{crit}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{idle}, \text{idle} \rangle : \frac{\text{crit} \xrightarrow{\text{exit}}_1 \text{idle}}{\langle \text{crit}, \text{idle} \rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{idle} \rangle} : \text{SOS}_1$$

$$\langle \text{idle}, \text{crit} \rangle \xrightarrow{\text{enter}} \langle \text{idle}, \text{idle} \rangle : \frac{\text{crit} \xrightarrow{\text{exit}}_2 \text{idle}}{\langle \text{idle}, \text{crit} \rangle \xrightarrow{\text{exit}} \langle \text{crit}, \text{idle} \rangle} : \text{SOS}_2$$

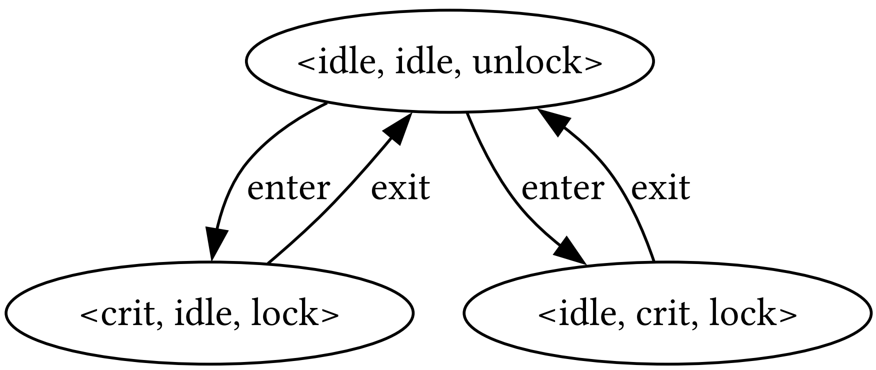
$$\langle \text{crit}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{crit} \rangle : \frac{\text{idle} \xrightarrow{\text{enter}}_2 \text{crit}}{\langle \text{crit}, \text{idle} \rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{crit} \rangle} : \text{SOS}_2$$

$$\langle \text{idle}, \text{crit} \rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{crit} \rangle : \frac{\text{idle} \xrightarrow{\text{enter}}_1 \text{crit}}{\langle \text{idle}, \text{crit} \rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{crit} \rangle} : \text{SOS}_1$$

$$\langle \text{crit}, \text{crit} \rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{crit} \rangle : \frac{\text{crit} \xrightarrow{\text{exit}}_1 \text{idle}}{\langle \text{crit}, \text{crit} \rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{crit} \rangle} : \text{SOS}_1$$

$$\langle \text{crit}, \text{crit} \rangle \xrightarrow{\text{exit}} \langle \text{crit}, \text{idle} \rangle : \frac{\text{crit} \xrightarrow{\text{exit}}_2 \text{idle}}{\langle \text{crit}, \text{crit} \rangle \xrightarrow{\text{exit}} \langle \text{crit}, \text{idle} \rangle} : \text{SOS}_2$$

Part B



Transitions

All relations are formed via SOS_3, which is the rule:

$$\frac{s \xrightarrow[1]{\alpha} s' \wedge q \xrightarrow[2]{\alpha} q'}{\langle s, q\rangle \xrightarrow{\alpha} \langle s', q'\rangle}$$

$$\langle \text{idle}, \text{idle}, \text{unlock}\rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{idle}, \text{lock}\rangle : \frac{\langle \text{idle}, \text{idle}\rangle \xrightarrow{\text{enter}}_1 \langle \text{crit}, \text{idle}\rangle \wedge \text{unlock} \xrightarrow{\text{enter}}_2 \text{lock}}{\langle \text{idle}, \text{idle}, \text{unlock}\rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{idle}, \text{lock}\rangle} : \text{SOS}_3$$

$$\langle \text{idle}, \text{idle}, \text{unlock}\rangle \xrightarrow{\text{enter}} \langle \text{idle}, \text{crit}, \text{lock}\rangle : \frac{\langle \text{idle}, \text{idle}\rangle \xrightarrow{\text{enter}}_1 \langle \text{crit}, \text{idle}\rangle \wedge \text{unlock} \xrightarrow{\text{enter}}_2 \text{lock}}{\langle \text{idle}, \text{idle}, \text{unlock}\rangle \xrightarrow{\text{enter}} \langle \text{crit}, \text{idle}, \text{lock}\rangle} : \text{SOS}_3$$

$$\langle \text{idle}, \text{crit}, \text{lock}\rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{idle}, \text{unlock}\rangle : \frac{\langle \text{idle}, \text{crit}\rangle \xrightarrow{\text{exit}}_1 \langle \text{idle}, \text{idle}\rangle \wedge \text{lock} \xrightarrow{\text{exit}}_2 \text{unlock}}{\langle \text{idle}, \text{crit}, \text{lock}\rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{idle}, \text{unlock}\rangle} : \text{SOS}_3,$$

$$\langle \text{crit}, \text{idle}, \text{lock}\rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{idle}, \text{unlock}\rangle : \frac{\langle \text{crit}, \text{idle}\rangle \xrightarrow{\text{exit}}_1 \langle \text{idle}, \text{idle}\rangle \wedge \text{lock} \xrightarrow{\text{exit}}_2 \text{unlock}}{\langle \text{crit}, \text{idle}, \text{lock}\rangle \xrightarrow{\text{exit}} \langle \text{idle}, \text{idle}, \text{unlock}\rangle} : \text{SOS}_3,$$

Exercise 5: Hardware Circuit

$$r' = x \wedge (\neg r)$$

$$y = x \vee r$$