Cyber Physical Systems - Discrete Models
Exercise Sheet 8 Solution

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December 10, 2023

Exercise 1: Prefixes and Closure I

Part A

$$P \subseteq \mathrm{cl}(P)$$

Solution

Let $\omega \in P$, then $\mathrm{pref}(\omega) \subseteq \mathrm{pref}(P)$ trivially. Since $\mathrm{pref}(\omega) \subseteq \mathrm{pref}(P)$ is the predicate for closure, then we can conclude $\forall \omega \in P \to \omega \in \mathrm{cl}(P)$, thefore $P \subseteq \mathrm{cl}(P)$.

Part B

$$\mathrm{pref}(\mathrm{cl}(P)) = \mathrm{pref}(P)$$

Solution

If we prove both $\mathrm{pref}(\mathrm{cl}(P)) \subseteq \mathrm{pref}(P)$ and $\mathrm{pref}(P) \subseteq \mathrm{pref}(\mathrm{cl}(P))$, then we can conclude $\mathrm{pref}(\mathrm{cl}(P)) = \mathrm{pref}(P)$.

Direction 1: $\mathrm{pref}(\mathrm{cl}(P)) \subseteq \mathrm{pref}(P)$
Let $\omega \in \mathrm{pref}(\mathrm{cl}(P))$, then $\exists \sigma \in \mathrm{cl}(P) \to \omega \in \mathrm{pref}(\sigma)$. By the definition of closure, $\forall \sigma \in \mathrm{cl}(P) \to \mathrm{pref}(\sigma) \subseteq \mathrm{pref}(P)$. Hence, $w \in \mathrm{pref}(P)$. Which concludes that $\forall \omega \in \mathrm{pref}(\mathrm{cl}(P)) \to \omega \in \mathrm{pref}(P)$.

Direction 2: $\mathrm{pref}(P) \subseteq \mathrm{pref}(\mathrm{cl}(P))$
Let $\omega \in \mathrm{pref}(P)$, then $\exists \sigma \in P \to \omega \in \mathrm{pref}(\sigma)$. By proof in part a, we can claim $\forall \sigma \in P \to \sigma \in \mathrm{cl}(P)$. Therefore, $\omega \in \mathrm{pref}(P) \to \omega \in \mathrm{pref}(\mathrm{cl}(P))$.

Exercise 2: Prefixes and Closure II

Part A

$$P_1 = \{A_0A_1... \mid \exists S \subseteq N \cdot (|S| = 1 \land \forall i \in S a \in A_i)\}$$

$$P_2 = \left\{A_0A_1... \mid \forall i \cdot \left(a \in A_i \to b \in A_{i+1}\right)\right\}$$

$$P_3 = \left\{A_0A_1... \mid \exists i \cdot \left(\forall j \cdot \left(j \geq i \to a \notin A_j\right)\right)\right\}$$

$$P_4 = \left\{A_0A_1... \mid a \in A_0 \land \overset{\infty}{\exists} i \cdot a \in A_i\right\}$$

Part B

$$\text{pref}(P_1) = \{A_0A_1...A_k \mid \exists S \subseteq N \cdot (|S| \leq 1 \land \forall i \in S a \in A_i)\}$$

$$\text{pref}(P_2) = \left\{A_0A_1...A_k \mid \forall i \cdot \left(i < k \land a \in A_i \to b \in A_{i+1}\right)\right\}$$

$$\text{pref}(P_3) = \{A_0A_1...A_k \mid \text{true}\}$$

$$\text{pref}(P_4) = \{A_0A_1...A_k \mid a \in A_0\}$$

Part C

$$\text{cl}(P_1) = \{A_0A_1... \mid \exists S \subseteq N \cdot (|S| \leq 1 \land \forall i \in S a \in A_i)\}$$

$$\text{cl}(P_2) = \left\{A_0A_1... \mid \forall i \cdot \left(a \in A_i \to b \in A_{i+1}\right)\right\}$$

$$\text{cl}(P_3) = \{A_0A_1... \mid \text{true}\}$$

$$\text{cl}(P_4) = \{A_0A_1... \mid a \in A_0\}$$

Exercise 3: Safety & Liveness Properties

Part A

P_1
- Is an invariant.
- Invariation condition: $a \notin S$.

P_2
- Not an invariant.
- Example trace: $\left(w = \{a\}^{\omega}\right) \notin P_2$. However, $\forall \sigma \in w \cdot \sigma = \{a\}$, and we can give trace $\{a\}\{b\}^{\omega} \in P_2$ as a counter example.

P_3
- Is an invariant.
- Invariant condition: $a \in S \to b \in S$.

P_4
- Not an invariant.
- Example trace: $\left(w = \{b\}^{\omega}\right) \notin P_4$. However, $\forall \sigma \in w \cdot \sigma = \{b\}$, and we can give trace $\{b\}\{a\}^{\omega} \in P_2$ as a counterexample.

Part B

P_1
- Is a safety property.
- Set of bad prefixes: $\mathrm{BadPref} = \{A_0 A_1 ... A_n \mid \exists i \in \{0, ..., n\} \cdot a \in A_i\}$.

P_2
- Not a safety property.
- Example trace: $\left(\sigma = \{a\}^{\omega}\right) \in \left(2^{\mathrm{AP}}\right)^{\omega} \setminus P_2$. But $\forall w \in \mathrm{pref}(\sigma)$, we can always extend it with $\sigma\{b\}^{w} \in P_2$ so no σ can be a bad prefix.

P_3
- Is a safety property.
- Set of bad prefixes:
  $\mathrm{BadPref} = \{A_0 A_1 ... A_n \mid \exists i \in \{0, ..., n\} \cdot b \in A_i \land a \notin A_i\}$.

P_4
- Is a safety property.
- Set of bad prefixes:
  $\mathrm{BadPref} = \{A_0 A_1 ... A_n \mid \exists S \subseteq \{0, ..., n\} \cdot |S| > 1 \land (\forall i \in S \cdot b \in S_i)\}$

Part C

P_1

- Is not a liveness property.
- Example bad prefix: $\{a\}$. $\forall w \in \left(2^{\mathrm{AP}}\right)^{\omega} \cdot (\{a\}w) \notin P_1$.

P_2

- Is a liveness property.
- We can extend any finite trace with $\{b\}^{\omega}$. $\forall w \in \left(2^{\mathrm{AP}}\right)^{*} \cdot w\{b\}^{\omega} \in P_2$.

P_3

- Is not a liveness property.
- Example bad prefix: $\{b\}$. $\forall w \in \left(2^{\mathrm{AP}}\right)^{\omega} \cdot (\{b\}w) \notin P_3$.

P_4

- Is not a liveness property.
- Example bad prefix: $\{b\}\{b\}$. $\forall w \in \left(2^{\mathrm{AP}}\right)^{\omega} \cdot (\{b\}\{b\}w) \notin P_4$.