ELECTRONIC ENGINEERING DEPARTMENT

MATH 214 NUMERICAL METHODS

2020 – 2021 FALL

PROJECT 5

Student ID :1901022039

Name Surname:Alperen Arslan

Preperation date: 6.01.2021

Upload date: 6.01.2021

1. Problem Definition and Formulation

The problem is plotting the graph for the given measured signal level data with using linear least squares polynomial method, least squares polynomial of degree two method and least squares method degree three method. After that error values must be calculated.

The least squares method basically aims to plot a polynomial graph which is as close as possible in every point of given data. Since the polynomial has more proper graph then the actual graph, it can give more reliable results and it helps for analyzing the data.

When the degree of the polynomial increases, it will be expected that the error decreases and the graph of polynomial is getting closer to actual curve.

The general formula of least squares method is given in equation system (1).

$$\begin{array}{l} a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + a_3 \sum_{i=1}^{m} x_i^3 + \cdots + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} x_i x_i^0 \\ a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + a_3 \sum_{i=1}^{m} x_i^4 + \cdots + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} x_i x_i^1 \\ \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \\ \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \\ \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \\ a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + a_3 \sum_{i=1}^{m} x_i^{n+3} + \cdots + a_n \sum_{i=1}^{m} x_i^{2n} = \sum_{i=1}^{m} x_i x_i^n \end{array} \tag{1}$$

Here x and y are the given data set, m is amount of data and n is degree of the polynomial.

The general formula of least squares polynomial is given in (2).

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0 \tag{2}$$

And the error is given in (3)

$$E = \sum_{i=1}^{m} (y_i - P_n(x_i))^2 \tag{3}$$

i. **Linear Least Squares Polynomial**

The degree of linear least squares polynomial is one, therefore n=1. Since n is 1, there is only two equation in this equation system.

$$a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 = \sum_{i=1}^{m} x_i x_i^0$$

$$a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i x_i^1 \tag{4}$$

The general of linear least squares polynomial method is given in equation system 4. The a_0 and a_1 coefficients can found by solving this system.

$$a_0 = \frac{(\sum_{i=1}^{m} x_i^2 \ . \sum_{i=1}^{m} y_i) - (\sum_{i=1}^{m} x_i y_i \, . \sum_{i=1}^{m} x_i)}{m(\sum_{i=1}^{m} x_i^2) - (\sum_{i=1}^{m} x_i)^2}$$

$$a_1 = \frac{m. \sum_{i=1}^{m} x_i y_i - (\sum_{i=1}^{m} y_i \, . \sum_{i=1}^{m} x_i)}{m(\sum_{i=1}^{m} x_i^2) - (\sum_{i=1}^{m} x_i)^2} \tag{5}$$

After finding the coefficients, the polynomial formula is given in (5)

$$P_1 = a_1 x + a_0 \tag{6}$$

ii. Least Squares Polynomial of Degree Two

Since the degree is two n is 2 and there are three equations. The Equation system is given in (7).

$$\begin{aligned}
a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 &= \sum_{i=1}^{m} x_i x_i^0 \\
a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 &= \sum_{i=1}^{m} x_i x_i^1 \\
a_0 \sum_{i=1}^{m} x_i^2 + a_1 \sum_{i=1}^{m} x_i^3 + a_2 \sum_{i=1}^{m} x_i^4 &= \sum_{i=1}^{m} x_i x_i^2
\end{aligned} \tag{7}$$

For solving this equation system, first, given system can turn into a matrix system.

$$\begin{bmatrix} \sum_{i=1}^{m} x_i^0 & \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 \\ \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 \\ \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 & \sum_{i=1}^{m} x_i^4 \end{bmatrix} . \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i x_i^0 \\ \sum_{i=1}^{m} x_i x_i^1 \\ \sum_{i=1}^{m} x_i x_i^2 \end{bmatrix} \tag{8}$$

$$M.a = B$$
$$a = b.M^{-1}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i x_i^0 \\ \sum_{i=1}^{m} x_i x_i^1 \\ \sum_{i=1}^{m} x_i x_i^2 \end{bmatrix} \cdot \begin{bmatrix} \sum_{i=1}^{m} x_i^0 & \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 \\ \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 \\ \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 & \sum_{i=1}^{m} x_i^4 \end{bmatrix}^{-1} \tag{9}$$

After a coefficients are found, the polynomial can be written as given below in (10).

$$P_2(x) = a_2 x^2 + a_1 x + a_0 \tag{10}$$

iii. Least Squares Polynomial of Degree Three

The degree is three, therefore n is 3 and there are 4 equations in this equation system.

$$\begin{aligned}
a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + a_3 \sum_{i=1}^{m} x_i^3 &= \sum_{i=1}^{m} x_i x_i^0 \\
a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + a_3 \sum_{i=1}^{m} x_i^4 &= \sum_{i=1}^{m} x_i x_i^1 \\
a_0 \sum_{i=1}^{m} x_i^2 + a_1 \sum_{i=1}^{m} x_i^3 + a_2 \sum_{i=1}^{m} x_i^4 + a_3 \sum_{i=1}^{m} x_i^5 &= \sum_{i=1}^{m} x_i x_i^2 \\
a_0 \sum_{i=1}^{m} x_i^3 + a_1 \sum_{i=1}^{m} x_i^4 + a_2 \sum_{i=1}^{m} x_i^5 + a_3 \sum_{i=1}^{m} x_i^6 &= \sum_{i=1}^{m} x_i x_i^3
\end{aligned} \tag{11}$$

If this equation system turned into a matrix, the a coefficients can be found.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i x_i^0 \\ \sum_{i=1}^{m} x_i x_i^1 \\ \sum_{i=1}^{m} x_i x_i^2 \\ \sum_{i=1}^{m} x_i x_i^3 \end{bmatrix} \cdot \begin{bmatrix} \sum_{i=1}^{m} x_i^0 & \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 \\ \sum_{i=1}^{m} x_i^1 & \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 & \sum_{i=1}^{m} x_i^4 \\ \sum_{i=1}^{m} x_i^2 & \sum_{i=1}^{m} x_i^3 & \sum_{i=1}^{m} x_i^4 & \sum_{i=1}^{m} x_i^5 \\ \sum_{i=1}^{m} x_i^3 & \sum_{i=1}^{m} x_i^4 & \sum_{i=1}^{m} x_i^5 & \sum_{i=1}^{m} x_i^6 \end{bmatrix}^{-1} \tag{12}$$

After the coefficients are founded, the formula of polynomial can be written as given below in (13).

$$P_3(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{13}$$

2. Code and Inputs

All Results and figures were generated using code which given in appendix section. The input data pr5data is imported and then, the sum of x, sum of x^2, sum of x^3, sum of x^4, sum of x^5, sum of x^6, sum of x^0y, sum of x^1y, sum of x^2y and sum of x^3y are founded by using given pr5data. After that the voltage versus distance graphs are plotted by using linear least squares polynomial formula which given in equation system (4), least squares polynomial of degree two which given in equation system (7) and least squares polynomial of degree three which given in equation system (11). Then, by solving this equation systems by multiplying by inverse method, the coefficients are found. After finding the coefficients the polynomials functions are wrote and the graphs which belongs these polynomials are plotted.

Finally an error analysis is made by using equation (3), the coefficients of polynomials and the error values are printed and all of the graphs plotted in same figure for better comparison.

When finding least squares method degree three, the code warned us: "Warning: Matrix is close to singular or badly scaled. Results may be inaccurate." This warning occurs because of inverse matrix. For calculating the inverse matrix, determinant must be calculated, and if a matrix's determinant is 0, this means that matrix doesn't have inverse. In this case, the determinant of matrix isn't zero but it is very close to zero, therefore there may occurs mistakes and the inverse of matrix may not calculated properly.

For solving this problem, in least squares polynomial of degree three, the gaussian elimination method could have used for solving the equation system instead of multiplying by inverse method. This would give us more reliable result since it is not required inverse matrix for solving an equation system.

3. Project Results and Discussions

A) Plotting the Real Graph

In this subsection, the graph is plotted with using given data pr5data for seeing the real graph as shown in Figure 1.

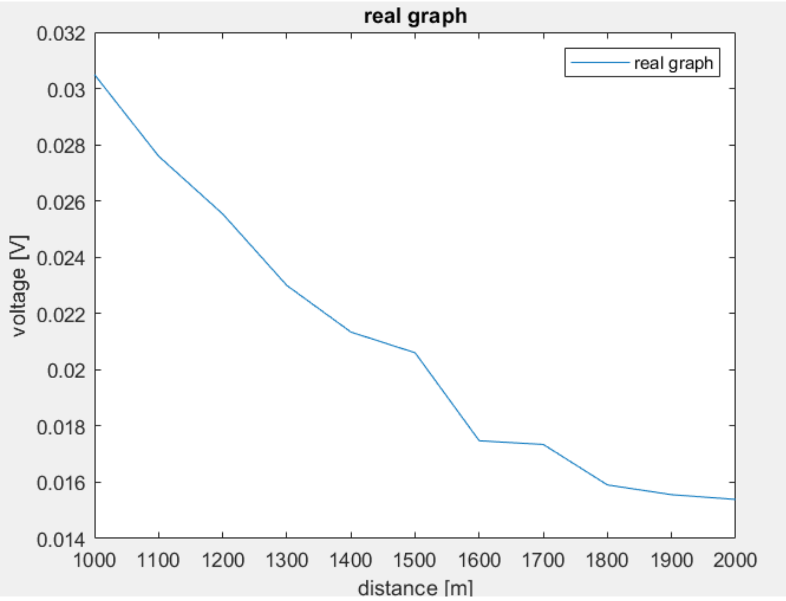

Figure 1. Real voltage versus distance graph

B) Plotting the Linear Least Squares Polynomial Graph

In this subsection, linear least squares polynomial graph was plotted as shown in Figure 2. Since the degree of polynomial is one, the curve is linear.

The line is close to real points, but since it's only degree one, there are relatively huge gaps between the line and the real graph at certain points.

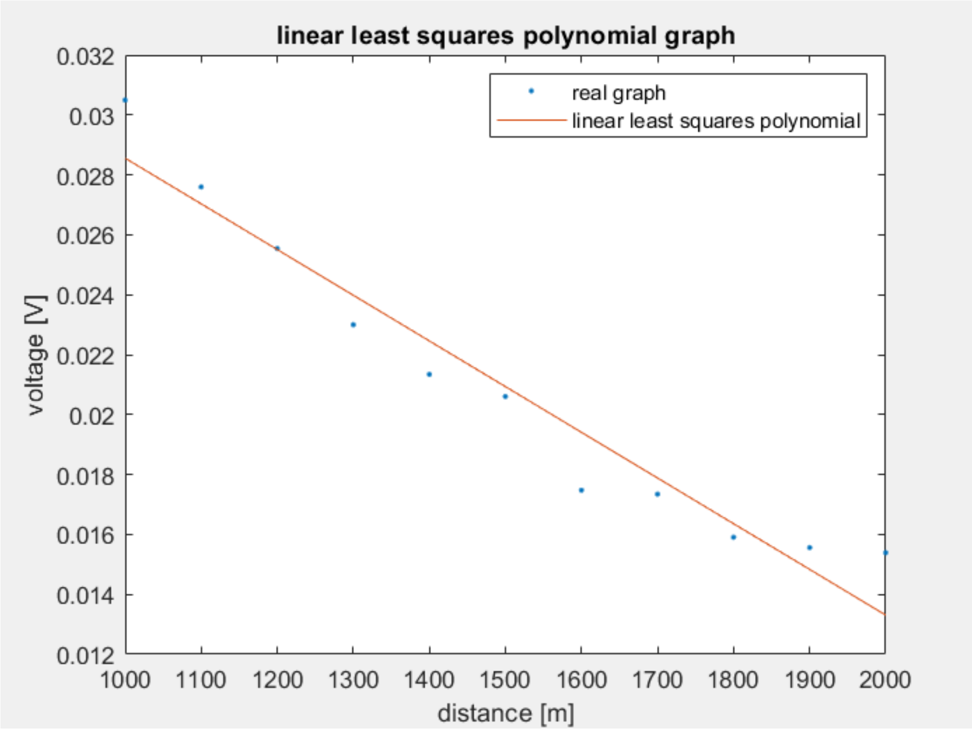

Figure 2. Linear least squares polynomial graph

The coefficients of linear least squares polynomial are:

$$a_0 = 0.043805655363636 \qquad a_1 = -0.000015249526545$$

And the polynomial is $P_1(x) = 0.043805655363636 - 0.000015249526545x$

C) Plotting the Least Squares Polynomial of Degree Two

In this subsection least squares polynomial of degree two was plotted as shown in Figure 3. Since the degree is two, now the curve is parabola instead of a line.

As expected, the accuracy of the curve is increases as the degree of the polynomial increases. Now, it looks like the curve is passes trough very close at every point of the real graph.

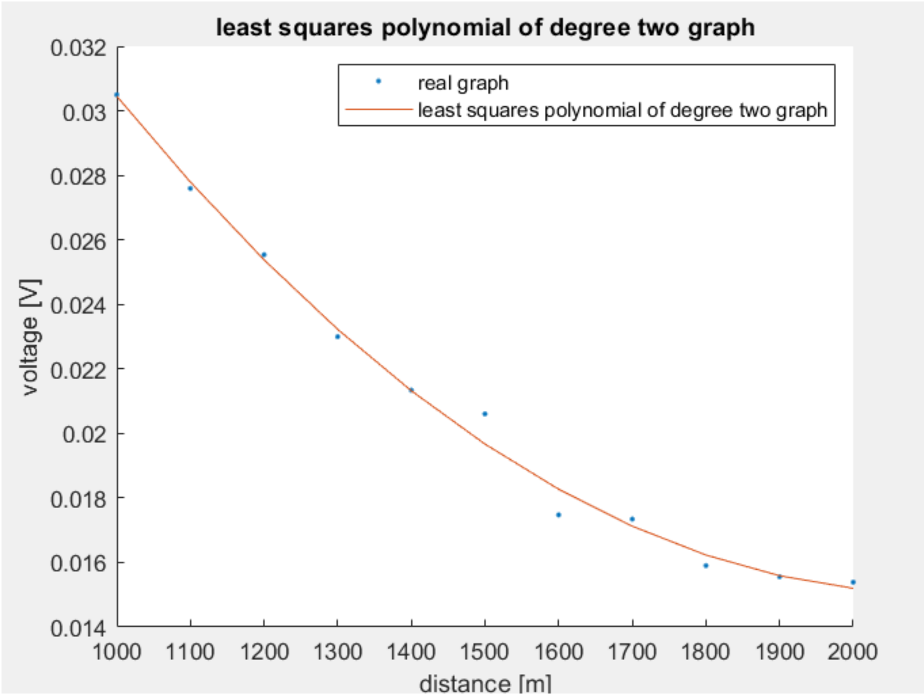

Figure 3. Least squares polynomial of degree two graph

The coefficients of the polynomial are:

$$a_0 = 0.070946826867136$$
$$a_1 = -0.000053120928643$$
$$a_2 = 0.000000012623801$$

And the polynomial is:

$$P_2(x) = 0.070946826867136 - 0.000053120928643\,x + 0.000000012623801\,x^2$$

D) Plotting the Least Squares Polynomial of Degree Three

In this subsection, least squares polynomial of degree three was plotted as shown in Figure 4. Unlike the difference between the linear and degree two, it is hard to tell the difference between degree two and degree three. They seem like the same.

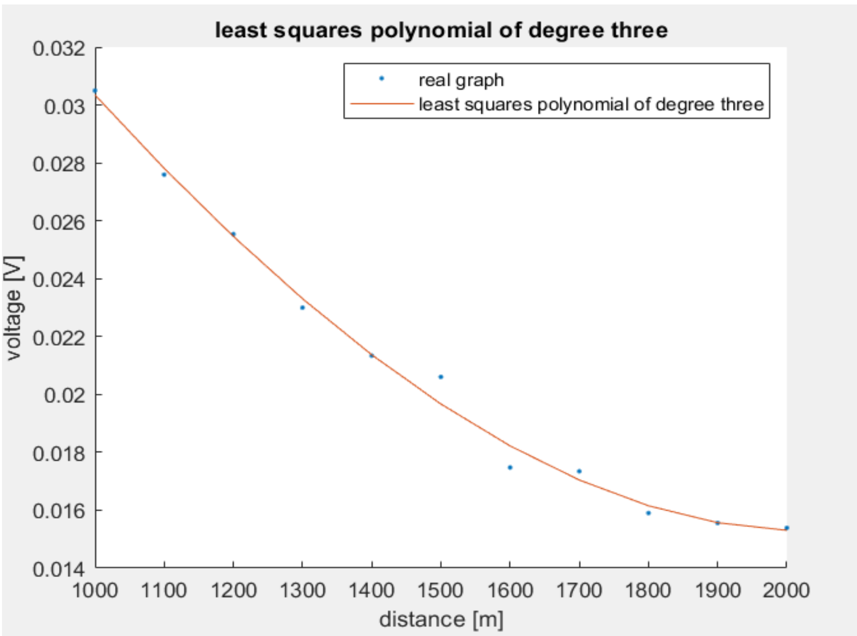

Figure 4. Least squares polynomial of degree three graph

The coefficients of the Least squares polynomial degree three are:

$$a_0 = 0.061828754205294$$
$$a_1 = -0.000033840370879$$
$$a_2 = -0.000000000578042$$
$$a_3 = 0.000000000002934$$

And the polynomial is:

$$P_3(x) = 0.061828754205294 - 0.000033840370879x - 0.000000000578042x^2 + 0.000000000002934x^3$$

E) Comparison of Graphs and Error Analysis

All of the curves are plotted at the same figure in Figure 5, for better comparison.

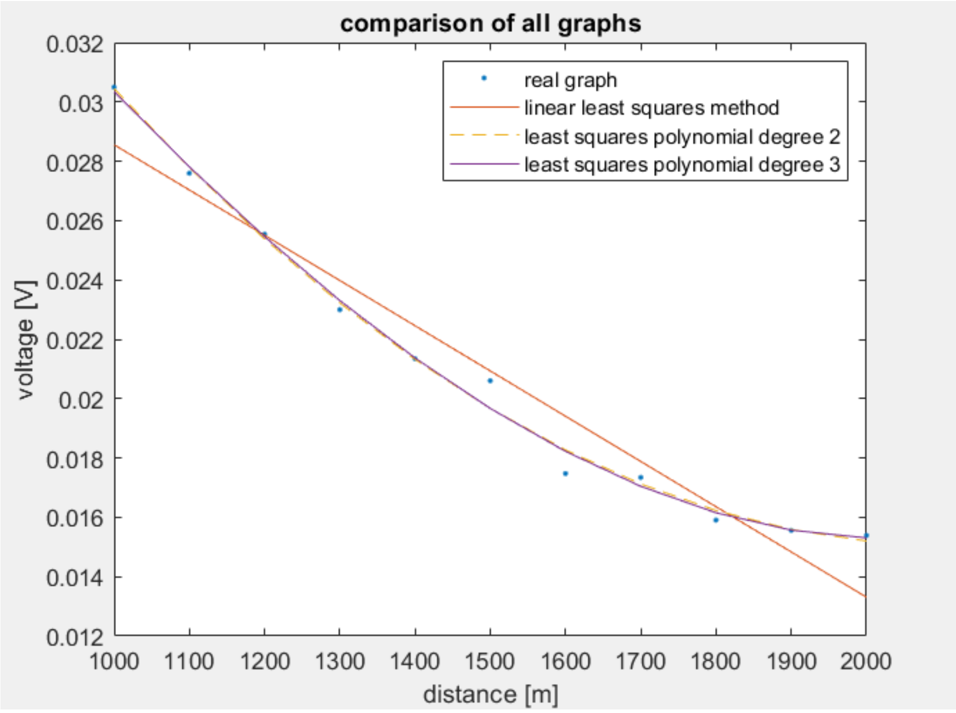

Figure 5. Comparison of all graphs

As shown in Figure 5, There is very tiny difference between the least squares polynomial of degree two and least squares polynomial of degree three. And they both seems more proper than linear least squares polynomial.

The general Error formula of the least squares polynomial method is given in equation (3). The result is given below when the formula is applied for each of the polynomials.

the error of linear least squares polynomial is 0.000015487670060
the error of least squares polynomial of degree 2 is 0.000001814552536
the error of least squares polynomial of degree 3 is 0.000001761382879

4. Conclusion

From analyzing error values, it can be concluded that the least squares polynomial of degree three is the most accurate polynomial and the least squares polynomial of degree two is the second most accurate one. There is really small difference between these two polynomials. And the linear least squares polynomial is the least accurate one.

5. Appendix

The code of the project is given below:

```
clear all; clc;
load pr5data.dat;
format long

 figure
plot(pr5data(:,1),pr5data(:,2));
 title("real graph")
 legend("real graph");
 xlabel("distance [m]");
 ylabel("voltage [V]");
%% linear least squares polynomial

sum_x2=0;
for i=1:11
   sum_x2=sum_x2 + pr5data(i,1)*pr5data(i,1);
end
sum_x1=0;
for i=1:11
   sum_x1=sum_x1 + pr5data(i,1);
end

sum_y=0;
for i=1:11
   sum_y=sum_y + pr5data(i,2);
end

sum_y2=0;
for i=1:11
   sum_y2=sum_y2 + pr5data(i,2)*pr5data(i,2);
end

sum_x1y=0;
for i=1:11
   sum_x1y=sum_x1y + pr5data(i,1)*pr5data(i,2);
end

a1(1) = ((sum_x2*sum_y)-(sum_x1y*sum_x1))/(11*(sum_x2)-(sum_x1*sum_x1));
a1(2) = (  11*sum_x1y    -   (sum_x1*sum_y)        )/(11*(sum_x2)-(sum_x1*sum_x1));
k=pr5data(1:11,1);
 p1=a1(2)*k + a1(1);

 figure
plot(k,pr5data(1:11,2),".");
hold on;
 plot(k,p1);
 hold off;
 title("linear least squares polynomial graph")
 legend("real graph","linear least squares polynomial");
 xlabel("distance [m]");
 ylabel("voltage [V]");

 error_linear=0;
 for i=1:11
```

```
    error_linear=error_linear + (pr5data(i,2)-p1(i))^2;%(a1*pr5data(i,1)+a0)^2;
 end

 %% least squares polynomial of degree 2

 sum_x0=0;
 for i=1:11
    sum_x0 = sum_x0 + 1;
 end

 sum_x3=0;
 for i=1:11
    sum_x3 = sum_x3 +  pr5data(i,1)^3;
 end

 sum_x4=0;
 for i=1:11
    sum_x4=sum_x4 + pr5data(i,1)^4;
 end


 matrix2=[sum_x0 sum_x1 sum_x2; sum_x1 sum_x2 sum_x3;sum_x2 sum_x3 sum_x4];

 sum_x0y=0;
 for i=1:11
    sum_x0y= sum_x0y + pr5data(i,2);
 end

 sum_x2y=0;
 for i=1:11
   sum_x2y = sum_x2y + pr5data(i,2)*pr5data(i,1)*pr5data(i,1);
 end
 results2=[sum_x0y;sum_x1y;sum_x2y];

a2=inv(matrix2)*results2;
p2=a2(1) + a2(2)*k + a2(3)*k.^2;

figure;
hold on;
plot(pr5data(:,1),pr5data(:,2),".");
plot(k,p2);
 xlabel("distance [m]");
 ylabel("voltage [V]");
 legend("real graph","least squares polynomial of degree two graph");
 title("least squares polynomial of degree two graph");
error_degree2=0;
for i=1:11
   error_degree2 = error_degree2 + (pr5data(i,2)-p2(i))^2;
end

%% least square polynomial of degree 3
sum_x5=0;
for i=1:11
   sum_x5 = sum_x5 + pr5data(i,1)^5;
end

sum_x6=0;
for i=1:11
   sum_x6 = sum_x6 + pr5data(i,1)^6;
end

 matrix3=[sum_x0 sum_x1 sum_x2 sum_x3; sum_x1 sum_x2 sum_x3 sum_x4;sum_x2 sum_x3
sum_x4 sum_x5; sum_x3 sum_x4 sum_x5 sum_x6];
sum_x3y=0;
for i=1:11
   sum_x3y = sum_x3y + pr5data(i,2)*pr5data(i,1)^3;
end
 results3=[sum_x0y ; sum_x1y ; sum_x2y;sum_x3y];
```

```
 a3=inv(matrix3)*results3;

 p3= a3(1) +a3(2)*k + a3(3)*k.^2 + a3(4)*k.^3;
 figure
hold on;
plot(pr5data(:,1),pr5data(:,2),".");
plot(k,p3);
hold off;
title("least squares polynomial of degree three");
legend("real graph","least squares polynomial of degree three")
 xlabel("distance [m]");
 ylabel("voltage [V]");
 %% comparison of graphs and error analysis
figure
plot(pr5data(:,1),pr5data(:,2),".");
hold on;
plot(k,p1);
plot(k,p2,"--");
plot(k,p3);
title("comparison of all graphs")
 xlabel("distance [m]");
 ylabel("voltage [V]");
 legend("real graph","linear least squares method","least squares polynomial degree
2","least squares polynomial degree 3");
error_degree3=0;
for i=1:11
   error_degree3 = error_degree3 + (pr5data(i,2)-p3(i))^2;
end
fprintf("\n\n");
fprintf("the coefficients for linear least squares polynomial method\na0 = %.15f \t
a1 = %.15f\n",a1(1),a1(2));
fprintf("the coefficients for least squares polynomial of degree two method\na0 =
%.15f \t a1 = %.15f \t a2 = %.15f\n",a2(1),a2(2),a2(3));
fprintf("the coefficients for least squares polynomial of degree three method\na0 =
%.15f \t a1 = %.15f \t a2 = %.15f \t a3 = %.15f\n",a3(1),a3(2),a3(3),a3(4));
fprintf("\n\n\n");
fprintf("the error of linear least squares polynomial is %.15f\n",error_linear);
fprintf("the error of least squares polynomial of degree 2 is
%.15f\n",error_degree2);
fprintf("the error of least squares polynomial of degree 3 is
%.15f\n",error_degree3);
```