Model-Free Approaches for VIX Products

Alperen Canbey

June 2021

Abstract

This paper investigates the volatility derivatives and their role in determining the index volatility and theoretical bounds for VIX futures. The paper starts with introducing the volatility derivatives and providing background information. Then, a comparison among different pricing and replication methods of variance swaps and variance swap forwards is conducted. In order to conduct the comparison analysis, model-independent approaches of Carr and Lee (2007) [4], Demeterfi et al. (1999) [8], Fukasawa et al. (2011) [13] and CBOE white paper [7] are revisited, besides the novel at-the-money (ATM) implied volatility (IV) [12] and the new Rolloos and Arslan [19] approaches. While 30-day variance swap strikes are used to determine the VIX, the paper determines the upper bounds for VIX futures by using the variance swap forward rates and Jensen's inequality. Empirical and theoretical convexity adjustment under the Heston model will be discussed at the end of the paper.

Contents

1 Introduction

Volatility is a measure to represent how much an asset's price fluctuates and it defines how risky is to hold the asset in a portfolio. This measure is used to balance the risk of the portfolio by choosing different assets which are negatively correlated to the given asset or simply adding the volatility derivatives to the portfolio to get protection from the asset's volatility. Moreover, the negative correlation between the asset prices and the volatility forces the investors to trade these volatility derivatives to protect their portfolios from downside risk. Besides hedging and risk management purposes, volatility derivatives are traded for speculative purposes. Investors, who want to trade following the future direction of the volatility can take advantage of these volatility derivatives by taking long or short positions on them. For instance, at the end of uncertain periods such as election periods, earning calls, and index inclusions; the variability in the concerning assets is generally inclined to decrease. Therefore, the volatility derivatives can be traded for speculative and hedging purposes.

Variance swaps and volatility swaps are some of the most popular volatility products traded in the financial market. The working mechanism of these products is similar to the other type of swaps such as interest rate swaps and currency swaps. These volatility products allow different parties to exchange the cash-flows created by the volatility in the market. While one side of the trade gets the pre-determined fixed strike rate, the other party gets the realized rate during the pre-determined period. In total, the first party pays the difference between the realized rate and the fixed strike rate times the notional value.

In 1993, the Chicago Board Options Exchange (CBOE) released volatility products such as VIX, however, the CBOE revised the VIX in 2003 and it becomes the option-implied expected variation of S&P500 over the following 30 days. In 2006, the CBOE released VIX futures as a new product. These products become popular quickly among the investors who want to protect their portfolio from risk and who want to speculate market volatility.

The relation between some of these products and the variance swaps is simple. The square root of the 30-day fixed strike of the variance swap strikes corresponds to the VIX while the fixed strike rate of the variance swap forwards determines an upper bound for the VIX futures. On the other hand, the fixed strike of volatility swap determines the lower bound for the VIX by Jensen's inequality. These relations will be explained in detail in the following chapters.

Since there are various types of replication schemes to approximate the variance swaps and volatility swaps, it enables us to replicate the VIX and find the upper bounds of the VIX futures. The main motivation of this paper is to replicate the variance swaps, volatility swaps and to compare the performances of different methodologies. Secondly, the necessity of hedging for institutional investors, dealers, and retail investors sometimes creates inefficiencies in the volatility market. Since the market reacts to the supply and demand dynamics, during times of over-demand on VIX futures dealers may absorb the demand pressure. This pushes the prices over the corresponding upper bounds indicated by the variance swap forwards. In other words, market dynamics may create the law of one price violations, as well as arbitrage opportunities in the market. Thus, variance swap forwards will be calculated for monthly maturities between 2013 and 2020, to determine the upper bounds for the VIX futures and the violations will be documented.

2 Literature Review

The results in this paper contribute to the literature on model-free approaches to replicate the variance swaps and volatility swaps. The assets traded in the financial markets come with risk factors. This risk is generally measured by the asset's volatility and its correlation with the market portfolio. However, in recent years volatility itself has become an asset type with its volatility and fair price. So, to hedge and to invest better with the volatility products, a need for research has arisen in this field. The works of Neuberger [18], Dupire [10], Carr and Madan [3] demonstrated the pricing dynamics of volatility. On the other hand, because of the connection between the volatility and the asset price several stochastic models are proposed by Heston [16], Duffie et al. [9] and Madan et al. [17] . Also, some of the irregularities that cannot be explained by the Black-Scholes-Merton model [2], such as the implied volatility skew or smile, resulted in a necessity of local volatility models [11].

This paper revisits some of the well-known papers in the literature that propose model-independent approaches and approximation techniques to replicate the variance swap and volatility swap rates.

These papers are Carr and Lee (2007) [4], Demeterfi et al. [8], Fukasawa et al. [13] and CBOE white paper [7] for variance swaps, while the new approach of Rolloos and Arslan [19] and at-the-money implied volatility method [12] will be visited to replicate volatility swap rates. The methods will be called Carr-Lee, Demeterfi, Fukasawa, CBOE, Vanna-Vomma (VV) and ATM IV respectively from now.

3 Background Information

In financial markets, there are various types of products that enable investors to manage their investment risk better. Swap contracts such as currency swaps, interest rate swaps, and volatility swaps are some examples of them. As the main concern of this paper is to investigate volatility products, the emphasis will be on them.

3.1 Set Up

Throughout the paper, the following asuumptions and notations will be used.

- B_t is a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, Q)$ where Q is the risk neutral probability measure.
- $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration generated by the Brownian motion.
- E_t denotes the conditional expectation with respect to $\mathcal{F}_t$ under Q.
- Under Black-Scholes model and risk-neutral probability, the price process of a risky asset S_t is $dS_t = S_t(\sigma_t dB_t + r dt)$, where σ_t is a square integrable stochastic process adapted to $(\mathcal{F}_t)_{t\geq 0}$.
- $r > 0$ is the constant interest rate.

Additionally, the realized volatility will be denoted by $R_{\tau,T}$ for the period $[\tau, T]$. Normally, volatility can be defined continuously and discretely. However, tracking the continuous process of the volatility is not feasible in real world. Therefore, the realized volatility in period $[\tau, T]$ can be defined as:

$$R_{\tau,T} := \frac{1}{T-\tau}\sum_{t=0}^{n-1}\left(\log\left(\frac{S_{t+1}}{S_t}\right)\right)^2,$$

where n is the number of trading days t in the specified period. When the trading days in a period goes to infinity, it can be thought as infinite partitions in a period converging to a continuous time process:

$$R_{\tau,T} := \frac{1}{T-\tau}\sum_{t=0}^{n-1}\left(\log\left(\frac{S_{t+1}}{S_t}\right)\right)^2 \to^{a.s.} \frac{1}{T-\tau}\int_{\tau}^{T}\sigma_t^2 dt, \textbf{ as } n \to \infty.$$

3.2 Variance Swaps

A variance swap is a contract that pays a notional amount N_{var} decided by the agreed parties times $R^2_{0,T} - K_{var}$ where $R^2_{0,T}$ is the realized variance over the period $[0,T]$ and K_{var} is the fixed strike of the variance swap. The strike of the variance swap is the expected value of the variance over the period. If this period is one month, the strike is equal to the VIX value. Therefore, replication of the variance swaps also provides a good approximation to VIX if the time to maturity is 1 month. The replication can be done with a portfolio created by a strip of options that are weighted to ensure a fixed value of volatility for different values of the strike price. These contracts are traded over-the-counter(OTC). In a variance swap contract, one of the counterparties gets $N_{var} * K_{var}$ while the other side gets $N_{var} * R^2_{0,T}$ at the end. The fixed strike of a variance swap is the expected value of the future volatility in period $[0,T]$.

$$K_{var} = E[R^2_{0,T}].$$

If this period is one month, then the variance swap strike is equal to the VIX.

$$VIX_T = \sqrt{E_T[R^2_{T,T+1}]},$$

where VIX_T denotes the VIX index at time T and $R^2_{T,T+1}$ represents the realized variance over a one month period.

3.3 Volatility Swaps

The structure of volatility swaps is similar with the variance swaps. The only difference is that the fixed strike of volatility swaps are the expectation of the realized volatility over a predetermined period.

In a volatility swap contract, the payoff at time T is $N_{vol}(R_{0,T} - K_{vol})$. where

$$K_{vol} = E[R_{0,T}],$$

The one-month volatility swap strike is lower than the VIX because of Jensen's inequality.

$$VIX_T = \sqrt{E_T[R^2_{T,T+1}]} \geq E_T[\sqrt{R^2_{T,T+1}}] = E_T[R_{T,T+1}]$$

3.4 Variance Swap Forwards

To see how the VIX Futures are bounded above by variance swap forward rate, we need to use the definition of VIX and the Jensen's inequality. Recall that,

$$VIX_T = \sqrt{E_T[R^2_{T,T+1}]},$$

$$Fut_{t,T} = E_t[VIX_T] = E_t[\sqrt{E_T[R^2_{T,T+1}]}] \leq \sqrt{E_t[E_T[R^2_{T,T+1}]]} = \sqrt{E_t[R^2_{T,T+1}]},$$

where $Fut_{t,T}$ represents the VIX futures value at time t which expires at time T and $E_t[R^2_{T,T+1}]$ represents the 1-month variance swap forward rate.

4 Replication of Variance Swaps

In this section, different model-free approaches with respect to volatility process σ_t will be demonstrated to replicate the variance swaps. These approaches will later be compared with each other in accordance with their pricing performance of the VIX. Although further methods are developed by Neuberger [18], and later improved in Demeterfi, Derman et al. [8], there are some differences in approximation methods and application of theories. Applying Itô's lemma on the log asset price process:

$$d(\log S_t) = 0 + \Big(\frac{r_t S_t}{S_t}\Big)dt + \frac{\sigma_t S_t}{S_t}dB_t = \Big(r_t - \frac{\sigma_t^2}{2}\Big)dt + \sigma_t dB_t.$$

To isolate the variance, we need to take the difference of these two processes as the following:

$$\frac{dS_t}{S_t} - d(\log S_t) = \sigma_t dB_t + r_t dt - \Big(r_t - \frac{\sigma_t^2}{2}\Big)dt - \sigma_t dB_t = \frac{\sigma_t^2}{2}dt.$$

By integrating the variance process throughout the time T:

$$R_{0,T} = \frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\int_0^T \frac{dS_t}{S_t} - \frac{2}{T}\log(\frac{S_T}{S_0}),$$

which gives the formula for fair volatility of the variance swap:

$$K_{var} = \frac{2}{T}E\Big[\int_0^T \frac{dS_t}{S_t} - \log(\frac{S_T}{S_0})\Big]. \tag{1}$$

This equation suggests that sum of a dynamic long position which has $1/S_t$ shares of S_t at each time $t \in [0,T]$ and a static short position in a contract paying the log amount of the return on $[0,T]$ correspond to the fixed strike of the variance swap. Since the log contracts are not traded, we need to replicate the payoff of a log contract in order to approximate the fixed strike of a variance swap.

To replicate the log component, Carr-Madan [3] proposes that any continuous, twice differentiable contract f can be replicated as the following:

$$f(S_T) = f(\kappa) + f'(\kappa)(S_T - \kappa) + \int_\kappa^{S_T} f''(K)(S_T - K)dK,$$

$$\int_\kappa^{S_T} f''(K)(S_T - K)dK = \int_{S_T}^{\kappa} f''(K)(K - S_T)^+ dK + \int_\kappa^{S_T} f''(K)(S_T - K)^+ dK,$$

where κ can be any number.

Since $f''(K)(K - S_T)^+ = 0$ for $K \le S_T$ the first term can be written as $\int_0^\kappa f''(K)(K - S_T)^+$. Similarly the second term can be written as $\int_\kappa^\infty f''(K)(S_T - K)^+ dK$. Combining these two results in the equation:

$$f(S_T) = f(\kappa) + f'(\kappa)(S_T - \kappa) + \int_0^\kappa f''(K)(K - S_T)^+ dK + \int_\kappa^\infty f''(K)(S_T - K)^+ dK.$$

Assuming that the risk neutral expectation of the variance over the period [0,T] and the risk-free rate is constant and replacing $f(s)$ with $-\log(s)$ will result in :

$$
\begin{aligned}
-\log(S_T) &= -\log(\kappa) - \frac{S_T}{\kappa} + 1 + \int_0^{\kappa} \frac{(K-S_T)^+}{K^2}dK + \int_0^{\infty} \frac{(S_T-K)^+}{K^2}dK, \\
-\log(\frac{S_T}{S_0}) &= -\log(\frac{\kappa}{S_0}) - \frac{S_T}{\kappa} + 1 + \int_0^{\kappa} \frac{(K-S_T)^+}{K^2}dK + \int_0^{\infty} \frac{(S_T-K)^+}{K^2}dK, \\
E\big[-\log(\frac{S_T}{S_0})\big] &= E\big[-\log(\frac{\kappa}{S_0}) - \frac{S_T}{\kappa} + 1 + \int_0^{\kappa} \frac{(K-S_T)^+}{K^2}dK + \int_0^{\infty} \frac{(S_T-K)^+}{K^2}dK\big] \\
&= -\log(\frac{\kappa}{S_0}) - \frac{S_0}{\kappa}e^{rT} + 1 + e^{rT}\Big[\int_0^{\kappa} \frac{P_0(K)}{K^2}dK + \int_0^{\infty} \frac{C_0(K)}{K^2}dK\Big], \qquad (2)
\end{aligned}
$$

where $e^{rT}\int_0^{\kappa}\frac{P_0(K)}{K^2}dK$ is the expected payoff of a series of put options which consists the put options with strikes smaller than κ and weights of $1/K^2$. Similarly, $e^{rT}\int_0^{\infty}\frac{C_0(K)}{K^2}dK$ is the expected payoff of a series of call options where strike prices are bigger than κ and weights are $1/K^2$. Under risk neutral expectation of the realized variance and assuming that the risk-free rate is constant,

$$
E\Big[\int_0^T \frac{dS_t}{S_t}\Big] = rT. \qquad (3)
$$

So, the fixed leg of the variance swap contract can be written by using (1), (2) and (3):

$$
K_{var} = \frac{2}{T}\Big(rT - \frac{S_0}{\kappa}e^{rT} + 1 - \log\frac{\kappa}{S_0} + e^{rT}\Big[\int_0^{\kappa} \frac{P_0(K)}{K^2}dK + \int_0^{\infty} \frac{C_0(K)}{K^2}dK\Big]\Big). \qquad (4)
$$

If κ is fixed to initial forward price $F_0 = S_0e^{rT}$, then the equation (4) simplifies to,

$$
K_{var} = E[R_{0,T}^2] = \frac{2}{T}e^{rT}\Big[\int_0^{\kappa} \frac{P_0(K)}{K^2}dK + \int_0^{\infty} \frac{C_0(K)}{K^2}dK\Big].
$$

4.1 Carr-Lee Method

In this section, the approach of Carr and Lee [4] will be followed to create the synthetic variance swaps. To be able to express the equation (4) in terms of trade positions in a portfolio, notice that:

$$
\Big(1 - \frac{S_0}{\kappa}e^{rT}\Big) = \Big(1 - \frac{F_0}{\kappa}\Big) = e^{rT}\Big(\frac{1}{F_0} - \frac{1}{\kappa}\Big)S_0.
$$

Therefore the following positions in a portfolio defines the synthetic variance swap at time 0:

- $e^{-rT}\big(2r + \frac{2}{T}\log(\frac{F_0}{\kappa})\big)$ cash position
- $(\frac{1}{F_0} - \frac{1}{\kappa})$ long position on shares
- $\frac{2}{TK^2}dK$ European puts and calls where calls struck at K $>$ κ and puts struck at K $<$ κ

In a discrete structure, the contribution option stripe on the portfolio cost can be calculated as

$$\frac{2}{T}\Big[\sum_{i=1}^{n}\frac{(K_i - K_{i-1})}{K_i^2}P(K_i) + \sum_{i=n+1}^{N}\frac{(K_i - K_{i-1})}{K_i^2}C(K_i)\Big],$$

where K_i is the strike price on the i'th order when ordered in ascending order, $P(K_i)$ and $C(K_i)$ are the prices of the put and call options for the strike K_i.

4.2 Demeterfi Method

Although the method proposed by Demeterfi et al. [8] has the same theoretical base with the Carr-Lee method, the approximation that is used to replicate logarithmic payoffs differs. This small difference in the approximation can be followed by below steps:

By Equation (1):

$$K_{var} = \frac{2}{T}E\Big[\int_0^T \frac{dS_t}{S_t} - \log(\frac{S_T}{S_0})\Big].$$

The logarithm term can be written as:

$$\log\Big(\frac{S_T}{S_0}\Big) = \log\Big(\frac{S_T}{\kappa}\Big) + \log\Big(\frac{\kappa}{S_0}\Big), \tag{5}$$

where $\log(\frac{\kappa}{S_0})$ is constant and independent of the final price of the stock. Thus, only the first term is replicated.

$$-\log(\frac{S_T}{\kappa}) = -\frac{S_T - \kappa}{\kappa} + \int_0^\kappa \frac{(K - S_T)^+}{K^2}dK + \int_\kappa^\infty \frac{(S_T - K)^+}{K^2}dK$$

To replicate the log component of the variance swap, a portfolio needs to consist:

- a short position of $\frac{1}{\kappa}$ forward contracts struck at κ,

- a long position of $\frac{1}{K^2}$ put options struck at K, for all strikes from 0 to κ,
- a long position of $\frac{1}{K^2}$ call options struck at K, for all strikes from κ to ∞

In order to approximate the weights of the vanilla options, the strike prices greater than κ are chosen.

$$K_0 = \kappa < K_{1c} < K_{2c} < K_{3c} < ...$$

Set

$$f(S_T) = \frac{2}{T}\Big[\frac{S_T - \kappa}{\kappa} - \log\frac{S_T}{\kappa}\Big].$$

The slope between the κ and the first option which has the smallest strike determines the number of options for that interval.

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0},$$

where $w_c(K)$ represents the weight of the call option struck at K, and f is a function mapping from strike price to option price.

This time the slope between the next two options determines the total weight obtained that far. Thus, the weights calculated before need to be subtracted from the new value.

$$w_c(K_1) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w_c(K_0),$$

$$w_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c}).$$

The weights of the put option can be approximated in a similar manner.

$$K_0 = \kappa > K_{1p} > K_{2p} > K_{3p} > ...$$

which result in:

$$w_p(K_{n,p}) = \frac{f(K_{n+1,p}) - f(K_{n,p})}{K_{n,p} - K_{n+1,c}} - \sum_{i=0}^{n-1} w_p(K_{i,p}).$$

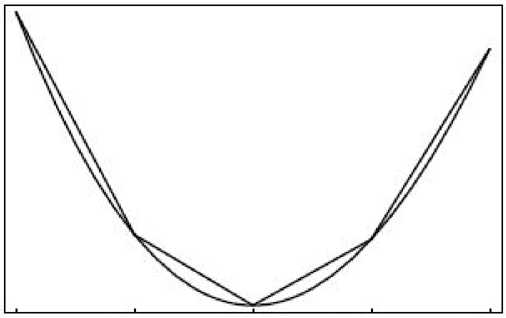

Figure 1: Approximation to the Variance Swap portfolio by Demeterfi.

4.3 CBOE Approach

According to the VIX definition of CBOE white paper [7]:

$$\begin{aligned}
\frac{VIX_0^2 T}{2} &= \int_0^F \frac{dK}{K^2} P(K) + \int_F^\infty \frac{dK}{K^2} C(K) \\
&= \int_0^{K_0} \frac{dK}{K^2} P(K) + \int_{K_0}^\infty \frac{dK}{K^2} C(K) + \int_{K_0}^F dK K^2 (P(K) - C(K)) \\
&=: \int_0^\infty \frac{dK}{K^2} Q(K) + \int_{K_0}^F \frac{dK}{K^2} (K - F) \ \textit{by put-call parity} \\
&\approx \int_0^\infty \frac{dK}{K^2} Q(K) + \frac{1}{K_0^2} \int_{K_0}^F \frac{dK}{K^2} (K - F) \\
&= \int_0^\infty \frac{dK}{K^2} Q(K) + \frac{1}{K_0^2} \frac{(K_0 - F)^2}{2}.
\end{aligned}$$

Discretizing the last expression follows by the VIX definition of CBOE:

$$VIX_0^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} Q_i(K_i) - \frac{1}{T} \Big[\frac{F}{K_0} - 1 \Big]^2,$$

where K_0 is the highest strike below the forward price, and the Q_i is the price of OTM option with strike price K_i.

4.4 Fukasawa Method

The model-free approximations to the VIX index assume the existence of options for a continuum of strike prices, however, it is not the case in the real world. The approximations generally discretize the option space which forces non-existent options to have approximate contributions obtained from the calculations on the nearest existent options. This problem leads to overvaluation in the replication of the log portfolio in equation (2). The negative contribution of the log portfolio in the equation (4) results in an undervaluation of the VIX replication scheme.

To eliminate this problem, Fukasawa et al. [13] suggest a new method. This new method enables the creation of a continuum of options to better approximate the contributions of non-existent options. Although, the origin of the theory is the same as the Carr-Lee approach, in this method we will deal with the d_2 and option-implied variance instead of the option prices and strike price.

Setting the log-moneyness as $k = \log(K/F)$, where K is strike price and F is forward price:

$$d_2(k,\sigma) := -\frac{k}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}.$$

Given a semi-martingale process for asset price S,

$$\frac{1}{T}E[< \log(S) >_T] = \int \sigma(g(z))^2\phi(z)dz,$$

where ϕ is the standard normal density, σ is the implied volatility as a function of the log-moneyness, and g is the inverse function of the mapping $d_2(k) = d_2(k,\sigma(k))$.

Although, the Black-Scholes implied volatility will be used to calculate the expected future quadratic variation it does not imply that the model is dependent on a specific option theory. The BS implied volatility just represents the nonlinear transformation of the prices.

Fukasawa method approximates the annualized expected quadratic variation ($-\frac{2}{T}E[\log(S_T/F)]$). This value also corresponds to what we have seen in the Carr-Lee and Demeterfi methods. Following the computations above, we have:

$$\frac{1}{T}E[< \log(S) >_T] = -\frac{2}{T}E[\log(S_T/F)] = \int \sigma(g(z))^2\phi(z)dz.$$

It is because by equations (1) and (5), we have:

$$K_{var} = \frac{2}{T}E\Big[\int_0^T \frac{dS_t}{S_t} - \log\big(\frac{S_T}{\kappa}\big) - \log\big(\frac{\kappa}{S_0}\big)\Big].$$

When we replace κ with F,

$$K_{var} = -\frac{2}{T}E\Big[\log(S_T/F)\Big] + \frac{2}{T}E\Big[\int_0^T \frac{dS_t}{S_t} - \log\big(\frac{F}{S_0}\big)\Big].$$

Under risk neutral expectation of the realized variance and assuming that the risk-free rate is constant, the second term is equal to 0.

To apply the Fukasawa method, we need to define a slope function $y'(x_j)$ where x_j is the d_2 of the option j, and y_j is the implied variance of the option j.

$$y'(x_j) = -\big(\frac{x_{j+1}-x_j}{l_{j+1}} - \frac{x_j - x_{j-1}}{l_j}\big)/\big(\frac{y_{j+1}-y_j}{l_{j+1}} - \frac{y_j - y_{j-1}}{l_j}\big),$$
$$l_j = \sqrt{(x_j - x_{j-1})^2 + ((y_j - y_{j-1})^2}.$$

The polynomial suggested to calculate the implied variance of a point between x_j and x_{j+1} is

$$y(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^2, x_j \leq x \leq x_{j+1},$$

where

$$a_j = y_j, \quad b_j = y'(x_j),$$
$$c_j = (3\Delta y_j - \Delta x_j y'(x_{j+1} - 2\Delta x_j y'(x_j))/(\Delta x_j^2),$$
$$d_j = (\Delta y_j - \Delta x_j y'(x_j - c_j \Delta x_j^2)/(\Delta x_j^3),$$

where $\Delta x_j = x_{j+1} - x_j$ and $\Delta y_j = y_{j+1} - y_j$. For the polynomials on $(-\infty, x_0]$ and $[x_M, \infty)$, $y = x_0$ and $y = x_M$ are used, respectively. The total annualized expected quadratic variation is the result of the following integral.

$$\int_{-\infty}^{\infty} \sigma(g(z))^2\phi(z)dz = \sum_{j=1}^{M-1}\int_{x_j}^{x_{j+1}} \sigma(g(z))^2\phi(z)dz + \int_{-\infty}^{x_1} \sigma(g(z))^2\phi(z)dz + \int_{x_M}^{\infty} \sigma(g(z))^2\phi(z)dz,$$

where the sum of the intervals can be found by the following polynomial.

$$\begin{aligned}
\int_{x_j}^{x_{j+1}} \sigma(g(z))^2 \phi(z)dz \approx& \int_{x_j}^{x_{j+1}} y(z)\phi(z)dz \\
=& \int_{x_j}^{x_{j+1}} (a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^2)\phi(z)dz \\
=& a_j A_j + b_j B_j + c_j C_j + d_j D_j,
\end{aligned}$$

where

$$\begin{aligned}
A_j :=& \Phi(x_{j+1}) - \Phi(x_j), \\
B_j :=& -(\phi(x_{j+1}) - \phi(x_j)) - x_j(\Phi(x_{j+1}) - \Phi(x_j)), \\
C_j :=& -(x_{j+1}\phi(x_{j+1}) - x_j\phi(x_j)) + 2x_j(\phi(x_{j+1}) - \phi(x_j)) + (1 + x_j^2)\Phi(x_{j+1}) - \Phi(x_j), \\
D_j :=& (1 - x_{j+1}^2)\phi(x_{j+1}) - (1 - x_j^2)\phi(x_j) + 3x_j(x_{j+1}\phi(x_{j+1}) - x_j\phi(x_j)) - 3(1 + x_j^2)(\phi(x_{j+1}) - \\
& \phi(x_j)) - x_j(3 + x_j^2)(\Phi(x_{j+1}) - \Phi(x_j)).
\end{aligned}$$

Therefore, the implied variance contribution of the options can be approximated using the above formula for the each interval and the summation of the results.

5 Replication of Volatility Swaps

In this section, the model-free approaches to approximate the volatility swap strike will be investigated. The methods provided will give volatility swap strike calculated at time $t = 0$ which corresponds to the risk-neutral expectation of future realized volatility. The volatility strike also determines a lower bound for the variance swap strike by Jensen's inequality.

$$K_{vol} = E\Big[\sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt}\Big] \leq \sqrt{E\Big[\frac{1}{T}\int_0^T \sigma_t^2 dt\Big]} = K_{var}$$

.

5.1 Vanna-Vomma Method

Vanna-Vomma model is proposed by Rolloos and Arslan [19] in 2017. The main advantage of this method is being independent of the correlation between the asset price and its volatility, unlike the ATM implied volatility method which assumes zero correlation. Therefore, this model is more practical and realistic compared to the ATM implied volatility.

Before starting to look at the theory behind the method, it is beneficial to know some of the first and second order BlackScholes greeks of an option. Assuming $\bar{\sigma}$ is the implied volatility of the option:

$$\Delta_C = \frac{\partial C}{\partial S} = e^{(-r)T} N(d_1),$$

$$\Delta_P = \frac{\partial P}{\partial S} = e^{(-r)T} [N(d_1) - 1],$$

$$\nu = \frac{\partial C}{\partial \bar{\sigma}} = \frac{\partial P}{\partial \bar{\sigma}} = S e^{(-r)T} \phi(d_1)\sqrt{T},$$

$$va = \frac{\partial^2 C}{\partial S \partial \bar{\sigma}} = \frac{\partial^2 P}{\partial S \partial \bar{\sigma}} = \frac{e^{(-r)T} d_2}{\bar{\sigma}} \phi(d_1),$$

$$vo = \frac{\partial^2 C}{\partial \bar{\sigma}^2} = \frac{\partial^2 P}{\partial \bar{\sigma}^2} = \frac{d_1 d_2}{\bar{\sigma}} \nu,$$

where N is the cumulative distribution function of the normal distribution.

As observed from the equations the vanna and the vomma of the BlackScholes option are directly proportional to d_2. Rolloos and Arslan suggest that the option which corresponds to d_2 that makes the vanna and the vomma equal to zero also corresponds to the implied volatility which is equal to the fair volatility swap rate.

This option can be found by solving the below equation.

$$K_{d_2} = S \exp\Big[\Big(-\frac{1}{2}\bar{\sigma}_{d_2}^2\Big)T\Big].$$

K_{d_2} express which option has the corresponding d_2 and the implied volatility, $\bar{\sigma}_{d_2}$. The main motivation of the model is to make the first and the second order terms in the following approximation zero. At the end, the volatility found with VV model and the fair strike of a volatility swap need to be equivalent. Setting option-implied volatility in period [0,T] as $\sigma_{0,T}$:

$$C(S, K_{d_2}, \bar{\sigma}_{d_2}, T) \approx \\ E[C(S, K_{d_2}, \sigma_{0,T}, T)] + \rho S E\Big(\Delta(S, K_{d_2}, \sigma_{0,T}, T) \int_0^T \sigma dW_\sigma\Big] + O(\rho^2),$$

where ρ is the correlation coefficient between W and B, and W is the Brownian Motion of the volatility process. The Taylor expansions of the C and Δ is given as:

$$C(S, K_{d_2}, \sigma_{0,T}, T) \approx C(S, K_{d_2}, \bar{\sigma}_{d_2}, T)] + \nu(S, K_{d_2}, \bar{\sigma}_{d_2}, T)](\sigma_{0,T} - \bar{\sigma}_{d_2}) \\ + \frac{1}{2} vo(S, K_{d_2}, \bar{\sigma}_{d_2}, T)](\sigma_{0,T} - \bar{\sigma}_{d_2})^2,$$

$$\Delta(S, K_{d_2}, \sigma_{0,T}, T) \approx \Delta(S, K_{d_2}, \bar{\sigma}_{d_2}, T)] + va(S, K_{d_2}, \bar{\sigma}_{d_2}, T)](\sigma_{0,T} - \bar{\sigma}_{d_2}).$$

Taking the expectations and fixing va and vo to 0, the deltas will be equivalent. The call price approximation will simplify to

$$C(S, K_{d_2}, \sigma_{0,T}, T) \approx C(S, K_{d_2}, \bar{\sigma}_{d_2}, T)] + \nu(S, K_{d_2}, \bar{\sigma}_{d_2}, T)](\sigma_{0,T} - \bar{\sigma}_{d_2}),$$

where the unique solution to this equation will be

$$K_{vol} = E[\sigma_{0,T}] \approx \bar{\sigma}_{d_2}.$$

5.2 At-the-money Implied Volatility Method

One of the other model-independent approaches to approximate the volatility swap strike is using the ATM (at-the-money) implied volatility. Although, the method is practical and easy to implement, it is valid under the zero correlation assumption between the asset volatility and the underlying price processes. The logic is simple and easy to apply. The ATM strike found directly corresponds to the fair volatility swap rate.

$$K_{vol} \approx \bar{\sigma}_{ATM}$$

6 Model Performance

In this section, a detailed analysis is conducted for the years between 2011 and 2020 by utilizing the theoretical information and approximation methods provided so far. The different approaches are applied to replicate the VIX and to assign an upper bound to VIX futures by taking advantage of their relation with variance swap forwards. There will be comparisons of the methods and comments on the model assumptions.

6.1 Data

The following dataset is used to analyze the performances of the mentioned methods:

SP500 Prices: The data is extracted with tidyquant package in R which uses the Yahoo Finance dataset. The data consists of the close prices of SP500 from January 2011 to December 2020.

3-month Treasury Bills: The daily data is scraped from *treasury.org* and used to determine the risk-free interest rates from January 2011 to December 2020.

VIX Prices: This data is also extracted with tidyquant package in R from Yahoo Finance. The data consists of close and adjusted prices of the VIX.

VIX Futures: The historical data is taken from the *cboe.com/us/futures.* Each excel file is merged in R to get the whole dataset from January 2013 to December 2020

Option Data: The data is provided by OptionMetrics which consists of the dates from January 2011 to December 2020. The dataset consists of variables such as bid and ask prices of the options, volume, last trading date, expiration date, type of the option, implied volatility, and first-order greeks of the options. Only the traditional monthly options are used for the analysis in the paper. Options with the symbols starting with "SPXW", "SPXQ" and "SPXPM" are eliminated to exclude weekly, quarterly, and PM settled options. This was also required because of the computational restrictions of the computer.

All the maturity dates, 3^{rd} Friday of each month, are stored at first to determine the initial dates to calculate variance swap and volatility swap strikes. To be consistent with the VIX price calculations, 30 days anterior of each maturity is selected as the starting day of the calculations which also corresponds to the maturity of VIX futures. For these days, maturities between 10 days to 1 year are selected to make the approximations for VIX, variance swaps, and volatility swaps.

To ensure the reliability of the option data, the illiquid options and options with unknown data points are eliminated with the following criterion. Options are eliminated if

1) Volume is lower than 10.

2) Last trading day is not the same as the date of the data (as well as the initial date.)

3) Implied volatility data point is NA.

4) Best offer or best bid is NA.

5) The difference between the best offer and best bid is higher than 2.

6.2 Methods

VIX is replicated with the methods proposed by Carr and Lee [4] Fukasawa et al. [13], Demeterfi et al. [8], and the CBOE VIX white paper [7]. Although the theoretical ground of the methods is the same, there are small differences in applying the theory. Most importantly the Fukasawa method approaches the issue by using the distribution of implied variance throughout to d_2 and they suggest an interpolation scheme to ensure continuity in the options data. For the sake of conducting a fair comparison, the SPX options data is eliminated under similar conditions unless the methodology directly depends on a distinct elimination process. The ATM strike is found by choosing the strike price where the call and put option prices have the minimum difference. Then, the forward price is calculated in accordance with the theoretical put-call parity given as:

$$F = K_0 + e^{rt}(C(K_0) - P(K_0)),$$

where $C(K_0)$ and $P(K_0)$ are the price of at-the-money call and put options used to calculate the forward price. Just for the CBOE approach, the maximum strike price below the forward price is chosen as stated in the CBOE white paper [7]. The put and call options which are used for the approximation to variance swaps are chosen if they are OTM given the ATM strike price. Then, the call options and put options are sorted separately by their d_2 values and the options that break the order are eliminated. At the end, if there is more than five put options and call options, the different methodologies mentioned in section (4) are applied.

Although there is data for both implied volatility and bidask prices, the price data are determined as the mean of the bid and ask prices instead of applying the Black-Scholes pricing formula on implied volatility when the price data is needed. For the Fukasawa method, the implied volatility data is used directly as suggested.

Throughout the code, there are checkpoints to ensure the reliability of the results which sometimes makes the calculations inconclusive. For instance, if the number of put options or call options is less than 5 for a date-maturity pair, then the code does not give any results. Similarly, if the code cannot find a forward price, it may be because of a problem in finding the ATM strike price, then it does not produce a result. This can also be observed from Figure 2 where the VIX makes peaks throughout the given period.

The volatility swap rates are calculated by utilizing the Vanna-Vomma and ATM implied volatility methods. Same data elimination techniques are utilized and the theories on sections 5.1 and 5.2 are applied.

One another point to discuss in this part is the VIX futures. Because of the concavity, variance swap rates cannot replicate the VIX futures, while calculating the variance swap forwards determines the upper bound for the VIX futures. One problem here is the approximation to the VIX futures. Although the time horizon between the initial days and the first maturity is fixed to 30 days as in the definition of the VIX, the number of days between the first maturity and the second maturity is either 28 or 35. However, for a better estimation of VIX futures, the day which is 30 days anterior to the second maturity (day D from now on) should be used. In other words, for a fixed day the variance swap strike for day D, and the second maturity should be calculated to extract the value of VIX futures which has the expiration at day D.

6.3 Results

The results on replication of VIX and the determination of upper bounds for the VIX futures will be shared in this section. Moreover, there will be a comparison between the theoretical and empirical convexity adjustment on variance swap rates and volatility swap rates.

6.3.1 Replication of VIX

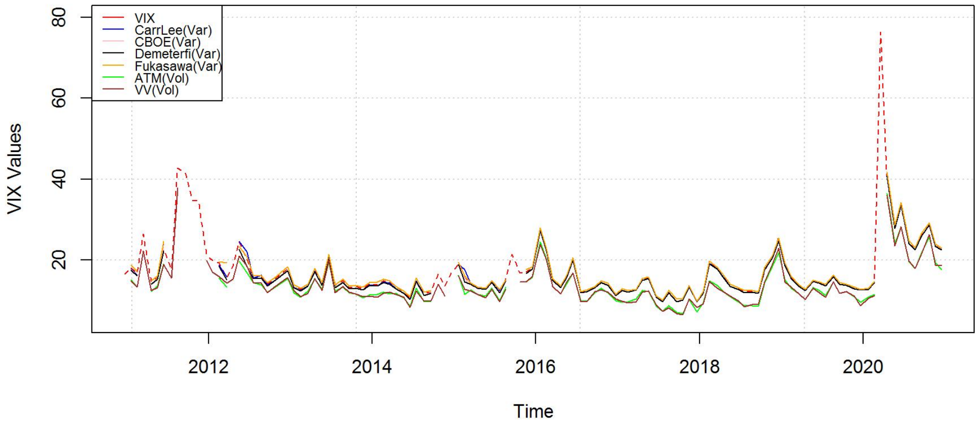

Figure 2: VIX vs Variance Swap Rates

Figure 2 illustrates the VIX values with the 30-day variance swap rates and the volatility swap rates, while Figure 3 shows the absolute differences between the concerning fixed strike value and the VIX. All four variance swap rate calculations are pretty close to the VIX values. However, some of the data points cannot be calculated where the VIX makes a peak. It is because the sudden increments in the price of the options make it harder to find a feasible ATM strike price which ensures to have more than five put options or call options to reach a reliable result. This problem can be observed in 14 data points among the 120.

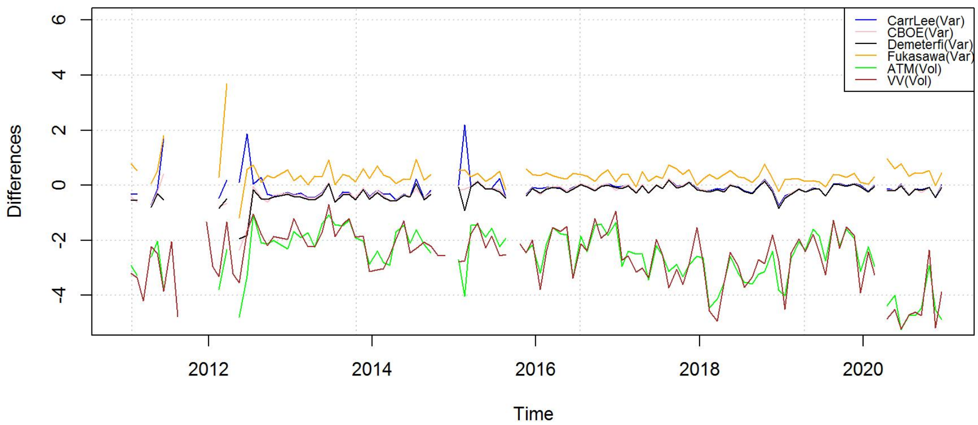

Figure 3: Difference between VIX and Variance Swap Rates

Table 1 documents the deviation metrics of the variance swap rates from the corresponding VIX

	Carr-Lee	Demeterfi	Fukasawa	CBOE
Total Deviation	11.13	31.52	-41.25	29.86
Total Absolute Deviation	27.48	33.14	44.65	33.11
Total Squared Deviation	19.97	25.42	37.68	44.83
Number of NAs	14	14	14	14
Number of Undershooting	80	93	7	87
Number of Overshooting	26	13	99	19

Table 1: Statistics of Different Methods

values. Although, the Carr-Lee method seems to perform better than the others, it does not seem to be significantly different from the others. Besides, the replication scheme may sometimes create problems because of the discrete structure of the options traded in the markets. As discussed before, this structure results in an undervaluation of the replication portfolio. Therefore, one should expect that the replication portfolio estimated with interpolation of the options should be higher than the prior which was also supported by the results. Table 1 documents that the values calculated with Fukasawa generally overshoots the VIX, while other methods generally underestimate it.

Figure 4 illustrates the percentage deviations of 30-day variance swap strikes calculated by Carr-Lee method from the VIX. The deviations resulting from the other methods can be found in Appendix A in Figures 9, 10, 11; and the underestimation of other methods relative to the Fukasawa method can be clearly observed.

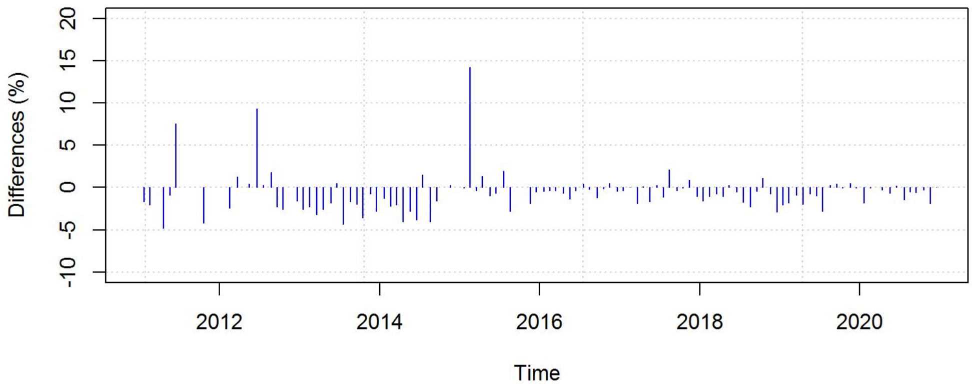

Figure 4: Percentage Difference between VIX and Carr-Lee Method

As discussed previously, the volatility swap rates calculated with the Vanna-Vomma and ATM implied volatility method give the lower bound for the VIX values, because of the Jensen's inequality on concave functions. It can be seen from Figure 12 that there is just one lower bound violation using the estimations of ATM implied volatility while all the values of VIX are higher than the

30-day volatility swap strike estimated with the Vanna-Vomma method as in Figure 13.

6.3.2 Assigning an Upper-Bound for VIX Futures

As discussed in section 6, variance swap forwards determines the upper bound for VIX futures if both are estimated for the same period. In Table 2 the summary statistics of the upper bounds are given. Since the total deviation is the sum of $(F_t - Fwd_t)$ for each of the methods, as expected they have a minus sign. Although the absolute deviations among the methods are more or less the same, the squared deviation of the Fukasawa method is much higher than the others. However, we do not look for accuracy here. The important statistics which validate the theory given in section 6 is the number of overshooting by each method. Therefore, we can validate that they act as upper bounds. Besides, Table 2 reports 14, 33 and 21 upper bound violations of VIX futures between 2013 and 2020 for the Carr-Lee, Demeterfi, and Fukasawa methods respectively. In Van Tassel (2020) [21], further analysis is made on no-arbitrage violations in the scope of the law of one price in volatility markets.

Table 3, and Figure 5 demonstrate examples of November 2018 and November 2019 VIX Futures and their upper bounds calculated by the concerning model-free approaches. As expected, variance forward swaps give the upper bound for the VIX Futures. However, Figure 6 illustrates the example for upper bound violations. The VIX futures of December 2020 went above their theoretical upper bound calculated with all three methodologies in September. Van Tassel [21] suggests that a portfolio consisting of a short position in VIX futures and a long position in a corresponding variance forward contract would benefit from the arbitrage opportunity for this period.

	Carr-Lee	Demeterfi	Fukasawa
Total Deviation	-232.65	-131.99	-225.16
Total Absolute Deviation	260.16	182.64	248.67
Total Squared Deviation	731.06	451.25	2890.85
Number of NAs	96	96	96
Number of UB Violations	14	33	21
Number of Overshooting	172	153	165

Table 2: Statistics of Different Methods

	2018-08-22	2018-09-19	2018-10-17
Close Price	15.35	15.1	17.13
Demeterfi UB	15.81	16.32	18.11
Carr-Lee UB	15.99	16.63	18.34
Fukasawa UB	16.11	16.29	18.17

Table 3: VIX Futures of November 2018

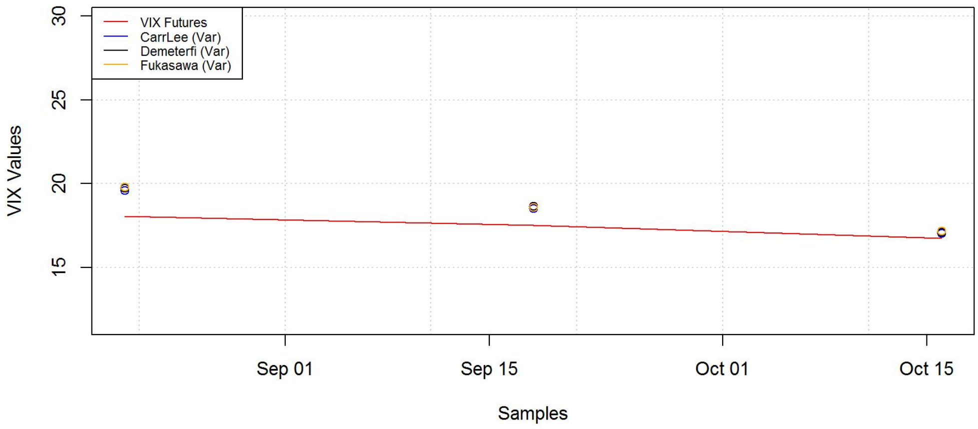

Figure 5: VIX Futures 2019-11

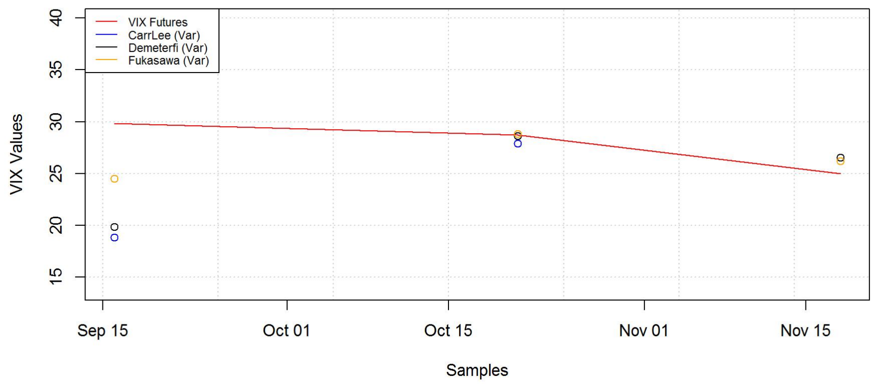

Figure 6: VIX Futures 2020-12

6.3.3 Convexity Adjustment under the Heston Model

Because of the concavity of the square root function the strike of a volatility swap is always smaller than the strike of a variance swap. The analysis that has been made so far provides the opportunity to assess the empirical value of convexity adjustment between the variance swap strikes and the volatility swap strikes. For this assessment, the values obtained by the Fukasawa method and the Vanna-Vomma method will be used for the date "2016-04-20", where the Heston model is assumed.

The Heston model is introduced by Heston [16] which proposes two different but correlated processes for the asset price and the volatility.

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dB_t \\
d\sigma_t^2 &= \lambda(\theta - \sigma_t^2)dt + \eta\sigma_t dW_t \\
E[dB_t, dW_t] &= \rho dt,
\end{aligned}
$$

where S and σ^2 are price and variance. The parameters λ, θ and η corresponds to mean-reversion rate, the mean variance and volatility of volatility, respectively.

The below Laplace transformation documented by Gatheral [14] and found by Schürger [20] results in the exact value of a volatility swap strike by using the variance swap strike

$$E[\sigma_{0,T}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E[e^{-z\sigma_{0,T}^2}]}{z^{3/2}} dz, \tag{6}$$

where Heston model allows for a closed form solution in the form of $E[e^{-z\sigma_{0,T}^2}] = Ae^{-z\sigma^2 B}$, where

$$A = \Big[\frac{2\phi e^{(\phi+\lambda)T/2}}{(\phi+\lambda)(e^{\phi T}-1)+2\phi}\Big]^{2\lambda\theta/\eta^2}, \quad B = \frac{2(e^{(\phi T}-1)}{(\phi+\lambda)(e^{\phi T}-1)+2\phi},$$

with $\phi = \sqrt{\lambda^2 + 2z\eta^2}$. Thanks to this closed form solution we will be able to convert the variance swap strike to the volatility swap strike. On the other hand, the expectation of the total variance in period $[0, T]$ is equal to

$$E[\sigma_{0,T}^2] = E\Big[\int_0^T \sigma_t^2 dt\Big] = \frac{1 - e^{-\lambda T}}{\lambda}(\sigma_0^2 - \theta) + \theta T,$$

where σ_0^2 is the instantaneous variance. Therefore, the expected annualized variance is:

$$\frac{1}{T} E[\sigma_{0,T}^2] = \frac{1 - e^{-\lambda T}}{\lambda T}(\sigma_0^2 - \theta) + \theta. \tag{7}$$

So finally, the convexity adjustment can be found by comparing the difference

$$\sqrt{E[\sigma_{0,T}^2]} - E\Big[\sqrt{\sigma_{0,T}^2}\Big],$$

by using the formulas (6) and (7).

To calibrate the model, one possibility would be to use the parameters documented by Bakshi, Cao and Chen (1997) [1]. Since the documented values are calibrated for the data before 2000s, it does not give a good approximation to the date that used. Thus, we need to find the variance swap strikes by calibrating a Heston model. To calibrate it, some assumptions and approximations are used to ease and fasten the process.

For example, the parameters stated in Bakshi et al. are $\sigma_0^2 = 0.04$, $\theta = 0.04$, $\lambda = 1.15$ and $\eta = 0.39$. Because the mean reverting phenomena is stable across the years, $\lambda = 1.15$ is fixed to use for the given date. Then, the instantaneous variance are approximated by using the VIX value as $(VIX/100)^2$, which corresponds to $\sigma_0^2 \approx 0.01836$ In order to optimize the other two parameters η and θ, the following procedure is followed:

- θ is guessed by calculating the one year volatility before the date "2016-04-20".
- Fixing $\lambda = 1.15$ and $\sigma_0^2 = 0.01836$, θ is optimized by minimizing a loss function for the difference between the real-life option implied volatilities and the ones created by using formula (3.17) of Gatheral [14]. η is not used here as volatility of volatility does not affect the value of the mean variance.
- In both steps, $\theta \approx 0.011$.
- Empirical convexity adjustment is calculated for the maturity "2016-05-20", by subtracting the volatility swap rate from the square root of the variance swap rate.
- With the 3 parameters that are already fixed, the difference between the empirical convexity adjustment and the theoretical convexity adjustment are minimized for the maturity "2016-05-20" by choosing the $\eta = 0.86$.

In Table 4, CA is defined as

$$CA = \sqrt{E[\sigma_{0,T}^2]} - E\left[\sqrt{\sigma_{0,T}^2}\right],$$

As stated before, the date 2016-04-20 is chosen and the values of variance and volatility strike in

Date	Demeterfi	CarrLee	Fukasawa	ATM	VV	FukasawaVV	CA
2016-05-20	13.16	13.19	13.55	11.51	11.59	1.96	1.55
2016-06-17	14.93	15.02	15.19	12.71	12.63	2.56	2.4
2016-07-15	16.03	16.45	16.3	13.27	13.26	3.03	2.93
2016-08-19	17.31	18.38	17.34	13.96	13.88	3.46	3.35
2016-09-16	17.55	18.22	18.26	14.39	14.29	3.97	3.58

Table 4: Swap Strikes as of 2016-04-20 and Convexity Adjustment Values

the following months are documented to demonstrate the results. In Table 4 first five columns represent these values. The column FukasawaVV shows the values for the difference between the third and fifth columns. To validate these values from the theoretical aspect, the Heston model for the volatility σ_t is assumed. Then, the closed from of $E(e^{-z\sigma^2_{0,T}})$ is used, which is the term that appears in the Laplace transform to convert the variance swap strike value into the volatility swap strike in equation (6).

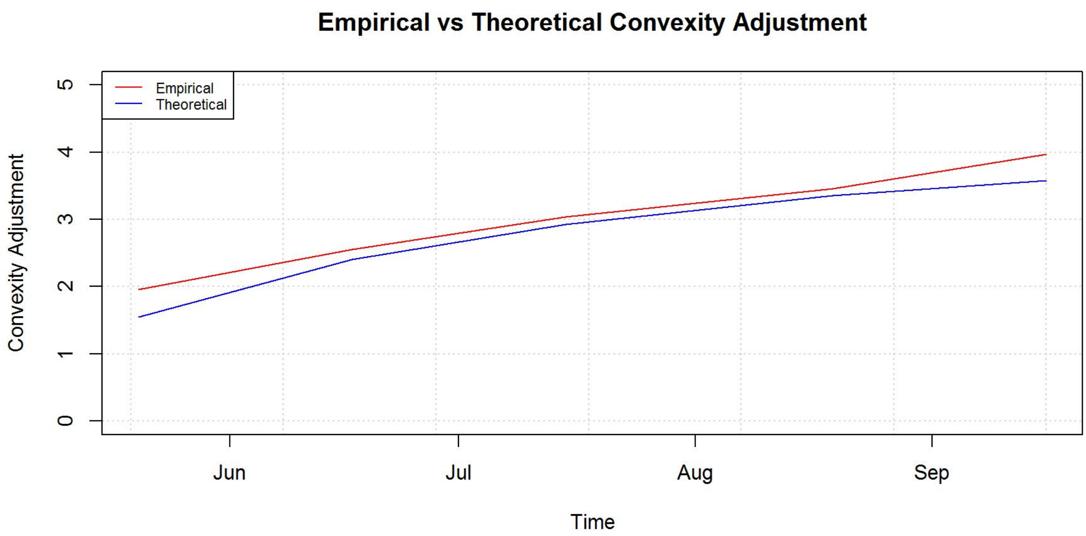

Figure 7: Convexity Adjustment with Discrete Points

Figure 7 illustrates the discrete values for theoretical and empirical convexity adjustment whenever there is a data point for the variance swap strikes, while the Figure 8 is drawn with the continuous values of the theoretical convexity adjustment.

7 Conclusion

The analysis shows us all the models used to replicate the VIX performed well, however, these models are unable to replicate the VIX during times of high volatility. One of the problems in the replication of the variance swaps is the discrete surface of the SPX options which may cause an

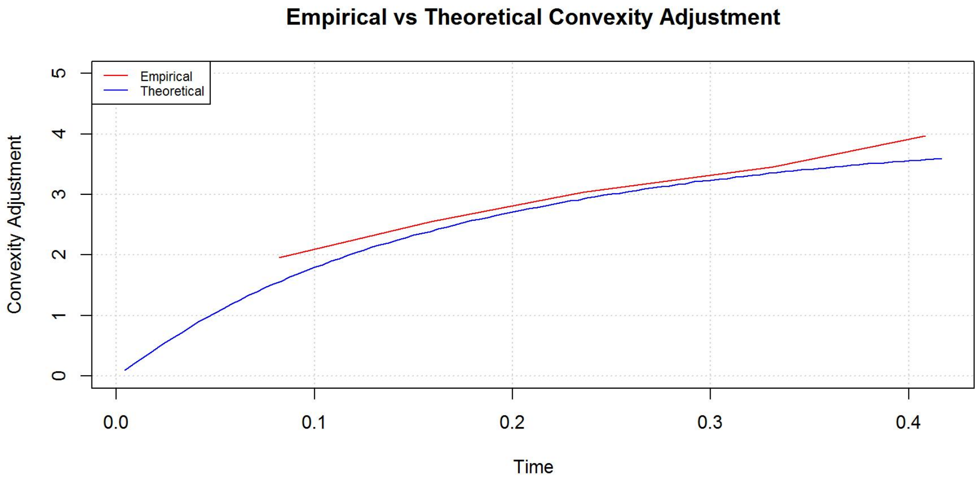

Figure 8: Convexity Adjustment with Continuous Theoretical Values

undervaluation of the strikes. The interpolation techniques can solve this problem and help better approximate the variance swap forwards. Since VIX is calculated with the CBOE procedure and the variance swaps are traded over-the-counter, we cannot directly claim that the Fukasawa method can better approximate either the VIX or variance swap forwards. This study also validates that the volatility swap strikes behave as a lower bound for the variance swap strikes. Furthermore, we observed that VIX futures may violate their upper bounds drawn by the variance swap forwards. A further study can be conducted to analyze the reasons for these discrepancies. Furthermore, the empirical and theoretical values of the convexity adjustment are compared in this paper. Although, we have very approximate results in between them, more sophisticated optimization methods as in Guillaume and Schoutens (2014) [15] can be used in the further studies in order to calibrate the parameter set.

References

[1] **Bakshi, Gurdip, Charles Cao, and Zhiwu Chen.** 1997. "Empirical performance of alternative option pricing models." *The Journal of Finance.*

[2] **Black, Fischer, and Myron Scholes.** 1973. "The pricing of options and corporate liabilities." *Journal of Political Economy.*

[3] **Carr, Peter, and Dilip Madan.** 2008. "Towards a theory of volatility trading." *Option Pricing, Interest Rates and Risk Management, Handbooks in Mathematical Finance.*

[4] **Carr, Peter, and Roger Lee.** 2007. "Realized volatility and variance: Options via swaps." *Risk.*

[5] **Carr, Peter, and Roger Lee.** 2008. "Robust Replication of Volatility Derivatives." *PRMIA award for Best Paper in Derivatives, MFA 2008 Annual Meeting.*

[6] **Carr, Peter, and Roger Lee.** 2009. "Volatility Derivatives." *Annual Review of Financial Economics.*

[7] **Chicago Board Options Exchange.** 2009. "The CBOE Volatility Index-VIX." *White Paper.*

[8] **Demeterfi, Kresimir, Emanuel Derman, Michael Kamal, and Joseph Zou.** 1999. "More than you ever wanted to know about volatility swaps." *Goldman Sachs Quantitative Strategies Research Notes.*

[9] **Duffie, Darrel., Jun Pan, and Kenneth Singleton.** 2000. "Transform analysis and asset pricing for affine jump-diffusions." *Econometrica.*

[10] **Dupire, Bruno.** 1993. "Model art." *Risk.*

[11] **Dupire, Bruno.** 1994. "Pricing with a smile." *Risk.*

[12] **Feinstein, Steven Phillip.** 1989. "The Black-Scholes formula is nearly linear in sigma for at-the-money options: Therefore implied volatilities from at-the-money options are virtually unbiased." *Federal Reserve Bank of Atlanta.*

[13] **Fukasawa, Masaaki, Isao Ishida, Nabil Maghrebi, Kosuke Oya, Masato Ubukata, Kazutoshi Yamazaki.** 2011. "Model-Free Implied Volatility: From Surface to Index." *International Journal of Theoretical and Applied Finance.*

[14] **Gatheral, Jim.** 2006. "The volatility surface: a practitioner's guide." *John Wiley* & *Sons.*

[15] **Guillaume, Florence, and Wim Schoutens** 2014. "Heston Model: the Variance Swap Calibration" *Journal of Optimization Theory and Applications.*

[16] **Heston, Steven L.** 1993. "A closed-form solution for options with stochastic volatility with applications to bond and currency options." *Review of financial studies.*

[17] **Madan, Dilip, Peter Carr, and Eric Chang** 1998. "The variance gamma process and option pricing." *Review of Finance.*

[18] **Neuberger, Anthony J.** 1990. "Volatility trading." *Institute of Finance and Accounting, London Business School.*

[19] **Rolloos, Frido, Melih Arslan.** 2017. "Taylor-made volatility swaps." *Wilmott.*

[20] **Schürger, Klaus.** 2002. "Laplace transforms and suprema of stochastic processes." *In Advances in Finance and Stochastics.*

[21] **Van Tassel, Peter.** 2020. "The Law of One Price in Equity Volatility Markets." *Federal Reserve Bank of New York Staff Reports.*

Appendix

A Figures and tables

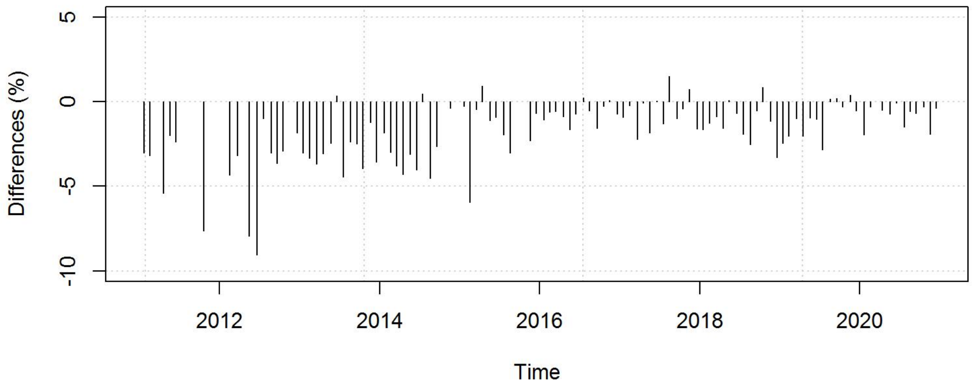

Figure 9: Percentage Difference between VIX and Demeterfi Method

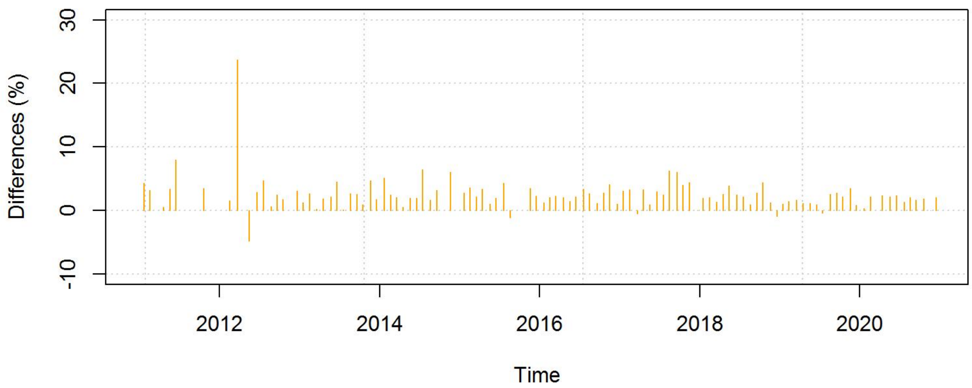

Figure 10: Percentage Difference between VIX and Fukasawa Method

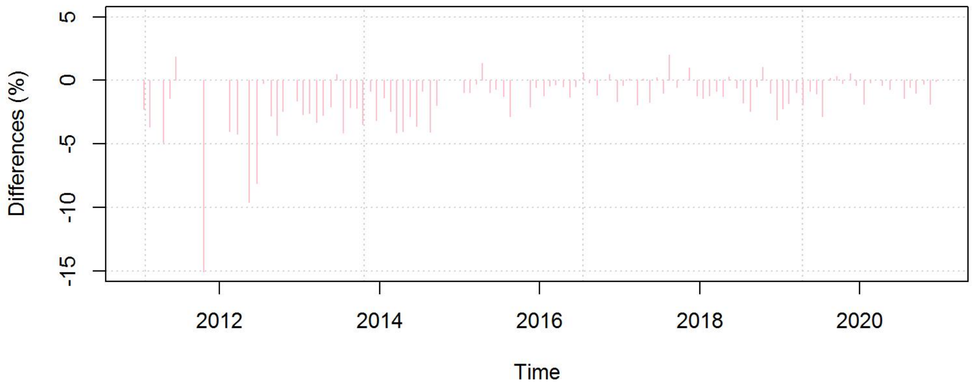

Figure 11: Percentage Difference between VIX and CBOE Procedure

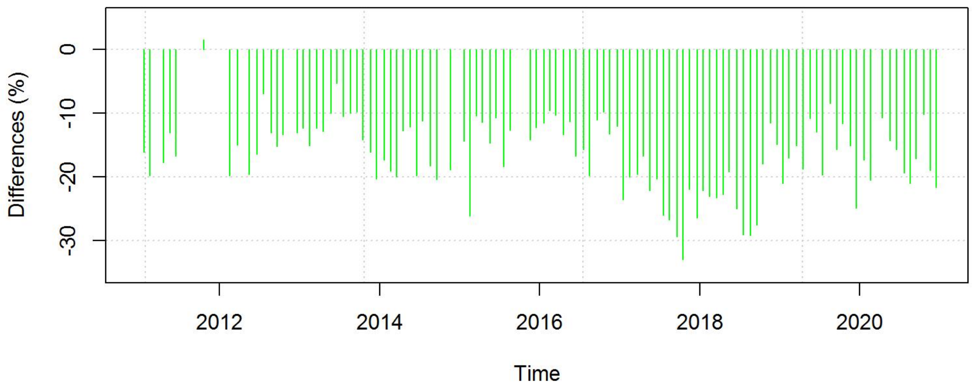

Figure 12: Percentage Difference between VIX and ATM Method

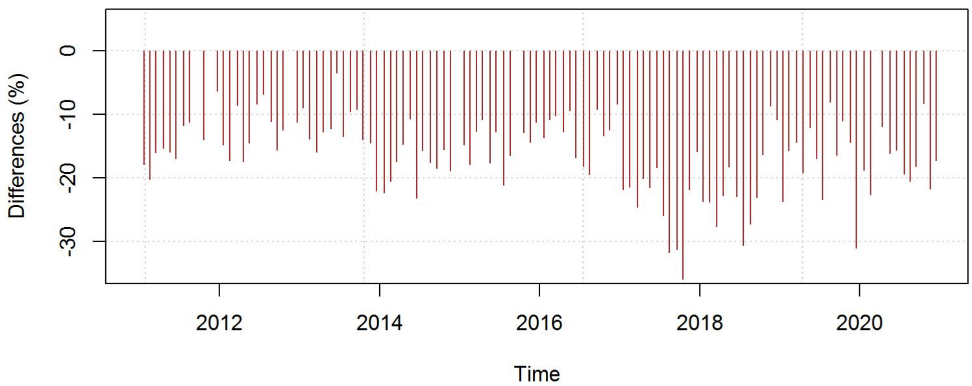

Figure 13: Percentage Difference between VIX and VV Method