Model-Free Approaches for VIX Products

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June 18, 2021

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Introduction

- Why is volatility important?
- What is VIX? Why is it important?
- Can we trade volatility?

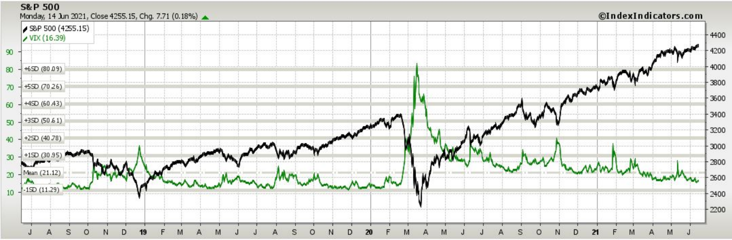

Figure: S&P500 vs VIX

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Research Purposes

- Comparison among the VIX replication methods:
  - To detect the performance of different model-independent approaches.
  - To demonstrate the discrepancy between using the discrete and continuous option strip while approximating a variance swap.
  - To conduct an ordinal comparison between the fixed strikes of Variance Swaps and Volatility Swaps.
- The relation between the variance swap forwards and VIX futures:
  - To identify the theoretical upper bound for VIX futures.
  - Detecting the law-of-one-price violations in equity volatility markets.
- Theoretical and empirical convexity adjustment comparison under Heston model
  - To identify the fit of the Heston model in real life.

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Literature Review

This paper revisits some of the well-known papers in the literature that propose model-independent approaches and approximation techniques to replicate the variance swap and volatility swap rates.

- CarrLee Method by Carr and Lee (2007) [1]
- Demeterfi Method by Demeterfi, Derman et al. (1999) [3]
- Fukasawa Method by Fukasawa, Ishida et al. (2011) [5]
- CBOE Approach [2]
- Vanna-Vomma Method by Rolloos and Arslan (2017) [6]
- At-the-money Implied Volatility Method by Feinstein (1989) [4]

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Set Up

- B_t is a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, Q)$.
- $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration generated by the Brownian motion.
- E_t denotes the conditional expectation with respect to $\mathcal{F}_t$ under Q.
- Under Black-Scholes model and risk-neutral probability Q, the price process of a risky asset S_t is $dS_t = S_t(\sigma_t dB_t + rdt)$, where σ_t is a square integrable stochastic process adapted to $(\mathcal{F}_t)_{t\geq 0}$.
- $r > 0$ is the constant interest rate.
- $R^2_{\tau,T}$ is a consistent estimator of $\frac{1}{T-\tau}\int_\tau^T \sigma_t^2 dt$.

$$R^2_{\tau,T} := \frac{1}{T-\tau}\sum_{t=0}^{n-1}\left(\log\left(\frac{S_{t+1}}{S_t}\right)\right)^2,$$

where n is the number of trading days t in the specified period and $R^2_{\tau,T}$ is the realized variance.

Trading Volatility

Figure: Variance and Volatility Swaps

- The fixed strikes are: $K_{var} = E[R^2_{0,T}]$, $K_{vol} = E[R_{0,T}]$.
- The relation with the VIX index is: $VIX_T = \sqrt{E_T[R^2_{T,T+1}]}$.
- The one-month volatility swap strike is lower than the VIX because of Jensen's inequality.

$$VIX_T = \sqrt{E_T[R^2_{T,T+1}]} \geq E_T\left[\sqrt{R^2_{T,T+1}}\right] = E_T[R_{T,T+1}]$$

Variance Swap Forwards

By Jensen's inequality, we get:

$$Fut_{\tau,T} = E_\tau[VIX_T] = E_\tau\left[\sqrt{E_T[R^2_{T,T+1}]}\right]$$
$$\leq \sqrt{E_\tau[E_T[R^2_{T,T+1}]]} = \sqrt{E_\tau[R^2_{T,T+1}]},$$

$$VIX_T = \sqrt{E_T[R^2_{T,T+1}]},$$

where $Fut_{t,T}$ represents the VIX futures value at time τ which expires at time T and $E_\tau[R^2_{T,T+1}]$ represents the 1-month variance swap forward rate.

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Theory

The price process of a risky asset S_t is:

$$dS_t = S_t(\sigma_t dB_t + rdt)$$

Applying Itô's lemma, we can show that:

$$R^2_{0,T} = \frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\int_0^T \frac{dS_t}{S_t} - \frac{2}{T}\log(\frac{S_T}{S_0}).$$

Taking the expectations, the final equation is:

$$K_{var} = \frac{2}{T}\Big(rT - \frac{S_0}{\kappa}e^{rT} + 1 - \log\frac{\kappa}{S_0} + e^{rT}\Big[\int_0^\kappa \frac{P_0(K)}{K^2}dK + \int_\kappa^\infty \frac{C_0(K)}{K^2}dK\Big]\Big).$$

Carr-Lee Method

$$\left(1 - \frac{S_0}{\kappa}e^{rT}\right) = \left(1 - \frac{F_0}{\kappa}\right) = e^{rT}\left(\frac{1}{F_0} - \frac{1}{\kappa}\right)S_0.$$

Therefore the following positions in a portfolio defines the synthetic variance swap at time 0:

- $e^{-rT}\left(2r + \frac{2}{T}\log(\frac{F_0}{\kappa})\right)$ cash position
- $(\frac{1}{F_0} - \frac{1}{\kappa})$ long position on shares
- $\frac{2}{TK^2}dK$ European puts and calls where calls struck at K $\geq \kappa$ and puts struck at K $\leq \kappa$

In a discrete structure, the contribution option stripe on the portfolio cost can be calculated as

$$\frac{2}{T}\Big[\sum_{i=1}^{n}\frac{(K_i - K_{i-1})}{K_i^2}P(K_i) + \sum_{i=n+1}^{N}\frac{(K_i - K_{i-1})}{K_i^2}C(K_i)\Big],$$

Demeterfi Method

$$K_0 = \kappa < K_{1c} < K_{2c} < K_{3c} < \dots$$

Set

$$f(S_T) = \frac{2}{T}\left[\frac{S_T - \kappa}{\kappa} - \log\frac{S_T}{\kappa}\right].$$

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0},$$

where $w_c(K)$ represents the weight of the call option struck at K.

$$w_c(K_1) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w_c(K_0),$$

$$w_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c}).$$

CBOE Approach

According to the VIX definition of CBOE white paper [2]:

$$\frac{VIX_0^2 T}{2} = \int_0^\infty \frac{dK}{K^2} Q(K) + \frac{1}{K_0^2} \frac{(K_0 - F)^2}{2}.$$

Discretizing the expression follows by the VIX definition of CBOE:

$$VIX_0^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} Q_i(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2,$$

where $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$, K_0 is the highest strike below the forward price, and the Q_i is the price of OTM option with strike price K_i.

Fukasawa Method

Define a slope function $y'(x_j)$ where x_j is the d_2, and y_j is the implied variance of the option j, where l_j is the Euclidean distance between options j and j-1.

$$y'(x_j) = -(\frac{x_{j+1} - x_j}{l_{j+1}} - \frac{x_j - x_{j-1}}{l_j})/(\frac{y_{j+1} - y_j}{l_{j+1}} - \frac{y_j - y_{j-1}}{l_j}).$$

The polynomial to calculate the implied variance of a point between x_j and x_{j+1} is,

$$y(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^2, x_j \leq x \leq x_{j+1},$$

The total annualized expected quadratic variation is the result of the following:

$$\sum_{j=1}^{M-1} \int_{x_j}^{x_{j+1}} \sigma(g(z))^2 \phi(z) dz + \int_{-\infty}^{x_1} \sigma(g(z))^2 \phi(z) dz + \int_{x_M}^{\infty} \sigma(g(z))^2 \phi(z) dz,$$

where $\int_{x_j}^{x_{j+1}} \sigma(g(z))^2 \phi(z) dz$ can be approximated by a polynomial which consists a sum of terms involving only $\phi(x_j)$, $\phi(x_{j+1})$, x_j, Δx_j, y_j, Δy_j and $y'(x_j)$.

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ATM Implied Volatility Method

- The method is practical and easy to implement.
- It is valid under the zero correlation assumption between the asset volatility and the underlying price processes.
- The implied volatility of an option which has the ATM strike corresponds to the fair volatility swap rate which draw the lower bound for variance swap strike.

$$K_{vol} \approx \bar{\sigma}_{ATM}$$

Vanna-Vomma Method

Assuming $\bar{\sigma}$ is the implied volatility of the option, the Black&Scholes greeks of an option are:

$$
\begin{aligned}
\Delta_C &= \frac{\partial C}{\partial S} = e^{(-r)T} N(d_1), \\
\Delta_P &= \frac{\partial P}{\partial S} = e^{(-r)T} [N(d_1) - 1], \\
\nu &= \frac{\partial C}{\partial \bar{\sigma}} = \frac{\partial P}{\partial \bar{\sigma}} = S e^{(-r)T} \phi(d_1) \sqrt{T}, \\
va &= \frac{\partial^2 C}{\partial S \partial \bar{\sigma}} = \frac{\partial^2 P}{\partial S \partial \bar{\sigma}} = \frac{e^{(-r)T} d_2}{\bar{\sigma}} \phi(d_1), \\
vo &= \frac{\partial^2 C}{\partial \bar{\sigma}^2} = \frac{\partial^2 P}{\partial \bar{\sigma}^2} = \frac{d_1 d_2}{\bar{\sigma}} \nu,
\end{aligned}
$$

This option can be found by solving the below equation.

$$
K_{d_2} = S \exp \left[\left(-\frac{1}{2} \bar{\sigma}_{d_2}^2 \right) T \right].
$$

$$
K_{vol} \approx \bar{\sigma}_{d_2}
$$

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Data

- SP500 Prices: Yahoo Finance, from January 2011 to December 2020.
- 3-month Treasury Bills: *treasury.org*, from January 2011 to December 2020.
- VIX Prices: Yahoo Finance, from January 2011 to December 2020.
- VIX Futures: *cboe.com/us/futures*, from January 2013 to December 2020
- Option Data: OptionMetrics, from January 2011 to December 2020.

Options are eliminated if
1) Volume is lower than 10.
2) Last trading day is not the same as the date of the data (as well as the initial date.)
3) Implied volatility data point is NA.
4) Best offer or best bid is NA.
5) The difference between the best offer and best bid is higher than 2.

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Replication of VIX

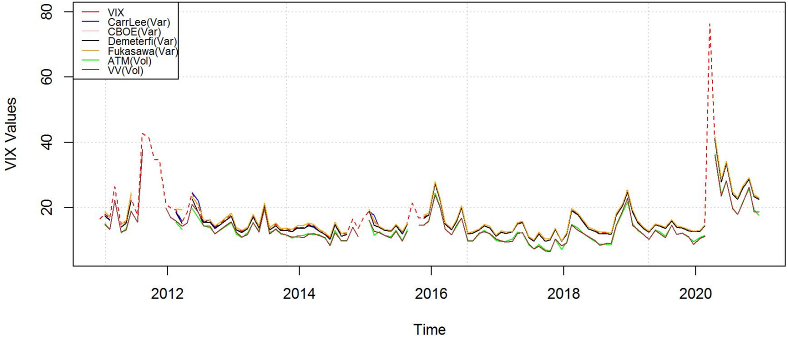

Figure: VIX vs Variance Swap Rates

Replication of VIX

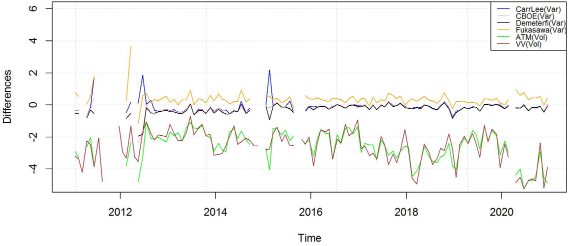

Figure: Difference between VIX and Variance Swap Rates

	Carr-Lee	Demeterfi	Fukasawa	CBOE
Total Deviation	11.13	31.52	-41.25	29.86
Total Absolute Deviation	27.48	33.14	44.65	33.11
Total Squared Deviation	19.97	25.42	37.68	44.83
Number of NAs	14	14	14	14
Number of Undershooting	80	93	7	87
Number of Overshooting	26	13	99	19

Table: Statistics of Different Methods

Replication of VIX

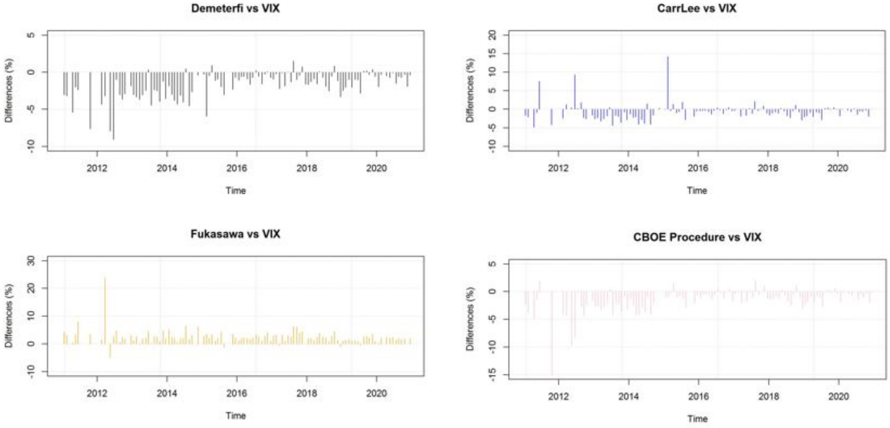

Figure: Percentage Differences between VIX and Replication Methods

Replication of VIX

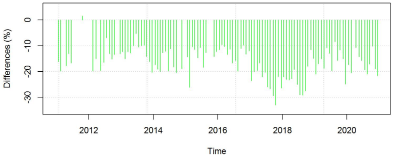

Figure: Percentage Difference between VIX and ATM Method

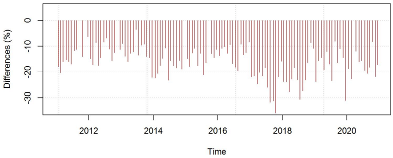

Figure: Percentage Difference between VIX and VV Method

Assigning an Upper-Bound for VIX Futures

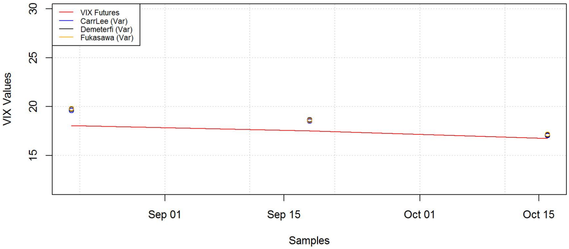

Figure: VIX Futures 2019-11

	2018-08-22	2018-09-19	2018-10-17
Close Price	15.35	15.1	17.13
Demeterfi UB	15.81	16.32	18.11
Carr-Lee UB	15.99	16.63	18.34
Fukasawa UB	16.11	16.29	18.17

Table: VIX Futures of November 2018

- Violation of Upper-Bound:

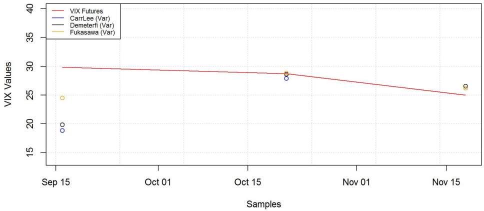

Figure: VIX Futures 2020-12

	Carr-Lee	Demeterfi	Fukasawa
Total Deviation	-232.65	-131.99	-225.16
Total Absolute Deviation	260.16	182.64	248.67
Total Squared Deviation	731.06	451.25	2890.85
Number of NAs	96	96	96
Number of UB Violations	14	33	21
Number of Overshooting	172	153	165

Table: Statistics of Different Methods

Convexity Adjustment under Heston Model

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dB_t \\
d\sigma_t^2 &= \lambda(\theta - \sigma_t^2)dt + \eta\sigma_t dW_t \\
E[dB_t, dW_t] &= \rho dt,
\end{aligned}
$$

$$
E[\sigma_{0,T}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E[e^{-z\sigma_{0,T}^2}]}{z^{3/2}} dz, \tag{1}
$$

$$
E[\sigma_{0,T}^2] = E\left[\int_0^T \sigma_t^2 dt\right] = \frac{1 - e^{-\lambda T}}{\lambda}(\sigma_0^2 - \theta) + \theta T, \tag{2}
$$

So finally, the convexity adjustment can be found by comparing the difference

$$
\sqrt{E[\sigma_{0,T}^2]} - E\left[\sqrt{\sigma_{0,T}^2}\right],
$$

by using the formulas (4) and (5).

Convexity Adjustment under Heston Model

In Table 4, CA is defined as

$$CA = \sqrt{E[\sigma^2_{0,T}]} - E\left[\sqrt{\sigma^2_{0,T}}\right],$$

Date	Demeterfi	CarrLee	Fukasawa	ATM	VV	FukasawaVV	CA
2016-05-20	13.16	13.19	13.55	11.51	11.59	1.96	1.55
2016-06-17	14.93	15.02	15.19	12.71	12.63	2.56	2.4
2016-07-15	16.03	16.45	16.3	13.27	13.26	3.03	2.93
2016-08-19	17.31	18.38	17.34	13.96	13.88	3.46	3.35
2016-09-16	17.55	18.22	18.26	14.39	14.29	3.97	3.58

Table: Swap Strikes as of 2016-04-20 and Convexity Adjustment Values

Convexity Adjustment under Heston Model

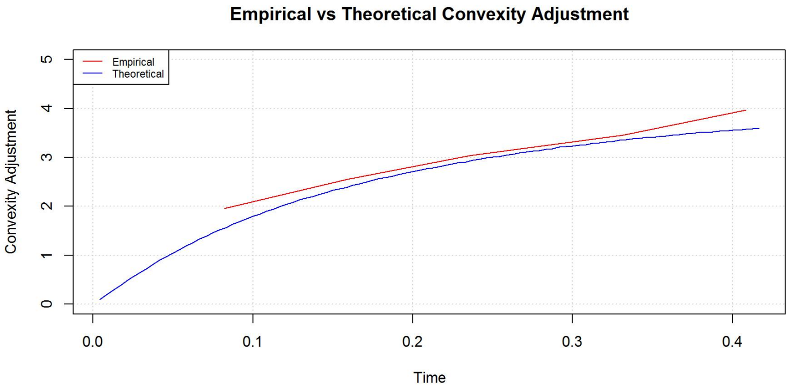

Figure: Convexity Adjustment with Continuous Theoretical Values

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Concluding Remarks

- All methods approximate the VIX quite well.
- Unable to replicate the VIX during times of high volatility.
- An approximation with discrete option stripe undershoots the variance swap strike compared to interpolated option stripe.
- The 1-month volatility swap strike draws the lower-bound for the VIX.
- VIX futures may violate their theoretical upper bound which is driven by the Jensen's inequality.
- The reasons for these discrepancies should be investigated in a further research.
- Performance of a portfolio consisting a long position in variance swap forwards and a short position in VIX futures can be investigated during the times of upper-bound violations.
- Heston model approximates well into empirical values of convexity adjustment.
- More sophisticated calibration methods can be used in a further research.

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