RESEARCH STATEMENT

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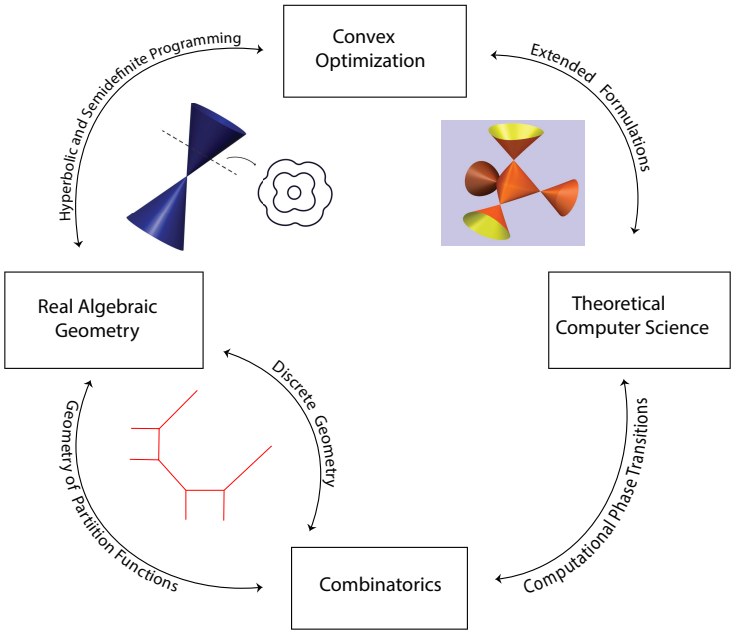
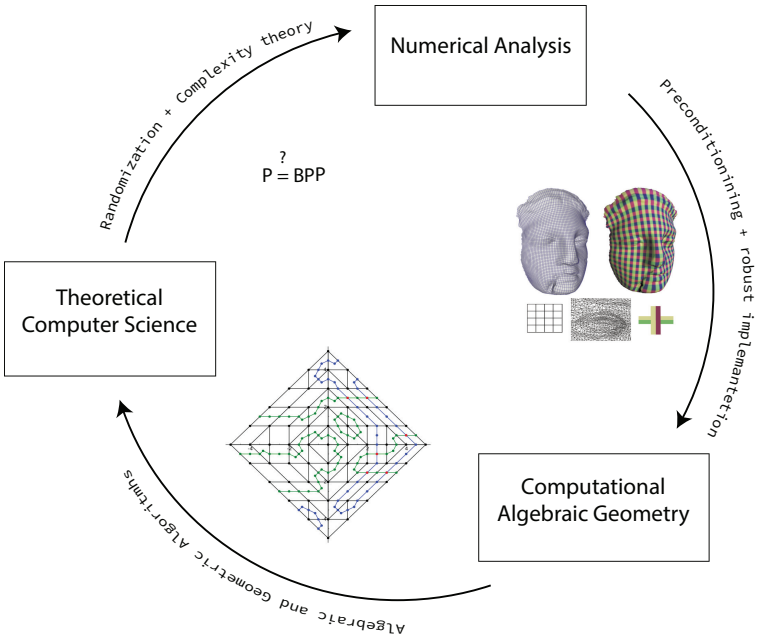
Real algebraic geometry is the theory of shapes that are cut out by polynomial equations and inequalities. I am fundamentally interested in understanding the descriptive power of real algebraic geometry and developing prescriptive capacity for this theory. Descriptive power refers to following kind of questions: Can we infer how hard it is to do linear optimization on (or create a uniform sample from) a compact space X from its algebraic description? If a space X admits a low-complexity real algebraic description, does it mean this space has few connected components? Prescriptive capacity refers to the following: Given real algebraic modeling of a problem, can we infer which algorithm should be used to solve it?

Descriptive power of real algebraic geometry plays a central role in a myriad of important developments in the theory of computing: the quest toward unique games conjecture [Rag08], construction of Ramanujan graphs [MSS15, MSS18], latest breakthrough on traveling salesman problem [KKG21], and the proof of Mihail-Vazirani conjecture [ALGV19] to name a few. Problems from diverse application domains such as 3D-reconstruction in computer vision [KBP11], geometry processing of curves and surfaces [SZ02], solving kinematic constraint systems [SJS18], high dimensional estimation [RSS18], and modeling of biochemical reaction networks [Dic20] all demand efficient inference on real algebraic sets (of which only few we are able to deliver).



I am interested in developing an algorithmic theory of real algebraic geometry that is guided by questions in theoretical computer science and problems from application domains. This naturally connects real algebraic geometry to combinatorics, convex geometry, discrete geometry, high dimensional probability, numerical analysis, computational phase transitions, and optimization. Above and below, I provide two figures[1] for schematic summaries of these interactions, then we delve into concrete problems and projects. Hope you enjoy the read.

[1]The Cayley cubic, a.k.a. samosa, is registered in creative commons by Salix alba, rest of the images are home-made.



1. At the crossroads of real algebraic geometry, complexity, and biology

Consider the following univariate polynomial given by sums of products of sparse polynomials:

$$f(x) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{ij}(x)$$

where f_{ij} are sparse polynomials with at most t terms. Real-tau conjecture of Koiran asserts that the number of real roots of f is bounded by a polynomial function of k, m and t [Koi11]. The surprising fact is that this innocent counting conjecture has deep consequences in complexity theory: If real-tau conjecture is proved to be correct this shows strong lower bounds for the hardness of fundamental counting problems [Koi11, Val79].

Now consider a system of multivariate polynomials $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ where every polynomial has a prescribed support set $A_i \subset \mathbb{Z}^n$, that is the monomials that appear in f_i are $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in A_i$. Suppose the cardinality of A_i are at most t. Kushnirenko's conjecture (1980) claims that the number of common real zeros of the system $\mathbf{f}$ is $O(t^n)$. Observe that if we were to count complex zeros neither Koiran's nor Kushnirenko's conjecture have any chance to hold: $x^d - 1 = 0$ have two terms and d complex zeros. It is the special nature of real algebraic geometry that allows us to conjecture that the description complexity controls complexity of the corresponding space [Kho83]. The case $n = 1$ of Kushnirenko's conjecture is confirmed by Descartes' rule of signs in 17th century, and for each $n \geq 2$ it remains open [KPT15].

Here is the biochemistry angle of this story: The mass kinematics equation that governs biochemical reaction networks (such as enzyme networks inside the cells) is the following:

$$\frac{dx_i}{dt} = \sum_{\alpha \in A} c_\alpha^{(i)} y^\alpha, \quad i = 1, 2, \ldots, n,$$

where $A \subset \mathbb{Z}^n$ is a finite set of lattice points (the support set). When one wants to count steady (equilibrium) states of the biochemical reaction network with species x_i, one simply

needs to write

$$\frac{dx_i}{dt} = \sum_{\alpha \in A} c_\alpha^{(i)} y^\alpha = 0\,, \quad i = 1, 2, \ldots, n.$$

Here the solutions to the above equation with densities $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$ represent the steady states of the dynamical system, thus we want to count the number of solutions [Dic20].

Our inability to prove or refute Kushnirenko's conjecture for decades, and the pressing need from applications, motivates the following question: Can we show Kushnirenko's conjecture actually holds for most systems? After a series of papers, we managed to prove the answer is affirmative. Indeed, the systems with more than $O(t^n)$ zeros, if they exist, are rare [BETC19, Bur23, ETTC23]. Our strongest result [ETTC23, Thm 1.9] requires specific terminology from discrete geometry, so I'd like to state an illustrative earlier result [BETC19] which corresponds to the special case $A = A_1 = A_2 = \ldots = A_n$.

Fix a finite subset $A \subseteq \mathbb{Z}^n$ of cardinality t together with a map $\sigma : A \to \mathbb{R}_+$. To this data, we assign the random polynomial system:

$$f_1(x) := \sum_{\alpha \in A} \sigma(\alpha)\, \xi_{1,\alpha} x^\alpha, \ldots, f_n(x) := \sum_{\alpha \in A} \sigma(\alpha)\, \xi_{n,\alpha} x^\alpha,$$

where the $\xi_{i,\alpha} \sim \mathcal{N}(0,1)$ are independent identically distributed (i.i.d.) standard Gaussian random variables. This amounts to considering random polynomial systems, $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, where each f_i has the support A, and the coefficients occurring in f_i are independent centered Gaussian coefficients, whose variances are given by $\sigma(\alpha)^2$ for $\alpha \in A$. For a given *support* A and a *system of variances* σ we denote by $\mathbb{E}N(A, \sigma)$ the *expected number of nondegenerate zeros* of the random system $f(x) = 0$ in $\mathbb{R}^n_+$.

Theorem 1.1 (Kushnirenko's Conjecture for Probabilistic Input)**.** *For any support $A \subseteq \mathbb{Z}^n$ of cardinality t and any system of variances σ, we have*

$$\mathbb{E}N(A, \sigma) \leq 2^{-n+1} \binom{t}{n} \leq \frac{t^n}{2^{n-1}}.$$

Note that this theorem holds for *any* variance vector σ. The next natural question in this line of work is the following:

Question 1.1. *Let S^{t-1} denote the unit sphere in $\mathbb{R}^t$, do we have*

$$\max_{\sigma \in S^{t-1}} \mathbb{E}N(A, \sigma) \geq \frac{(t-n)^{\frac{n}{2}}}{2^n}?$$

By Erdös first moment method, such a lower bound would imply existence of polynomial systems with $O(t^{\frac{n}{2}})$ positive real zeros, and this is crucial for biology applications.

Counting real zeros of sparse polynomial systems is NP-Hard (and likely much harder). So, from a complexity theory point of view the natural question is to understand where the hardness lies and to see if the problem is hard for average-case. I have worked on using techniques from tropical geometry (Viro's patchworking method) to count and find real zeros of input systems where coefficients have certain convexity properties [EW22]. Several colleagues reached out and asked the following question:

Question 1.2. *Define the polynomial systems that satisfy the criterion in* [EW22] *as patchworked systems. If a sparse polynomial system is created randomly is it a patchworked polynomial system?*

This question needs to be addressed due to its importance, and I will work on it in 2024. I am currently more active on the following problem.

Question 1.3. *Let $A_1, A_2, \dots, A_n$ be support sets with at most t elements, and assume the diameter of these sets is at most d. Is there an algorithm that is polynomial in $n, \ln d, t$ which for a given input polynomial system supported on $A_1, A_2, \dots, A_n$ detects if the system has $O(n)$ real zeros or $\Omega(t^n)$ real zeros?*

This question has two motivations: (1) An algorithm that can separate systems with many zeros from the ones with few or no zeros is useful for applications, and (2) from a complexity theory standpoint this is a natural tractable relaxation of the intractable real zero counting problem. For this project, I plan to use techniques from computational phase transitions, see Section 5.

2. A problem from Smale's list

At the beginning of the new millennium, International Mathematical Union made a call for a list of problems for the 21st century –in the spirit of Hilbert's problems for 20th century–. Stephen Smale answered with a list of problems which included the following [Sma98]:

Question 2.1 (Smale's 17th Problem)**.** *Can a zero of n complex polynomials in n unknowns be found approximately on the average in polynomial time with a uniform algorithm?*

Approximate zero in the text means a point that is in the basin of attraction for Newton's method, and uniform algorithm means an algorithm that admits inputs of any degree. Smale did not precisely define the probabilistic input model to conduct average analysis. It was taken to be the following by researchers in the field: Consider a system of $n-1$ homogeneous polynomials $\mathbf{p} = (p_1, p_2, \dots, p_{n-1})$ with degrees $d_1, d_2, d_3, \dots, d_{n-1}$ as follows:

$$p_i = \sum_{|\alpha|=d_i} \binom{d_i}{\alpha}^{\frac{1}{2}} \psi_\alpha x^\alpha \text{ for } i = 1, 2, 3, \dots, n-1$$

where ψ_α are Gaussian i.i.d. random coefficients, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. This model assumes the input polynomials have all $\binom{n+d_i-1}{d_i}$ many monomials with non-zero coefficients. The model has two main technical features/restrictions:

(1) This model of randomness is invariant under the change of variables.
(2) This model of randomness is isotropic with respect to the standard inner product (Bombieri's inner product) on the vector space of homegenous polynomials.

Smale's 17th problem literature furnishes beautiful results [BC11, BP09, Lai17, Lai20] all of which assume the two features above. The main catch of the two assumptions: all points on projective space are equally viable to be a zero due to invariance assumption, and all directions in polynomial space are equally viable due to isotropic assumption.

The technical machinery developed in Smale's 17th problem literature is not capable of handling the question on any input model that does not have the invariance properties above. On the other hand, in theory and in practice, we are to deal with polynomials that are sparse and structured. Sparse polynomials do not satisfy the first assumption: the input model is not invariant under change of variables. Moreover, Bombieri inner product is not suitable to work with structured polynomials.

Given a lattice polytope $Q \subset \mathbb{Z}^n$, we say a polynomial $p = \sum c_\alpha x^\alpha$ has Newton polytope Q if for all non-zero terms x^α of p we have $\alpha \in Q$. I am interested in the following refined variant of Smale's problem.

Question 2.2 (Sparse Smale's 17th Problem)**.** *Let $Q \subset \mathbb{Z}^n$ be a lattice polytope, and suppose the cardinality of the set $Q \cap \mathbb{Z}^n$ is m. Let d denote the diameter of Q. Is there an algorithm that computes $\boldsymbol{a}$ zero of a system of n equations that have the Newton polytope Q on average polynomial time with respect to $n, \ln(d), m$?*

My published works towards sparse Smale's 17th problem, with different sets of co-authors, are the following: Developing a probabilistic toolbox that works for general models of random polynomials without any invariance assumption [EPR19, EPR21], developing numerical algorithms using a collection of norms instead of defaulting to Bombieri inner product [CETC22]. I am collaborating with Jonathan Leake (Waterloo) to solve sparse Smale's 17th problem. A basic difference with earlier work is we have to search for zeros on a toric variety; this is the space that captures sparsity of the input polynomials. Using some strong machinery, we have developed new algorithms to create a uniform sample on a toric variety. This allows us to define and compute with L_2 inner-products appropriate for sparse polynomials. The second basic difference is we cannot assume any invariance in the probability model for input polynomials, this is not a problem using my earlier work. Our current focus is to find the natural search algorithm, Newton's method if you will, on toric varieties.

3. Preconditioning: 3D-reconstruction and beyond

A basic problem in computer vision is 3D-reconstruction of objects from 2D images [HZ03]. A simplified version of the pipeline is as follows: One collects several images from different cameras and then match the images, that is, 2D-points in different images that correspond to same 3D point are marked. To be able to continue with reconstruction we need a certain number of marked points, and when we operate with the minimal number of points, we have a minimal problem. Minimal problems, and minimal solvers, are subject of a large literature [KP]. Here is a simple example, called 7-point problem, from this family: $x_1, \ldots, x_7 \in \mathbb{R}^2$ and $y_1, y_2, \ldots, y_7 \in \mathbb{R}^2$ are seven image points where x_i and y_i correspond to each other. Let x_i' and y_i' be homogenized image points, that is, $x_i' = (x_i, 1)$. The seven point problem is to find a 3×3 fundamental matrix F such that

$$x_i' F y_i' = 0 \ , \ \det F = 0$$

Stability and accuracy of minimal solvers has been an issue since the early days: A famous paper by Hartley suggested a trick to stabilize seven and eight point algorithms, and saved the latter from being abandoned[Har97].

In numerical analysis tradition, one defines a condition number to measure sensitivity of the solution map on a given input. High condition number means higher computational cost, and it could potentially mean no numerical method would solve the problem accurately. One would then try to find transformations that are cheap to undo, but very effective to reduce condition number of the input. These kind of transformations are called preconditioner.

Question 3.1 (Preconditioners on Minimal Solvers)**.** *Does Hartley preconditioner* [Har97] *work on all inputs? What is the average-case improvements in conditioning provided by Hartley's method? What are the natural generalizations of the method to other minimal problems?*

Despite the crucial importance of stability, and after a few decades since Hartley's paper, a rigorous analysis of the method is still lacking. Together with Rekha Thomas and Erin Connely (UW Seattle), Sameer Agarwal (Google) we are working on conditioning and preconditioning of minimal solvers. Describing the image data that gives rise to instability

(ill-posed locus) turned out to be an interesting problem with connections to classical algebraic geometry and invariant theory [CAET23, CTV23]. Our upcoming paper will give rigorous explanation for Hartley's trick and would provide a framework for condition numbers and preconditioners on minimal solvers[CAET24].

The more general research problem here is the following:

Question 3.2 (Preconditioning Real Algebraic Sets)**.** *Let* $\mathbf{p} = (p_1, p_2, \dots, p_m)$ *be a system polynomial equations with* n *unknowns. Consider the action of* $\mathbb{SL}(m)$ *on polynomial equations, and the action of* $\mathbb{SL}(n)$ *on variables. Can one design an efficient algorithm that finds the optimal group action* $g \in \mathbb{SL}(m) \times \mathbb{SL}(n)$ *on* $\mathbf{p}$ *to minimize the condition number?*

To the best of my knowledge there is no rigorous preconditioner on higher degree algebraic equations. The surprising fact is that the optimization problem in Question 3.2 can be tackled using geodesically convex optimization. We understand this after recent works in computational invariant theory [BFG+19, GGOW19, BFG+18]. This is a joint project with Levent Doğan who is writing a PhD dissertation in the field [BDM+21].

The dominant perspective in numerical analysis is to prove guarantees for preconditioners in all possible inputs. This worst-case perspective limits our ability to differentiate between methods that are very useful most of the time than the ones that are midly useful all the time. I'll only mention one project in this direction due to lack of space: A simple case of the general idea that is explained after Question 3.2 applied on linear systems (based on matrix scaling) is called equilibriate in Matlab. Despite emprical evidence, the performance of this preconditioner is still not well-understood see e.g. [AZLOW17].

Question 3.3 (Probabilistic Analysis of Equilibrate Preconditioner)**.** *Analyze the effects of equlibriate on Skeel condition number* [Ske79] *of Gaussian random matrices.*

4. Interface with convex optimization

We start with a fundamental result from [Put93]:

Theorem 4.1 (Putinar's Positivstellensatz)**.** *Consider the following semialgebraic set*

$$\mathcal{V} := \{x \in \mathbb{R}^n : g_1(x) \geq 0, g_2(x) \geq 0, \dots, g_m(x) \geq 0\}$$

If g_i *creates an Archimedean quadratic module (a condition similar to assuming* $\mathcal{V}$ *is compact), then*

$$f(x) > 0 \text{ for all } x \in \mathcal{V} \Leftrightarrow f(x) = u_o(x) + \sum_{i=1}^{m} g_i(x) u_i(x)$$

where u_i *are sum of squares (of polynomials).*

In simple terms, this theorem states that positivity on a general collection of domains like $\mathcal{V}$ can be certified by using sum of squares (SOS) of polynomials. In this sense, this theorem is a basic tool for inference: A very broad range of inference and optimization tasks can be formulated within the framework of this basic result, see e.g. [RSS18]. The connection to convex optimization comes from the fact that the search for sum of squares polynomials can be done via semi-definite programming. The main question is to understand how large is the degree of sum of squares polynomials u_i in a certificate for a given computational task. This general question is important for applications [Par00] and also for fundamental quests in theory of computing such as the unique games conjecture [FKP+19]. My published work in the field are [Erg18, Erg19]. For the sake of brevity I'll only state two projects below; hope this gives the flavor of my research goals in the field.

Question 4.1. *Given a randomly generated symmetric d-tensor of rank m with n-variables, what is the expected degree in SOS hierarchy to estimate its maxima (with constant error) over the unit sphere?*

Here the low-rank assumption is critical as the behaviour of SOS method on structured input is far from being well-understood.

Question 4.2 (Average-case positivstelle for quadratics)**.** *Given a system of random quadratic equations $q = (q_1, q_2, \ldots, q_m)$ with n variables, understand the computational thresholds for the number of equations $m > n$ and the lowest possible level d in sum of squares hierarchy to certify that there is no real solution to $q_1(x) = q_2(x) = \ldots = q_m(x) = 0$.*

I had observed that for the case $d = 2$ available results on concentration of mixed-volumes gives a sharp threshold for m. This observation is included as Section 5 in [HK22] and the paper also gives near-sharp bounds for $d = 4$. The problems remains open for any other d.

Real algebraic geometry pays back to convex optimization as follows: it offers a vast family of 'efficient" convex domains where the boundary is described by low-degree polynomial inequalities. When we can decide if a query point $x \in \mathbb{R}^n$ is in a convex domain K efficiently, then we can hope to optimize a linear function over K efficiently. A general question is to compare the expressive power of these efficient convex real algebraic domains to existing convex optimization paradigms such as semidefinite, second-order, or entropy programming. Here we focus on a very specific group of such domains given by *hyperbolic polynomials*.

Definition 4.2 (Hyperbolic Polynomial)**.** *A homogeneous polynomial p is hyperbolic with respect to direction $e \in \mathbb{R}^n$ if for all $x \in \mathbb{R}^n$ all zeros of the univariate polynomial $p(x + te)$ are real. We call p hyperbolic if there is such a direction e.*

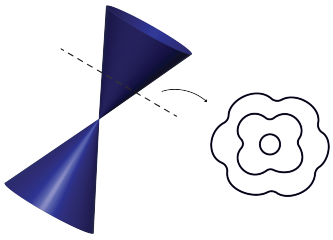


FIGURE 1. *A degree 6 hyperbolic polynomial with 3 variables*

Given a hyperbolic polynomial p of degree $2d$ with n variables we have the following facts [Ren06]: (1) the zero set of p is d nested ovals (cones), (2) the inner-most oval is convex, and its interior is given by all directions e so that p is hyperbolic with respect to e, (3) $\log p$ is a self-concordant barrier function for the convex inner oval. Due to third fact, one can use interior point method to optimize a linear function on the inner convex oval of a hyperbolic polynomial. This is called hyperbolic programming, and it includes semidefinite programming as the special case for $p = \det X$ and $e = \mathbb{I}$.

Although hyperbolic programming is formally more general than semidefinite programming, it is not clear if it effectively creates a more powerful framework. Suppose we are given a hyperbolic polynomial p of degree d with n variables, is it possible to express the inner convex oval of p approximately as a *projection of feasible region of a small semidefinite program*? If the answer to this question is affirmative, then semidefinite programming is as powerful as hyperbolic programming.

Question 4.3 (Semidefinite Representation of Hyperbolic Programs)**.** *Can every hyperbolic program given by a degree d homogeneous n-variate polynomial be ε-approximated, in Banach-Mazur distance, by projection of a section of the cone of $M \times M$ positive-semidefinite (PSD) matrices where $M \leq f(n, d, \log(\frac{1}{\varepsilon}))$ for a polynomial f?*

This is joint work with Grigoris Paouris and Petros Valettas. If we do not allow projections but only work with sections than the expressive power of hyperbolic programming changes drastically see e.g. [SP15], and in that restricted setting the question is settled [RRSW19].

5. Interface with computational phase transitions

Suppose we have a graph $G = (V, E)$ with n nodes, that is $V = \{1, \dots, n\} := [n]$. We would like to count or sample a cut on this graph, so we are thinking about functions $\sigma : V \to \{+1, -1\}$. For every edge $(i, j) \in E$ we attach a weight $\omega_{ij}(t)$, and the entire collection of weights is parametrized by $t > 0$. The setup is quite general as one can pick the weight functions $\omega_{ij}(t)$ as one pleases. The main question we would like to understand is for which parameter regimes for t sampling a cut is easy and for which it is very hard. This type of threshold phenomenon appears in all our basic algorithms problems, we typically ask where lies the hardness? In statistical physics, these type of threshold phenomenon has been studied to understand phase transtions in physical state of a system, e.g, ferromagnetic or anti-ferromagnetic, gas, liquid, or solid etc. The main mathematical construct in this world is the partition function, and for our running example it looks like this:

$$Z(u, t) = \sum_{S \subset [n]} u^S \prod_{i \in S, j \notin S} \omega_{ij}(t)$$

where $u = (u_1, u_2, \dots, u_n)$ represents the weights given to nodes and $u^S := \prod_{i \in S} u_i$. It is perhaps clear to the reader that $Z(u, t)$ encodes cuts of the graph G. After some manipulation and specification of u_i as functions of t, $Z(u, t)$ becomes the Ising model that explains the phase transitions in magnetic behaviour, see [Bar16, Section 7.4.4].

A surprising fact is that the partition functions that show up in questions like the one above are either hyperbolic polynomials, or they have hyperbolic like properties. Understanding the zero set of the partition function gives a way to design algorithms: if one could connect two configurations of edges with a line segment without crossing the zero set, one could track the change in the quantity of interest numerically [Bar16]. This connection to real algebraic geometry sparked my interest, and I have been learning the subject for a few years.

I'll briefly sketch my current interests and results in the subject: I am lucky to be able to collaborate with world-renowned subject expert Amin Coja-Oghlan to combine his deep knowledge of statistical physics techniques with my background in real algebraic geometry. Together with Amin and his collaborators we managed to combine algebraic machinery with cavity method from statistical physics to settle a conjecture of Lelarge [Lel13, COEG$^+$23]. To be more precise: Lelarge conjectured an asymptotic rank formula for sparse random matrices on $\mathbb{F}_2$, and we showed the formula holds for any field. Our next project with Amin is to apply real algebraic techniques on Anderson-Edwards model.

I am also interested in the reverse direction of this connection: Defining and using an appropriate partition function to design an algorithm that answers Question 1.3. My current line of attack is based on defining a partition function that comes from a localized version of Kac-Rice formula we obtained here [ETTC23], and using the basic approximation tools build by Barvinok [Bar16].

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