Reaction networks and a generalization of Descartes' rule of signs to hypersurfaces

Máté L. Telek[†]
joint work with Elisenda Feliu [†]

May 4, 2023

[†] Department of Mathematical Sciences
University of Copenhagen

THE PLAN FOR TODAY

- History and generalizations of Descartes' rule of signs

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- Motivation from reaction network theory

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(Classical) Descartes' rule of signs

Descartes' rule of signs

A univariate real polynomial cannot have more positive real roots than the number of sign changes in its coefficients sequence.

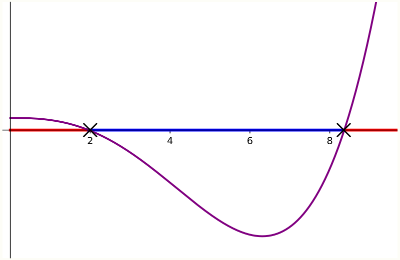

Figure 1: $59 + x - 4x^2 - 8x^3 + x^4$

History

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- 1918 - Curtiss [5]: proof for real exponents
- 1999 - Grabiner [6]: Descartes' bound is sharp

Generalization to polynomial equation systems

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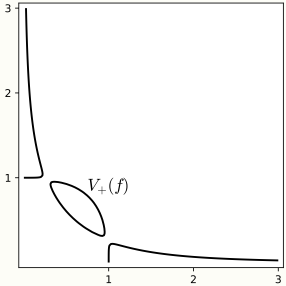
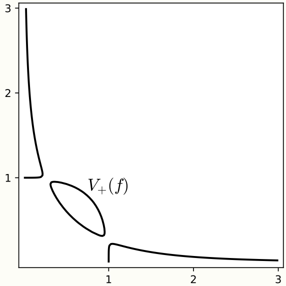


Figure 2: f(x,y) = 1 - x - y + x$^4 y + xy^4$

Generalizations to hypersurfaces

- 1991 - Khovanskii [9]: bound on the sum of Betti numbers of the positive real zero set of a polynomial, based on the number of variables and the number of monomials



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- 2009 - Bihan and Sottile [10] improve Khovanskii's bound on the sum of the Betti numbers

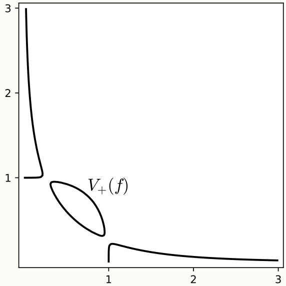

Figure 2: f(x,y) = 1 - x - y + $x^4y + xy^4$

Generalizations to hypersurfaces

- 2017 - Forsgård, Nisse and Rojas [11]: bound on the number of connected components of the positive real zero set of a polynomial, based on the number of variables and the number of monomials

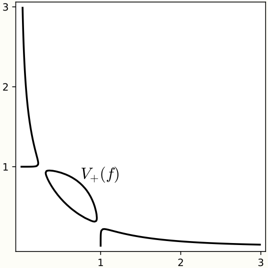

Figure 3: f(x,y) = 1 - x - y + $x^4y + xy^4$

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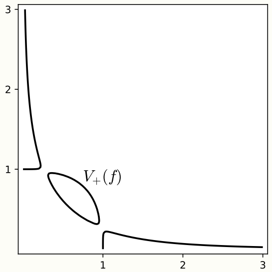

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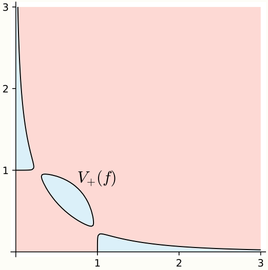

Figure 4: $f(x,y) = 1 - x - y + x^4y + xy^4$

Descartes' rule of signs for hypersurfaces

Problem

Consider a signomial $f\colon \mathbb{R}^n_{>0} \to \mathbb{R}$ with $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$, and $\sigma(f) \subseteq \mathbb{R}^n$ a finite set, called the support of f.

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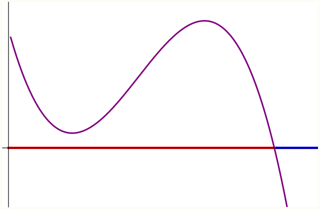
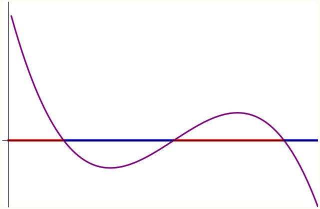
Find a (sharp) upper bound on the number of connected components $b_0\big(f^{-1}(\mathbb{R}_{<0})\big)$, based on the sign of the coefficients and the geometry of $\sigma(f)$.



(a) $15 - 20x + 9x^2 - x^3$



(b) $15 - 23x + 9x^2 - x^3$

MOTIVATION COMES FROM REACTION NETWORK THEORY

species $\{X_1, \dots, X_n\}$

reactions $\left\{ \sum_{i=1}^{n} a_{ij} X_i \xrightarrow{\kappa_i} \sum_{i=1}^{n} b_{ij} X_i \right\}_{j=1,\dots,r}$

running example

$\mathrm{X}_1, \quad \mathrm{X}_2$

$\mathrm{X}_1 \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} \mathrm{X}_2, \quad 2\mathrm{X}_1 + \mathrm{X}_2 \xrightarrow{\kappa_3} 3\mathrm{X}_1$

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reaction rate function $v_\kappa(x) = \kappa * x^A$

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Notation

The evolution of the concentrations of the species over time is modeled by the ODE system:

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The ODE system (1) is forward invariant on stoichiometric compatibility classes

$$\mathcal{P}_c = \{x \in \mathbb{R}^n_{\geq 0} \mid Wx = c\},$$

where $c \in \mathbb{R}^d, W \in \mathbb{R}^{d \times n}$ is a fullrank matrix such that $WN = 0$.

Running example

$$X_1 \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} X_2 \qquad 2X_1 + X_2 \xrightarrow{\kappa_3} 3X_1$$

$$\begin{aligned}
\dot{x}_1 &= \kappa_3 x_1^2 x_2 - \kappa_1 x_1 + \kappa_2 x_2 \\
\dot{x}_2 &= -\kappa_3 x_1^2 x_2 + \kappa_1 x_1 - \kappa_2 x_2
\end{aligned}$$

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$$W = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$x_1 + x_2 = c$$

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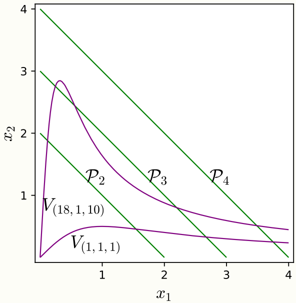
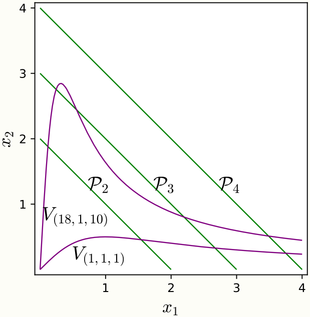


The parameter region of multistationarity

A parameter pair (κ, c) enables multistationarity, if the intersection of V_κ and $\mathcal{P}_c$ contains at least two positive points.

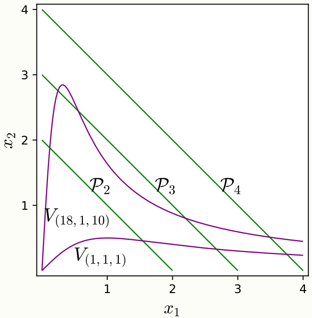


The parameter region of multistationarity

A parameter pair (κ, c) enables multistationarity, if the intersection of V_κ and $\mathcal{P}_c$ contains at least two positive points.

We call the set of all parameter pairs that enable multistationarity the parameter region of multistationarity.

$$\Omega := \left\{ (\kappa, c) \in \mathbb{R}^r_{>0} \times \mathbb{R}^d \mid \#\left(V_\kappa \cap \mathcal{P}_c \cap \mathbb{R}^n_{>0} \right) \geq 2 \right\}$$



Question:

What is the shape of the parameter region of multistationarity?

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Is it a connected set?

Criterion for connectivity

Theorem [2, Feliu, T.]

Under some assumption on the network, there exists a polynomial

$$q\colon \mathbb{R}^n_{>0} \times \mathbb{R}^\ell_{>0} \to \mathbb{R}$$

such that if $q^{-1}(\mathbb{R}_{<0})$ is connected and its closure equals $q^{-1}(\mathbb{R}_{\leq 0})$, then the set containing the parameter pairs (κ, c) which enable multistationarity is connected.

Running example

$$X_1 \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} X_2 \qquad 2X_1 + X_2 \xrightarrow{\kappa_3} 3X_1$$

$$q(h, \lambda) = h_1\lambda_2 - h_1\lambda_1 + h_2\lambda_1 + h_2\lambda_2$$

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$$q(h, \lambda) = h_1\lambda_2 - h_1\lambda_1 + h_2\lambda_1 + h_2\lambda_2$$

One can check that $q^{-1}(\mathbb{R}_{<0})$ is path connected and its closure equals $q^{-1}(\mathbb{R}_{\leq 0})$. So we can conclude that the parameter region of multistationarity is path connected.

Why do we care about such a generalization of Descartes' rule of signs?

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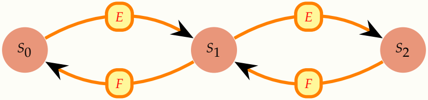
Example: 2-site phosphorylation system



$$\mathrm{S}_0 + \mathrm{E} \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} \mathrm{ES}_0 \xrightarrow{\kappa_3} \mathrm{S}_1 + \mathrm{E} \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} \mathrm{ES}_1 \xrightarrow{\kappa_9} \mathrm{S}_2 + \mathrm{E}$$

$$\mathrm{S}_2 + \mathrm{F} \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} \mathrm{FS}_2 \xrightarrow{\kappa_{12}} \mathrm{S}_1 + \mathrm{F} \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} \mathrm{FS}_1 \xrightarrow{\kappa_6} \mathrm{S}_0 + \mathrm{F}$$

2-SITE PHOSPHORYLATION SYSTEM

$$S_0 + E \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} ES_0 \xrightarrow{\kappa_3} S_1 + E \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} ES_1 \xrightarrow{\kappa_9} S_2 + E$$

$$S_2 + F \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} FS_2 \xrightarrow{\kappa_{12}} S_1 + F \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} FS_1 \xrightarrow{\kappa_6} S_0 + F$$

- Is $q^{-1}(\mathbb{R}_{<0})$ connected?

$q(h,\lambda) = -\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_6h_7 -$
$\lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_6h_7 - \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_6h_7 -$
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$\lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_6h_7 -$
$\lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_6h_7 - \lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_1h_2h_4h_6h_7 -$
$\lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_2h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_2h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_5^2h_0h_1h_2h_4h_6h_7$

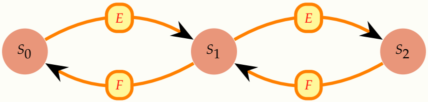
$-\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_1h_2h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_2h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_2h_4h_6h_7 - \lambda_2^3\lambda_4\lambda_5^2h_0h_1h_2h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_1h_2h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_5^3h_0h_1h_2h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_5^3h_0h_1h_2h_4h_6h_7 - \lambda_2^3\lambda_5^3h_0h_1h_2h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_1h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_3h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_5^2h_0h_1h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_1h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_3h_4h_6h_7 - \lambda_2^3\lambda_4\lambda_5^2h_0h_1h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_1h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_5^3h_0h_1h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_5^3h_0h_1h_3h_4h_6h_7 - \lambda_2^3\lambda_5^3h_0h_1h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_5^2h_0h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_4\lambda_5^2h_0h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_5^3h_0h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_5^3h_0h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_5^3h_0h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_1h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_3\lambda_5^2h_1h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_1h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_4\lambda_5^2h_1h_2h_3h_4h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_5^3h_1h_2h_3h_4h_6h_7 - \lambda_0\lambda_2^2\lambda_5^3h_1h_2h_3h_4h_6h_7 - \lambda_1\lambda_2^2\lambda_5^3h_1h_2h_3h_4h_6h_7 - \lambda_2^3\lambda_5^3h_1h_2h_3h_4h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_6h_7 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_1h_2h_3h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_5h_6h_7 + \lambda_2^3\lambda_3\lambda_5^2h_1h_2h_3h_5h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_1h_2h_3h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_5h_6h_7 + \lambda_2^3\lambda_4\lambda_5^2h_1h_2h_3h_5h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_5^3h_1h_2h_3h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_5^3h_1h_2h_3h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_5^3h_1h_2h_3h_5h_6h_7 + \lambda_2^3\lambda_5^3h_1h_2h_3h_5h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_1h_3h_4h_5h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_3h_4h_5h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_3h_4h_5h_6h_7 - \lambda_2^3\lambda_3\lambda_4\lambda_5h_1h_3h_4h_5h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_1h_3h_4h_5h_6h_7 - \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_1h_3h_4h_5h_6h_7 - \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_1h_3h_4h_5h_6h_7 - \lambda_2^3\lambda_3\lambda_5^2h_1h_3h_4h_5h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_1h_3h_4h_5h_6h_7 - \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_1h_3h_4h_5h_6h_7 - \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_1h_3h_4h_5h_6h_7 - \lambda_2^3\lambda_4\lambda_5^2h_1h_3h_4h_5h_6h_7 -$
$\lambda_0\lambda_1\lambda_2\lambda_5^3h_1h_3h_4h_5h_6h_7 - \lambda_0\lambda_2^2\lambda_5^3h_1h_3h_4h_5h_6h_7 - \lambda_1\lambda_2^2\lambda_5^3h_1h_3h_4h_5h_6h_7 - \lambda_2^3\lambda_5^3h_1h_3h_4h_5h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_7 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_7 + \lambda_2^3\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_7 +$
$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_7$

$$
\begin{aligned}
&+\lambda_2^3\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_7 + \lambda_0\lambda_1\lambda_2\lambda_5^3h_2h_3h_4h_5h_6h_7 + \lambda_0\lambda_2^2\lambda_5^3h_2h_3h_4h_5h_6h_7 + \lambda_1\lambda_2^2\lambda_5^3h_2h_3h_4h_5h_6h_7 + \\
&\lambda_2^3\lambda_5^3h_2h_3h_4h_5h_6h_7 + \lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_5h_8 + \\
&\lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_2h_3h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_2h_3h_5h_8 + \\
&\lambda_2^3\lambda_3\lambda_5^2h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_2h_3h_5h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_2h_3h_5h_8 + \\
&\lambda_2^3\lambda_4\lambda_5^2h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_1h_2h_3h_5h_8 + \lambda_0\lambda_2^2\lambda_5^3h_0h_1h_2h_3h_5h_8 + \lambda_1\lambda_2^2\lambda_5^3h_0h_1h_2h_3h_5h_8 + \lambda_2^3\lambda_5^3h_0h_1h_2h_3h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_1h_2h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_1h_2h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_2h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_2h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_5^2h_0h_1h_2h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_1h_2h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_2h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_2h_4h_5h_8 + \lambda_2^3\lambda_4\lambda_5^2h_0h_1h_2h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_1h_2h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_5^3h_0h_1h_2h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_5^3h_0h_1h_2h_4h_5h_8 + \lambda_2^3\lambda_5^3h_0h_1h_2h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_1h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_1h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_1h_3h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_5^2h_0h_1h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_1h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_1h_3h_4h_5h_8 + \lambda_2^3\lambda_4\lambda_5^2h_0h_1h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_1h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_5^3h_0h_1h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_5^3h_0h_1h_3h_4h_5h_8 + \lambda_2^3\lambda_5^3h_0h_1h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_5^2h_0h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_4\lambda_5^2h_0h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_5^3h_0h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_5^3h_0h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_5^3h_0h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_1h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_1h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_3\lambda_5^2h_1h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_1h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_4\lambda_5^2h_1h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_1h_2h_3h_4h_5h_8 + \lambda_0\lambda_2^2\lambda_5^3h_1h_2h_3h_4h_5h_8 + \lambda_1\lambda_2^2\lambda_5^3h_1h_2h_3h_4h_5h_8 + \lambda_2^3\lambda_5^3h_1h_2h_3h_4h_5h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_2h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_4h_5h_6h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_2h_4h_5h_6h_8
\end{aligned}
$$

$$+\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_2h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_4h_5h_6h_8 + \lambda_2^3\lambda_3\lambda_5^2h_0h_2h_4h_5h_6h_8 +$$
$$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_2h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_4h_5h_6h_8 + \lambda_2^3\lambda_4\lambda_5^2h_0h_2h_4h_5h_6h_8 +$$
$$\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_2h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_5^3h_0h_2h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_5^3h_0h_2h_4h_5h_6h_8 + \lambda_2^3\lambda_5^3h_0h_2h_4h_5h_6h_8 +$$
$$2\,\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_3h_4h_5h_6h_8 + 2\,\lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_3h_4h_5h_6h_8 +$$
$$2\,\lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_3h_4h_5h_6h_8 + 2\,\lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_3h_4h_5h_6h_8 +$$
$$2\,\lambda_2^3\lambda_3\lambda_5^2h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_3h_4h_5h_6h_8 + 2\,\lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_3h_4h_5h_6h_8 +$$
$$2\,\lambda_2^3\lambda_4\lambda_5^2h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_2^2\lambda_5^3h_0h_3h_4h_5h_6h_8 + 2\,\lambda_1\lambda_2^2\lambda_5^3h_0h_3h_4h_5h_6h_8 +$$
$$2\,\lambda_2^3\lambda_5^3h_0h_3h_4h_5h_6h_8 + \lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_8 +$$
$$\lambda_2^3\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_8 +$$
$$\lambda_2^3\lambda_3\lambda_5^2h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_8 +$$
$$\lambda_2^3\lambda_4\lambda_5^2h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_1\lambda_2\lambda_5^3h_2h_3h_4h_5h_6h_8 + \lambda_0\lambda_2^2\lambda_5^3h_2h_3h_4h_5h_6h_8 + \lambda_1\lambda_2^2\lambda_5^3h_2h_3h_4h_5h_6h_8 +$$
$$\lambda_2^3\lambda_5^3h_2h_3h_4h_5h_6h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_7h_8 + 2\,\lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_7h_8 +$$
$$2\,\lambda_2^3\lambda_3\lambda_4\lambda_5h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_5h_7h_8 + 2\,\lambda_1\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_3h_5h_7h_8 +$$
$$2\,\lambda_2^3\lambda_3\lambda_5^2h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_5h_7h_8 + 2\,\lambda_1\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_3h_5h_7h_8 +$$
$$2\,\lambda_2^3\lambda_4\lambda_5^2h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_1\lambda_2\lambda_5^3h_1h_2h_3h_5h_7h_8 + 2\,\lambda_0\lambda_2^2\lambda_5^3h_1h_2h_3h_5h_7h_8 + 2\,\lambda_1\lambda_2^2\lambda_5^3h_1h_2h_3h_5h_7h_8 +$$
$$2\,\lambda_2^3\lambda_5^3h_1h_2h_3h_5h_7h_8 + \lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_1h_2h_4h_5h_7h_8 +$$
$$\lambda_2^3\lambda_3\lambda_4\lambda_5h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_1h_2h_4h_5h_7h_8 +$$
$$\lambda_2^3\lambda_3\lambda_5^2h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_1h_2h_4h_5h_7h_8 +$$
$$\lambda_2^3\lambda_4\lambda_5^2h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_1\lambda_2\lambda_5^3h_1h_2h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_1h_2h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_1h_2h_4h_5h_7h_8 + \lambda_2^3\lambda_5^3h_1h_2h_4h_5h_7h_8 +$$
$$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_7h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_2h_3h_4h_5h_7h_8 +$$
$$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_2h_3h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_5h_7h_8 + \lambda_2^3\lambda_3\lambda_5^2h_2h_3h_4h_5h_7h_8 +$$
$$\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_2h_3h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_5h_7h_8 + \lambda_2^3\lambda_4\lambda_5^2h_2h_3h_4h_5h_7h_8 +$$
$$\lambda_0\lambda_1\lambda_2\lambda_5^3h_2h_3h_4h_5h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_2h_3h_4h_5h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_2h_3h_4h_5h_7h_8 + \lambda_2^3\lambda_5^3h_2h_3h_4h_5h_7h_8 -$$
$$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_6h_7h_8 - \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_6h_7h_8 - \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_2h_3h_6h_7h_8 - \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_2h_3h_6h_7h_8 -$$
$$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_2h_3h_6h_7h_8 - \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_3h_6h_7h_8 - \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_2h_3h_6h_7h_8 - \lambda_2^3\lambda_3\lambda_5^2h_0h_2h_3h_6h_7h_8$$

$$
\begin{aligned}
&-\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_2h_3h_6h_7h_8 - \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_3h_6h_7h_8 - \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_2h_3h_6h_7h_8 - \lambda_2^3\lambda_4\lambda_5^2h_0h_2h_3h_6h_7h_8 - \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_2h_3h_6h_7h_8 - \lambda_0\lambda_2^2\lambda_5^3h_0h_2h_3h_6h_7h_8 - \lambda_1\lambda_2^2\lambda_5^3h_0h_2h_3h_6h_7h_8 - \lambda_2^3\lambda_5^3h_0h_2h_3h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_0h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_0h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_0h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_0h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_0h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_0h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_5^2h_0h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_0h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_0h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_0h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_4\lambda_5^2h_0h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_0h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_0h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_0h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_5^3h_0h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_2h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_2h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_5^2h_2h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_2h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_4\lambda_5^2h_2h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_2h_3h_4h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_2h_3h_4h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_2h_3h_4h_6h_7h_8 + \lambda_2^3\lambda_5^3h_2h_3h_4h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_2h_3h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_3h_5h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_2h_3h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_2h_3h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_2h_3h_5h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_5^2h_2h_3h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_2h_3h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_2h_3h_5h_6h_7h_8 + \lambda_2^3\lambda_4\lambda_5^2h_2h_3h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_2h_3h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_2h_3h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_2h_3h_5h_6h_7h_8 + \lambda_2^3\lambda_5^3h_2h_3h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_2h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_2h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_2h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_2h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_2h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_2h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_5^2h_2h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_2h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_2h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_2h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_4\lambda_5^2h_2h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_2h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_2h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_2h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_5^3h_2h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5h_3h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_4\lambda_5h_3h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_4\lambda_5h_3h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_4\lambda_5h_3h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_3\lambda_5^2h_3h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_3\lambda_5^2h_3h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_3\lambda_5^2h_3h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_3\lambda_5^2h_3h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_4\lambda_5^2h_3h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_4\lambda_5^2h_3h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_4\lambda_5^2h_3h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_4\lambda_5^2h_3h_4h_5h_6h_7h_8 + \\
&\lambda_0\lambda_1\lambda_2\lambda_5^3h_3h_4h_5h_6h_7h_8 + \lambda_0\lambda_2^2\lambda_5^3h_3h_4h_5h_6h_7h_8 + \lambda_1\lambda_2^2\lambda_5^3h_3h_4h_5h_6h_7h_8 + \lambda_2^3\lambda_5^3h_3h_4h_5h_6h_7h_8
\end{aligned}
$$

2-SITE PHOSPHORYLATION SYSTEM

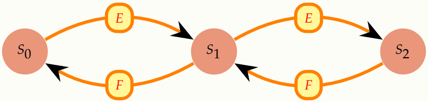


$$S_0 + E \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} ES_0 \xrightarrow{\kappa_3} S_1 + E \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} ES_1 \xrightarrow{\kappa_9} S_2 + E$$

$$S_2 + F \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} FS_2 \xrightarrow{\kappa_{12}} S_1 + F \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} FS_1 \xrightarrow{\kappa_6} S_0 + F$$

- number of variables of $q = 15$
- $\#\sigma(q) = 400$

2-SITE PHOSPHORYLATION SYSTEM



$$S_0 + E \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} ES_0 \xrightarrow{\kappa_3} S_1 + E \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} ES_1 \xrightarrow{\kappa_9} S_2 + E$$

$$S_2 + F \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} FS_2 \xrightarrow{\kappa_{12}} S_1 + F \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} FS_1 \xrightarrow{\kappa_6} S_0 + F$$

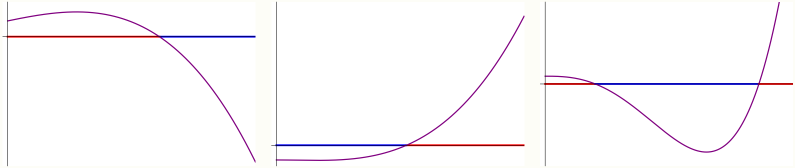
- number of variables of $q = 15$
- $\#\sigma(q) = 400$
- Is $q^{-1}(\mathbb{R}_{<0})$ connected?

DESCARTES' RULE OF SIGNS FOR HYPERSURFACES

Problem

Consider a signomial $f\colon \mathbb{R}^n_{>0} \to \mathbb{R}$ with $f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$, and $\sigma(f) \subseteq \mathbb{R}^n$ a finite set, called the support of f.

Find a (sharp) upper bound on the number of connected components $b_0\big(f^{-1}(\mathbb{R}_{<0})\big)$, based on the sign of the coefficients and the geometry of $\sigma(f)$.



(a) $50 + 20x - x^2 - x^3$
$+ \cdots + - \cdots -$

(b) $-42 - 2x^2 + x^3$
$- \cdots - + \cdots +$

(c) $59 + x - 4x^2 - 8x^3 + x^4$
$+ \cdots + - \cdots - + \cdots +$

Positive and negative support

Definition

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}, x \mapsto f(x) = \sum_{\mu \in \sigma(f)} c_\mu x^\mu$ be a signomial. The positive (resp. negative) support of f is defined as follows:

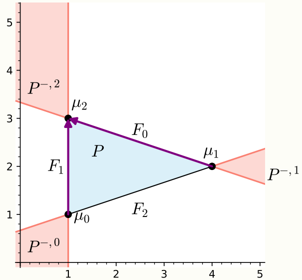
$$\sigma_+(f) := \{\mu \in \sigma(f) \mid c_\mu > 0\},$$
$$\sigma_-(f) := \{\mu \in \sigma(f) \mid c_\mu < 0\}.$$

Negative vertex cones

Definition

Given an n-simplex $P \subseteq \mathbb{R}^n$ with vertices $\{\mu_0, \ldots, \mu_n\}$, we define the *negative vertex cone* at the vertex μ_k as

$$P^{-,k} := \mu_k + \mathrm{Cone}(\mu_k - \mu_0, \ldots, \mu_k - \mu_n)$$



Negative vertex cones

Definition

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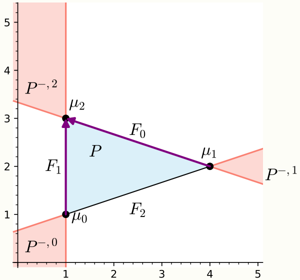
$$P^{-,k} := \mu_k + \operatorname{Cone}(\mu_k - \mu_0, \ldots, \mu_k - \mu_n)$$

we write $P^{-} := \bigcup_{i=0}^{n} P^{-,k}$



NEGATIVE VERTEX CONES

Theorem [1, Feliu, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If there exists a n-simplex P such that $\sigma_-(f) \subset P$ and $\sigma_+(f) \subset P^-$, then $f^{-1}(\mathbb{R}_{<0})$ is either empty or contractible. In particular, $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) \leq 1$.

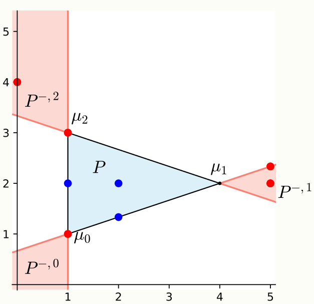


Figure 7: $f(x,y) = x^5y^{\frac{7}{3}} + x^5y^2 - x^2y^2 + xy^3 + y^4 - 2\,x^2y^{\frac{4}{3}} - 2\,xy^2 + xy$

Theorem [1, Feliu, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If there exists a n-simplex P such that $\sigma_-(f) \subset P$ and $\sigma_+(f) \subset P^-$, then $f^{-1}(\mathbb{R}_{<0})$ is either empty or contractible. In particular, $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) \leq 1$.

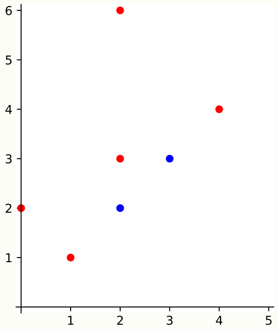


Figure 8: $x_1^4x_2^4 + x_1^2x_2^6 + x_1x_2 + x_2^2 - 5x_1^3x_2^3 - 3x_1^2x_2^2$

Negative vertex cones

Theorem [1, Feliu, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If there exists a n-simplex P such that $\sigma_-(f) \subset P$ and $\sigma_+(f) \subset P^-$, then $f^{-1}(\mathbb{R}_{<0})$ is either empty or contractible. In particular, $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) \leq 1$.

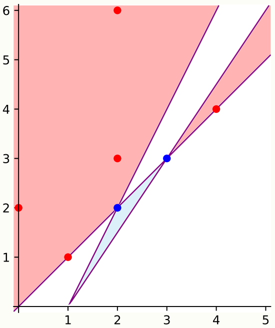
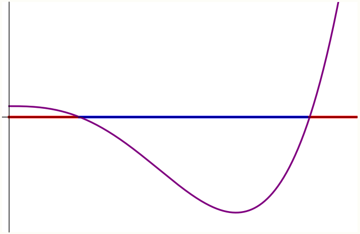


Figure 9: $x_1^4x_2^4 + x_1^2x_2^6 + x_1x_2 + x_2^2 - 5x_1^3x_2^3 - 3x_1^2x_2^2$

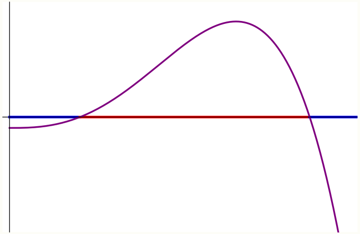
CAN WE FLIP THE SIGNS?



(a) f(x) = $59 + x - 4x^2 - 8x^3 + x^4$

$+ \cdots + - \cdots - + \cdots +$

$f^{-1}(\mathbb{R}_{<0})$ is connected



(b) -f(x) = $-59 - x + 4x^2 + 8x^3 - x^4$

$- \cdots - + \cdots + - \cdots -$

$(-f)^{-1}(\mathbb{R}_{<0})$ is not connected

Can we flip the signs?

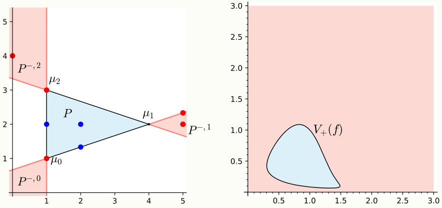


Figure 11: $f(x,y) = x^5y^{\frac{7}{3}} + x^5y^2 - x^2y^2 + xy^3 + y^4 - 2\,x^2y^{\frac{4}{3}} - 2\,xy^2 + xy$

Can we flip the signs?

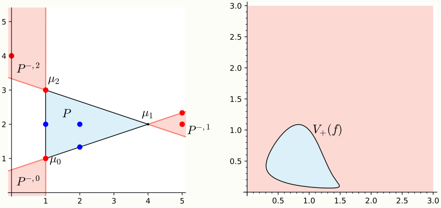


Figure 11: $f(x,y) = x^5y^{\frac{7}{3}} + x^5y^2 - x^2y^2 + xy^3 + y^4 - 2\,x^2y^{\frac{4}{3}} - 2\,xy^2 + xy$

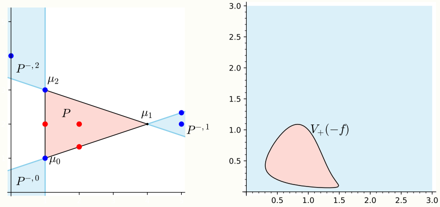


Figure 12: $-f(x,y) = -x^5y^{\frac{7}{3}} - x^5y^2 + x^2y^2 - xy^3 - y^4 + 2\,x^2y^{\frac{4}{3}} + 2\,xy^2 - xy$

Can we flip the signs?

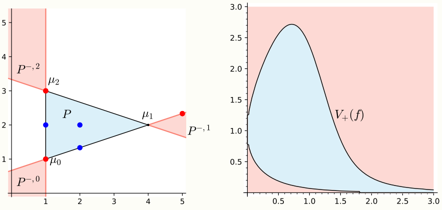


Figure 13: $f(x,y) = x^5y^{\frac{7}{3}} - x^2y^2 + xy^3 - 2\,x^2y^{\frac{4}{3}} - 2\,xy^2 + xy$

Can we flip the signs?

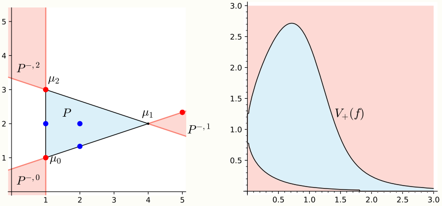


Figure 13: $f(x,y) = x^5y^{\frac{7}{3}} - x^2y^2 + xy^3 - 2\,x^2y^{\frac{4}{3}} - 2\,xy^2 + xy$

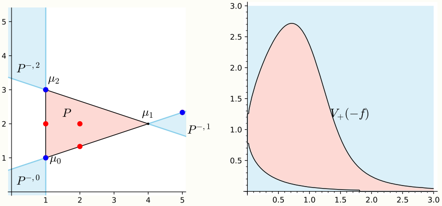
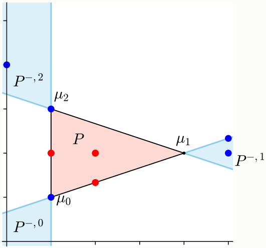


Figure 14: $-f(x,y) = -x^5y^{\frac{7}{3}} + x^2y^2 - xy^3 + 2\,x^2y^{\frac{4}{3}} + 2\,xy^2 - xy$

Negative vertex cones

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial such that $n \geq 2$. Assume that there exists an n-simplex $P \subset \mathbb{R}^n$ such that $\sigma_+(f) \subset P$ and $\sigma_-(f) \subset P^-$. If $\sigma_-(f) \cap \operatorname{int}(P^-) \neq \emptyset$, then $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) = 1$.



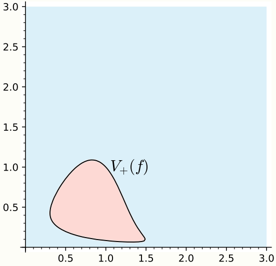


Figure 15: $f(x,y) = x^5y^{\frac{7}{3}} + x^5y^2 - x^2y^2 + xy^3 + y^4 - 2\,x^2y^{\frac{4}{3}} - 2\,xy^2 + xy$

Theorem [1, Feliu, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If f has at most one negative coefficient, then $\mathrm{Log}(f^{-1}(\mathbb{R}_{<0}))$ is a convex set. In particular, $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) \leq 1$.

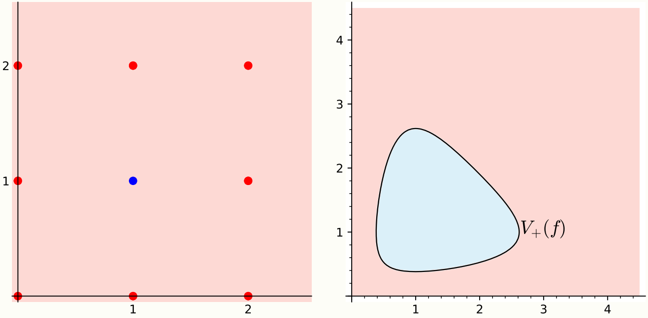


Figure 16: $f(x_1, x_2) = x_1^2x_2^2 + x_1^2x_2 + x_1x_2^2 + x_1^2 - 11x_1x_2 + x_2^2 + x_1 + x_2 + 1$

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial such that $n \geq 2$. If f has at most one positive coefficient, then $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) = 1$.

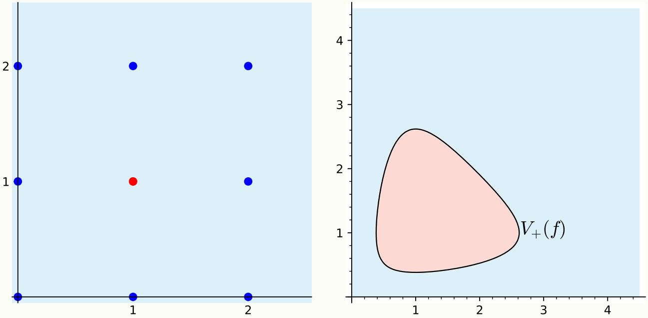
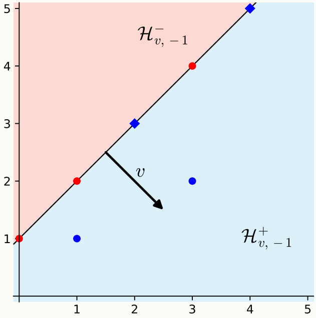


Figure 17: $f(x_1, x_2) = -x_1^2x_2^2 - x_1^2x_2 - x_1x_2^2 - x_1^2 + 11x_1x_2 - x_2^2 - x_1 - x_2 - 1$

Separating vector of the support

Theorem [1, Feliu, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If there exists a strict separating vector of $\sigma(f)$, then $f^{-1}(\mathbb{R}_{<0})$ is non-empty and contractible. In particular, $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) = 1$



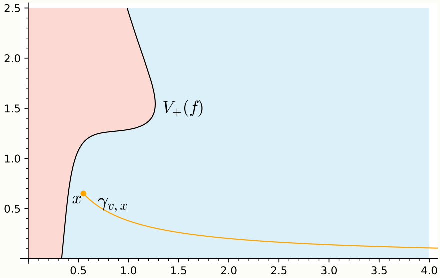


Figure 18: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$

Paths on logarithmic scale

Given $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n_{>0}$, we consider continuous paths

$$\gamma_{v,x} \colon [1, \infty) \to \mathbb{R}^n_{>0}, \qquad t \mapsto (t^{v_1} x_1, \ldots, t^{v_n} x_n).$$

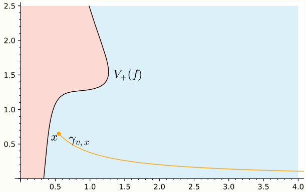


Figure 19: $f(x_1, x_2) = 3x_1^3 x_2^4 + x_1 x_2^2 + x_2 - x_1^2 x_2^3 - x_1^4 x_2^5 - 3x_1 x_2 - x_1^3 x_2^2$

Paths on logarithmic scale

Given $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n_{>0}$, we consider continuous paths

$$\gamma_{v,x} \colon [1, \infty) \to \mathbb{R}^n_{>0}, \qquad t \mapsto (t^{v_1} x_1, \dots, t^{v_n} x_n).$$

these paths are transformed into half-lines $[0, \infty) \to \mathbb{R}^n, s \mapsto sv + \mathrm{Log}(x)$, under the coordinate-wise logarithm

$$\mathrm{Log} \colon \mathbb{R}^n_{>0} \to \mathbb{R}^n, \qquad (x_1, \dots, x_n) \mapsto (\log(x_1), \dots, \log(x_n)),$$

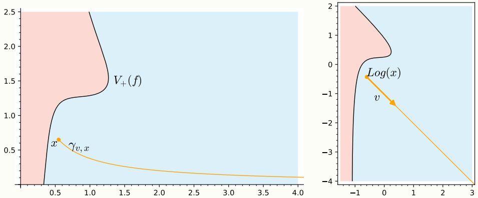


Figure 19: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$

Paths on logarithmic scale

Given $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n_{>0}$, we consider continuous paths

$$\gamma_{v,x} \colon [1, \infty) \to \mathbb{R}^n_{>0}, \qquad t \mapsto (t^{v_1} x_1, \ldots, t^{v_n} x_n).$$

Composing a signomial f with $\gamma_{v,x}$ we have a function in one variable:

$$f_{v,x} := f \circ \gamma_{v,x} \colon [1, \infty) \to \mathbb{R}, \qquad t \mapsto \sum_{\mu \in \sigma(f)} (c_\mu x^\mu) t^{v \cdot \mu}.$$

Example: $v = (1, -1)$

$f(x_1, x_2) = 3x_1^3 x_2^4 + x_1 x_2^2 + x_2 - x_1^2 x_2^3 - x_1^4 x_2^5 - 3x_1 x_2 - x_1^3 x_2^2$

$f_{v,x}(t) = -x_1^3 x_2^2 t^1 - 3x_1 x_2 t^0 + (3x_1^3 x_2^4 + x_1 x_2^2 + x_2 - x_1^2 x_2^3 - x_1^4 x_2^5) t^{-1}$

AFFINE HYPERPLANES

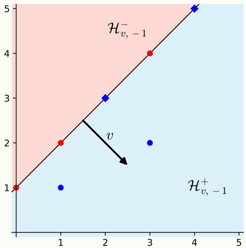
Definition

A vector $v \in \mathbb{R}^n \setminus \{0\}$ and $a \in \mathbb{R}$ define *an affine hyperplane*

$$\mathcal{H}_{v,a} := \{x \in \mathbb{R}^n \mid v \cdot x = a\},$$

and two *half-spaces*:

$$\mathcal{H}^+_{v,a} := \{x \in \mathbb{R}^n \mid v \cdot x \geq a\}, \quad \mathcal{H}^-_{v,a} := \{x \in \mathbb{R}^n \mid v \cdot x \leq a\}.$$



Definition

We say that $v \in \mathbb{R}^n \setminus \{0\}$ is a *separating vector* of $\sigma(f)$ if for some $a \in \mathbb{R}$ we have:

$$\sigma_-(f) \subseteq \mathcal{H}^+_{v,a}, \qquad \sigma_+(f) \subseteq \mathcal{H}^-_{v,a}.$$

The affine hyperplane $\mathcal{H}_{v,a}$ is called a *separating hyperplane* of $\sigma(f)$.

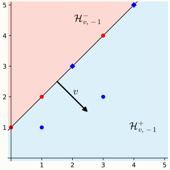


Figure 20: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$, $f_{v,x}(t) = -x_1^3x_2^2t^1 - 3x_1x_2t^0 + (3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5)t^{-1}$

Paths on logarithmic scale

Definition

We say that $v \in \mathbb{R}^n \setminus \{0\}$ is a *separating vector* of $\sigma(f)$ if for some $a \in \mathbb{R}$ we have:

$$\sigma_-(f) \subseteq \mathcal{H}^+_{v,a}, \qquad \sigma_+(f) \subseteq \mathcal{H}^-_{v,a}.$$

The affine hyperplane $\mathcal{H}_{v,a}$ is called a *separating hyperplane* of $\sigma(f)$. A separating vector v is called *strict separating vector* if

$$\sigma_-(f) \cap \mathcal{H}^{+,\circ}_{v,a} \neq \emptyset.$$

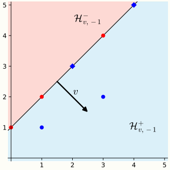
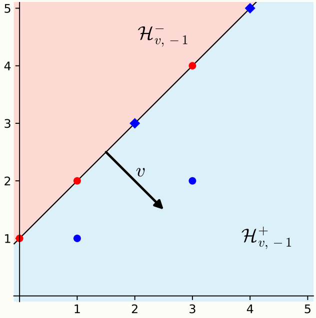


Figure 20: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$, $f_{v,x}(t) = -x_1^3x_2^2t^1 - 3x_1x_2t^0 + (3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5)t^{-1}$

Lemma

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial and $x \in f^{-1}(\mathbb{R}_{<0})$. If $v \in \mathbb{R}^n$ is a separating vector of $\sigma(f)$, then $f_{v,x}(t) < 0$ for all $t \in [1, \infty)$.



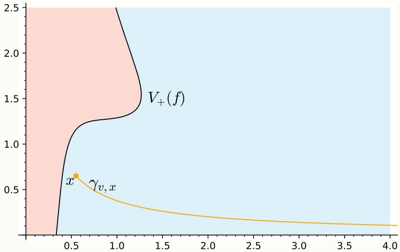
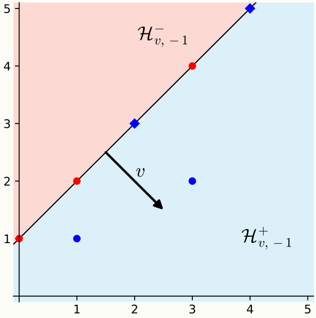


Figure 21: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$
$f_{v,x}(t) = -x_1^3x_2^2t^1 - 3x_1x_2t^0 + (3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5)t^{-1}$

Theorem [1, Feliu, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. If there exists a strict separating vector of $\sigma(f)$, then $f^{-1}(\mathbb{R}_{<0})$ is non-empty and contractible. In particular, $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) = 1$.



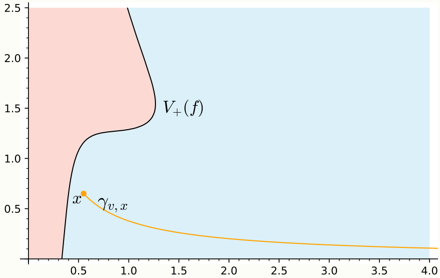


Figure 22: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$

Separating vector of the support

Sketch of proof:

- Consider the signomial

$$\tilde{f}(x) := \sum_{\alpha \in \sigma_+(f)} c_\alpha x^\alpha + \sum_{\beta \in \sigma_-(f) \cap \mathcal{H}^{+;\circ}_{v,a}} c_\beta x^\beta$$

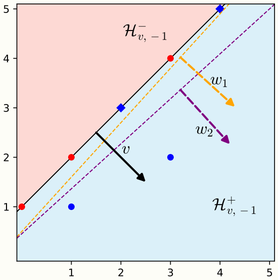


Figure 23: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$,
$\tilde{f}(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - 3x_1x_2 - x_1^3x_2^2$

Sketch of proof:

- Consider the signomial

$$\tilde{f}(x) := \sum_{\alpha \in \sigma_+(f)} c_\alpha x^\alpha + \sum_{\beta \in \sigma_-(f) \cap \mathcal{H}_{v,a}^{+,\circ}} c_\beta x^\beta$$

- Find linearly independent separating vectors $w_1, \ldots, w_n$ of $\sigma(\tilde{f})$ such that $v \in \mathrm{Cone}^\circ(w_1, \ldots, w_n)$

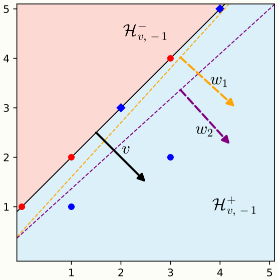


Figure 23: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$, $\tilde{f}(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - 3x_1x_2 - x_1^3x_2^2$

Separating vector of the support

Sketch of proof:

- $\tilde{f}^{-1}(\mathbb{R}_{<0}) \subset f^{-1}(\mathbb{R}_{<0})$

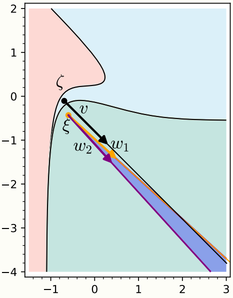


Figure 24: $f(x_1,x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$,
$\tilde{f}(x_1,x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - 3x_1x_2 - x_1^3x_2^2$

Separating vector of the support

Sketch of proof:

- $\tilde{f}^{-1}(\mathbb{R}_{<0}) \subset f^{-1}(\mathbb{R}_{<0})$
- for all $x \in \tilde{f}^{-1}(\mathbb{R}_{<0})$ holds $\xi + \mathrm{Cone}(w_1, \ldots, w_n) \subset \mathrm{Log}(f^{-1}(\mathbb{R}_{<0}))$,

where $\xi := \mathrm{Log}(x)$.

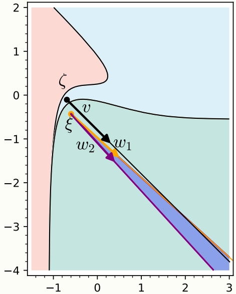


Figure 24: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$,
$\tilde{f}(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - 3x_1x_2 - x_1^3x_2^2$

Sketch of proof:

- $\tilde{f}^{-1}(\mathbb{R}_{<0}) \subset f^{-1}(\mathbb{R}_{<0})$
- for all $x \in \tilde{f}^{-1}(\mathbb{R}_{<0})$ holds $\xi + \mathrm{Cone}(w_1, \ldots, w_n) \subset \mathrm{Log}(f^{-1}(\mathbb{R}_{<0}))$, where $\xi := \mathrm{Log}(x)$.
- connect each $\zeta = \mathrm{Log}(y) \in \mathrm{Log}(f^{-1}(\mathbb{R}_{<0}))$ to $\xi + \mathrm{Cone}(w_1, \ldots, w_n)$ via the path $\mathrm{Log} \circ \gamma_{v,y}$.

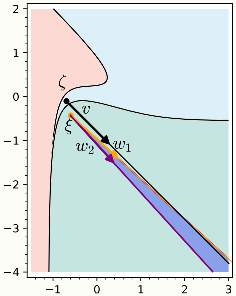
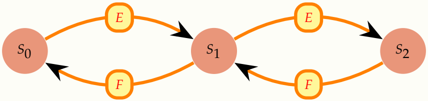


Figure 24: $f(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - x_1^2x_2^3 - x_1^4x_2^5 - 3x_1x_2 - x_1^3x_2^2$,
$\tilde{f}(x_1, x_2) = 3x_1^3x_2^4 + x_1x_2^2 + x_2 - 3x_1x_2 - x_1^3x_2^2$

2-site phosphorylation system

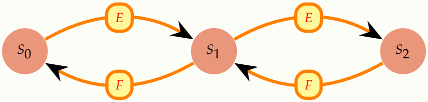


$$\mathrm{S_0 + E} \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} \mathrm{ES_0} \xrightarrow{\kappa_3} \mathrm{S_1 + E} \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} \mathrm{ES_1} \xrightarrow{\kappa_9} \mathrm{S_2 + E}$$

$$\mathrm{S_2 + F} \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} \mathrm{FS_2} \xrightarrow{\kappa_{12}} \mathrm{S_1 + F} \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} \mathrm{FS_1} \xrightarrow{\kappa_6} \mathrm{S_0 + F}$$

- number of variables of $q = 15$
- $\#\sigma_+(q) = 288, \#\sigma_-(q) = 112$

2-site phosphorylation system



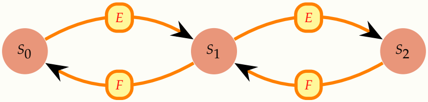
$$S_0 + E \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} ES_0 \xrightarrow{\kappa_3} S_1 + E \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} ES_1 \xrightarrow{\kappa_9} S_2 + E$$

$$S_2 + F \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} FS_2 \xrightarrow{\kappa_{12}} S_1 + F \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} FS_1 \xrightarrow{\kappa_6} S_0 + F$$

- number of variables of $q = 15$
- $\#\sigma_+(q) = 288, \#\sigma_-(q) = 112$
- $\sigma(q)$ has a strict separating hyperplane $(0.28\ s)$

APPLICATIONS

2-site phosphorylation system

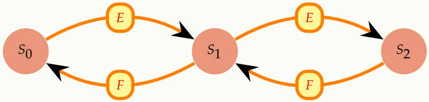


$$\mathrm{S}_0 + \mathrm{E} \underset{\kappa_2}{\overset{\kappa_1}{\rightleftharpoons}} \mathrm{ES}_0 \xrightarrow{\kappa_3} \mathrm{S}_1 + \mathrm{E} \underset{\kappa_8}{\overset{\kappa_7}{\rightleftharpoons}} \mathrm{ES}_1 \xrightarrow{\kappa_9} \mathrm{S}_2 + \mathrm{E}$$

$$\mathrm{S}_2 + \mathrm{F} \underset{\kappa_{11}}{\overset{\kappa_{10}}{\rightleftharpoons}} \mathrm{FS}_2 \xrightarrow{\kappa_{12}} \mathrm{S}_1 + \mathrm{F} \underset{\kappa_5}{\overset{\kappa_4}{\rightleftharpoons}} \mathrm{FS}_1 \xrightarrow{\kappa_6} \mathrm{S}_0 + \mathrm{F}$$

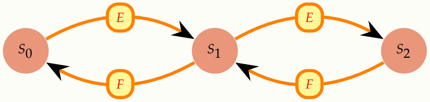
- number of variables of $q = 15$
- $\#\sigma_+(q) = 288, \#\sigma_-(q) = 112$
- $\sigma(q)$ has a strict separating hyperplane $(0.28\ s)$
- the set containing the parameter pairs (κ, c) which enable multistationarity is connected.

Other examples



	n	r	ℓ	$\#\sigma_+(q)$	$\#\sigma_-(q)$	t. comp. q	t. find sep. hyp.
HHK	6	6	2	17	2	0.03 s	0.01 s
2-site	9	12	6	288	112	0.99 s	0.28 s
3-site	12	18	9	2560	1536	1 m 24 s	4.4 s
4-site	15	24	12	??	??	∞	??
2-site w.irr.	13	20	10	1020	228	43.11 s	does not exist
2 site F_i	10	12	6	304	48	1.84 s	0.4 s
2 substr.	12	15	8	5088	224	35.68 s	10.36 s
ERK	12	18	9	15040	3432	4 m 4 s	49 s

Other examples



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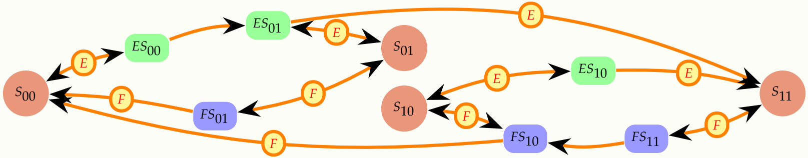
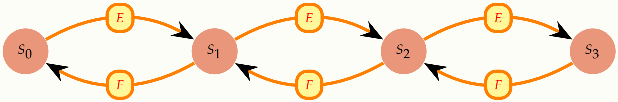


Other examples



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2-site w.irr.	13	20	10	1020	228	43.11 s	does not exist
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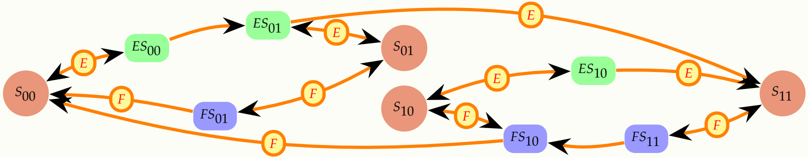
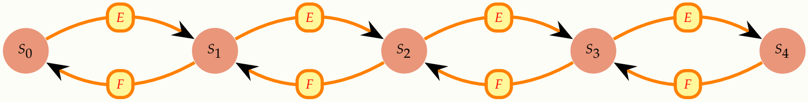


Other examples



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4-site	15	24	12	??	??	∞	??
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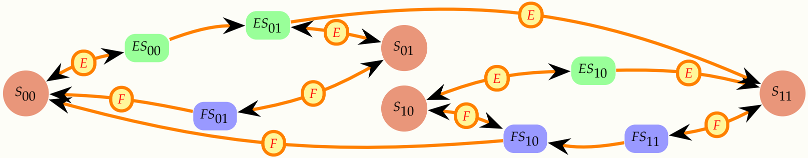
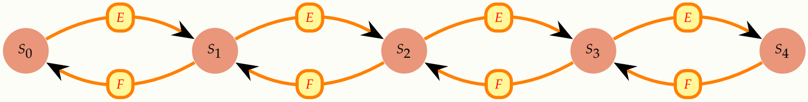


Other examples



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2-site	9	12	6	288	112	0.99 s	0.28 s
3-site	12	18	9	2560	1536	1 m 24 s	4.4 s
4-site	15	24	12	75	54	0.53 s	does not exist
2-site w.irr.	13	20	10	1020	228	43.11 s	does not exist
2 site F_i	10	12	6	304	48	1.84 s	0.4 s
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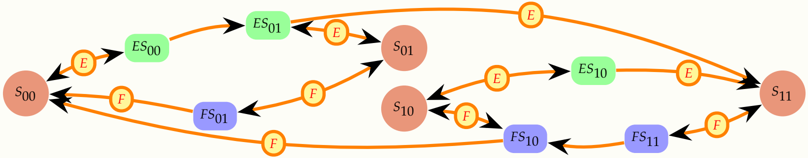
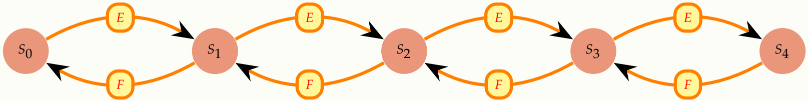


Other examples



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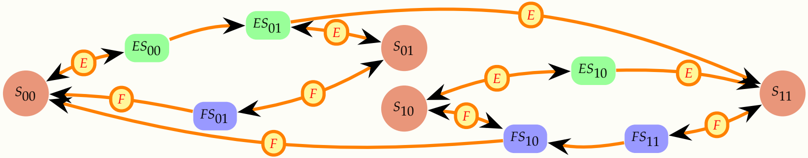

Faces of the Newton polytope

Definition

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. The Newton polytope of f is

$$\mathrm{N}(f) := \mathrm{Conv}\left(\sigma(f)\right).$$

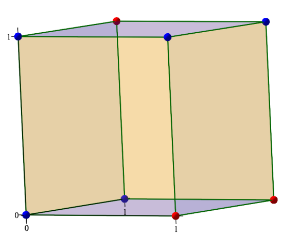

Figure 25: Newton polytope of
$f(x, y, z) = x + xy - y - 1 + yz - z - xz - xyz$

Faces of the Newton polytope

Definition

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial. The Newton polytope of f is

$$\mathrm{N}(f) := \mathrm{Conv}\left(\sigma(f)\right).$$

A face of $\mathrm{N}(f)$ is a set of the form

$$\mathrm{N}(f)_v := \left\{p \in \mathrm{N}(f) \mid v \cdot p = \max_{\mu \in \mathrm{N}(f)} v \cdot \mu\right\}, \quad \text{for } v \in \mathbb{R}^n.$$

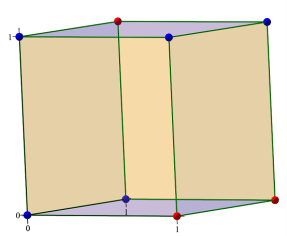

Figure 25: Newton polytope of
$f(x,y,z) = x + xy - y - 1 + yz - z - xz - xyz$

Reduction to a negative face of the Newton polytope

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$, $x \mapsto \sum_{\mu \in \sigma(f)} c_\mu x^\mu$ be a signomial. If there exists a face F of the Newton polytope $\mathrm{N}(f)$ such that $\sigma_-(f) \subseteq F$, then

$$b_0\big(f^{-1}(\mathbb{R}_{<0})\big) = b_0\big(f_{|F}^{-1}(\mathbb{R}_{<0})\big),$$

where $f_{|F}(x) = \sum_{\mu \in \sigma(f) \cap F} c_\mu x^\mu$.

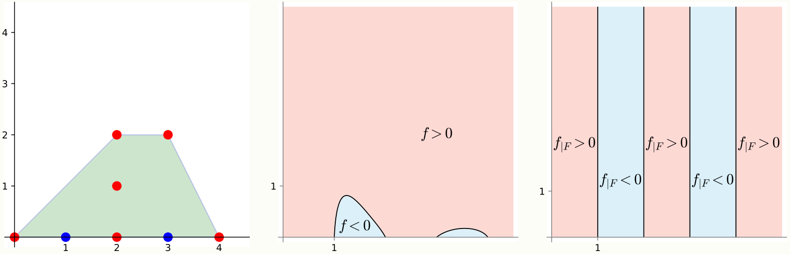

Figure 26: $2x^4 - 20x^3 + 70x^2 - 100x + 48 + 0.5x^3y^2 + +0.5x^2y^2 + 0.5x^2y$

Reduction to a negative face of the Newton polytope

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Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$, $x \mapsto \sum_{\mu \in \sigma(f)} c_\mu x^\mu$ be a signomial. If there exists a face F of the Newton polytope $\mathrm{N}(f)$ such that $\sigma_-(f) \subseteq F$, then

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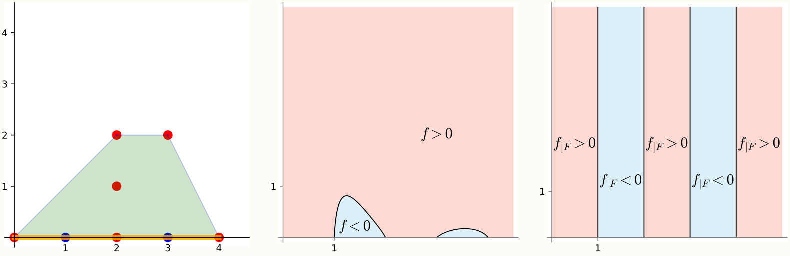


Figure 27: $2x^4 - 20x^3 + 70x^2 - 100x + 48 + 0.5x^3y^2 + +0.5x^2y^2 + 0.5x^2y$

Reduction to parallel faces of the Newton polytope

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial Assume that there exists $v \in \mathbb{R}^n$ such that $\sigma(f) \subseteq \mathrm{N}(f)_v \cup \mathrm{N}(f)_{-v}$.

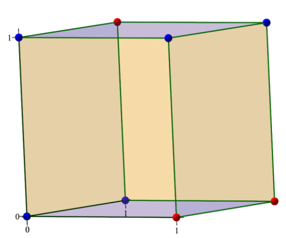


Figure 28: Newton polytope of
$f(x, y, z) = x + xy - y - 1 + yz - z - xz - xyz$

Reduction to parallel faces of the Newton polytope

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial Assume that there exists $v \in \mathbb{R}^n$ such that $\sigma(f) \subseteq \mathrm{N}(f)_v \cup \mathrm{N}(f)_{-v}$. If

- $b_0\big(f^{-1}_{|N(f)_v}(\mathbb{R}_{<0})\big) = b_0\big(f^{-1}_{|N(f)_{-v}}(\mathbb{R}_{<0})\big) = 1$,

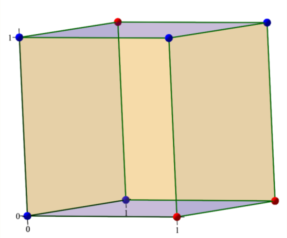


Figure 28: Newton polytope of
$f(x,y,z) = x + xy - y - 1 + yz - z - xz - xyz$

Reduction to parallel faces of the Newton polytope

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial Assume that there exists $v \in \mathbb{R}^n$ such that $\sigma(f) \subseteq \mathrm{N}(f)_v \cup \mathrm{N}(f)_{-v}$. If

- $b_0\big(f^{-1}_{|N(f)_v}(\mathbb{R}_{<0})\big) = b_0\big(f^{-1}_{|N(f)_{-v}}(\mathbb{R}_{<0})\big) = 1$, and
- $f^{-1}_{|N(f)_v}(\mathbb{R}_{<0}) \cap f^{-1}_{|N(f)_{-v}}(\mathbb{R}_{<0}) \neq \emptyset$,

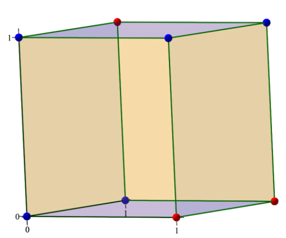


Figure 28: Newton polytope of $f(x,y,z) = x + xy - y - 1 + yz - z - xz - xyz$

Reduction to parallel faces of the Newton polytope

Theorem [3, T.]

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial Assume that there exists $v \in \mathbb{R}^n$ such that $\sigma(f) \subseteq \mathrm{N}(f)_v \cup \mathrm{N}(f)_{-v}$. If

- $b_0\big(f^{-1}_{|N(f)_v}(\mathbb{R}_{<0})\big) = b_0\big(f^{-1}_{|N(f)_{-v}}(\mathbb{R}_{<0})\big) = 1$, and
- $f^{-1}_{|N(f)_v}(\mathbb{R}_{<0}) \cap f^{-1}_{|N(f)_{-v}}(\mathbb{R}_{<0}) \neq \emptyset$,

then $b_0\big(f^{-1}(\mathbb{R}_{<0})\big) = 1$.

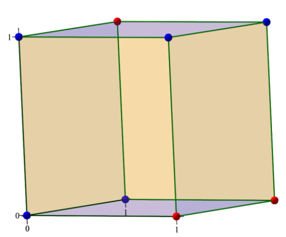


Figure 28: Newton polytope of $f(x,y,z) = x + xy - y - 1 + yz - z - xz - xyz$

Reduction to faces of the Newton polytope

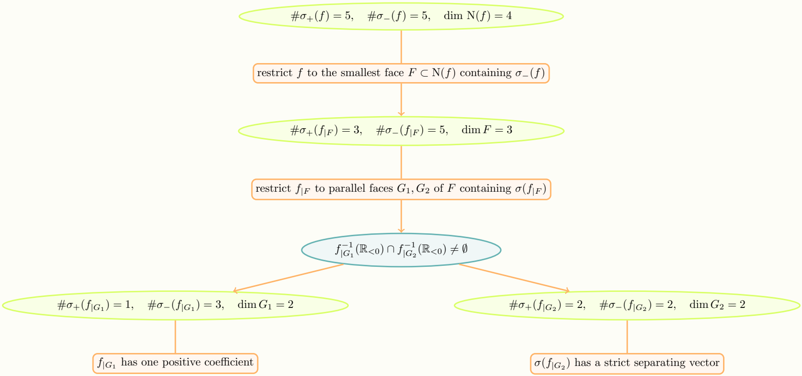
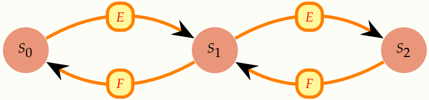


Figure 29: $f(x,y,z,w) = x + xy - y - 1 + yz - z - xz - xyz + w^3 + xw$

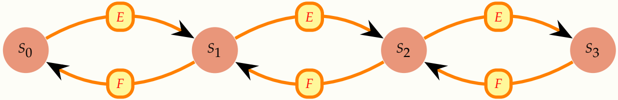
$F = \mathrm{N}(f)_v, \quad v = (0,0,0,-1), \quad G_1 = F_{v_1}, \quad v_1 = (0,0,1,-1)$

$G_2 = F_{v_2}, \quad v = (0,0,-1,-1)$

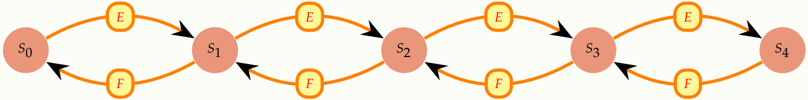
Applications - 4-site phosphorylation system



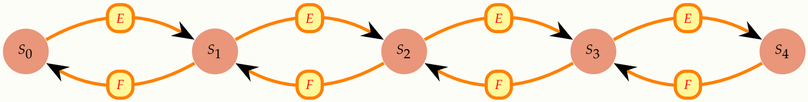
Applications - 4-site phosphorylation system

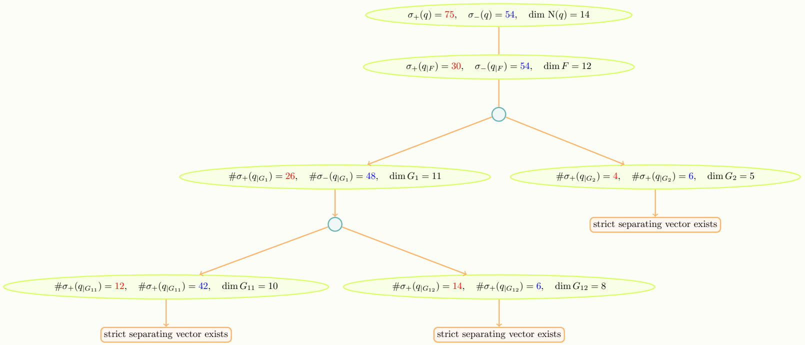


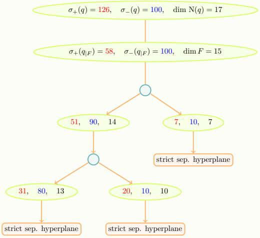
Applications - 4-site phosphorylation system



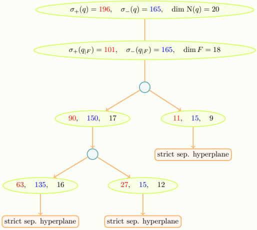
Applications - 4-site phosphorylation system



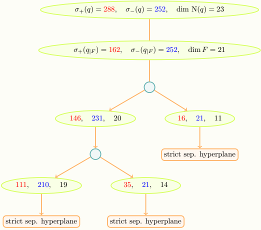



$\sigma_+(q) = 126, \quad \sigma_-(q) = 100, \quad \dim \mathrm{N}(q) = 17$
$\sigma_+(q_{|F}) = 58, \quad \sigma_-(q_{|F}) = 100, \quad \dim F = 15$
51, 90, 14
7, 10, 7
strict sep. hyperplane
31, 80, 13
20, 10, 10
strict sep. hyperplane
strict sep. hyperplane

(a) $m = 5$

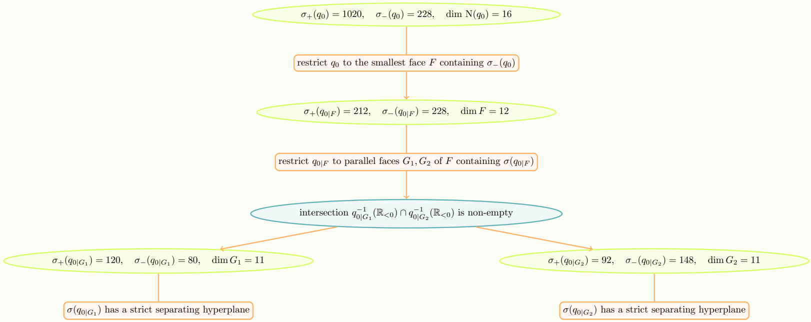

$\sigma_+(q) = 196, \quad \sigma_-(q) = 165, \quad \dim \mathrm{N}(q) = 20$
$\sigma_+(q_{|F}) = 101, \quad \sigma_-(q_{|F}) = 165, \quad \dim F = 18$
90, 150, 17
11, 15, 9
strict sep. hyperplane
63, 135, 16
27, 15, 12
strict sep. hyperplane
strict sep. hyperplane

(b) $m = 6$

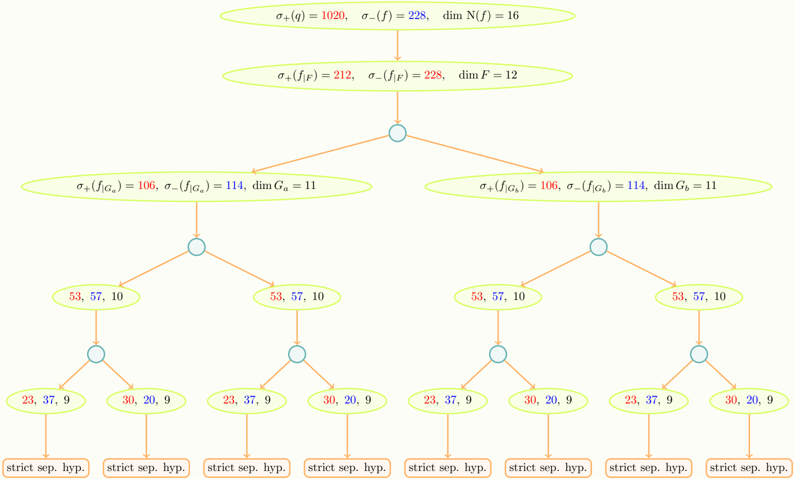

$\sigma_+(q) = 288, \quad \sigma_-(q) = 252, \quad \dim \mathrm{N}(q) = 23$
$\sigma_+(q_{|F}) = 162, \quad \sigma_-(q_{|F}) = 252, \quad \dim F = 21$
146, 231, 20
16, 21, 11
strict sep. hyperplane
111, 210, 19
35, 21, 14
strict sep. hyperplane
strict sep. hyperplane

(c) $m = 7$

Applications
2-site weakly irreversible phosphorylation system



APPLICATIONS
2-SITE WEAKLY IRREVERSIBLE PHOSPHORYLATION SYSTEM



Thank you for your attention!

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Convex signomials

Lemma

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ be a signomial, and let $\Delta_n := \mathrm{Conv}(\{0, e_1, \ldots, e_n\})$ be the standard n-simplex in $\mathbb{R}^n$.

If $\sigma_-(f) \subseteq \Delta_n$ and $\sigma_+(f) \subseteq \Delta_n^-$, then f is a convex function.

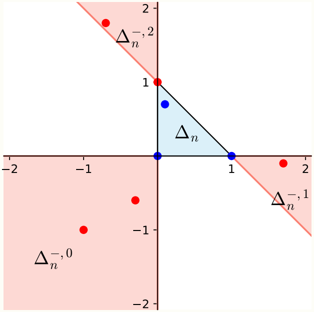
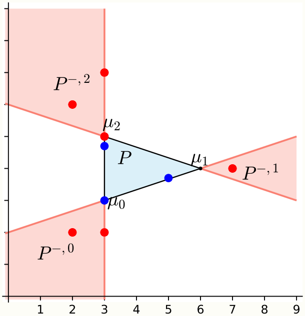


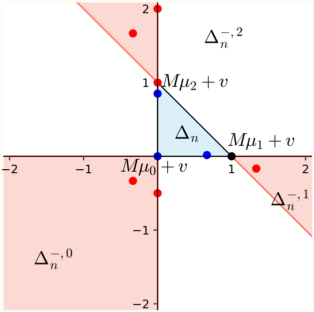
Figure 31: $x_1^{-1}x_2^{-1} + x_1^{-0.3}x_2^{-0.6} - 2x_1 - x_1^{0.1}x_2^{0.7} - 5 + 2x_1^{1.7}x_2^{-0.1} + x_2 + x_1^{-0.7}x_2^{1.8}$

DESCARTES' RULE OF SIGNS FOR HYPERSURFACES

Sketch of proof:

- Find $M \in \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{R}^n$ such that $MP + v = \Delta_n$

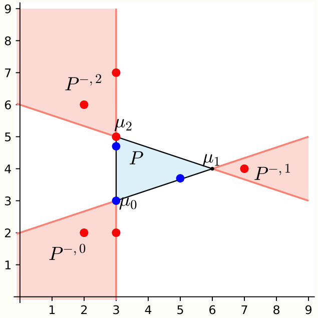


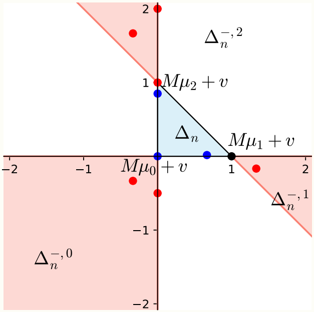


Sketch of proof:

- Find $M \in \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{R}^n$ such that $MP + v = \Delta_n$
- For $F\colon \mathbb{R}^n_{>0} \to \mathbb{R}, x \mapsto x^v f(x^{M_1}, \ldots, x^{M_n})$ it holds that

$\sigma_-(F) = M\sigma_-(f) + v \subseteq \Delta_n$ and $\sigma_+(F) = M\sigma_+(f) + v \subseteq \Delta_n^-$.





DESCARTES' RULE OF SIGNS FOR HYPERSURFACES

Sketch of proof:

- Find $M \in \mathrm{GL}_n(\mathbb{R}), v \in \mathbb{R}^n$ such that $MP + v = \Delta_n$
- For $F\colon \mathbb{R}^n_{>0} \to \mathbb{R}, x \mapsto x^v f(x^{M_1}, \ldots, x^{M_n})$ it holds that

$\sigma_-(F) = M\sigma_-(f) + v \subseteq \Delta_n$ and $\sigma_+(F) = M\sigma_+(f) + v \subseteq \Delta_n^-$.

- $f^{-1}(\mathbb{R}_{<0})$ is homeomorphic to $F^{-1}(\mathbb{R}_{<0})$