RESEARCH STATEMENT

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Polynomials with real coefficients model important phenomena in a diverse array of subjects including optimization (semidefinite-hyperbolic programming), extremal combinatorics (incidence geometry), statistical physics (partition functions), biology (distance geometry), chemistry (multistationary reaction networks), and statistical learning theory (moment estimation). Real polynomials represent a large family of shapes (semi-algebraic geometry), and are used for geometric applications (computer aided design). Despite their ubiquity, a 21^{st} century theory of real polynomials that incorporates modern computational perspective is yet to be developed.

I work on problems arising from optimization and combinatorics with a real algebraic perspective, and on problems in real algebraic geometry with a computational perspective. I have thus developed a research program at the interface of algebraic geometry and computer science connecting basic structural questions on real polynomials to computation and complexity theory. This program, if successful, would change our view on some fundamental problems in real algebraic geometry, and make a qualitative leap in our computational power for a variety of application domains. Four concrete directions that I plan to pursue in the near future are the following:

(1) Combinatorial Structures in Real Polynomials
If a real polynomial has a simple expression does the real zero set of this polynomial have to have low (topological) complexity?
(2) Real Algebraic Geometry and Optimization
How can we exploit the rich algebraic structure of the convex bodies arising from optimization, such as the cone of positive semidefinite matrices, for designing faster and more stable algorithms? In the opposite direction, how can we use methods from convex optimization for speeding up polynomial computations?
(3) Beyond Worst Case Analysis for Algorithms in Algebraic Geometry
How can we develop theoretical tools that predict practical behavior of geometric algorithms?
(4) Real Algebraic Geometry in Combinatorics
Is there an underlying algebraic structure behind the extremal configurations in combinatorial geometry? On the other edge of the spectrum, can we use the real algebraic structure of partition functions to understand random combinatorial objects?

1. Combinatorial Structures in Real Polynomials

Consider the equation $x^d - 1 = 0$, how many solutions does this equation have? The fundamental theorem of algebra says it has d solutions over the complex numbers. Over the reals, however, the equation has at most two solutions regardless of its degree d. This a simple instance of a general philosophy in real algebraic geometry: The description complexity of a real algebraic set controls its topological complexity (i.e., the number of zeros, holes, Betti numbers). Here is a concrete question along these lines:

Question 1.1 (Kushnirenko's Conjecture)**.** *Let $f_1, f_2, \ldots, f_n$ be real polynomials with n variables, and at most t terms in each. Is the number of (non-degenerate) zeros of the system $(f_1, f_2, \ldots, f_n)$ bounded by ct^n where c is an absolute constant?*

Descartes' rule of signs from 1636 says a univariate polynomial with t terms has at most $2t-1$ real zeros, so this settles Question 1.1 for the case $n=1$. As of September 2019, it remains open for all $n \geq 2$ [KPT15]. Besides its obvious geometric appeal, and important applications in life sciences [MFR⁺16], Question 1.1 also has a complexity theoretic motivation: proving tight upper bounds for the number of real zeros of combinatorially structured univariate polynomials settles major conjectures in complexity theory [Koi].

The state of the art motivated us (with Peter Bürgisser and Josue Tonelli-Cueto) to work on the average case of Question 1.1. Fix a finite subset $A \subseteq \mathbb{Z}^n$ of cardinality t together with a map $\sigma : A \to \mathbb{R}_+$. We assign to this data the random polynomial system

$$f_1(x) := \sum_{\alpha \in A} \sigma(\alpha)\, \xi_{1,\alpha} x^{\alpha}, \ldots, f_n(x) := \sum_{\alpha \in A} \sigma(\alpha)\, \xi_{n,\alpha} x^{\alpha},$$

where the $\xi_{i,\alpha} \sim \mathcal{N}(0,1)$ are independent identically distributed (i.i.d.) standard Gaussian random variables. For a given support set A and a system of variances σ we denote by $\mathbb{E}N(A,\sigma)$ the expected number of (nondegenerate) real zeros of the random system $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$.

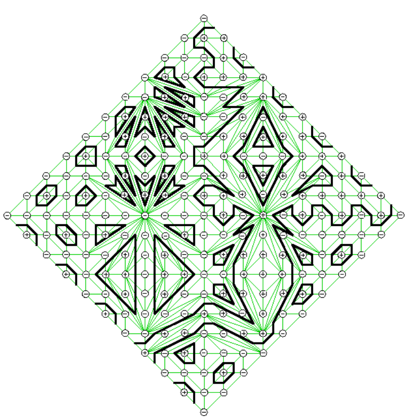
Theorem 1.1 (Bürgisser, Ergür, Tonelli-Cueto [BETC])**.** *We have*

$$\mathbb{E}N(A,\sigma) \leq 2\binom{t}{n}$$

for any support $A \subseteq \mathbb{Z}^n$ of cardinality t and any system of variances σ.

Note that the bound in Theorem 1.1 only depends on the cardinality of the set A and confirms Kushnirenko's conjecture for average instances. Now, I would like to pass to another fundamental question in real algebraic geometry, namely Hilbert's 16th problem.

Question 1.2 (Hilbert's 16th problem for Sparse Polynomials)**.** *Given a set of lattice points $A := \{a_1, a_2, \ldots, a_t\} \subset \mathbb{Z}^n$, classify (up to isotopy) possible topology types of real zero sets $Z_f(\mathbb{R})$ of non-singular real hypersurfaces $f = \sum_{i=1}^t c_i x^{a_i}$.*



Suppose we take a polynomial $p(x) = \sum_{i=1}^t c_i x^{a_i}$, pick a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_t)$, and consider the parametric family of equations $p(s,x) = \sum_{i=1}^t c_i e^{s\omega_i} x^{a_i}$ where $s \in [0,\infty)$. How does the topology of $p(s,x)$ change as s goes from 0 to ∞?

First, miraculously, the asymptotic shape of this one parameter family i.e. the isotopy type of $p(s,x)$ as $s \to \infty$ can be understood by purely combinatorial techniques! This goes under the name of tropical geometry or Viro's patchworking method [Vir01]. For instance, the discrete picture on the left due to Ilia Itenberg shows the existence of a degree 10 plane curve with 32 ovals and refutes the famous Ragsdale conjecture.

Now that we know the combinatorial structure at the limit, we would like to understand topology changes (phase transitions) along the deformation. Together with Boulos El-Hilany we learned to use Morse theory to handle these transitions. If we could prove good upper bounds for the number of phase transitions along the way, we could answer Question 1.1. So far this effort had little success. This led me to turn to an easier question: For what kind of polynomials is there no phase transition at all? If there would be no change in the topology (number of zeros for instance) one can start from the limit shape and track back the deformation with numerical methods. These type of combinatorially structured polynomial systems (ones with no phase transition) are called patchworked polynomial systems.

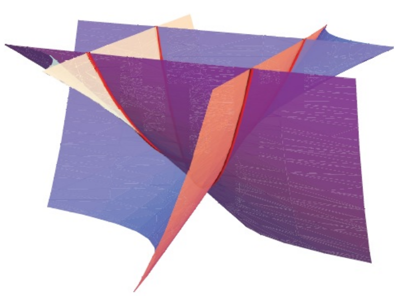
Theorem 1.2 (Ergür, de Wolff [EdW])**.** *Suppose we are given support sets $A_1, \ldots, A_n \subset \mathbb{Z}^n$ with at most t elements in each, and a system of polynomials $f_1, f_2, \ldots, f_n$ where f_i is supported with A_i. There exist an algorithm which does less than $O(t^{n+1})$ arithmetic operations and certifies if the given polynomial system $f_1, f_2, \ldots, f_n$ is a patchworked polynomial system. Moreover, in this case there at most $O(t^n)$ common real zeros of $f_1, f_2, \ldots, f_n$, and there exist an optimal homotopy continuation algorithm to find these zeros.*

We conclude with a project related to distance geometry that I plan to pursue this year.

Project 1.1 (Combinatorial Preconditioning for Quadratic Equations)**.** *The objective of combinatorial preconditioning is to use combinatorial tools to develop systematic preconditioners that replace ad hoc numerical heuristics. This idea is successfully used for solving large systems of linear equations* [Spi10]*. My goal is to develop a combinatorial preconditioning scheme for finding common real zeros of a system of quadratic equations $x^T Q_1 x, \ldots, x^T Q_n x$ where $x \in \mathbb{R}^n$.*

2. Real Algebraic Geometry and Optimization

Let $H_{n,2d}$ denote the vector space of degree $2d$ homogeneous polynomials (forms) with n variables, and define $P(n, 2d) := \{f \in H_{n,2d} : f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\}$.



The image on the left, due to Pablo Parrilo, plots the points (a, b, c) where there is a change in the number of zeros of the corresponding polynomial $p(x, y) = x^4 + ax^3y + bx^2y^2 + cxy^3 + y^4$. The convex region in the middle of the plot corresponds to $P(2, 4)$.

Main objective of (unconstrained) polynomial optimization is to test membership of a query point to the set $P(n, 2d)$. This task is NP-Hard for $d \geq 2$; the cone $P(n, 2d)$ is hopelessly complicated [Nie12]. Now consider a subspace $E \subset H_{n,2d}$ of structured polynomials, and let $P(E)$ denote the cone of nonnegative polynomials in E. For some linear spaces E, such as the space of even symmetric sextics, $P(E)$ is strikingly simple: it is a cone over a regular n-gon [CLR87]. This contrast inspired me to study polyhedral approximations to the cone of nonnegative polynomials. To state the result, let us define the following hyperplane in $H_{n,2d}$: $L := \{f \in H_{n,2d} : \int_{S^{n-1}} p(x)\, \sigma(x) = 1\}$. Then we say a cone C approximates the cone $P(n, 2d)$ by ratio $\alpha > 1$ if

$$(C \cap L - r) \subset (P(n, 2d) \cap L - r) \subset \alpha\, (C \cap L - r)$$

where $r = (x_1^2 + \ldots + x_n^2)^d$.

Theorem 2.1 (Ergür)**.** *For every $\alpha > 1$, and for all $d \geq 1$, any polyhedral cone that approximates $P(n, 2d)$ with ratio α^d has to have at least $a_0 e^{\frac{a_1 n}{\alpha}}$ many facets where a_o and a_1 are universal constants. On the other hand, for any linear space $E \subset H_{n,2d}$ with $(x_1^2 + \ldots + x_n^2)^d \in E$ and $\dim(E) = m$ there exists a polyhedral cone with $O(n^{m-2})$ many facets that approximates $P(E)$ with ratio $\alpha = (1 + \frac{n}{m})^{\frac{3m}{n}}$.*

The proof of Theorem 2.1 is based on the recent solution of Kadison-Singer problem, hence it is not constructive. I also developed a randomized construction with $O(n^m)$ facets achieving the same level of approximation (please see the last section of [Erga] for details).

Theorem 2.1 suggests it is easier to optimize polynomials when we have a fixed dimensional subspace such as the space of symmetric polynomials. This is somewhat restrictive; one wants a more general notion of sparsity. On the other hand, the current popular algorithms based on based on sums of squares hierarchy do not seem to improve when the objective function is sparse! As a first step to understand this issue I worked on the quantitative analysis of sums of squares relaxation for multihomogenous polynomials [Ergb]. This generalizes earlier work of Greg Blekherman [Ble06], and has implications in quantum information theory [KMŠZ17]. Currently I am interested in understanding power of relative entropy programs for sparse polynomial optimization.

Project 2.1 (Relative Entropy Programming for Sparse Polynomial Optimization)**.** *Relative entropy programs are a family of convex programs that can be solved efficiently via interior point methods* [CS17]*. Despite their efficiency, the expressive power these programs and their relation to semidefinite representability is not well understood. This project, aims to study limits of relative entropy programming for sparse polynomial optimization with a special focus on copositive programming.*

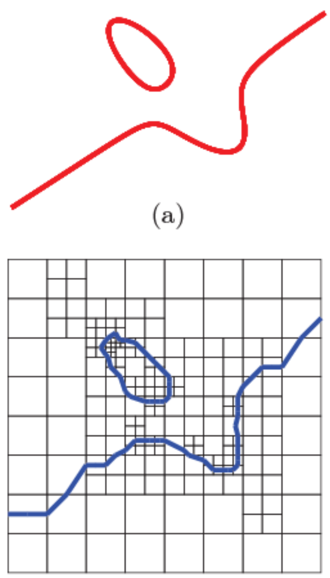
The standard notion of sparsity does not seem to couple well with the current workhorse sums of squares algorithm. An alternative way is SOS-sparsity; the certificates that can be obtained by few steps in the SOS hierarchy. The following project joint with Venkat Gruswami and Pravesh Kothari, aims to employ this idea for learning theory applications.

Project 2.2 (Polynomial Optimization for Robust Regression)**.** *This project aims to use efficient polynomial nonnegativity certificates for some problems in learning theory. Specifically, we are planning to study sums of squares relaxation for covariance estimation in the list-decodable setting.*

3. Beyond Worst Case Analysis for Algorithms in Algebraic Geometry

Developing models to explain behavior of algorithms is a highly non-trivial task. A classical example is the ellipsoid algorithm for linear programming compared to the simplex method: the ellipsoid algorithm has worst case polynomial time complexity and a very unsatisfactory practical performance, whereas the simplex method has exponential time worst case complexity but solves large instances of linear programming rather fast. A similar case is present in real algebraic geometry: for the case of real algebraic surfaces Cylindrical Algebraic Decomposition (CAD) has polynomial time worst case complexity but almost never works in practice, on the other hand subdivision based methods has exponential time worst case complexity with quite satisfying practical performance. The example of CAD is just the tip of the iceberg: in a myriad of instances of polynomial solving, the worst case complexity analysis fails to explain behavior of algorithms.

One general idea is to use random or semi-random instances to model a typical input. In the realm of polynomial system solving, this translates into average-case and smoothed analysis. A celebrated line of research starting from Smale's list of problems for 21st century [Sma98] conducts average-case analysis of homotopy continuation algorithms for a specific type of random input [SS93b, SS93a, SS94, BP11, BC13, BC11, Lai17]. The main criticism for this line of research was the restricted random input model that has a very specific variance structure, obstructing analysis of the algorithms for sparse polynomial systems. My work with Grigoris Paouris and J. Maurice Rojas provided probabilistic condition number estimates that hold for a general family distributions allowing analysis of the algorithms for sparse inputs and furthermore allowing adversarial random models such as smoothed analysis [EPR, EGM]. Later, with Felipe Cucker and Josue Tonelli-Cueto, we used the technology developed in [EPR, EGM] to explain behavior of a fast in practice but doubly exponential in worst case meshing algorithm due to Plantinga and Vegter (PV) [CETC]. I am involved in several running projects in this realm, joint with Elias Tsigaridas, Felipe Cucker, and Josue Tonelli-Cueto, two of these projects are listed below.


(a) (b)

Project 3.1 (Adaptive Subdivision Methods for Core Tasks in Real Algebraic Geometry)**.** *This project envisions a comprehensive study on design and analysis of refined adaptive subdivision methods for core tasks in real algebraic geometry such as finding real zeros, meshing curves and surfaces, and computing Betti numbers of semialgebraic sets.*

The picture on left, due to Elias Tsigaridas and Michael Burr, shows a) real zero set of $f(x,y) = 3y^3 + 3xy^2 - 2x^3 - 3y^2 + xy + 3x^2 - 3y + 3x + 2$ b) PV algorithm's output.

Project 3.2 (Smoothed Analysis for Symbolic Computation)**.** *We aim to conduct smoothed analysis of symbolic root isolation algorithms for univariate polynomials. We perturb the coefficients using a discrete random variable with rational values, thus relating to symbolic computation. The motivation is that even for this basic case, the worst case complexity analysis fails to distinguish fast and slow algorithms.*

4. Real Algebraic Geometry in Combinatorics

Imagine a set M of points in the real plane together with a set N of lines. How many incidences can happen between the points in M and the lines in N? The famous Szemeredi-Trotter theorem answers this question. Now imagine a finite set of points on the real line, and create a grid in $\mathbb{R}^3$ by taking the cartesian product of the set with itself (three times). Let p be a degree d polynomial with 3 variables, how many zeros can p have on the grid? This question is answered by another famous result named Schwartz-Zippel Lemma. Together with Levent Dogan, Jake Mundo, and Elias Tsigaridas we proved a common generalization of these two famous results:

Theorem 4.1 (Multivariate Schwartz-Zippel Lemma [DEMT])**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-partition of n, let $S_i \subseteq \mathbb{C}^{\lambda_i}$ be finite sets, and let $S := S_1 \times S_2 \times \ldots \times S_m$ be the multi-grid defined by S_i. Then for a λ-irreducible polynomial p of degree $d \geq 2$, and for every $\varepsilon > 0$ we have*

$$|Z(p) \cap S| = O_{n,\varepsilon}\left(d^3 \prod_{i=1}^{m} |S_i|^{1-\frac{1}{\lambda_i+1}+\varepsilon} + d^{n^4} \sum_{i=1}^{m} \prod_{j \neq i} |S_j| \right)$$

where $O_{n,\varepsilon}$ notation only hides constants depending on ε and n.

We skip mildly technical definition of λ-irreducibility but just note that our paper includes an effective symbolic algorithm to detect λ-reducible polynomials. Returning back to Szemeredi-Trotter theorem, consider $p(x_1, x_2, x_3, x_4, x_5) = x_1x_4 + x_2x_5 + x_3$, and let the set of lines be represented by $x_1u + x_2v + x_3 = 0$ where $(x_1, x_2, x_3) \in S_1$, the set of points be $(x_4, x_5) \in S_2$, then $|Z(p) \cap S|$ is the number of incidences between the sets of points and lines.

Theorem 4.1 belongs to a rapidly growing field called "the polynomial method" which uses tools of real algebraic geometry for questions arising from extremal combinatorics (see, e.g., [Gut16, Tao14]). There are also similar developments in probabilistic combinatorics mostly based on exploiting the real algebraic structure of partition functions. I recently wrote my first paper on this line of research with Amin Coja-Oghlan, Samuel Hetterich, and Maurice Rolvien [COEHR]. Our paper concerns the partition function coming from uniform distribution over the kernel of a random matrix, and provides a rank formula that holds over any field. The main merit of the paper is to combine algebraic insight with techniques coming from statistical physics, where earlier work in the field was based on deliberate combinatorial arguments that work only over finite fields.

We plan to continue our collaboration with Amin on using real algebraic tools for probabilistic combinatorics problems, and our next project is the following:

Project 4.1 (Barvinok Polynomial Method for Anderson-Edwards Model)**.** *Barvinok's method is a general idea for approximately computing partition functions based on homotopy deformation of "hard" polynomials into "simple" ones* [Bar16]*. We aim to study the limits of Barvinok's method for approximately computing a partition function coming from the well-known Anderson-Edwards model in statistical physics.*

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