Toward an Algorithmic Theory of Polynomials

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Outline of today's Colloquium

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1. Examples from a variety of subjects with connections to polynomials

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2. Basic Algebraic Geometry with a computational lens

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2. Basic Algebraic Geometry with a computational lens
3. Revisiting the introductory examples with an algebra-geometric perspective

Part I
Examples

Extremal Combinatorics

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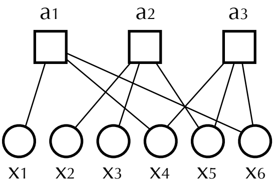

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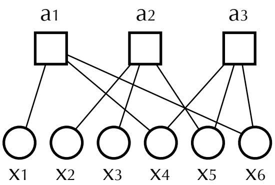

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Define a random bipartite graph with m check nodes and n variable nodes as follows

- each check node has degree k, and each variable node has degree $\frac{km}{n}$
- edges have weights from an arbitrary distribution over $\mathbb{F}^*$ for a field $\mathbb{F}$

Let A be the $m \times n$ random matrix defined by this graph. What is the rank of A?

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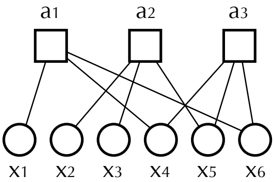

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$A \succ 0$ means A is a PSD matrix.

A map $\phi : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is positive if $\phi(A) \succ 0$ for all $A \succ 0$.

Positive maps are used to detect entanglement.

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ϕ is k-positive if the map

$$\phi^k : \mathbb{R}^{k \times k} \otimes \mathbb{R}^{n \times n} \to \mathbb{R}^{k \times k} \otimes \mathbb{R}^{n \times n}$$

$$\phi^k(M \otimes A) = M \otimes \phi(A)$$

is positive. ϕ is completely positive if it is positive for all k.

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Send ϕ to a biquadratic polynomial by $p_\phi(x, y) = y^T \phi(xx^T) y$. Then,

- ϕ is positive if $p_\phi(x, y) \geq 0$ for all x, y.
- ϕ is completely positive if $p_\phi(x, y) = \sum h_i(x, y)^2$ for some h_i.

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What percentage of positive maps are completely positive?

What percentage of biquadratic nonnegative polynomials are sum of squares?

Biochemical Reaction Networks

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We have two input data: $A = \{a_1, a_2, \ldots, a_t\} \subset \mathbb{Z}^n$, $n \times t$ matrix κ.
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How many $x \in \mathbb{R}_+^n$ are there with $f_1(x) = f_2(x) = \ldots = f_n(x) = 0$?

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (f_1(x), f_2(x), \ldots, f_n(x))$$

Mass kinematics equation models the dynamical system, we want to count equlibrium (steady) states.

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- Develop structure aware theorems and algorithms
- Distinguish between the real and complex geometry
- Go beyond the worst case and understand typical instances

Part II

Basic Algebraic Geometry with a Computational Lens

Going back to the beginnings

We go back to univariate polynomials: Let $0 < a_1 < \ldots < a_t$ be a sequence of integers, and consider the following univariate polynomial

$$p(x) = c_0 + c_1 x^{a_1} + \ldots + c_t x^{a_t}$$

where c_i are real numbers. How many zeros does p have?

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1. Fundamental Theorem of Algebra: p has a_t many zeros over the complex numbers.
2. Descartes Rule of Signs: The number of real zeros of p depends on the coefficients, but it is at most $2t$.

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Example

Consider $p(x) = 3 + 5x + 7x^{100}$. It has at most 4 real zeros, and 100 complex zeros.

Basic Algebraic Geometry

Theorem (Bézout)

Let $p = (p_1, \ldots, p_{n-1})$ be a system of homogenous polynomials with n variables where p_i have degree d. Then the polynomial system p has at most d^{n-1} many non-degenerate zeros, and this bound is generically exact.

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Example

Let A be a $n \times n$ matrix, and consider the eigenpair problem:

$$(A - \lambda)x = 0$$

This is a quadratic system of equations in (λ, x).
Bézout's theorem gives the bound 2^n.

Basic Algebraic Geometry

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Theorem (Kushnirenko, 70's)

Let $A = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{Z}^n$ be set of lattice points. Consider the following polynomial equations

$$p_1(x) = \sum_{i=1}^{m} c_{1i} x^{a_i}, \ldots, p_n(x) = \sum_{i=1}^{m} c_{ni} x^{a_i}$$

Then the system of equations $p = (p_1, p_2, \ldots, p_n)$ has at most $|conv(A)|$ many non-degenerate zeros (where $|.|$ denotes volume), and this bound is generically exact.

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There is a series of papers titled complexity of Bézout's theorem, but there is no paper (yet) titled complexity of Kushnirenko's theorem.

An example

Example

$$
\begin{aligned}
f_1 &= 10500t - t^2u^{492} - 3500u^{463}v^5w^5 \\
f_2 &= 10500t - t^2 - 3500u^{691}v^5w^5 \\
f_3 &= 14000 - 2t + tu^{492} - 2500v \\
f_4 &= 14000 + 2t - tu^{492} - 3500w
\end{aligned}
$$

How many zeros are there?

1. Bezout bound: 82 billion in $\mathbb{C}^4$
2. Volume (Kushnirenko) bound: 7663 in $(\mathbb{C}^*)^4$
3. Number of positive real zeros: 6 in $(\mathbb{R}_+)^4$

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$$2^{\binom{t-1}{2}} (n+1)^{t-1}$$

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Consider a system of two polynomials, two variables, with 8 terms in each. Khovanskii says $2^{21} \times 3^7$, Kushnirenko claims 64.

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In general, this conjecture is open for any fixed $n \geq 2$. Descartes solved $n = 1$ in 1636.

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Let $A \subset \mathbb{Z}^n$ be a set of cardinality t, and let $\sigma : A \to \mathbb{R}_+$ be a function. We consider the following system of random polynomials:

$$f_1(x) = \sum_{\alpha \in A} \sigma(\alpha)\xi_{1\alpha}x^\alpha, \ldots, f_n(x) = \sum_{\alpha \in A} \sigma(\alpha)\xi_{n\alpha}x^\alpha$$

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Theorem (Bürgisser, Ergür, Tonelli-Cueto, 19)

Let $E(A, \sigma)$ denote the expected number of non-degenerate real zeros of $f = (f_1, f_2, \ldots, f_n)$. Then, we have

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This confirms Kushnirenko's conjecture for average instances.

Complexity of Bézout's Theorem

Smale's 17th Problem

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Smale adds: Similar but harder questions can be asked over the reals.

Complexity of Bézout's Theorem

Theorem (Shub, Smale, Beltran, Pardo, Bürgisser, Cucker, Lairez)

There exists an algorithm that computes a zero of a system of equations $(p_1, p_2, \dots, p_n)$ with degrees $d_1, d_2, \dots, d_n$ on average time

$$O(nd^{\frac{3}{2}}N^2) \text{ where } N = \sum_{i=1}^{n} \binom{n+d_i}{d_i}, d = \max_i d_i$$

and average is with respect to Kostlan-Shub-Smale ensemble.

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Complexity of Bézout series started in Berkeley, 1992, and the solution of Smale's 17th problem was completed in Berlin, 2015.

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- $N = \sum_{i=1}^{n} \binom{n+d_i}{d_i}$ is huge!
- Practically works well for finding all complex zeros: Bertini, PSS5, PHCPack, JuliaHomotopy

Complexity of Kushnirenko's Theorem

Sparse Smale's 17th Problem

Let $A \subset \mathbb{Z}^n$ be a set of t lattice points with bounded degree d. Is there an algorithm that finds a complex zero of a of system of n polynomial equations with support set A on average time $poly(t, n, \log(d))$?

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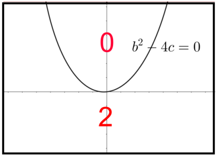
Necessary tool box under development: Ergür, Rojas, Paouris (18 and 19), Malajovich (17 and 19), Ergür and Malajovich (ongoing work).

Smale's Question over the Reals



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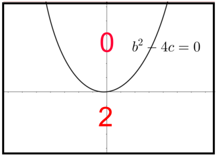

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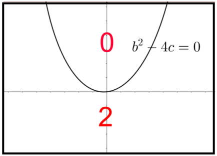

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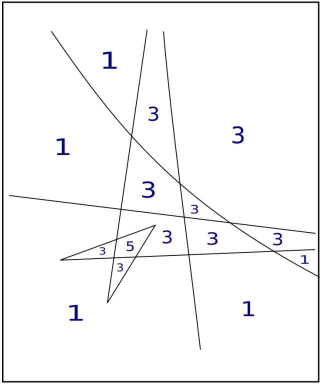
The corresponding discriminant equation has degree 36 with very large coefficients, it fills pages.

Handling the Discriminant Variety



X is a complex algebraic variety in $(\mathbb{C}^*)^m$.

$$Log : X \to \mathbb{R}^m \ , \ Log(z_1, \ldots, z_n) = (log|z_1|, \ldots, log|z_n|)$$

This is called amoeba of a variety.

Real Homotopy Continuation Algorithm

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If p is located in the unbounded components of the complement of discriminant amoeba, then p has at most $O(t^n)$ many real zeros. Moreover, there exist a real homotopy algorithm to compute the real zeros of p.

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The algorithm also provides a certificate of being in the unbounded components of the amoeba complement.

Part III

Revisiting the examples with an algebra-geometric perspective

Multihomogenous Nonnegative Polynomials

$H_{n,2d}$: $2d$ homogenous in first n variables, and $2d$ homogenous in the second n variables.

$$p(x) = x_1^2 x_3 x_4 + x_2^2 x_3^2 + x_1 x_2 x_4^2 \quad p \in H_{2,2}$$

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Theorem (Ergür, 2018)

$$c_1 \pi^{-2d} (\frac{n}{2} + 2d)^{-2d} \leq \frac{|\Sigma_{n,2d} \cap L|}{|P_{n,2d} \cap L|} \leq c_2 n^{\frac{1}{2}} (\frac{n}{d} + 1)^{-2d}$$

Sum of Squares Hierarchy

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Consider the following semialgebraic set

$$\mathcal{V} := \{x \in \mathbb{R}^n : g_1(x) \geq 0, g_2(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

If g_i creates an Archimedean quadratic module (a technical condition a bit stronger then assuming $\mathcal{V}$ is compact), then

$$f(x) > 0 \text{ for all } x \in \mathcal{V} \Leftrightarrow f(x) = u_o(x) + \sum_{i=1}^{m} g_i(x) u_i(x)$$

where u_i are sum of squares.

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I am interested in bounds on the degrees of u_i for typical situations.

Grids and Incidence Geometry

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Let $n = \lambda_1 + \lambda_2 + \ldots + \lambda_m$ be an m partition of n. Let $S_i \subseteq \mathbb{C}^{\lambda_i}$ be finite sets.

$$S := S_1 \times S_2 \times \ldots \times S_m \subset \mathbb{C}^n$$

be a multigrid.

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Example

Let $S_1 \subset \mathbb{C}^2$ represent points, let $S_2 \subset \mathbb{C}^2$ represent lines $ay + bz + 1 = 0$. Define the polynomial $p(x) = x_3x_1 + x_4x_2 + 1$.

$$|Z(p) \cap S_1 \times S_2| = \text{ number of incidences between points and lines}$$

The Multivariate Schwartz-Zippel Lemma

Theorem (Dogan, Ergür, Tsigaridas, Mundo, 19)

Let p be a degree d and λ-irreducible polynomial, then for any $\varepsilon > 0$ we have

$$|Z(p) \cap S| = O_{d,\varepsilon}\left(\prod_{i=1}^{m} |S_i|^{1+\varepsilon-\frac{1}{\lambda_i}} + \sum_{i} \prod_{j \neq i} |S_j|\right)$$

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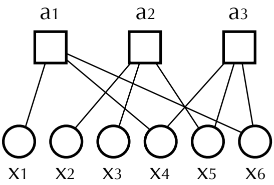
For $m = 2$, we have

$$|Z(p) \cap S| = O_{d,\varepsilon}\left(|S_1|^{1-\frac{1}{\lambda_1}+\varepsilon} |S_2|^{1-\frac{1}{\lambda_2}+\varepsilon} + |S_1| + |S_2|\right)$$

A Sparse Random Matrix Model



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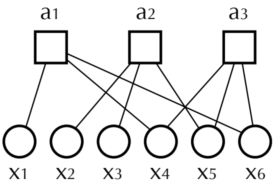

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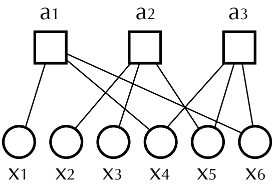

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- with $d = \mathbb{E}[\mathbf{d}]$, $k = \mathbb{E}[\mathbf{k}]$ and $m \sim Po(dn/k)$ and given

$$\sum_{i=1}^{n} \mathbf{d}_i = \sum_{i=1}^{m} \mathbf{k}_i$$

generate a random bipartite graph G with degrees $\mathbf{d}_i, \mathbf{k}_i$

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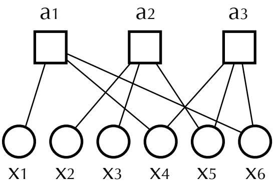

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  generate a random bipartite graph G with degrees $\mathbf{d}_i, \mathbf{k}_i$
- insert entries from a distribution on $\mathbb{F}^*$ to obtain A.

Lots of Work on Different Ground Fields

- full rank: $\mathbb{F}_2$, $\mathbf{d} \sim Po(d)$, $\mathbf{k} = k$ [DM03,DGMMPR10,PS16]
- rank for $\mathbb{F}_2$, $\mathbf{d} \sim Po(d)$, $\mathbf{k} = k$ [CFP18]
- full rank: $\mathbb{F}_3$, $\mathbf{d} \sim Po(d)$, $\mathbf{k} = k$ [FG12]
- rank for $\mathbb{F}_q$, $\mathbf{d} \sim Po(d)$, $\mathbf{k} = k$ [ACOGM17]
- dense matrices [K96,BKW97,CV10,...]

The Rank of Sparse Random Matrices

Theorem (Coja-Oghlan, Ergür, Gao, Hettereich, Rolvien, 19)

Let $D(x)$, $K(x)$ the probability generating functions of $\mathbf{d}, \mathbf{k}$, let

$$\Phi(z) = D\left(1 - K'(z)/k\right) + \frac{d}{k}\left(K(z) + (1-z)K'(z) - 1\right)$$

Then,

$$\lim_{n\to\infty} rank(A)/n = 1 - \max_{\alpha\in[0,1]} \Phi(\alpha)$$

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The proof is a combination of algebraic insight with statistical physics techniques.

This is the first step of a long term project on using real algebraic tools for approximating partition functions.

Thank you for your attention!