Derivations for replication of Kaplan & Violante (2010)

This document contains derivations of Euler equations using the budget constraints in the paper, as well as a more standard version that does not scale assets by survival probability.

Functional forms employed

Income process

$$y_{it} = \exp(\kappa_t + z_{it} + \varepsilon_{it})$$
$$z_{it} = z_{i,t-1} + \eta_{it}$$

Utility Function (Omitting i subscripts)

$$u(c_t) = \frac{1}{1-\gamma} c^{1-\gamma}$$
$$u_c(c_t) = c_t^{-\gamma}$$

Standard budget constraint

The following derivations are those implemented in the code KV 2010.jl. I omit i subscripts.

Kaplan & Violante's budget constraint

The following derivations are following my interpretation of what is presented in the paper. I omit i subscripts.

Budget constraints

$$c_t \frac{\xi_t}{\xi_{t+1}} a_{t+1} = (1+r)a_t + P(\tilde{y})$$

$$c_t + a_{t+1} = (1+r)a_t + y_t, \quad ift < T^{ret} c_t + \left(\frac{\xi_t}{\xi_{t+1}}\right) a_{t+1} = (1+r)a_t + P\left(\tilde{\mathbf{Y}}_i\right), \quad ift \geq T^{ret}$$

Retirement (FOC for T$_{ret}$:T-1) $\mathrm{V}_t(a_{t+1}, \tilde{y}) = \max_{c_t, a_{t+1}} u(c_t) + \beta \frac{\xi_{t+1}}{\xi_t} E_t[V_{t+1}(a_{t+2}, \tilde{y})]$
$s.t. c_t = (1+r)a_t + P(\tilde{y}) - \frac{\xi_t}{\xi_{t+1}} a_{t+1}$

$$\frac{\partial u(c_t)}{\partial a_{t+1}} = -\frac{\xi_t}{\xi_{t+1}} c_t^{-\gamma}$$

$$\frac{\partial V_{t+1}(\cdot)}{\partial a_{t+1}} = (1+r) c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t^{-\gamma} = \beta \left(\frac{\xi_{t+1}}{\xi_t}\right)^2 (1+r) c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t = \left(\beta \left(\frac{\xi_{t+1}}{\xi_t}\right)^2 (1+r)\right)^{-1/\gamma} c_{t+1}$$

FOC for retirement:

$$\frac{\xi_t}{\xi_{t+1}} c_t^{-\gamma} = \beta \frac{\xi_t}{\xi_{t+1}} (1+r) c_{t+1}^{-\gamma}$$

Following Giovanni's notes, would have: $\mathrm{V}_t(a_{t+1}, \tilde{y}) = \max_{c_t, a_{t+1}} u(c_t) + \beta \frac{\xi_{t+1}}{\xi_t} E_t[V_{t+1}(a_{t+2}, \tilde{y})]$
$s.t. c_t = (1+r)a_t + P(\tilde{y}) - a_{t+1}$

$$\Rightarrow c_t^{-\gamma} = \beta \frac{\xi_{t+1}}{\xi_t}(1+r)c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t = \left(\beta \frac{\xi_{t+1}}{\xi_t}(1+r)\right)^{-1/\gamma} c_{t+1}$$

Working years (FOC for T=1:T$_{ret}$-2) $\mathrm{V}_t(c_t + a_{t+1}, z_t) = \max_{c_t, a_{t+1}} u(c_t) + \beta E_t[V_{t+1}(c_{t+1} + a_{t+2}, z_{t+1})]$
$s.t. c_t = (1+r)a_t + y_t - a_{t+1}$

$$\Rightarrow c_t^{-\gamma} = \beta(1+r)E_t[c_{t+1}^{-\gamma}]$$

$$E_t[c_{t+1}^{-\gamma}] = E_t[((1+r)a_{t+1} + y_{t+1} - a_{t+2})^{-\gamma}] = E_t[((1+r)a_{t+1} + \exp(\kappa_{t+1} + z_{t+1} + \varepsilon_{t+1}) - a_{t+2})^{-\gamma}] = \sum_{z_{t+1}|z_t} \sum_{\varepsilon_t} \pi_z \tag{1}$$

$$\Rightarrow c_t = \left(\beta(1+r)E_t[c_{t+1}^{-\gamma}]\right)^{-\frac{1}{\gamma}} \tag{2}$$

$$a_t = \frac{1}{1+r}\left(c_t + a_{t+1} - y_t\right) = \frac{1}{1+r}\left(c_t + a_{t+1} - \exp(\kappa_t + z_t + \varepsilon_t)\right) \tag{3}$$