Derivations for replication of Kaplan & Violante (2010)

This document contains derivations of Euler equations using the budget constraints in the paper, as well as a more standard version that does not scale assets by survival probability. Notice that I may have used slightly different timing, as T_{ret} is the last working year for my derivations, and T is the last year of life.

Functional forms employed

Income process

$$y_{it} = \exp(\kappa_t + z_{it} + \varepsilon_{it})$$

$$z_{it} = z_{i,t-1} + \eta_{it}$$

Utility Function (Omitting i subscripts)

$$u(c_t) = \frac{1}{1-\gamma} c^{1-\gamma}$$

$$u_c(c_t) = c_t^{-\gamma}$$

Standard budget constraint

The following derivations are those implemented in the code KV 2010.jl. I omit i subscripts.

Budget constraints

$$\begin{aligned} c_t + a_{t+1} &= (1+r)a_t + y_t, \quad \text{if } t \leq T^{ret} \\ c_t + a_{t+1} &= (1+r)a_t + P(\tilde{y}), \quad \text{if } t > T^{ret} \end{aligned}$$

Retirement ($t \in [T_{ret}+1, T]$)

$$\begin{aligned} V_t(a_{t+1}, \tilde{y}) &= \max_{c_t, a_{t+1}} u(c_t) + \beta \frac{\xi_{t+1}}{\xi_t} V_{t+1}(a_{t+2}, \tilde{y}) \\ \text{s.t. } c_t &= (1+r)a_t + P(\tilde{y}) - a_{t+1} \end{aligned}$$

$$\frac{\partial u(c_t)}{\partial a_{t+1}} = -c_t^{-\gamma}$$

$$\frac{\partial V_{t+1}(\cdot)}{\partial a_{t+1}} = (1+r)c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t^{-\gamma} = \beta \left(\frac{\xi_{t+1}}{\xi_t}\right)(1+r)c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t = \left(\beta \left(\frac{\xi_{t+1}}{\xi_t}\right)(1+r)\right)^{-1/\gamma} c_{t+1}$$

Working periods ($t \in [1, T_{ret} - 1]$)

$$
\begin{aligned}
V_t(c_t + a_{t+1}, z_t) = \max_{c_t, a_{t+1}} u(c_t) + \beta \mathbb{E}_t[V_{t+1}(c_{t+1} + a_{t+2}, z_{t+1})] \\
\text{s.t. } c_t = (1+r)a_t + y_t - a_{t+1} \\
\Rightarrow c_t^{-\gamma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\gamma}]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_t[c_{t+1}^{-\gamma}] &= \mathbb{E}_t[((1+r)a_{t+1} + y_{t+1} - a_{t+2})^{-\gamma}] \\
&= \mathbb{E}_t[((1+r)a_{t+1} + \exp(\kappa_{t+1} + z_{t+1} + \varepsilon_{t+1}) - a_{t+2})^{-\gamma}] \\
&= \sum_{z_{t+1}|z_t} \sum_{\varepsilon_t} \pi_{z,t}(z_{t+1}|z_t)\pi_\varepsilon(\varepsilon_{t+1}) \left((1+r)a_{t+1} + e^{(\kappa_{t+1} + z_{t+1} + \varepsilon_{t+1})} - a_{t+2}\right)^{-\gamma}
\end{aligned} \tag{1}
$$

$$
\Rightarrow c_t = \left(\beta(1+r)\mathbb{E}_t[c_{t+1}^{-\gamma}]\right)^{-\frac{1}{\gamma}} \tag{2}
$$

$$
\begin{aligned}
a_t &= \frac{1}{1+r}\left(c_t + a_{t+1} - y_t\right) \\
&= \frac{1}{1+r}\left(c_t + a_{t+1} - \exp(\kappa_t + z_t + \varepsilon_t)\right)
\end{aligned} \tag{3}
$$

Kaplan & Violante's budget constraint

The following derivations are following my interpretation of what is presented in the paper. I omit i subscripts.

Budget constraints

$$
\begin{aligned}
c_t + a_{t+1} &= (1+r)a_t + y_t, \quad \text{if } t < T^{ret} \\
c_t + \left(\frac{\xi_t}{\xi_{t+1}}\right) a_{t+1} &= (1+r)a_t + P\left(\tilde{y}\right), \quad \text{if } t \geq T^{ret}
\end{aligned}
$$

Retirement ($t \in [T_{ret}, T]$)

$$
\begin{aligned}
V_t(a_{t+1}, \tilde{y}) = \max_{c_t, a_{t+1}} \xi_t u(c_t) + \beta \xi_{t+1} \mathbb{E}_t[V_{t+1}(a_{t+2}, \tilde{y})] \\
\text{s.t. } c_t = (1+r)a_t + P(\tilde{y}) - \frac{\xi_t}{\xi_{t+1}} a_{t+1} \\
\frac{\partial u(c_t)}{\partial a_{t+1}} = -\xi_t \frac{\xi_t}{\xi_{t+1}} c_t^{-\gamma} \\
\frac{\partial V_{t+1}(\cdot)}{\partial a_{t+1}} = (1+r)c_{t+1}^{-\gamma} \\
\Rightarrow c_t^{-\gamma} = \beta \left(\frac{\xi_{t+1}}{\xi_t}\right)^2 (1+r) c_{t+1}^{-\gamma} \\
\Rightarrow c_t = \left(\beta \left(\frac{\xi_{t+1}}{\xi_t}\right)^2 (1+r)\right)^{-1/\gamma} c_{t+1}
\end{aligned}
$$