Income process

$$Y_{it} = \exp(\kappa_t + z_{it} + \varepsilon_{it})$$

$$z_{it} = z_{i,t-1} + \eta_{it}$$

Utility Function   (Omitting i subscripts)

$$u(c_t) = \frac{1}{1-\gamma} c^{1-\gamma}$$

$$u_c(c_t) = c_t^{-\gamma}$$

Retirement (FOC for T_{ret}-1:T-2)   From KV2010:

$$V_t(a_{t+1}, \tilde{y}) = \max_{c_t, a_{t+1}} u(c_t) + \beta \frac{\xi_{t+1}}{\xi_t} \mathbb{E}_t[V_{t+1}(a_{t+2}, \tilde{y})]$$

$$\text{s.t. } c_t = (1+r)a_t + P(\tilde{y}) - \frac{\xi_t}{\xi_{t+1}} a_{t+1}$$

$$\frac{\partial u(c_t)}{\partial a_{t+1}} = -\frac{\xi_t}{\xi_{t+1}} c_t^{-\gamma}$$

$$\frac{\partial V_{t+1}(\cdot)}{\partial a_{t+1}} = (1+r)c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t^{-\gamma} = \beta \left(\frac{\xi_{t+1}}{\xi_t}\right)^2 (1+r)c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t = \left(\beta \left(\frac{\xi_{t+1}}{\xi_t}\right)^2 (1+r)\right)^{-1/\gamma} c_{t+1}$$

FOC for retirement:

$$\frac{\xi_t}{\xi_{t+1}} c_t^{-\gamma} = \beta \frac{\xi_t}{\xi_{t+1}} (1+r)c_{t+1}^{-\gamma}$$

Following Giovanni's notes, would have:

$$V_t(a_{t+1}, \tilde{y}) = \max_{c_t, a_{t+1}} u(c_t) + \beta \frac{\xi_{t+1}}{\xi_t} \mathbb{E}_t[V_{t+1}(a_{t+2}, \tilde{y})]$$

$$\text{s.t. } c_t = (1+r)a_t + P(\tilde{y}) - a_{t+1}$$

$$\Rightarrow c_t^{-\gamma} = \beta \frac{\xi_{t+1}}{\xi_t} (1+r)c_{t+1}^{-\gamma}$$

$$\Rightarrow c_t = \left(\beta \frac{\xi_{t+1}}{\xi_t} (1+r)\right)^{-1/\gamma} c_{t+1}$$

Working years (FOC for T=1:T$_{ret}$-2)

$$V_t(c_t + a_{t+1}, z_t) = \max_{c_t, a_{t+1}} u(c_t) + \beta \mathbb{E}_t[V_{t+1}(c_{t+1} + a_{t+2}, z_{t+1})]$$

$$\text{s.t. } c_t = (1+r)a_t + y_t - a_{t+1}$$

$$\Rightarrow c_t^{-\gamma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\gamma}]$$

$$\begin{aligned}
\mathbb{E}_t[c_{t+1}^{-\gamma}] &= \mathbb{E}_t[((1+r)a_{t+1} + y_{t+1} - a_{t+2})^{-\gamma}] \\
&= \mathbb{E}_t[((1+r)a_{t+1} + \exp(\kappa_{t+1} + z_{t+1} + \varepsilon_{t+1}) - a_{t+2})^{-\gamma}] \\
&= \sum_{z_{t+1}|z_t} \sum_{\varepsilon_t} \pi_{z,t}(z_{t+1}|z_t)\pi_\varepsilon(\varepsilon_{t+1})\left((1+r)a_{t+1} + e^{(\kappa_{t+1}+z_{t+1}+\varepsilon_{t+1})} - a_{t+2}\right)^{-\gamma}
\end{aligned}$$

$$\tag{1}$$

$$\Rightarrow c_t = \left(\beta(1+r)\mathbb{E}_t[c_{t+1}^{-\gamma}]\right)^{-\frac{1}{\gamma}} \tag{2}$$

$$\begin{aligned}
a_t &= \frac{1}{1+r}(c_t + a_{t+1} - y_t) \\
&= \frac{1}{1+r}(c_t + a_{t+1} - \exp(\kappa_t + z_t + \varepsilon_t))
\end{aligned} \tag{3}$$