CS 577 | Study Group – Intractability | Fall 2022

Name: ____________________ Wisc id: ____________________

Basics

1. Highly intelligent aliens land on Earth and tell us the following two things and then leave immediately.

    1. The 3-Coloring problem (which is NP-complete) is solvable in worst-case $O(n^9)$ time, where n denotes the number of vertices in the graph.
    2. There is no algorithm for 3-Coloring that runs faster than $\Omega(n^7)$ time in the worst case.

    Assuming these two facts, for each of the following assertions, indicate whether it can be inferred from the information the aliens have given us. (In all cases, time complexities are understood to be *worst-case* running time.) Provide a short justification in each case.

    (a) All NP-complete problems are solvable in polynomial time.

    Solution:
    yes

    (b) All problems in NP, even those that are *not* NP-complete, are solvable in polynomial time.

    Solution:
    yes

    (c) All NP-hard problems are solvable in polynomial time.

    Solution:
    no , NP-Hard harder than complete

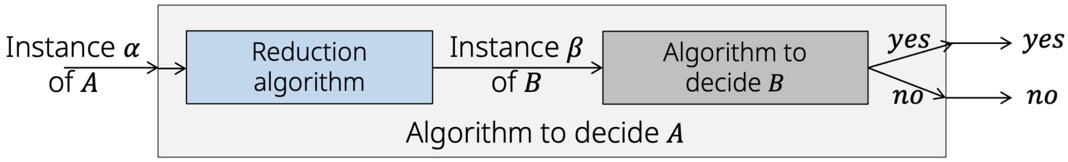
    (d) All NP-complete problems are solvable in $O(n^9)$ time.

    Solution:
    no

    (e) No NP-complete problem can be solved faster than $\Omega(n^7)$ time.

    Solution:
    no

Proving NPC

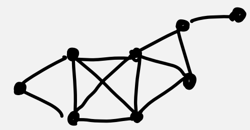


To prove B is NP-Complete:

- Prove $B \in NP$
- Prove $B \in NP\text{-}hard$
  - Select a known NPC problem A
  - Construct a reduction f transforming every instance of A to an instance of B
  - Prove that B outputs 1 if and only if A outputs 1
    That is, for all α, $Sol\text{-}A(\alpha) = 1 \iff Sol\text{-}B(\beta) = 1$
  - Prove that f is a polynomial time transformation

A clique in $G = (V, E)$ is a subset of V

- Each pair of vertices in a clique is connected by an edge in E
- Size of a clique = Number of vertices it contains



$|V| - V' = k$

Clique 4

CLIQUE: Is there a clique of size k in G?

A vertex cover of $G = (V, E)$ is a subset $V' \subseteq V$ such that if $(w, v) \in E$, then $w \in V'$ or $v \in V'$ or both.

VERTEX-COVER: Is there a vertex-cover of size k in G?

2. Prove that VERTEX-COVER is NP-Complete.

Solution:

(1) show NP

given a graph G and a subset of the set of vertices V' to check if vertex cover w/ a size k. for each vertex in V' check if the union of all vertices reaches all vertices in V. its polynomial

do a linear search for each $v \in V'$ and compare all to V

$O(n \cdot n)$

(2) show NP complete clique $\leq_p$ vertex cover

graph $G = (V, E)$ w/ a subset V' is a clique w/ size k $\Leftrightarrow$ graph $G^c = (V, E^c)$ is a vertex cover size $|V| - k$

($\Rightarrow$) Graph G has a k size clique w/ subset V'.

- if $(a,b) \in V'$, $(a,b) \in E$
- if $(a,b) \notin E$, $a \notin V'$ + $b \notin V'$
- if $(a,b) \in E^c$, $a \in V - V'$ + $b \in V - V'$
- $|V| - |V'| = k$, $|V| - k = |V'|$ $\therefore$ $V - V'$ is a vertex cover for G^c size $|V| - k$

($\Leftarrow$) Graph G^c has a size $|V| - k$ VC w/ V' vertexes in VC

- if $(a,b) \in V'$, $(a,b) \in EC$
- if $(a,b) \notin E^c$, $(a,b) \notin V'$
- if $(a,b) \in E$, $(a,b) \in V - V'$
- $|V| - |V| + k = k$, $V - V'$ is a graph G clique

A Hamiltonian cycle of $G = (V, E)$ is a cycle that visits every vertex exactly once.

HAM-CYCLE: Given a graph G, is there a Hamiltonian cycle in G?

Travelling Salesperson (TSP): Given a set of cities and their pairwise distances, find a tour → cycle of cost (travelled distance) at most k that visits each city exactly once.

always complete graph

3. Prove that TSP is NP-Complete.

Solution:

(1) show NP

(2) show NPcomplete

TSP $\in$ NP

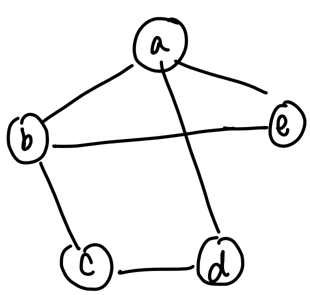
a path $\leq K$

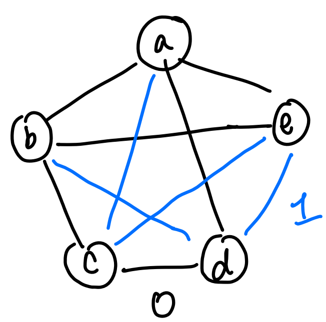
- goes through all nodes
- cost $\leq K$

$O(|E|)$ → same as $O(|V^2|)$? complete graph

② Given G from Ham, construct G' for TSP s.t.

G has a ham cycle $\iff$ G' has a TSP tour $\leq K$, $K = 0$





→ so can't go on the 1 edges

4. You are given a directed graph $G = (V, E)$ with weights $w(e)$ on its edges $e \in E$. The weights can be negative or positive. The **Zero-Weight-Cycle** Problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-complete.

Solution: