Numerical Derivatives:

$$\frac{df}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ (forward difference)

truncation error:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \mathcal{O}(h^3)$$

$$f'(x) \Rightarrow \frac{f(x+h)-f(x)}{h} = f'(x) + \underbrace{\frac{h}{2} f''(x) + \frac{h^2}{6} f'''(x) + \ldots}$$

truncation error increases w/ h
roundoff error decreases w/ h

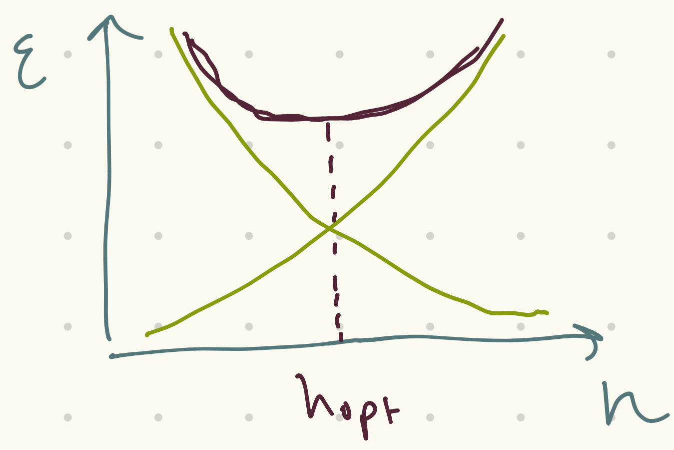
optimizing h:

$$\text{roundoff error} \sim \varepsilon_r \sim \varepsilon_f \frac{|f|}{h}$$

↳ single precision 10^{-7}
double precision 10^{-16}

total error:

$$\varepsilon = \varepsilon_t + \varepsilon_r \sim h f'' + \varepsilon_f \frac{|f|}{h}$$



$$\frac{d\varepsilon}{dh} = 0 = f'' - \varepsilon_f f/h^2$$

$$\Rightarrow \boxed{h_{opt} = \sqrt{\frac{\varepsilon_f f}{f''}}}$$

Centered difference method:

$$\frac{df}{dx} = f'(x) = \frac{f(x+h/2) - f(x-h/2)}{h}$$

... taylor series ... $= f'(x) + \frac{h^2}{24} f'''(x) + \mathcal{O}(h^3)$

↑ no second derivative term

$$e \sim h^2 f''' + \varepsilon_f \, f/h$$

optimal h: $\frac{de}{dh} = 0 = 2hf''' - \varepsilon_f \, f/h^2$

$$\boxed{h_{opt} \sim \sqrt[3]{\varepsilon_f \, (f/f''')}}$$ larger h_{opt}, smaller ε

a more precise method:

$$f'(x) = \frac{f(x+h/4) - f(x-h/4)}{h/2} = f' + \frac{h^2}{96} f''' + \mathcal{O}h^4$$

$$\Rightarrow h_{opt} \sim \varepsilon_f^{1/5} \sqrt{f/f'''}$$

$e_t \Rightarrow$ under our control