Gaussian Quadrature Method:

arbitrary w_i & x_i

$$\int_a^b f(x)\, w(x)\, dx \sim \sum_{k=1}^{N} w_k f(x_k)$$

we have 2N parameters: N weights & N choices of x's

* exact for polynomials up to degree $\leq 2N-1$ *

$$f(x) = P(x) + r(x)$$

$$P(x) = \sum_{k=1}^{N} f(x_k) \prod_{j\neq k}^{N} \frac{x - x_j}{x_k - x_j} = \sum_{k=1}^{N} f(x_k) \frac{\alpha(x)}{(x-x_k)\,\alpha'(x_k)}$$

where $\left[\alpha(x) = \prod_{i=1}^{N} (x - x_i)\right]$

for small integral of remainder:

$$\int_a^b f(x)\, w(x)\, dx = \int_a^b P(x)\, w(x) = \sum_{k=1}^{N} \frac{f(x_k)}{\alpha'(x_k)} \int_a^b \frac{\alpha(x)}{(x - x_k)}\, w(x)\, dx$$

$$\boxed{w_k \equiv \frac{1}{\alpha'(x_k)} \int_a^b \frac{\alpha(x)\, w(x)}{x - x_k}\, dx}$$

Need: $\alpha(x)$ is orthogonal to any polynomial of degree $\leq 2N-1$

↳ in order for this to be an exact representation

let $Q_1(x)$ be any polynomial of degree $\leq N-1$

$$\Rightarrow \int_a^b f(x) w(x)\, dx = \int_a^b \alpha(x) Q_1(x) w(x)\, dx$$

$$= \sum_{n=1}^{N} w_n \alpha(x_n) Q_1(x_n) = 0 \qquad (\alpha(x_n)=0) \checkmark$$

$\Rightarrow \alpha(x)$ orthogonal

assuming $\alpha(x)$ is orthogonal to all polynomials $N-1$

$\Rightarrow$ show that this formula is exact

$$\underset{2N-1}{f(x)} = \underbrace{\alpha(x) Q_2(x)}_{2N-1} + \underbrace{\rho(x)}_{N-1}$$

$$\Rightarrow \int_a^b f(x) w(x)\, dx = \underbrace{\int_a^b \alpha(x) Q_2(x) w(x)\, dx}_{=0} + \underbrace{\int_a^b \rho(x) w(x)\, dx}_{\text{exact if use formula because } \rho \text{ is } N-1 \text{ order}}$$

$\Rightarrow$ interpolation is exact $\checkmark$

$$\boxed{\int_a^b f(x) w(x)\, dx = \sum_{n=1}^{N} w_n p(x_n) = \sum_{n=1}^{N} w_n f(x_n)} \checkmark$$

mapping from [-1, 1]

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

$$t = \frac{2x-(a+b)}{a-b} \quad \text{or} \quad x = \frac{(b-a)t+(b+a)}{2}$$

popular choices for weights

$w(x) = 1 \Rightarrow$ Legendre Polynomials $P_N(x)$

$\Rightarrow P_N(x_n) = 0$

$w(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow$ Chebyshev Polynomial

$w(x) = x^{\alpha} e^{-x} \Rightarrow$ Laguerre

$w(x) = e^{-x^2} \Rightarrow$ Hermite

x_n's are then the roots of these polynomials

Random Numbers:

random real numbers : r
in the range $0 \leq r \leq 1$

$$r' = a + (b-a)\, r$$

integers between 0 & $m-1$

$$i_m = f(i_{m-1}) \leftarrow k \text{ values}$$

$$i_n = \underline{f(i_{n-1}, i_{n-2}, \ldots)} \Rightarrow m^k$$