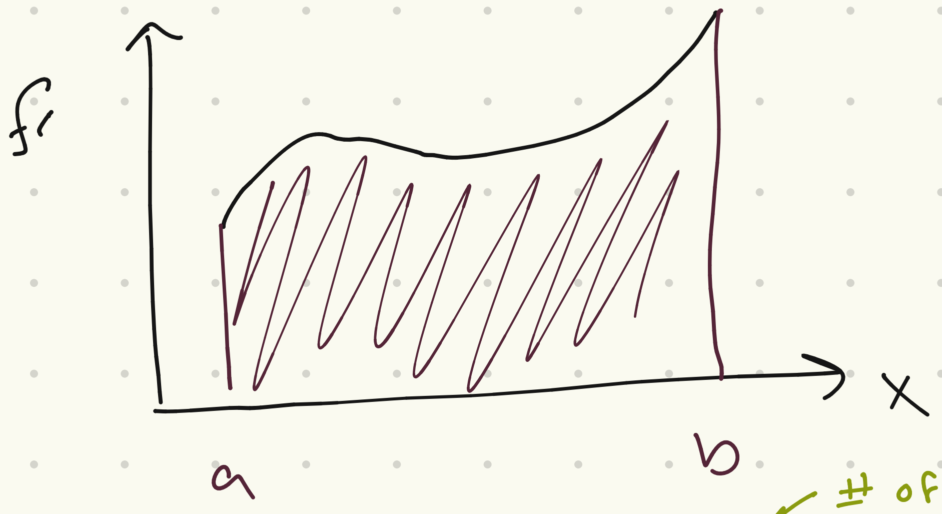
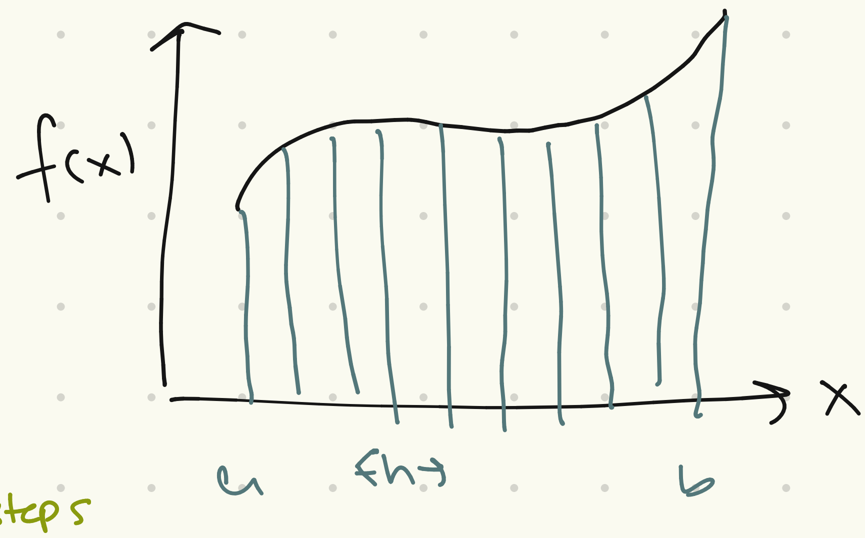
Numerical Integration:





$$\int_a^b f(x)\,dx = \lim_{h\to 0}\left[h \sum_{i=1}^{(b-a)/h} f(x_i)\right]$$

\# of steps

replace w/ finite sum

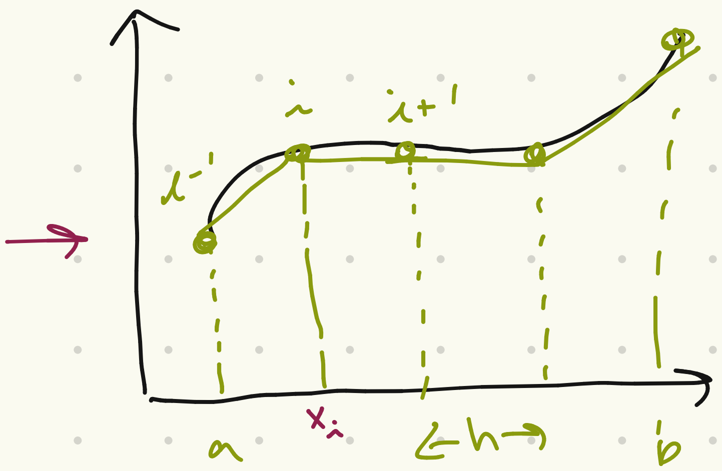
$$\int_a^b f(x)\,dx \sim \sum_{i=1}^{N} f(x_i)\, w_i$$

Two categories:

1) x_i's are equally spaced → trapezoid, Simpson

2) x_i's not equally spaced → gausian quadrature

Trapezoid Method



N-1 intervals:

$$h = \frac{b-a}{N-1}$$

$$x_i = a + (i-1)h$$

$$i = 1, \dots, N$$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \frac{f_{i+1} - f_i}{}\left(\frac{h}{2}\right) = \frac{h}{2} f_i + \frac{h}{2} f_{i+1}$$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{h}{2}\left[f_1 + 2f_2 + 2f_3 + \dots + 2f_{N-1} + f_N\right]$$

$$\boxed{w_i\text{'s} = h\left\{\frac{1}{2}, 1, 1, \dots, 1, \frac{1}{2}\right\}}$$

$$\int f(x)\,dx = \sum f_i w_i$$

Simpson's Rule: (quadratic interpolation)

$$f(x) \sim ax^2 + bx + c$$

for each interval:

$$\int_{x_i}^{(x_i)+h} (ax^2 + bx + c)\, dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Big|_{x_i}^{x_i+h}$$

$$\int_{-1}^{1} (ax^2 + bx + c)\, dx = \frac{2}{3}a + 2c = \frac{1}{3} f(-1) + \frac{4}{3} f(0) + \frac{1}{3} f(1)$$

$$\left.\begin{aligned} f(-1) &= a - b + c \\ f(0) &= c \\ f(1) &= a + b + c \end{aligned}\right\}$$

$$a = \frac{f(1) + f(-1)}{2} + f(0)$$

$$b = \frac{f(1) - f(-1)}{2}$$

$$c = f(0)$$

choose odd N $(N-1)$ is even

$$\int_{x_i - h}^{x_i + h} f(x)dx) = \frac{h}{3} f_{i-1} + \frac{4}{3} h f_i + \frac{h}{3} f_{i+1} \Leftarrow$$

$$\int_a^b f(x)\,dx = \frac{h}{3} f_1 + \frac{4h}{3} f_2 + \frac{2h}{3} f_3 + \frac{4h}{3} f_4 + \ldots + \frac{4h}{3} f_{N-1} + \frac{h}{3} f_N$$

$$\boxed{w_i = h \left\{ \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \ldots, \frac{4}{3}, \frac{1}{3} \right\}}, \quad \sum_{i+1}^{N} w_i = (N-1)h$$

Error Estimates:

$$e_{trap} \sim \mathcal{O}\left(\frac{(b-a)^3}{N^2}\right)\frac{f^{(2)}}{I} \qquad I \equiv \int_a^b f(x)dx$$

$$e_{simp} \sim \mathcal{O}\left(\frac{(b-a)^5}{N^4}\right)\frac{f^{(4)}}{I}$$

$$e_{roun} \sim \sqrt{N}\,\varepsilon_m$$

$$e = e_t + e_r \quad @ \quad e_t \sim e_r$$

trap: $\sqrt{N}\,\varepsilon_m \sim \frac{(b-a)^3}{N^2}\frac{f^{(2)}}{I} \Rightarrow \boxed{N_{trap} \sim \varepsilon_m^{-2/5}}$ f

$N \sim 10^6$

simps: $\boxed{N_{simp} \sim \varepsilon_m^{-2/9}}$ $N \sim 2000$

total error:

$$e_{trap} \sim \sqrt{N}\,\varepsilon_m \sim \varepsilon_m^{4/5}$$

$$e_{simp} \sim \sqrt{N}\,\varepsilon_m \sim \varepsilon_m^{8/9}$$

10^{-12}

5×10^{-14}

Gaussian Quadrature Method:

arbitrary w_i & x_i

$$\int_a^b f(x)\,w(x)\,dx \sim \sum_{k=1}^{N} w_k f(x_k)$$

we have $2N$ parameters: N weights & N choices of x's

* exact for polynomials up to degree $\leq 2N-1$ *

$$f(x) = P(x) + r(x)$$

$$P(x) = \sum_{k=1}^{N} f(x_k) \prod_{j\neq k}^{N} \frac{x-x_j}{x_k-x_j} = \sum_{k=1}^{N} f(x_k) \frac{\alpha(x)}{(x-x_k)\alpha'(x_k)}$$

where $\left\{ \alpha(x) = \prod_{i=1}^{N} (x-x_i) \right]$

for small integral of remainder:

$$\int_a^b f(x)\,w(x)\,dx = \int_a^b P(x)\,w(x) = \sum_{k=1}^{N} \frac{f(x_k)}{\alpha'(x_k)} \int_a^b \frac{\alpha(x)}{(x-x_k)}\,w(x)\,dx$$

$$\boxed{w_k \equiv \frac{1}{\alpha'(x_k)} \int_a^b \frac{\alpha(x)\,w(x)}{x-x_k}\,dx}$$

Need: $\alpha(x)$ is orthogonal to any polynomial of degree $\leq 2N-1$
↳ in order for this to be an exact representation

let $Q_1(x)$ be any polynomial of degree $\leq N-1$

$$\Rightarrow \int_a^b f(x)\omega(x)\,dx = \int_a^b \alpha(x)\,Q_1(x)\,\omega(x)\,dx$$

$$= \sum_{n=1}^{N} \omega_n \alpha(x_n)\,Q_1(x_n) = 0 \qquad (\alpha(x_n)=0) \checkmark$$

$\Rightarrow \alpha(x)$ orthogonal

assuming $\alpha(x)$ is orthogonal to all polynomials $N-1$
$\Rightarrow$ show that this formula is exact

$$\underset{2N-1}{f(x)} = \underbrace{\alpha(x)\,Q_2(x)}_{2N-1} + \underset{N-1}{\underline{\rho(x)}}$$

$$\Rightarrow \int_a^b f(x)\omega(x)\,dx = \underbrace{\int_a^b \alpha(x)\,Q_2(x)\,\omega(x)\,dx}_{=0} + \underbrace{\int_a^b \rho(x)\,\omega(x)\,dx}_{\text{exact if use formula}}$$

because ρ is $N-1$ order

$\Rightarrow$ interpolation is exact $\checkmark$

$$\boxed{\int_a^b f(x)\,\omega(x)\,dx = \sum_{n=1}^{N} \omega_n\, p(x_n) = \sum_{n=1}^{N} \omega_n\, f(x_n)} \checkmark$$

mapping from $[-1, 1]$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

$$t = \frac{2x - (a+b)}{a-b} \quad \text{or} \quad x = \frac{(b-a)t + (b+a)}{2}$$

popular choices for weights

$w(x) = 1 \Rightarrow$ Legendre Polynomials $P_N(x)$
$\Rightarrow P_N(x_k) = 0$

$w(x) = \dfrac{1}{\sqrt{1-x^2}} \Rightarrow$ Chebyshev Polynomial

$w(x) = x^\alpha e^{-x} \Rightarrow$ Laguerre

$w(x) = e^{-x^2} \Rightarrow$ Hermite

x_k's are then the roots of these polynomials