Homework 2, section I.7

7.2

$$\begin{aligned} f(1,1,1) &= 1^2(1) + 2(1)1^2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} |f(x,y,z)-3| &= \left|x^2y + 2xz^2 - 3\right| \\ &= \left|\left(x^2y - y\right) + (y-1) + \left(2xz^2 - 2x\right) + (2x-2)\right| \\ &\leq \left|x^2y - y\right| + |y-1| + \left|2xz^2 - 2x\right| + |2x-2| \\ &= |y|\left|x^2-1\right| + |y-1| + 2|x|\left|z^2-1\right| + 2|x-1| \\ &= |y||x+1||x-1| + |y-1| + 2|x||z+1||z-1| + 2|x-1| \end{aligned}$$

Let $0 < \delta < 1$. Then $|(x,y,z)-(1,1,1)| < \delta$ implies:

$$\begin{aligned} |x-1| &< 1 \\ |x| &< 2 \\ |x+1| &< 3 \\ |y-1| &< 1 \\ |y| &< 2 \\ |z-1| &< 1 \\ |z+1| &< 3 \end{aligned}$$

$$\begin{aligned} |f(x,y,z)-3| &< 2(3)|x-1| + |y-1| + 2(2)(3)|z-1| + 2|x-1| \\ &= 8|x-1| + |y-1| + 12|z-1| \end{aligned}$$

Let $\epsilon > 0$ and $\delta = \min\left(1, \frac{\epsilon}{21}\right)$. Then $|(x,y,z)-(1,1,1)| < \delta$ implies $|f(x,y,z)-3| < 21\delta = \epsilon$. Thus $\lim\limits_{\mathbf{x}\to(1,1,1)} f(\mathbf{x}) = 3 = f(1,1,1)$, so f is continuous at $(1,1,1)$.

7.3

$$\begin{aligned} f(x,x) &= \frac{x^2-x^2}{x^2+x^2} \\ &= 0 \end{aligned}$$

But,

$$\begin{aligned} f(x,2x) &= \frac{x^2-(2x)^2}{x^2+(2x)^2} \\ &= -\frac{3x^2}{5x^2} \\ &= -\frac{3}{5} \end{aligned}$$

Since $0 \neq -\frac{3}{5}$, f cannot be continuous at $(0,0)$.