Homework 11

28.

$$
\begin{aligned}
f(x,y,z) &= x^2 + 2y - z^2 \\
g_1(x,y,z) &= 2x - y \\
g_2(x,y,z) &= y + z \\
\nabla f &= \begin{bmatrix} 2x \\ 2 \\ -2z \end{bmatrix} \\
\nabla g_1 &= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\
\nabla g_2 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
2x &= 2\lambda_1 \\
2 &= -\lambda_1 + \lambda_2 \\
-2z &= \lambda_2 \\
0 &= 2x - y \\
0 &= y + z
\end{aligned}
$$

According to WolframAlpha, this results in a critical point at $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ with $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{8}{3}$.

$$
\begin{aligned}
&\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
&\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
&\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= s \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\nabla h &= \nabla f - \lambda_1 \nabla g_1 - \lambda_2 \nabla g_2 \\
&= \nabla f - \frac{2}{3}\nabla g_1 - \frac{8}{3}\nabla g_2 \\
&= \begin{bmatrix} 2x - \frac{4}{3} \\ 2 + \frac{2}{3} - \frac{8}{3} \\ -2z - \frac{8}{3} \end{bmatrix} \\
H_h &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\
q\left(-\frac{s}{2}, -s, s\right) &= \begin{bmatrix} -\frac{s}{2} & -s & s \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -\frac{s}{2} \\ -s \\ s \end{bmatrix} \\
&= 2\left(-\frac{s}{2}\right)^2 - 2s^2 \\
&= -\frac{3}{2}s^2
\end{aligned}
$$

So q is negative definite on the tangent space, thus $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ is a local maximum.

ACoSV II

8.2 (a)

$$
\begin{aligned}
q(x, y, z) &= 2x^2 + 5y^2 + 2z^2 + 2xz \\
&= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
|A_1| &= \begin{vmatrix} 2 \end{vmatrix} = 2 > 0 \\
|A_2| &= \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} = 10 > 0 \\
|A_3| &= \begin{vmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 15 > 0
\end{aligned}
$$

Thus by theorem 8.8, q is positive definite.

(c)

$$
\begin{aligned}
0 &= \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} \\
&= (5-\lambda)((2-\lambda)(2-\lambda) - 1) \\
&= (5-\lambda)\left(3 - 4\lambda + \lambda^2\right) \\
&= (5-\lambda)(3-\lambda)(1-\lambda)
\end{aligned}
$$

Eigenvalues, $1, 3, 5$, are all positive, so q is positive definite.