Homework 1

1. (a)

$$\begin{aligned}
\mathbf{v} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n \\
\langle \mathbf{v}, \mathbf{u}_j \rangle &= \langle c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n, \mathbf{u}_j \rangle \\
&= \langle c_1\mathbf{u}_1, \mathbf{u}_j \rangle + \cdots + \langle c_n\mathbf{u}_n, \mathbf{u}_j \rangle \\
&= c_1\langle \mathbf{u}_1, \mathbf{u}_j \rangle + \cdots + c_n\langle \mathbf{u}_n, \mathbf{u}_j \rangle \\
&= c_j\langle \mathbf{u}_j, \mathbf{u}_j \rangle && \text{(since } \mathbf{u} \text{ is pairwise orthogonal)} \\
c_j &= \frac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{|\mathbf{u}_j|^2}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{0} &= c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n \\
c_j &= \frac{\langle \mathbf{0}, \mathbf{u}_j \rangle}{|\mathbf{u}_j|^2} \\
&= 0
\end{aligned}$$

Thus all coefficients must be 0, showing that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent.

(c) $\mathbf{U}^{-1}$ is the matrix with rows, in order, $\frac{\mathbf{u}_1}{|\mathbf{u}_1|^2}, \ldots, \frac{\mathbf{u}_n}{|\mathbf{u}_n|^2}$. So, each cell of $\mathbf{U}^{-1}\mathbf{U}$ is

$$\left\langle \frac{\mathbf{u}_i}{|\mathbf{u}_i|^2}, \mathbf{u}_j \right\rangle = \frac{\langle \mathbf{u}_i, \mathbf{u}_j \rangle}{|\mathbf{u}_i|^2}$$

If $i = j$, then the cell is along the diagonal, and $\frac{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}{|\mathbf{u}_i|^2} = \frac{|\mathbf{u}_i|^2}{|\mathbf{u}_i|^2} = 1$. If $i \neq j$, then the cell is not on the diagonal, and $\frac{\langle \mathbf{u}_i, \mathbf{u}_j \rangle}{|\mathbf{u}_i|^2} = \frac{0}{|\mathbf{u}_i|^2} = 0$.
This verifies that $\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$ holds.