
I.7.2 **Claim.** $f(x, y, z) = x^2y + 2xz^2$ is continuous at $(1, 1, 1)$.

Proof. Check the value of f at $(1, 1, 1)$.

$$\begin{aligned} f(1,1,1) &= 1^2(1) + 2(1)1^2 \\ &= 3 \end{aligned}$$

Compute the limit of f at $(1, 1, 1)$.

$$\begin{aligned} |f(x,y,z) - 3| &= \left|x^2y + 2xz^2 - 3\right| \\ &= \left|\left(x^2y - y\right) + (y-1) + \left(2xz^2 - 2x\right) + (2x-2)\right| \\ &\le \left|x^2y - y\right| + |y-1| + \left|2xz^2 - 2x\right| + |2x-2| \\ &= |y|\left|x^2 - 1\right| + |y-1| + 2|x|\left|z^2-1\right| + 2|x-1| \\ &= |y||x+1||x-1| + |y-1| + 2|x||z+1||z-1| + 2|x-1| \qquad (1) \end{aligned}$$

Let $0 < \delta < 1$. Then $|(x, y, z) - (1, 1, 1)| < \delta$ implies:

$$\begin{aligned} |x-1| &< \delta \\ |x| &< 2 \\ |x+1| &< 3 \\ |y-1| &< \delta \\ |y| &< 2 \\ |z-1| &< \delta \\ |z+1| &< 3 \end{aligned}$$

Substituting these facts into (1) results in:

$$\begin{aligned} |f(x,y,z) - 3| &< 2(3)\delta + \delta + 2(2)(3)\delta + 2\delta \\ &= 21\delta \end{aligned}$$

Let $\varepsilon > 0$ and $\delta = \min\left(1, \frac{\varepsilon}{21}\right)$. Then $|(x, y, z) - (1, 1, 1)| < \delta$ implies $|f(x, y, z) - 3| < 21\delta \le \varepsilon$.

Thus,

$$\begin{aligned} \lim_{(x,y,z)\to(1,1,1)} f(x,y,z) &= 3 \\ &= f(1,1,1) \end{aligned}$$

And so f must be continuous at $(1, 1, 1)$. $\square$
