
Homework 8

18.

$$G(x,y) = \left(x, y, Ax^2 + By^2\right)$$
$$G'(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2Ax & 2By \end{bmatrix}$$
$$G'(x,y)^{\perp} = s\begin{bmatrix} 2Ax \\ 2By \\ -1 \end{bmatrix}$$

$$P_1 : \quad \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \end{bmatrix}$$
$$P_2 : \quad \begin{bmatrix} 2Aa & 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} a \\ 0 \\ Aa^2 \end{bmatrix}\right) = \begin{bmatrix} 0 \end{bmatrix}$$
$$P_3 : \quad \begin{bmatrix} 0 & 2Bb & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ b \\ Bb^2 \end{bmatrix}\right) = \begin{bmatrix} 0 \end{bmatrix}$$

19.

$$f(x,y,z) = xyz$$
$$\nabla f = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$
$$G(x,y,z) = \begin{bmatrix} 2yz + 2xz + 2xy - 10 \\ x - 2y \end{bmatrix}$$
$$\nabla G = \begin{bmatrix} 2y + 2z & 1 \\ 2x + 2z & -2 \\ 2x + 2y & 0 \end{bmatrix}$$
$$yz = \lambda_1(y + z) + \lambda_2$$
$$xz = \lambda_1(x + z) - 2\lambda_2$$
$$xy = \lambda_1(x + y)$$
$$x = 2y$$
$$10 = 2yz + 2xz + 2xy$$
$$2y^2 = 3\lambda_1 y$$
$$y = \frac{3\lambda_1}{2}$$
$$x = 3\lambda_1$$

5.4 Extreme distances from $\frac{x^2}{9} + \frac{y^2}{4} = 1$ to $(1, 1)$.

$$\begin{aligned}
g(x,y) &= \frac{x^2}{9} + \frac{y^2}{4} - 1 \\
\nabla g &= \begin{bmatrix} \frac{2x}{9} \\ \frac{y}{2} \end{bmatrix} \\
f(x,y) &= (x-1)^2 + (y-1)^2 && (\text{dist}^2 \text{ maximizes dist}) \\
\nabla f &= \begin{bmatrix} 2x-2 \\ 2y-2 \end{bmatrix} \\
2x - 2 &= \frac{\lambda 2x}{9} && (\nabla f = \lambda \nabla g) \\
9x - 9 &= \lambda x \\
\lambda &= 9 - \frac{9}{x} \\
2y - 2 &= \frac{\lambda y}{2} && (\nabla f = \lambda \nabla g) \\
y(4 - \lambda) &= 4 \\
y\left(\frac{9}{x} - 5\right) &= 4
\end{aligned}$$

Input into WolframAlpha:

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solve {y*(9/x - 5) = 4, x^2/9 + y^2/4 = 1}
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$$x \approx -2.90702 \text{ and } y \approx -0.494074 \qquad (\text{max})$$
$$x \approx 1.24999 \text{ and } y \approx 1.81812 \qquad (\text{min})$$

5.6 Box of maximum volume within $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\begin{aligned}
g(x,y,z) &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \\
\nabla g &= \begin{bmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \\ \frac{2z}{c^2} \end{bmatrix} \\
f(x,y,z) &= 8xyz \\
\nabla f &= \begin{bmatrix} 8yz \\ 8xz \\ 8xy \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
yz &= \frac{\lambda x}{a^2} && (\nabla f = \lambda \nabla g) \\
xyz &= \frac{\lambda x^2}{a^2} \\
xz &= \frac{\lambda y}{b^2} && (\nabla f = \lambda \nabla g) \\
xyz &= \frac{\lambda y^2}{b^2} \\
\frac{\lambda x^2}{a^2} &= \frac{\lambda y^2}{b^2} \\
\frac{x^2}{a^2} &= \frac{y^2}{b^2} \\
xy &= \frac{\lambda z}{c^2} && (\nabla f = \lambda \nabla g) \\
xyz &= \frac{\lambda z^2}{c^2} \\
\frac{x^2}{a^2} &= \frac{z^2}{c^2} \\
1 &= \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} \\
\frac{a^2}{3} &= x^2 \\
x &= \frac{a}{\sqrt{3}}
\end{aligned}
$$

So box of maximum volume has a corner at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$, and is symmetric about the origin.
