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Thermodynamics of Rubber Band Heat Engine

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Abstract

In this paper, we study the thermodynamics of a rubber band and calculate the efficiency of Archibald's rubber band heat engine. The efficiency of this rubber engine is compared with the efficiency of a Carnot rubber band engine. The efficiency of the Archibald's engine is in the order 10^{-3}, which is very less compared to the efficiency of ideal heat engine.

INTRODUCTION

Most of us are familiar with the very interesting characteristic property of a rubber band that unlike most of the solid material that extends when heated, a rubber band contracts when it gets heated. In 19th century, the English physicist James Prescott Joule noted this phenomenon and then very sooner, another scientist Paul Archibald used this interesting phenomenon to make his own model of a heat engine. In order to give a thermodynamic description of the Archibald rubber band heat engine, we will be using an Ideal rubber band law for its equation of state. The equation of state of a real rubber band is very complex. But as we have seen in the case of gases, the ideal state approximation reduces the complexity of the equation of state of gases, we will also assume an ideal rubber band whose equation of state would be simpler to describe. In order to study the thermodynamics of heat engines, the one modelled by Archibald has a number of advantages over many other heat engines as it is very easy to build and demonstrate. Also, the engine once placed in hot water can easily continue to run for an hour or so. But, sometimes hot water can decrease the rotation speed and can even damage the rubber bands.

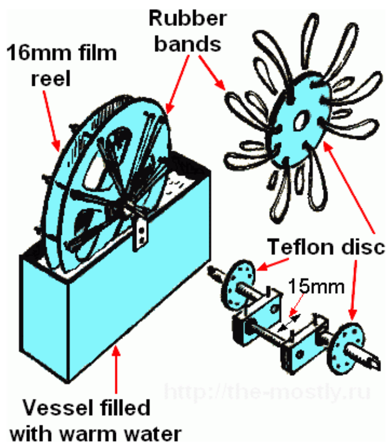

Figure-1: Archibald rubber band heat engine model.

So far we have been studying p, V, T systems in thermodynamics, but the rubber band system is one of the non p, V, T system where thermodynamics can be applied, which makes it even more desirable to study. When we treat this engine to be reversible, we see that its efficiency is less than that of the ideal Carnot engine which arises because of the heat capacity of the rubber bands. If this heat capacity could be neglected and other assumptions could hold, our heat engine would approach ideal Carnot limit for efficiency. In this paper, the analysis of the torques and other forces, and the change in the Internal Energy and entropy are readily taken [4].

BASIC DESIGN OF A RUBBER BAND HEAT ENGINE

The Archibald rubber band heat engine consist of a series of rubber bands attached to a 16 mm film reel as a wheel. While the other end of the rubber bands are fastened by a teflon washer that are used as bearings. The wheel and the attached bands rotate when submerged in hot water, while both the reel axis and the rubber band axis are fixed and do not move. By just examining one of the rubber bands on the assembly, one can analyze the basic working of engine.

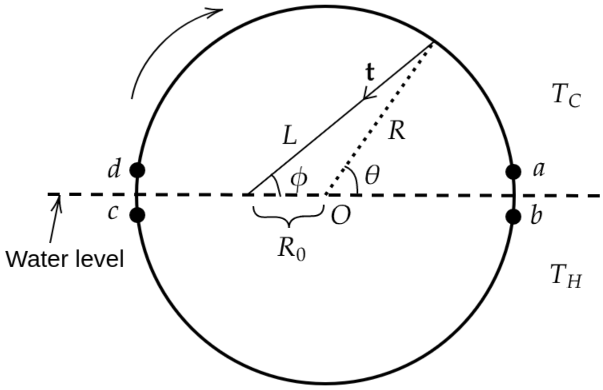

Figure-1: Schematic representation of a rubber band heat engine. Here, O and R are the centre and radius of the wheel, respectively. L is the length of the rubber band and R_o is the offset distance between the wheel axis and rubber band axis. T_C and T_H are the respective temperature above and below the water level and t is the tension force on the wheel.

The rubber band that lies inside the hot water gets hot, and thus contracts that increases the tension in the band which causes a difference in tensions between the bands above and below the water level. Thus, this difference in tension forces causes a net torque about the central axis of the wheel, which causes the rotation of the wheel. Here, we will also be assuming that the bands can quickly reach to the temperature of the surrounding reservoir, since they are very thin and the rotation speed is also very slow. Thus, when the rubber band crosses from a to b, its temperature will quickly change from T_C to T_H.

THERMODYNAMICS OF RUBBER BAND

Some of the state parameters of the rubber band system that we will be using are the Length of the rubber band (L), radius of the wheel (R), absolute temperature of the rubber band (T). Also, we know that the tension due to the rubber band is a function of L and T [1]. Now, we'll do a first order Taylor expansion to determine the Force law for the given system. We'll be doing that about the bands' mean length, radius of the wheel and the average temperature $\bar{T}$ which is given as,

$$\bar{T} = \frac{T_C + T_H}{2}$$

Thus, we shall write the Tension in the rubber band as;

$$t(L,T) = t(R,\bar{T}) + \rho(L-R) + \sigma(t-\bar{T}) \quad (1)$$

where ρ and σ are written as;

$$\rho = \frac{\partial t}{\partial L}\Big|_T$$

$$\sigma = \frac{\partial t}{\partial T}\Big|_L$$

that are evaluated at $(R,\bar{T})$. Clearly, these factors depends upon the rubber bands and the engine parameter used. Equation 1 should give us a good description of our rubber band engine, since,

$$\frac{|L-R|}{R} << 1 \; and \; \frac{|T-\bar{T}|}{\bar{T}} << 1$$

Now, we know that the change in heat for a reversible process is given by,

$$dQ_{rev} = TdS$$

where the Entropy is a function of T and L, which can be written as,

$$dS = \frac{\partial S}{\partial T}\Big|_L dT + \frac{\partial S}{\partial L}\Big|_T dL$$

Thus, the above equation for heat can be rewritten as,

$$TdS = T\frac{\partial S}{\partial T}\Big|_L dT + T\frac{\partial S}{\partial L}\Big|_T dL$$

where $\frac{\partial S}{\partial T}\big|_L = C_L$ which is the heat capacity of the rubber band at constant length. Also, we know that the Maxwell's equation says;

$$\frac{\partial S}{\partial P}\Big|_T = -\frac{\partial V}{\partial T}\Big|_P$$

Now, if we substitute $P \to L$ and $V \to -t$, the equation for TdS can be rewritten as;

$$TdS = C_L dT - T\frac{\partial t}{\partial T}\Big|_L dL$$

$$TdS = C_L dT - T\sigma dL \tag{2}$$

From the above equation, we also note that C_L is a function of T only and it does not depend on L. This can be shown by the following argument,

$$\frac{\partial C_L}{\partial L}\Big|_T = \frac{\partial}{\partial L}\left(T\frac{\partial S}{\partial T}\Big|_L\right)\Bigg|_T$$

inverting the order of differentiation and using the Maxwell's relation, we find,

$$\frac{\partial C_L}{\partial L}\Big|_T = -T\frac{\partial^2 t}{\partial T^2}\Big|_L$$

Here, the RHS of the above equation vanishes since we have used the linear force law. This clearly implies that C_L must only depend on T and not on L if Equation 1 is a good representation of the equation of state.

RUBBER BAND ENGINE EFFICIENCY

We know that the efficiency of a Carnot's heat engine is defined as,

$$\eta = \frac{W_o}{Q_i}$$

where W_o is the work obtained from the engine per band per revolution and Q_i is the heat put in the engine per band per revolution. Now, from the first law of thermodynamics we know that,

$$đQ = dU + đW \tag{3}$$

where $đQ$ is the heat gained by the rubber band system, dU is the change in the Internal energy of the system and $đW$ is the work done by the system. For a reversible process, the first law can be written as,

$$TdS = dU - tdL$$

where $đW = -tdL$ is analogous to pdV in the $p,\ V,\ T$ system.
Now, analogous to PV diagram analysis that we do in case of an ideal gas Carnot engine, we will be studying the tL plot for our rubber band heat engine. Here, in order to determine the W_o and Q_i, we will be studying the four basic process for one of the rubber bands when it goes from $a \to b$, $b \to c$, $c \to d$ and $d \to a$, as labeled on Figure-2. Now, we will calculate the heat gained by one rubber band in all these processes.
When our rubber band absorbs heat while going from a to b, the length L remains constant if the distance between a and b is much less than the wheel radius R (which is to say that the angle aOb is small compared to π). As we have already assumed our rubber band to be thin enough, the time required to heat the band from T_C to T_H will be short and thus, the process could be treated as a reversible process. But, for this to be strictly true, our process would have to be quasi-static, for which we will require an infinite number of heat reservoirs. This is the fact that restricts our heat engine from being an ideal Carnot engine. Although, in a limiting case, the engine will approach the Carnot efficiency. Now, by using equation-2 for $dL = 0$, we can determine the heat gained by the system in moving from a and b as;

$$Q_{ab} = \int_{T_C}^{T_H} C_L(T)dT \tag{4}$$

Now, while going from b to c, the rubber band goes through an isothermal contraction at temperature T_H. Thus, again by using Equation-2, the heat gained in this process can be written as;

$$Q_{bc} = -T_H \int_{R+R_o}^{R-R_o} \frac{\partial t}{\partial T}\Big|_L dL$$

$$Q_{bc} = 2\sigma R_o T_H \tag{5}$$

Similarly, for the path c to d, we will get a constant length heat process and for d to a, we will get an isothermal expansion of the rubber band at temperature T_C. Thus, the change in heat in the rubber band suring these two processes is given by;

$$Q_{CD} = \int_{T_C}^{T_H} C_L(T)dT \tag{6}$$

$$Q_{da} = -2\sigma R_o T_H \tag{7}$$

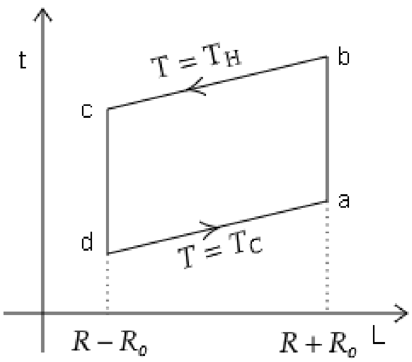

Figure-2: The figure shows the work cycle for the Archibald's rubber band heat engine.

Since, we are studying a cyclic process, the change in the Internal Energy over the whole cycle would be zero, i.e. $\Delta U = 0$. Thus, using the first law of the thermodynamics, we get,

$$W_o = Q_{ab} + Q_{bc} + Q_{cd} + Q_{da} \tag{8}$$

Also, we have,

$$Q_i = Q_{ab} + Q_{bc} \tag{9}$$

Now, on substituting the results of Equations-(4-7) into Equation-10 and 11, we get the efficiency as;

$$\eta = \frac{\left(1 - \frac{T_C}{T_H}\right)}{1 + \frac{\int_{T_C}^{T_H} C_L(T)dT}{2\sigma R_o T_H}} \tag{10}$$

which will be exactly equal to the efficiency of a Carnot engine if,

$$\frac{\bar{C_L}}{2\sigma R_o} << 1 \tag{11}$$

where $\bar{C_L}$ is the temperature averaged specific heat.

As it was very obvious, this condition does not hold true for an Ideal engine. The σ value calculated for a Janis No. 12 bands comes out to be $\sigma = 1 \times 10^{-2}\ N/K$. Now, if we take $C_L = 0.4\ cal/g\ K \times 0.2\ g$ and $R_o = 2\ cm$, we fine (Mullen et al., 1974),

$$\frac{\bar{C_L}}{2\sigma R_o} \approx 10^3$$

Here, we can clearly see that this value is far away from our ideal limit, i.e. 1. Thus, the efficiency of our rubber band heat engine is 10^{-3}, which is very less in comparison to other engines available nowadays. Hence, it is not a very good idea to make heat engines using rubber band for our practical purposes.

COMPARISON WITH A CARNOT ENGINE

Now, we'll move on to compare the efficiency of our heat engine calculated above to the efficiency of a Carnot engine made up by rubber bands. In order to make this comparison, we determine the t v/s L relations for both isotherms and adiabats for rubber bands obeying equation-1. From equation-1, the isotherms can be written as;

$$t - \rho L = K(T) \tag{12}$$

where $K(T)$ is a constant for a given isotherm and relates to the temperature as;

$$K(T) = t(R, \bar{T}) - \rho R + \sigma(T - \bar{T}) \tag{13}$$

Now, the equation for the tension as a function of L for the adiabats can be found from Eqs-1 and 2, with $dS = 0$ and C_L regarded as temperature independent over some relevant temperature range. The family of adiabats are described by the relation,

$$t = K'(S)e^{\frac{\sigma L}{C_L}} + \rho L + K_o \tag{14}$$

where $K_o = t(R, \bar{T} - \rho R - \sigma\bar{T})$. Now, we can clearly see that when $\frac{\sigma L}{C_L} >> 1$, the t v/s L curves for the adiabats goes vertical. This is why we get the Carnot efficiency in the limit of condition (11), which is equivalent to the above condition.

Following is the Carnot cycle for a rubber band;

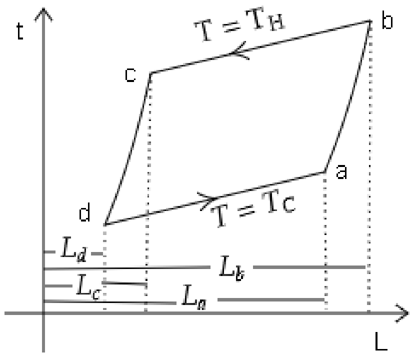

Figure-3: The figure shows the work cycle for the Archibald's rubber band under going a Carnot cycle.

Again, the processes ab and cd are adiabatic and processes bc and da are isothermic for the rubber band Carnot engine work cycle. For this cycle,

$$Q_{ab} = 0 = Q_{cd},$$

$$Q_{bc} = -\sigma T_H(L_c - L_b),$$

$$Q_{da} = -\sigma T_C(L_a - L_d),$$

Also, from equation-1, we can show that,

$$L_b - L_a = L_c - L_d = \frac{C_L}{\sigma} ln(\frac{T_H}{T_C}) \qquad (15)$$

By using these equations, we can easily calculate the efficiency of the rubber band Carnot engine which comes out to be,

$$\eta = 1 - \frac{T_C}{T_H} \qquad (16)$$

This directly confirms that for any reversible engine, $\Delta S_{universe} = 0$. The difference in the value we got for both the efficiencies also emphasizes that reversible engines do not always gives the Carnot efficiency. If our engine would have been a two heat reservoir system, we would have got the Carnot efficiency. This also follows from equation- 11 which does not seems to likely occur practically.

Limitations of the Carnot engine lie in value of ΔT in the adiabatic process which does not go past the yield point of the rubber band used. From equation-15, we can calculate η as,

$$\eta \approx \frac{\sigma(L_a - L_b)}{\bar{C_L}}$$

,

which implies that rubber band Carnot engine can not be much more efficient than the Archibald engine, because the stretch in the length of the rubber band can never exceed $2R_o$ without exceeding its yield point. Thus, the Carnot engine with rubber band is possible to make but it is totally impractical.

Conclusion

The heat engine made by rubber bands is easy to construct and it follows simple thermodynamics analysis. We found the efficiency in the reversible limit in equation-10, that under ideal conditions functions as a Carnot engine. Because of the limit in the temperature range of the working of a rubber band, without getting deteriorated, indicates that even in Carnot limit this engine would not be very efficient.

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