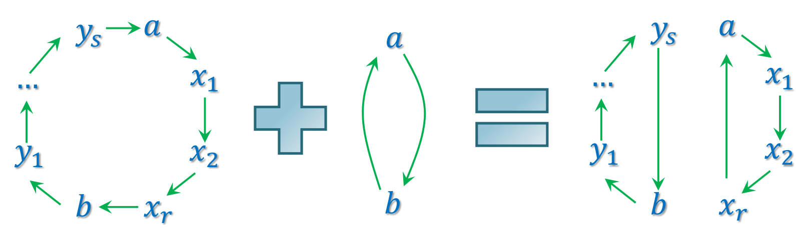
Ma/CS 6a

Class 17: More Permutations



By Adam Sheffer

Reminder: The Permutation Set S_n

- S_n – The set of permutations of $\mathbb{N}_n = \{1,2,3,\ldots,n\}$.
- The set S_3:

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{matrix}$$

Reminder: Cycle Notation

- We can consider a permutation as a set of cycles.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 6 & 3 & 2 & 1 & 4 \end{matrix}$$

$$1 \to 5 \qquad 2 \to 6 \to 4 \qquad 3$$

- We write this permutation as
$$(1\ 5)(2\ 6\ 4)(3).$$

Reminder: Classification of Permutations

- Both $(1\ 2\ 4)(3\ 5)$ and $(1\ 2\ 3)(4\ 5)$ are of the same ***type***: one cycle of length 3 and one of length 2.
  - We denote this type as $[2\ 3]$
- In general, we write a type as $[1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} \ldots]$.

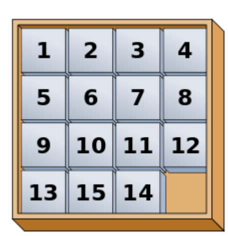
The 15 Puzzle and Permutations

- How a configuration of the puzzle can be described as a permutation?
  - Denote the missing tile as 16.
  - The board below corresponds to the permutation

$1\ 16\ 3\ 4\ 6\ 2\ 11\ 10\ 5\ 8\ 7\ 9\ 14\ 12\ 15\ 13$

The 15 Puzzle Revisited

- What kind of permutations describe a move in the 15 Puzzle?
  - Permutations that switch 16 with an element that was adjacent to it.



Transpositions

- *Transposition*: a permutation that interchanges two elements and leaves the rest unchanged.
  - $(1)(3)(5)(2\ 4)$

$$1\ 2\ 3\ 4\ 5$$

$$1\ 4\ 3\ 2\ 5$$

Decomposing a Cycle

- **Problem.** Write the cycle $(1\ 2\ 3)$ as a composition of transpositions.

$$1\ 2\ 3$$

$(1\ 2)(3)$

$$2\ 1\ 3$$

$(1\ 3)(2)$

$$2\ 3\ 1$$

$$\boxed{(1\ 2\ 3) = (1\ 3)(1\ 2)}$$

Decomposing a Cycle (2)

- **Problem.** Write the cycle $(x_1\ x_2\ \dots\ x_k)$ as a composition of transpositions.

$$x_1\ x_2\ x_3\ \dots\ x_k$$

$$(x_1\ x_2)$$

$$x_2\ x_1\ x_3\ \dots\ x_k$$

$$(x_1\ x_3)$$

$$x_2\ x_3\ x_1\ \dots\ x_k$$

$$\boxed{\begin{array}{c}(x_1\ x_2\ \dots\ x_k) = \\ (x_1\ x_k)(x_1\ x_{k-1})\cdots(x_1\ x_2)\end{array}}$$

$$x_2\ x_3\ x_4\ \dots\ x_1$$

Decomposing a Permutation

- **Problem.** Can every permutation be written as a composition of transpositions?

- Yes! Write the permutation in its cycle notation and decompose each cycle.

$$(1\ 3\ 6)(2\ 4\ 5\ 7)$$
$$= (1\ 6)(1\ 3)(2\ 7)(2\ 5)(2\ 4)$$

Unique Representation?

- **Problem.** Does every permutation have a unique decomposition into transpositions (up to their order)?

- $(1\ 2\ 3)(4\ 5\ 6)$:
  - $(1\ 3)(1\ 2)(4\ 6)(4\ 5)$.
  - $(1\ 4)(1\ 6)(1\ 5)(3\ 4)(2\ 4)(1\ 4)$.
  - No. But the different decompositions of a permutation have a common property.

Composing a Permutation with a Transposition

- **Problem.**
  - α – a permutation of S_n that consists of c cycles in its cycle notation.
  - τ – a transposition of S_n.
  - What can we say about the number of cycles in $\tau\alpha$? And of $\alpha\tau$?

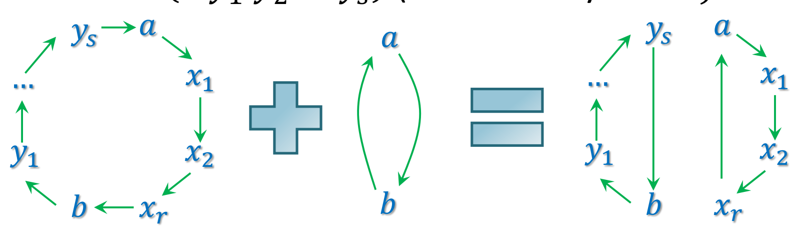
Solution: Case 1

- Write $\tau = (a\ b)$.
- First, assume that a, b are in the same cycle of α.
  - Write the cycle as
    $$(a\ x_1\ x_2\ \ldots\ x_r\ b\ y_1\ y_2\ \ldots y_s).$$
  - Then $\tau\alpha$ contains the cycles $(a\ x_1\ x_2\ \ldots\ x_r)$ and $(b\ y_1\ y_2\ \ldots y_s)$ (and similarly for $\alpha\tau$).

$$\begin{array}{cccccccc} a & x_1 & \cdots & x_r & b & y_1 & \cdots & y_s \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & \cdots & a & y_1 & y_2 & \cdots & b \end{array}$$

Solution: Case 1

- Write $\tau = (a\ b)$.
- First, assume that a, b are in the same cycle of α.
  - Write the cycle as
    $$(a\ x_1\ x_2\ \ldots\ x_r\ b\ y_1\ y_2\ \ldots y_s).$$
  - Then $\tau\alpha$ contains the cycles $(a\ x_1\ x_2\ \ldots\ x_r)$ and $(b\ y_1\ y_2\ \ldots y_s)$ (and similarly for $\alpha\tau$).



Solution: Case 2

- Write $\tau = (a\ b)$.
- Assume that a and b are in different cycles of α.
  - Write the cycles as $(a\ x_1\ x_2\ \dots\ x_r)$ and $(b\ y_1\ y_2\ \dots y_s)$.
  - Then $\tau\alpha$ contains the cycle
    $(a\ x_1\ x_2\ \dots\ x_r\ b\ y_1\ y_2\ \dots y_s)$.

$$\begin{array}{cccccccc}
a & x_1 & \dots & x_r & b & y_1 & \dots & y_s \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
x_1 & x_2 & \dots & b & y_1 & y_2 & \dots & a
\end{array}$$

Solution

- α – a permutation of S_n that consists of c cycles in its cycle notation.
- τ – a transposition of S_n.
- The number of cycles in $\tau\alpha$ (or $\alpha\tau$) is either $c+1$ or $c-1$.

Parity of a Permutation

- **Theorem.** Consider a permutation $\alpha \in S_n$. Then
  - Either every decomposition of α into transpos. consists of an even number of elements,
  - or every such decomposition consists of an odd number of elements.

- $(1\ 2\ 3)(4\ 5\ 6)$:
  - $(1\ 3)(1\ 2)(4\ 6)(4\ 5)$.
  - $(1\ 4)(1\ 6)(1\ 5)(3\ 4)(2\ 4)(1\ 4)$.

Proof

- c – the number of cycles in α.
- (WLOG) Assume that n is even.
- Consider a decomposition $\alpha = \tau_1 \tau_2 \cdots \tau_k$.
  - The number of cycles in the cycle structure of τ_k is $n - 1$.
  - The number of cycles in $\tau_{k-1}\tau_k$ is even.
  - The number of cycles in $\tau_{k-2}\tau_{k-1}\tau_k$ is odd.
  - …
  - The number of cycles in $\tau_1\tau_2 \cdots \tau_k$ is c.
- Thus, k has the same parity as c.

Even and Odd Permutations

- We say that a permutation is *even* or *odd* according to the parity of the number of transpositions in its decompositions.

Finding the Parity

- What is the parity of

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{array}$$

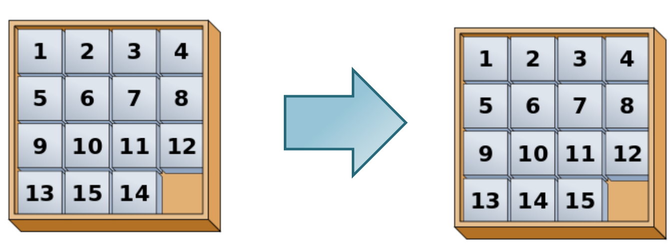
- Cycle notation: $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$.
- We saw that a cycle of length k can be expressed as composition of $k-1$ transpositions.
  - 8 transpositions = even.

Parity of Inverse

- **Problem.** Prove that any permutation $\alpha \in S_n$ has the same parity as its inverse α^{-1}.
- **Proof.**
  - Decompose α into transpositions $\tau_1 \tau_2 \cdots \tau_k$.
  - We have $\alpha^{-1} = \tau_k \cdots \tau_2 \tau_1$, since the product of these two permutation is obviously id.

The 15 Puzzle

- **Problem.** Start with the configuration on the left and move the tiles to obtain the configuration on the right.



Solution (Finally!)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Even permutation

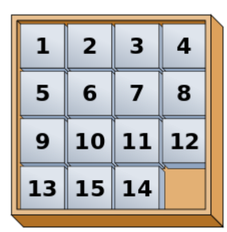
1 2 3 4 5 6 7 8 9 10 11 12 13 15 14 16

Odd permutation

- The number of moves is even since
  - For every time that we move the empty tile left/up, we must move it back right/down.

Solution (Finally!)

Even permutation



Odd permutation

- The number of moves/ transpositions is even.
- To move from an even transposition to an odd one, there must be an odd number of transpositions.

Contradiction!

Even and Odd Permutations of S_5

Type	Example	Number	
$[1^5]$	id	1	*Even*
$[1^3 2]$	$(1\ 2)(3)(4)(5)$	10	*Odd*
$[1^2 3]$	$(1\ 2\ 3)(4)(5)$	20	*Even*
$[12^2]$	$(1\ 2)(3\ 4)(5)$	15	*Even*
$[14]$	$(1\ 2\ 3\ 4)(5)$	30	*Odd*
$[23]$	$(1\ 2\ 3)(4\ 5)$	20	*Odd*
$[5]$	$(1\ 2\ 3\ 4\ 5)$	24	*Even*

Even: 60 *Odd: 60*

Even and Odd Permutations of S_n

- **Theorem.** For any integer $n \geq 2$, half of the permutations of S_n are even and half are odd.

Proof

- τ – an arbitrary transposition of S_n.
- If $\alpha \in S_n$ is even, then $\tau\alpha$ is odd.
- If $\alpha \in S_n$ is odd, then $\tau\alpha$ is even.
- For any $\alpha \in S_n$, we have $\tau\tau\alpha = \alpha$.
- τ defines a *bijection* between the set of even permutations of S_n and the set of odd permutations of S_n.
  - Thus, the two sets are of the same size.

Example: The Bijection in S_3

- Let $\tau = (1\ 2) \in S_3$.

Even		*Odd*
$(1)(2)(3)$	$\longleftrightarrow$	$(1\ 2)(3)$
$(1\ 2\ 3)$	$\longleftrightarrow$	$(1)(2\ 3)$
$(3\ 2\ 1)$	$\longleftrightarrow$	$(1\ 3)(2)$

$(1\ 2)(1\ 3) = (1\ 2\ 3)$

The End: The First Math Theorem Proved in a TV Script?

- The 10^{th} episode of 6^{th} season of the TV show *Futurama* is about people switching bodies.
- This is a permutation of people, and a property of the permutations is used as a *plot twist*!
- You can also see a complete mathematical proof for a second.