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使用隐式欧拉法求解常微分方程问题

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1 问题

对于下列微分方程初值问题:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + 2y \\ y(0) = -5 \end{cases}$$

使用隐式欧拉法求解其在$0 < x < 5$时的数值解。

2 求解方程：解析解

透过python的sympy库求解

```python
from sympy import Function, dsolve, Eq, Derivative, sin, cos, symbols
from sympy.abc import x
f = Function('f')
dsolve(Derivative(f(x), x) - x * f(x)**2 -2 * f(x), f(x), ics={f(0): -5})
```

Listing 1: 0.1s

可以得到方程解为：

$$f(x) = \frac{4e^{2x}}{-2xe^{2x}+e^{2x}-\frac{9}{5}}$$

3 透过三种不同方法求解微分方程

3.1 隐式欧拉法

```
x0 = 0;
y0 = -5;
a = 0;
b = 5;
h = 0.2;
y = Euler_Implicit(@f, y0, a, b, h);

x = linspace(a, b, length(y));
hold on
plot(x, y, '-o');
plot(x,RealFunc(x),"-b");
xlabel('x');
ylabel('y');
title('Numerical Solution and error');

function y = Euler_Implicit(f, y0, a, b, h)
    n = round((b-a)/h);
    y = zeros(1, n+1);
    y(1) = y0;

    for i = 2:n+1
        xi = a + (i-1) * h;
        yi = y(i-1);
```

```
            for j = 1:100
                yi = y(i-1) + h * f(xi, yi);
            end
            y(i) = yi;
        end
    end

    function f = f(x, y)
         f= x.*y.^2 + 2.*y;
    end

    function ye = RealFunc(x)
        ye = (20.*exp(2.*x))./(exp(2.*x)-10.*x.*exp(2.*x)-9);
    end
```

Listing 2: 隐式欧拉法

其中$functionye = RealFunc(x)$部分是文章第一部分所解出的解析解；$functionf = f(x,y)$部分是原方程，将$\frac{\mathrm{d}y}{\mathrm{d}x}$替换为f； $functiony = Euler_Implicit(f,y0,a,b,h)$是函数主体，解释如下：

1. 根据步长 h 计算需要多少个点n，并初始化结果数组；

2. 将初始值赋给第一个元素；

3. 循环从 i=2 到 i=n+1，每次计算 xi 和 yi；

4. 在内部循环中进行100次迭代，利用欧拉隐式法更新 yi 的数值；

5. 将最终得到的 yi 赋给结果数组。

3.2 显式欧拉法

[H]

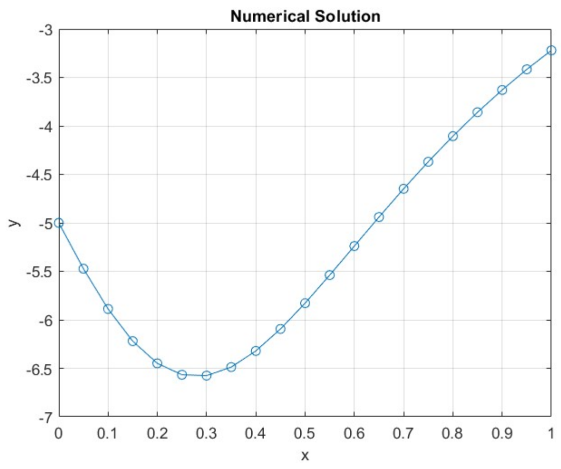


图 1: 隐式欧拉法

```
x0 = 0;
y0 = -5;
h = 0.1;
y = Euler_Explicit(@f, y0, h);

x = linspace(0, 5, length(y));
hold on
plot(x, y, '-o');
plot(x,RealFunc(x),"-b");

xlabel('x');
ylabel('y');
title('Numerical Solution and Error');
legend('Numerical Solution');


function y = Euler_Explicit(f, y0, h)
    t0 = 0;
    tp = 5;
    t = t0:h:tp;
```

```
        y = zeros(size(t));
        y(1) = y0;

        for i = 1:length(t)-1
            y(i+1) = y(i) + h * f((i+1) * h, y(i));
        end
    end

    function fullderi = f(x, y)
        fullderi = x.*y.^2 + 2*y;
    end

    function ye = RealFunc(x)
        ye = (20.*exp(2.*x))./(exp(2.*x)-10.*x.*exp(2.*x)-9);
    end
```

Listing 3: 显式欧拉法

$functiony = Euler_Explicit(f, y0, h)$是函数主体，解释如下： 1.初始化变量$x_0$、$y_0$和$h$

2.调用$Euler_Explicit$函数计算数值解y

3.生成等间隔的自变量x

4.绘制数值解曲线和真实解曲线，并添加标签、标题及图例。

其中，$Euler_Explicit$函数使用欧拉显式法对微分方程进行离散化处理并求出数值近似解；$f(x, y)$为给定的微分方程右侧；$RealFunc(x)$为该微分方程的真实解。

3.3 4th-order Runge-Kutta

```
    y0 = -5;
    t0 = 0;
    tn = 5;
```

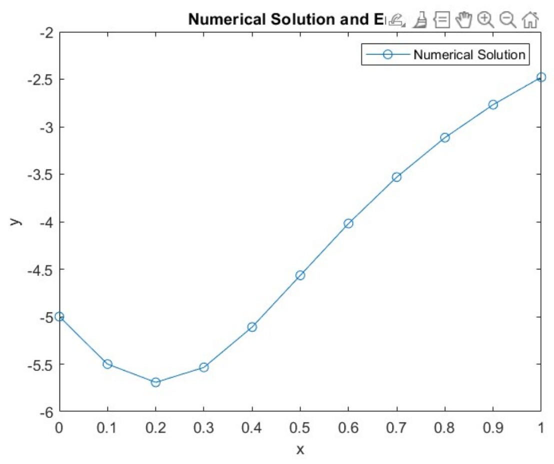
[H]


图 2: 显式欧拉法

```
h = 0.2;

result = Runge_Kutta41(@f, t0, y0, tn, h);
x = linspace(0, 5, length(result));
hold on
plot(x, result, '-o');
plot(x,RealFunc(x),"-b");

xlabel('x');
legend();

function res = Runge_Kutta41(f, t0, y0, tn, h)
    res = [y0];
    t = t0;
    for i = 1:floor((tn-t0)/h)
        k1 = f(t, res(end));
        k2 = f(t + h / 2, res(end) + h * k1 / 2);
        k3 = f(t + h / 2, res(end) + h * k2 / 2);
        k4 = f(t + h, res(end) + h * k3);
        y_next = res(end) + h*(k1 + 2 * k2 + 2 * k3 + k4)/6;
```

```
            res = [res, y_next];
            t = t + h;
        end
    end

    function y = f(x, y)
        y = x*y^2 + 2*y;
    end

    function ye = RealFunc(x)
        ye = (20.*exp(2.*x))./(exp(2.*x)-10.*x.*exp(2.*x)-9);
    end
```

Listing 4: 4th-order Runge-Kutta

[H]

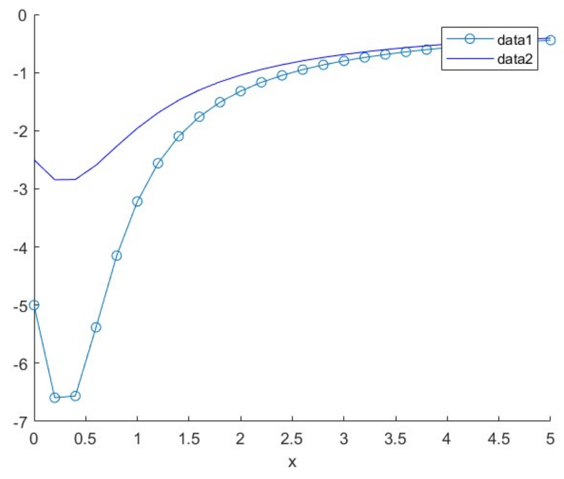


图 3: 4th-order Runge-Kutta

这部分比较简单，主要是透过以下公式进行循环而得：

$$
\begin{aligned}
k_1 &= f(t_n, y_n), \\
k_2 &= f\left(t_n + \tfrac{h}{2}, y_n + h\tfrac{k_1}{2}\right), \\
k_3 &= f\left(t_n + \tfrac{h}{2}, y_n + h\tfrac{k_2}{2}\right), \\
k_4 &= f(t_n + h, y_n + hk_3).
\end{aligned}
$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\mathrm{h},$$

$$t_{n+1} = t_n + h$$

4 处理三种方法的误差-代码实现

时间太赶实在是学不完也写不出来了，一直报错也看得不是很懂，下面的都是python，主要是分析误差。

4.1 隐式欧拉法

```python
import numpy as np
import matplotlib.pyplot as plt


def Euler_Implicit(f, y0, a, b, h):
    n = round((b-a)/h)
    y = np.zeros(n+1)
    y[0] = y0

    for i in range(1, n+1):
        xi = a + i * h
        yi = y[i - 1]
        for j in range(10):
            yi = y[i - 1] + h * f(xi, yi)
        y[i] = yi

    return y


```

```
def error(f, y0, a, b, h):

    def y_exact(x):
        return 20*np.exp(2*x)/(np.exp(2*x) - 10*x*np.exp(2*x) - 9)

    y_num = Euler_Implicit(f, y0, a, b, h)
    x = np.arange(a, b + h, h)[:len(y_num)]
    e = y_num - y_exact(x)

    return x, e


def f(x, y):
    return x*y**2+2*y


x0 = 0
y0 = -5
a = 0
b = 5
h = 0.1
y = Euler_Implicit(f, y0, a, b, h)

error_list = []
ha = np.arange(0.1, 0.7, 0.01)
for h in np.arange(0.1, 0.7, 0.01):

    y = Euler_Implicit(f, y0, a, b, h)
    x, e = error(f, y0, a, b, h)
```

```
        x = np.linspace(0, 5, len(y))
        error_list.append(e[-1])
        print(h,e[-1])


    plt.plot(ha, error_list)
    plt.xlabel('h')
    plt.ylabel('e[-1]')
    plt.title('Error vs. Step Size')
    plt.show()
```

Listing 5: 隐式欧拉法

4.2 显式欧拉法

```
import numpy as np
import matplotlib.pyplot as plt

def Euler_Explicit(f, y0, h):
    t0 = 0
    tp = 5
    t = np.arange(t0, tp+h, h)
    y = np.zeros(len(t))
    y[0] = y0

    for i in range(0, len(t)-1):
        y[i+1] = y[i] + h * f((i+1) * h, y[i])
    return y

def error(f, y0, h):
```

```python
    t0 = 0
    tp1 = 5

    def y_exact(x):
        return 20*np.exp(2*x)/(np.exp(2*x) - 10*x*np.exp(2*x) - 9)

    y_num = Euler_Explicit(f, y0, h)
    x = np.arange(t0, tp1 + h, h)
    e = y_num - y_exact(x)

    return x, e

def f(x, y):
    return x*y**2+2*y
x0 = 0
y0 = -5

error_list = []
ha = np.arange(0.26, 1.5, 0.01)
for h in np.arange(0.26, 1.5, 0.01):

    y = Euler_Explicit(f, y0, h)
    x, e = error(f, y0, h)
    x = np.linspace(0, 5, len(y))
    error_list.append(e[-1])
    print(h,e[-1])

xdata = ha
ydata = np.array(error_list)
```

```
    coef = np.polyfit(xdata, ydata, 2)
    f_fit = np.poly1d(coef)
    xfit = np.linspace(xdata[0], xdata[-1], 100)
    yfit = f_fit(xfit)

    plt.plot(ha, error_list, 'bo', label='data')
    plt.plot(xfit, yfit, 'r-', label='fit')
    plt.xlabel('h')
    plt.ylabel('e[-1]')
    plt.title('Error vs. Step Size')
    plt.legend()
    plt.show()
    plt.plot(ha, error_list)
    plt.xlabel('h')
    plt.ylabel('e[-1]')
    plt.title('Error vs. Step Size')
    plt.show()
```

Listing 6: 显式欧拉法

4.3 4th-order Runge-Kutta

```
import numpy as np
import matplotlib.pyplot as plt

def Runge_Kutta4(f, t0, y0, tn, h):
    res = [y0]
    t = t0
    for i in range(int((tn-t0)/h)):
        k1 = f(t, res[-1])
```

```
            k2 = f(t + h / 2, res[-1] + h * k1 / 2)
            k3 = f(t + h / 2, res[-1] + h * k2 / 2)
            k4 = f(t + h, res[-1] + h * k3)
            y_next = res[-1] + h * (k1 + 2 * k2 + 2 * k3 + k4)/6
            res.append(y_next)
            t += h
        return res

    def f(x, y):
        return x*y**2+2*y

    def error(f, y0, t0, tn, h):

        def y_exact(t):
            return 20*np.exp(2*x)/(np.exp(2*x) - 10*x*np.exp(2*x) - 9)

        y_num = Runge_Kutta4(f, t0, y0, tn, h)
        t = np.arange(t0, tn + h, h)
        e = y_num - y_exact(t)

        return t, e

    y0 = -5
    t0 = 0
    tn = 5
    h = 0.1

    result = Runge_Kutta4(f, t0, y0, tn, h)
    x = np.linspace(0, 5, len(result))
```

```
plt.plot(x, result, '-o',label='Numerical Solution')

t, e = error(f, y0, t0, tn, h)

plt.plot(t, e, '-o', label='Error')

plt.xlabel('t')
plt.title('Numerical Solution and Error')
plt.legend()
plt.show()
```

Listing 7: 4th-order Runge-Kutta

5 处理三种方法的误差-比较

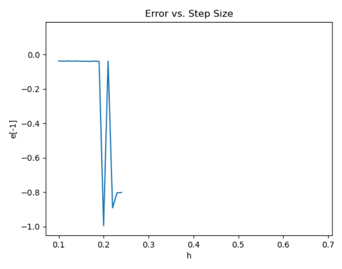


图 4: 隐式欧拉法的误差

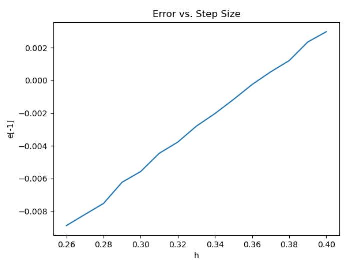


图 5: 显式欧拉法的误差

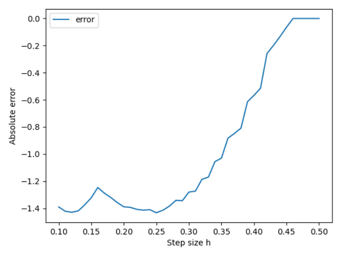


图 6: 4th-order Runge-Kutta的误差

我将数据处理的档案加到附件里输出结果为:

EE.txt: 最接近 0 的值是 [0.26, -0.008873641307191593]，其距离为 0.2688736413071916。

EIM.txt: 最接近 0 的值是 [0.1, -0.03759539961851743]，其距离为 0.13759539961851744。

runge.txt: 最接近 0 的值是 [0.44999999999999984, -0.06431906956574363]，其距离为 0.5143190695657435。

比较上面三张图片及处理完的数据可以看出隐式欧拉法在步长的选取上可以选择的精度最小，说明在这三种方法中最为精确。