双delta势垒

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1 双delta势

现在假设有一势$V = \alpha[\delta(x+a) + \delta(x-a)]$

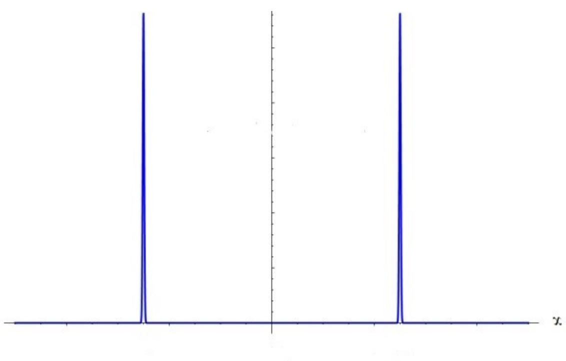

图 1: $V = \alpha[\delta(x+a) + \delta(x-a)]$

1.1 求解

对于薛定谔方程：

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi(x,t)$$

分离变量 $\Psi(x,t) = \psi(x)\phi(t)$

$$i\hbar\psi\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\phi\frac{d^2\psi}{dx^2} + V\psi\phi$$

$$\Rightarrow i\hbar\frac{\dot{\phi}}{\phi} = -\frac{\hbar^2}{2m}\frac{\ddot{\psi}}{\psi} + V = E$$

$$\begin{cases} i\hbar\frac{\dot{\phi}}{\phi} = E \\ -\frac{\hbar^2}{2m}\frac{\ddot{\psi}}{\psi} + V = E \end{cases}$$

将势能V带入,得 $\begin{cases} i\hbar\frac{\dot{\phi}}{\phi} = E \\ -\frac{\hbar^2}{2m}\frac{\ddot{\psi}}{\psi} + \alpha[\delta(x+a)+\delta(x-a)] = E \end{cases}$

E为满足边界条件的特征值,

$$\frac{\partial^2\psi}{\partial x^2} - \frac{2m}{\hbar^2}\left\{\alpha[\delta(x+a)+\delta(x-a)] - E\right\}\psi = 0 \ (1)$$

如果$x \neq \pm a$, $\Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{2mE}{\hbar^2}\psi = 0$ 会有以下解

$$\psi(x) = \begin{cases} Ae^{kx}, x < -a \\ Ce^{kx} + De^{-kx}, -a < x < a \\ Fe^{-kx}, x > a \end{cases}, k = \frac{\sqrt{-2mE}}{\hbar}$$

并且此时有边界条件:

$$\lim_{x\to -a^-}\psi(x) = \lim_{x\to -a^+}\psi(x)$$

$$\lim_{x\to +a^-}\psi(x) = \lim_{x\to +a^+}\psi(x)$$

这样可得: $\begin{cases} Ae^{-ka} = Ce^{-ka} + De^{ka} \\ Fe^{-ka} = Ce^{ka} + De^{-ka} \end{cases}$

而其他的条件可对(1)式积分,范围从$-a-\epsilon$到$-a+\epsilon$

$$\int_{-a-\epsilon}^{-a+\epsilon}\left\{\frac{\partial^2\psi}{\partial x^2} - \frac{2m}{\hbar^2}\left\{\alpha[\delta(x+a)+\delta(x-a)] + E\right\}\psi\right\}dx = 0$$

$$\int_{-a-\epsilon}^{-a+\epsilon}\frac{\partial^2\psi}{\partial x^2}dx - \frac{2m}{\hbar^2}\left\{\alpha\int_{-a-\epsilon}^{-a+\epsilon}[\delta(x+a)+\delta(x-a)]\psi(x)dx + E\int_{-a-\epsilon}^{-a+\epsilon}\psi(x)dx\right\} = 0$$

$$\frac{\mathrm{d}\psi}{\mathrm{dx}}\bigg|_{a-\epsilon}^{a+\epsilon} - \frac{2m}{\hbar^2}\left[\alpha\psi(a) + E\psi(a)\int_{a-\epsilon}^{a+\epsilon}\mathrm{dx}\right] = 0$$

$$\frac{\mathrm{d}\psi}{\mathrm{dx}}\bigg|_{a-\epsilon}^{a+\epsilon} - \frac{2m}{\hbar^2}[\alpha\psi(a) + E\psi(a)(2\epsilon)] = 0 \text{ , 令}\epsilon \to 0$$

$$\frac{\mathrm{d}\psi}{\mathrm{dx}}\bigg|_{-a^-}^{-a^+} - \frac{2m\alpha}{\hbar^2}\psi(-a) = 0$$

$$\frac{2m\alpha}{\hbar^2}\psi(-a) = \lim_{x\to -a^-}\frac{\mathrm{d}\psi}{\mathrm{dx}} - \lim_{x\to -a^+}\frac{\mathrm{d}\psi}{\mathrm{dx}}$$

$$\frac{2m\alpha}{\hbar^2}Fe^{-ka} = \left[kCe^{ka} - kDe^{-ka} + kFe^{-ka}\right]$$

这样就有四个条件可以求解了，$\begin{cases} Ae^{-ka} = Ce^{-ka} + De^{ka} \\ Fe^{-ka} = Ce^{ka} + De^{-ka} \\ \frac{2m\alpha}{\hbar^2} Fe^{-ka} = \left[kCe^{ka} - kDe^{-ka} + kFe^{-ka} \right] \\ \frac{2m\alpha}{\hbar^2} Ae^{-ka} = \left[kAe^{-ka} - kCe^{-ka} + kDe^{ka} \right] \end{cases}$

最终可以得到

$D = (\frac{\hbar^2 k}{m\alpha} - 1)^2 De^{4ka}$

$1 = (\frac{\hbar^2 k}{m\alpha} - 1)e^{2ka}$

$\begin{cases} \frac{k\hbar^2}{m\alpha} - 1 = e^{-2ka}, even \\ 1 - \frac{k\hbar^2}{m\alpha} = e^{-2ka}, odd \end{cases}$