

Mean-Field Game Analysis of SIR Model with Social Distancing

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Abstract

As the current COVID-19 outbreak shows, disease epidemic is a complex problem which must be tackled with the right public policy. One of the main tools of the policymakers is to control the contact rate among the population, commonly referred to as social distancing, in order to reduce the spread of the disease. We pose a mean-field game model of individuals each choosing a dynamic strategy of making contacts, given the trade-off of utility from contacts but also risk of infection from those contacts. We compute and compare the mean-field equilibrium (MFE) strategy, which assumes individuals acting selfishly to maximize their own utility, to the socially optimal strategy, which is the strategy maximizing the total utility of the population. We prove that the infected always want to make more contacts than the level at which it would be socially optimal, which reinforces the need to reduce contacts of the infected (e.g. quarantining, sick paid leave). Additionally, we compute the socially optimal strategies, given the costs to incentivize people to change from their selfish strategies. We find that if we impose limited resources, curbing contacts of the infected is more important after the peak of the epidemic has passed. Lastly, we compute the price of anarchy of this system, to understand the conditions under which large discrepancies between the MFE and socially optimal strategies arise, which is when public policy would be most effective.

1 Introduction

The current COVID-19 pandemic has shown that posing a public policy to fight such an outbreak is an extremely difficult and interdisciplinary task. A big part of such policy is urging people to practice social distancing, which is to reduce interpersonal contacts to slow down the spread of infection. However, it is uncertain whether people are incentivized enough to practice social distancing, when there are clear benefits to making contacts, such as earning wages from jobs, as well as the general human need for social relationships.

Given this trade-off between the additional utility from making more contacts, and the additional chance of infection from those contacts, we will compare 1) the selfish strategies, in which individuals make contacts to optimize only their own utilities and 2) socially optimal strategies, in which individuals make contacts which optimize the utility of the total population.

Previous work has explicitly included contact behaviors as control variables into the classical SIR model and have shown that different dynamics can emerge with adaptive behavior [1–3]. Disease dynamics has also been studied with game theory, focusing mostly on steady-state problems or those related to vaccination [4–6]. Optimal control theory is also often used to study policy interventions on infectious disease dynamics [7, 8]. Our work here builds on these previous models to pose a mean-field game problem of social distancing, explicitly modeling the feedback between the individuals and the population structure. The individual, which is susceptible (S), infected (I), or recovered (R), each chooses a dynamic contact strategy to maximize its accumulated utility over the time period.

Mean-field games is a recently developing field, studying dynamic game theoretic problems of large number of players [9, 10]. Our model is a relatively simple formulation of the mean-field

games, in which we have a deterministic game with 3 discrete states (S,I,R) in continuous time. Depending on the number of contacts each individual makes, the dynamics of the population follow the SIR model, and the anticipated population structure influences the computation of an individual's optimal strategy. We take mean-field assumptions such that 1) the number of individuals in the population is large ($N \rightarrow \infty$), 2) the individuals are homogeneous within each compartment and the population is well-mixed, and 3) individuals engage in symmetric interactions.

2 Model

2.1 SIR dynamics

Let x_z be the fraction of the population in compartment $z \in \{S, I, R\}$. Then we can write the dynamics of the SIR system as

$$\begin{cases} \dot{x}_S = -C(\cdot)\beta x_S x_I \\ \dot{x}_I = C(\cdot)\beta x_S x_I - \mu x_I \\ \dot{x}_R = \mu x_I \end{cases} \quad (1)$$

$C(\cdot)$ is the rate that an S individual and an I individual make contact. Let a z individual at time t choose $c_z(t)$ number of contacts. There are two intuitive ways of writing an expression for $C(\cdot)$ in terms of the contact strategies. First is frequency-dependent, in which each contact made by an S individual is with an I individual with probability, $\frac{c_I(t)x_I}{c_S(t)x_S + c_I(t)x_I + c_R(t)x_R}$. Therefore in the frequency-dependent case, we have $C(\cdot) = \frac{c_S(t)c_I(t)}{c_S(t)x_S + c_I(t)x_I + c_R(t)x_R}$. Second is density-dependent, in which each contact made by a susceptible individual has probability of being with an infected individual of $\frac{c_I(t)}{c_S^{max}c_I^{max}}$. In this case, $C(\cdot) = \frac{c_S(t)c_I(t)}{c_S^{max}c_I^{max}}$. For the rest of this paper, we will assume density-dependent contact rates. β is the likelihood that infection happens given the contact between S and I, and μ is the rate of recovery from infection.

2.2 Utility of contacts

$c_z(t)$ is the contact strategy which is explicitly chosen by an individual. Large $c_z(t)$ means going out to work and socializing, while smaller $c_z(t)$ means refraining from those activities, and practicing social distancing. $u_z(c_z)$ denotes the utility received by a z individual from making c_z contacts, as a combination of economic gains and personal well-being. The realistic conditions that we impose are the following. Starting from zero contacts, increasing the number of contacts results in increased utility, reflecting human need for social interactions as well as the economic gain of spending time at work. We assume that the marginal increase in utility is decreasing with more contacts, until eventually, more contacts become detrimental.

Therefore, the appropriate functional form of $u_z(c_z)$ must be a concave function with an interior maximum. One appropriate function form with interpretations of the parameters is $u_z(c_z) = (b_z c_z - c_z^2)^\gamma - a_z$ where $c_z \in [0, b_z]$ [2]. b_z is a parameter of how the disease impacts the marginal economic productivity of the individual. Large b_z means that the individual has the choice to make more contacts, as well as receive higher marginal utility from making contacts. a_z is a parameter of the baseline cost of being infected, such as an individual's general propensity to be healthy. These two parameters attempt to decompose the effect of the disease into economic cost and health cost. Some assumptions on these parameters are that $b_S = b_R > b_I$ and $a_S = a_R < a_I$, since the infected become impaired in economic productivity, as well as suffer the health cost compared to the S and R. γ changes the concave shape of the function, and $\gamma \in (0, 1]$ ensures that $u_z(c_z)$ is concave everywhere in the domain.

From this utility function, we see that each individual has some optimal level of social contacts. The utility of a z individual is maximized at $\frac{b_z}{2}$, which is each individual's optimal contact strategy in the absence of adaptive behavior in response to the risk of infectious disease.

2.3 Value function

Over a time period, $t \in [0, T]$, an individual's total utility is the sum of utility gained at each time point. For example, a recovered individual, who makes continuous contact decision $c(t)$, receives



Figure 1: $u_z(c_z)$ is shown where $z \in \{S, R\}$ are healthy (green) and $z \in \{I\}$ is infected (red) and the parameters are $b_S = b_R = 10$, $b_I = 6$, $a_S = a_R = 0$, $a_I = 1$, and $\gamma = 0.25$. A healthy individual gains utility from making more contacts, but eventually does not want to make more, when $c_S = 5$. The infected suffers the baseline cost, but also gains some utility from making contacts, although at a lower rate compared to the healthy.

total utility $\int_0^T u_R(c(t))dt$. Let us define $V_R(t)$ to be the total future utility expected by the R individual at time t until T . From this formulation, it follows that $V_R(T) = 0$, and

$$V_R(t) = \int_t^T u_R(c(t))dt = u_R(c(t))dt + \int_{t+dt}^T u_R(c(t))dt = u_R(c(t))dt + V_R(t+dt) \quad (2)$$

Similarly for S and I individuals, we write $V_S(t)$ and $V_I(t)$, which depends on the rate at which the individuals move between the SIR states. (Fig. 2)

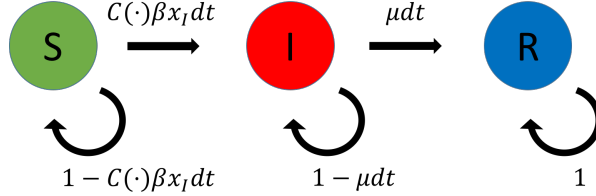


Figure 2: Between time t and $t+dt$, S, I, and R individuals move between the states at these rates. The S individual becomes infected at a larger rate with higher contact rates. The I individual becomes recovered at a constant rate of μdt . The recovered individual remains in the state.

From these transition rates, we can write the Bellman equations, which gives the value functions for an individual in S, I, and R state as:

$$\begin{aligned} V_S(t) &= \max_{c_S} \left\{ \int_t^{t+dt} u_S(c_S)dt + (1 - C(\cdot)\beta x_I dt)V_S(t+dt) + C(\cdot)\beta x_I dt V_I(t+dt) \right\} \\ V_I(t) &= \max_{c_I} \left\{ \int_t^{t+dt} u_I(c_I)dt + (1 - \nu dt)V_I(t+dt) + \nu dt V_R(t+dt) \right\} \\ V_R(t) &= \max_{c_R} \left\{ \int_t^{t+dt} u_R(c_R)dt + V_R(t+dt) \right\} \end{aligned} \quad (3)$$

with terminal conditions, $V_S(T) = V_I(T) = V_R(T) = 0$.

2.4 Mean Field Equilibrium solution

The SIR dynamics and the Bellman equations are coupled by the contact strategies, c_S, c_I, c_R , and the population, x_S, x_I, x_R . Therefore, the solution to this problem is the mean-field equilibrium

(MFE), which is the fixed point $(c_S^{eq}, c_I^{eq}, c_R^{eq}, x_S, x_I, x_R)$ such that 1) the strategies $c_S^{eq}(t)$, $c_I^{eq}(t)$, $c_R^{eq}(t)$ are the optimal solutions in equations 3 given x_S, x_I , and x_R and 2) x_S, x_I , and x_R are solutions to the system of ODEs in equation 1 given the optimal strategies.

2.5 Socially Optimal solution

In contrast, we can also characterize the socially optimal solution. If a central planner can choose the contact strategies of the population to maximize the utility of the entire population, the resulting socially optimal solution solves the dynamic optimization problem,

$$c_S^{opt}, c_I^{opt}, c_R^{opt} = \arg \max \int_0^T x_S(t)u_S(c_S(t)) + x_I(t)u_I(c_I(t)) + x_R(t)u_R(c_R(t))dt \quad (4)$$

3 Results

3.1 Mean Field Equilibrium solution

Proposition 1. *The optimal strategy for an R individual is $c_R^{eq} = 0.5b_R$ and the corresponding optimal value function is $V_R(t) = -u_R^{max}t + u_R^{max}T$.*

Proof. We can substitute the Taylor expansion $V_R(t + dt) = V_R(t) + \dot{V}_R(t)dt$, and take only the first order dt terms.

$$V_R(t) = \max_{c_R} \left\{ \int_t^{t+dt} u_R(c_R)dt + V_R(t) + \dot{V}_R(t)dt \right\} \quad (5)$$

$$\implies -\dot{V}_R(t) = \max_{c_R} \left\{ u_R(c_R) \right\} \quad (6)$$

u_R attains its maximum, u_R^{max} , when $c_R^{eq} = 0.5b_R$, so we have

$$\dot{V}_R(t) = -u_R^{max} \implies V_R(t) = -u_R^{max}t + C \quad (7)$$

With the terminal condition, $V_R(T) = 0$, we have $V_R(t) = -u_R^{max}t + u_R^{max}T$. \square

In our model, the recovered individuals do not become reinfected, so they do not have any reason to decrease contacts. Therefore, they always choose maximum contact rates.

Proposition 2. *The optimal strategy for an individual in the infected state is $c_I^{eq} = 0.5b_I$ and the corresponding optimal value function is $V_I(t) = V_R(T) - \frac{u_R^{max} - u_I^{max}}{\mu}(1 - e^{\mu(t-T)})$*

Proof. Substituting the first order Taylor expansion and taking only the first order terms,

$$V_I(t) = \max_{c_I} \left\{ \int_t^{t+dt} u_I(c_I)dt + (1 - \mu dt)(V_I(t) + \dot{V}_I(t)dt) + \mu dt(V_R(t) + \dot{V}_R(t)dt) \right\} \quad (8)$$

$$\implies V_I(t) = \max_{c_I} \left\{ \int_t^{t+dt} u_I(c_I)dt + V_I(t) + \dot{V}_I(t)dt - \mu V_I(t)dt + \mu V_R(t)dt \right\} \quad (9)$$

$$\implies -\dot{V}_I(t) = \max_{c_I} \left\{ u_I(c_I) + \mu(V_R(t) - V_I(t)) \right\} \quad (10)$$

Because $u_I(c_I)$ is the only term dependent on c_I , the value function attains its maximum, u_I^{max} , when $c_I^{eq} = 0.5b_I$.

$$-\dot{V}_I(t) = u_I^{max} + \mu(V_R(t) - V_I(t)) \quad (11)$$

$V_R(t)$ is known from Proposition 1, and this is a first-order linear ordinary differential equation that can be explicitly solved with integrating factor. Using the terminal condition, $V_I(T) = 0$, we can find

$$V_I(t) = V_R(t) - \frac{u_R^{max} - u_I^{max}}{\mu}(1 - e^{\mu(t-T)}) \quad (12)$$

\square

The infected individuals, similar to the recovered individuals, do not have a reason to decrease contacts. The corresponding value function, $V_I(t)$ is bounded by $V_R(t)$, and the difference is bigger for smaller μ , since it implies longer time spent as an infected.

Proposition 3. *The optimal strategy for a susceptible individual is $c_S^{eq}(t) < 0.5b_S$ during time $0 \leq t < T$.*

Proof. Substituting the Taylor expansion and taking only the first order terms,

$$V_S(t) = \max_{c_S} \int_t^{t+dt} u_S(c_S)dt + \left(1 - c_S c_I \beta x_I dt\right) V_S(t+dt) + c_S c_I \beta x_I dt V_I(t+dt) \quad (13)$$

$$\Rightarrow V_S(t) = \max_{c_S} \int_t^{t+dt} u_S(c_S)dt + V_S(t) + \dot{V}_S(t)dt - c_S c_I \beta x_I dt (V_S(t) - V_I(t)) \quad (14)$$

$$\Rightarrow -\dot{V}_S(t) = \max_{c_S} u_S(c_S) - c_S c_I \beta x_I (V_S(t) - V_I(t)) \quad (15)$$

The objective function is concave, so we set the derivative equal to 0 to find c_S^* .

$$\left. \frac{du_S}{dc_S} \right|_{c_S=c_S^{eq}} = 0.5b_I \beta x_I (V_S(t) - V_I(t)) \quad (16)$$

u_S is concave, so c_S^{eq} can be uniquely found. Also, we have $V_S(t) > V_I(t)$ for all $t < T$. This is because starting from the terminal condition $V_S(T) = V_I(T) = 0$, if at any time t we are sufficiently close to $V_S = V_I$, $\dot{V}_S(t) = -u_S^{max} < -u_I^{max} = \dot{V}_I(t)$. Therefore, $0.5b_I \beta x_I (V_S - V_I) > 0$ for $t < T$, and $c_S^{eq} < 0.5b_S$. \square

Note that c_S^{eq} is smaller for bigger values of $b_I \beta x_I (V_S(t) - V_I(t))$, which means that the susceptibles should decrease contact if 1) infected population gets large, 2) the disease spreads well, 3) cost of being infected is large, or 4) if the disease minimally affects the ability of the infected.

Numerical results

We use discrete time steps Δt for the equations. With some initial $c^0 = (c_S^0, c_I^0, c_R^0)$, we compute $x^0 = (x_S^0, x_I^0, x_R^0)$ using the ODE forward equations. Then, c^1 can be computed via backward induction with given x^0 . We continue this until we find c^k and x^k such that $c^k = c^{k-1}$ and $x^k = x^{k-1}$, which is the MFE solution.

With disease parameters $\beta = 0.03$ and $\mu = 0.1$, and utility parameters $b_S = b_R = 10$, $b_I = 6$, $a_S = a_R = 0$, $a_I = 4$, and $\gamma = 0.25$, Fig. 3 shows the computed MFE. We see that the spread of infection is mitigated by the behavioral changes by the susceptible population, compared to the classical case in which no adaptive behavior is considered. As we showed in Proposition 1 and 2, the recovered and infected population have no incentive to lower their contact rates, and so they continue at their maximum level of activities. The susceptible population lowers their contact rate to balance their immediate utilities and their expected cost of possibly getting infected.

Fig. 4 shows the cumulative epidemic curve of the MFE solution and the classical SIR for different parameter values of β, μ, b_I , and a_I . For any set of parameter values, the MFE solution, by considering the adaptive behaviors, results in smaller final size as well as more gradual spread of the epidemic. If the disease is more infectious (large β), the MFE solution, just like the classical SIR solution, shows faster spread as well as larger final size, although mitigated compared to what the classical SIR model would predict (Fig. 4a). If the disease has longer recovery time (small μ), the classical model predicts faster spread, but we see that the MFE solution does not always move in the same direction. For example, when $\mu = 0.02$, it might be intuitive to predict that since μ is small, the population will get infected faster, but because we explicitly consider the decision of each individual in our model, we predict a very gradual epidemic, as it becomes optimal to further suppress contacts to avoid getting infected in the first place (Fig. 4b). b_I and a_I are parameters which would not be included in the classical SIR model, but they will change the predicted epidemic curves. If a disease does not particularly affect the infected individual's productivity (high b_I), the disease will spread as if its transmission rate β is higher (Fig. 4c). The epidemic spread as predicted by the classical SIR model would not change with a_I , but the MFE solution shows smaller epidemic and flatter growth rate for high a_I , as individuals choose to make less contacts to avoid the high cost of becoming infected (Fig. 4d).

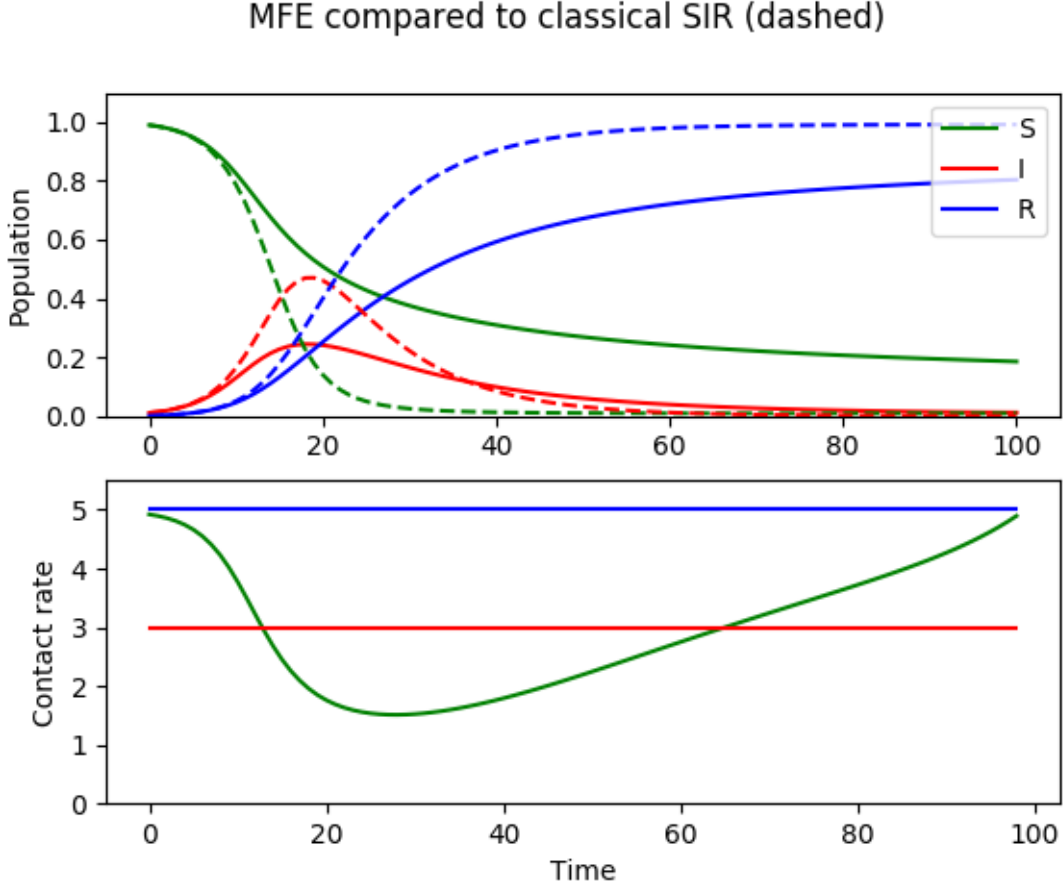


Figure 3: The top figure shows x_z for $z \in \{S, I, R\}$, the SIR dynamics for MFE solution (solid lines) and the classical case without adaptive behaviors (dashed lines). Bottom figure shows $c_z^{eq}(t)$ for $z \in \{S, I, R\}$.

3.2 Socially optimal solution

We pose a centralized control problem, in which we find $(c_S^{opt}, c_I^{opt}, c_R^{opt})$ to maximize the utility of the entire population. Therefore we solve

$$c_S^{opt}(t), c_I^{opt}(t), c_R^{opt}(t) = \arg \max \int_0^T x_S(t)u_S(c_S(t)) + x_I(t)u_I(c_I(t)) + x_R(t)u_R(c_R(t))dt \quad (17)$$

$$\text{subject to: } \begin{cases} \dot{x}_S = -c_S c_I \beta x_S x_I \\ \dot{x}_I = c_S c_I \beta x_S x_I - \nu x_I \\ \dot{x}_R = \nu x_I \end{cases} \quad (18)$$

In order to solve the optimal control problem, we use Pontryagin's maximization principle, which gives the necessary conditions for the optimal controls, given the evolving dynamics of the system.

Theorem 3.1 (Pontryagin's Maximization Principle). *Let $\mathbf{x} = [x_S, x_I, x_R]^T$ and $\mathbf{c} = [c_S, c_I, c_R]^T$. For the given deterministic dynamics, $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{c})$, the Hamiltonian is defined as*

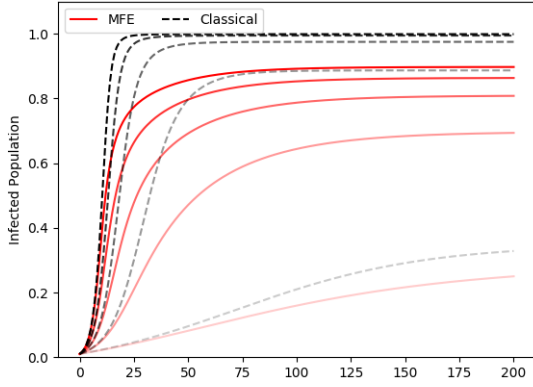
$$H(\mathbf{x}, \mathbf{c}, \boldsymbol{\lambda}, t) := L(\mathbf{x}, \mathbf{c}) + \boldsymbol{\lambda}^T f(\mathbf{x}, \mathbf{c})$$

where $\boldsymbol{\lambda}(t)$ is the costate trajectory. If $\mathbf{x}(t)$, $\mathbf{c}^{opt}(t)$ is the optimal trajectory in $0 \leq t \leq T$ from $\mathbf{x}(0)$, then $\boldsymbol{\lambda}(t)$ satisfies

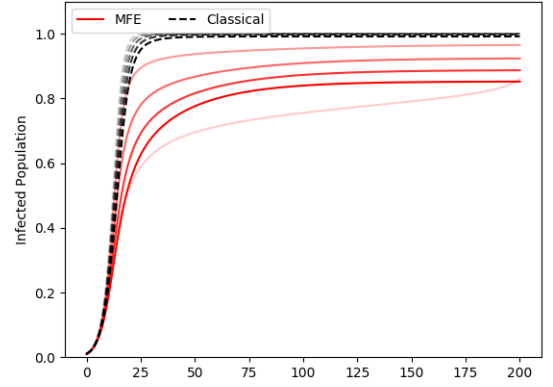
$$-\dot{\boldsymbol{\lambda}} = H_{\mathbf{x}}(\mathbf{x}^{opt}, \mathbf{c}^{opt}, \boldsymbol{\lambda}, t) = L_{\mathbf{x}}(\mathbf{x}^{opt}, \mathbf{c}^{opt}) + \boldsymbol{\lambda}^T f_{\mathbf{x}}(\mathbf{x}^{opt}, \mathbf{c}^{opt})$$

and \mathbf{c}^{opt} is the solution to the optimization problem,

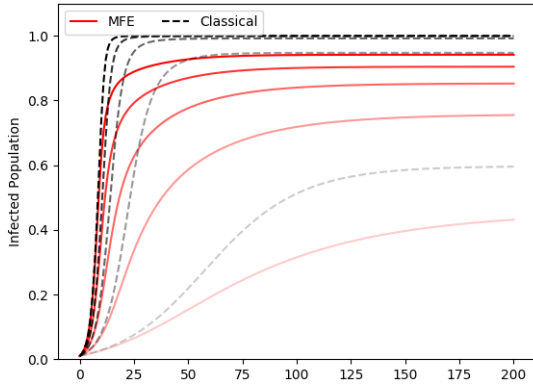
$$\mathbf{c}^{opt} = \arg \max_{\mathbf{c}} H(\mathbf{x}^{opt}, \mathbf{c}, \boldsymbol{\lambda})$$



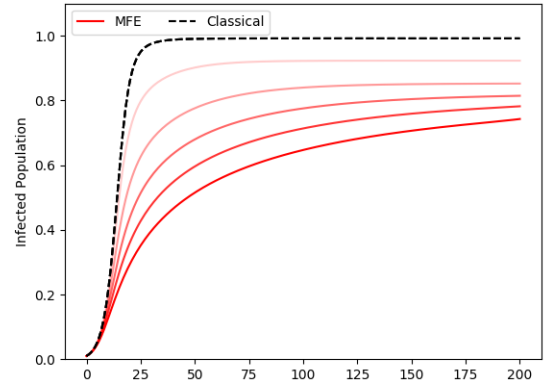
(a) Cumulative curves ($0.01 \leq \beta \leq 0.05$)



(b) Cumulative curves ($0.02 \leq \mu \leq 0.1$)



(c) Cumulative curves ($2 \leq b_I \leq 10$)



(d) Cumulative curves ($2 \leq a_I \leq 10$)

Figure 4: The cumulative epidemic size is shown for the MFE solution (red) and the classical SIR (black) for the range of given parameter. For each (a)-(d), five cumulative curves are shown corresponding to five values of the given parameter in the range, where the more transparent lines are smaller parameter values.

Proposition 4. *The socially optimal contact rate of the infected, c_I^{opt} must always be less than the MFE contact rate, c_I^{eq} during time $0 \leq t < T$.*

Proof. We can apply the Pontryagin's maximization principle, which gives the necessary condition for optimality. If $c_S^{opt}, c_I^{opt}, c_R^{opt}$ are optimal solutions, then there exist Lagrangian multipliers, $\lambda_S(t), \lambda_I(t), \lambda_R(t)$, such that $\lambda_S(T) = 0, \lambda_I(T) = 0, \lambda_R(T) = 0$, and for $t < T$, they satisfy:

$$-\dot{\lambda}_S = c_S^{opt} c_I^{opt} \beta x_I (\lambda_I - \lambda_S) + u_S(c_S^{opt}) \quad (19)$$

$$-\dot{\lambda}_I = c_S^{opt} c_I^{opt} \beta x_S (\lambda_I - \lambda_S) + \mu (\lambda_R - \lambda_I) + u_I(c_I^{opt}) \quad (20)$$

$$-\dot{\lambda}_R = u_R(c_R^{opt}) \quad (21)$$

$$c_S^{opt}, c_I^{opt}, c_R^{opt} = \arg \max c_S c_I \beta x_S x_I (\lambda_I - \lambda_S) + \mu x_I (\lambda_R - \lambda_I) + \sum_z x_z u_z(c_z) \quad (22)$$

Note the similarities between our expressions for $\lambda_S, \lambda_I, \lambda_R$ and V_S, V_I, V_R from the MFE, aside from the additional term in (20). Intuitively, this additional term represents the I individual caring about the consequences of its contact strategy on the population, which the selfish I individual does not take into account in its objective function. It can be interpreted as the I individual's internalized negative externalities (i.e. thinking about our actions and their impact on others). First, we see that the c_R term in the objective function can be separated, and so we can find the maximizer, $c_R^{opt} = 0.5b_R = c_R^{eq}$, which gives $\lambda_R(t) = V_R(t)$ for all t . Therefore, the optimization

problem for c_S^* and c_I^* is

$$c_S^{opt}, c_I^{opt} = \arg \max x_S u_S(c_S) + x_I u_I(c_I) - c_S c_I \beta x_S x_I (\lambda_S - \lambda_I) \quad (23)$$

Clearly, $\beta x_S x_I > 0$ for all t , assuming nontrivial initial conditions, $x_S(0) > 0$ and $x_I(0) > 0$. Also, $\lambda_S - \lambda_I > 0$ for $t < T$. Starting from the given terminal conditions $\lambda_S(T) = \lambda_I(T) = 0$, if at any point we get sufficiently close to $\lambda_S = \lambda_I$, we see that $\dot{\lambda}_S < \dot{\lambda}_I$, so λ_S will be strictly larger than λ_I in $0 \leq t < T$. Therefore, $\beta x_S x_I (\lambda_S - \lambda_I) > 0$ for all $0 \leq t < T$.

With the utility function $u_z(c_z) = (b_z c_z - c_z^2)^\gamma - a_z$, we will prove that $c_I^{opt} < 0.5b_I$ for all t by dividing the problem into two cases: i) $\gamma < 1$ and ii) $\gamma = 1$.

i) $\gamma < 1$

From (23), $c_I^{opt} = 0.5b_I$ only if $c_S^{opt} \beta x_S x_I (\lambda_S - \lambda_I) = 0$, which is only true if $c_S^{opt} = 0$. However, we see that for some given c_I^{opt} , c_S^{opt} can be computed to be

$$\left. \frac{du_S}{dc_S} \right|_{c_S=c_S^{opt}} - c_I^{opt} \beta x_I (\lambda_S - \lambda_I) = 0 \quad (24)$$

$$\implies \frac{\gamma(b_S - 2c_S^{opt})}{(b_S c_S^{opt} - c_S^{opt2})^{1-\gamma}} = c_I^{opt} \beta x_I (\lambda_S - \lambda_I) \quad (25)$$

Since the left side monotonically decreases from ∞ to 0 in the domain $[0, 0.5b_I]$, c_S^{opt} can be uniquely found and since the right side is bounded, $c_S^{opt} > 0$. Therefore, $c_I^{opt} < 0.5b_I$ must be true.

ii) $\gamma = 1$

Plugging in $\gamma = 1$ gives the optimization problem as

$$c_S^{opt}, c_I^{opt} = \arg \max x_S (b_S c_S - c_S^2) + x_I (b_I c_I - c_I^2) - c_S c_I \beta x_S x_I (\lambda_S - \lambda_I) \quad (26)$$

Assume for contradiction that $c_I^{opt} = 0.5b_I$ at some $t < T$. This is only possible if $c_S^{opt} = 0$, which implies that $b_S \leq c_I^{opt} \beta x_I (\lambda_S - \lambda_I)$ holds true. Since the optimal contact rates at time t is given as $c_S^{opt}(t) = 0$ and $c_I^{opt}(t) = 0.5b_I$, we can plug these in to get

$$\dot{\lambda}_S(t) = 0 \quad (27)$$

$$\dot{\lambda}_I(t) = -\mu(\lambda_R(t) - \lambda_I(t)) - u_I^{max} \quad (28)$$

With the known dynamics of λ_S, λ_I , and x_I , we can use the first order Taylor expansion to find

$$x_I(t+dt) \left(\lambda_S(t+dt) - \lambda_I(t+dt) \right) = (x_I - \mu x_I dt) \left(\lambda_S - \lambda_I + \mu(\lambda_R - \lambda_I) dt + u_I^{max} dt \right) \quad (29)$$

$$= x_I \left(\lambda_S - \lambda_I + \mu(\lambda_R - \lambda_I) dt + u_I^{max} dt - \mu(\lambda_S - \lambda_I) dt \right) > x_I (\lambda_S - \lambda_I) \quad (30)$$

where the last inequality is due to λ_R being the upper bound of λ_S . We see that $c_I^{opt} \beta x_I (\lambda_S - \lambda_I)$ is monotonically increasing in time, and therefore, if $(c_S^{opt}, c_I^{opt}) = (0, 0.5b_I)$ is the optimal solution at time t , it is also the solution at time $t + dt$. This means that $\lambda_S - \lambda_I$ must be increasing in time, but then it cannot satisfy the transversality condition, $\lambda_S(T) = \lambda_I(T) = 0$. Therefore, by contradiction, $c_I^* < 0.5b_I$ for all $0 \leq t < T$. \square

This proposition implies that the selfish behavior of the infected is never optimal for the population, and so the appropriate policy in all cases is to decrease the contacts of the infected below the level that they want to selfishly make.

Numerical results

For the same set of parameters as in Fig. 3, we can compute the socially optimal solution. The optimal solution as shown in Fig. 5b is to completely suppress the contacts of the I, so that no additional infection can take place, while the susceptible population resumes normal activities. In fact, this is most often the socially optimal strategy. An exception is if a large part of the population was already infected initially (Fig. 5c,d). Here, 70 percent of the population is already infected, and so the optimal contact strategy is to completely isolate the susceptible until the number of infected decreases to some level, at which, we go back to complete quarantining of the I and the resuming of normal activities by the S.

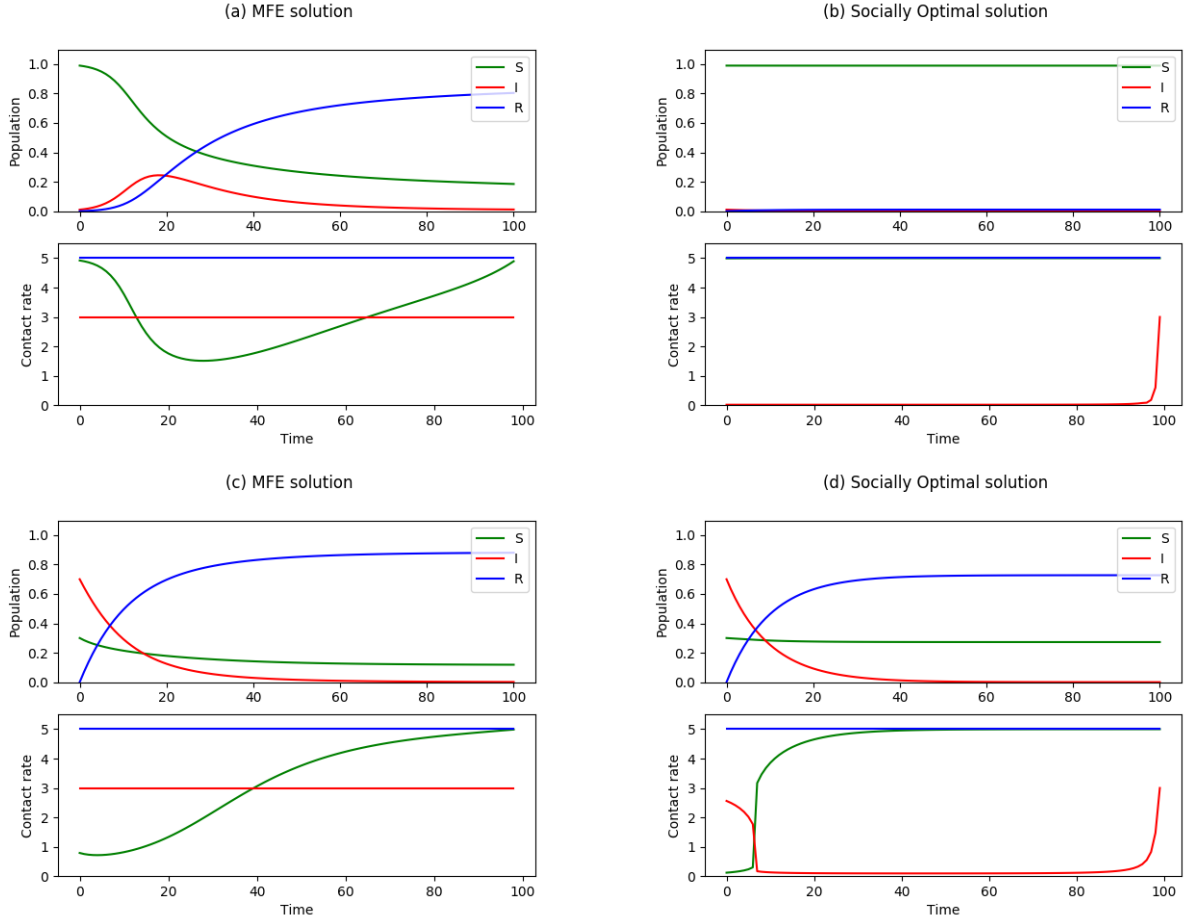


Figure 5: (a) shows the same MFE solution as in Figure 3. (b) shows the socially optimal solution, which maximizes the total utility of the population. (c) and (d) show the MFE solution and socially optimal solution for a different initial population, $(x_S(0), x_I(0)) = (0.3, 0.7)$.

3.3 Cost of central planning

While we assume in computing our socially optimal solution that the central planner can freely choose contact rates, it is not realistic. For example, in the optimal solutions of Fig. 5b, complete curbing of contact rates of I does not come free. Whether it is done through quarantining or providing proper incentives to keep infected people from making social contacts, they cost resources. Additionally, people do not generally like their choices to be decided by an authority, so this cost can also include the "loss in freedom," which the central planner would want to minimize. Therefore, a realistic addition to the problem is a cost for deviating away from the population's MFE.

$$c_S^{opt}(t), c_I^{opt}(t), c_R^{opt}(t) = \underset{c}{\operatorname{argmax}} \int_0^T \left[x_S(t)u_S(c_S(t)) + x_I(t)u_I(c_I(t)) + x_R(t)u_R(c_R(t)) - \frac{1}{2}k \sum x_z(c_z - c_z^{eq})^2 \right] dt \quad (31)$$

$k \geq 0$ is the parameter balancing the competing objectives of maximizing total utility and minimizing the deviations from selfish strategies. When $k = 0$, the solution to the objective function is the socially optimal solution, and when k is large, the solution is the MFE solution. As k is increased from 0, we can find the solution which balances the tradeoff.

As we see in Fig. 6, for different values of k , we compute the socially optimal contact rates of the population. Another interpretation is that given limited resources to control contact rates, we find how it should be distributed during $t \in [0, T]$. We see a common result among the range of values for k , which is that when susceptibles have low contact rates, it is less important to keep the infected contact rates as low. Instead, it is more optimal use of resources to make sure the infected

contact rates after the peak of the epidemic is lowered because this is when the susceptibles start ramping up to normal activities, thus posing higher risk of a second epidemic. Another reason is that because the number of infected is smaller at this time and the imposed cost is fixed per capita, the same amount of resources is more efficiently used by controlling the smaller number of I rather than the larger number during the peak.

3.4 Price of Anarchy

We compute the price of anarchy (PoA), which is a measure of how much the system degrades due to the selfish strategies of each compartment. In the context of mean-field games, it is the ratio of the total utility of the population adopting MFE strategies to the total utility of the population adopting socially optimal strategies [11]. The PoA is given by

$$\text{PoA} = \frac{V_{opt}(0)}{x_S(0)V_S(0) + x_I(0)V_I(0) + x_R(0)V_R(0)} \quad (32)$$

where V_{opt} is the maximum of the objective function in (17). The PoA, computed for ranges of β, μ, b_I , and a_I , is shown in Fig. 7. While one parameter is changed, the others were kept fixed with the values from Fig. 3.

As β gets larger, PoA increases because each selfish behavior of the infected becomes magnified by each infecting more susceptible population per contact (Fig. 7a). We see a similarly increasing trend as μ gets smaller, because it increases the time spent infected (Fig. 7b). However, increase in β results in decreasing marginal gain in PoA while decrease in μ results in increasing marginal gain in PoA. Even for unreasonably large β , the overall utility suffers by around 11%, while small μ can cause a decline of 45%.

Fig. 7c shows that an intermediate value of b_I results in the largest PoA. This non-monotonic relationship is because of the trade-offs of large b_I . On one hand, large b_I means that the disease does not affect the day-to-day productivity of the infected as severely, and so the infected are not as penalized. On the other hand, this also means that the infected are able to be more active and making more contacts, which infects more susceptibles. These two opposing effects are balanced near $b_I = 5$, where the PoA is at its maximum. When a_I is large (Fig. 7d), the baseline cost of getting infected is larger, which results in less utility at each time point.

4 Discussion

4.1 Possible additions

The public policy response to COVID-19 is an extremely complex problem with many factors which this paper has not covered. This model and analysis are, in many ways, the simplest baseline case from which we can make more realistic to fit a particular disease. First, we can pose the problem with different compartmental models. For example, if we take the SIS model, we would see different optimal strategies since the infected also face the burden of social distancing. In the case of COVID-19, it will be most useful to include a compartment with asymptomatic transmission, which behaves like S from the central planner's perspective. The socially optimal strategies depended on being able to distinguish the S and the I, when it is not always the case. The lack of available testing of COVID-19, for example, provides the uncertainty within the population as well as from the central planner's perspective. Second, we can add heterogeneity to the population by including state variables such as age, socioeconomic status, or level of prosociality. By explicitly adding the different subpopulations, we can understand the game theoretic dilemma at play between the old vs. the young, the financially stable vs. the unstable, or the prosocial vs. antisocial. Then, more specific policies may be proposed that target the contact strategies of a particular group.

4.2 Some conclusions

While our analysis does not include many important factors, we can still make some general conclusions.

Selfish strategies still "flatten the curve."

By including adaptive behavior of individuals, our model predicts epidemic curves with flatter growth rate, compared to the classical counterparts. In our simple model, the curve is flattened naturally because of susceptibles who weigh the trade-off between current utility of making contacts and the future cost of getting infected. The recovered and the infected do not have any trade-offs to decrease contacts. Even if only the susceptibles are practicing social distancing, it still decreases the number of contacts of the system, and so the infected population reaches a smaller peak (Fig. 3, 4). It should be emphasized that the curve is flattened because individuals anticipate future growth in infections and decrease their contacts to avoid being exposed to the infected individuals. If the possible outbreak is flat-out denied by the media, then individuals will not adapt their behaviors, causing an unmitigated large peak in the infected population. Therefore, it is the responsibility of the policymakers to clearly communicate the existence and extent of the spreading disease.

Selfish strategy of the infected is never socially optimal.

We prove that the socially optimal strategy of the infected, c_I^{opt} , is always less than c_I^{eq} . Therefore, policies in response to the epidemic should decrease the contact rates of the infected. We see examples of policies with this aim such as quarantining the infected or granting paid sick leave to individuals who tested positive. Both policies respectively decrease the contact rates of the infected directly or indirectly by decreasing b_I , reducing the potential gain in utility of making more contacts. Because reducing the contacts of the infected is so important, policymakers might consider even more aggressive policies.

It is important to control the infected contact rates, following the peak of the epidemic.

If cost is imposed to the central planner in changing the contact rates of the individuals, we find the new socially optimal contact rates, depending on k , which is the per capita unit cost of changing contact rates. For $k = 0$ and $k \gg 1$ respectively, we find the cost-imposed socially optimal solution to be the previously computed c^{opt} and c^{eq} . For k values in between (Fig. 7), we commonly see that when it is too costly to decrease the infected contact rate for the whole time period, it is most beneficial to at least focus on decreasing after the epidemic has subsided. An assumption here is that cost of central planning is constant in time, when it may not be in real world situations. When outbreak is at its peak, more public attention is on the disease, and it may be easier to implement social distancing or secure funding for quarantining. However, when the disease has subsided, it might be harder to convince the public to behave differently.

This result reinforces the need for formal social distancing policy which goes beyond the peak of the epidemic. When the disease is prevalent, social distancing can be naturally favored due to individual optimization, but to sustain it for longer requires centralized public policy to prevent second peaks. This general result is in agreement with other studies of COVID-19 policies which mention the likely possibility of second peaks [12–14]. Additional work, using more realistic central planning cost depending on time and population structure, will help us better understand how such long-term social distancing policies should be implemented.

Policies are most needed for diseases with low μ , high a_I , high β and intermediate b_I .

By computing the price of anarchy, we can measure the effect of different parameters on how much the system is degraded by the selfish behaviors. Diseases with low μ , high a_I , high β , intermediate b_I , in this order, seem to most affect the population such that their selfish behaviors will degrade the system more compared to central intervention.

An interesting future work will be to put different diseases on the spectrum of these 4 variables, depending on its epidemiological characteristics as well as its economic and health effects on the infected. Then, we can roughly categorize diseases which need to be centrally intervened in the case of an outbreak.

References

- [1] Timothy C Reluga. Game theory of social distancing in response to an epidemic. *PLoS computational biology*, 6(5), 2010.
- [2] Eli P Fenichel, Carlos Castillo-Chavez, M Graziano Ceddia, Gerardo Chowell, Paula A Gonzalez Parra, Graham J Hickling, Garth Holloway, Richard Horan, Benjamin Morin, Charles Perrings, et al. Adaptive human behavior in epidemiological models. *Proceedings of the National Academy of Sciences*, 108(15):6306–6311, 2011.
- [3] Benjamin R Morin, Eli P Fenichel, and CARLOS CASTILLO-CHAVEZ. Sir dynamics with economically driven contact rates. *Natural resource modeling*, 26(4):505–525, 2013.
- [4] Peter J Francis. Optimal tax/subsidy combinations for the flu season. *Journal of Economic Dynamics and Control*, 28(10):2037–2054, 2004.
- [5] Timothy C Reluga. An sis epidemiology game with two subpopulations. *Journal of Biological Dynamics*, 3(5):515–531, 2009.
- [6] Josu Doncel, Nicolas Gast, and Bruno Gaujal. A mean-field game analysis of sir dynamics with vaccination. 2017.
- [7] Suzanne Lenhart and John T Workman. *Optimal control applied to biological models*. CRC press, 2007.
- [8] Rachael Miller Neilan and Suzanne Lenhart. An introduction to optimal control with an application in disease modeling. In *Modeling Paradigms and Analysis of Disease Transmission Models*, pages 67–82, 2010.
- [9] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.
- [10] Minyi Huang, Roland P Malhamé, Peter E Caines, et al. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.
- [11] René Carmona, Christy V Graves, and Zongjun Tan. Price of anarchy for mean field games. *ESAIM: Proceedings and Surveys*, 65:349–383, 2019.
- [12] Francesco Di Lauro, István Z Kiss, and Joel Miller. The timing of one-shot interventions for epidemic control. *medRxiv*, 2020.
- [13] Dylan Morris, Joshua B Plotkin, Fernando W Rossine, and Simon Levin. Optimal, near-optimal, and robust epidemic control. *OSF Preprints*. April, 4, 2020.
- [14] Neil Ferguson, Daniel Laydon, Gemma Nedjati Gilani, Natsuko Imai, Kylie Ainslie, Marc Baguelin, Sangeeta Bhatia, Adhiratha Boonyasiri, ZULMA Cucunuba Perez, Gina Cuomo-Dannenburg, et al. Report 9: Impact of non-pharmaceutical interventions (npis) to reduce covid19 mortality and healthcare demand. 2020.

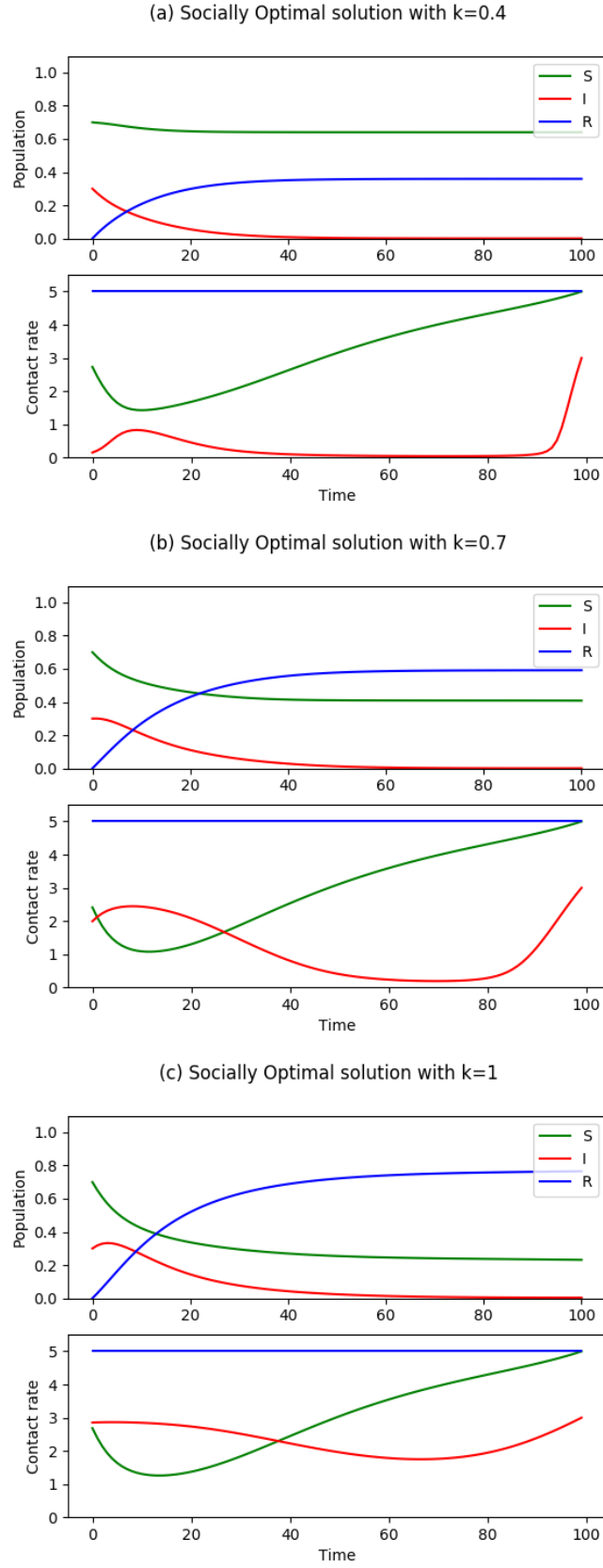
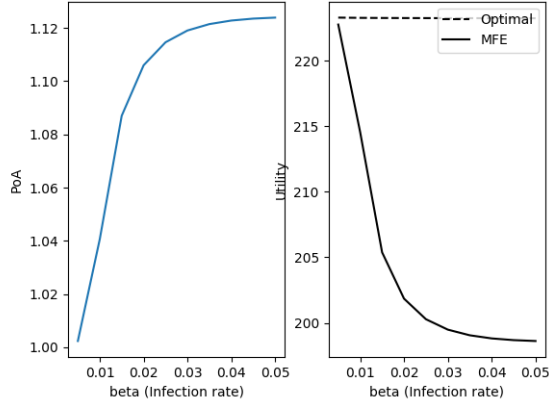
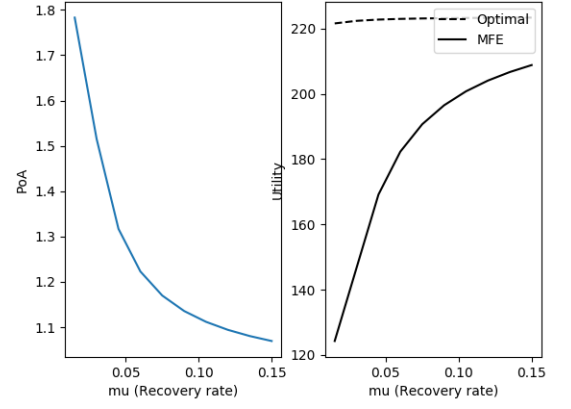


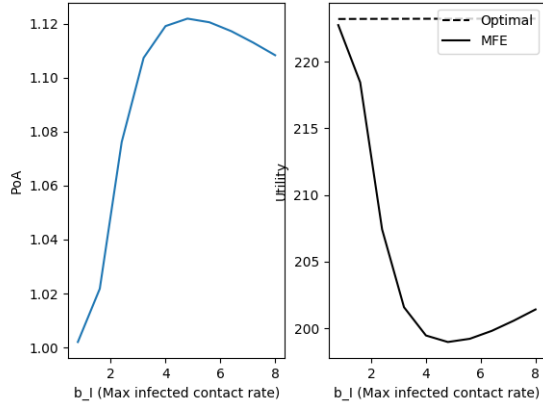
Figure 6: With initial condition $x_I(0) = 0.3$, we compute the socially optimal contact strategies and corresponding population dynamics for $k = 0.4, 0.7$, and 1 . As k is increased, it becomes more expensive to shift the infected strategies from the selfish strategy, and so the socially optimal contacts of the infected is larger, as the cost of lowering it begins to outweigh the utility benefits.



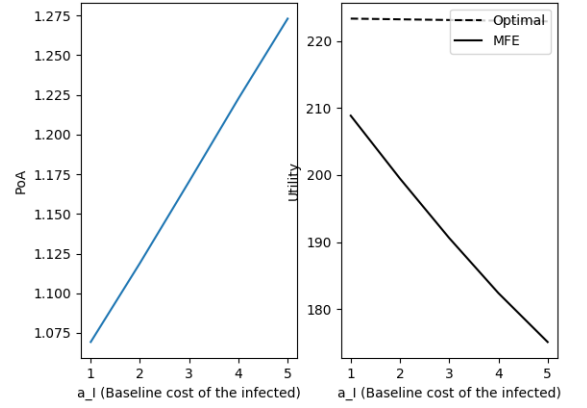
(a) PoA of $\beta \in [0.005, 0.05]$



(b) PoA of $\mu \in [0.015, 0.15]$



(c) PoA of $b_I \in [1, 8]$



(d) PoA of $a_I \in [1, 5]$

Figure 7: We vary the model parameters and compute the price of anarchy (left) and the total population utility of the socially optimal strategy (right, dashed) and the MFE strategy (right, solid). The parameter space where PoA is high is where intervening public policy would be most needed, since this is the case in which selfish strategies are most degrading the total utility.