Lectures Week #1 and #2 Notes Summary

Prepared by Laurence Davies

Topics Covered

I Introduction

There are three main parts to this course:

1. Geometry (extra Euclidean geometry)
   - Centres of triangles (mean, circumcentre, and so on)
   - Circles
2. Transformations in geometry
   - (Rotations, reflections, glide reflections, similarities)
3. Groups (abstract algebra)
   - e.g. Groups of symmetries: Consider the reflective symmetries of an equilateral triangle in figure 1.
   - There will also be material on freeze groups and wallpaper groups.

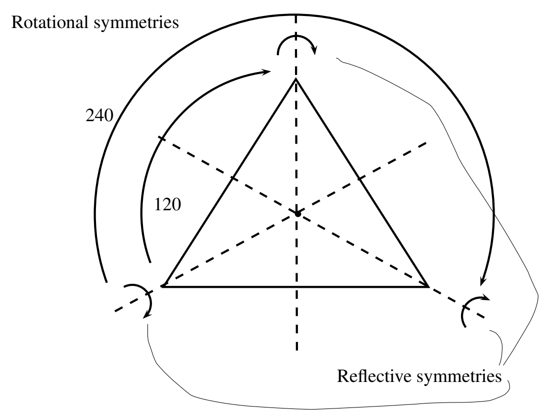
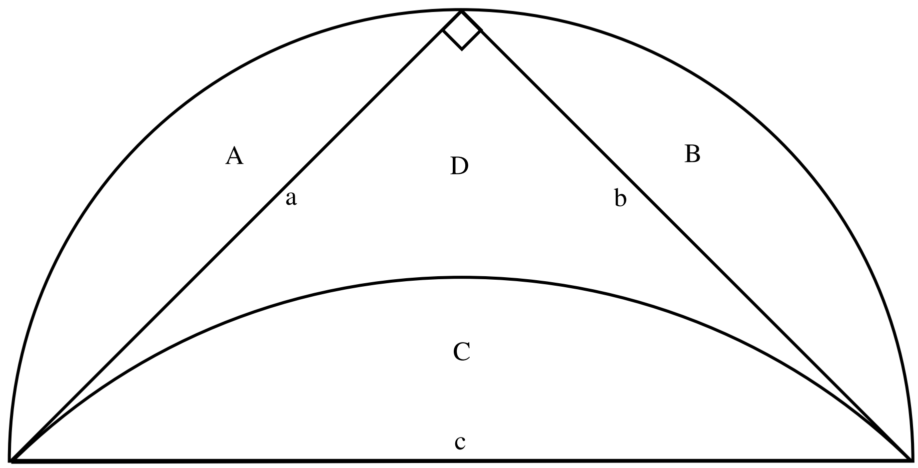


Figure 1: Symmetries of an equilateral triangle

II Historical Example: Lunes

The following considers an ancient text on Hippocrates Lunes (˜450BC).

Find a square with the same area as a curved lune.



Examine areas A, B, C and D. This example will show that $C = A + B$ and hence the area of the lune $A + D + B = C + D$ which is the area of the triangle $\frac{1}{2} \times base \times height$.

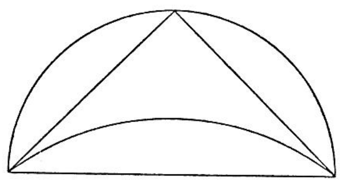
Consider the Pythagorean relationship $c^2 = a^2 + b^2$.

The relationship between an area and the corresponding chord length is a quadrature i.e. $A = \lambda a^2$ where λ is the same for all three segments.

GREEK MATHEMATICS

“ Καὶ οἱ τῶν μηνίσκων δὲ τετραγωνισμοὶ δόξαντες εἶναι τῶν οὐκ ἐπιπολαίων διαγραμμάτων διὰ τὴν οἰκειότητα τὴν πρὸς τὸν κύκλον ὑφ’ Ἱπποκράτους ἐγράφησάν τε πρώτου καὶ κατὰ τρόπον ἔδοξαν ἀποδοθῆναι· διόπερ ἐπὶ πλέον ἁψώμεθά τε καὶ διέλθωμεν. ἀρχὴν μὲν οὖν ἐποιήσατο καὶ πρῶτον ἔθετο τῶν πρὸς αὐτοὺς χρησίμων, ὅτι τὸν αὐτὸν λόγον ἔχει τά τε ὅμοια τῶν κύκλων τμήματα πρὸς ἄλληλα καὶ αἱ βάσεις αὐτῶν δυνάμει. τοῦτο δὲ ἐδείκνυεν ἐκ τοῦ τὰς διαμέτρους δεῖξαι τὸν αὐτὸν λόγον ἐχούσας δυνάμει τοῖς κύκλοις.

“ Δειχθέντος δὲ αὐτῷ τούτου πρῶτον μὲν ἔγραφε μηνίσκου τὴν ἐκτὸς περιφέρειαν ἔχοντος ἡμικυκλίου



τίνα τρόπον γένοιτο ἂν τετραγωνισμός. ἀπεδίδου δὲ τοῦτο περὶ τρίγωνον ὀρθογώνιόν τε καὶ ἰσοσκελὲς ἡμικύκλιον περιγράψας καὶ περὶ τὴν βάσιν τμῆμα κύκλου τοῖς ὑπὸ τῶν ἐπιζευχθεισῶν ἀφαιρουμένοις ὅμοιον. ὄντος δὲ τοῦ περὶ τὴν βάσιν τμήματος ἴσου τοῖς περὶ τὰς ἑτέρας ἀμφοτέροις, καὶ κοινοῦ προστεθέντος τοῦ μέρους τοῦ τριγώνου τοῦ ὑπὲρ τὸ τμῆμα τὸ περὶ τὴν βάσιν, ἴσος ἔσται ὁ μηνίσκος τῷ τριγώνῳ. ἴσος οὖν ὁ μηνίσκος τῷ τριγώνῳ δειχθεὶς τετραγωνίζοιτο ἄν. οὕτως μὲν

238

HIPPOCRATES OF CHIOS

“ The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful to his purpose, that similar segments of circles have the same ratios as the squares on their bases.[a] And this he proved by showing that the squares on the diameters have the same ratios as the circles.[b]

“ Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides.[c] Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking

[a] Lit. “ as the bases in square.”

[b] This is Eucl. xii. 2 (see *infra*, pp. 458-465). Euclid proves it by a method of exhaustion, based on a lemma or its equivalent which, on the evidence of Archimedes himself, can safely be attributed to Eudoxus. We are not told how Hippocrates effected the proof.

[c] As Simplicius notes, this is the problem of Eucl. iii. 33 and involves the knowledge that similar segments contain equal angles.

239

III Euclid’s Elements

As a brief introduction to Euclids Elements, this section will discuss one or two items from the definitions, postulates, common notions and propositions[1].

In proposition 1[2] **”To construct an equilateral triangle on a given finite straight line”** there are areas that can be subject to critique, such as the notion that the circles must intersect. Without this intersection, C cannot be described and thus the proposition does not hold, but Euclid does not explicitly explain how they must intersect. This aspect as well as others are the subject of deeper mathematics that was explored by Hilbert and Gödel (in his Incompleteness Theorems).

Aristotle (who predated Euclid) proposed that every science be organised as a set of propositions deduced from *self-evident* axioms, definitions and postulates. If it is organised in such a way, everything that follows is certain and there would be no need for measurement or observation.

Unfortunately, this did not work for most sciences but it did work for mathematics, in particular in number theory.

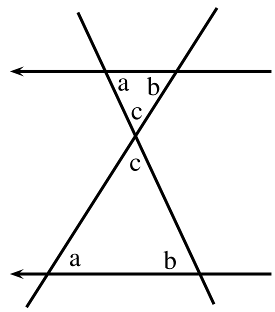
There are two common definitions of geometry. The first is pure (Euclidean) geometry, and the second is

[1] For a more comprehensive list of definitions, postulates and common notions, see *http://aleph0.clarku.edu/˜djoyce/java/elements/elements.html*

[2] *http://aleph0.clarku.edu/ djoyce/java/elements/bookI/propI1.html*

the shape of space (as explored in physics). This course aims to focus on the first.

Consider the preliminary example of two triangles described by two lines intersecting between two parallel lines as below.



These triangles can be said to be similar because it can be shown that the corresponding angles are equal. The corresponding angle pairs a and b are alternate angles in parallel lines, and the angle pair c are simply opposite angles described by intersecting straight lines. It can be then stated that these similar triangles have corresponding sides in the same ratio.

Geometry of Circles

Consider the figure below which describes the angles at the centre and circumference of a circle described by a chord.

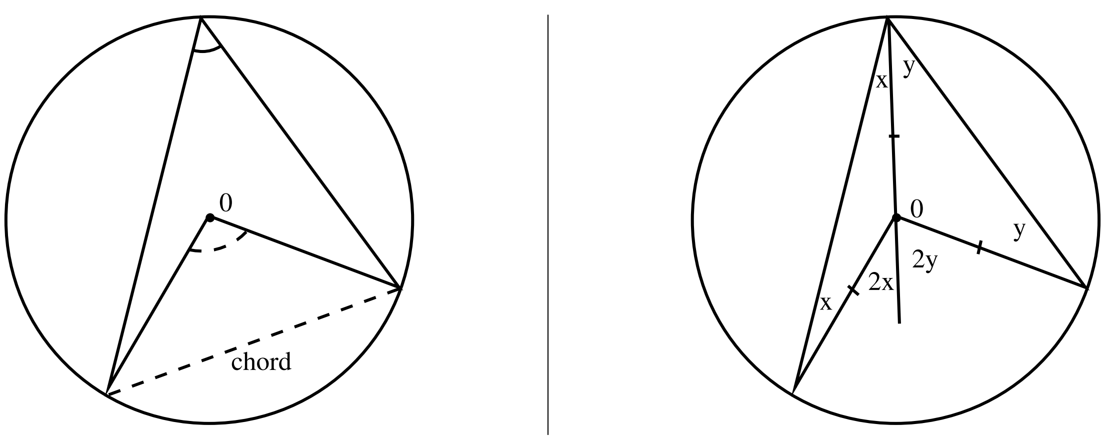


Figure 2: The angle at the centre is twice the angle at the circumference subtended by the same chord

It is required to prove that **the angle at the centre is twice the angle at the circumference subtended by the same chord**.

1. Construct a line from the vertex of the angle on the circumference passing through the centre of the circle.
2. The segment of the line from the vertex to the centre is the radius, and thus forms two isosceles triangles as depicted below
3. The exterior angle of a triangle is the sum of the opposing two interior angles (shown later). Thus the angle at the centre is $2x + 2y$, which is twice the angle at the circumference $x + y$.

Two **corollaries** can be derived from this proof:

1. All angles subtended at the circumference by a chord are equal.
2. The angle in a semi-circle is a right angle.

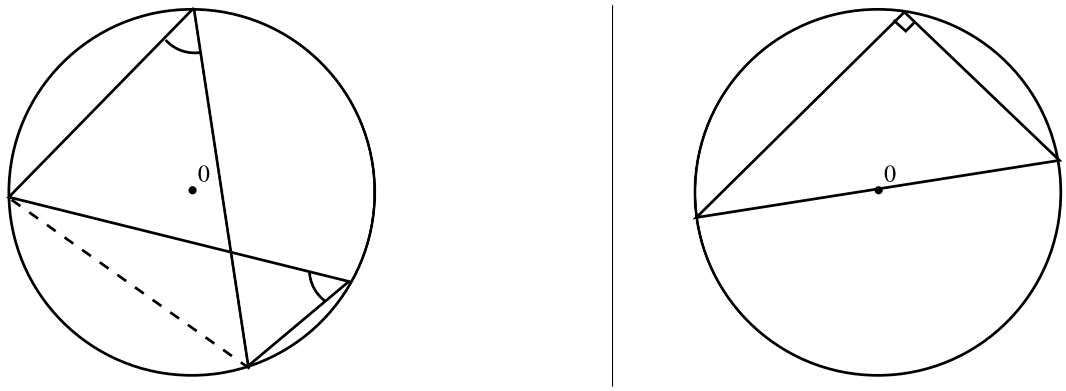
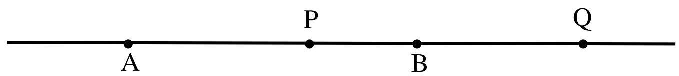


Figure 3: Corollaries of the circumference angle property

IV Ceva's Theorem and Menelaus' Theorem

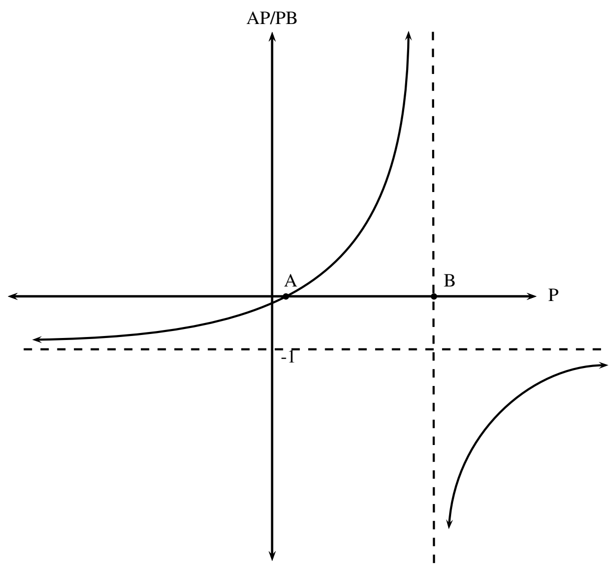
First, some conventions that allow *negative ratios of lengths*. Given a line from A to B, the length AB is the negative of the length from B to A, i.e. $AB = -BA$.

AB is a length from A to B, starting at point A. For P on the line extended from AB,



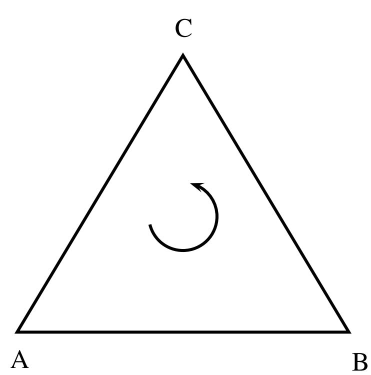
- $\frac{AP}{PB}$ is **positive** because P is between A and B.
- $\frac{AQ}{QB}$ is **negative** because Q is exterior to the line between A and B.

The graph of the ratio $\frac{AP}{PB}$ for any P on the extended line AB is described as follows:

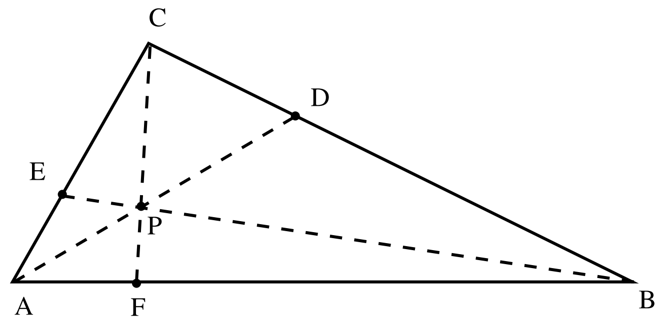


No two values of P give the same $\frac{AP}{PB}$, i.e. this ratio *determines* P.

Convention: Given a trangle, lengths defined in the counter-clockwise direction are positive. For example, in the below figure, AB, BC, CA are positive lengths, but BA, CB, AC are negative.



Ceva's Theorem 1. *(Italy, 1678) ("Chayva") For a triangle* $\triangle ABC$ *with* $\triangle DEF$ *on the sides* BC, CA, AB *or their extensions, then* AD, BE, CF *are concurrent if and only if* $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$ *or* $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$
Note tht concurrent means literally "run together" so here it signifies that all lines intersect at the same point.



Proof. First suppose that AD, BE, CF are concurrent at point P.
Case 1: Point of intersection is inside the triangle $\triangle ABC$.
Extend CF and BE such that they meet a line running through A parallel to BC at M and N respectively.

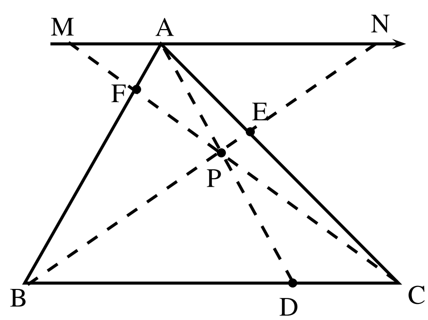


Figure 4: Proof of Ceva's theorem in the first case where P is interior to $\triangle ABC$

$$\frac{AF}{FB} = \frac{AM}{CB} \text{ because } \triangle AFM \sim \triangle BFC \tag{1}$$

$$\text{Opposite angles are equal, alternate angles are equal.} \tag{2}$$

$$\frac{CE}{EA} = \frac{CB}{NA} \text{ because } \triangle CEB \sim \triangle AEN \tag{3}$$

$$\text{For the same reasons as above.} \tag{4}$$

$$\tag{5}$$

Additionally,

$$\frac{BD}{NA} = \frac{DP}{AP} \text{ because } \triangle BDP \sim \triangle NAP \tag{6}$$
$$\frac{AP}{DP} = \frac{AM}{DC} \text{ because } \triangle DCP \sim \triangle AMP \tag{7}$$
$$\tag{8}$$

So

$$\frac{BD}{DC} = \frac{BD}{NA} \cdot \frac{AM}{DC} \cdot \frac{NA}{AM} = \frac{DP}{AP} \cdot \frac{AP}{DP} \cdot \frac{NA}{AM} = \frac{NA}{AM} \tag{9}$$

By substituting each of the above we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AM}{CB} \cdot \frac{CB}{NA} \cdot \frac{NA}{AM} \tag{10}$$
$$= 1 \tag{11}$$

Conversely, suppose that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \tag{12}$$

Define $P = BE \cap CF$, i.e. the intersection of just two lines BE and CF.

Then let $D' = AP \cap BC$. (The aim is to show $D' = D$).

We cam apply what we proved above to D'. We know that

$$\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} = 1 \tag{13}$$

And we assumed

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \tag{14}$$

So by cancellation $\frac{BD'}{D'C} = \frac{BD}{DC}$ and hence $D = D'$ from the earlier defintion that the ratio at which a point divides a line uniquely determines the point.

We've shown in case 1 that iff $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ then BE, AD, CF are concurrent.

Case 2: Point of intersection is outside the triangle $\triangle ABC$.

Follow the same proof as before, but note that some of the ratios are negative.

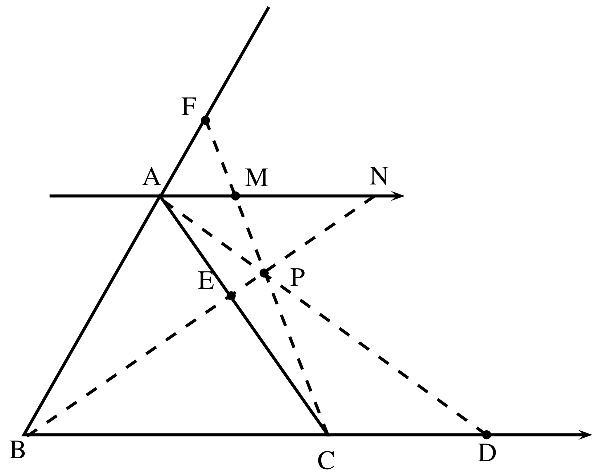


Figure 5: Proof of Ceva's theorem in the second case where P is exterior to $\triangle ABC$

Figure 6:

In the following example where P is not enclosed by $\triangle ABC$:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -ve \cdot -ve \cdot +ve$$

□

Menelaus's Theorem 1. *Let $\triangle ABC$ be a triangle with D, E, F on BC, CA, AB or their extensions, Then D, E, F are **collinear** if and only if*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1 \tag{15}$$

To proved that D, E, F are collinear, use similar right-angled triangles.

Proof.

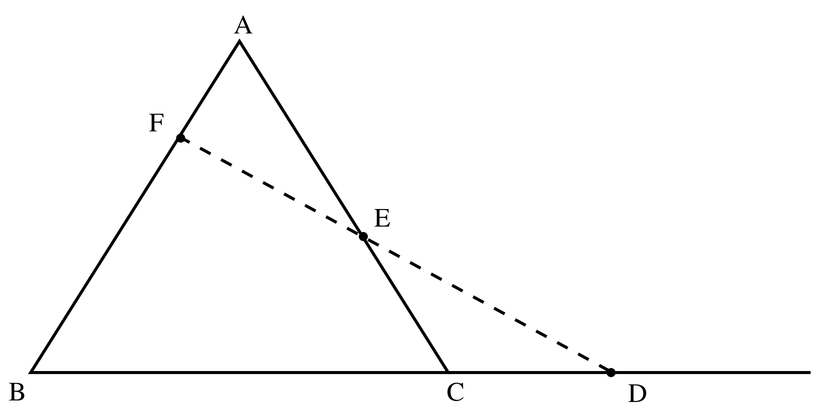


Figure 7: Menelaus's Theorem on $\triangle ABC$

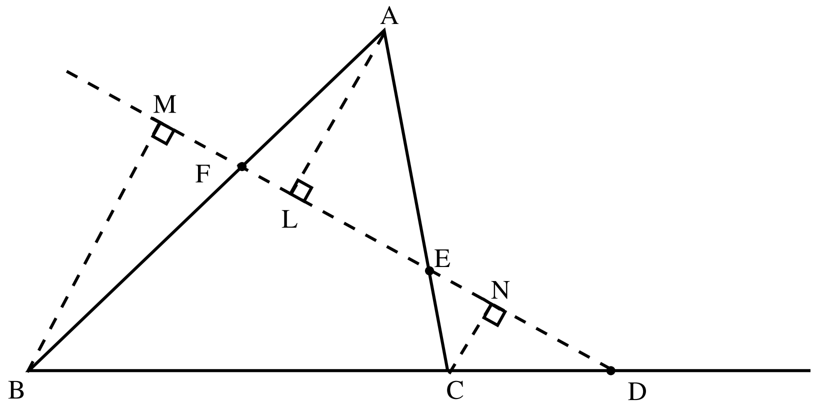


Figure 8: Construct three similar right angled triangles

First suppose D, E, F are collinear. Then

$$\frac{AF}{FB} = \frac{AL}{MB} \text{ since } \triangle AFL \sim \triangle BFM \tag{16}$$
$$\frac{BD}{DC} = \frac{BM}{NC} \text{ since } \triangle BDM \sim \triangle CDN \tag{17}$$
$$\frac{CE}{EA} = \frac{CN}{LA} \text{ since } \triangle CEN \sim \triangle AEL \tag{18}$$

Multiplying the above together gives

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AL}{MB} \cdot \frac{BM}{NC} \cdot \frac{CN}{LA} \tag{19}$$
$$= (-1)^3 = -1 \tag{20}$$

Case 2: Where DEF doesn't meet the triangle $\triangle ABC$.

The proof for this is the same as above. This is left as an exercise to the reader.

Conversely, suppose

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1 \tag{21}$$

Define $EF \cap BC = D'$ (Aim to show $D = D'$).

Then by the first part of this proof,

$$\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} = -1 \tag{22}$$

Equating the assumption and property above and cancelling

$$\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \tag{23}$$

$$\frac{BD}{DC} = \frac{BD'}{D'C} \tag{24}$$

Hence $D = D'$ using the property that the ratio determines the point.

Therefore if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$ then D, E, F are collinear.

$\square$