# Chapter 3

# Partial differential equations of the first-order

Although the major focus of these introductory volumes is on linear partial differential equations of second- and higher-order, it is instructive to examine some aspects of first-order partial differential equations. These can arise in a number of engineering applications including the study of waves in shallow water, traffic flow, gas dynamics, isothermal plug flow reactors in chemical engineering and in the analysis of the thermal efficiency of heat exchangers.

## 3.1 General concepts

We focus attention on the quasi-linear first-order partial differential equation which has the form

$$A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u) \tag{3.1}$$

where $x$ and $y$ are the independent variables. The solution of (3.1) is a surface in the $(x,y,u)$ space. Let us consider a point $P(x,y,u)$ on the solution surface $u = u(x,y)$ and we move along a direction given by the vector $\mathbf{t} = \{A(x,y,u), B(x,y,u), C(x,y,u)\}$. However, at any point on the surface, the direction of the normal is given by (Figure 3.1) $\mathbf{n} = \{\partial u/\partial x, \partial u/\partial y, -1\}$.

From these it is clear that $\mathbf{n}$ and $\mathbf{t}$ are orthogonal. Therefore any solution surface must be tangent to a vector with components $\{A, B, C\}$ and since this vector is tangential we do not leave the solution surface. Also, the total differential $du$ is given by

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy \tag{3.2}$$

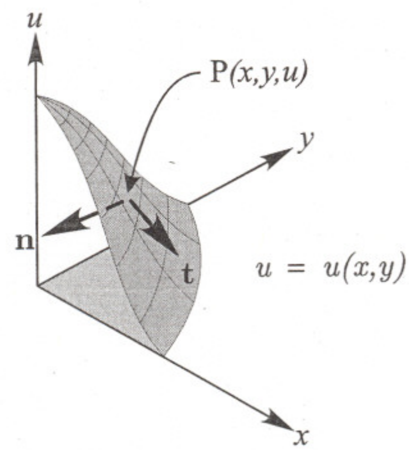


**Figure 3.1**: Solution surface for $u\,(x, y)$.

Consequently, from (3.1) and (3.2)

$$\{A, B, C\} \equiv \{dx, dy, du\} \tag{3.3}$$

The solution to equation (3.1) can be obtained by using the following theorem.

## THEOREM 3.1

The general solution of the quasi-linear PDE

$$A(x,y,u)\frac{\partial u}{\partial x}+B(x,y,u)\frac{\partial u}{\partial y}=C(x,y,u) \tag{3.4}$$

can be written in the form

$$\mathcal{F}(F,G)=0 \tag{3.5}$$

where $\mathcal{F}$ is an arbitrary function and $F(x,y,u)=k_1$, and $G(x,y,u)=k_2$ form a solution of the equations

$$\frac{dx}{A(x,y,u)}=\frac{dy}{B(x,y,u)}=\frac{du}{C(x,y,u)} \tag{3.6}$$

## PROOF

The equations (3.6) consist of a set of two independent ODE's (i.e. a two parameter family of curves in space). Also, one set can be written as

$$\frac{dy}{dx}=\frac{B(x,y,u)}{A(x,y,u)} \tag{3.7}$$

which is referred to as a "characteristic curve".

(i) If $A=A(x,y)$ and $B=B(x,y)$ then (3.7) is a function in the $(x,y)$ space. This is referred to as a base curve.

(ii) When $A$ and $B$ are constants (3.7) defines a set of parallel lines in the $(x,y)$ space.

In both cases (i) and (ii) (3.7) can be evaluated without a knowledge of $u(x,y)$. In the quasi-linear case (3.7) cannot be evaluated until $u(x,y)$ is known.

Returning to the equations (3.1) and (3.2) we can write

$$\begin{bmatrix} A & B \\ dx & dy \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = \begin{bmatrix} C \\ du \end{bmatrix} \tag{3.8}$$

Both equations must hold on the solution surface and one can interpret each equation as a plane element, where they intersect on a line along which $\partial u/\partial x$ and $\partial u/\partial y$ may exist; i.e. $\partial u/\partial x$ and $\partial u/\partial y$ are themselves indeterminate along the intersection line but they are related to or determinate to each other since (3.8) must hold.

Using a principle in linear algebra, if a square coefficients matrix of a set of $n$ simultaneous equations has a vanishing determinant, a necessary condition for the existence of finite solutions is that when the right hand side is substituted for any column, the determinants of the resulting coefficients matrices must also vanish, i.e.

$$\begin{vmatrix} A & B \\ dx & dy \end{vmatrix} = \begin{vmatrix} A & C \\ dx & du \end{vmatrix} = \begin{vmatrix} C & B \\ du & dy \end{vmatrix} = 0 \tag{3.9}$$

Evaluating the determinants we have

$$\frac{dx}{A(x,y,u)} = \frac{dy}{B(x,y,u)} = \frac{du}{C(x,y,u)} \tag{3.10}$$

## 3.2 Examples involving first-order equations

In this section we shall examine some basic first-order equations and their characteristic curves.

### Example 3.1

Examine the characteristic curves for the simplest possible first-order partial differential equation

$$\frac{\partial u}{\partial x} = 0 \tag{3.11}$$

### Solution

The general solution of equation (3.11) is

$$u = f\,(y) \tag{3.12}$$

where $f\,(y)$ is a completely arbitrary function of $y$. The characteristic curves in the $x, y$ plane are straight lines (Figure 3.2).

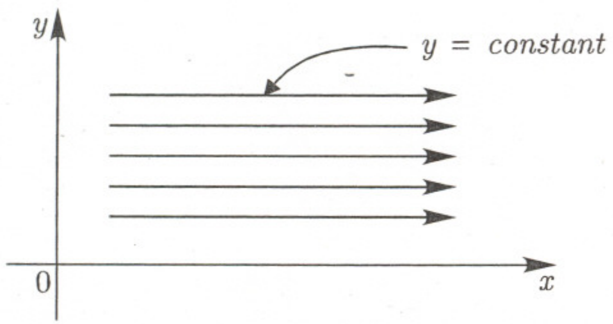


**Figure 3.2**: Characteristic curves for $\partial u/\partial x = 0$.

---

**Example 3.2**

---

Consider the first-order linear homogeneous partial differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{3.13}$$

Examine the characteristic curves associated with this equation.

**Solution**

Comparing this with (3.1) we have $A = 1$ and $B = 1$; and (3.10) gives

$$dx = dy \tag{3.14}$$

Integrating (3.14) we have

$$x - y = \text{const} \tag{3.15}$$

Consequently, the general form of the solution is

$$u = F\,(x - y) \tag{3.16}$$

where $F$ is an arbitrary function. Therefore if $y = x$ or $y = x + \Delta$, then $u = \text{const}$. Hence there is a characteristic direction in the $x, y$ plane along which $u$ is constant (Figure 3.3).

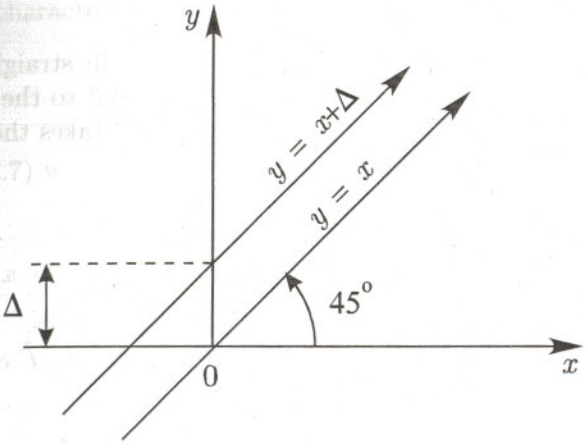


**Figure 3.3**: Characteristic curves for $\partial u/\partial x + \partial u/\partial y = 0$.

The general solution (3.16) can be made specific by imposing additional conditions.

e.g. If $u(x, y) = \eta x$, where $\eta$ is a constant, when $y = 0$, the specific solution for (3.16) is

$$u(x, y) = \eta(x - y) \tag{3.17}$$

---

**Example 3.3**

---

Generalize the results presented in connection with Example 3.2 to include the case

$$A_0 \frac{\partial u}{\partial x} + B_0 \frac{\partial u}{\partial y} = 0 \tag{3.18}$$

where $A_0$ and $B_0$ are both *constants*.

### Solution

We can prove that the characteristic curves are still straight lines except that they are inclined at an angle $\alpha = \tan^{-1}(B_0/A_0)$ to the $x$-axis (Figure 3.4). Avoiding details, the general solution of (3.18) takes the form

$$u(x, y) = \widetilde{F}\left(\frac{x}{A_0} - \frac{y}{B_0}\right) \tag{3.19}$$

where $\widetilde{F}$ is an arbitrary function. Again specific forms for $\widetilde{F}$ can be obtained by assigning additional conditions.

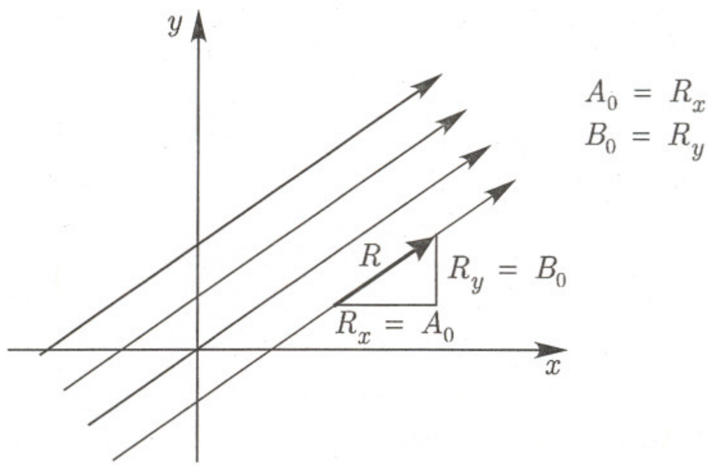


**Figure 3.4**: Characteristic curves for $A_0(\partial u/\partial x) + B_0(\partial u/\partial y) = 0$.

---

### Example 3.4

---

Consider the first-order partial differential equation (3.18) which can be further generalized to the following form

$$A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} = 0 \tag{3.20}$$

Examine the characteristic curves associated with the partial differential equation

**Solution**

Considering (3.7) we have

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)} \tag{3.21}$$

Since $A(x,y)$ and $B(x,y)$ are arbitrary functions, the orientations of the tangent vector $\mathbf{R}$ to the characteristic lines are still parallel to the $x, y$ plane (i.e. independent of $u$) but has a variable orientation in the $x, y$ plane (Figure 3.5).

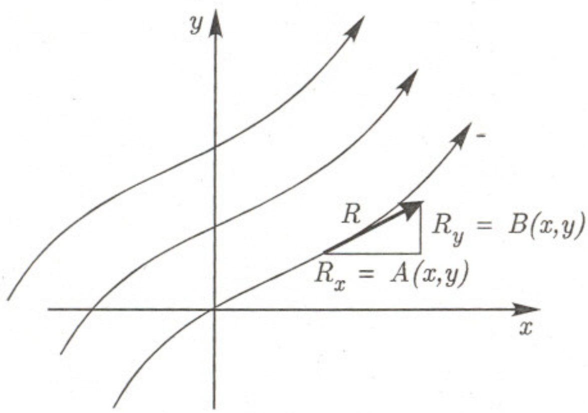


**Figure 3.5**: Characteristic curves $A(x,y)(\partial u/\partial x) + B(x,y)(\partial u/\partial y) = 0$.

We can rewrite (3.21) as

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} = ds \tag{3.22}$$

Using (3.2) and (3.22) we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = ds\left[A\left(x,y\right)\frac{\partial u}{\partial x} + B\left(x,y\right)\frac{\partial u}{\partial y}\right] = 0 \tag{3.23}$$

The result (3.23) confirms that along a characteristic curve $du = 0$, irrespective of $A\left(x,y\right)$ and $B\left(x,y\right)$.

---

**Example 3.5**

---

Examine a specific case of the general first-order homogeneous partial differential equation (3.20);

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = 0 \tag{3.24}$$

Obtain a solution such that $u(x,0) = \mu x^4$ where $\mu$ is a constant.

**Solution**

A comparison with (3.20) gives

$$A\left(x,y\right) = \frac{1}{x} \quad ; \quad B\left(x,y\right) = \frac{1}{y} \tag{3.25}$$

and using (3.25) in (3.21) and integrating we have

$$\int \frac{dx}{A\left(x,y\right)} = \int \frac{dy}{B\left(x,y\right)}$$

or

$$\int x \, dx = \int y \, dy$$

or

$$x^2 - y^2 = -C_0 \tag{3.26}$$

where $C_0$ is a constant. Therefore the most general functional form of the solution of (3.24) is

$$u(x, y) = F^* \left(x^2 - y^2\right) \tag{3.27}$$

Substituting (3.27) in (3.24) it is evident that the latter is satisfied for all choices of $F^*$. Again, a specific form of (3.27) can be obtained by imposing additional conditions on $u(x, y)$. e.g. If $u(x, 0) = \mu x^4$ is it evident that the specific solution of $u(x, y)$ is

$$u(x, y) = \mu \left(x^2 - y^2\right)^2 \tag{3.28}$$

where $\mu$ is a constant.

• • •

Let us now consider the inhomogeneous first-order partial differential equation posed by a reduced form of (3.1). i.e.

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = C(x, y) \tag{3.29}$$

The characteristic curves occupy the $(x, y, u)$ space except that a characteristic line does not lie in a plane $u = \text{const}$. Consider (3.10) and (3.22)

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} = \frac{du}{C(x,y)} = ds \tag{3.30}$$

Of these, two equations can be taken to be independent, i.e.

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \quad ; \quad \frac{dy}{B(x,y)} = \frac{du}{C(x,y)} \tag{3.31}$$

and as stated previously, if these equations are integrable, they can be evaluated in the form

$$F(x,y) = C_1 \quad ; \quad G(x,y) = C_2 \tag{3.32}$$

and the general solution of the partial differential equation (3.29) can be written in the form

$$\Omega(F,G) = 0 \tag{3.33}$$

---

**Example 3.6**

---

As a specific case of (3.29), consider the non-homogeneous partial differential equation

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = \frac{1}{y} \tag{3.34}$$

Develop a solution such that $u = \frac{1}{2}\left(3 - x^2\right)$ when $y = 1$.

**Solution**

Comparing (3.29) and (3.34) we have

$$A\left(x,y\right)=\frac{1}{x}\;;\;\;B\left(x,y\right)=\frac{1}{y}\;\;;\;\;C\left(x,y\right)=\frac{1}{y} \tag{3.35}$$

The associated ordinary differential equations corresponding to (3.31) are

$$\frac{dx}{(1/x)}=\frac{dy}{(1/y)}\;\;;\;\;\frac{dy}{(1/y)}=\frac{du}{(1/y)} \tag{3.36}$$

Integrating (3.36) we have

$$F=\left(x^2-y^2\right)=C_1\;\;;\;\;G=\left(u-y\right)=C_2 \tag{3.37}$$

and the general form of the function is

$$\Omega\left[\left(x^2-y^2\right),\left(u-y\right)\right]=0 \tag{3.38}$$

Hence for $y=1$, (3.38) gives

$$\Omega\left[\left(x^2-1\right),\left(u-1\right)\right]=2u-3+x^2=2\left(u-1\right)+\left(x^2-1\right) \tag{3.39}$$

Therefore the general relationship equivalent to (3.33) is obtained by replacing $\left(x^2-1\right)$ by $\left(x^2-y^2\right)$ and $\left(u-1\right)$ by $\left(u-y\right)$ in (3.39), i.e.

$$\Omega\left[\left(u-y\right),\left(x^2-y^2\right)\right]=2\left(u-y\right)+\left(x^2-y^2\right) \tag{3.40}$$

which gives

$$u\left(x,y\right)=\frac{1}{2}\left[2y-\left(x^2-y^2\right)\right] \tag{3.41}$$

It can be verified by substitution that (3.41) satisfies the first-order partial differential equation (3.34) subject to the additional condition (3.39) when $y = 1$.

## 3.3 Advective transport in reactor column

In the preceding, we have examined the various types of first-order partial differential equations in a very abstract sense without reference to any engineering application. We now apply the general theory concerning the first-order non-homogeneous partial differential equation to a problem which has applications in both *environmental engineering* and

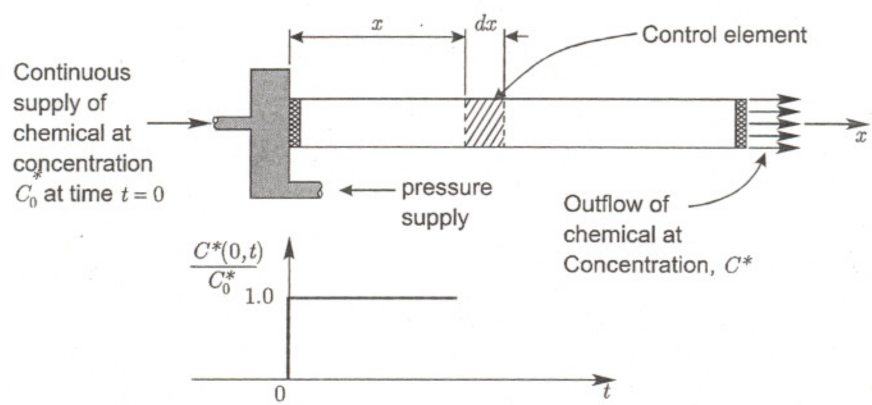


**Figure 3.6**: Advective transport in column.

*chemical engineering*. The problem deals with the so called "*advective transport*" of a species (such as a chemical, biological medium, effluent, etc.) along a reactor column. The reactor column is basically a tube with a circular cross sectional area which contains either a reactive or non-reactive porous solid (Figure 3.6). A chemical species is introduced into the flow regime and the "*concentration*" of the chemical or species is maintained constant at the start of the experiment. The objective of the analysis is to determine the variation of the concentration of the species along the tube as a function of time.

### 3.3.1 Governing equation - one dimensional case

Before attempting to formulate the governing equation, it is necessary to define the parameters governing the problem. The porous medium is characterized by continuous flow paths within the void space. A measure of the void space is the scalar quantity *porosity* ($n^*$) . This is defined by

$$n^* = \frac{\text{volume of voids}}{\text{total volume}} = \frac{V_v}{V} \tag{3.42}$$

where $V_v$ is the volume of voids and $V$ is the total volume.

Fluid flow which results in advection of the species takes place in the void space. The average local velocity of flow in the void space at any cross section is defined by $\overline{v}$. We define an average advective velocity over the entire cross section as $\widehat{v}$. In the case of one dimensional flow, both these velocities would be aligned in the $x$-direction (Figure 3.6). For conservation of mass we require.

$$\widehat{v}A = \overline{v}A_v \tag{3.43}$$

For most porous media with an *isotropic* or *direction independent* fabric or structure,

$$\frac{A_v}{A} \simeq \frac{V_v}{V} = n^* \tag{3.44}$$

We define the concentration $C(x,t)$ as the mass of the species per *unit volume of the fluid* conveying the species. We can also define a concentration $\widehat{C}(x,t)$ as the mass of the species per *unit total volume of the entire porous medium.* If the porous medium is *saturated*, the volume of the fluid is identically equal to the volume of the voids. From these definitions we have,

$$C(x,t) = \frac{\text{mass of species}}{\text{volume of fluid}} = \frac{\text{mass of species}}{\text{volume of voids}} = \frac{\text{mass of species}}{V_v}$$

or

$$\frac{\text{mass of species}}{\text{total volume}} = \frac{C(x,t)V_v}{V} = n^* C(x,t) = \widehat{C}(x,t) \tag{3.45}$$

Consider the one-dimensional flow in the tube, where, prior to the introduction of the species, a pressure gradient induces a uniform average fluid velocity of $\widehat{v}$. The cross sectional area of the tube is $a_0$. The mass of species transported in the tube by "*advection*", per unit cross sectional area per unit time is $\widehat{v} n^* C(x,t)$.

The total mass of the species which enters the control element of cross sectional area $a_0$ and length $dx$ is

$$F_x = a_0 \widehat{v} n^* C(x,t) \tag{3.46}$$

The total mass of the species leaving the control element

$$= F_x + \frac{\partial F_x}{\partial x} \cdot dx \tag{3.47}$$

The conservation of mass equation for the control element can be stated as follows:

$$\begin{aligned} \begin{bmatrix} \text{net rate of change} \\ \text{of mass of species} \\ \text{within the element} \end{bmatrix} &= \begin{bmatrix} \text{flux of species} \\ \text{out of the element} \end{bmatrix} \\ &\quad - \begin{bmatrix} \text{flux of species} \\ \text{into the element} \end{bmatrix} \\ &\quad \pm \begin{bmatrix} \text{loss or gain of the} \\ \text{mass of the species} \\ \text{to reactions} \end{bmatrix} \end{aligned} \tag{3.48}$$

From (3.48) we have

$$-\frac{\partial}{\partial t}(n^* C a_0 dx) = \frac{\partial}{\partial x}(\widehat{v} n^* C a_0)\, dx \pm \xi n^* C a_0 dx \tag{3.49}$$

where $\xi$ is the rate of accumulation (+) or loss (-) of chemicals within the control element due to reactions. We can reduce (3.49) to the form

$$\frac{\partial C}{\partial t} + \widehat{v}\frac{\partial C}{\partial x} = \pm\xi C \tag{3.50}$$

For the simple one-dimensional advective flow problem, which includes a reaction term, the governing first-order differential equation is identical in form to (3.1). The boundary and initial conditions governing (3.50) can be written as follows:

$$C = 0 \quad ; \quad t = 0 \ ; \quad x \geq 0 \tag{3.51}$$

$$C = C_0 \ ; \quad t > 0 \ ; \quad x = 0 \tag{3.52}$$

In the instance where there is loss of the chemical concentration due to reactive processes we take the negative component of the right hand side of (3.50). Furthermore we introduce a change of variables such that

$$\eta = \frac{C}{C_0} \ ; \quad \theta = \xi t \ ; \quad \tau = \frac{x\xi}{\widehat{v}} \tag{3.53}$$

Consequently we can write

$$\frac{\partial \eta}{\partial \theta} + \frac{\partial \eta}{\partial \tau} = -\eta \tag{3.54}$$

with the appropriate boundary conditions being

$$\eta = 0 \quad ; \quad \theta = 0 \quad ; \quad \tau \geq 0 \tag{3.55}$$

$$\eta = 1 \quad ; \quad \theta > 0 \quad ; \quad \tau = 0 \tag{3.56}$$

The characteristic equations are

$$\frac{d\theta}{1} = \frac{d\tau}{1} = -\frac{d\eta}{\eta} \tag{3.57}$$

which gives

$$\frac{d\tau}{d\theta} = 1 \quad ; \quad \theta \geq 0 \quad ; \quad \tau \geq 0 \tag{3.58}$$

$$\frac{d\eta}{d\tau} = -\eta \quad ; \begin{array}{l} \eta = 0 \quad ; \quad \theta = 0 \quad ; \quad \tau \geq 0 \\ \eta = 1 \quad ; \quad \theta > 0 \quad ; \quad \tau = 0 \end{array} \tag{3.59}$$

These linear ordinary differential equations can be integrated to give the solution

$$\frac{C}{C_0} = 0 \quad ; \quad x > \widehat{v}t \tag{3.60}$$

and

$$\frac{C}{C_0} = e^{-x\xi/\widehat{v} \quad ; \quad x<\widehat{v}t} \tag{3.61}$$

In this simple example we have considered only "advective transport" processes of the species in the porous medium. There is a further mechanism called "diffusive transport" which accounts for transport by virtue of a gradient of the concentration of the species at a point. These processes can occur simultaneously and the degree to which one process is dominant will depend on the velocity of flow and the mechanical properties which govern the diffusion processes. These will be discussed in a later chapter.

### 3.3.2 Governing equation - generalized formulation

The formulation of the advective transport problem presented in the previous section is useful from the point of view of illustrating the distinct factors that should be taken into consideration in the development of the final governing equation in one-dimension. It is also possible to present a generalized formulation of the advective transport problem applicable to three dimensions.

The fundamental definitions required for the three dimensional formulation will now be briefly reviewed. Consider an arbitrary region $V$ of a porous medium with surface $S$. (Figure 3.7)

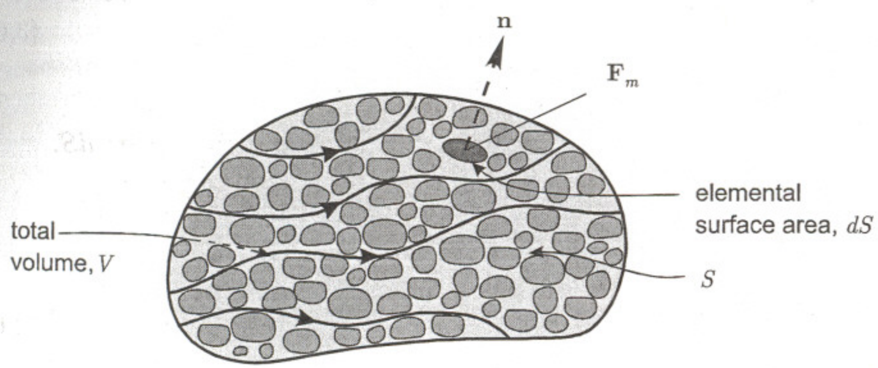


**Figure 3.7**: Advective transport in a porous medium.

The porosity measure $n^*$ defined by (3.42) and (3.44) is equally applicable to the three dimensional case. The concentration of the species per unit volume of the fluid is defined by $C(\mathbf{x}, t)$; indicating that the concentration is a function of the position vector $\mathbf{x}$. Similarly the concentration of the species per unit total volume of the porous medium is defined by $\widehat{C}(\mathbf{x}, t)$. Again at