1 Direct and exchange local terms

We define creation and annihilation operators $a^\dagger_{s,i}(\vec{n})$. This operator creates a particle on the lattice site $\vec{n}$ with spin s and isospin i. The particle density is $\hat{\rho}(\vec{n}) = a^\dagger_{s,i}(\vec{n}) a_{s,i}(\vec{n})$ where we sum over double indices. A most general two-body interaction on the lattice can be written with this definition as

$$\begin{aligned} V &= c \sum_{\vec{r}_{...}, I, S, s_{...}, i_{...}} : a^\dagger_{s_4,i_4}(\vec{r}_1) a_{s_2,i_2}(\vec{r}_2) a^\dagger_{s_3,i_3}(\vec{r}_3) a_{s_1,i_1}(\vec{r}_4) : \times \\ & \qquad V^{s_1,s_2,s_3,s_4,i_1,i_2,i_3,i_4}(\vec{r_1} - \vec{r_3}, \vec{r_2} - \vec{r_4}) \ . \end{aligned} \tag{1}$$

The direct and exchange local terms of this general interaction

$$\begin{aligned} V^{\text{direct}}(\vec{x}) &= \sum_{s_{13}, s_{24}, i_{13}, i_{24}} V^{s_{13},s_{24},s_{13},s_{24},i_{13},i_{24},i_{13},i_{24}}(\vec{x}, \vec{x}) \\ V^{\text{exchange}}(\vec{x}) &= - \sum_{s_{14}, s_{23}, i_{14}, i_{23}} V^{s_{14},s_{23},s_{14},s_{23},i_{14},i_{23},i_{14},i_{23}}(-\vec{x}, \vec{x}) \end{aligned}$$

are connected to the effective interaction between two α-clusters and therefore to the binding of heavier nuclei. We illustrate this connection for the case of the interactions from leading order $\not\pi$-EFT which are smeared. There are four possible point-like interactions in leading order $\not\pi$-EFT.

$$\begin{aligned} V_1 &= c_0 \sum_{\vec{n}, s_{...}, i_{...}} : a^\dagger_{s_2,i_2}(\vec{n}) a_{s_2,i_2}(\vec{n}) a^\dagger_{s_1,i_1}(\vec{n}) a_{s_1,i_1}(\vec{n}) : \ , \qquad (2) \\ V_2 &= c_i \sum_{\vec{n}, I, s_{...}, i_{...}} : a^\dagger_{s_2,i_4}(\vec{n}) \, [\tau]^I_{i_4,i_2} \, a_{s_2,i_2}(\vec{n}) a^\dagger_{s_1,i_3}(\vec{n}) \, [\tau]^I_{i_3,i_1} \, a_{s_1,i_1}(\vec{n}) : \ , \\ V_3 &= c_s \sum_{\vec{n}, S, s_{...}, i_{...}} : a^\dagger_{s_4,i_2}(\vec{n}) \, [\sigma]^S_{s_4,s_2} \, a_{s_2,i_2}(\vec{n}) a^\dagger_{s_3,i_1}(\vec{n}) \, [\sigma]^S_{s_3,s_1} \, a_{s_1,i_1}(\vec{n}) : \ , \\ V_4 &= c_{si} \sum_{\vec{n}, I, S, s_{...}, i_{...}} : a^\dagger_{s_4,i_4}(\vec{n}) \, [\sigma]^S_{s_4,s_2} \, [\tau]^I_{i_4,i_2} \, a_{s_2,i_2}(\vec{n}) a^\dagger_{s_3,i_3}(\vec{n}) \, [\sigma]^S_{s_3,s_1} \, [\tau]^I_{i_3,i_1} \, a_{s_1,i_1}(\vec{n}) : \ . \end{aligned}$$

Only two of them are linearly independent. This can be shown using Fierz transformation. The relevant result of the transformation which allows to proof this is

$$NN^\dagger = -\frac{1}{4} N^\dagger N - \frac{1}{4} N^\dagger \sigma^S N \sigma^S - \frac{1}{4} N^\dagger \tau^I N \tau^I - \frac{1}{4} N^\dagger \sigma^S \tau^I N \sigma^S \tau^I \tag{3}$$

where the nucleon field N is defined as $N^\dagger = (a^\dagger_{00}, a^\dagger_{01}, a^\dagger_{10}, a^\dagger_{11})$ and σ^S and τ^I the corresponding matrices in spin $\bigotimes$ isospin space. In order to give these point-like interactions some range we smear them out.

Let us introduce the smeared operators

$$\hat{a}^\dagger_{s,i}(\vec{n}) = a^\dagger_{s,i}(\vec{n}) + s\sum_{|\vec{l}|=1} a^\dagger_{s,i}(\vec{n}+\vec{l}) \ .$$

Thereby is s a numerical smearing parameter. With this definition some parts of the particle are on the neighboring sites of $\vec{n}$ and the particle has some spatial extension. We smear the interaction (2) in a *non-local* way by replacing the operators $a^\dagger_{s,i}(\vec{n})$ with the smeared ones. The smearing is non-local because then the smeared particle density $\hat{\rho}(\vec{n}) = \hat{a}^\dagger_{s,i}(\vec{n})\hat{a}_{s,i}(\vec{n})$ contains terms which allow the particle to hop, e.g. $a^\dagger_{s,i}(\vec{n})a_{s,i}(\vec{n}+\vec{1})$. A similar *local* smearing is defined by

$$\hat{\rho}(\vec{n}) = \rho(\vec{n}) + s'\sum_{|\vec{l}|=1} \rho(\vec{n}+\vec{l}) \ .$$

Here the particle has also some extension but there are no hopping terms. The effective range of the interaction is controlled by the smearing parameter s or respectively by s'. In order to write the smeared interaction in a simple way we define

$$f_{\vec{a}} = \begin{cases} 1, & \vec{a}=0 \\ s, & |\vec{a}|=1 \\ 0, & \text{else} \ . \end{cases}$$

Then smeared the contact interaction V_1 becomes

$$\begin{aligned} V_1^{\text{non-local}} &= c_0 \sum_{\vec{n},s_\ldots,i_\ldots} :\hat{a}^\dagger_{s_2,i_2}(\vec{n})\hat{a}_{s_2,i_2}(\vec{n})\hat{a}^\dagger_{s_1,i_1}(\vec{n})\hat{a}_{s_1,i_1}(\vec{n}): \\ &= c_0 \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d},s_\ldots,i_\ldots} f_{\vec{a}}f_{\vec{b}}f_{\vec{c}}f_{\vec{d}} : a^\dagger_{s_2,i_2}(\vec{n}+\vec{a})a_{s_2,i_2}(\vec{n}+\vec{b})a^\dagger_{s_1,i_1}(\vec{n}+\vec{c})a_{s_1,i_1}(\vec{n}+\vec{d}): \\ V_1^{\text{local}} &= c_{L0} \sum_{\vec{n},\vec{a},\vec{c},s_\ldots,i_\ldots} f_{\vec{a}}f_{\vec{c}} : a^\dagger_{s_2,i_2}(\vec{n}+\vec{a})a_{s_2,i_2}(\vec{n}+\vec{a})a^\dagger_{s_1,i_1}(\vec{n}+\vec{c})a_{s_1,i_1}(\vec{n}+\vec{c}): \ . \end{aligned}$$

The Fierz transformation (3) is unchanged if we allow the fields to be on different space points

$$\begin{aligned} N(\vec{x})N^\dagger(\vec{y}) &= -\frac{1}{4}N^\dagger(\vec{y})N(\vec{x}) - \frac{1}{4}N^\dagger(\vec{y})\sigma^S N(\vec{x})\sigma^S - \frac{1}{4}N^\dagger(\vec{y})\tau^I N(\vec{x})\tau^I \\ &\quad -\frac{1}{4}N^\dagger(\vec{y})\sigma^S\tau^I N(\vec{x})\sigma^S\tau^I \ . \end{aligned}$$

If the transformation is applied to the non-local interaction (dropping the detailed

notation with all the spin/isospin indices)

$$\begin{aligned} V_1^{\text{non-local}} &= c_0 \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} : N^\dagger(\vec{n}+\vec{a}) N(\vec{n}+\vec{b}) N^\dagger(\vec{n}+\vec{c}) N(\vec{n}+\vec{d}) : \\ &= -\frac{c_0}{4} \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} : N^\dagger(\vec{n}+\vec{a}) N(\vec{n}+\vec{c}) N^\dagger(\vec{n}+\vec{b}) N(\vec{n}+\vec{d}) + \dots \end{aligned}$$

we can see that the calculation will be the same as in the not smeared case since we are summing over all vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and can simply rename $\vec{b}$ to $\vec{c}$ and vice versa. However, for the local smeared interaction

$$\begin{aligned} V_1^{\text{local}} &= c_{L0} \sum_{\vec{n},\vec{a},\vec{c}} f_{\vec{a}} f_{\vec{c}} : N^\dagger(\vec{n}+\vec{a}) N(\vec{n}+\vec{a}) N^\dagger(\vec{n}+\vec{c}) N(\vec{n}+\vec{c}) : \\ &= -\frac{c_{L0}}{4} \sum_{\vec{n},\vec{a},\vec{c}} f_{\vec{a}} f_{\vec{c}} : N^\dagger(\vec{n}+\vec{a}) N(\vec{n}+\vec{c}) N^\dagger(\vec{n}+\vec{a}) N(\vec{n}+\vec{c}) + \dots \end{aligned}$$

we get a new structure by applying the Fierz transformation since we are not summing over all vector indices. Therefore we have two linearly independent non-locally smeared interactions while all four locally smeared interactions are linearly independent.

We can rewrite the interactions with Pauli matrices using the relation

$$\sum_I [\tau]^I_{i_1,i_2} [\tau]^I_{i_3,i_4} = 2\delta_{i_1,i_4}\delta_{i_2,i_3} - \delta_{i_1,i_2}\delta_{i_3,i_4} \tag{4}$$

for both τ and σ. Then e.g. the non-local smeared V_2 interaction is

$$\begin{aligned} V_2^{\text{non-local}} &= c_i \sum_{\vec{n},I} : \hat{a}^\dagger_{s_2,i_4}(\vec{n}) [\tau]^I_{i_4,i_2} \hat{a}_{s_2,i_2}(\vec{n}) \hat{a}^\dagger_{s_1,i_3}(\vec{n}) [\tau]^I_{i_3,i_1} \hat{a}_{s_1,i_1}(\vec{n}) : \\ &= 2c_i \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} : a^\dagger_{s_2,i_1}(\vec{n}+\vec{a}) a_{s_2,i_2}(\vec{n}+\vec{b}) a^\dagger_{s_1,i_2}(\vec{n}+\vec{c}) a_{s_1,i_1}(\vec{n}+\vec{d}) : \\ &\quad -c_i \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} : a^\dagger_{s_2,i_2}(\vec{n}+\vec{a}) a_{s_2,i_2}(\vec{n}+\vec{b}) a^\dagger_{s_1,i_1}(\vec{n}+\vec{c}) a_{s_1,i_1}(\vec{n}+\vec{d}) : . \end{aligned}$$

Thereby the second term is proportional to the contact interaction V_1 and the first term is an isospin exchange term. Similarly the other two interactions V_3 and V_4 will include terms where upon interaction the particles will exchange their spin for V_3 or their spin and isospin simultaneously for V_4. The corresponding terms from locally smeared interactions have $\vec{a} = \vec{b}$ and $\vec{c} = \vec{d}$.

$$\begin{aligned} \text{(contact)} &= : a^\dagger_{s_2,i_2}(\vec{n}+\vec{a}) a_{s_2,i_2}(\vec{n}+\vec{b}) a^\dagger_{s_1,i_1}(\vec{n}+\vec{c}) a_{s_1,i_1}(\vec{n}+\vec{d}) : \\ \text{(isospin exchange)} &= : a^\dagger_{s_2,i_1}(\vec{n}+\vec{a}) a_{s_2,i_2}(\vec{n}+\vec{b}) a^\dagger_{s_1,i_2}(\vec{n}+\vec{c}) a_{s_1,i_1}(\vec{n}+\vec{d}) : \\ \text{(spin exchange)} &= : a^\dagger_{s_1,i_2}(\vec{n}+\vec{a}) a_{s_2,i_2}(\vec{n}+\vec{b}) a^\dagger_{s_2,i_1}(\vec{n}+\vec{c}) a_{s_1,i_1}(\vec{n}+\vec{d}) : \\ \text{(spin+isospin exchange)} &= : a^\dagger_{s_1,i_1}(\vec{n}+\vec{a}) a_{s_2,i_2}(\vec{n}+\vec{b}) a^\dagger_{s_2,i_2}(\vec{n}+\vec{c}) a_{s_1,i_1}(\vec{n}+\vec{d}) : \end{aligned}$$

Now let us consider the matrix elements for a two α-particle state:

$$|a^\dagger_{00}(0)a^\dagger_{01}(0)a^\dagger_{10}(0)a^\dagger_{11}(0)a^\dagger_{00}(\vec{x})a^\dagger_{01}(\vec{x})a^\dagger_{10}(\vec{x})a^\dagger_{11}(\vec{x})\rangle \ .$$

The distance $\vec{x}$ is hereby the distance between the two α's and is assumed to be non-zero. The interactions V_i are two-body like and cannot move the α-particles but there can be some effective interaction between the α-particles due to the smearing of V_i. The direct interaction leaves all the particles at the same place as before the interaction while the exchange interaction swipes two constituent particles between the α's. The other 6 constituent particles are not interacting. We are not considering any self interaction between the α's, so there are always 16 possible contractions (choosing one particle from the first α and one particle from the second) which could be zero. For example all possible contractions are non-zero for the direct contact interaction. For the exchange interaction there are only 4 contraction. This happens because this interaction would exchange the place of two particles but if the particles do not have same spin and isospin the α state becomes zero due to Pauli exclusion. In the figure 1 we show this consideration for a spin-up proton for all terms and in table 1 summarize the number of possible contractions.

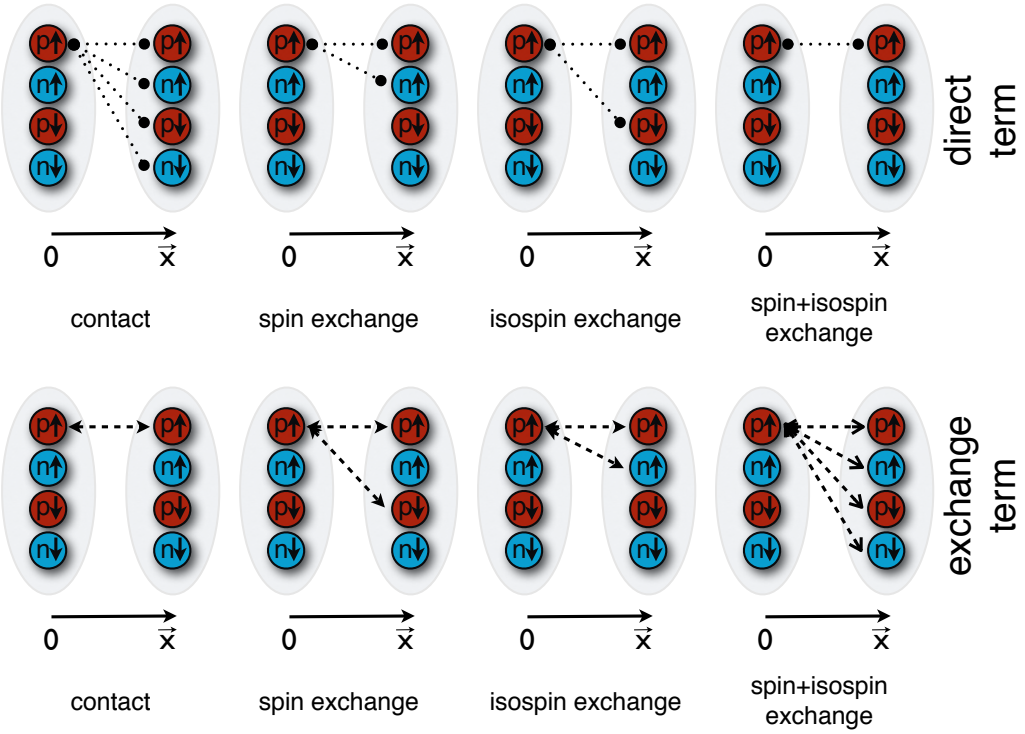

Figure 1: Illustration of possible interactions of a spin-up proton in the first α-particle with the constituents of the second α particle via different interaction terms.

Interaction	term	Possible non-zero contractions
direct	contact	16
direct	isospin exchange	8
direct	isospin + spin exchange	4
exchange	contact	4
exchange	isospin exchange	8
exchange	isospin + spin exchange	16

Table 1: The number of possible non-zero contractions due to spin/isospin structure.

Furthermore there are the following conditions from the space structure

$$\begin{aligned}
\text{direct, local smearing:} \quad & \vec{a}-\vec{c} \;=\; \vec{x} \;=\; \vec{a}-\vec{c} \\
\text{direct, non-local smearing:} \quad & \vec{b}-\vec{d} \;=\; \vec{x} \;=\; \vec{a}-\vec{c} \\
\text{exchange, local smearing:} \quad & \vec{a}-\vec{c} \;=\; \vec{x} \;=\; \vec{c}-\vec{a} \\
\text{exchange, non-local smearing:} \quad & \vec{b}-\vec{d} \;=\; \vec{x} \;=\; \vec{c}-\vec{a}\ .
\end{aligned}$$

Thereby $|\vec{a}| = 0, 1$ and the same is true for $\vec{b}, \vec{c}$ and $\vec{d}$. The above condition mean that the exchange term for the local smearing is always zero because $|\vec{x}| \neq 0$. The conditions for the direct and exchange term in the non-local smearing are the same with switched $\vec{a}$ and $\vec{c}$. The following table shows the possible non-zero solution for the direct, local smearing.

smearing	possible values for $\vec{a}, \vec{c}$		
	$\vec{x} = \vec{e}_x$	$\vec{x} = \vec{e}_x + \vec{e}_y$	$\vec{x} = 2\vec{e}_x$
direct, local	$\vec{a} = \vec{e}_x, \vec{c} = 0$ $\vec{a} = 0, \vec{c} = -\vec{e}_x$	$\vec{a} = \vec{e}_x, \vec{c} = -\vec{e}_y$ $\vec{a} = \vec{e}_y, \vec{c} = -\vec{e}_x$	$\vec{a} = \vec{e}_x, \vec{c} = -\vec{e}_x$

The solutions for the non-local smearing are similar. Since the solutions for $\vec{a}, \vec{c}$ are independent of $\vec{b}, \vec{d}$ the number of possible solutions is squared for the non-local smearing for each $\vec{x}$. For all $\vec{x}$ that are not listed in the table there are no solutions. The whole interactions V_1, V_2, V_3 and V_4 have the following structure

$$\begin{aligned}
V_1^{\text{non-local}} &= \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} \Big(c_0(\text{contact}) \Big) \\
V_2^{\text{non-local}} &= \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} \Big(2c_i(\text{isospin exchange}) - c_i(\text{contact}) \Big) \\
V_3^{\text{non-local}} &= \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} \Big(2c_s(\text{spin exchange}) - c_s(\text{contact}) \Big)
\end{aligned}$$

$$V_4^{\text{non-local}} = \sum_{\vec{n},\vec{a},\vec{b},\vec{c},\vec{d}} f_{\vec{a}} f_{\vec{b}} f_{\vec{c}} f_{\vec{d}} \Big(4c_{si}(\text{spin+isospin exchange}) + c_{si}(\text{contact}) \\ -2c_{si}(\text{isospin exchange}) - 2c_{si}(\text{spin exchange}) \Big) \ .$$

For the whole matrix element we have to combine the results from spin/isospin structure summarized in table 1 and from the space structure. Different solutions from the space structure combined with coefficients $f_{\vec{x}}$ result in different powers of the smearing factor s in the matrix elements. Thereby the exchange term for the local smearing is zero because of the space structure while V_2, V_3, V_4 matrix elements from the direct term are zero because of the spin/isospin structure. The minus sign in the exchange terms results from the antisymmetry by exchanging the place of two particles.

Term	$\lvert\vec{x}\rvert$	$V_1^{n.-l.}$	$V_2^{n.-l.}$	$V_3^{n.-l.}$	$V_4^{n.-l.}$	$V_1^{l.}$	$V_2^{l.}$	$V_3^{l.}$	$V_4^{l.}$
direct	1	$64s^2c_0$	0	0	0	$32s'c_{L0}$	0	0	0
	$\sqrt{2}$	$64s^4c_0$	0	0	0	$32s'^2c_{L0}$	0	0	0
	2	$16s^4c_0$	0	0	0	$16s'^2c_{L0}$	0	0	0
exch.	1	$-16s^2c_0$	$-48s^2c_i$	$-48s^2c_s$	$-144s^2c_{si}$	0	0	0	0
	$\sqrt{2}$	$-16s^4c_0$	$-48s^4c_i$	$-48s^4c_s$	$-144s^4c_{si}$	0	0	0	0
	2	$-4s^4c_0$	$-12s^4c_i$	$-12s^4c_s$	$-36s^4c_{si}$	0	0	0	0

Table 2: Direct and exchange term of smeared two-body $\not\pi$-EFT interactions with smearing parameter s between two α-cluster states with spatial separation $\vec{x}$.

In the Monte Carlo calculation we have chosen tree different interaction. All of them reproduce the phase shifts in the two-body sectors. However, one is a pure non-locally smeared interaction with a very small numerical parameter for the direct and exchange local terms. The second and the third interaction have both locally and non-locally smeared parts but the numerical value for the direct and exchange local terms are different.

interaction	c_{1S0}	c_{3S1}	$c_{L,1S0} = c_{L,3S1}$	s	s'
1	-0.231	-0.319	0	0.08	0.7
2	-0.077	-0.186	-0.046	0.08	0.7
3	-0.119	-0.223	-0.027	0.08	0.81

Table 3: Coefficients in lattice units for different interactions reproducing the same two-body phase shifts.

The relation between c_{1S0}, c_{3S1}, $c_{L,1S0}$ and the coefficients in the table 2 is

$$c_0 = \frac{3c_{1S0} + c_{3S1}}{4}, \qquad c_i = \frac{c_{1S0} - c_{3S1}}{4}, \qquad c_{L0} = \frac{3}{8} c_{L1S0} \ . \tag{5}$$

Now we calculate with the help of table 2 the numerical coefficients for the different interactions.

interaction	$\|x\|$	direct term, non-local smearing	exchange term, non-local smearing	direct term, local smearing	whole local term
1	1	-0.10360	0.01910	0	-0.08450
	$\sqrt{2}$	-0.00066	0.00012	0	-0.00054
	2	-0.00017	0.00003	0	-0.00014
2	1	-0.04288	0.00235	-0.38640	-0.42693
	$\sqrt{2}$	-0.00027	0.00002	-0.27048	-0.27074
	2	-0.00007	4×10^{-6}	-0.13524	-0.13531
3	1	-0.05929	0.00681	-0.26244	-0.31492
	$\sqrt{2}$	-0.00038	0.00004	-0.21258	-0.21291
	2	-0.00009	0.00001	-0.10629	-0.10637

Table 4: Direct and exchange terms in lattice units for different interactions reproducing the same two-body phase shifts.