11 Testing the code

To check the properties of operators and their effects in sign problem, let us compute the phase factor of $\det X(s)$ for various combination of interactions.

$$
\begin{aligned}
M^{(n_t)}_{LP,aux}(s, s_S, s_I, s_{S,I}, \pi'_I) \quad = \quad & : \exp\Bigg\{ -H_{free}\alpha_t - \frac{g_A \alpha_t}{2 f_\pi \sqrt{q_\pi}} \sum_{\boldsymbol{n},S,I} \Delta_S \pi'_I(\boldsymbol{n}, n_t) \rho_{S,I}(\boldsymbol{n}) \\
& + \sqrt{-C\alpha_t} \sum_{\boldsymbol{n}} s(\boldsymbol{n}, n_t)\rho(\boldsymbol{n}) + i\sqrt{C_{S^2}\alpha_t} \sum_{\boldsymbol{n},S} s_S(\boldsymbol{n}, n_t)\rho_S(\boldsymbol{n}) \\
& + i\sqrt{C_{I^2}\alpha_t} \sum_{\boldsymbol{n},I} s_I(\boldsymbol{n}, n_t)\rho_I(\boldsymbol{n}) + i\sqrt{C_{S^2,I^2}\alpha_t} \sum_{\boldsymbol{n},S,I} s_{SI}(\boldsymbol{n}, n_t)\rho_{SI}(\boldsymbol{n}) \Bigg\} :
\end{aligned} \tag{196}
$$

Also we add $SU(4)$ symmetric transfer matrix

$$
\begin{aligned}
M^{(n_t)}_4 \quad &= \quad : \exp[-H_4 \alpha_t] :, \quad H_4 = H_{free} + \frac{1}{2} C_4 \sum_{\boldsymbol{q}} f(\boldsymbol{q}) : \rho(\boldsymbol{q})\rho(-\boldsymbol{q}) :, \quad C_4 < 0, \\
M^{(n_t)}_{4,aux}(s) \quad &= \quad : \exp\left\{ -H_{free}\alpha_t + \sqrt{-C_4\alpha_t} \sum_{\boldsymbol{n}} s(\boldsymbol{n}, n_t)\rho(\boldsymbol{n}) \right\} :
\end{aligned} \tag{197}
$$

Though the interpolating Hamiltonian was defined as

$$
H = d_h H_{LO} + (1 - d_h) H_4, \tag{198}
$$

all following results are for $d_h = 1$.

- all calculations are done for ^4He.
- $L = 6$, $L_t = 2 * L_{t,out} + L_{t,in}$, $L_{t,out} = 10$, $L_{t,in} = 4$, $1/a = 100$ MeV, $1/a_t = 150$ MeV.
- Only some of interactions are turned on/off by changing overall coupling strength. However, not all combinations are considered.
- we set $d_h = 1.0$.
- Here, $SU(4)$ represents the $SU(4)$ symmetric transfer matrix acting as a filter at the beginning and ending time steps. This is different from the above C_4 in interpolating Hamiltonian.
- c_0 represents the coupling $C(d_h) = d_h C + (1 - d_h) C_4$. In similar way, c_S, c_I,c_{SI} and g_A corresponds to $C_{S^2}(d_h)$, $C_{I^2}(d_h)$, $C_{S^2,I^2}(d_h)$ and $g_A(d_h)$.

$$
\begin{aligned}
& C_0 = -0.192 \times 10^{-4}, C_S = 0.4 \times 10^{-5}, C_I = 0.87 \times 10^{-5}, \\
& C_{SI} = 0.64 \times 10^{-5}, C_4 = -0.7 \times 10^{-4}, \text{ in MeV}^{-2}, \\
& g_A = 1.29
\end{aligned} \tag{199}
$$

My opinions on results in Table.1 are

- Because there is a change of Hamiltonian between $L_{t,in}$ and $L_{t,out}$ and $L_{t,out}$ is rather large, the presence of $SU(4)$ interaction have large effects on the results. So, the comparison between the case with SU(4) interaction 'on' and 'off' may not be clear. Instead, probably, to study the effects of operator forms, wouln't it be better to set $L_{t,out} = 0$ and $L_{t,in}$ to be larger?

- Without SU(4) interaction, phase is usually small. However, because the SU(4) interaction does not make complex phase, I expected SU(4) may affect binding energy much but does not make much change on the phase factor. The difference of prior (to other interaction) SU(4) and post SU(4) may be the reason?
- Looking at the case with only one interaction or absence of one interaction, it seems to be the C_{SI} interaction have most important contribution to the phase.
- Looking at the case with only two interactions (and SU(4)) are on, the main origin of phase is from the interference between c_{SI} and g_A. When both c_{SI} and g_A are not on, phase are usually small.
- As we discussed before, the phase is very small(or zero) even in case three interactions(with SU(4)) are on except for c_{SI} interaction.
- The smallest value(or largest phase) occurs when only c_0 is off. It may be because c_0 interaction provides attractive $SU(4)$ interaction to the system to lessen the sign problem.

Table 1: Results of various couplings combinations.

index	$SU(4)$	c_0	c_S	c_I	c_{SI}	g_A	$Re\langle e^{i\theta}\rangle$	B.E.	raw amplitude
full	O	O	O	O	O	O	0.922	-27.3(3)	0.641(3)
1	O	X	X	X	X	X	1.0	28.2(2)	1.46(3)
13	O	X	O	X	X	X	1.0	25.8(2)	1.41
14	O	X	X	O	X	X	1.0	21.5(2)	1.33
15	O	X	X	X	O	X	0.992	13.1(1)	1.18
16	O	X	X	X	X	O	1.0	29.8(1)	1.48
2	O	O	X	X	X	X	1.0	13.5(3)	1.20(5)
3	O	O	O	X	X	X	1.0	7.74(2)	1.11(3)
4	O	O	X	O	X	X	1.0	-0.4(3)	0.999(5)
5	O	O	X	X	O	X	0.991	-14.5(3)	0.817(4)
6	O	O	X	X	X	O	1.0	18.1(2)	1.27
29	O	X	X	X	O	O	0.951	18.6(1)	1.22
30	O	X	O	O	X	X	1.0	19.2(1)	1.29
31	O	X	O	X	O	X	0.991	7.77(9)	1.09
32	O	X	O	X	X	O	1.0	28.1(1)	1.45
33	O	X	X	O	O	X	0.989	0.1(1)	0.99
34	O	X	X	O	X	O	1.0	24.8(1)	1.39
17	O	O	O	O	X	X	1.0	-4.5(3)	0.942(5)
18	O	O	O	X	O	X	0.991	-21.6(3)	0.742(3)
19	O	O	O	X	X	O	1.0	13.6(2)	1.19
20	O	O	X	O	O	X	0.990	-31.9(3)	0.647(3)
21	O	O	X	O	X	O	0.998(1)	8.5(6)	1.12(1)
22	O	O	X	X	O	O	0.945	-5.2(3)	0.881(3)
23	O	O	O	O	O	X	0.9799	-37.7(3)	0.592(3)
24	O	O	O	O	X	O	0.998(1)	3.3(7)	1.04(1)
25	O	O	O	X	O	O	0.942	-12.2(3)	0.801(4)
26	O	O	X	O	O	O	0.938	-21.5(3)	0.704(2)
27	O	X	O	O	O	O	0.917	3.6(1)	0.96
0	X	X	X	X	X	X	1.0	0.	1.0
7	X	O	X	X	X	X	1.0	-1.58(6)	0.979(1)
8	X	X	O	X	X	X	1.0	-0.339(9)	0.955(1)
11	X	X	X	O	X	X	1.0	-0.88(1)	0.988
12	X	X	X	X	O	X	0.999	-1.65(2)	0.977
9	X	X	X	X	X	O	1.0	-0.10(1)	0.998
28	X	X	X	X	O	O	0.997	-1.63(2)	0.975
10	X	O	O	O	O	O	0.995	-5.07(7)	0.929