Cluster(Halo) Effective Field Theory

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1 General

Review article of 1702.08605 and thesis by E. Ryberg. **Interpreted in my way. Thus it may be not correct.**

- Let us consider a system of core (denoted as c)and valence (denoted as v) particles. (Thus, mass, charge, momentum and so on of core are denoted as m_c, Z_c, p_c.. Similarly m_v, Z_v, p_v...). Also only consider core and valence are different particles.

- **Lagrangian of free core and valence field** is

$$\mathcal{L}_{cv}^{free} = c_s^\dagger \left(iD_0 + \frac{\boldsymbol{D}^2}{2m_c} \right) c_s + v_{s'}^\dagger \left(iD_0 + \frac{\boldsymbol{D}^2}{2m_v} \right) v_{s'} \tag{1}$$

  where s denotes possible spin quantum number of core and valence, with charge operator Q,$e > 0$, $D_\mu = \partial_\mu + eQA_\mu$,

$$\begin{aligned} D_0 &= \frac{\partial}{\partial t} + eQA_0, \\ \boldsymbol{D} &= (\nabla - eQ\boldsymbol{A}), \\ \boldsymbol{D}^2 &= \nabla^2 - 2eQ\boldsymbol{A}\cdot\nabla - eQ(\nabla\cdot\boldsymbol{A}) + e^2Q^2\boldsymbol{A}^2 \end{aligned} \tag{2}$$

  For typical low momentum scale $Q \sim M_{lo}$ in non-relativistic limit, we would have scaling dimension $[\partial_t] \sim Q^2$, $[\frac{p^2}{2m}] \sim Q^2$ (nucleon mass does not scale with Q). This implies particle field will scale as $[c] \sim [v] \sim Q^{3/2}$. And EM fields are counted as $A_0 \sim Q^2$ and $\boldsymbol{A} \sim Q$.

  In Coulomb gauge, and if consider only one-external photon, Feynman rule for charge and vector potential becomes

$$-ie\hat{Q}A_0 - i\frac{e\hat{Q}\boldsymbol{p}}{m}\cdot\boldsymbol{A} \tag{3}$$

- **Natural** case interaction Lagrangian:

$$\begin{aligned} \mathcal{L} &= -\frac{C_0^c}{4}c^\dagger cc^\dagger c - \frac{C_2^c}{4}(c^\dagger\nabla c)^2 + \cdots \\ &\quad -\frac{C_0^v}{4}v^\dagger vv^\dagger v - \frac{C_2^v}{4}(c^\dagger\nabla c)^2 + \cdots \\ &\quad -C_0c^\dagger cv^\dagger v - C_2(c^\dagger\nabla c)(v^\dagger\nabla v) + \cdots \end{aligned} \tag{4}$$

  But, because we only consider one core and one valence , first two lines are unnecessary. In natural case, which does not need fine-tuning of coupling, C_0 and C_2 will scale with M_{hi} as naive dimensional analysis,$C_0 \sim \frac{1}{mM_{hi}}$,$C_2 \sim \frac{1}{mM_{hi}^3}$. (Why mass m is required ? Argument in the Review paper) And they would be treated as perturbation. And consequently, $a_0 \sim r_0 \sim 1/M_{hi}$.

- **Unnatural** case : However, if scattering length is very large or there is a shallow bound state in S-wave, we have two scales $a \sim 1/M_{lo}$ and $r \sim 1/M_{hi}$. In this case, we need a fine-tuning of coupling strength such that $C_0 \sim \frac{1}{mM_{lo}}$ and should be included in the LO non-perturbatively.($C_2 \sim \frac{1}{mM_{lo}^2 M_{hi}}$ is a perturbative correction to LO.) [1] This resummation, sum of bubble diagrams, can be done efficiently by introducing di-cluster fields.

- For unnatural channels X, introducing dimer fields for those channels,[2].

$$\mathcal{L}_d = \sum_X d_X^\dagger \left[\Delta_X + \sum_{n=0}^{N} \nu_{n,X} \left(iD_0 + \frac{\boldsymbol{D}^2}{2M_{tot}} \right)^n \right] d_X \tag{5}$$

  which interacts with clusters as[3]

$$\mathcal{L}_{dcv} = \sum_X -g_X d_X^\dagger [ci\nabla v]_X + h.c. \tag{6}$$

  where $[ci\nabla v]_X$ is a particular combination of cluster fields having the same total angular momentum as di-cluster field. We don't know yet how to count the di-cluster field and its interaction. From the relation between diagrams in Lagrangian with/without dimer fields, we can expect that $C_0 \sim g_0^2/\Delta_0 \sim \frac{1}{mM_{lo}}$ for S-wave.

  Because we need the interactions at leading order ,Q^5, can we regard $d_{l=0} \sim Q^2$, $d_{l=1} \sim Q^1$? Not sure.

- At LO, we expect error to be order of k_{lo}/k_{hi}, and at NLO error of $(k_{lo}/k_{hi})^2$. Scaling dimension 5 determines LO. (according to Ryberg.) There are some arguments about power counting and scaling of each terms in Ryberg. However, it is difficult to follow why non-canonical dimension $[d_0] = 2$ and $[d_1] = 1$. According to this non-cannonical dimension, one can see that $L = 0$ case, Δ_0 term ($d_{l=0}\Delta_0 d_{l=0} \sim Q^4$) and g_0 term ($d_{l=0}cn \sim Q^5$) should be included in LO. But, ν_0 term($d_{l=0}\partial_t d_{l=0} \sim Q^6$) is NLO. On the other hand, P-wave case, g_1 term($d_1 c\nabla n \sim Q^5$), Δ_1 term($d_1 d_1 \sim Q^2$) and ν_1 term $d_1\partial_t d_1 \sim Q^4$ are treated as LO. Let us forget it for now.

For elastic scattering, at low energy, EFT calculation can be matched with effective range expansion,

$$k^{2l+1} \cot \delta_l \sim -\frac{1}{a_l} + \frac{1}{2} r_l k^2 - \frac{P_l}{4} k^4 + \cdots \tag{7}$$

In natural case, with only one scale M_{hi}, we expect the expansion can be treated as

$$k^{2l+1} \cot \delta_l \sim M_{hi}^{2l+1} + M_{hi}^{2l-1} k^2 + M_{hi}^{2l-3} k^4 + \cdots \tag{8}$$

So that each terms are correction to previous one by $(\frac{M_{lo}}{M_{hi}})^2$. However, in unnatural case, where new low energy scale M_{lo} appears, it can change. According to the following analysis of loop corrections, for renormalization, a is necessary for S-wave at LO, while a and r for P-wave and a, r, P for D-wave. In other words, we would count

$$\begin{aligned} k \cot \delta_0 &\sim M_{lo} + M_{hi}^{-1} k^2 + M_{hi}^{-3} k^4 + \ldots, \\ k^3 \cot \delta_1 &\sim M_{lo}^3 + M_{lo}^1 k^2 + M_{hi}^{-1} k^4 + \ldots, \\ k^5 \cot \delta_2 &\sim M_{lo}^5 + M_{lo}^3 k^2 + M_{lo}^1 k^4 + \ldots, \end{aligned} \tag{9}$$

This corresponds to a situation where scattering length is very large.

Normally in S-wave case, effective range correction would be $\frac{k_{lo}^2}{k_{hi}^2}$ order(NNLO). However, it is only $\frac{k_{lo}}{k_{hi}}$ (Thus, NLO) in unnatural case.

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[1] In some case, even effective range may be counted as $r_0 \sim 1/k_{lo}$, so that effective range part can be considered as Leading order. Then, the power counting also have to be changed.

[2] required power of n is determined by power counting.

[3] If we consider mass dimension(units), $[c] \sim [v] \sim m^{3/2}$ but also $[d] \sim m^{3/2}$. (if $[\nu] \sim m^0$. this implies $[\nu'] \neq m^0$) . For s-wave, Then, $[g_0] \sim m^{-1/2}$. Considering both mass dimension and scaling dimension, $g_0 \sim \frac{1}{\sqrt{k_{hi}}}$.

1.1 Partial wave combination of clusters

If core and valence have spin, clusters of s-wave can be written as

$$[ci\overleftrightarrow{\nabla}v]_j^{l=0} = \sum_{s_c,s_v} C^j_{s_c,s_v} c_{s_c} v_{s_v} \tag{10}$$

where $C^j_{s_c,s_v} = C^{j_d j}_{j_c s_c, j_v s_v}$ is a Clebsch-Gordan coefficient and angular momentum j_d, j_c, j_v are suppressed for convenience. In case of p-wave,

$$[ci\overleftrightarrow{\nabla}v]_j^{l=1} = \sum_{s_c,s_v,S} C^j_{kS} C^S_{s_c,s_v} c_{s_c} i\overleftrightarrow{\nabla}_k v_{s_v} \tag{11}$$

For example, for core spin $j_c = 0$ and valence spin $j_v = \frac{1}{2}$, we can write

$$\begin{aligned}\mathcal{L} \quad = \quad & -g_0 d_s^{(1/2)\dagger} c v_s + h.c. \\ & -g_1 d_s^{(1/2)\dagger} \sum C^{\frac{1}{2}s}_{1k\frac{1}{2}s''} C^{\frac{1}{2}s''}_{00,\frac{1}{2}s'} c(i\overleftrightarrow{\nabla}_k) n_{s'} + h.c. \\ & -g_1 d_s^{(3/2)\dagger} \sum C^{\frac{3}{2}s}_{1k\frac{1}{2}s''} C^{\frac{1}{2}s''}_{00,\frac{1}{2}s'} c(i\overleftrightarrow{\nabla}_k) n_{s'} + h.c.\end{aligned} \tag{12}$$

To have Galilean invariance, the ∇ have to correspond to relative momentum between clusters. This can be done by using definition

$$ci\overleftrightarrow{\nabla}v = c\left(\frac{m_c i\overrightarrow{\nabla} - m_v i\overleftarrow{\nabla}}{M_{tot}}\right)v \tag{13}$$

(relative momentum is defined as

$$\boldsymbol{p} = \frac{M_c \boldsymbol{p}_v - m_v \boldsymbol{p}_c}{M_{tot}} \tag{14}$$

so, in c.m. frame, $\boldsymbol{p}_v = \boldsymbol{p} = -\boldsymbol{p}_c$ and $c\overleftrightarrow{\nabla}v \to c(i\boldsymbol{p})v)$

In a similar way, we may introduce d-wave

$$[ci\overleftrightarrow{\nabla}v]_j^{l=2} = \sum_{s_c,s_v,LS} C^j_{LS} C^S_{s_c,s_v} C^L_{\alpha\beta} \times c_{s_c} \frac{1}{2}(i\overleftrightarrow{\nabla}_\alpha i\overleftrightarrow{\nabla}_\beta + i\overleftrightarrow{\nabla}_\beta i\overleftrightarrow{\nabla}_\alpha) v_{s_v} \tag{15}$$

Note that the components are written in spherical coordinates. The same expression may be written in Cartesian coordinates as

$$\frac{1}{2}\left(\overleftrightarrow{\nabla}_i \overleftrightarrow{\nabla}_j + \overleftrightarrow{\nabla}_j \overleftrightarrow{\nabla}_i\right) - \frac{1}{d-1}\overleftrightarrow{\nabla}^2 \delta_{ij}. \tag{16}$$

(Check that this gives only 5 independent combination as it should be. See Appendix for explicit proof)

Can we show that $\hat{L}^2[ci\nabla v]^l_j = l(l+1)[ci\nabla v]^l_j$? Can we agree that $[cv]$ is a $l=0$ wave? If it is true, $[c\nabla v]$ must be $l=1$ wave which will also have correct parity for p-wave. But, is it ?

Another point: According to argument around eq.(66) of Review paper of Hammer et.al. , if $j = 3/2$ channel does not have large scattering length, p-wave interactions are suppressed and one can treat it as perturbation.(no need of di-cluster field) What about $^{12}C + p$ or $^{12}C + n$?

2 Neutral case

From now on we consider systems of unnaturally large scattering length with neutral particles. (without Coulomb interaction.)

Consider spinless particles scattering first. ($j_c = 0$, $j_v = 0$ case. Non-zero spin case would be the same except the total angular momentum combination factors.)

single cluster propagator is

$$iS_y(p_0, \boldsymbol{p}) = \frac{i}{p_0 - \frac{\boldsymbol{p}^2}{2m_y} + i\epsilon}, \quad y = c, v. \tag{17}$$

bare di-cluster propagator is(there will be additional δ_{ij} for $l \neq 0$.)

$$iD_x^{(bare)}(E, \boldsymbol{P}) = \frac{i}{\Delta_x + \nu_x(E - \frac{P^2}{2M_{tot}} + i\epsilon)} \tag{18}$$

And from the self energy correction $-i\Sigma_x(E, \boldsymbol{P})$, full propagator becomes

$$\begin{aligned} iD_x(E, \boldsymbol{P}) &= iD^{(bare)} + iD^{(bare)}(-i\Sigma)(iD^{bare}) + \cdots \\ &= \frac{i}{\Delta + \nu(E - \frac{P^2}{2M_{tot}} + i\epsilon) - \Sigma(E, \mathrm{P})} \end{aligned} \tag{19}$$

Total momentum and relative momentum are

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{p}_c + \boldsymbol{p}_v, \quad \boldsymbol{p}_{rel} = \frac{m_c}{M_{tot}}\boldsymbol{p}_v - \frac{m_v}{M_{tot}}\boldsymbol{p}_c, \\ \boldsymbol{p}_v &= \frac{m_v}{M_{tot}}\boldsymbol{P}_{tot} + \boldsymbol{p}_{rel}, \quad \boldsymbol{p}_c = \frac{m_c}{M_{tot}}\boldsymbol{P}_{tot} - \boldsymbol{p}_{rel} \end{aligned} \tag{20}$$

From now on, we will only consider c.m. frame. $\boldsymbol{P} = 0$.

T-matrix may be expressed in partial waves(for spinless particles) ,[4]

$$T(k, \cos\theta) = \sum_{l \geq 0} T_l(k, \cos\theta) = -\frac{2\pi}{m_R} \sum_{l \geq 0} \frac{2l+1}{k \cot\delta_l - ik} P_l(\cos\theta). \tag{22}$$

where we may express the denominator using Effective Range Expansion(ERE),

$$k^{2l+1} \cot\delta_l = -\frac{1}{a_l} + \frac{1}{2} r_l k^2 - \frac{P_l}{4} k^4 + \cdots. \tag{23}$$

(Note that dimension of a_l, r_l, P_l depends on l.)

3 neutral s-wave (spinless) elastic scattering

Self energy[5] with total Momentum P,

$$\begin{aligned} -i\Sigma(E, P) &= (-ig)^2 \int \frac{d^4p}{(2\pi)^4} iS_c(p_0, \frac{m_c}{M_{tot}} P_{tot} + \boldsymbol{p}) iS_v(E - p_0, \frac{m_v}{M_{tot}} P_{tot} - \boldsymbol{p}) \\ &= -g^2 \int \frac{d^3p}{(2\pi)^3} iS_{tot}(E - \frac{P^2}{2M_{tot}}, \boldsymbol{p}), \quad iS_{tot}(p_0, \boldsymbol{p}) = \frac{i}{p_0 - \frac{\boldsymbol{p}^2}{2m_R} + i\epsilon}. \end{aligned} \tag{25}$$

[4] the sign of T-matrix can be dependent on the convention. But, it have to be related with scattering amplitude as

$$f(\theta) \sim \pm \frac{m_R}{2\pi} T(\theta) \tag{21}$$

Since the potential appears as $-iV$ for Feynman diagram, we may regard Feynman diagram corresponds to $-iT$.

[5]

$$\int \frac{dp_0}{2\pi} \frac{i}{p_0 - \frac{\boldsymbol{p}_c^2}{2m_c} + i\epsilon} \frac{i}{E - p_0 - \frac{\boldsymbol{p}_v^2}{2m_v} + i\epsilon} = \frac{(-2\pi i)}{2\pi} \frac{i^2}{E - \frac{\boldsymbol{p}_c^2}{2m_c} - \frac{\boldsymbol{p}_v^2}{2m_v} + i\epsilon}. \tag{24}$$

where $m_R = \frac{m_c m_v}{m_c+m_v}$ is a reduced mass of m_c and m_v.

In fact, in all diagram we can replace the two lines of core and valence particle $iS_c(p_0,p_c)iS_v(E-p_0,p_v)$ (with $\int d^4p$ integral) into $iS_{tot}(E_{rel},p_{rel})$ (with $\int d^3p$ integral) $E_{rel} = E - \frac{P^2}{2M_{tot}}$. In other words, we can treat the diagram as time-ordered fashion with only relative kinematics. I.e. we can treat it as a two-nucleon Green's function in time-ordered diagram (Right?).

$$iS_{tot}(E,\boldsymbol{p}) = \langle \boldsymbol{p}|iG_0(E)|\boldsymbol{p}\rangle \tag{26}$$

By using $k = \sqrt{2m_R E + i\epsilon}$, if $P = 0$

$$\begin{aligned}
-i\Sigma_0(E,0) &= (-ig^2)\int \frac{d^3p}{(2\pi)^3} S_{tot}(E,\boldsymbol{p}) = (-ig^2)\frac{(-2m_R)}{2\pi^2}\int_0^\infty dp \frac{p^2}{p^2-2m_RE-i\epsilon} \\
&= i\frac{(2g^2m_R)}{2\pi^2}\left(\int_0^\infty dp + 2m_RE\int_0^\infty dp\frac{1}{(p-k-i\epsilon)(p+k+i\epsilon)}\right) \\
&= i\frac{(2g^2m_R)}{2\pi^2}\left(L_1 + k^2\frac{1}{2}(2\pi i)\frac{1}{2p}|_{p=k+i\epsilon}\right) \\
&= i\frac{g^2m_R}{\pi^2}\left(L_1 + i\frac{\pi}{2}\sqrt{2m_RE+i\epsilon}\right)
\end{aligned} \tag{27}$$

where divergent part is written as

$$L_n = \int_0^\infty dp p^{n-1} \tag{28}$$

This can be expressed in various regularization scheme(Cutoff , dimensional, PDS). [6]

Thus,

$$\Sigma_0(E) = -\frac{g^2m_R}{2\pi}\left(\frac{2}{\pi}L_1 + i\sqrt{2m_RE+i\epsilon}\right) \tag{30}$$

If total momentum were non-zero, $k = \sqrt{2m_R(E-\frac{P^2}{2M_{tot}})+i\epsilon} = \sqrt{2m_RE_{rel}+i\epsilon}$,

$$\Sigma_0(E,P) = -\frac{g^2m_R}{2\pi}\left(\frac{2}{\pi}L_1 + i\sqrt{2m_R(E-\frac{P^2}{2M_{tot}})+i\epsilon}\right) \tag{31}$$

full propagator is

$$\begin{aligned}
iD(E,0) &= \frac{i}{\Delta+\nu(E+i\epsilon)+\frac{g^2m_R}{2\pi}(\frac{2}{\pi}L_1+ik)} \\
&= -\left(\frac{2\pi}{g^2m_R}\right)\frac{i}{-\left(\frac{2\pi\Delta}{g^2m_R}+\frac{2}{\pi}L_1\right)+\frac{1}{2}\left(-\frac{2\nu\pi}{g^2m_R^2}\right)k^2-ik}
\end{aligned} \tag{32}$$

And if total momentum is non-zero,

$$iD(E,P) = iD(E-\frac{P^2}{2M_{tot}},0) = iD(E_{rel},0) \tag{33}$$

[6] In fact, in dimensional and PDS, one have to take into account angular integral part together. For $D=4$, dimensional regularization gives integral

$$I = \left(\frac{\mu}{2}\right)^{4-D}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\frac{i}{E-\frac{p^2}{2\mu}+i\epsilon} = (\frac{\mu}{2})^{4-D}(-i)^{D-1}i(2\mu)(2\mu E)^{\frac{D-3}{2}}\frac{\pi^{\frac{D-1}{2}}}{(2\pi)^{D-1}}\Gamma(-\frac{D-3}{2}) \tag{29}$$

In other words, only relative energy matters. This would be necessary when computing electromagnetic process or reaction.

And scattering amplitude

$$
\begin{aligned}
-iT_0(E) &= (-ig)^2 iD(E,0) \\
&= \left(\frac{2\pi}{m_R}\right)\frac{i}{-\left(\frac{2\pi\Delta}{g^2 m_R}+\frac{2}{\pi}L_1\right)+\frac{1}{2}\left(-\frac{2\nu\pi}{g^2 m_R^2}\right)k^2 - ik}
\end{aligned} \tag{34}
$$

Compare with, in ERE,

$$
-iT_0(E) = i\frac{2\pi}{m_R}\frac{1}{k\cot\delta_0(E) - ik}, \tag{35}
$$

we get renormalization at LO,

$$
\begin{aligned}
\frac{1}{a_0} &= \frac{2\pi}{g^2 m_R}\Delta + \frac{2}{\pi}L_1, \\
r_0 &= -\frac{2\nu\pi}{g^2 m_R^2}.
\end{aligned} \tag{36}
$$

Note that ν determines the sign of effective range. Also if $\nu = 0$, we are in the limit of ignoring effective range correction. Thus, Δ_0 is at leading order of S-wave and ν is higher order.

After renormalization, we can write down,

$$
\begin{aligned}
iD_0(E,0) &= -\left(\frac{2\pi}{g^2 m_R}\right)\frac{i}{-\frac{1}{a_0}+\frac{1}{2}r_0 k^2 - ik}, \\
T_0(E) &= -\frac{2\pi}{m_R}\left(\frac{1}{-\frac{1}{a_0}+\frac{1}{2}r_0 k^2 - ik}\right)
\end{aligned} \tag{37}
$$

[7]

In term of usual partial-wave S-matrix, $S_0 = e^{2i\delta_0}$, we would have

$$
T_0(E) = -\frac{2\pi}{m_R}\frac{S_0 - 1}{2ik} \to S_0 = 1 - (2ik)\frac{m_R}{2\pi}T_0(E). \tag{39}
$$

(In other words, $T_0(E)$ is not the same as partial wave t-matrix.)

If we consider LO and ignore r_0, and we expand around pole position $k = i\gamma_B$(this is a binding momentum of s-wave bound state if $\gamma_B > 0$ or virtual state if $\gamma_B < 0$)[8] ,

$$
S_0(E) = 1 - (2ik)\frac{m_R}{2\pi}T_0(E) = \frac{-1/a + ik}{-1/a - ik} = -\frac{k + i\gamma_B}{k - i\gamma_B}, \quad E \to -B \tag{40}
$$

where $B = \frac{\gamma_B^2}{2m_R}$, $\gamma_B = \frac{1}{a_0}$. In fact, with NLO perturbative correction, S-matrix becomes

$$
S_0(E) = 1 - (2ik)\frac{m_R}{2\pi}T_0(E) = 1 + \frac{2ik}{-\frac{1}{a_0} - ik} - (2ik)\frac{1}{2}\frac{r_0 k^2}{(-1/a - ik)^2} + \ldots \tag{41}
$$

[7]This amplitude can be used for cross section calculation by using

$$
\begin{aligned}
f(\theta) &= \sum_l (2l+1) f_l(k) P_l(\cos\theta), \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2, \\
f_l(k) &= \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k} = \frac{1}{k\cot\delta_l - ik} \simeq \left(\frac{1}{-\frac{1}{a_0}+\frac{1}{2}r_0 k^2 - ik}\right)
\end{aligned} \tag{38}
$$

[8]large negative scattering length implies $\gamma_B < 0$ and thus virtual state.

Bound state wave function renormalization factor (for LSZ reduction) as

$$\mathcal{Z}_0 = \left[\frac{d}{dE}D_0(E)^{-1}\right]^{-1}_{E=-B} = \frac{2\pi\gamma_B}{m_R^2 g_0^2}\frac{1}{1-\gamma_B r_0} \tag{42}$$

where $\gamma_B = \sqrt{2m_R B}$. Note that bare constant factor canclels out in LSZ reduction. (In strict sense, perturbative corrections cannot change pole position. But, it looks like r_0 is treated non-perturbatively and pole is found as $-\frac{1}{a_0} - \frac{1}{2}r_0\gamma_B^2 + \gamma_B = 0$. I am not sure whether it is okay.) Note that $\mathcal{Z}_0$ can be related to the ANC of s-wave bound state[9]

$$\psi_0(\boldsymbol{r}) \quad = \quad A_B Y_{00}(\hat{r})\frac{\exp(-\gamma_B r)}{r}, \tag{49}$$

$$A_B = \sqrt{\frac{m_R^2 g_0^2}{\pi}\mathcal{Z}_0} = \sqrt{\frac{2\gamma_B}{1-\gamma_B r_0}} \tag{50}$$

Thus instead of fixing LECs from ERE parameters one can use ANC.

We have not considered particles having spins. But, for $l=0$, we only need to consider factors related with $C^{SS_z}_{j_c m_c j_v m_v}$. But, they are straightforward and simply gives Kronecker deltas $\delta^{S_z S_z'}$ for full propagator and scattering amplitude.

3.1 spin consideration

In case of non-zero spin, we have to take into account the C-G. for specific total angular momentum. For simplification let us consider scattering of nucleon over spin-zero core. Then, an amplitude $c + n_m \to c + n_{m'}$ if the spin and orbital angular momentum does not change during scattering ,

$$\begin{aligned} T(E,\theta)_{m,m'} &= \frac{2\pi}{m_R}\sum_{LSJM}\sqrt{4\pi}\sqrt{2L+1}C^{Sm}_{00\frac{1}{2}m}C^{Sm'}_{00\frac{1}{2}m'}C^{JM}_{L0,Sm}C^{JM}_{LM-m',Sm'}Y_{L,M-m}(\theta)\left(\frac{1}{k\cot\delta_{LSJ}-ik}\right) \\ &= \frac{2\pi}{m_R}\sum_{LJM}\sqrt{4\pi}\sqrt{2L+1}C^{Jm}_{L0,\frac{1}{2}m}C^{JM}_{Lm-m',\frac{1}{2}m'}Y_{L,m-m'}(\theta)\left(\frac{1}{k\cot\delta_{L\frac{1}{2}J}-ik}\right) \end{aligned} \tag{51}$$

[9] Following the Review article 1702.08605,

$$\frac{1}{E-H} = \frac{1}{E-H_0} + \frac{1}{E-H_0}t(E)\frac{1}{E-H_0}. \tag{43}$$

Near $E=-B$ bound state, left-hand side will be

$$\langle \boldsymbol{k}'|\frac{1}{E-H}|\boldsymbol{k}\rangle = \sum_n \langle \boldsymbol{k}'|\psi_n\rangle\langle\psi_n|\frac{1}{E-H}|\psi_n\rangle\langle\psi_n|\boldsymbol{k}\rangle = \frac{\psi_0(\boldsymbol{k}')\psi_0^*(\boldsymbol{k})}{E+B} + (\text{regular}) \tag{44}$$

with

$$\psi(\boldsymbol{k}) = \int d^3\boldsymbol{r} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\frac{A_B}{\sqrt{4\pi}}\frac{e^{-\gamma_B r}}{r} = \sqrt{4\pi}A_B\int_0^\infty dr r j_0(kr)e^{-\gamma_B r} = \frac{\sqrt{4\pi}A_B}{\gamma_B^2+k^2} \tag{45}$$

On right-hand side, any bound state pole is on the second piece. Because,

$$\langle \boldsymbol{k}'|t_0(E)|\boldsymbol{k}\rangle = \frac{2\pi}{m_R}\frac{1}{1/a_0 - \frac{1}{2}r_0k^2 + ik} \sim \frac{g^2\mathcal{Z}_0}{E+B} + (regular) \tag{46}$$

near the pole $E=-B$, righ-hand side becomes,

$$\sim \frac{1}{B+E_{k'}}\frac{g^2\mathcal{Z}_0}{E+B}\frac{1}{B+E_k} = \frac{2m_R}{\gamma_B^2+k^2}\frac{g^2\mathcal{Z}_0}{E+B}\frac{2m_R}{\gamma_B^2+k^2} \tag{47}$$

Thus, we get relation

$$4\pi A_B^2 = (2m_R)^2 g_0^2 \mathcal{Z}_0 \tag{48}$$

It may be strange that the proof involves integration over r, even though the wave function form is only valid for large separation. But, the integration only comes from the F.T. If we had started from position space, $\langle r'|G(E)|r\rangle$ at large r, the matching would not involve short distance part of wave function.

For $L=0$, only $m=m'$ is possible and the final expression is the same as spin-less case. Note that if $L\geq 1$, one have to take into account $m\neq m'$ (spin-flip).

4 neutral p-wave

Ref: C.A. Bertulani et al. Nucl.Phys.A712(2002)37-58. for $n\alpha$ scattering.

p-wave case is almost the same as s-wave except that the vertex have momentum factors.

For two-spinless particles, we can write

$$\mathcal{L} = g_1 d_i^\dagger (ci\overleftrightarrow{\nabla}_i v) + h.c. \tag{52}$$

where i are Cartesian components(instead of spherical component) for convenience.

Self energy diagram is

$$\begin{aligned}[-i\Sigma_1(E)]_{ij} &= \int \frac{d^4p}{(2\pi)^4}\left(ig_1\boldsymbol{p}_i\right)\left(ig_1\boldsymbol{p}_j\right) iS_c(p_0,\boldsymbol{p})iS_v(E-p_0,-\boldsymbol{p}) \\ &= -g_1^2\int\frac{d^3p}{(2\pi)^3}\boldsymbol{p}_i\boldsymbol{p}_j iS_{tot}(E,\boldsymbol{p})\end{aligned} \tag{53}$$

The index should be δ_{ij}, and we get [10]

$$\begin{aligned}(-i\Sigma_1(E))_{ij} &= -\frac{g_1^2}{3}\delta_{ij}\frac{1}{2\pi^2}\int dp p^4 iS_{tot}(E,\boldsymbol{p}) \\ &= i\frac{g_1^2 m_R}{6\pi}\delta_{ij}\left(\frac{2}{\pi}L_3+\frac{2}{\pi}L_1k^2+ik^3\right)\end{aligned} \tag{55}$$

And full propagator

$$(iD_1(E))_{ij} = \frac{i\delta_{ij}}{\Delta_1+\nu_1(E+i\epsilon)+\frac{g_1^2m_R}{6\pi}\left(\frac{2}{\pi}L_3+\frac{2}{\pi}L_1k^2+ik^3\right)} \tag{56}$$

Then, T-matrix becomes (spinless case has no external index for scattering.)

$$\begin{aligned}iT_1(E;\theta) &= (-ig_1\boldsymbol{k}_i)(-ig_1\boldsymbol{k}'_j)\frac{i\delta_{ij}}{\Delta_1+\nu_1(E+i\epsilon)+\frac{g_1^2m_R}{6\pi}\left(\frac{2}{\pi}L_3+\frac{2}{\pi}L_1k^2+ik^3\right)} \\ &= i\frac{6\pi}{m_R}\frac{k^2\cos\theta}{-\left(\frac{6\pi\Delta_1}{g_1^2m_R}+\frac{2}{\pi}L_3\right)-\left(\frac{3\pi\nu_1}{g_1^2m_R}+\frac{2}{\pi}L_1\right)k^2-ik^3}\end{aligned} \tag{57}$$

Comparison with

$$T_1(k) = \frac{6\pi}{m_R}\frac{k^2}{k^3\cot\delta_1(k)-ik^3} \tag{58}$$

$$\begin{aligned}\frac{1}{a_1} &= \frac{6\pi\Delta_1}{g_1^2m_R}+\frac{2}{\pi}L_3, \\ r_1 &= -\frac{6\pi\nu_1}{g_1^2m_R}-\frac{4}{\pi}L_1.\end{aligned} \tag{59}$$

---

[10]

$$\begin{aligned}\int dp\frac{p^4}{E-\frac{\boldsymbol{p}^2}{2m_R}+i\epsilon} &= (-2m_R)\int dp\frac{p^4}{p^2-k^2-i\epsilon} \\ &= (-2m_R)\int dp\frac{p^2(p^2-k^2)+k^2(p^2-k^2)+k^4}{p^2-k^2-i\epsilon} \\ &= (-2m_R)\left(L_3+k^2L_1+k^4\int_0^\infty dp\frac{1}{p^2-k^2-i\epsilon}\right) = (-2m_R)\left(L_3+k^2L_1+i\frac{\pi}{2}k^3\right)\end{aligned} \tag{54}$$

Then for wave function renormalization,

$$D_1(E) = \frac{6\pi}{g_1^2 m_R} \frac{1}{\frac{1}{a_1} - \frac{1}{2} r_1 k^2 + i k^3} \tag{60}$$

$$\begin{aligned} \mathcal{Z}_1 &= -\frac{6\pi}{m_R^2 g_1^2} \frac{1}{r_1 + 3\gamma_1} \\ &= \frac{1}{r_1 + 3\gamma_1} \left(\frac{r_1}{\nu_1} - \frac{4}{\pi} L_1 \right). \end{aligned} \tag{61}$$

For pole position, $k = \kappa$,

$$-\frac{1}{a_1} + \frac{r_1}{2}\kappa^2 - i\kappa^3 = 0. \tag{62}$$

When, $a_1, r_1 < 0$, poles are at (bound state and resonance)

$$\kappa_1 = i\gamma_1, \quad \kappa_\pm = i(\gamma \pm i\tilde{\gamma}), \tag{63}$$

where

$$\begin{aligned} \gamma_1 &= \frac{1}{6}(|r_1| + \frac{|a_1|^{1/3}|r_1|^2}{v} + \frac{v}{|a_1|^{1/3}}), \\ \gamma &= \frac{1}{6}(|r_1| - \frac{|a_1|^{1/3}|r_1|^2}{2v} - \frac{v}{2|a_1|^{1/3}}), \\ \tilde{\gamma} &= -\frac{\sqrt{3}}{12}(\frac{|a_1|^{1/3}|r_1|^2}{v} - \frac{v}{|a_1|^{1/3}}), \\ v &= (108 + |a_1||r_1|^3 + 108\sqrt{1 + |a_1||r_1|^3/54})^{1/3}. \end{aligned} \tag{64}$$

This gives

$$S_1 = e^{2i\delta_1} = -\frac{k+\kappa_1}{k-\kappa_1}\frac{k+\kappa_+}{k-\kappa_+}\frac{k+\kappa_-}{k-\kappa_-} = -\frac{k+i\gamma_1}{k-i\gamma_1}\frac{E-E_0-\frac{i}{2}\Gamma(E)}{E-E_0+\frac{i}{2}\Gamma(E)} \tag{65}$$

where

$$E = \frac{k^2}{2m_R}, \quad E_0 = \frac{\gamma^2 + \tilde{\gamma}^2}{2\mu}, \quad \Gamma(E) = -4\gamma\sqrt{\frac{E}{2\mu}}, \quad B = \frac{\gamma_1^2}{2m_R}. \tag{66}$$

and

$$\delta_1 = \frac{1}{2i}\ln S_1 = \delta_s(E) - \arctan(\frac{\Gamma(E)}{2(E-E_0)}), \quad \delta_s(E) = \frac{1}{2}\arctan(\frac{2\sqrt{EB}}{E-B}) \tag{67}$$

In other words, in p-wave, two LO parameters determine resonance position and width(and bound state).

4.1 spin consideration

Consider $n + \alpha$ scattering case.

Feynman rule of vertex are (Be careful to distinguish upper and lower index. See Appendix.)

$$\begin{aligned} c(\boldsymbol{p}) + v_{s'}(-\boldsymbol{p}) \to d_s(0) \quad &\Rightarrow \quad -ig_1 \sum_\alpha C^s_{1\alpha,\frac{1}{2}s'} \boldsymbol{p}_\alpha \\ d_s(0) \to c(\boldsymbol{p}) + v_{s'}(-\boldsymbol{p}) \quad &\Rightarrow \quad -ig_1 \sum_\alpha C^s_{1\alpha,\frac{1}{2}s'} \boldsymbol{p}^\alpha \end{aligned} \tag{68}$$

Self energy diagram for di-cluster with spin j,

$$
\begin{aligned}
(-i\Sigma_1(E))_{ss'} &= \int \frac{d^4p}{(2\pi)^4}\left(-ig_1 C^{js}_{1\alpha,\frac{1}{2}m}\boldsymbol{p}_\alpha\right)\left(-ig_1 C^{js'}_{1\beta,\frac{1}{2}m}\boldsymbol{p}^\beta\right) iS_c(p_0,\boldsymbol{p})iS_v(E-p_0,-\boldsymbol{p}) \\
&= -g_1^2 C^{js}_{1\alpha,\frac{1}{2}m}C^{js'}_{1\beta,\frac{1}{2}m}\int \frac{d^4p}{(2\pi)^4}\boldsymbol{p}_\alpha\boldsymbol{p}^\beta iS_c(p_0,\boldsymbol{p})iS_v(E-p_0,-\boldsymbol{p}) \\
&= -g_1^2 C^{js}_{1\alpha,\frac{1}{2}m}C^{js'}_{1\beta,\frac{1}{2}m}\int \frac{d^3p}{(2\pi)^3}\boldsymbol{p}_\alpha\boldsymbol{p}^\beta iS_{tot}(E,\boldsymbol{p}) \qquad (69)
\end{aligned}
$$

After angular integration, we would have $\boldsymbol{p}_\alpha\boldsymbol{p}^\beta \to \frac{1}{3}\delta_{\alpha\beta}p^2$(See Appendix.)[11] .

$$
\begin{aligned}
(-i\Sigma_1(E))_{ss'} &\to -\frac{1}{3}g_1^2 C^{js}_{1\alpha,\frac{1}{2}m}C^{js'}_{1\alpha,\frac{1}{2}m}\int \frac{d^3p}{(2\pi)^3}p^2 iS_{tot}(E,\boldsymbol{p}) \\
&= -\frac{1}{3}g_1^2\delta_{ss'}\int \frac{d^3p}{(2\pi)^3}p^2 iS_{tot}(E,\boldsymbol{p}) \qquad (71)
\end{aligned}
$$

Because self energy have to be diagonal, it is convenient to define

$$
\begin{aligned}
\Sigma_1(E) &= \frac{\delta_{mm'}}{2j+1}(\Sigma_1(E))_{mm'} = \frac{g_1^2}{6\pi^2}\int dpp^4 S_{tot}(E,p) \\
&= -\frac{g_1^2 m_R}{6\pi}\left(\frac{2}{\pi}L_3 + \frac{2}{\pi}L_1k^2 + ik^3\right). \qquad (72)
\end{aligned}
$$

The full propagator should also be diagonal,

$$
\begin{aligned}
iD_1(E) &= \frac{\delta_{mm'}}{2j+1}[iD_1(E)]_{mm'} \\
&= \frac{i}{\Delta_1 + \nu_1(E+i\epsilon) + \frac{g_1^2 m_R}{6\pi}\left(\frac{2}{\pi}L_3 + \frac{2}{\pi}L_1k^2 + ik^3\right)}. \qquad (73)
\end{aligned}
$$

So, the expression are the same for both j case.

However, scattering amplitude $(iT_1)_{mm'}$ does not have to be diagonal for p-wave! Where $m = \pm\frac{1}{2}$ is the spin projection of a nucleon.

On-shell scattering amplitude $|\boldsymbol{k}'| = |\boldsymbol{k}| = k$,

$$
[iT(E;\boldsymbol{k}',\boldsymbol{k})]_{mm'} = \sum_{jM\alpha\beta}\left(-ig_1 C^{jM}_{1\alpha,\frac{1}{2}m}\boldsymbol{k}'_\alpha\right)\left(-ig_1 C^{jM}_{1\beta,\frac{1}{2}m'}\boldsymbol{k}^\beta\right)(iD_1(E)) \qquad (74)
$$

$$
[T_{p_{3/2}}(E,\theta)]_{mm'} = [F_{p_{3/2}}(\theta) + \boldsymbol{\sigma}\cdot\hat{n}G_{p_{3/2}}(\theta)]_{mm'} \qquad (75)
$$

Results from C.A. Bertulani et al. Need explicit verification from above expression.

The differential cross section for spin 1/2 particle scattering is

$$
\frac{d\sigma}{d\Omega} = |F(k,\theta)|^2 + |G(k,\theta)|^2 \qquad (76)
$$

$$
T_{mm'} = \frac{2\pi}{m_R}(F + i\boldsymbol{\sigma}\cdot\hat{n}G)_{mm'} \qquad (77)
$$

[11] In fact, considering $\delta_{ss'}$, sum of C-G. coefficient implies

$$
\frac{\delta_{ss'}}{2j+1}\left(C^{js}_{1\alpha,\frac{1}{2}m}\right)\left(C^{js'}_{1\beta,\frac{1}{2}m}\right) = \frac{1}{2j+1}\frac{2j+1}{3}\delta_{\alpha\beta} \qquad (70)
$$

$$F(k,\theta) = \sum_{l\geq 0}[(l+1)f_{l+}(k) + lf_{l-}(k)]P_l(\cos\theta),$$
$$G(k,\theta) = \sum_{l\geq 1}[f_{l+}(k) - f_{l-}(k)]P_l^1(\cos\theta), \tag{78}$$

$$P_l^1(x) = (1-x^2)^{1/2}\frac{d}{dx}P_l(x) \tag{79}$$

where $\hat{n} = \boldsymbol{k}\times\boldsymbol{k}'/|\boldsymbol{k}\times\boldsymbol{k}'|$. ($f_{l+} = f_{l-}$ corresponds to spin-independent case. $l\pm$ means $j = l\pm 1/2$ channel.)

For $p_{3/2}$ channel, [12]

$$T^{LO}_{p_{3/2}}(k,\theta) = \frac{2\pi}{m_R}k^2(2\cos\theta + i\boldsymbol{\sigma}\cdot\hat{n}\sin\theta)\left[-\left(\frac{6\pi\Delta_1}{g_1^2 m_R} + \frac{2}{\pi}L_3\right) - \left(\frac{3\pi\nu_1}{g_1^2 m_R} + \frac{2}{\pi}L_1\right)k^2 - ik^3\right]^{-1} \tag{80}$$

Need a direct verification of relation between (51), eq. (74) and (80).

5 neutral d-wave

REF: J. Braun et al., 1803.02169

For self energy calculation, it is convenient to use Cartesian components of di-cluster d_{ij},

$$\mathcal{L} = -g_2 d^\dagger_{ij}\left[c(\frac{1}{2}(\nabla_i\nabla_j + \nabla_j\nabla_i) - \frac{1}{d-1}\nabla^2\delta_{ij})n\right] + h.c. \tag{81}$$

This gives Feynman rule, $c(p) + n(-p) \to d_{ij}$,

$$-ig_2(p_ip_j - \frac{1}{3}p^2\delta_{ij}) \tag{82}$$

Then, the self energy of di-cluster is

$$\begin{aligned}(-i\Sigma_2(E))_{ij,op} &= \int\frac{d^4p}{(2\pi)^4}(-ig_2(p_ip_j - \frac{1}{3}p^2\delta_{ij}))(-ig_2(p_op_p - \frac{1}{3}p^2\delta_{op}))iS_c(p_0,\boldsymbol{p})iS_v(E-p_0,-\boldsymbol{p}) \\ &= -g_2^2\int\frac{d^3p}{(2\pi)^3}(p_ip_jp_op_p - \frac{1}{3}p^2p_ip_j\delta_{op} - \frac{1}{3}\delta_{ij}p^2p_op_p + \frac{1}{9}p^4\delta_{ij}\delta_{op})iS_{tot}(E,\boldsymbol{p})\end{aligned} \tag{83}$$

We may expect/define,

$$(-i\Sigma_2(E))_{ij,op} = (-i\Sigma_2(E))\frac{\delta_{io}\delta_{jp} + \delta_{ip}\delta_{jo} - \frac{2}{3}\delta_{ij}\delta_{op}}{2} \tag{84}$$

$$\begin{aligned}-i\Sigma_2(E) &= \frac{1}{5}\frac{\delta_{io}\delta_{jp} + \delta_{ip}\delta_{jo} - \frac{2}{3}\delta_{ij}\delta_{op}}{2}(-i\Sigma_2(E))_{ij,op} \\ &= -g_2^2\frac{2}{15}\frac{1}{2\pi^2}\int dp p^6 iS_{tot}(E,\boldsymbol{p})\end{aligned} \tag{85}$$

The integral can be done as

$$\begin{aligned}\int dpp^6 iS_{tot}(E,\boldsymbol{p}) &= \int dp\frac{-i(2m_R)p^6}{p^2-k^2-i\epsilon} \\ &= -i(2m_R)\int dp\frac{p^4(p^2-k^2) + k^2p^2(p^2-k^2) + k^4(p^2-k^2) + k^6}{p^2-k^2-i\epsilon} \\ &= -i(2m_R)(L_5 + L_3k^2 + L_1k^4 + k^5\frac{\pi}{2})\end{aligned} \tag{86}$$

[12]Compared to spinless case factor $\frac{6\pi}{m_R}$, we may interpret the factor $\frac{2\pi}{m_R}2$ as $\frac{6\pi}{m_R}\frac{4}{4+2}$ for $p_{3/2}$. In the same way, we would have $\frac{6\pi}{m_R}\frac{2}{4+2}$ for $p_{1/2}$.

Thus, we get

$$
\begin{aligned}
-i\Sigma_2(E) &= -g_2^2 \frac{2}{15}\frac{1}{2\pi^2}(-i(2m_R))(L_5 + L_3 k^2 + L_1 k^4 + k^5 i\frac{\pi}{2}) \\
&= -i(-\frac{g_2^2 m_R}{15\pi})(\frac{2}{\pi}L_5 + \frac{2}{\pi}L_3 k^2 + \frac{2}{\pi}L_1 k^4 + ik^5)
\end{aligned} \tag{87}
$$

In the same way, we get the full propagator

$$
[iD_2(E)]_{ij,op} = iD_2(E)\frac{\delta_{io}\delta_{jp} + \delta_{ip}\delta_{jo} - \frac{2}{3}\delta_{ij}\delta_{op}}{2} \tag{88}
$$

$$
\begin{aligned}
iD_2(E) &= \frac{i}{\Delta_2 + \nu_1(E+i\epsilon) + \nu_2(E+i\epsilon)^2 - \Sigma_2(E)} \\
&= \frac{15\pi}{g_2^2 m_R} i \left[\left(\frac{15\pi}{g_2^2 m_R}\Delta_2 + \frac{2}{\pi}L_5 \right) + \left(\frac{15\pi}{g_2^2 2m_R^2}\nu_1 + \frac{2}{\pi}L_3 \right) k^2 + \left(\frac{15\pi}{g_2^2 4m_R^3}\nu_2 + \frac{2}{\pi}L_1 \right) k^4 + ik^5 \right]^{-1}
\end{aligned} \tag{89}
$$

We can see that ν_2 term is necessary for renormalization in D-wave.

For scattering of spinless fermions, $k = |k| = |k'|$

$$
\begin{aligned}
iT_2(E,\theta) &= (-ig_2(k'_i k'_j - \frac{1}{3}k'^2\delta_{ij}))(-ig_2(k_o k_p - \frac{1}{3}k^2\delta_{op}))[iD_2(E)]_{ij,op} \\
&= -i\frac{15\pi}{m_R}k^4(\cos^2\theta - \frac{1}{3}) \left[\left(\frac{15\pi}{g_2^2 m_R}\Delta_2 + \frac{2}{\pi}L_5 \right) + \left(\frac{15\pi}{g_2^2 2m_R^2}\nu_1 + \frac{2}{\pi}L_3 \right) k^2 + \left(\frac{15\pi}{g_2^2 4m_R^3}\nu_2 + \frac{2}{\pi}L_1 \right) k^4 + ik^5 \right]^{-1}
\end{aligned}
$$

Which is to be compared with

$$
T_2(E) = \frac{15\pi}{m_R}\frac{k^4(\cos^2\theta - \frac{1}{3})}{-1/a_2 + \frac{1}{2}r_2 k^2 - \frac{1}{4}P_2 k^4 - ik^5} \tag{90}
$$

This determines the renormalization of couplings in terms of ERE parameters. Near a bound state pole,

$$
\begin{aligned}
D_2(E) &= \frac{Z_2}{E+B} + (regular), \\
Z_2 &= -\frac{15\pi}{m_R^2 g_2^2}\frac{1}{r_2 + P_2\gamma_2^2 - 5\gamma_2^3}.
\end{aligned} \tag{91}
$$

Pole positions are given at $k = i\gamma$,

$$
\frac{1}{a_2} + \frac{1}{2}r_2\gamma^2 + \frac{1}{4}P_2\gamma^4 = 0. \tag{92}
$$

5.1 spin consideration

In fact, if we are only considering elastic scattering, the T-matrix will be matched to ERE and thus, we already know the answer for any L, S, J, **eq.** (51). (Only difference between ERE and EFT for elastic scattering is the consistent power counting.) We just need to replace the phase shift part using ERE(up to which order is decided by power counting.) for each (L, S, J) channels.

6 Charged case

For charged particles, we need to take into account long range Coulomb interaction. Coulomb introduce a new scale, $k_C = Z_c \alpha_{EM} m_R$ and can be characterized by Sommerfeld parameter, $\eta = k_C/p$.

Coulomb Green's function can be written in terms of momentum space Coulomb four-point function χ, with V_c Coulomb interaction,

$$
\begin{aligned}
G_C^{(+)}(E) &= \frac{1}{E + i\epsilon - H_c} = G_0^{(+)}(E) + G_0^{(+)}(E) V_c G_C^{(+)}(E), \\
\langle \boldsymbol{k} | G_C(E) | \boldsymbol{p} \rangle &= -G_0(E, \boldsymbol{k}) \chi(\boldsymbol{k}, \boldsymbol{p}; E) G_0(E, \boldsymbol{p}).
\end{aligned} \tag{93}
$$

In coordinate space, the Coulomb Green's function can be expressed in spectral representation as (Because of Coulomb interaction, one have to insert eigen-states of Coulomb Hamiltonian. Also, because Coulomb is repulsive here, we don't need discrete bound state contributions.)

$$
\begin{aligned}
\langle \boldsymbol{r} | G_C^{(+)}(E) | \boldsymbol{r}' \rangle_r &= \int \frac{d^3p}{(2\pi)^3} \langle \boldsymbol{r} | \psi_p^{(+)} \rangle \langle \psi_p^{(+)} | G_C^{(+)}(E) | \psi_p^{(+)} \rangle \langle \psi_p^{(+)} | \boldsymbol{r} \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{\psi_p^{(+)}(\boldsymbol{r}) \psi_p^{(+)*}(\boldsymbol{r}')}{E - \frac{\boldsymbol{p}^2}{2m_R} + i\epsilon}
\end{aligned} \tag{94}
$$

where, notation $|a\rangle_r$ indicate a coordinate space and $\psi_p^{(+)}(\boldsymbol{r})$ is a Coulomb wave function, with $\rho = pr$, $\sigma_l = \arg \Gamma(l + 1 + i\eta)$,

$$
\begin{aligned}
\psi_p^{(+)}(\boldsymbol{r}) &= e^{-\frac{1}{2}\pi\eta} \Gamma(1 + i\eta) M(-i\eta, 1, ipr - i\boldsymbol{p} \cdot \boldsymbol{r}) e^{i\boldsymbol{p} \cdot \boldsymbol{r}} \\
&= \sum_{l=0}^{\infty} (2l + 1) i^l e^{i\sigma_l} \frac{F_l(\eta, \rho)}{\rho} P_l(\cos\theta)
\end{aligned} \tag{95}
$$

where M is a Kummer function or confluent hypergeometric function. [13]

Alternative expression in terms of Whittaker M and W functions are

$$
\begin{aligned}
F_l(\eta, \rho) &= A_l(\eta) M_{i\eta, l+1/2}(2i\rho), \\
A_l(\eta) &= \frac{|\Gamma(l + 1 + i\eta)| \exp(-\pi\eta/2 - i(l + 1)\pi/2)}{2(2l + 1)!}.
\end{aligned} \tag{97}
$$

and

$$
\begin{aligned}
G_l(\eta, \rho) &= iF_l(\eta, \rho) + B_l(\eta) W_{i\eta, l+1/2}(2i\rho), \\
B_l(\eta) &= \frac{\exp(\pi\eta/2 + il\pi/2)}{\arg \Gamma(l + 1 + i\eta)}.
\end{aligned} \tag{98}
$$

Asymptotic form of Coulomb wave function is better represented by Coulomb Hankel function,

$$
H_L^{\pm}(\eta, \rho) = G_L(\eta, \rho) \pm iF_L(\eta, \rho) \tag{99}
$$

Which becomes for large ρ

$$
H_L^{\pm}(\eta, \rho) \sim e^{\pm i(\rho - L\pi/2 + \sigma_L(\eta) - \eta \ln(2\rho))} \tag{100}
$$

For bound state case, which have a pole of S-matrix at $k = ik_I$, $\eta = i\eta_I$, $\eta_I = -k_C/k_I$ asymptotic solution should be proportional to $H_L^{(+)}(i\eta_I, \rho)$. But $H_l^{+}(\eta, \rho)$ is not suitable for bound state, because

[13] $(\pm)$ of Coulomb wave function represent the boundary condition. Formally, it can be written as

$$
|\psi_p^{(\pm)}\rangle = (1 + G_C^{(\pm)} V_c)|p\rangle = G_C^{(\pm)} G_0^{-1} |p\rangle \tag{96}
$$

it is not well defined everywhere. Instead of H_l^+, Whittaker function is used. Asymptotic wave function with ANC,

$$\psi_B(\boldsymbol{r}) \sim A\frac{W_{-i\eta,l+1/2}(2\gamma r)}{r}[Y_l \otimes \chi_s]_{jm}, \quad r > R, \tag{101}$$

(Following expression from Thompson's book seems to be wrong.? In terms of Whittaker function is rescaled H_l^+.

$$\begin{aligned} W_{i\eta,l+1/2}(2i\rho) &= e^{-\pi\eta/2}e^{-i(\pi l/2+\sigma_L)}H_l^+(\eta,\rho) \to e^{i\rho-\pi\eta/2-i\eta\ln(2\rho)}? \\ W_{-\eta_I,l+1/2}(-2k_I r) &\to e^{-k_I r+\eta_I\ln(2k_I r)}? \end{aligned} \tag{102}$$

However, Above expression may have error. NIST says $W_{\kappa,\mu}(z) \sim e^{-\frac{1}{2}z}z^k$ as $z \to \infty$.)

The Gamow-Sommerfeld factor is given as[14]

$$C_0^2(\eta) = |\psi_p(0)|^2 = \frac{2\pi\eta}{e^{2\pi\eta}-1} = e^{-\pi\eta}\Gamma(1+i\eta)\Gamma(1-i\eta). \tag{103}$$

Later Coulomb-modified effective range expansion parameters are defined as

$$k^{2l+1}C_l(\eta)^2(\cot\delta_l - i) + 2k_C h_l(\eta) = -\frac{1}{a_l} + \frac{1}{2}r_l k^2 + \cdots \tag{104}$$

where[15]

$$\begin{aligned} C_l(\eta)^2 &= \exp(-\pi\eta)\Gamma(l+1+i\eta)\Gamma(l+1-i\eta), \\ h_l(\eta) &= p^{2l}\frac{C_l(\eta)^2}{C_0(\eta)^2}\left(\psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta)\right). \end{aligned} \tag{105}$$

For a check, in the limit of $\eta = k_C/p \to 0$,

$$\lim_{\eta\to 0} 2k_C h_0(\eta) = ip \tag{106}$$

(Also $\mathrm{Im}\psi(i\eta) = \frac{1}{2\eta} + \frac{\pi}{2}\coth\pi\eta$, $\mathrm{Im}h_0(\eta) = \frac{C_\eta^2}{2\eta}$.)

One can also obtain partial wave expansion of Coulomb Green's function,

$$\begin{aligned} \langle \boldsymbol{r}_1|G_C(E)|\boldsymbol{r}_2\rangle &= \sum_{l=0}^{\infty}(2l+1)G_C^{(l)}(E;r_1,r_2)P_l(\hat{r}_1\cdot\hat{r}_2), \\ G_C^{(l)}(E;r_1,r_2) &= \int\frac{d^3p}{(2\pi)^3}\frac{F_l(\eta,\rho_1)F_l(\eta,\rho_2)^*}{\rho_1\rho_2}\frac{1}{E-\frac{\boldsymbol{p}^2}{2m_R}} \end{aligned} \tag{107}$$

For bound state, to satisfy the boundary condition of $r \to 0$ and $r \to \infty$, one needs to use combination of two Coulomb functions. Thus, to have $e^{-\gamma r}$ behavior at large r, At the binding energy $\rho = i\gamma r$,

$$G_C^{(l)}(-B;\rho',\rho) = -\left[\frac{m_R p}{2\pi}\frac{F_l(\eta,\rho')[iF_l(\eta,\rho)+G_l(\eta,\rho)]}{\rho'\rho}\right]_{p=i\gamma} \tag{108}$$

(Does it mean always it must be $r' < r$?)

$$iF_l(\eta,\rho) + G_l(\eta,\rho) = \exp(i\sigma_l + \pi\eta/2 - il\pi/2)W_{-i\eta,l+1/2}(-2i\rho), \tag{109}$$

[14] From $M(a,b,0)=1$, we get $\psi_p(0) = e^{-\pi\eta/2}\Gamma(1+i\eta) = e^{-\pi\eta/2}|\Gamma(1+i\eta)|e^{i\sigma_0} = C_0(\eta)e^{i\sigma_0}$.

[15] $C_l(\eta)$ seems to be different from the definition given in the Thompson's book. Explicitly,

$$\begin{aligned} C_1^2(\eta) &= (1+\eta^2)C_0^2(\eta) \\ C_2^2(\eta) &= (4+\eta^2)(1+\eta^2)C_0^2(\eta) \end{aligned}$$

From,

$$\begin{aligned}
\lim_{\rho\to 0}\frac{F_0(\eta,\rho)}{\rho} &= \exp(-\pi\eta/2)\sqrt{(\gamma(1+i\eta))\Gamma(1-i\eta)},\\
\lim_{\rho\to 0}\frac{F_1(\eta,\rho)}{\rho^2} &= \frac{1}{3}\exp(-\pi\eta/2)\sqrt{(\gamma(2+i\eta))\Gamma(2-i\eta)},
\end{aligned} \tag{110}$$

We have

$$\begin{aligned}
G_C^{(0)}(-B;0,\rho) &= -\frac{m_R p}{2\pi}\Gamma(1+i\eta)\frac{W_{-i\eta,1/2}(-2i\rho)}{\rho}?\\
\lim_{\rho'\to 0}\frac{G_C^{(1)}(E;r',r)}{\rho'} &= i\frac{m_R p}{6\pi}\Gamma(2+i\eta)\frac{W_{-i\eta,3/2}(-2i\rho)}{\rho}.
\end{aligned} \tag{111}$$

on-shell S-matrix is defined as

$$S(\boldsymbol{p},\boldsymbol{p}') = \langle\Psi_{\boldsymbol{p}'}^{(-)}|\Psi_{\boldsymbol{p}}^{(+)}\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{p}) - 2\pi i\delta(E'-E)T(\boldsymbol{p}',\boldsymbol{p}) \tag{112}$$

where $|\Psi_{\boldsymbol{p}}^{(\pm)}\rangle$ is a full scattering solution with outgoing/incoming boundary condition. If we separate the Coulomb part and strong part, by using two-potential formula, we get

$$\begin{aligned}
T(\boldsymbol{p}',\boldsymbol{p}) &= T_C(\boldsymbol{p}',\boldsymbol{p}) + T_{CS}(\boldsymbol{p}',\boldsymbol{p}),\\
T_C(\boldsymbol{p}',\boldsymbol{p}) &= \langle\boldsymbol{p}'|V_C|\psi_{\boldsymbol{p}}^{(+)}\rangle,\\
T_{CS}(\boldsymbol{p}',\boldsymbol{p}) &= \langle\psi_{\boldsymbol{p}'}^{(-)}|V_S|\Psi_{\boldsymbol{p}}^{(+)}\rangle.
\end{aligned} \tag{113}$$

where $|\psi_{\boldsymbol{p}}^{(\pm)}\rangle$ is a solution of Coulomb Hamiltonian($|\boldsymbol{p}\rangle$ is a free solution.). Partial wave expansion is

$$T_C(\boldsymbol{p}',\boldsymbol{p}) = \langle\boldsymbol{p}'|V_c|\psi_{\boldsymbol{p}}^{(+)}\rangle = -\frac{2\pi}{m_R}\sum_{l=0}^{\infty}(2l+1)\left(\frac{e^{2i\sigma_l}-1}{2ip}\right)P_l(\cos\theta) \tag{114}$$

$$T_{SC}(\boldsymbol{p}',\boldsymbol{p}) = \langle\psi_{\boldsymbol{p}'}^{(-)}|V_S|\Psi_{\boldsymbol{p}}^{(+)}\rangle = -\frac{2\pi}{m_R}\sum_{l=0}^{\infty}(2l+1)e^{2i\sigma_l}\left(\frac{e^{2i\delta_l}-1}{2ip}\right)P_l(\cos\theta) \tag{115}$$

If leading order part of the potential should be treated non-perturbative, We need a full summation of diagrams

$$T_{CS}^{LO}(\boldsymbol{p}',\boldsymbol{p}) = \langle\psi_{\boldsymbol{p}'}^{(-)}|V_{LO} + V_{LO}G_CV_{LO} + V_{LO}(G_CV_{LO})^2 + \dots|\psi_{\boldsymbol{p}}^{(+)}\rangle \tag{116}$$

Note that the sum in the first line corresponds to the resummation of bubble diagrams in pionless EFT. ($V \to -iC_0$)

$$-iT_{CS}^{LO}(\boldsymbol{p}',\boldsymbol{p}) = \langle\psi_{\boldsymbol{p}'}^{(-)}| - iC_0(1 + iG_C(-iC_0) + (iG_C(-iC_0))^2\dots)|\psi_{\boldsymbol{p}}^{(+)}\rangle \tag{117}$$

The second line form corresponds to resummation of di-cluster propagator. $V \to (-ig_0)iD_0(-ig_0)$

$$\begin{aligned}
-iT_{CS}^{LO}(\boldsymbol{p}',\boldsymbol{p}) &= \langle\psi_{\boldsymbol{p}'}^{(-)}|(-ig_0)iD_0(-ig_0) + (-ig_0)iD_0(-i\Sigma_{CS})iD_0(-ig_0) + \dots|\psi_{\boldsymbol{p}}^{(+)}\rangle\\
&= \langle\psi_{\boldsymbol{p}'}^{(-)}|(-ig_0)\left(i\frac{D_0}{1-\Sigma_{CS}D_0}\right)(-ig_0)|\psi_{\boldsymbol{p}}^{(+)}\rangle\\
&= \langle\psi_{\boldsymbol{p}'}^{(-)}|(-ig_0)iD_{CS}(-ig_0)|\psi_{\boldsymbol{p}}^{(+)}\rangle
\end{aligned} \tag{118}$$

where Σ_{CS} have both strong and Coulomb interaction.

Higher order of potential V_S may be treated as a perturbative correction.

7 s-wave

The self energy of s-wave di-cluster is

$$
\begin{aligned}
-i\Sigma(E) &= (-ig)^2 \int \frac{d^3k d^3k'}{(2\pi)^6} \langle \boldsymbol{k}|iG_C(E)|\boldsymbol{k}'\rangle \\
&= -ig^2 \langle 0|G_C(E)|0\rangle_r
\end{aligned}
\tag{119}
$$

This can be evaluated in dimensional regularization ($d \to 4$) as

$$
\begin{aligned}
\Sigma(E) &= g^2(\frac{\mu}{2})^{4-d} \int \frac{d^{d-1}p}{(2\pi)^3} \frac{\psi_p(0)\psi_p^*(0)}{E - \frac{\boldsymbol{p}^2}{2m_R} + i\epsilon} \\
&= -2m_R g^2 (\frac{\mu}{2})^{4-d} \int \frac{d^{d-1}p}{(2\pi)^3} \frac{e^{-\pi\eta_p}\Gamma(1-i\eta_p)\Gamma(1+i\eta_p)}{p^2 - k^2 - i\epsilon} \\
&= -g^2 \frac{k_C m_R}{\pi} h_0(\eta) - \Sigma^{div},
\end{aligned}
\tag{120}
$$

where(See Appendix for detail.)

$$
\begin{aligned}
h_0(\eta) &= \psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta), \\
&\to \frac{1}{12\eta^2} + \frac{1}{120\eta^4} + \cdots + \frac{i\pi}{e^{2\pi\eta} - 1}, \quad \eta \gg 1.
\end{aligned}
\tag{121}
$$

where digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\eta = k_c/k$ with $E = k^2/(2m_R)$ and divergent part (in PDS) is , with Euler constant $C_E = 0.5772$,

$$
\Sigma^{div} = -\frac{g^2 k_C m_R}{\pi} \left[\frac{1}{3-d} + \log(\frac{\sqrt{\pi}\mu}{2k_C}) + 1 - \frac{3}{2} C_E \right] + \frac{g^2 m_R \mu}{2\pi}. \tag{122}
$$

Note that here divergent part contains two terms depends on scale μ.
The full propagator of di-cluster becomes,

$$
iD_0(E) = \frac{i}{\Delta_0 + \nu_0(E + i\epsilon) + \frac{g^2 k_C m_R}{\pi} h_0(\eta) + \Sigma^{div}} \tag{123}
$$

The s-wave scattering amplitude have both Coulomb scattering amplitude and Coulomb-modified strong scattering amplitude,

$$
iT_0(E; \boldsymbol{k}, \boldsymbol{k}') = iT_{C,0}(E) + iT_{CS,0}(E). \tag{124}
$$

(See appendix) The Coulomb scattering amplitude is well known. T_{CS} for s-wave can be obtained as

$$
iT_{CS,0}(E) = (-ig\psi_p^{(-)*}(0)) iD_0(E) (-ig\psi_p^{(+)}(0)) \tag{125}
$$

where ($\mp$) represent incoming/outgoing boundary condition. From

$$
\lim_{r\to 0} \psi_p^{(+)}(r) = e^{i\sigma_0(E)} C_0(\eta), \quad C_0(\eta) = \sqrt{\frac{2\pi\eta}{e^{2\pi\eta} - 1}}, \tag{126}
$$

We get

$$
\begin{aligned}
iT_{CS,0}(E) &= -ig^2 e^{2i\sigma_0(E)} C_0(\eta)^2 D_0(E) \\
&= -\frac{2\pi}{m_R} \frac{e^{2i\sigma_0} C_0(\eta)^2}{\frac{2\pi}{g^2 m_R}(\Delta_0 + \Sigma^{div}) + \frac{\pi\nu_0}{g^2 m_R^2} k^2 + 2k_C h_0(\eta)}
\end{aligned}
\tag{127}
$$

This would be matched with

$$T_{CS,0}(E) = \frac{2\pi}{m_R}\frac{e^{2i\sigma_0}}{k(\cot\delta_0 - i)} \simeq \frac{2\pi}{m_R}\frac{e^{2i\sigma_0}C_0(\eta)^2}{-\frac{1}{a_0}+\frac{1}{2}r_0k^2 - 2k_c h_0(\eta)} \tag{128}$$

Thus, renormalization is

$$\begin{aligned}\frac{1}{a_0} &= \frac{2\pi}{g^2 m_R}\left(\Delta_0 + \Sigma^{div}\right),\\ r_0 &= -\frac{2\pi\nu_0}{g^2 m_R^2}.\end{aligned} \tag{129}$$

The residue of full propagator at the bound state, $E = -B = -\frac{\gamma^2}{2m_R}$, defines (Need verification. This has opposite sign with the Ryberg thesis.)

$$\begin{aligned}D_0(E\to -B) &= \frac{\mathcal{Z}_0}{E+B} + (regular),\\ \mathcal{Z}_0 &= \frac{6\pi k_C}{g^2 m_R^2}\frac{-1}{\tilde{h_0}(\gamma,k_C) - 3k_C r_0},\\ \tilde{h_0}(\gamma,k_C) &= \frac{6k_C^2}{m_R}\frac{d}{dE}h_0(\eta)|_{E=-B}\\ &\to 1 - \frac{\gamma^2}{5k_C^2} + \frac{\gamma^4}{7k_C^4} + \cdots, \quad \gamma \ll k_C.\end{aligned} \tag{130}$$

This wave function renormalization is related with ANC of asymptotic wave function as $k = i\gamma_B$

$$u_l(r) = A_B W_{-i\eta,l+1/2}(2\gamma r), \tag{131}$$

$$\mathcal{Z} = \frac{-\pi}{g^2 m_R^2 |\Gamma(1+k_C/\gamma)|^2} A_B^2 \quad (verification?) \tag{132}$$

Compared to neutron halo case,

$$\mathcal{Z}_0 = \frac{2\pi\gamma}{g^2 m_R^2}\frac{1}{1-\gamma r_0}, \tag{133}$$

where low energy physics is governed by γ, proton halo seems to be governed by $3k_C$.

proof of above relation? As like the neutral case, the relation between ANC and wave function renormaliatoin factor may be related by using

$$G = G_0 + G_0(T_C + T_{CS})G_0 = G_C + G_0 T_{CS} G_0. \tag{134}$$

Since the bound state pole will comes from the second part, we may equate for $k^2 = |\boldsymbol{k}|^2 = |\boldsymbol{k}'|^2$,

$$\frac{\phi_B(\boldsymbol{k})\phi_B^*(\boldsymbol{k}')}{E+B} \sim \frac{-2\mu}{\gamma^2+k^2}\frac{(-g^2 e^{2i\sigma_0}C_0(\eta)^2)\mathcal{Z}}{E+B}\frac{-2\mu}{\gamma^2+k^2} \tag{135}$$

Thus, one needs

$$\begin{aligned}\phi_B(\boldsymbol{k}) &= \sqrt{4\pi}\int drr^2 j_0(kr) A_B \frac{W_{-\eta_I,1/2}(2\gamma r)}{r}\\ &\sim \frac{\sqrt{4\pi}A_B e^{i\sigma_0}C_0(\eta)}{k^2+\gamma^2}?\end{aligned} \tag{136}$$

[16]

[16]How to confirm these? If I use Asymptotic form of Whittaker function, the relevant integral is

$$\begin{aligned}&\int_0^\infty drj_0(kr)e^{-\gamma r}(2\gamma r)^{\eta_I} r\\ &= \frac{(2\gamma)^{\eta_I}\Gamma(1+\eta_I)}{2ik}\left((\gamma - ik)^{-1-\eta_I} - (\gamma+ik)^{-1-\eta_I}\right)\end{aligned} \tag{137}$$

8 p-wave(spinless)

By changing to p-wave vertex, self energy of p-wave di-cluster is (i, j are Cartesian index)

$$\begin{aligned}
[-i\Sigma(E)]_{ij} &= (-ig)^2 \int \frac{d^3k d^3k'}{(2\pi)^6} \boldsymbol{k}_i \boldsymbol{k'}_j \langle \boldsymbol{k} | iG_C(E) | \boldsymbol{k'} \rangle \\
&= -ig^2 \int \frac{d^3k d^3k'}{(2\pi)^6} \boldsymbol{k}_i \boldsymbol{k'}_j \int d^3r d^3r' e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \langle \boldsymbol{r} | G_C(E) | \boldsymbol{r'} \rangle e^{i\boldsymbol{k'}\cdot\boldsymbol{r'}} \\
&= -ig^2 \int \frac{d^3p}{(2\pi)^3} \frac{\left(\int \frac{d^3k}{(2\pi)^3} \psi^{(+)}_{\boldsymbol{p}}(\boldsymbol{k}) \boldsymbol{k}_i\right) \left(\int \frac{d^3k}{(2\pi)^3} \psi^{(+)}_{\boldsymbol{p}}(\boldsymbol{k}) \boldsymbol{k}_j\right)^*}{E - \frac{p^2}{2m_R} + i\epsilon} \qquad (138)
\end{aligned}$$

From Ryberg,

$$\boldsymbol{X}(E_p) = \int \frac{d^3k}{(2\pi)^3} \boldsymbol{k} \psi_{\boldsymbol{p}}(\boldsymbol{k}) = -\boldsymbol{p} e^{i\sigma_1(E_p)} C_1(\eta_p) \qquad (139)$$

and replace $\boldsymbol{p}_i \boldsymbol{p}_j \to \frac{1}{3} p^{\delta}_{ij}$ in the integral,

$$\begin{aligned}
-i\Sigma(E_k) &= -ig^2 \frac{1}{3} \int \frac{d^3p}{(2\pi)^3} \frac{p^2 C_1(\eta_p)^2}{E - \frac{p^2}{2m_R} + i\epsilon} \\
&= -ig^2 \frac{2m_R}{6\pi^2} \int dp \frac{p^4 C_1(\eta_p)^2}{k^2 - p^2 + i\epsilon} \\
&= -ig^2 \frac{m_R}{3\pi^2} \int dp \frac{p^4 (\eta_p^2 + 1) C_0(\eta_p)^2}{k^2 - p^2 + i\epsilon} \\
&= -ig^2 \frac{m_R}{3\pi^2} \int dp C_0(\eta_p)^2 \frac{p^2(p^2 - k^2) + (k^2 + k_C^2)(p^2 - k^2) + k^2(k^2 + k_C^2)}{k^2 - p^2 + i\epsilon} \qquad (140)
\end{aligned}$$

where we used $C_1(\eta)^2 = (\eta^2 + 1) C_0(\eta)^2$. If we introduce divergent intgeral

$$L_n = \int dp C_0(\eta)^2 p^{n-1} \qquad (141)$$

$$-i\Sigma(E_k) = -ig^2 \frac{m_R}{3\pi^2} \left[-L_3 - (k^2 + k_C^2) L_1 + k^2 (k^2 + k_C^2) \int dp \frac{C_0(\eta_p)^2}{k^2 - p^2 + i\epsilon} \right] \qquad (142)$$

The last term is finite,

$$J^{fin}(k) = -\frac{m_R}{3\pi^2} \int dp \frac{C_0(\eta_p)^2}{k^2 - p^2 + i\epsilon} = \frac{k_C m_R}{3\pi k^2} h_0(\eta_k) \qquad (143)$$

Thus,

$$\begin{aligned}
\Sigma(E) &= -\frac{g^2 m_R}{3\pi^2} \left(-L_3 - (k^2 + k_C^2) L_1 - (k^2 + k_C^2) k_C \pi h_0(\eta) \right) \\
&= -\frac{g^2 m_R}{6\pi} \left(-\frac{2}{\pi} L_3 - \frac{2}{\pi} (k^2 + k_C^2) L_1 - 2k_C h_1(\eta) \right) \qquad (144)
\end{aligned}$$

Full propagator

$$\begin{aligned}
iD(E) &= \frac{\delta_{ij}}{3} [iD(E)]_{ij} \\
&= \frac{i}{\Delta_1 + \nu_1(E + i\epsilon) + \frac{g^2 m_R}{6\pi} \left(-\frac{2}{\pi} L_3 - \frac{2}{\pi}(k^2 + k_C^2) L_1 - 2k_C h_1(\eta) \right)} \\
&= \frac{6\pi}{g^2 m_R} \frac{i}{\left(\frac{6\pi}{g^2 m_R} \Delta_1 - \frac{2}{\pi} L_3 - \frac{2}{\pi} k_C^2 L_1 \right) + \left(\frac{6\pi}{g^2 m_R} \nu_1 - \frac{2}{\pi} L_1 \right) k^2 - 2k_C h_1(\eta)} \qquad (145)
\end{aligned}$$

Scattering amplitude(sign check required)

$$\begin{aligned} iT_1(E) &= ig^2 D(E)(\boldsymbol{X}^{(+)})^* \cdot \boldsymbol{X}^{(-)}(E) \\ &= ig^2 D(E) e^{2i\sigma_1} k^2 C_1(\eta)^2 \end{aligned} \tag{146}$$

This to be compared with

$$T_1(E) = \frac{6\pi}{m_R} \frac{k^2 C_1(\eta)^2 e^{2i\sigma_1}}{-\frac{1}{a_1} + \frac{1}{2} r_1 k^2 - 2k_C h_1(\eta)} \tag{147}$$

All the rest would follow the similar procedure as previous section. And most of relevant expressions can be found in the Ryberg's papers.

9 d-wave

For elastic scattering, we already know what it should look like. This part should be done for renormalization.

$$(-i\Sigma_2(E))_{ij,op} = \int \frac{d^3p d^3p'}{(2\pi)^6} (-ig_2(p_i p_j - \frac{1}{3}p^2\delta_{ij}))(-ig_2(p'_o p'_p - \frac{1}{3}p'^2\delta_{op}))\langle \boldsymbol{p}|iG_C(E)|\boldsymbol{p}'\rangle \tag{148}$$

$$\boldsymbol{X}_{ij}(E_p) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} (\boldsymbol{k}_i \boldsymbol{k}_j - \frac{1}{3}\boldsymbol{k}^2\delta_{ij})\psi_{\boldsymbol{p}}(\boldsymbol{k}) = \tag{149}$$

10 spin consideration

In fact, for elastic scattering, as long as we use ERE expansion parameters and no tensor interactions, the result is already known,

$$T_{CS}(E,\theta)_{m,m'} = \frac{2\pi}{m_R} \sum_{LJM} \sqrt{4\pi}\sqrt{2L+1} C^{Jm}_{L0,\frac{1}{2}m} C^{JM}_{Lm-m',\frac{1}{2}m'} Y_{L,m-m'}(\theta) \left(\frac{e^{2i\sigma_L}}{k\cot\delta_{L\frac{1}{2}J} - ik} \right) \tag{150}$$

To use the Coulomb modified ERE, we can express

$$\begin{aligned} \frac{1}{k\cot\delta_{L\frac{1}{2}J} - ik} &= \frac{k^{2l}C_l(\eta)^2}{k^{2l+1}C_l(\eta)^2(\cot\delta_{L\frac{1}{2}J} - i) + 2k_c h_l(\eta) - 2k_c h_l(\eta)}, \\ &= \frac{k^{2l}C_l(\eta)^2}{-\frac{1}{a_l^{(j)}} + \frac{1}{2}r_l^{(j)}k^2 + \cdots - 2k_c h_l(\eta)} \end{aligned} \tag{151}$$

Note that up to which order to include non-perturbatively depends on the power counting and renormalization scheme of EFT.

The differential cross section becomes

$$(\frac{d\sigma}{d\Omega})_{mm'} = |f_c(\theta)\delta_{mm'} + \frac{m_R}{2\pi}[T_{CS}(\theta)]_{mm'}|^2 \tag{152}$$

11 E.M. Form factor

We can consider electromagnetic form factor of bound state.

In addition to the minimal substitution in iD_μ, we have EM correction Lagrangian

$$\begin{aligned}\mathcal{L}_{EM} &= -n_s^\dagger \frac{e\rho_n^2}{6}[\nabla^2 A_0 - \partial_0(\nabla\cdot\boldsymbol{A})]n_s - c^\dagger \frac{e\rho_c^2}{6}[\nabla^2 A_0 - \partial_0(\nabla\cdot\boldsymbol{A})]c \\ &\quad -\sigma_s^\dagger \frac{\nu_0 Z_{tot} e\rho_\sigma^2}{6}[\nabla^2 A_0 - \partial_0(\nabla\cdot\boldsymbol{A})]\sigma_s - \pi_s^\dagger \frac{\nu_1 Z_{tot} e\rho_\sigma^2}{6}[\nabla^2 A_0 - \partial_0(\nabla\cdot\boldsymbol{A})]\pi_s + \ldots\end{aligned} \tag{153}$$

For Charge Form factor,

$$F_C(\boldsymbol{Q}^2) = \frac{1}{eZ_{tot}}\langle \boldsymbol{k}'|J^0_{EM}|\boldsymbol{k}\rangle, \tag{154}$$

where $\boldsymbol{Q} = \boldsymbol{k}' - \boldsymbol{k}$ and $\boldsymbol{k}$ is a momentum of C.M of di-cluster. The Leading term comes from $d^\dagger iD_0 d$ term of Lagrangian. Which gives 1 for $F_C(Q)$. The next correction comes from point nucleon/core term $n^\dagger iD_0 n$ of loop diagram. Another correction comes from ρ_n, ρ_c terms in the same loop diagram and also from ρ_d term. These corrections corresponds to change charge to $\sim e \to e(1 - r_c h^2 Q^2/6)$.

For S-wave, we get charge radius corrections as

$$r_{ch}^2 = \frac{1}{1-\gamma_0 r_0}(r_{pt,LO}^2 + \rho_c^2 + \frac{1}{Z_c}\rho_n^2 - \gamma_0 r_0 \rho_\sigma^2) \tag{155}$$

여기서, $\frac{1}{1-\gamma_0 r_0}$ 는 wave function renormalization factor $\mathcal{Z}_0$ 로 부터 온다. 이 때, $1/g_0^2$ of $\mathcal{Z}_0$ 는 loop diagram의 vertex g_0^2 와 cancel되거나, di-cluster Lagrangian의 ν_0 term과 만나 (ν_0/g_0^2) effective range 로 바뀐다. $r_{pt,LO}^2$ 는 loop diagram으로 부터 오고 나머지는 $\mathcal{L}_{EM}$ correction. 여기서,

$$r_{pt,LO}^2 = \frac{f^2}{2\gamma_0^2}. \tag{156}$$

따라서, if we assume $1/r_{pt,LO} \sim \gamma_0 \sim k_{lo}$ and $1/\rho_c \sim 1/\rho_\sigma \sim 1/r_0 \sim k_{hi}$, it can be expanded as

$$r_{ch}^2 = \underbrace{r_{pt,LO}^2}_{k_{lo}^{-2}} + \underbrace{\gamma_0 r_0 r_{pt,LO}^2}_{k_{lo}^{-2}(k_{lo}/k_{hi})} + \underbrace{(\gamma_0 r_0)^2 r_{pt,LO}^2 + \rho_c^2}_{k_{lo}^{-2}(k_{lo}/k_{hi})^2} + \underbrace{\gamma_0 r_0(\rho_c^2 - \rho_\sigma^2)}_{k_{lo}^{-2}(k_{lo}/k_{hi})^3} + \ldots \tag{157}$$

We can determine ρ_c for core-particle. However, ρ_σ is undetermined. Thus, the theory is unpredictive at N^3LO.

For P-wave,

$$\begin{aligned} r_{ch}^2 &= r_{pt}^2 + \tilde{\rho}_\pi^2 + \ldots, \\ r_{pt}^2 &= -\frac{5f^2}{2\gamma_1(3\gamma_1 + r_1)}, \quad \text{for P-wave,} \\ \tilde{\rho}_\pi^2 &= -\nu_1 \mathcal{Z}_1 \rho_\pi^2. \end{aligned} \tag{158}$$

Because $\mathcal{Z}_1$ contains divergence, it have to be absorbed to ρ_π and $\tilde{\rho}_\pi^2$ is finite. In fact, charge radius correction of core also requires the inclusion of ρ_π^2 term. But, problem is that it means that one cannot make prediction even at NLO for P-wave.

In case of proton-Halo, the loop diagram have Coulomb Green's function. And (without derivation)

$$\begin{aligned} r_{ch}^2 &= -\frac{3}{e(Z_c+1)}\mathcal{Z}\Gamma''_{loop,full}(0) \\ &= r_{pt}^2 + \frac{1}{1-3k_C r_0}\left(\frac{Z_c}{Z_c+1}\rho_c^2 + \frac{1}{Z_c+1}\rho_p^2 - 3k_c r_0 \rho_d^2\right), \\ r_{pt}^2 &= -\frac{3}{e(Z_c+1)}\mathcal{Z}\Gamma''_{loop}(0) \end{aligned} \tag{159}$$

Thus proton Halo is only predictive to NLO. (ρ_d is not known.)

To be more specific, E.M. Charge current can be computed from Feynman diagrams. First consider neutron-core case. In this case, we can think the energy and momentum of dicluster before EM, $(E, +\frac{Q}{2})$ and after EM $(E, -\frac{Q}{2})$.(Energy transfer of photon is zero in Berit frame?) Within the loop, if we set momentum of neutron as $\boldsymbol{p}_n$, the momentum of core changes from $-\boldsymbol{p}_n + \frac{Q}{2}$ to $-\boldsymbol{p}_n - \frac{Q}{2}$. In terms of relative momentum, it changes from $\boldsymbol{p} = (1-f)\boldsymbol{p}_n - f\boldsymbol{p}_c = \boldsymbol{p}_n - f\frac{q}{2}$ to $\boldsymbol{p}' = \boldsymbol{p}_n + f\frac{q}{2}$, with $f = m_n/M_{tot}$.(the relative momentum changes by the interaction with photon! Photon should be virtual!)

$$(-ig\sqrt{\mathcal{Z}})^2 \int \frac{d^3p_n}{(2\pi)^3} \langle \boldsymbol{p}_n - f\frac{q}{2}|iG_0(-B)|\boldsymbol{p}_n - f\frac{q}{2}\rangle(-ieZ_c)\langle \boldsymbol{p}_n + f\frac{q}{2}|iG_0(-B)|\boldsymbol{p}_n + f\frac{q}{2}\rangle \tag{160}$$

One can do explicit calculation of this integral as like the pionless case. Let us skip this part.

In case of proton halo, one have to replace $iG_0(-B) \to iG_C(-B)$ and also have to consider possible change of relative momentum by Coulomb interaction. Thus,

$$\begin{aligned} i\Gamma_{LO}(|\boldsymbol{q}|) &= (-ig\sqrt{\mathcal{Z}})^2 \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \langle \boldsymbol{k}_3|iG_C(-B)|\boldsymbol{k}_2 - f\frac{q}{2}\rangle(-ieQ_c)\langle \boldsymbol{k}_2 + f\frac{q}{2}|iG_C(-B)|\boldsymbol{k}_1\rangle \\ &\quad +[(f\to 1-f),(Q_c\to Q_p)] \end{aligned} \tag{161}$$

In coordinate space,

$$\begin{aligned} i\Gamma_{LO}(|\boldsymbol{q}|) &= (-ig\sqrt{\mathcal{Z}})^2 \int d^3r \langle \boldsymbol{r}=0|iG_C(-B)|\boldsymbol{r}\rangle(-ieQ_c)e^{if\boldsymbol{q}\cdot\boldsymbol{r}}\langle \boldsymbol{r}|iG_C(-B)|\boldsymbol{r}=0\rangle \\ &\quad +[(f\to 1-f),(Q_c\to Q_p)] \end{aligned} \tag{162}$$

We have already,

$$G_C^{(0)}(-B;0,r) = -\frac{m_R}{2\pi}\Gamma(1+\eta_I)\frac{W_{-\eta_I,1/2}(2\gamma r)}{r} \tag{163}$$

Integration can be done numerically.(Or Analytic expression?) Additional correction from ρ_X can be included by replacing $\exp(if\boldsymbol{q}\cdot\boldsymbol{r}) \to \exp(if\boldsymbol{q}\cdot\boldsymbol{r})(1 - \frac{Z_X\rho_X^2 q^2}{6Z_{tot}})$.

12 Radiative Capture

Radiative capture of proton Halo at LO have Coulomb corrections. One is from Coulomb Green's function. And the other is from Coulomb scattering(In position space, this becomes a Coulomb wave function.)

Appendix: Spherical and Cartesian tensor

Spherical tensor and Cartesian tensor relation

$$\begin{aligned}
A_{+1} &= -A^{-1} = -\frac{1}{\sqrt{2}}(A_x + iA_y), \\
A_0 &= A^0 = A_z, \\
A_{-1} &= -A^{+1} = \frac{1}{\sqrt{2}}(A_x - iA_y).
\end{aligned} \tag{164}$$

$$\begin{aligned}
&[A^{(1)} \otimes B^{(1)}]^{(0)}_0 = -\frac{1}{\sqrt{3}}\vec{A}\cdot\vec{B} = -\frac{1}{\sqrt{3}}[A^{(1)}_0 B^{(1)}_0 - A^{(1)}_{+1}B^{(1)}_{-1} - A^{(1)}_{-1}B^{(1)}_1], \\
&[A^{(1)} \otimes B^{(1)}]^{(1)}_0 = \frac{1}{\sqrt{2}}(A_1B_{-1} - A_{-1}B_1) = \frac{i}{\sqrt{2}}(A_xB_y - A_yB_x) \\
&[A^{(1)} \otimes B^{(1)}]^{(1)}_{\pm1} = \frac{1}{\sqrt{2}}(A_{\pm1}B_0 \mp A_0B_{\pm1}) = \frac{1}{2}(A_zB_x - A_xB_z \pm i(A_zB_y - A_yB_z)) \\
&[A^{(1)} \otimes B^{(1)}]^{(2)}_{\pm2} = A^{(1)}_{\pm1}B^{(1)}_{\pm1} = \frac{1}{2}(A^x \pm iA^y)(B^x \pm iB^y) \\
&[A^{(1)} \otimes B^{(1)}]^{(2)}_{\pm1} = \frac{1}{\sqrt{2}}\left(A^{(1)}_0 B^{(1)}_{\pm1} + A^{(1)}_0 B^{(1)}_{\pm1}\right) = -\frac{1}{2}A^z(B^x \pm iB^y) - \frac{1}{2}B^z(A^x \pm iA^y) \\
&[A^{(1)} \otimes B^{(1)}]^{(2)}_0 = \sqrt{\frac{1}{6}}(A_{+1}B_{-1} + 2A_0B_0 + A_{-1}B_{+1}) = \sqrt{\frac{1}{6}}(3A^zB^z - \vec{A}\cdot\vec{B})
\end{aligned} \tag{165}$$

For a angular integration, in Cartesian coordinate, one can replace

$$\int d^3p p_i p_j \to \frac{1}{3}\delta_{ij}\int d^3p p^2 \tag{166}$$

In case of spherical components,

$$p_\alpha p_\beta \to \frac{1}{3}p^2(\delta_{\alpha 0}\delta_{\beta 0} + \delta_{\alpha 1}\delta_{\beta -1} + \delta_{\alpha -1}\delta_{\beta 1}) \tag{167}$$

Then,

$$\sum_{\alpha\beta m} C^{js}_{1\alpha\frac{1}{2}m} C^{js'}_{1\beta\frac{1}{2}m} \int d^3p p_\alpha p_\beta \to \delta_{ss'}\frac{(C^{js}_{10\frac{1}{2}s})^2}{3}\int d^3p p^2 \tag{168}$$

On the other hand , from explicit forms, we can show

$$p_\alpha p^\beta \to \frac{1}{3}p^2\delta^\beta_\alpha \tag{169}$$

Appendix: Details on the integral for Coulomb

The self energy diagram involving Coulomb interaction requires following integral,

$$\begin{aligned}
J_0(k) &= 2m_R\int \frac{d^3p}{(2\pi)^3}\frac{e^{-\pi\eta_p}\Gamma(1-i\eta_p)\Gamma(1+i\eta_p)}{k^2 - p^2 + i\epsilon} \\
&= 2m_R\int \frac{d^3p}{(2\pi)^3}\frac{C_0(\eta_p)^2}{k^2 - p^2 + i\epsilon} \\
&= 2m_R\int \frac{d^3p}{(2\pi)^3}\frac{2\pi\eta_p}{e^{2\pi\eta_p} - 1}\left(\frac{1}{p^2}\frac{(p^2 - k^2) + k^2}{k^2 - p^2 + i\epsilon}\right)
\end{aligned} \tag{170}$$

In the last line, we separated the finite part and divergent part of the integral.

$$\begin{aligned} J_0^{fin}(k) &= 2m_R \int \frac{d^3p}{(2\pi)^3} \frac{2\pi\eta_p}{e^{2\pi\eta_p}-1} \frac{k^2}{p^2(k^2-p^2+i\epsilon)} \\ J_0^{div}(k) &= -2m_R \int \frac{d^3p}{(2\pi)^3} \frac{2\pi\eta_p}{e^{2\pi\eta_p}-1} \frac{1}{p^2} \end{aligned} \tag{171}$$

The finite part can be calculated by using

$$\int_0^\infty dx \frac{x}{(e^x-1)(x^2+a^2)} = \frac{1}{2}\left(\ln(\frac{a}{2\pi}) - \frac{\pi}{a} - \psi(\frac{a}{2\pi})\right), \tag{172}$$

we get

$$J_0^{fin}(k) = -\frac{m_R k_C}{\pi} h_0(\eta_k) \tag{173}$$

The divergent part can be obtained in dimensional regularization by using

$$\int_0^\infty dx \frac{x^{\epsilon-1}}{e^x-1} = \Gamma(\epsilon)\zeta(\epsilon) \tag{174}$$

and in the limit of $\epsilon \to 0$,

$$\begin{aligned} \zeta(\epsilon) &= -\frac{1}{2}[1+\epsilon\ln 2\pi] + O(\epsilon^2), \quad \epsilon \to 0 \\ \zeta(1+\epsilon) &= \frac{1}{\epsilon} + C_E + O(\epsilon). \end{aligned} \tag{175}$$

$$J_0^{div}(k) = -\frac{m_R k_C}{\sqrt{\pi}} \left(\frac{\mu m_R}{2k_C\sqrt{\pi}}\right)^{3-d} \frac{\Gamma(3-d)\zeta(3-d)}{\Gamma(\frac{d}{2})}, \quad d=3 \tag{176}$$

In PDS, one have to subtract divergence in $d \to 2$ and then divergence in $d \to 3$.

13 Appendix: Formal scattering theory

Denote free state $|k\rangle$, ($H_0|k\rangle = \frac{k^2}{2\mu}|k\rangle$) and scattering state $|k\rangle^{(\pm)}$, And Suppose relation,

$$(E_k \pm i\epsilon - \hat{H})|\boldsymbol{k}\rangle^{(\pm)} = \pm i\epsilon|\boldsymbol{k}\rangle^{(\pm)} = \pm i\epsilon|\boldsymbol{k}\rangle. \tag{177}$$

We have naturally

$$\begin{aligned} G_0(E \pm i\epsilon)|\boldsymbol{k}\rangle &= \frac{1}{E \pm i\epsilon - E_k}|\boldsymbol{k}\rangle \\ G(E \pm i\epsilon)|\boldsymbol{k}\rangle^{(\pm)} &= \frac{1}{E \pm i\epsilon - E_k}|\boldsymbol{k}\rangle^{(\pm)} \end{aligned} \tag{178}$$

$$|\boldsymbol{k}\rangle^{(\pm)} = \lim_{\epsilon\to 0} \frac{\pm i\epsilon}{E_k \pm i\epsilon - \hat{H}}|\boldsymbol{k}\rangle = \lim_{\epsilon \to 0} \pm i\epsilon G(E_k \pm i\epsilon)|\boldsymbol{k}\rangle. \tag{179}$$

From the property of matrix, $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$, we can show [17]

$$G = G_0 + GVG_0 = G_0 + G_0VG \tag{181}$$

[17]

$$\begin{aligned} G - G_0 &= G(G_0^{-1} - G^{-1})G_0 = GVG_0 \\ G_0 - G &= G_0(G^{-1} - G_0)G = -G_0VG \end{aligned} \tag{180}$$

This gives the LS equation

$$|\boldsymbol{k}\rangle^{(\pm)} = |\boldsymbol{k}\rangle + G_0^{(\pm)}\hat{V}|\boldsymbol{k}\rangle^{(\pm)} = |\boldsymbol{k}\rangle + G^{(\pm)}\hat{V}|\boldsymbol{k}\rangle \tag{182}$$

S-matrix and T-matrix is defined as[18]

$$\begin{aligned}
\langle\boldsymbol{p}'|\hat{S}|\boldsymbol{p}\rangle &\equiv {}^{(-)}\langle\boldsymbol{p}'|\boldsymbol{p}\rangle^{(+)} = \langle\boldsymbol{p}'|\boldsymbol{p}\rangle^{(+)} + \langle\boldsymbol{p}'|VG^{(+)}(E_{p'})|\boldsymbol{p}\rangle^{(+)} \\
&= \langle\boldsymbol{p}'|\boldsymbol{p}\rangle + \langle\boldsymbol{p}'|G_0^{(+)}(E_p)V|\boldsymbol{p}\rangle^{(+)} + \langle\boldsymbol{p}'|VG^{(+)}(E_{p'})|\boldsymbol{p}\rangle^{(+)} \\
&= \langle\boldsymbol{p}'|\boldsymbol{p}\rangle + \frac{\langle\boldsymbol{p}'|V|\boldsymbol{p}\rangle^{(+)}}{E_p + i\epsilon - E_{p'}} + \frac{\langle\boldsymbol{p}'|V|\boldsymbol{p}\rangle^{(+)}}{E_p' + i\epsilon - E_p} \\
&= \langle\boldsymbol{p}'|\boldsymbol{p}\rangle - 2\pi i\delta(E_p' - E_p)t(\boldsymbol{p}',\boldsymbol{p}), \\
t(\boldsymbol{p}',\boldsymbol{p}) &\equiv \langle\boldsymbol{p}'|V|\boldsymbol{p}\rangle^{(+)} = {}^{(-)}\langle\boldsymbol{p}'|V|\boldsymbol{p}\rangle = \langle\boldsymbol{p}'|\hat{T}|\boldsymbol{p}\rangle.
\end{aligned} \tag{183}$$

This definition implies

$$\hat{V}|\boldsymbol{k}\rangle^{(+)} = \hat{T}|\boldsymbol{k}\rangle, \quad {}^{(-)}\langle\boldsymbol{k}|\hat{V} = \langle\boldsymbol{k}|\hat{T}, \tag{184}$$

and thus,

$$\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{T} = \hat{V} + \hat{T}\hat{G}_0\hat{V} \tag{185}$$

This implies also

$$\hat{G}\hat{V} = \hat{G}_0\hat{T}, \quad \hat{V}\hat{G} = \hat{T}\hat{G}_0 \tag{186}$$

thus, we can write

$$G = G_0 + G_0TG_0, \quad T = \hat{V} + \hat{V}\hat{G}\hat{V} \tag{187}$$

Two potential formula

$$t(\boldsymbol{p}',\boldsymbol{p}) = \langle\boldsymbol{p}'|V_C + V_S|\boldsymbol{p}\rangle^{(+)} = t_C(\boldsymbol{p}',\boldsymbol{p}) + {}^{(-)}\langle\phi_{\boldsymbol{p}'}|V_S|\boldsymbol{p}\rangle^{(+)} \tag{188}$$

where $|\phi\rangle$ is a solution for H_C. Also, we can write

$$G = G_0 + G_0(T_C + T_{CS})G_0 = G_C + G_0T_{CS}G_0 \tag{189}$$

$$T_C(\boldsymbol{p}',\boldsymbol{p}) = \langle\boldsymbol{p}'|V_c|\psi_{\boldsymbol{p}}^{(+)}\rangle = -\frac{2\pi}{m_R}\sum_{l=0}^{\infty}(2l+1)\left(\frac{e^{2i\sigma_l}-1}{2ip}\right)P_l(\cos\theta) \tag{190}$$

$$T_{SC}(\boldsymbol{p}',\boldsymbol{p}) = \langle\psi_{\boldsymbol{p}'}^{(-)}|V_S|\Psi_{\boldsymbol{p}}^{(+)}\rangle = -\frac{2\pi}{m_R}\sum_{l=0}^{\infty}(2l+1)e^{2i\sigma_l}\left(\frac{e^{2i\delta_l}-1}{2ip}\right)P_l(\cos\theta) \tag{191}$$

[18] Be careful for the normalization. Here normalization is chosen as $\langle\boldsymbol{p}|\boldsymbol{p}'\rangle = \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$.