Chapter 2. Scattering Theory

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1 Scattering cross sections and scattering amplitudes

In non-relativistic quantum mechanics, we consider a scattering process that a uniform monoenergetic beam (projectiles) of momentum $\hbar\mathbf{k}$, current density $\mathbf{J}_{inc}$ and energy E are incident on a target, and some of them will be scattered and detected at the asymptotic region through a surface element $d\mathbf{S}$. Let the scattered current denote $\mathbf{J}_{sc}$. The differential scattering cross section, $d\sigma$, is defined as

$$\mathbf{J}_{inc}d\sigma = \mathbf{J}_{sc} \cdot d\mathbf{S} = r^2 \mathbf{J}_{sc} \cdot d\mathbf{\Omega}$$

where $d\mathbf{\Omega}$ is the solid angle in the radial direction, $\mathbf{e}_r$. The scattering outgoing wave can be written as

$$\varphi_{sc} = f(\theta)\frac{e^{ikr}}{r}$$

where the modulation factor, $f(\theta)$, is called the scattering amplitude.

The number of particles scattered into $d\mathbf{\Omega}$ can be obtained from the radial component of $\mathbf{J}_{sc}$

$$\begin{aligned} J_{sc,r} &= \frac{\hbar}{2mi}[\varphi_{sc}^* \frac{\partial}{\partial r}\varphi_{sc} - \varphi_{sc}\frac{\partial}{\partial r}\varphi_{sc}^*] \\ &= \frac{\hbar}{2mi}[f^*(\theta)\frac{e^{-ikr}}{r}f(\theta)\frac{e^{ikr}}{r}(ik - \frac{1}{r}) - f(\theta)\frac{e^{ikr}}{r}f^*(\theta)\frac{e^{-ikr}}{r}(-ik - \frac{1}{r})] \\ &= \frac{\hbar k}{mr^2}|f(\theta)|^2 \end{aligned}$$

The incident beam along z-axis has a magnitude of

$$J_{inc} = \frac{\hbar k}{m}$$

From the definition of $d\sigma$ given, we obtain

$$\begin{aligned} J_{inc}d\sigma &= \frac{\hbar k}{m}d\sigma = \mathbf{J}_{sc} \cdot d\mathbf{S} = r^2 J_{sc,r} d\Omega \\ &= r^2 \frac{\hbar k}{mr^2}|f(\theta)|^2 d\Omega \\ d\sigma &= |f(\theta)|^2 d\Omega \end{aligned}$$

It implies that the scattering cross section depends only on the scattering angle θ and not on ϕ. It is due to the rotational symmetry of the scattering about the incident beam axis and the assumed quality of the central interaction. Thus the total cross section can be written as

$$\sigma = \int d\sigma = 2\pi \int_0^\pi |f(\theta)|^2 \sin\theta d\theta$$

2 Partial wave expansions

The wave function for the steady-state scattering, at positions far from the scattering target will contain a plane wave incident component and an outgoing scattering component,

$$\varphi(r,\theta) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

By the partial wave expansion of each term at the asymptotic region, we show that the scattering amplitude becomes

$$f(\theta) \;=\; \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$$

where δ_ℓ is the phase shift that the partial wave incurs in scattering.

At the asymptotic region where the interaction potential $V(r)$ is rapidly approaching zero, one may expect the solution given by asymptotic form of the free particle solution $j_\ell(kr)$,

$$R_{k\ell}^{free} = j_{k\ell} \sim \frac{1}{kr}\sin(kr - \frac{\ell\pi}{2})$$

If $V(r)$ decreases faster than r^{-1}, this free particle asymptotic form remains intact save for a change in argument through a phase shift δ_ℓ,

$$R_{k\ell} \sim \frac{1}{kr}\sin(kr - \frac{\ell\pi}{2} + \delta_\ell)$$

The partial wave expansion of the wave function given at fixed k can be written

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell R_{k\ell} P_\ell(\cos\theta)$$

This wave function should match the asymptotic form of the right hand side of the equation given. With the expression for e^{ikz},

$$e^{ikz} \sim \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\frac{1}{kr}\sin(kr - \frac{\ell\pi}{2})P_\ell(\cos\theta)$$

we obtain the matching equation,

$$\sum_{\ell=0}^{\infty} B_\ell \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr}P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\frac{\sin(kr - \ell\pi/2)}{kr}P_\ell(\cos\theta) + f(\theta)\frac{e^{ikr}}{r}$$

We now have

$$\begin{aligned}
F &= \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} \\
&= \frac{1}{2ikr}[e^{-ikr}e^{i(\ell\pi/2-\delta_\ell)} - e^{ikr}e^{-i(\ell\pi/2-\delta_\ell)}] \\
&= \frac{1}{2ikr}e^{-i\delta_\ell}[e^{-ikr}e^{i\ell\pi/2} - e^{ikr}e^{-i\ell\pi/2}e^{2i\delta_\ell}]
\end{aligned}$$

We use the relation

$$e^{2i\delta_\ell} = 1 + 2ie^{i\delta_\ell}\sin\delta_\ell, \qquad e^{-i\ell\pi/2} = (-i)^\ell = (\frac{1}{i})^\ell$$

Then F becomes

$$
\begin{aligned}
F &= \frac{1}{2ikr}e^{-i\delta_\ell}\{[e^{-ikr}e^{i\ell\pi/2} - e^{ikr}e^{-i\ell\pi/2}] - e^{ikr}2i(\frac{1}{i})^\ell e^{i\delta_\ell}\sin\delta_\ell\} \\
&= e^{-i\delta_\ell}\frac{\sin(kr-\ell\pi/2)}{kr} - \frac{e^{ikr}}{kr}\frac{1}{i^\ell}\sin\delta_\ell
\end{aligned}
$$

The matching equation yields

$$
\begin{aligned}
&\sum_{\ell=0}^{\infty} B_\ell[e^{-i\delta_\ell}\frac{\sin(kr-\ell\pi/2)}{kr} - \frac{e^{ikr}}{r}\frac{1}{ki^\ell}\sin\delta_\ell]P_\ell(\cos\theta) \\
&\quad = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\frac{\sin(kr-\ell\pi/2)}{kr}P_\ell(\cos\theta) + f(\theta)\frac{e^{ikr}}{r}
\end{aligned}
$$

Regrouping the terms with $\frac{\sin(kr-\ell\pi/2)}{kr}$ and $\frac{e^{ikr}}{r}$ gives

$$
\begin{aligned}
B_\ell &= i^\ell(2\ell+1)e^{i\delta_\ell} \\
f(\theta) &= \frac{1}{k}\sum_{\ell=0}^{\infty}\frac{B_\ell}{i^\ell}\sin\delta_\ell P_\ell(\cos\theta)
\end{aligned}
$$

Finally, we obtain

$$
f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)
$$

The total cross section is

$$
\begin{aligned}
\sigma &= \int d\sigma \\
&= 2\pi\int_0^\pi |f(\theta)|^2\sin\theta d\theta \\
&= 2\pi\int_0^\pi |\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)|^2 d(\cos\theta) \\
&= \frac{2\pi}{k^2}\sum_{\ell=0}^{\infty}\sum_{\ell_0=0}^{\infty}(2\ell+1)(2\ell_0+1)\sin\delta_\ell\sin\delta_{\ell_0}\int_0^\pi P_\ell(\cos\theta)P_{\ell_0}(\cos\theta)d(\cos\theta) \\
&= \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell
\end{aligned}
$$

where we use the orthogonality of $P_\ell(\cos\theta)$,

$$
\int_0^\pi P_\ell(\cos\theta)P_{\ell_0}(\cos\theta)d(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell_0}
$$

In this partial wave expansion method, formulas for the scattering amplitude and total cross section necessitate knowing all the phase shift δ_ℓ. These phase shifts can be calculated from the radial equation if the potential $V(r)$ is known. This equation must be solved separately for each value of ℓ. Thus this method is attractive when there is a sufficiently small number of non-zero phase shifts. For a finite range interaction $V(r)$, the phase shifts are negligible for $\ell > \ell_c = kr_0$, where r_0 is the finite range. Thus, only δ_ℓ values will contribute for which $\ell < kr_0$. For low energy scattering with $kr_0 \ll 1$, only S-wave ($\ell = 0$) phase shift will differ appreciably from zero.

3 Optical theorem

Setting $\theta = 0$ in $f(\theta)$ of the previous section gives

$$\begin{aligned} f(0) &= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(1) \\ &= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\cos\delta_\ell\sin\delta_\ell + \frac{i}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \end{aligned}$$

Comparison with σ in the previous section yields

$$\sigma = \frac{4\pi}{k}\ \mathrm{Im}\ [f(0)]$$

which is known as the *optical theorem.*

4 Low energy S-wave scattering

4.1 Square well

Consider a scattering of particles of mass m and a very low incident energy E by the attractive spherical well of depth V_0 (positive, > 0) and radius a.

$$V(r) = \begin{cases} -V_0, & 0 < r < a \\ 0, & r > a \end{cases}$$

Let us obtain the S-wave ($\ell = 0$) phase shift.

When energies are sufficiently low that $ka \ll 1$, we need only look at the S-wave scattering. The radial Schrödinger equation is given

$$[\frac{1}{r}\frac{d^2}{dr^2}r - \frac{\ell(\ell+1)}{r^2} + \frac{2m(E-V)}{\hbar^2}]R_{k\ell}(r) = 0$$

Setting $\ell = 0$ and $u = u_{k\ell}(r) \equiv rR_{k\ell}(r)$ gives, for $r < a$ (Region I),

$$\frac{d^2u_I}{dr^2} + k_I^2 u_I = 0, \qquad \frac{\hbar^2 k_I^2}{2m} = E + V_0$$

The solution to this equation with a boundary condition $u_I(0) = 0$ is

$$u_I = A\sin k_I r \qquad (r < a)$$

For $r > a$ (Region II) where $V = 0$, we obtain the general solution

$$\begin{gathered} \frac{d^2u_{II}}{dr^2} + k^2 u_{II} = 0, \qquad \frac{\hbar^2 k^2}{2m} = E \\ u_{II} = B\sin(kr + \delta_0) \qquad (r > a) \end{gathered}$$

Matching conditions at $r = a$ give

$$\begin{aligned} A\sin k_I a &= B\sin(ka + \delta_0) \\ Ak_I\cos k_I a &= Bk\cos(ka + \delta_0) \end{aligned}$$

or

$$\begin{gathered}\cot(ka+\delta_0) = \frac{k_I}{k}\cot k_I a \equiv \frac{1}{\alpha} \\ \frac{\sin ka\cos\delta_0 + \cos ka\sin\delta_0}{\cos ka\cos\delta_0 - \sin ka\sin\delta_0} = \alpha \\ \sin ka + \cos ka\tan\delta_0 = \alpha(\cos ka - \sin ka\tan\delta_0)\end{gathered}$$

We thus obtain

$$\tan\delta_0 = \frac{\alpha - \tan ka}{1+\alpha\tan ka} = \frac{(k/k_I)\tan k_I a - \tan ka}{1+(k/k_I)\tan k_I a\tan ka}$$

If we define $\tan qa \equiv \alpha = (k/k_I)\tan k_I a$, then

$$\begin{aligned}\tan\delta_0 &= \frac{\tan qa - \tan ka}{1+\tan qa\tan ka} = \tan(qa-ka) \\ \delta_0 &= \tan^{-1}(\frac{k}{k_I}\tan k_I a) - ka\end{aligned}$$

At very low energies, using $\tan x \approx x$ for $x \ll 1$, we get

$$\tan\delta_0 \approx \delta_0 \approx ka(\frac{\tan k_I a}{k_I a} - 1)$$

At very low energies the cross section only has the $\ell = 0$ contribution to it,

$$\begin{aligned}\sigma_0 &= \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 \\ &\approx \frac{4\pi}{k^2}(ka)^2(\frac{\tan k_I a}{k_I a} - 1)^2 \\ &\approx 4\pi a^2(\frac{\tan k_I a}{k_I a} - 1)^2\end{aligned}$$

When $k_I a$ passes through $n(\pi/2)$, which is just the condition that the well be deep enough for a bound state to develop, then $\tan k_I a \to \infty$ and

$$\tan\delta_0 \to \infty$$

that is, δ_0 goes through $n(\pi/2)$ and thus $\sin\delta_0 \approx 1$. It yields that the cross section has maximum and the scattering shows resonances.

$$\sigma_{0,max} = \frac{4\pi}{k^2}, \qquad k_I a = n(\pi/2)$$

When the beam energy satisfies the transcendental relation

$$\tan k_I a = k_I a, \qquad \tan\delta_0 \to 0$$

δ_0 becomes zero. It implies that the attractive well scattering reveals transparent. Such resonant transparency is experimentally corroborated in the scattering of low-energy electrons by rare gas atoms. It is called the *Ramsauer Effect.*

4.2 Repulsive sphere scattering

Consider scattering of particles of mass m and a very low incident energy E by the repulsive sphere of height V_0 and radius a.

$$V(r) = \begin{cases} V_0, & 0 < r < a \\ 0, & r > a \end{cases}$$

Let us obtain the S-wave ($\ell = 0$) phase shift.

Solutions for the repulsive sphere can be obtained by simply replacing k_I by $i\kappa$ in the previous spherical well problem. For the interior solution we obtain

$$u_I = A \sinh \kappa r \quad (r < a), \qquad \frac{\hbar^2 \kappa^2}{2m} = V_0 - E > 0$$

In the exterior region, the sinusoidal solution should be maintained.

$$u_{II} = B \sin(kr + \delta_0) \quad (r > a) \quad \frac{\hbar^2 k^2}{2m} = E$$

The matching conditions at $r = a$ give the S-wave phase shift

$$\delta_0 \approx ka(\frac{\tan i\kappa a}{i\kappa a} - 1) \approx ka(\frac{\tanh \kappa a}{\kappa a} - 1)$$

where $\tanh x = -i \tan ix$ is used.

The total cross section becomes

$$\sigma_0 \quad \approx \quad 4\pi a^2 (\frac{\tanh \kappa a}{\kappa a} - 1)^2$$

In the limit of $V_0 \to \infty$ (impenetrable and hard sphere), and thus $\kappa a \to \infty$,

$$\tanh \kappa a \to 1 \quad \Rightarrow \quad \frac{\tanh \kappa a}{\kappa a} \to 0$$

We thus obtain

$$\sigma_0 = 4\pi a^2$$

which is four times the geometric cross section πa^2. Low-energy scattering means a very large wavelength scattering and we do not necessarily expect a classically reasonable result.

5 Hard sphere scattering

Consider scattering of particles of mass m and incident energy E by the impenetrable hard sphere of radius a.

$$V(r) = \begin{cases} \infty, & 0 < r < a \\ 0, & r > a \end{cases}$$

The solution for $r < a$ must vanish, and it must also match onto the most general free-particle solution in the spherical coordinates for a given partial wave for $r > a$, that is,

$$R_\ell = \alpha_\ell j_\ell(kr) + \beta_\ell n_\ell(kr)$$

which gives

$$\frac{\beta_\ell}{\alpha_\ell} = -\frac{j_\ell(ka)}{n_\ell(ka)}$$

At the large r limit, the solution is of form

$$\begin{aligned} R_{\ell,sc} &\sim \gamma_\ell \frac{1}{kr}\sin(kr - \frac{\ell\pi}{2} + \delta_\ell) \\ &\sim \gamma_\ell \frac{1}{kr}[\sin(kr - \frac{\ell\pi}{2})\cos\delta_\ell + \cos(kr - \frac{\ell\pi}{2})\sin\delta_\ell] \end{aligned}$$

By using the asymptotic behavior of j_ℓ and n_ℓ, we obtain

$$R_\ell = \gamma_\ell[\cos\delta_\ell j_\ell(kr) - \sin\delta_\ell n_\ell(kr)]$$

Comparing with the first equation above yields

$$\frac{\beta_\ell}{\alpha_\ell} = -\frac{\gamma_\ell\sin\delta_\ell}{\gamma_\ell\cos\delta_\ell} = -\frac{j_\ell(ka)}{n_\ell(ka)}$$

We thus obtain the partial wave phase shifts

$$\tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}, \quad \text{or} \quad \sin^2\delta_\ell = \frac{j_\ell^2(ka)}{j_\ell^2(ka) + n_\ell^2(ka)}$$

The total cross section becomes

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\frac{j_\ell^2(ka)}{j_\ell^2(ka) + n_\ell^2(ka)}$$

By using the asymptotic behavior of the spherical Bessel and Neumann functions

$$j_\ell(x\to 0) \sim \frac{x^\ell}{(2\ell+1)!!}, \qquad n_\ell(x\to 0) \sim \frac{(2\ell-1)!!}{x^{\ell+1}}$$

we obtain the partial phase shifts for $ka \ll 1$,

$$\tan\delta_\ell = \frac{(ka)^{2\ell+1}}{(2\ell+1)[(2\ell-1)!!]^2}$$

It obviously shows that only the $\ell = 0$ wave will contribute appreciably to the scattering for low energies. The total cross section becomes

$$\sigma_0 \approx \frac{4\pi}{k^2}\delta_0^2 = 4\pi a^2$$

since $\sin\delta_0 \approx \delta_0 \approx ka$.

At the high energy limit where $ka \gg 1$, we obtain

$$\tan\delta_\ell = -\frac{\sin(ka - \ell\pi/2)}{\cos(ka - \ell\pi/2)}$$

so that $\delta_\ell \to -(ka - \ell\pi/2) \gg 1$ in magnitude. It implies many partial waves contribute to the total cross section. Each contributing factor of $\sin^2\delta_\ell$ with a large argument (rapidly oscillating function) will average to 1/2. The summation will be cut off at a value of $\ell_{max} \approx ka$. We thus approximate

$$\sum_{\ell=0}^{\infty}(2\ell+1) = \int_0^{\ell_{max}}(2\ell+1)d\ell \approx \ell_{max}^2 \approx (ka)^2$$

The total cross section would then be

$$\sigma_0 \approx \frac{4\pi}{k^2}\frac{1}{2}(ka)^2 = 2\pi a^2$$

which is two times the geometric cross section πa^2. The origin of the factor of 2 can be understood that a half (πa^2) is just from the reflected contribution without interference from different ℓ and another half called the shadow term is originated from deflected waves. See J. J. Sakurai Section 7.6 for details.

6 Scattering length

The scattering length b is defined as the negative of the limiting value of the scattering amplitude as the energy of the incident particle goes to zero.

$$b = -\lim_{k\to 0} f(\theta)$$

At sufficiently low energy only the S-wave ($\ell = 0$) term in the partial wave analysis will be significant. Then the scattering amplitude becomes

$$f(\theta) = \frac{1}{k} e^{i\delta_0} \sin\delta_0$$

Furthermore, for relatively small phase shift

$$e^{i\delta_0} \approx 1, \quad \sin\delta_0 \approx \delta_0 \quad \Rightarrow \quad f(\theta) \approx \frac{\delta_0}{k}$$

We obtain

$$b = -\lim_{k\to 0} f(\theta) = -\lim_{k\to 0} \frac{\delta_0}{k}$$

The total cross section becomes

$$\sigma = \frac{4\pi}{k^2} \sin^2\delta_0 \approx 4\pi \frac{\delta_0^2}{k^2} = 4\pi b^2$$

We saw that the total cross section from the impenetrable rigid sphere is from IV-6

$$\sigma = 4\pi a^2$$

where a is the radius of the sphere. By comparing this with the result of the total cross section, we obtain

$$b = a$$

7 Born approximation

7.1 Scattering amplitude and cross section in the Born approximation

The scattering amplitude $f(\theta)$ by a central potential in the Born approximation is given as

$$\begin{aligned} f_B(\theta) &= -\frac{m}{2\pi\hbar^2}\int V(r)\exp[i\mathbf{r}\cdot(\mathbf{k_i}-\mathbf{k_f})]d\mathbf{r} \\ &= -\frac{m}{2\pi\hbar^2}\int V(r)e^{-i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r} \end{aligned}$$

where $\mathbf{k_i}$ ($\mathbf{k_f}$) denotes the incident (scattered) momentum, $\mathbf{q}=\mathbf{k_f}-\mathbf{k_i}$ is the momentum transfer of the particle, and the minus sign in the front is just conventional. Assume that scattered particles suffer no loss in energy, i.e., $k_0 = k$.

Owing to the equal magnitudes of $\mathbf{k_i}$ and $\mathbf{k_f}$, we may set

$$q = 2k\sin(\theta/2) = q_{max}\sin(\theta/2)$$

where θ is the scattering angle. Let us take the polar axis to be coincident with $\mathbf{q}$. The volume integration becomes

$$\begin{aligned} f_B(\theta) &= -\frac{m}{2\pi\hbar^2}2\pi\int_0^\infty drr^2V(r)\int_0^\pi d\vartheta\sin\vartheta e^{-iqr\cos\vartheta} \\ &= -\frac{m}{\hbar^2}\int_0^\infty drr^2V(r)\int_{-1}^1 d\eta e^{-iqr\eta} \\ &= -\frac{m}{\hbar^2}\int_0^\infty drr^2V(r)\frac{1}{-iqr}(e^{-iqr}-e^{iqr}) \\ &= -\frac{2m}{q\hbar^2}\int_0^\infty drrV(r)\sin qr \end{aligned}$$

where ϑ in the first line is the angle between $\mathbf{q}$ and $\mathbf{r}$. There are several comments;
(1) $f_B(\theta)$ is always real,
(2) $f_B(\theta)$ or $d\sigma/d\Omega$ is a function of q^2 only, since the term

$$\frac{\sin qr}{q} = \frac{1}{q}(qr - \frac{1}{3!}q^3r^3 + ...) = r - \frac{1}{3!}q^2r^3 + ...$$

and thus depends on the energy $E = \hbar^2k^2/2m$ and the scattering angle only through the combination of $\hbar^2q^2 = 8mE\sin^2(\theta/2)$. Furthermore, q^2 is a proper Lorentz scalar.
(3) $d\sigma/d\Omega$ is independent of the sign of $V(r)$.

Low energies imply that $\mathbf{r}\cdot\mathbf{q}\to 0$, so that

$$f_B(\theta) \approx -\frac{m}{2\pi\hbar^2}\int V(r)d\mathbf{r} \approx -\frac{m}{2\pi\hbar^2}V(0)d^3r$$

where d^3r is the volume over which the potential is nonvanishing. It obviously shows that the scattering is independent of angle for low energies.

At large energies, the argument of the exponential phase, $2kr_0\sin(\theta/2)$ where r_0 is the interaction range, becomes large, and thus the phase factor oscillates rapidly. Thus the contribution from the total integral in that region becomes small due to the cancelations. A rough cutoff for when this happens is when

$$2kr_0\sin(\theta/2) \;\Rightarrow\; kr_0\theta \approx 1$$

As the energy increases ($k = k_i$ gets bigger), θ becomes smaller, and the differential cross section becomes nonnegligible only for angles satisfying

$$\theta_{max} \le \frac{1}{kr_0}$$

It explains more forward peaked angular distributions at higher energies. The total cross section can then be roughly written at large energies

$$\begin{aligned}\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \approx 2\pi \int_0^{1/kr_0} |f_B(\theta)|^2 \sin\theta d\theta \\ &\approx 2\pi |f_B(0)|^2 \int_0^{1/kr_0} \theta d\theta \approx \pi |f_B(0)|^2 \frac{1}{k^2 r_0^2} \\ &\approx \frac{\pi\hbar^2}{2mr_0^2} |f_B(0)|^2 \frac{1}{E}\end{aligned}$$

The total cross section thus decreases with energy.

7.2 Examples

7.2.1 Spherical well

Consider a scattering of particles of mass m and incident energy E by the spherical well with potential

$$V(r) = \begin{cases} -V_0, & 0 < r < a \\ 0, & r > a \end{cases}$$

The scattering amplitude becomes

$$\begin{aligned}f_B(\theta) &= \frac{2m}{\hbar^2 q} \int_0^a drrV_0 \sin qr \\ &= \frac{2mV_0}{\hbar^2 q}[-\frac{1}{q} r \cos qr|_0^a + \int_0^a \frac{1}{q} \cos qrdr] \\ &= \frac{2mV_0}{\hbar^2 q}[-\frac{1}{q} a \cos qa + \frac{1}{q^2} \sin qa] \\ &= \frac{2mV_0 a^3}{\hbar^2 (qa)} (\frac{\sin qa}{(qa)^2} - \frac{\cos qa}{(qa)}) \\ &= \frac{2mV_0 a^3}{\hbar^2} \frac{j_1(qa)}{qa}\end{aligned}$$

where $j_n(x)$ is the spherical Bessel function and $\hbar q = 2\sqrt{2mE}\sin(\theta/2)$. Thus, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = (\frac{2mV_0}{\hbar^2})^2 (\frac{j_1(qa)}{qa})^2$$

Furthermore,

$$x \to 0, \quad j_\ell(x) \sim \frac{x^\ell}{(2\ell+1)!!}, \quad j_1(x) \sim \frac{x}{3}$$

As $qa \to 0$,

$$f_B(\theta) = \frac{2mV_0 a^3}{\hbar^2} \frac{j_1(qa)}{qa} \sim \frac{2mV_0 a^3}{3\hbar^2}$$

without any angle dependence.

7.2.2 Yukawa Potential and Coulomb Potential

Let us consider the Yukawa potential of the form

$$V(r) = \pm V_0 \frac{e^{-\alpha r}}{r}$$

Substituting this Yukawa potential into the scattering amplitude formula in the previous section yields

$$\begin{aligned} f_B(\theta) &= \mp \frac{2mV_0}{q\hbar^2} \int_0^\infty dr e^{-\alpha r} \sin qr \\ &= \mp \frac{2mV_0}{\hbar^2} \frac{1}{q^2 + \alpha^2} \end{aligned}$$

where $\hbar q = 2k \sin(\theta/2) = 2\sqrt{2mE}\sin(\theta/2)$. The differential scattering cross section becomes

$$\frac{d\sigma}{d\Omega} = (\frac{2mV_0}{\hbar^2})^2 \frac{1}{(q^2+\alpha^2)^2}$$

The total cross section becomes in an integral form

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_0^\pi |f_B(\theta)|^2 \sin\theta d\theta \\ &= 2\pi (\frac{2mV_0}{\hbar^2})^2 \int_0^\pi \frac{1}{(q^2+\alpha^2)^2} \sin\theta d\theta \end{aligned}$$

By setting

$$\mu = \sin\frac{\theta}{2}, \quad d\mu = \frac{1}{2}\cos\frac{\theta}{2} d\theta, \quad q^2 = 4k^2\mu^2$$

we obtain the integration

$$\begin{aligned} I &= \int_0^\pi \frac{1}{(q^2+\alpha^2)^2} \sin\theta d\theta \\ &= \int_0^1 \frac{4\mu d\mu}{(4k^2\mu^2+\alpha^2)^2} = \int_0^1 \frac{2d\nu}{(4k^2\nu+\alpha^2)^2} \\ &= \int_{\alpha^2}^{4k^2+\alpha^2} \frac{d\nu'}{2k^2\nu'^2} = \frac{-1}{2k^2\nu'}\Big|_{\alpha^2}^{4k^2+\alpha^2} \\ &= \frac{1}{2k^2}(\frac{1}{\alpha^2} - \frac{1}{4k^2+\alpha^2}) \\ &= \frac{2}{\alpha^2(4k^2+\alpha^2)} \end{aligned}$$

Thus, we obtain the total cross section

$$\sigma = 4\pi (\frac{2mV_0}{\hbar^2})^2 \frac{1}{\alpha^2(4k^2+\alpha^2)}$$

When $\alpha \to 0$, with $V_0 = Ze^2$ and $E = \hbar^2 k^2/2m$,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= (\frac{2mZe^2}{\hbar^2})^2 \frac{1}{q^4} = (\frac{2mZe^2}{\hbar^2})^2 \frac{1}{16k^4 \sin^4(\theta/2)} \\ &= (\frac{Ze^2}{4E})^2 \frac{1}{\sin^4(\theta/2)} \end{aligned}$$

which is the precise expression for the Rutherford cross section. It implies that the Coulomb potential can be considered to be a Yukawa potential of infinite range. Note also that $\sigma \to \infty$, as $\alpha \to 0$.

Somewhat amazingly, this is the same result obtained using purely classical treatments. The fact that this cross section does not contain any explicit factors of $\hbar$ when expressed in terms of the observable energy E is special to the case of Coulomb scattering, and is also suggestive of the unique correspondence of the classical result.

7.2.3 Gaussian potential

We obtain the differential and total scattering cross section in the Born approximation for scattering of particles of mass m from the attractive Gaussian potential

$$V(r) = -V_0 \exp[-(\frac{r}{a})^2]$$

The scattering amplitude for the attractive Gaussian potential in the Born approximation is

$$\begin{aligned} f_B(\theta) &= \frac{2m}{\hbar^2 q}\int_0^\infty dr r V_0 \exp[-(\frac{r}{a})^2]\sin qr \\ &= \frac{2mV_0}{\hbar^2 q}\{-\frac{a^2}{2}\exp[-(\frac{r}{a})^2]\sin qr|_0^\infty + \frac{qa^2}{2}\int_0^\infty dr \exp[-(\frac{r}{a})^2]\cos qr]\} \\ &= \frac{\sqrt{\pi} m V_0 a^3}{2\hbar^2}\exp[-\frac{q^2a^2}{4}] \end{aligned}$$

where we use

$$\int_0^\infty e^{-\alpha^2x^2}\cos bx dx = \frac{\sqrt{\pi}}{2}e^{-b^2/4\alpha^2}$$

Thus, the total cross section becomes in an integral form

$$\sigma = \frac{\pi^2}{2}(\frac{mV_0a^3}{\hbar^2})^2\int_0^\pi \exp[-\frac{q^2a^2}{2}]\sin\theta d\theta$$

By setting

$$\mu = \sin\frac{\theta}{2}, \quad d\mu = \frac{1}{2}\cos\frac{\theta}{2}d\theta, \quad q^2a^2 = 4k^2a^2\mu^2$$

we obtain the integration

$$\begin{aligned} I &= \int_0^\pi \exp[-\frac{q^2a^2}{2}]\sin\theta d\theta = \int_0^1 \exp[-2k^2a^2\mu^2]4\mu d\mu \\ &= -\frac{4}{4k^2a^2}\exp[-2k^2a^2\mu^2]_0^1 \\ &= \frac{1-e^{-2k^2a^2}}{k^2a^2} \end{aligned}$$

Thus, we obtain the total cross section

$$\sigma = \frac{\pi^2}{2}(\frac{mV_0a^3}{\hbar^2})^2\frac{(1-e^{-2k^2a^2})}{k^2a^2}$$

8 Form factors

The electrostatic potential in a point charge in the field of a charge density can be written

$$V(r) = Ze^2 \int \frac{\rho(\mathbf{r_0})d\mathbf{r_0}}{|\mathbf{r}-\mathbf{r_0}|}$$

where $\mathbf{r_0}$ is the internal coordinate of the target and the charge density $\rho(\mathbf{r_0})$ is normalized such that

$$\int d\mathbf{r_0}\rho(\mathbf{r_0}) = 1$$

with the total charge $Q = Ze$.

We substitute the given potential into the Born amplitude

$$\begin{aligned}
f_B(\theta) &= -\frac{m}{2\pi\hbar^2}\int V(r)e^{-i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r} \\
&= -\frac{mZe^2}{2\pi\hbar^2}\int d\mathbf{r}e^{-i\mathbf{q}\cdot\mathbf{r}}[\int d\mathbf{r_0}\frac{\rho(\mathbf{r_0})}{|\mathbf{r}-\mathbf{r_0}|}] \\
&= -\frac{mZe^2}{2\pi\hbar^2}[\int d\mathbf{r}\frac{\exp[-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r_0})]}{|\mathbf{r}-\mathbf{r_0}|}][\int d\mathbf{r_0}\rho(\mathbf{r_0})\exp(-i\mathbf{q}\cdot\mathbf{r_0})]
\end{aligned}$$

The first term is simply the Fourier transform of the Coulomb potential, that is, the Rutherford scattering amplitude. The second term is called the *form factor*, the Fourier transform of the charge density given,

$$F(\mathbf{q}) = \int d\mathbf{r}\rho(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}$$

It describes the modification to the Rutherford cross section due to the smearing out of the charge. We thus obtain the differential cross section

$$\frac{d\sigma}{d\Omega} = (\frac{d\sigma}{d\Omega})_R|F(\mathbf{q})|^2$$

We expand $F(\mathbf{q})$ for small q,

$$\begin{aligned}
F(\mathbf{q}) &= \int d\mathbf{r}\rho(\mathbf{r}) - i\sum_j q_j\int d\mathbf{r}x_j\rho(\mathbf{r}) - \frac{1}{2}\sum_{j,k}q_jq_k\int d\mathbf{r}x_jx_k\rho(\mathbf{r}) + ... \\
&= 1 - i\sum_j q_j < x_j > -\frac{1}{2}\sum_{j,k}q_jq_k < x_jx_k > +... \\
< f(\mathbf{r}) > &= \int d\mathbf{r}f(\mathbf{r})\rho(\mathbf{r})
\end{aligned}$$

For a spherically symmetric charge distribution with $\rho(\mathbf{r}) = \rho(r)$, the average values satisfy

$$< x_j >= 0, \quad \text{and} \quad < x_jx_k >= \frac{1}{3} < r^2 > \delta_{j,k}$$

since $< x^2 >=< y^2 >=< z^2 >$. This yields

$$F(q) = 1 - \frac{1}{6}q^2 < r^2 >$$

which implies that information of the form factor gives the average spatial extent of the charge distribution, square of the mean charge radius $< r^2 >$.

8.1 Nuclear charge form factor

The nuclear charge distributions are experimentally obtained from electron-nucleus scattering. The experimental angular distributions give the form factor of which inverse Fourier transform determines the charge distribution.

1. A constant nuclear charge density: First we obtain ρ_0 from the normalization

$$1 = 4\pi \int_0^R \rho(r) r^2 dr = \frac{4\pi R^3}{3}\rho_0, \quad \rho_0 = \frac{3}{4\pi R^3}$$

The charge form factor which is the Fourier transform of ρ becomes

$$\begin{aligned} F(q) &= \frac{4\pi}{q}\int_0^R drr\rho_0 \sin qr \\ &= 3\frac{j_1(qR)}{qR} = 3(\frac{\sin qR}{q^3R^3} - \frac{\cos qR}{q^2R^2}) \end{aligned}$$

which is a diffractive pattern. The root mean square radius becomes

$$< r^2 >= 4\pi\frac{3}{4\pi R^3}\int_0^R r^4 dr = \frac{3}{5}R^2$$

2. A Yukawa type charge distribution: We obtain ρ_0

$$1 = 4\pi \int_0^\infty \rho_0 \frac{e^{-\alpha r}}{r} r^2 dr = \frac{4\pi}{\alpha^2}\rho_0, \quad \rho_0 = \frac{\alpha^2}{4\pi}$$

The charge form factor becomes (See IV-10.)

$$\begin{aligned} F(q) &= \frac{4\pi}{q}\int_0^\infty drr\rho_0 \frac{e^{-\alpha r}}{r}\sin qr \\ &= \frac{4\pi}{q}\frac{\alpha^2}{4\pi}\frac{q}{q^2+\alpha^2} = \frac{\alpha^2}{q^2+\alpha^2} \end{aligned}$$

which is a smooth Lorentzian. The root mean square radius becomes

$$< r^2 >= 4\pi\frac{\alpha^2}{4\pi}\int_0^\infty \frac{e^{-\alpha r}}{r} r^4 dr = \alpha^2\frac{6}{\alpha^4} = \frac{6}{\alpha^2}$$

3. A Gaussian type charge distribution: We obtain ρ_0

$$1 = 4\pi \int_0^\infty \rho_0 e^{-(r/a)^2} r^2 dr = \frac{4\pi\sqrt{\pi}a^3}{4}\rho_0, \quad \rho_0 = (\frac{1}{a\sqrt{\pi}})^3$$

The charge form factor becomes (See IV-11.)

$$\begin{aligned} F(q) &= \frac{4\pi}{q}\int_0^\infty drr\rho_0 e^{-(r/a)^2}\sin qr \\ &= \frac{4\pi}{q}(\frac{1}{a\sqrt{\pi}})^3\frac{\sqrt{\pi}qa^3}{4}\exp[-\frac{q^2a^2}{4}] = \exp[-\frac{q^2a^2}{4}] \end{aligned}$$

which is a smooth Gaussian. The root mean square radius is

$$< r^2 >= 4\pi(\frac{1}{a\sqrt{\pi}})^3\int_0^\infty e^{-(r/a)^2} r^4 dr = \frac{3}{2}a^2$$

8.2 Proton form factor

The proton form factor obtained from electron scattering is fit rather well over the region $\hbar cq \approx 0 - 1.4$ GeV with a dipole form,

$$F(q^2) = \frac{1}{(1+(q/q_0)^2)^2}$$

where $\hbar c q_0 \approx 840$ MeV.

Using the dipole form, the proton charge distribution may be expressed as

$$\rho(r) = \rho_0 e^{-q_0 r}$$

We can prove it in the following way:

$$1 = 4\pi \int_0^\infty \rho_0 e^{-q_0 r} r^2 dr = \frac{8\pi}{q_0^3}\rho_0, \quad \rho_0 = \frac{q_0^3}{8\pi}$$

The charge form factor becomes

$$\begin{aligned}
F(q) &= \frac{4\pi}{q}\int_0^\infty dr r \rho_0 e^{-q_0 r}\sin qr \\
&= \frac{4\pi}{q}\frac{q_0^3}{8\pi}(-1)\frac{\partial}{\partial q_0}(\frac{q}{q^2+q_0^2}) \\
&= \frac{-q_0^3}{2q}(\frac{-2qq_0}{(q^2+q_0^2)^2}) \\
&= \frac{1}{(1+(q/q_0)^2)^2}
\end{aligned}$$

where we use a formula

$$\int_0^\infty dr r^n e^{-q_0 r}\sin qr = (-1)^n \frac{\partial^n}{\partial q_0^n}(\frac{q}{q^2+q_0^2})$$

The root mean square radius becomes

$$\begin{aligned}
< r^2 > &= 4\pi \frac{q_0^3}{8\pi}\int_0^\infty e^{-q_0 r} r^4 dr \\
&= \frac{q_0^3}{2}\frac{4!}{q_0^5} = \frac{12}{q_0^2} \\
&= \frac{12(\hbar c)^2}{(\hbar c)^2 q_0^2} = \frac{12\times 197.3^2}{840^2} \\
&= 0.662 \ \ \mathrm{F}^2 \\
\sqrt{< r^2 >} &\approx 0.81 \ \mathrm{F}, \quad 1\mathrm{F} = 10^{-15} \ \mathrm{m}
\end{aligned}$$

9 Identical particles scattering

In the scattering of two identical particles, there are two classical trajectories reaching the same signals in the detectors, namely, with the scattering angles (θ, ϕ) and $(\pi - \theta, \pi + \phi)$.

Since there are no spin wavefunctions, the scattering amplitude is simply the symmetric combination and yields the differential cross section

$$\begin{aligned}
(\frac{d\sigma}{d\Omega})_{J=0} &= |f(\theta) + f(\pi - \theta)|^2 \\
&= |f(\theta)|^2 + |f(\pi - \theta)|^2 \\
&\qquad + [f^*(\theta) f(\pi - \theta) + f(\theta) f^*(\pi - \theta)] \\
&= \frac{d\sigma(\theta)}{d\Omega} + \frac{d\sigma(\pi - \theta)}{d\Omega} + 2\Re[f(\theta) f^*(\pi - \theta)]
\end{aligned}$$

It leads to an enhancement at $\pi/2$.

$$(\frac{d\sigma}{d\Omega})_{J=0} = 4|f(\pi/2)|^2 = 2(\frac{d\sigma}{d\Omega})_{Classical}$$

Another interesting point is that only even partial wave amplitudes will contribute to the scattering, since $\cos(\pi - \theta) = -\cos(\theta)$ and the Legendre polynomials satisfy $P_\ell(-y) = (-1)^\ell P_\ell(y)$.

For the fermions the amplitude should reflect basic antisymmetry of the total wavefunction under the interchange of the two particles. If they are in a spin singlet state, then the spatial wave function is symmetric and

$$(\frac{d\sigma}{d\Omega})_{singlet} = |f(\theta) + f(\pi - \theta)|^2$$

while they are in a triplet state, then the spatial wave function is antisymmetric and

$$(\frac{d\sigma}{d\Omega})_{triplet} = |f(\theta) - f(\pi - \theta)|^2$$

In the scattering of two unpolarized fermions, all spin states are equally likely, and thus the probability of finding the two fermions in a triplet state is three times as large as finding them in a singlet state. We thus obtain

$$\begin{aligned}
(\frac{d\sigma}{d\Omega})_{J=1/2} &= \frac{1}{4}(\frac{d\sigma}{d\Omega})_{singlet} + \frac{3}{4}(\frac{d\sigma}{d\Omega})_{triplet} \\
&= \frac{1}{4}|f(\theta) + f(\pi - \theta)|^2 + \frac{3}{4}|f(\theta) - f(\pi - \theta)|^2 \\
&= |f(\theta)|^2 + |f(\pi - \theta)|^2 \\
&\qquad - \frac{1}{2}[f^*(\theta) f(\pi - \theta) + f(\theta) f^*(\pi - \theta)] \\
&= \frac{d\sigma(\theta)}{d\Omega} + \frac{d\sigma(\pi - \theta)}{d\Omega} - \Re[f(\theta) f^*(\pi - \theta)]
\end{aligned}$$

10 Lippmann-Schwinger equation - Formal scattering theory

Consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}e^{-\epsilon|t|/\hbar}$$

where $\hat{H}_0$ is the free-particle Hamiltonian with the eigenfunction φ, ϵ is the infinitesimal parameter and $\hat{V}$ is assumed to be independent of time.

The Schrödinger equation in the interaction picture has the integral form

$$|\psi_I(t) >= |\psi_I(t_0) > + \frac{1}{i\hbar}\int_{t_0}^{t} \hat{V}_I(t')|\psi_I(t') > dt'$$

where

$$\begin{aligned} |\psi_I(t) > &= e^{it\hat{H}_0/\hbar}|\psi(t) > \\ \hat{V}_I(t) &= e^{it\hat{H}_0/\hbar}\hat{V}(t)e^{-it\hat{H}_0/\hbar} \end{aligned}$$

For the scattering theory, $\hat{V}_I(t) \to 0$ in the limits $t \to \pm\infty$. At these asymptotic values, with the interaction vanishingly small, $|\psi(t) >$ loses its time dependence, and thus we may have the stationary free particle state φ as

$$|\varphi > \equiv |\psi_I(\pm\infty) >$$

Putting over the interval $t_0 = \pm\infty, t = 0$, and identifying the scattering states in the first equation

$$|\psi^{(\pm)} > \equiv |\psi(0) >$$

gives

$$|\psi^{(\pm)} >= |\varphi > + \frac{1}{i\hbar}\int_{\pm\infty}^{0} \hat{V}_I(t)|\psi_I(t) > dt$$

which represents scattered states in the domain of interaction. We identify $|\psi^{(+)} >$ relevant to the time interval $(-\infty \le t \le 0)$ with incoming incident waves, commonly called the "in" solution, while $|\psi^{(-)} >$ relevant to the interval $(\infty \le t \le 0)$, commonly called the "out" solution which is the time-reversed state of $|\psi^{(+)} >$. In the limit $\epsilon \to 0$, we take $|\psi^{(\pm)} >$ to be an eigenstate of the total Hamiltonian

$$\hat{H}|\psi^{(\pm)} >= E|\psi^{(\pm)} >$$

We now evaluate in the first equation

$$\begin{aligned} \hat{V}_I(t')|\psi_I(t') > &= e^{it\hat{H}_0/\hbar}\hat{V}e^{-\epsilon|t|/\hbar}e^{-it\hat{H}_0/\hbar}e^{it\hat{H}_0/\hbar}|\psi(t) > \\ &= e^{it\hat{H}_0/\hbar}\hat{V}e^{-\epsilon|t|/\hbar}e^{-it\hat{H}/\hbar}|\psi(0) > \\ &= e^{it\hat{H}_0/\hbar}\hat{V}e^{-\epsilon|t|/\hbar}e^{-it\hat{H}/\hbar}|\psi^{(\pm)} > \\ &= e^{it\hat{H}_0/\hbar}\hat{V}e^{-\epsilon|t|/\hbar}e^{-it\hat{E}/\hbar}|\psi^{(\pm)} > \end{aligned}$$

The first equation now becomes

$$\begin{aligned} |\psi^{(\pm)} > &= |\varphi > \\ &\quad + \frac{1}{i\hbar}\int_{\pm\infty}^{0} e^{it\hat{H}_0/\hbar}\hat{V}e^{-\epsilon|t|/\hbar}e^{-it\hat{E}/\hbar}|\psi^{(\pm)} > dt \end{aligned}$$

For "in" and "out" solutions, respectively, we encounter the time integral

$$\begin{aligned}
\hat{G}^{(+)} &= \frac{1}{i\hbar}\int_{-\infty}^{0} e^{it(E-\hat{H}_0+i\epsilon)/\hbar}dt = \frac{1}{E-\hat{H}_0+i\epsilon} \\
\hat{G}^{(-)} &= \frac{1}{i\hbar}\int_{\infty}^{0} e^{it(E-\hat{H}_0-i\epsilon)/\hbar}dt = \frac{1}{E-\hat{H}_0-i\epsilon}
\end{aligned}$$

Note that the integrals $\hat{G}^{(\pm)}$ do not converge without the presence of ϵ. Plugging these expressions into the first equation yields the *Lippmann-Schwinger* (LS) Equation.

$$|\psi^{(\pm)}>=|\varphi>+\frac{1}{E-\hat{H}_0\pm i\epsilon}|\psi^{(\pm)}>$$

By applying the operator $E-\hat{H}_0\pm i\epsilon$ to the LS equation, it is immediately established that in the limit $\epsilon\to 0$,

$$(H_0-E)|\psi^{(\pm)}>=V|\psi^{(\pm)}>$$

Let us define

$$\hat{G}_\pm \equiv \lim_{\epsilon\to 0}\ G^{(\pm)}$$

The LS equation may then be written

$$|\psi^{(\pm)}>=|\varphi>+\hat{G}_\pm\hat{V}|\psi^{(\pm)}>$$

We now express this equation in the coordinate representation. We here make the identifications

$$\begin{aligned}
<\mathbf{r}|\varphi> &= \varphi_{\mathbf{k}}(\mathbf{r}) \\
<\mathbf{r}|\psi^{(+)}> &= \psi_{\mathbf{k}}^{(+)}(\mathbf{r}) \\
<\mathbf{r}|\hat{V}\psi^{(+)}> &= V(\mathbf{r})\psi_{\mathbf{k}}^{(+)}(\mathbf{r})
\end{aligned}$$

We now operate $<\mathbf{r}|$ from the left in the LS equation

$$\begin{aligned}
|\psi_{\mathbf{k}}^{(\pm)}(\mathbf{r})> &= |\varphi_{\mathbf{k}}(\mathbf{r})>+<\mathbf{r}|\hat{G}_\pm\hat{V}|\psi^{(\pm)}> \\
&\equiv |\varphi_{\mathbf{k}}(\mathbf{r})>+I_\pm
\end{aligned}$$

where

$$\begin{aligned}
I_\pm &= <\mathbf{r}|\hat{G}_\pm\hat{V}|\psi^{(\pm)}> \\
&= \int d\mathbf{r}'\int d\bar{\mathbf{k}}<\mathbf{r}|\bar{\mathbf{k}}><\bar{\mathbf{k}}|\hat{G}_\pm|\mathbf{r}'><\mathbf{r}'|\hat{V}|\psi^{(\pm)}>
\end{aligned}$$

Here we employ the spectral resolution of unity, namely, the identity operation

$$\hat{I}=\sum_n|\varphi_n><\varphi_n|$$

We recall

$$<\mathbf{r}|\mathbf{k}>=\frac{1}{(2\pi)^{3/4}}e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad \hat{H}_0|\mathbf{k}>=\frac{\hbar^2k^2}{2m}|k>$$

whence

$$<\bar{\mathbf{k}}|\hat{G}_\pm|\mathbf{r}'>=\lim_{\epsilon\to 0}\frac{1}{E-(\hbar^2\bar{k}^2/2m)\pm i\epsilon}<\bar{\mathbf{k}}|\mathbf{r}'>$$

Thus we obtain

$$I_\pm = \frac{1}{(2\pi)^3} \int d\mathbf{r}' \int d\bar{\mathbf{k}} \frac{\exp[i\bar{\mathbf{k}} \cdot (\mathbf{r} - \mathbf{r}')]}{E - \frac{\hbar^2 \bar{k}^2}{2m} \pm i\epsilon} < \mathbf{r}'|\hat{V}|\psi^{(\pm)} >$$

Let us first consider the $\bar{\mathbf{k}}$ integration,

$$I_{\pm,\bar{\mathbf{k}}} \equiv \int d\bar{\mathbf{k}} \frac{e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}}{E_\pm}$$

where

$$\mathbf{x} \equiv \mathbf{r} - \mathbf{r}', \qquad E_\pm = E - \frac{\hbar^2 \bar{k}^2}{2m} \pm i\epsilon$$

With

$$\int d\bar{\mathbf{k}} = 2\pi \int_0^\infty \bar{k}^2 d\bar{k} \int_{-1}^{1} d\mu, \qquad \mu \equiv \cos\theta = \frac{\bar{\mathbf{k}} \cdot \mathbf{x}}{|\bar{\mathbf{k}} \cdot \mathbf{x}|}$$

integration over μ gives

$$\int_{-1}^{1} d\mu e^{i\bar{k}x\mu} = \frac{1}{i\bar{k}x}[e^{i\bar{k}x} - e^{-i\bar{k}x}]$$

As $\bar{k}/E_\pm$ is an odd function of $\bar{k}$, we find

$$\begin{aligned} I_{\pm,\bar{\mathbf{k}}} &= 2\pi \int_0^\infty d\bar{k}\bar{k}^2 \frac{1}{i\bar{k}x} \frac{[e^{i\bar{k}x} - e^{-i\bar{k}x}]}{E_\pm} \\ &= \frac{2\pi}{ix} \int_{-\infty}^{\infty} \frac{d\bar{k}\bar{k}e^{i\bar{k}x}}{E_\pm} \end{aligned}$$

Next we set

$$E = \frac{\hbar^2 k^2}{2m}$$

which, by conservation of energy, is the same as the free-particle energy of the incident wave, $\varphi_{\mathbf{k}}(\mathbf{r})$. We obtain

$$\begin{aligned} \frac{2m}{\hbar^2} E_\pm &= (k^2 - \bar{k}^2 \pm i\frac{2m\epsilon}{\hbar^2}) \\ &= (k - \bar{k} \pm i\bar{\epsilon})(k + \bar{k} \pm i\bar{\epsilon}) \end{aligned}$$

where $\bar{\epsilon} \equiv m\epsilon/\hbar^2 k$. Thus

$$\frac{\bar{k}}{E_\pm} = -\frac{2m}{\hbar^2}\left(\frac{1/2}{(k - \bar{k} \mp i\bar{\epsilon})} + \frac{1/2}{(k + \bar{k} \pm i\bar{\epsilon})}\right)$$

We are now prepared to integrate over $\bar{k}$ by contour integration. As the integrand contains the factor $\exp(i\bar{k}\Delta r)$, it must be closed in the domain Im $\bar{k} > 0$, that is, the upper half $\bar{k}$ plane. For $I_{+,\bar{\mathbf{k}}}$ there is a pole at

$$\bar{k} = k + i\bar{\epsilon}$$

whereas for $I_{-,\bar{\mathbf{k}}}$ there is a pole at

$$\bar{k} = k - i\bar{\epsilon}$$

Passing to the limit $\bar{\epsilon} \to 0$, we obtain

$$\begin{aligned} I_{\pm,\bar{\mathbf{k}}} &= -\frac{2\pi}{ix}\frac{2m}{\hbar^2}2\pi i(\frac{1}{2})e^{\pm ikx} \\ &= \frac{4\pi^2 m}{\hbar^2}\frac{e^{\pm ikx}}{x} \end{aligned}$$

whence

$$I_{\pm} = -\frac{1}{(2\pi)^3}\frac{4\pi^2 m}{\hbar^2}\int d\mathbf{r}'\frac{e^{\pm ikx}}{x} < \mathbf{r}'|\hat{V}|\psi^{(+)} >$$

Inserting this relation into the starting LS equation

$$\psi_{\mathbf{k}}^{(\pm)}(\mathbf{r}) = \varphi_{\mathbf{k}}(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{\exp[\pm ik|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}V(\mathbf{r}')\psi_{\mathbf{k}}^{(\pm)}(\mathbf{r}')d\mathbf{r}'$$

At large distances from the interaction region we may write

$$\begin{aligned} \frac{1}{|\mathbf{r}-\mathbf{r}'|} &\approx \frac{1}{r} + ... \\ k|\mathbf{r}-\mathbf{r}'| &= kr[1 + (\frac{r'}{r})^2 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2}]^{1/2} \\ &\approx kr[1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + ... \\ &\approx kr - \mathbf{k}'\cdot\mathbf{r}' + ... \end{aligned}$$

where we set

$$\mathbf{k}' \equiv \frac{k}{r}\mathbf{r}$$

Plugging these expansions into the LS equation in the coordinate representation yields

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \varphi_{\mathbf{k}}(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r} < \varphi_{\acute{\mathbf{k}}}|V|\psi_{\mathbf{k}}^{(+)} >$$

Comparing with the definition of the scattering amplitude gives

$$f(\theta) = -\frac{m}{2\pi\hbar^2} < \varphi_{\acute{\mathbf{k}}}|V|\psi_{\mathbf{k}}^{(+)} >$$