Chapter 6. Inelastic Scattering in the Coupled-Channel (CC) Method

Contents

This lecture is summarized in the paper by Taro Tamura, (*Review of Modern Physics*, **37**, 679-708 (1965)) entitled "Analysis of the Scattering of Nuclear Particles by Collective Nuclei in terms of the Coupled-channel Calculation" and in the technical report by B. T. Kim and T. Udagawa entitled "Coupled-channel Analysis for Heavy-ion Induced Scattering and Transfer Reactions", *Tech. Rep. No. 2, Nuclear Theory Group, Univ. of Texas (1977).*

1 Introduction

Most of elastic and inelastic scattering data have been successfully described by the distorted wave Born approximation (DWBA) which we have studied in the previous Lecture. The reason for the success of DWBA lies in the fact that the interaction which causes these reactions is usually comparatively weak, so the first-order perturbation treatment is a good approximation.

In treating inelastic scattering, special care must be exercised, compared to the treatment of transmutation reactions. In some cases the inelastic scattering cross sections have rather large magnitudes, indicating that the interactions involved there are sometimes much stronger than those in the transmutation reactions, and thus that DWBA might become a poor approximation. Such a situation is often met when the target nucleus is of a collective nature.

When the DWBA turns out to be poor, one conceivable improvement is to include the second and higher order Born approximations. However to perform such higher-order calculations is a rather involved task.

Another possible improvement is to DWBA is the coupled-channel (CC) calculation. In such calculations the interactions are considered to infinite order in the sense that, although one restricts the number of channels to be considered to a finite number, the interaction is treated exactly within the channels considered. As is expected, the restriction of the number of channels is not a bad approximation if we are interested in the explanation of the excitation of a limited number of lower excited states.

There is another big advantage in the use of the coupled-channel calculations over the DWBA. In many nuclei there are low-lying states that have strong collective natures, but the inelastic cross sections to them are not necessarily very large. These are, for example, the higher phonon states in vibrational nuclei, or the higher members of the rotational bands of deformed nuclei. The excitation of these states is caused via the excitation of the states that lie at lower energies. In other words, their excitation cannot be caused through single-step processes. If such multiple processes are to be treated in terms of perturbation theory, it has to be made through higher-order Born approximations. However the formulation and the programming become progressively more involved each time one increases the order of the approximation. In the coupled-channel calculations, such processes can be treated by simply increasing the number of coupled channels. Furthermore, in this calculations the formulation, and consequently the coding the computer program, is essentially the same irrespective of how many channels one takes into account and thus can be handled comparatively easily.

Tamura developed a computer program, called JUPITER, for the full CC calculations in 1967.

When two heavy-ions collide with a sufficiently large impact parameter, so that the two ions only slightly at their nuclear surfaces (peripheral collision), there occur rather simple processes such as elastic and inelastic scattering and few-nucleon transfer reactions leading to discrete low-lying states. It can be well described by the direct reaction approach. If two ions touch more closely, then more violent processes involving large energy, angular momentum, charge, and mass transfer, called deep inelastic scattering and fusion, occur. It can simply be treated as absorption when we are concerned with the description of the peripheral collisions.

Another important element in the heavy-ion reactions is the Coulomb interaction. Because of the large charge of ions participating in the reactions, the Coulomb potential energy becomes particularly large, and thus the Coulomb excitation processes has to be correctly taken into account. However, there are two reasons, which make the CC calculations including the effects of the Coulomb excitation prohibitively time consuming: (1) long-range Coulomb force requires integration of the CC equations out to large distances, and (2) a large number of partial waves must be considered.

Kim and Udagawa developed a computer program, called JPWKB, which allows the calculations to be made very fast without losing needed accuracy, for heavy-ion scattering in 1977. They solved CC equations exactly in the interior region based on the program JUPITER, while in the exterior region, a WKB approximation is used.

We discuss the formulation of the CC calculations according to the Tamura's paper and then present briefly the idea of Kim and Udagawa.

2 Coupled-channel differential equation

Let us assume that there are N_s states in the target nucleus which are coupled strongly (directly or indirectly) to the ground state by the coupling potential V_{coupl}. We label these states with $n = 1$ to N_s, $n = 1$ meaning the ground state, and we let the spin, parity, and energy of nth state be I_n, π_n, and ω_n, respectively. Then if the energy of the incident particle (in the center-of-mass system) is E_1, the energy of the particle which leaves the target in its nth state is $E_n = E_1 - \omega_n$.

In a channel corresponding to the nth state of the target consider a partial wave of a projectile of a spin s that has orbital angular momentum ℓ_n and total angular momentum $j_n (= \vec{\ell}_n + \vec{s})$. Couple this j_n to I_n vectorially and define an angular momentum J of the whole system together with a parity operator Π also of the whole system:

$$\vec{J} = \vec{j}_n + \vec{I}_n, \quad \text{and} \quad \Pi = \pi_n (-)^{\ell_n}$$

With the choice of the interaction potential V_{coupl} in the next section, it is easy to see that none of ℓ_n, j_n, I_n, π_n, or $(-)^{\ell_n}$ is a good quantum number, but J as well as Π is good. In other words, several partial waves whose ℓ_n and j_n satisfy the above equations for a given set of J and Π are coupled together through V_{coupl} to form a set of coupled differential equations. We denote the number of such set of (ℓ_n, j_n) by $n_c^{(n)}$ and call its maximum possible number as $N_c^{(n)}$. For a given value of I_n the value of $N_c^{(n)}$ is as follows:

$$\begin{aligned}
N_c^{(n)} &= I_n + 1 \quad \text{if } s = 0 \text{ and } I_n \text{ is an integer,} \\
&= I_n + 1/2 \quad \text{if } s = 0 \text{ and } I_n \text{ is a half-integer,} \\
N_c^{(n)} &= 2I_n + 1 \quad \text{if } s = 1/2 \quad \text{and all } I_n, \\
N_c^{(n)} &= 3I_n + 2 \quad \text{if } s = 1 \text{ and } I_n \text{ is an integer,} \\
&= 3I_n + 3/2 \quad \text{if } s = 1 \text{ and } I_n \text{ is a half-integer.}
\end{aligned}$$

Corresponding to $n_c^{(n)}$ and $N_c^{(n)}$ we define

$$n_c = \sum_{n=1}^{N_s} n_c^{(n)}, \quad \text{and} \quad N_c = \sum_{n=1}^{N_s} N_c^{(n)}$$

which give, respectively, the total number of the coupled partial waves $(\ell_n j_n)$ for a given J and Π, and its maximum possible value.

The terminology to be used as follows: If we say a "partial-wave channel", it means a channel corresponding to a particular set of values of ℓ_n and j_n. On the other hand if we say a "nth state channel" it means $n_c^{(n)}$ sets of values of $(\ell_n j_n)$, as a whole.

In the actual numerical calculations the values of J are to be varied for each Π from its minimum values J_{min}, which is equal to 0 or 1/2 depending, respectively, whether $I_n + s$ is an integer or a half-integer, to its maximum value J_{max}, which may be taken as equal to $I_1 + j_{max}$, where j_{max} is the maximum value of j of the partial wave that gives a non-negligible contribution to the (elastic) scattering.

We now introduce the Hamiltonian

$$\begin{aligned}
H &= T + H_t + V(r, \theta, \phi) \\
&= T + H_t + V_{diag} + V_{coupl}
\end{aligned}$$

where T is the kinetic energy of the incident particle, H_t the Hamiltonian of the internal motion of the target nucleus, and V the interaction potential which is divided into V_{diag} and V_{coupl} which will be defined in the next section in detail. The Schrödinger equation can then be written as

$$H\Psi = E_1 \Psi$$

where the total wave function Ψ may be written as

$$\begin{aligned}\Psi &= \frac{1}{r}\sum_{Jn\ell_n j_n} R_{Jn\ell_n j_n}(r)(\mathcal{Y}_{\ell_n j_n}\otimes\Phi_{I_n})_{JM} \\ &= \frac{1}{r}\sum_{Jn\ell_n j_n} R_{Jn\ell_n j_n}(r)\sum_{m_j M_n}(j_n m_j I_n M_n|JM)\mathcal{Y}_{\ell_n j_n m_j}\Phi_{I_n M_n}\end{aligned}$$

where

$$\mathcal{Y}_{\ell_n j_n m_j} = \sum_{m_\ell m_s}(\ell_n m_\ell s m_s|j_n m_j) i^\ell Y_{\ell_n m_\ell}\chi_{s m_s}$$

$\chi_{s m_s}$ being the spin wave function of the projectile, while $\Phi_{I_n M_n}$ is the wave function of the target nucleus in the nth state. By definition we have

$$H_t\Phi_{I_n M_n} = \omega_n\Phi_{I_n M_n}$$

Inserting the total wave function into the Schrödinger equation, multiplying it by $(\mathcal{Y}_{\ell_n j_n}\otimes\Phi_{I_n})^*_{JM}$ from the left, integrating over all the coordinates except the radial variable r, and finally dividing by E_n, one gets the following equation:

$$\begin{aligned}&(\frac{d^2}{d\rho_n^2} - \frac{\ell_n(\ell_n+1)}{\rho_n^2} - \frac{1}{E_n}V_{diag} + 1)R_{Jn\ell_n j_n}(r) \\ &\quad = \frac{1}{E_n}\sum_{n'\ell'_n j'_n} < (\mathcal{Y}_{\ell_n j_n}\otimes\Phi_{I_n})_{JM}|V_{coupl}|(\mathcal{Y}_{\ell'_n j'_n}\otimes\Phi_{I'_n})_{JM} > \; R_{Jn'\ell'_n j'_n}(r)\end{aligned}$$

where $\rho_n = k_n r$ with k_n the wave number. This equation represents a set of n_c coupled equations.

This CC equation is quite general and holds irrespective of the nature of the projectile or the target nucleus. Assumptions about the nuclear structure only affect the matrix elements that appear on the right-hand side of the CC equation, and the evaluation of that matrix elements is the most crucial part of the whole calculation. We will first present the coupling potential in the next section and then discuss the coupling matrix elements in the section VIII-4.

3 Interaction potential

As is easily seen, the coupled-channel calculation is required to be made only when the couplings between the excited state and the ground state channels are strong, that is to say only when the low-lying states of the target nucleus have a strong collective nature. Such collective motions can be described fairly well in terms of the (permanent or vibrational) deformation of the nuclear shape. In this view the description of the coupling potentials between channels can be made in a simple and rather unambiguous way.

Throughout this work, we use this phenomenological description, and thus assume that the whole interaction to which an incident particle is subject is described by an optical potential $V(r,\theta,\phi)$ which is, in general, nonspherical. This potential is usually complex and includes the spin-orbit interaction and also the Coulomb interaction if the incident particle is charged. As for the radial dependent of the potential, we assume that of the Woods-Saxon form and its derivative. Thus explicitly we assume that

$$\begin{aligned}V(r,\theta,\phi) &= -(V+iW)\frac{1}{1+\exp[(r-R)/a]} + V_{Coul} \\ &\quad -4iW_D\frac{1+\exp[(r-\bar{R})/\bar{a}]}{\{1+\exp[(r-\bar{R})/\bar{a}]\}^2} \\ &\quad -V_{SO}(\vec{\sigma}\cdot\vec{\ell})\bar{\lambda}_\pi^2\frac{1}{ar}\frac{1+\exp[(r-R)/a]}{\{1+\exp[(r-R)/a]\}^2}\end{aligned}$$

where $\bar{\lambda}_\pi$ is the π-meson Compton wavelength and the Coulomb interaction V_{Coul} is given explicitly later. If R and $\bar{R}$ are taken independent of the angle, then this potential is nothing but the optical model potential. However we make R and $\bar{R}$ dependent on the polar angles θ and ϕ, in accord with the phenomenological description mentioned above. Their dependence is to be determined according to the collective nature of the target nucleus considered. If the target nucleus is spherically symmetric, but is susceptible to vibrational around the spherical shape, R and $\bar{R}$ may be expressed as

$$\begin{aligned} R &= R_0(1+\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta,\phi)) \\ \bar{R} &= \bar{R}_0(1+\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta,\phi)) \end{aligned}$$

Here, $R_0 = r_0 A^{1/3}$ and $\bar{R}_0 = \bar{r}_0 A^{1/3}$, where A is the mass number of the target. On the other hand, if the target is an (axially symmetric) deformed nucleus, they are written as

$$\begin{aligned} R &= R_0(1+\sum_{\lambda}\beta_{\lambda}Y_{\lambda 0}(\theta')) \\ \bar{R} &= \bar{R}_0(1+\sum_{\lambda}\beta_{\lambda}Y_{\lambda 0}(\theta')) \end{aligned}$$

where the angle θ' refers to the body-fixed system.

Since it is impractical to solve the scattering subject to the interaction potential just discussed, we now make the following (plausible) approximations.

(1) For the vibrational nucleus we insert R into the interaction potential and expand the latter in powers of $\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}$. By defining e and $\bar{e}$ by

$$e = \exp[(r-R_0)/a] \quad \text{and} \quad \bar{e} = \exp[(r-\bar{R}_0)/\bar{a}]$$

and taking the series up to the second order of $\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}$, we get

$$\begin{aligned} V &= -(V+iW)(1+e)^{-1} - 4iW_D\bar{e}(1+\bar{e})^{-2} - V_{SO}(\vec{\sigma}\cdot\vec{\ell})\bar{\lambda}_\pi^2\frac{1}{ar}e(1+e)^{-2} \\ &\quad -\{(V+iW)(R_0/a)e(1+e)^{-2} - 4iW_D(\bar{R}_0/2\bar{a})\bar{e}(1-\bar{e})(1+\bar{e})^{-3}\}(\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}) \\ &\quad +\{(V+iW)(R_0^2/2a^2)e(1-e)(1+e)^{-3} - 4iW_D(\bar{R}_0^2/2\bar{a}^2)\bar{e}(1-4\bar{e}+\bar{e}^2)(1+\bar{e})^{-4}\}(\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu})^2 \\ &\quad +V_{Coul} \end{aligned}$$

We can use the following relation

$$(\sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu})^2 = \sum_{\lambda\lambda_1\lambda_2}\frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}\hat{\lambda}}(\lambda_1 0\lambda_2 0|\lambda 0)\ \sum_{\mu}Y_{\lambda\mu}(\alpha_{\lambda_1}\otimes\alpha_{\lambda_2})_{\lambda\mu}$$

where $\hat{\lambda}\equiv(2\lambda+1)^{1/2}$, while the symbol $\otimes$ means the vector addition, i.e.,

$$(\alpha_{\lambda_1}\otimes\alpha_{\lambda_2})_{\lambda\mu} = \sum_{\mu_1\mu_2}(\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu)\alpha_{\lambda_1\mu_1}\alpha_{\lambda_2\mu_2}$$

We now specify the Coulomb interaction following Satchler as

$$\begin{aligned} V_{Coul} &= ZZ'e^2\int\rho(r',\theta',\phi')(|\vec{r}-\vec{r}'|)^{-1}d\vec{r}' \\ &= 4\pi ZZ'e^2\sum_{\lambda\mu}\int\rho(r',\theta',\phi')\ \hat{\lambda}^{-2}r_<^{\lambda}r_>^{-(\lambda+1)}Y_{\lambda\mu}(\theta,\phi)Y^*_{\lambda\mu}(\theta',\phi')r'^2dr'd\Omega' \end{aligned}$$

Here Ze and $Z'e$ are the charges of the projectile and the target, respectively, while $\rho(r', \theta', \phi')$ gives the charge distribution in the target, which with a reasonable accuracy may be assumed to be constant within the Coulomb radius $R_C(\theta', \phi')$ and zero outside. Thus ρ is written as

$$\rho(r', \theta', \phi') = \frac{3}{4\pi R_C^3}\Theta(R_C(\theta', \phi') - r')$$

where $\Theta(r) = 1$ if $r > 0$ and $\Theta(r) = 0$ if $r < 0$.

Assuming again that

$$R_C(\theta', \phi') \;\; = \;\; R_C(1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta', \phi'))$$

V_{Coul} is expressed to second order in $\sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}$ as

$$\begin{aligned}
V_{Coul} &= \frac{ZZ'e^2}{2R_C}(3 - \frac{r^2}{R_C^2})\Theta(R_C - r) + \frac{ZZ'e^2}{r}\Theta(r - R_C) \\
&+ \sum_{\lambda\mu} \frac{3ZZ'e^2}{\hat{\lambda}^2}[\frac{r^\lambda}{R_C^{\lambda+1}}\Theta(R_C - r) + \frac{R_C^\lambda}{r^{\lambda+1}}\Theta(r - R_C)]\ (\alpha_{\lambda\mu} Y_{\lambda\mu}) \\
&+ \sum_{\lambda\mu} \frac{3ZZ'e^2}{2\hat{\lambda}^2}[\frac{(1-\lambda)r^\lambda}{R_C^{\lambda+1}}\Theta(R_C - r) + \frac{(\lambda+2)R_C^\lambda}{r^{\lambda+1}}\Theta(r - R_C)] \\
&\quad \times \sum_{\lambda_1\lambda_2} \frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}\hat{\lambda}}(\lambda_1 0 \lambda_2 0|\lambda 0)(\alpha_{\lambda_1} \otimes \alpha_{\lambda_2})_{\lambda\mu} Y_{\lambda\mu}
\end{aligned}$$

We thus have the interaction potential

$$V(r, \theta, \phi) = V_{diag} + V_{coupl}^{(v)}$$

where the superfix (v) meaning that we are considering vibrational nuclei as targets. The diagonal term V_{diag} and the coupling term V_{coupl} become

$$\begin{aligned}
V_{diag} &= -(V + iW)(1 + e)^{-1} - 4iW_D\bar{e}(1 + \bar{e})^{-2} - V_{SO}(\vec{\sigma} \cdot \vec{\ell})\bar{\lambda}_\pi^2 \frac{1}{ar} e(1 + e)^{-2} \\
&\quad + \frac{ZZ'e^2}{2R_C}(3 - \frac{r^2}{R_C^2})\Theta(R_C - r) + \frac{ZZ'e^2}{r}\Theta(r - R_C) \\
V_{coupl}^{(v)} &= \sum_{\lambda\mu} v_{cp;\lambda}^{(1)(v)} \alpha_{\lambda\mu} Y_{\lambda\mu} + \sum_{\lambda\mu} v_{cp;\lambda}^{(2)(v)} \frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}\hat{\lambda}}(\lambda_1 0 \lambda_2 0|\lambda 0) \sum_{\mu} (\alpha_{\lambda_1} \otimes \alpha_{\lambda_2})_{\lambda\mu} Y_{\lambda\mu} \\
v_{cp;\lambda}^{(1)(v)} &= -\{(V + iW)(R_0/a)e(1 + e)^{-2} - 4iW_D(\bar{R}_0/2\bar{a})\bar{e}(1 - \bar{e})(1 + \bar{e})^{-3}\} \\
&\quad + \frac{3ZZ'e^2}{\hat{\lambda}^2}[\frac{r^\lambda}{R_C^{\lambda+1}}\Theta(R_C - r) + \frac{R_C^\lambda}{r^{\lambda+1}}\Theta(r - R_C)] \\
v_{cp;\lambda}^{(2)(v)} &= \{(V + iW)(R_0^2/2a^2)e(1 - e)(1 + e)^{-3} - 4iW_D(\bar{R}_0^2/2\bar{a}^2)\bar{e}(1 - 4\bar{e} + \bar{e}^2)(1 + \bar{e})^{-4}\} \\
&\quad + \sum_{\lambda\mu} \frac{3ZZ'e^2}{2\hat{\lambda}^2}[\frac{(1-\lambda)r^\lambda}{R_C^{\lambda+1}}\Theta(R_C - r) \; + \; \frac{(\lambda+2)R_C^\lambda}{r^{\lambda+1}}\Theta(r - R_C)]
\end{aligned}$$

The potential V_{diag} is diagonal with respect to the total spin j of the projectile and the eigenspin I of the target nucleus, and is nothing but the usual optical model potential. On the other hand, $v_{cp;\lambda}^{(v)}$ gives the coupling potential between channels which have different j and I.

(2) For a rotational nucleus, if the resulting potential is again expanded in powers of $\sum_\lambda \beta_\lambda Y_{\lambda 0}(\theta')$, one gets exactly the same expression as in (1), except that $\alpha_{\lambda\mu}$ and $Y_{\lambda\mu}(\theta\phi)$, are replaced, respectively, by β_λ and $Y_{\lambda 0}(\theta') = \sum_\mu D^\lambda_{\mu 0}(\theta_i) Y_{\lambda\mu}(\theta,\phi)$. Here $D^\lambda_{\mu 0}(\theta_i)$ is a rotation matrix and θ_i stands for the Euler angles between the body-fixed and the space-fixed coordinates.

It was noticed that, while $v^{(1)(v)}_{cp;\lambda}$ for the vibrational nucleus has vanishing diagonal elements, the corresponding term in the rotational nucleus has non-vanishing diagonal elements, and this difference can give rise to a large difference in the calculated differential cross sections in some cases. Thus it is customary to adopt the expansion in terms of Legendre polynomials $P_\lambda(\cos\theta')$, or rather in terms of $Y_{\lambda 0}(\cos\theta') = (\hat{\lambda}/4\pi)^{1/2} P_\lambda(\cos\theta')$, and then replace $Y_{\lambda 0}(\cos\theta')$ by $\sum_\mu D^\lambda_{\mu 0}(\theta') Y_{\lambda\mu}(\theta,\phi)$, instead of the power series expansion in $\sum_\lambda \beta_\lambda Y_{\lambda 0}(\theta')$.

The calculation of the Legendre polynomial expansion is straightforward, and it is enough simply to give the final result. We again have

$$V(r,\theta,\phi) = V_{diag} + V^{(r)}_{coupl}$$

where $V^{(r)}_{coupl}$ is given by

$$V^{(r)}_{coupl} = \sum_{\lambda\mu,\lambda\neq 0} v^{(\lambda)(r)}_{cp} D^\lambda_{\mu 0} Y_{\lambda\mu}(\theta,\phi)$$

with

$$v^{(\lambda)(r)}_{cp} = 4\pi \int_0^1 \{ \frac{-(V+iW)}{1+e(\theta')} + \frac{-4iW_D \bar{e}(\theta')}{[1+\bar{e}(\theta')]^2} \}\ Y_{\lambda 0}(\theta') d(\cos\theta')$$

where

$$e(\theta') = \exp[(r - R_0(1 + \sum_{\lambda'} \beta_{\lambda'} Y_{\lambda' 0}(\theta'))/a]$$
$$\bar{e}(\theta') = \exp[(r - \bar{R}_0(1 + \sum_{\lambda'} \beta_{\lambda'} Y_{\lambda' 0}(\theta'))/a]$$

On the other hand, V_{diag} is the same as that of the case for the vibrational nucleus, except that the first two terms in it are to be replaced by $(4\pi)^{-1/2}$ times $v^{(0)(r)}_{cp}$ in the above equation. Here we gave only the nuclear part of the coupling potential. For the Coulomb part perhaps the power-series expansion is sufficiently good approximation. Therefore, the Coulomb interaction for a deformed nucleus can be taken into account by adding to $V^{(r)}_{coupl}$, after replacing $\alpha_{\lambda\mu} Y_{\lambda\mu}$ by $\beta_\lambda D^\lambda_{\mu 0} Y_{\lambda\mu}$ in $v^{(1)(v)}_{cp;\lambda}$ and $v^{(2)(v)}_{cp;\lambda}$.

4 Coupling matrix elements

We now discuss the coupling matrix elements. The coupling potential V_{coupl} in the previous section can still be in a general form as

$$V_{coupl} = \sum_{t,\lambda} v_\lambda^{(t)}(r)(Q_\lambda^{(t)} \cdot Y_\lambda)$$

where the superscript t conveniently discriminates terms of different character but the same tensorial rank λ, and $Q_\lambda^{(t)}$ means an operator which operates only on the coordinates of the target nucleus. The calculation of the matrix element is straightforward and the result is given by

$$\begin{aligned} M &= <(\mathcal{Y}_{\ell j} \otimes \Phi_I)_{JM}|V_{coupl}|(\mathcal{Y}_{\ell' j'} \otimes \Phi_{I'})_{JM}> \\ &\equiv <\ell j I|V_{coupl}|\ell' j' I'> \\ &= \sum_{t,\lambda} v_\lambda^{(t)}(r) < I||Q_\lambda^{(t)}||I' > A(\ell j I, \ell' j' I', \lambda J s) \end{aligned}$$

with

$$\begin{aligned} A(\ell j I, \ell' j' I', \lambda J s) &= (4\pi)^{-1/2}(-)^{J-s-I'+\ell+\ell'+1/2(\ell'-\ell)}\ \hat{\ell}\hat{\ell'}\hat{j}\hat{j'}(\ell 0 \ell' 0|\lambda 0) \\ &\quad \times\ W(jIj'I'; J\lambda)W(\ell j \ell' j'; s\lambda) \end{aligned}$$

The factor $A(\ell j I, \ell' j' I', \lambda J s)$ is completely geometrical.

Proof of $A(\ell j I, \ell' j' I', \lambda J s)$:

$$\begin{aligned} M &= \sum_{t,\lambda} v_\lambda^{(t)}(r) \sum_{all\ m's} < I M_I|Q_{\lambda\mu}^{(t)}|I' M_I' > \ [\int Y_{\ell m_\ell} Y^*_{\ell' m_\ell'} Y_{\lambda\mu} d\Omega]\ [\chi_{s m_s}\chi_{s' m_s'}] \\ &\quad \times\ i^\ell (j m_j I M_I|JM)(\ell m_\ell s m_s|j m_j)\ i^{-\ell'}(j' m_j' I' M_I'|JM)(\ell' m_\ell' s' m_s'|j' m_j') \end{aligned}$$

Now we have

$$\begin{aligned} P &\equiv < I M_I|Q_{\lambda\mu}^{(t)}|I' M_I' > \\ &= < I||Q_\lambda^{(t)}||I' > \hat{I}^{-1}(I' M_I' \lambda\mu|I M_I) \\ Q &\equiv \int Y_{\ell m_\ell} Y^*_{\ell' m_\ell'} Y_{\lambda\mu} d\Omega \\ &= \frac{\hat{\ell}\hat{\lambda}}{\sqrt{4\pi}\hat{\ell'}}(\ell 0 \lambda 0|\ell' 0)(\ell m_\ell \lambda\mu|\ell' m_{\ell'}) \\ &= \frac{(-)^\ell}{\sqrt{4\pi}}\hat{\ell}(\ell 0 \ell' 0|\lambda 0)(\ell m_\ell \lambda\mu|\ell' m_{\ell'}) \end{aligned}$$

which gives

$$\begin{aligned} A &= \sum_{all\ m's} \frac{(-)^\ell \hat{\ell}}{\sqrt{4\pi}\hat{I}} i^{\ell-\ell'}(\ell 0 \ell' 0|\lambda 0)(\ell m_\ell \lambda\mu|\ell' m_{\ell'})(I' M_I' \lambda\mu|I M_I)(j m_j I M_I|JM)(\ell m_\ell s m_s|j m_j) \\ &\quad \times\ (j' m_j' I' M_I'|JM)(\ell' m_\ell' s' m_s'|j' m_j') \end{aligned}$$

The products of the Clebsch-Gordan coefficients leave

$$\begin{aligned}
R &= \sum_{m_\ell m'_\ell m_s} (\ell m_\ell \lambda\mu|\ell' m'_\ell)(\ell m_\ell s m_s|j m_j)(\ell' m'_\ell s m_s|j' m'_j) \\
&= (-)^{\ell+\lambda-\ell'} \hat{\ell'}\hat{j}(\lambda\mu j m_j|j' m'_j) W(\lambda\ell j' s|\ell' j) \\
&= (-)^{\ell+\lambda-\ell'+\ell'+j-\lambda-s} \hat{\ell'}\hat{j}(\lambda\mu j m_j|j' m'_j) W(\ell j \ell' j'; s\lambda) \\
S &= \sum_{m_j m'_j M_I M'_I} (\lambda\mu j m_j|j' m'_j)(I' M'_I \lambda\mu|I M_I)\ (j m_j I M_I|J M)(j' m'_j I' M'_I|J M) \\
&= (-)^{j+I-J+j'+I'-J} \hat{I}\hat{j}' W(j\lambda J I'; j' I) \\
&= (-)^{j+I-J+j'+I'-J}(-)^{j'+I'-\lambda-J} \hat{I}\hat{j}' W(j I j' I'; J\lambda)
\end{aligned}$$

Combing together gives $A(\ell j I, \ell' j' I', \lambda J s)$. *Q.E.D.*

When $s = 1/2$, it reduces to

$$A(\ell j I, \ell' j' I', \lambda J s) = (4\pi)^{-1/2}(-)^{J-1/2-I'+\ell+\ell'+1/2(\ell'-\ell)}\ \hat{j}\hat{j}'(j, -\frac{1}{2} j' \frac{1}{2}|\lambda 0) W(j I j' I'; J\lambda)$$

by using a relation

$$\hat{\ell}\hat{\ell'}(\ell 0 \ell' 0|\lambda 0) W(\ell j \ell' j'; s\lambda) = (j, -\frac{1}{2} j' \frac{1}{2}|\lambda 0),$$

while if $s = 0$ it becomes

$$A(\ell j I, \ell' j' I', \lambda J s) = (4\pi)^{-1/2}(-)^{J-I'+\ell+\ell'+1/2(\ell'-\ell)}\ \hat{\ell}\hat{\ell'}(\ell 0 \ell' 0|\lambda 0) W(\ell I \ell' I'; J\lambda)\delta_{\ell j}\delta_{\ell' j'}$$

since

$$\hat{j}\hat{j}' W(\ell j \ell' j'; 0\lambda) = \delta_{\ell j}\delta_{\ell' j'}$$

The reduced matrix element $< I||Q^{(t)}_\lambda||I' >$ appearing in the coupling matrix element is defined by P in the proof of A, and contains all the dynamics involved in the problem. Here I stands for the all quantum numbers needed is specifying the state that is simply written as $|I >$.

Tamura gave the explicit form of the reduced matrix elements for various interesting cases. As an simple example, the reduced matrix element for the transition between the ground state and one-phonon (vibrational) states becomes

$$< 0; 0||Q^{(1)}_\lambda||1; I >= \delta_{\lambda I}\beta_\lambda$$

where β_λ is the deformation parameter. For the rotational excitation of the states which belong to the ground rotational band in he deformed nucleus,

$$< I||Q^{(1)}_\lambda||I' >= \hat{I}'(I' K \lambda 0|I K)$$

where K is the component of I along the symmetry axis in the body-fixed system.

5 Scattering amplitudes and cross sections

We are now ready to solve the CC equation for the radial wave function $R_{Jn\ell j}(r)$. If the solution thus obtained is matched at an appropriate matching radius, R_m, to its corresponding asymptotic solution, the scattering coefficient or the S matrix is obtained.

In order to write down the asymptotic solution explicitly, we first assume that initially both the projectile and the target are in some polarized states (including the unpolarized state as a special case), and that their polarized states are described by the amplitudes $a^{(i)}_{m_s}$ and $b^{(i')}_{M_1}$, respectively. Here the superscripts (i) and (i') specify different spin ensembles when more than one ensemble is needed for the description of the polarized state. It is understood that amplitudes for projectiles or targets which belong to different ensembles do not interfere, but the amplitudes for those which belong to the same ensemble but have different magnetic quantum numbers do interfere with each other.

For the unpolarized targets, $b^{(i')}_{m_I}$, when the z axis is parallel to the z' axis which we choose for polarization, becomes

$$b^{(i')}_{M_1} = b^{(N_1)}_{M_1} = \delta_{N_1 M_1} \hat{I}_1^{-1}$$

where N_1 is the magnetic quantum number in z'-axis, while for unpolarized beam of $s = \frac{1}{2}$ projectile, $a^{(i)}_{m_s}$

$$a^{(1)}_{\frac{1}{2}} = a^{(2)}_{-\frac{1}{2}} = (\frac{1}{2})^{1/2}, \quad \text{and} \quad a^{(1)}_{-\frac{1}{2}} = a^{(2)}_{\frac{1}{2}} = 0$$

However, it is desirable to give these amplitudes explicitly for the polarized targets or projectiles. See Tamura's Sec. IV-D for details.

Using the amplitudes $a^{(i)}_{m_s}$ and $b^{(i')}_{M_1}$, the asymptotic form of the wavefunction may be written as

$$\begin{aligned}
\Psi_{asymp} \;=\; & \frac{\sqrt{4\pi}}{rk_1\sqrt{v_1}} \sum_{(ii')m_s M_1} a^{(i)}_{m_s} b^{(i')}_{M_1} \sum_{\ell j J M} (\ell 0 s m_s | j m_s)(j m_s I_1 M_1 | J M) \\
& \times \sum_{n\ell' j'} [\hat{\ell}' \exp(i\sigma^{(1)}_{\ell'}) \delta_{n1} \delta_{\ell\ell'} \delta_{jj'} F^{(1)}_{\ell} + (\frac{k_1}{k_n})^{3/2} \hat{\ell}' \exp(i\sigma^{(n)}_{\ell'}) C^{J}_{\ell j; n\ell' j'} (G^{(n)}_{\ell'} + iF^{(n)}_{\ell'})] \\
& \times \; (\mathcal{Y}_{\ell' j'} \otimes \Phi_{I_n})_{JM}
\end{aligned}$$

where $F^{(n)}_{\ell}$ and $G^{(n)}_{\ell}$) are, respectively, regular and irregular Coulomb wave functions (at $r = R_m$), specified with the orbital angular momentum ℓ and the energy E_n, as well as the parameter $\eta_n = mZZ'\epsilon\hbar^2 k_n$ (m is the reduced mass of the projectile). $\sigma^{(n)}_{\ell}$ is the corresponding Coulomb phase shift.

All the expressions in the present section are given for charged projectiles. It is easy, however, to get the corresponding expressions appropriate for uncharged projectile by putting equal to zero the parameter η_n, the Coulomb phase shift $\sigma^{(n)}_{\ell}$ and the Rutherford amplitude $f_c(\theta)$ that appear below. We should further replace $F^{(n)}_{\ell}$ and $G^{(n)}_{\ell}$), respectively, $\rho_n j^{(n)}_{\ell}$ and $\rho_n n^{(n)}_{\ell}$), where $j^{(n)}_{\ell}$ and $n^{(n)}_{\ell}$) are spherical Bessel and Neumann functions, respectively, while $\rho_n = k_n R_m$.

Comparing the forms of radial equation discussed in Section VIII-2 and Ψ_{asymp} we can immediately write down the matching equation and the scattering coefficients $C^{J}_{\ell j; n\ell' j'}(G^{(n)}_{\ell'} + iF^{(n)}_{\ell'})$ are obtained by solving these equations.

In deriving the expression for the scattering cross section from Ψ_{asymp}, we first note that the first term of Ψ_{asymp} is just the incident wave Ψ_{inc} which is the sum of the (Coulomb distorted)

plane wave and the Coulomb scattered wave;

$$\Psi_{inc} = \frac{1}{\sqrt{v_1}}\{\exp[i(k_1 z - \eta_1 \ln k_1(r-z))](1 - \frac{\eta_1^2}{2k_1(r-z)}) + \frac{1}{r} f_c(\theta)\exp[i(k_1 r - \eta_1 \ln(2k_1 r))]\}$$
$$\times \sum_{(ii')m_s M_1} a_{m_s}^{(i)} b_{M_1}^{(i')} \chi_{sm_s} \Phi_{I_1 M_1}$$

where $f_c(\theta)$ is the Rutherford scattering amplitude, (See Lecture V-4.)

$$f_c(\theta) = \frac{\eta_1}{2k_1 \sin^2(\theta/2)} \exp[-i\eta_1 \ln(\sin^2(\theta/2)) + i\pi + 2i\sigma_0]$$

Corresponding to Ψ_{inc}, we rewrite the second term of Ψ_{asymp} by noting that asymptotically

$$G_\ell^{(n)} \longrightarrow_{r\to\infty} \cos(\Sigma_\ell^{(n)}),$$
$$F_\ell^{(n)} \longrightarrow_{r\to\infty} \sin(\Sigma_\ell^{(n)})$$

with

$$\Sigma_\ell^{(n)} = k_n - r\eta_n \ln(2k_n r) - (\ell\pi/2) + \sigma_\ell^{(n)}$$

Adding the second term of Ψ_{asymp} and Ψ_{inc}, the wave function for the scattered wave, Ψ_{scatt}, is now given as

$$\Psi_{scatt} = \sum_n \frac{\exp[i(k_n r - \eta_n \ln(2k_n r))]}{r v_n^{1/2}} \sum_{(ii')m_s M_1} a_{m_s}^{(i)} b_{M_1}^{(i')} \{f_c(\theta)\delta_{n1}\chi_{sm_s}\Phi_{I_1 M_1}$$
$$+ \sum_{\ell j n \ell' j'} \frac{\sqrt{4\pi}}{k_n} \hat{\ell}' \exp(i2\sigma_{\ell'}^{(n)}) C^J_{\ell j; n\ell' j'}(\ell 0 s m_s | j m_j)(j m_s I_1 M_1 | JM)(\mathcal{Y}_{\ell' j'} \otimes \Phi_{I_n})_{JM}\}$$

The scattering differential cross sections $\sigma_n(\theta,\phi)$, which leaves the target in its nth state is given as the absolute square of the second term of Ψ_{scatt}. Therefore we get

$$\sigma_n(\theta,\phi) = \sum_{(ii')m_s' M_n} | \sum_{m_s M_1} a_{m_s}^{(i)} b_{M_1}^{(i')} \{f_c(\theta)\delta_{n1}\delta_{m_s m_s'}\delta_{M_n M_1}$$
$$+ \sum_{\ell j n \ell' j' J m_\ell' m_j'} \frac{\sqrt{4\pi}}{k_n} \hat{\ell}' \exp(i2\sigma_{\ell'}^{(n)}) C^J_{\ell j; n\ell' j'}(\ell 0 s m_s | j m_j)(j m_s I_1 M_1 | JM)$$
$$(\ell' m_\ell' s' m_s' | j' m_j')(j' m_s' I_n M_n | JM) Y_{\ell' m_\ell'}(\theta,\phi)\} |^2$$

Note that $\sigma_n(\theta,\phi)$ depends in general on the azimuthal angle ϕ.

In order to express the differential cross section in a more compact form, we shall introduce an amplitude $X_{m_s M_1; m_s' M_n}(\theta,\phi)$ defined as

$$X_{m_s M_1; m_s' M_n}(\theta,\phi) = f_c(\theta)\delta_{n1}\delta_{m_s m_s'}\delta_{M_n M_1}$$
$$+ \sum_{\ell j n \ell' j' J m_\ell' m_j' M} \frac{\sqrt{4\pi}}{k_n} \hat{\ell}' \exp(i2\sigma_{\ell'}^{(n)}) C^J_{\ell j; n\ell' j'}(\ell 0 s m_s | j m_s)(j m_s I_1 M_1 | JM)$$
$$\times (\ell' m_\ell' s' m_s' | j' m_j')(j' m_s' I_n M_n | JM) Y_{\ell' m_\ell'}(\theta,\phi)$$

Then the differential cross section is reduced to

$$\sigma_n(\theta,\phi) = \sum_{(ii')m_s' M_n} | \sum_{m_s M_1} X_{m_s M_1; m_s' M_n}(\theta,\phi) a_{m_s}^{(i)} b_{M_1}^{(i')} |^2$$

The amplitude $X_{m_sM_1;m'_sM_n}(\theta,\phi)$ allows us to write down the expression for the polarization also in a compact form. The (vector) polarization $P_n(\theta,\phi)$, parallel to a given unit vector $\vec{n}$, of the particle scattered in the direction specified by the polar angles (θ,ϕ) and leaving the target in its nth state, is defined as the expectation value of the operator $(\vec{\sigma}\cdot\vec{n})$ with respect to the scattered amplitude, i.e., the second sum of Ψ_{scatt}, divided by $\sigma_n(\theta,\phi)$. We thus have

$$\begin{aligned}\sigma_n(\theta,\phi)P_n(\theta,\phi) &= \sum_{ii'}\sum_{m's} X^*_{m_sM_1;m'_sM_n}(\theta,\phi)X_{\bar{m}_s\bar{M}_1;\bar{m}'_s\bar{M}_n}(\theta,\phi) \\ &\times \ < m'_s|(\vec{\sigma}\cdot\vec{n})|\bar{m}'_s > a^{(i)*}_{m_s}b^{(i')*}_{M_1}a^{(i)}_{\bar{m}_s}b^{(i')}_{\bar{M}_1}\end{aligned}$$

where $\sum_{m's}$ means summation over all the magnetic quantum numbers that appear.

If the $n=1$ part of Ψ_{asymp} is written as Ψ_1 and the quantity

$$\frac{\hbar}{2im}\int[\Psi_1\frac{\partial\Psi_1^\dagger}{\partial r}-\Psi_1^\dagger\frac{\partial\Psi_1}{\partial r}]_r r^2 d\tau d\Omega$$

is computed, this is nothing but the reaction cross section σ_r. In the above equation $d\tau$ means summation and integration over all the internal coordinates, while $d\Omega$ means integration over all the angular variables of the relative motion. If the total elastic cross section σ_{el} [integral of $\sigma_n(\theta,\phi)$ over $d\Omega$] is added to σ_r, the result is the total cross section σ_t (which is to be compared with the experimental total cross section as obtained by the transmission experiment). If the incident particle is charged, σ_{el} and σ_t are both infinity and thus consideration on these cross sections has meaning only when a neutral particle is used as the projectile.

The expressions for σ_t and σ_r simplified by introducing an amplitude $Z_{m_sM_1;m'_sM'_1;\ell'}$, which is intimately related to the amplitude $X_{m_sM_1;m'_sM_n}(\theta,\phi)$, is defined as

$$Z_{m_sM_1;m'_sM'_1;\ell'} = \sum_{\ell jj'JMm'_\ell m'_j}(\ell 0sm_s|jm_s)(jm_sI_1M_1|JM)(\ell' 0sm'_s|j'm'_s)(j'm'_sI_1M'_1|JM)C^J_{\ell j;1\ell'j'}$$

Then

$$\sigma_t = \frac{4\pi}{k_1^2}\sum_{\ell'}(2\ell'+1)\{\Im\sum_{ii'm_sm'sM_1M'_1}a^{(i)}_{m_s}a^{(i)*}_{m'_s}b^{(i')}_{M_1}b^{(i')*}_{M'_1}Z_{m_sM_1;m'_sM'_1;\ell'}\}$$

and

$$\sigma_r = \sigma_t - \frac{4\pi}{k_1^2}\sum_{\ell'}(2\ell'+1)\sum_{ii'j'J}|\sum_{m_sM_1}a^{(i)}_{m_s}b^{(i')}_{M_1}\ (\ell 0sm_s|jm_s)(jm_sI_1M_1|JM)C^J_{\ell j;1\ell'j'}|^2$$

The equation for σ_t is nothing but the optical theorem in our case.

6 Numerical solution of CC equations

The CC equation can be written, in a matrix form, as

$$\frac{d^2R(\rho)}{d\rho^2} = A(\rho)R(\rho)$$

$R(\rho)$ is a column vector with elements $R_{Jn\ell_n j_n}(\rho_n)$ and $A(\rho)$ is a square matrix with elements,

$$\begin{aligned} A^J_{n\ell_n j_n, n'\ell'_n j'_n} &= [\frac{\ell_n(\ell_n+1)}{\rho_n^2} + \frac{V_{diag}}{E_n} - 1]\delta_{nn'}\delta_{\ell_n\ell'_n}\delta_{j_n j'_n} \\ &\quad + \frac{1}{E_n} < (\mathcal{Y}_{\ell_n j_n} \otimes \Phi_{I_n})_{JM} | V_{coupl} | (\mathcal{Y}_{\ell'_n j'_n} \otimes \Phi_{I'_n})_{JM} > \end{aligned}$$

It is well known that if there are N_c coupled equations, any solution of these equations can be written as a sum of N_c linearly independent solutions. It is also well known that the solutions obtained with linearly independent boundary solutions are linearly independent of each other at any radius. Since we know that $R(0) = 0$, we shall let the qth independent solution ($q = 1, 2, ..., N_c$) be the one that is obtained with the boundary condition that the column vector $R(h)$ has a nonvanishing value ($= c_q$) in the qth row and has zero in any other row. What value to be taken for c_q is immaterial as we shall see below, although it is desirable to choose a small value.

We shall label the (partial wave) channels $(\ell_n j_n)$ as $p = 1, 2, ..., N_n$. Then if we denote by u_{qp} the qth solution in the pth channel at the matching radius r_m, the internal solution in the pth channel can be written in the general form

$$R_{int,p}(r_m) = \sum_q a_q u_{qp}(r_m)$$

where unknown coefficient a_q is to be determined by matching $R_{int,p}(r_m)$ to an appropriate external (or asymptotic) solution. Before introducing the explicit expression of the asymptotic form, however, we remark that it is easy to see that a different choice of c_q is indeed immaterial, because it simply gives rise to a different value of a_q, so as to leave the value of $R_{int,p}(r_m)$ unchanged.

The asymptotic form of the scattered wave, Ψ_{scatt}, given in the previous section, reduces to the Coulomb distorted plane wave of a unit amplitude, if $I_n = s = 0$ and the scattering coefficient $C^{J\pi}_{\ell j, n'\ell' j'}$ is set equal to zero. Therefore what the quantity in the square bracket in Ψ_{scatt} describes is the radial part of the wave function in the channel $(n'\ell' j')$, when there is a unit amplitude incoming wave in the channel $(n\ell j)$. We can thus equate the the internal solution to the square bracket of Ψ_{scatt}, by understanding that the channel p stands for the channel $(n'\ell' j')$. Such equations taken for $p = 1, 2, ..., N_c$ and the corresponding equations for the derivative are simply the matching equations. The explicit form is

$$\begin{aligned} \sum_{q=1}^{N_q} a_q u_{qp} &= \hat{\ell}' \exp(i\sigma^{(n')}_{\ell'})[F^{(n')}_{\ell'}\delta_{nn'}\delta_{\ell_n\ell'_n}\delta_{j_n j'_n} + (\frac{k_1}{k_n})^{3/2} C^{J\pi}_{\ell j, n'\ell' j'}(G^{(n')}_{\ell'} + iF^{(n')}_{\ell'})] \\ \text{with} \quad & p = (n'\ell' j') \\ \sum_{q=1}^{N_q} a_q u'_{qp} &= \hat{\ell}' \exp(i\sigma^{(n')}_{\ell'})[F^{(n')'}_{\ell'}\delta_{nn'}\delta_{\ell_n\ell'_n}\delta_{j_n j'_n} + (\frac{k_1}{k_n})^{3/2} C^{J\pi}_{\ell j, n'\ell' j'}(G^{(n')'}_{\ell'} + iF^{(n')'}_{\ell'})] \end{aligned}$$

We see that this is a set of $2N_c$ linear equation for N_c unknown a_q and N_c unknown $C^{J\pi}_{\ell j, n'\ell' j'}$. Therefore, this set of CC equations is a well defined equation.

Instead of solving these CC equations just as it stands, we first reduce it to a set of N_c equations for the N_c unknown a_q. This can be done by multiplying the first equation and the second one,

respectively, by $(G_{\ell'}^{(n')'} + iF_{\ell'}^{(n')'})$, and $(G_{\ell'}^{(n')} + iF_{\ell'}^{(n')})$ and then taking the difference of the two equations thus obtained. The result is

$$\sum_{q=1}^{N_q} a_q[u_{qp}(G_{\ell'}^{(n')'} + iF_{\ell'}^{(n')'}) - u'_{qp}(G_{\ell'}^{(n')} + iF_{\ell'}^{(n')}) = \ -\hat{\ell}' \exp(i\sigma_{\ell'}^{(n')})\delta_{nn'}\delta_{\ell_n\ell'_n}\delta_{j_nj'_n}$$

If this is solved for a_q, then $C^{J\pi}_{\ell j,n'\ell'j'}$ can be obtained from the relation

$$(\frac{k_1}{k_n})^{3/2}\hat{\ell}' \exp(i\sigma_{\ell'}^{(n')})C^{J\pi}_{\ell j,n'\ell'j'} = \sum_{q=1}^{N_q} a_q[u_{qp}F_{\ell'}^{(n')'} - u'_{qp}F_{\ell'}^{(n')})]$$

which has been derived by the difference of the two equations after they are multiplied, respectively, by $F_{\ell'}^{(n')'}$ and $F_{\ell'}^{(n')}$. In this way our coupled equation has been solved completely.

7 Solution of CC equations with a strong Coulomb excitation

We now present a method to carry out CC calculations, including the Coulomb excitation effects, much faster. We return to the CC equation, neglecting the spin-orbit interaction, that is the case for the heavy-ion scattering,

$$(\frac{d^2}{d\rho_q^2} - \frac{\ell(\ell+1)}{\rho_q^2} - \frac{1}{E_q}V_{diag} + 1)\chi_{qq_0}(r) = \frac{1}{E_q}\sum_{q'} < (i^\ell Y_\ell \otimes \Phi_{I_A})_{JM}|V_{coupl}|(i^{\ell'}Y_{\ell'} \otimes \Phi_{I'_A})_{JM} > \ \chi_{q'q_0}(r)$$

The product function $|(i^\ell Y_\ell \otimes \Phi_{I_A})_{JM} >$ is called the channel wave function, which is characterized by a set of quantum number $q = (\ell_a, I_A)$, together with the total angular momentum J and the parity π,

$$\vec{J} = \vec{\ell} + \vec{I}_A, \quad \text{and} \quad \pi = (-)^\ell \pi_A.$$

$\chi_{qq_0}(r)$ is the radial wave function of the relative motion in the channel q generated by waves in an incident channel q_0 with a given J and π. $\rho_q = k_q r$ with k_q being the wave number in the channel q.

We solve CC equation using a WKB approximation. Our basic procedure is as follows. We first divide the whole interaction region into two parts: (1) an interior region, where the radial distance r is less than a separation distance R_S, and where both nuclear and Coulomb interactions have to be taken into account, and (2) an exterior region where $r > R_S$ and only Coulomb interactions need be considered. In the interior region, the CC equations are solved exactly, while in the exterior region a WKB approximation of Alder and Pauli (Nucl. Phys. **A128**, 193 (1969).) is used, which allows the calculations to be made very fast without losing needed accuracy.

For the moment, let us completely forget the effects of Coulomb excitation in the exterior region and try to solve the problem exactly. This is a well defined CC problem, and we denote the S-matrix thus obtained as $S^{(in)}$. We already explained how to obtain $S^{(in)}$ in detail in Section VIII-6.

We next consider the effects of Coulomb excitation in the exterior region. The total S-matrix, S, can then be written as

$$S = U^T S^{(in)} U$$

where the transformation matrix U describes all the effects of the Coulomb excitation in the exterior region. In the rest of this section, we derive the the above relation and describe how to calculate U, by using the WKB method.

Under a WKB approximation, the radial part of the partial wave in the channel q is written as

$$u_{qq_0} = \frac{1}{\sqrt{k_q}}(h_q^{(-)} a_{qq_0}^{(-)} - h_q^{(+)} a_{qq_0}^{(+)})$$

where $h_q^{(\pm)}$ are Coulomb wave functions given by

$$h_q^{(\pm)} = G_q \pm iF_q$$

and k_q is the q-channel wave number and F_q and G_q are the regular and irregular Coulomb wave functions, respectively. $a_{qq_0}^{(\pm)}$ is a function of r to be solved for. Since the strong oscillation of the wave function is already described by $h_q^{(\pm)}$, we may assume that $a_{qq_0}^{(\pm)}$ is a very slowly varying function of r.

The CC equation for $a_{qq_0}^{(\pm)}$ is obtained by inserting u_{qq_0} into the CC equation, multiplying either $h^{(+)}$ or $h^{(-)}$ from the left, and then applying the following three approximations: (1) $h_q^{(\pm)}$ are approximated by those obtained by the WKB approximation; (2) the second derivative of $a_{qq_0}^{(\pm)}$ with respect to r is neglected; and (3) the rapidly oscillating terms like $h^{(-)}h^{(-)}$ and $h^{(+)}h^{(+)}$ are neglected. It is then easy to see that $a_{qq_0}^{(\pm)}$ satisfies,

$$\frac{da_{qq_0}^{(\pm)}}{dr} = \mp \sum_{q'} V_{qq'}^{(\pm)} a_{q'q_0}^{(\pm)}$$

where

$$\begin{aligned}
V_{qq'}^{(\pm)} &= \frac{1}{2i}\frac{P_q P_{q'}}{\sqrt{k_q k_{q'}}} v_{qq'}^J \exp[\pm i(\phi_q - \phi_{q'})] \\
P_q &= [1 - \frac{2\eta_q}{k_q r} - \frac{\ell(\ell+1)}{k_q^2 r^2}]^{-1/4} \\
\phi_q &= \frac{k_q r}{P_q^2} + \eta_q \log[\frac{\sqrt{\eta_q^2 + \ell(\ell+1)}}{k_q r - \eta_q + \sqrt{k_q r}/P_q}] - \sqrt{\ell(\ell+1)} \\
&\quad \times \cos^{-1}[\frac{\eta_q}{\sqrt{\eta_q^2 + \ell(\ell+1)}} + \frac{\ell(\ell+1)}{k_q r\sqrt{\eta_q^2 + \ell(\ell+1)}}]
\end{aligned}$$

with η_q being the Sommerfeld parameter and $a_{qq_0}^{(\pm)}$ the electric multipole matrix element. Since $a_{qq_0}^{(\pm)}$ are slowly varying, the differential equation for $a_{qq_0}^{(\pm)}$ can be integrated in large steps, which is the origin of the fast integration we desired.

We note here that $V_{qq'}^{(\pm)}$ has the property that

$$V_{qq'}^{(\pm)*} = \mp V_{qq'}^{(\mp)}$$

which leads to the relation

$$a_{qq'}^{(-)*} = a_{qq'}^{(+)}.$$

In other words, $a_{qq'}^{(+)}$ and $a_{qq'}^{(-)}$ are not independent of each other, and thus the differential equation for $a_{qq'}^{(\pm)}$ needs to solved only for one set, either $a_{qq'}^{(+)}$ or $a_{qq'}^{(-)}$. The integration is carried out n times (n being the dimension of the CC equations) from $r = R_S$ to $r = R_M$, the matching radius, with a different starting values given as

$$a_{qk}^{(-)}(r = R_S) = \delta_{qk}, \quad q = 1, 2, ..., n.$$

The solution u_{qq_0} which corresponds to the boundary condition at $r = R_M$, i.e., the condition that

$$u_{qq_0} \to \frac{1}{\sqrt{k_q}}(h_q^{(-)}\delta_{qq_0} - h_q^{(+)}S_{qq_0})$$

may be put in the form

$$u_{qq_0} = \frac{1}{\sqrt{k_q}}\sum_k (h_q^{(-)}a_{qq_0}^{(-)}\alpha_{kq_0}^{(-)} - h_q^{(+)}a_{qk}^{(+)}\alpha_{kq_0}^{(+)})$$

The unknown coefficients $\alpha_{kq_0}^{(\pm)}$ are determined by the condition of u_{qq_0} as well as the requirement that u_{qq_0} matches smoothly to the interior solution at $r = R_S$. The result is gives, in the matrix form, as

$$\begin{aligned} \alpha^{(-)} &= [a^{(-)}(r \to \infty)]^{-1} \equiv U \\ \alpha^{(+)} &= S^{(in)}\alpha^{(-)} = S^{(in)}U \end{aligned}$$

It is also seen that the total S-matrix can be written as

$$S = a^{(+)}(\infty)\alpha^{(+)}$$

Using the relations above, and the fact that $U^\dagger = U^{-1}$, it is easy to see that the equation of S is reduced to $S = U^T S^{(in)} U$; The S-matrix is obtained in terms of $S^{(in)}$ and the transformation matrix U determined by our new method. Once an S-matrix is obtained, the C-matrix can be calculated as,

$$C = \frac{1}{2i}(S - 1)$$

The differential cross section can be expressed in terms of the C-matrix, $C^J_{q'q}$, given by the above equation, as

$$\frac{d\sigma}{d\Omega} = \frac{1}{2I_A + 1}\sum_{M'_A M_A} |T_{M'_A M_A}(\theta)|^2$$

where

$$T_{M'_A M_A}(\theta) = \sum_{\ell_a \ell'_a} \frac{4\pi}{\sqrt{k'_q k_q}} \hat{\ell}'_a \hat{\ell}_a \exp[i(\sigma_{\ell'_a} + \sigma_{\ell_a})](\ell_a 0 I_A M_A | J M_A)(\ell'_a m_{\ell'_a} I'_A M'_A | J M_A) C^J_{q'q} Y_{\ell'_a m_{\ell'_a}}(\theta)$$

where σ_{ℓ_a} is the Coulomb phase shift.

8 (Homework Set #16) CC calculation

Reproduce the results of B. T. Kim, (Phys. Lett. 80, 353 (1979)) by using the CC program, JPWKB.