

Scattering Theory Note: Reaction(FRESCO를 중심으로.)

Young-Ho Song

February 9, 2021

Contents

1	Direct Reaction theory	7
1.1	Reaction의 종류	7
1.2	핵반응에서 측정하는 물리량	12
1.3	Conservation laws in nuclear reaction	12
1.4	Reaction Theory 기본 용어	12
1.5	Direct Reaction channel	13
1.5.1	Wave functions	15
1.5.2	Optical Model	15
1.5.3	Cross section	15
1.5.4	Fraunhofer and Fresnel diffraction	20
2	Coupled Channel equations	23
2.1	Terms	23
2.1.1	In one convention	26
2.2	Model space wave function	27
2.3	Coupled channel equations	28
3	Couplings, Interactions	31
3.1	Multipole expansion of interaction	31
3.2	Double folding model	32
3.2.1	density matrix	33
3.2.2	Double Folding using Fourier Transformation	33
3.2.3	Spin/Isospin dependent folding potential	36
3.3	Exchange term of Double folding potential	37
3.3.1	M3Y potential	41
3.3.2	Folding of DDM3Y	43
3.3.3	Parametrization of density,potential and F.T.	47
3.4	Sao-Paulo potential	48
3.5	DWBA matrix element	48
3.6	Coupled equation sets	51
3.6.1	Coupled equations in partial wave form	54
3.7	Optical Potential: Feschbach P	55
3.7.1	a model	57
3.8	Appendix: Dispersion relation	60
3.8.1	Kramers-Kronig relation	60
3.8.2	In response	60
3.8.3	In optical potential	61

4	DWBA	63
4.1	DWBA expression from B.T.Kim's lecture	63
4.2	DWBA in Fresco	63
4.3	DWBA in other book...	63
5	Couplings, Interactions	65
5.1	Typical Form of optical potential	65
5.2	Single Nucleon binding potential	68
5.3	Multipole expansion of interaction	68
5.4	FRESCO potentials	69
5.5	Transition potential	69
5.6	General Spin transfer coupling	70
5.7	Inelastic couplings	71
5.8	Collective Model	72
5.9	Coulomb deformation	73
5.10	Inelastic couplings: simple rotor model	75
5.10.1	Deformation of potential	79
5.11	Another attempt	81
5.11.1	rotational excitation	82
5.11.2	Vibrational excitation	83
5.11.3	Coulomb polarization potential	84
5.11.4	Nuclear Moments	85
5.11.5	Fresco: deformation	86
5.11.6	Inelastic scattering: single particle excitation	87
5.12	single particle transfer	87
5.12.1	Cluster Wave function: two particle	90
5.12.2	Knockout reactions	91
5.12.3	Continuum: Break up	91
5.12.4	Capture	91
5.12.5	Charge exchange reactions	91
5.12.6	Photo-nuclear couplings	92
5.13	Structure Models	92
5.13.1	Shell Model	92
5.13.2	Cluster Models	92
5.13.3	Mean Field Models	92
5.13.4	Collective nuclear-matter description	92
5.14	Overlap functions	93
5.14.1	Non-antisymmetrized theory: Pauli principle이 무시되는 경우	93
5.14.2	Antisymmetrized theory	95
5.15	Cluster overlaps	97
5.16	One photon coupling	98
5.16.1	vector potential	98
5.16.2	Photon Cross section	100
5.17	Appendix: Watanabe potential and JS adiabatic potential	101

6	Inelastic Excitations	105
6.1	Collective excitation	105
6.1.1	Vibrational excitation	106
6.1.2	Rotational excitation	107
6.2	Microscopic single particle excitation	108
6.3	Microscopic multi-particle excitation?	108
7	Transfer reaction	109
7.1	Projectile stripping	109
7.1.1	Classical picture	109
7.2	Projectile pickup (or target stripping)	109
7.3	Knock out reaction	109
8	R-matrix methods and Compound Nucleus Reaction	111
8.1	Very simple model	111
8.2	Compound Nucleus Reaction	113
8.3	Convention	115
8.4	One-channel R-matrix method	116
8.4.1	potential scattering and resonance scattering	120
8.4.2	Gaussian basis	120
8.4.3	Lagrange function basis	121
8.5	Phenomenology	122
8.5.1	single channel isolated resonance	123
8.6	Multichannel R matrix	123
8.7	Short summary for multilevel multichannel R-matrix	124
8.8	Relation to cross section	126
8.9	Compound nuclear reaction	126
9	Resonating Group Method(RGM) and Generator Coordinate Method(GCM)	129
10	Approximations	131
10.1	General	131
10.2	Adiabatic methods	132
10.2.1	Johnson Soper approximation for transfer	133
10.3	Eikonal methods	134
10.4	Eikonal Knockout code : momdis	136
11	Fusion and Fission	139
11.1	Icoming Wave Boundary condition: CCFULL case	140
11.1.1	Semi-classical (above barrier)	141
11.1.2	Parabolic Approximation (tunneling below barrier)	141
11.2	Simple fission Liquid drop model	143
12	FRESCO Examples	145
12.1	Elastic Scattering :(M) $4\text{He}+58\text{Ni}$	145
12.1.1	test potential READ IN	146
12.2	Inelastic scattering(Rotor Model): (B) $12\text{C}(\alpha, \alpha)12\text{C}(2^+)$	146
12.3	Inelastic scattering: (M) $^{16}\text{O} + ^{208}\text{Pb}$ (Projectile and Taget Excitation)	148
12.3.1	Another example for ^8Li excitation	151

12.3.2	Inelastic scattering: (T) 11Be+197Au	154
12.3.3	Inelastic scattering: (T) alpha+20Ne	154
12.3.4	Inelastic scattering: (T) 16O+58Ni	154
12.4	Inelastic scattering(Cluster Model): (M) $^{28}\text{Si}(^{19}\text{F}, ^{19}\text{F}^*)^{28}\text{Si}$ at 60 MeV . . .	155
12.5	Breakup (CDCC) : (B) $^8\text{B} + ^{208}\text{Pb}$	158
12.5.1	CDCC : (T) 11Be+4He	161
12.5.2	1-DWDCC: (T) 8B + 58Ni	162
12.6	Adiabatic CDCC:(T) $d + ^{208}\text{Pb}$	162
12.6.1	Adiabatic, R-matrix solution: (T) d+208Pb	163
12.7	1 nucleon Transfer reaction: (B) $^{14}\text{N}(^{17}\text{F}, ^{18}\text{Ne})^{13}\text{C}$	164
12.8	Transfer: (T) 136Xe(t, α)135I in ZR approximation	166
12.9	Transfer CCBA: (T) $^{28}\text{Si}(^{19}\text{F}, ^{16}\text{O})^{31}\text{P}$	167
12.10	Two Neutron Transfer: (T) 9Li(t,p)11Li	168
12.10.1	Another two neutron transfer example	172
12.11	CRC : (T) 16O+208Pb 80 MeV multi-channel path	176
12.12	Capture : (B) $^{14}\text{C}(n, \gamma)^{15}\text{C}$	180
12.13	Exchange transfer: $^7\text{Li}(^8\text{Li}, ^7\text{Li})^8\text{Li}$	181
12.14	Sfresco: (B) p+112Cd elastic scattering	183
12.15	Deuteron overlap using Reid potential	184
12.16	Runtime Error	185
12.17	Charge Exchange	185
13	Appendix	187
13.1	Center of mass correction	187
14	FRESCO INPUT CARDS	189
14.1	Definition Cards or &FRESCO	189
14.1.1	Card 0	189
14.1.2	Card 1	189
14.1.3	Card 1a : In case of Rmatch< 0	190
14.1.4	Card 2	190
14.1.5	Card 3	190
14.1.6	Card 4	191
14.1.7	Card 4.5: If IBLOCK<0, then read:	192
14.1.8	Card 5	193
14.2	Potentials	197
14.2.1	Card 9: Coulomb potential	197
14.2.2	Card 10	197
14.2.3	Card 11: For TYPEs 12-13 or TYPE=10-13 with SHAPE=7,8,9 (&STEP)	197
14.2.4	Card 12: Blank card for indication of end of potential	197

Chapter 1

Direct Reaction theory

Basic notation of reaction, When projectile a collides with target A , and produce heavy remnant X and ejectiles x_1, \dots , we denote this reactions as

$$A(a, x_1 x_2 \dots) X. \quad (1.1)$$

When particle is excited denote it as A^* . When projectile lose its kinetic energy denote it as a' . In many case, only the energy of projectile and ejectile are measured.

실제 Reaction은 많은 경우, Binary reaction이고, Entrance Channel에서 mass m_a , spin s_a 인 projectile 을 mass m_A , spin I_A 인 Target 과 충돌시킨 뒤 m_b, s_b 와 M_B, I_B 인 핵이 angle θ 방향으로 얼마나 생성되는 지를 알고싶다. 따라서, 보통 Lab frame에서의 projectile kinetic energy를 input으로 한다. 반면, 일반적으로 이론 계산은 C.M. frame 에서 계산하는 것이 편하므로, 먼저, input을 C.M. frame 에서의 양으로 변환하고, 이론적으로 계산한뒤, 다시 Lab frame에서의 결과로 변환해 주어야 한다.

1.1 Reaction의 종류

Direct Reaction

- Elastic scattering, $A(a, a)A$: projectile and target stay in their g.s. , zero Q-value, no internal structure change. Usually the projectile angular distribution is measured.
- Inelastic scattering, $A(a, a')A^*$ or $A(a, a^*)A^*$: projectile or target left in excited state. Projectile gives up some of its energy to excite target nucleus A. If nucleus a also complex nucleus, it can also be excited. [If energy resolution in detection of a not small enough to resolve g.s. of target from low-lying excited states then cross section will be sum of elastic and inelastic components. This is called **quasi-elastic scattering**].

Usually the projectile angular distribution is measured with momentum/energy.

- Transfer reaction, $A(a, b)B$: 1 or more nucleons moved to the other nucleus. Stripping or Pickup.
- Fragmentation/Breakup/Knockout: 3 or more nuclei/nucleons in the final state

Usually referring to breakup of projectile a into two or more fragments. This may be elastic breakup or inelastic breakup depending on whether target remains in ground state.

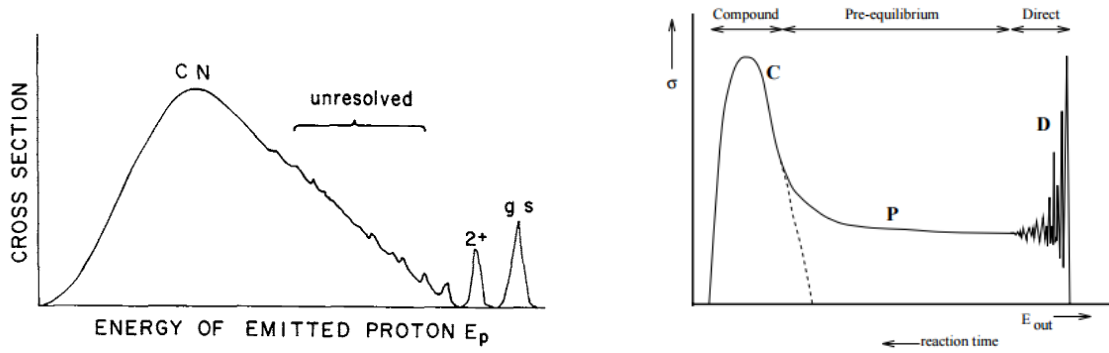


Figure 1.1: Reaction cross section for proton scattering. proton scattering 이 반드시 elastic scattering을 의미하지는 않는다. 그림에서 g.s.로 표시된 부분만이 elastic scattering에 해당한다. 주의할 것은 x축이 emitted particle energy라는 것이다.(즉, x축은 $E - E_{ex}$ 로 생각할 수 있다.) E_p 가 낮을 수록 남은 핵이 높은 에너지 상태에 있게되어 available level density가 높아진다. 이 경우에는 통계적인 관계식, Maxwell distribution을 잘 따르게 된다. 반면, E_p 가 높으면, 남은 핵의 에너지가 낮은 에너지 상태가 되어 각각의 resonance등의 level이 중요하게 된다. 여기서 낮은 에너지 쪽은 Coulomb에 의해 Maxwell distribution으로부터 distorted 되어 있다.

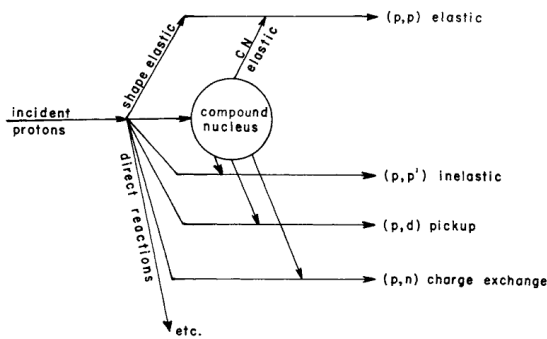


Fig. 1.2. Depiction of the processes that are typical of proton-nucleus interactions. (Adapted from P. E. Hodgson, 1971.)

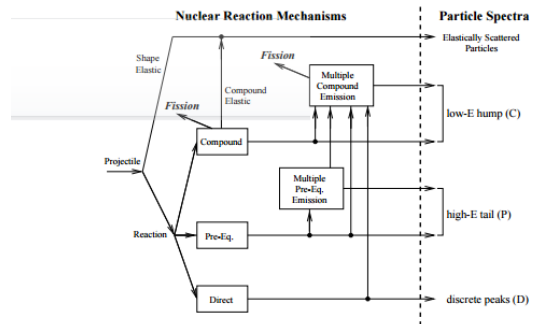


Figure 1.2: Competition between reactions for proton scattering. Note that compound nucleus can contribute any reaction channels.

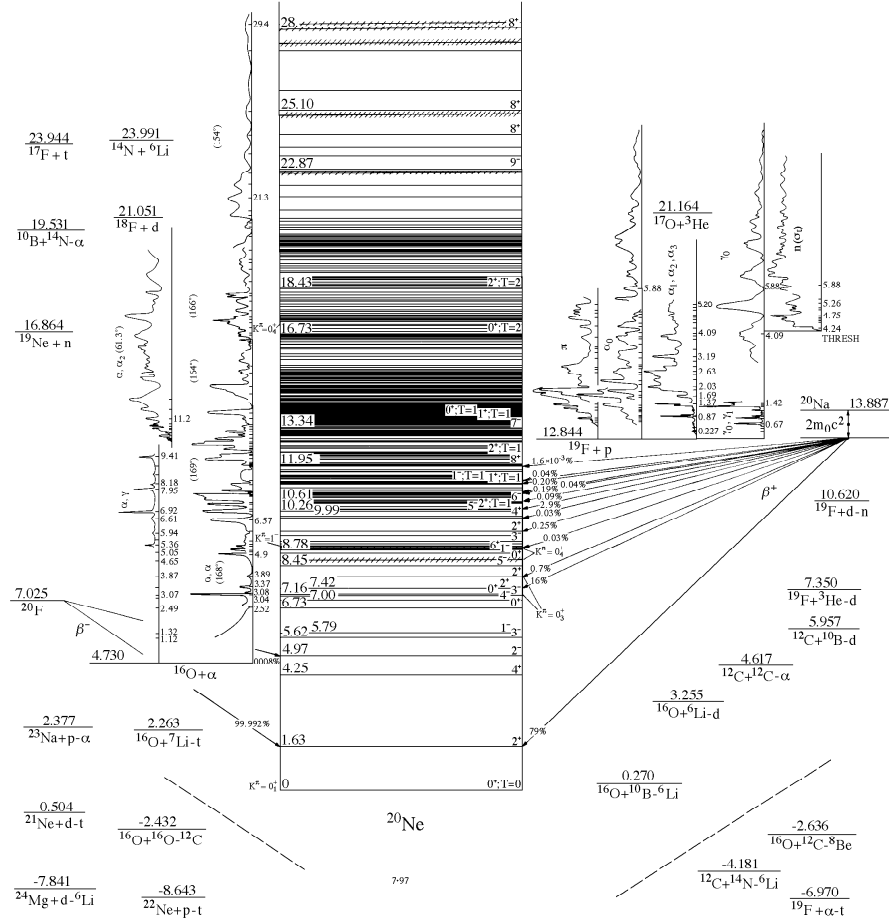


Figure 1.3: An Example of Energy levels. (Ne20 case). Consider the scattering of $^{16}\text{O} + \alpha$. Near threshold only several energy levels of ^{20}Ne are relevant. On the other hand, as increasing energy of scattering, very dense resonance states become relevant. In this case, statistical method should be used.

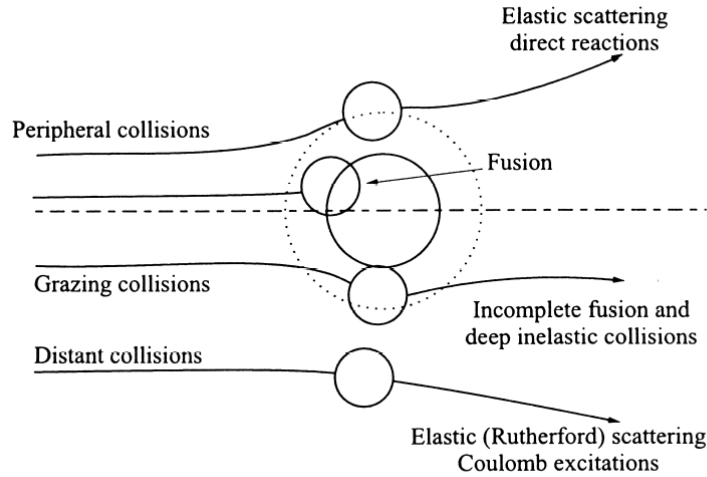


Figure 1.4: Various Heavy ion reaction depending on the impact parameter

In case of knockout reaction (in narrow definition), the projectile fragments into a core and a nucleon and the momentum of core particle is measured (target or fragment nucleon is not measured) with the gamma-ray emission measuring coincidences with decay gamma-rays emitted in flight. The momentum distribution gives the information of angular momentum and spectroscopic factor. (Difference between stripping and knockout? both refers a nucleon is removed from projectile. In case of stripping or inelastic breakup, the removed nucleon interacts with the target. But, in diffractive or elastic breakup, a nucleon is dissociated elastically?) Because only core is observed, both contribution of stripping and elastic breakup have to be summed.

MOMDIS code describes reaction $(c + v) + T \rightarrow c + X$.

- Charge Exchange: A is constant but Z (charge) varies, e.g. by pion exchange. mass numbers remain the same. Can be elastic or inelastic. It can occur in many ways. (1) directly changing the charge of both projectile and target. ($\Delta T = 1$ transition both in target and projectile) (2) knocking out one nucleon and replacing the projectile. ($\Delta T = 0$ transition vertex but exchanging two particles. In effect, the nuclear transition will be $\Delta T = 1$ though the vertex is $\Delta T = 0$.) (3) In a two step process by absorbing and emitting particles between target and projectile.
- Multistep Processes: intermediate steps can be any of the above ('virtual' rather than 'real')

Compound Reaction

- Deep inelastic collisions: Highly excited states produced
- Fusion: Nuclei stick together
- Fusion-evaporation: fusion followed by particle-evaporation and/or gamma emission
- Fusion-fission: fusion followed by fission

Compound Reaction과 Direct reaction의 차이

- Nuclei can coalesce to form highly excited Compound nucleus (CN) that lives for relatively long time. Long lifetime sufficient for excitation energy to be shared by all nucleons. If sufficient energy localised on one or more nucleons (usually neutrons) they can escape and CN decays. Independence hypothesis: CN lives long enough that it loses its memory of how it was formed. So probability of various decay modes independent of entrance channel.

Nuclei make ‘glancing’ contact and separate immediately, said to undergo Direct reactions(DI). Projectile may lose some energy, or have one or more nucleons transferred to or from it.

- Reaction이 일어나는 위치에 따라서 구분: CN reactions at small impact parameter, DI reactions at surface and large impact parameter. CN reaction involves the whole nucleus. DI reaction usually occurs on the surface of the nucleus. This leads to diffraction effects.

- Reaction에 걸리는 시간에 따라서 구분:

A typical nucleon orbits within a nucleus with a period of 10^{-22} sec. [as K.E. ~ 20 MeV]. If reaction complete within this time scale or less then no time for distribution of projectile energy around target) DI reaction occurred. CN reactions require 10^{-22} sec.

- Angular Distribution: In DI reactions differential cross section strongly forward peaked as projectile continues to move in general forward direction. Differential cross sections for CN reactions do not vary much with angle (not complete isotropy as still slight dependence on direction of incident beam).

Direct reactions at low energies need extra care. Are such energies high enough and timescales short enough for reactions to be fast and direct?

Let us consider an example of neutron incident reaction for different incoming neutron energies.

- Low energies ($E < 0.2$ MeV): Elastic scattering and capture. Since at low energy below 1st excited state of target, only possible reactions are (1) elastic scattering (2) capture (3) fission if target is fissionable.
- Low energies ($0.2 < E < 4$ MeV) : Inelastic scattering to discrete states. All reactions in previous energy are available and Compound reactions contributes to all channels. When the incident energy crosses an inelastic threshold, the compound inelastic contribution rises rapidly and predominates, whereas the direct component increases more gradually.
- High energy ($E > 4$ MeV): Pre-equilibrium reactions. reactions to the continuum also have a compound and a direct-like component. The latter are usually described by pre-equilibrium reactions which, by definition, include direct reactions to the continuum. Non-elastic channels contribute.
- High energy ($E > 8$ MeV): Multiple compound emission. At incident energies above about the neutron separation energy, the residual nuclides formed after the first binary reaction contain enough excitation energy to enable further decay by compound nucleus particle emission or fission.

- High energy ($E > 40$ MeV): Multiple pre-equilibrium emission. At still higher incident energies, above several tens of MeV, the residual nuclides formed after binary emission may contain so much excitation energy that the presence of further fast particles inside the nucleus becomes possible.

1.2 핵반응에서 측정하는 물리량

- 탄성산란의 미분단면적 : projectile energy(de Broglie wave length) and the size of the target affects the angle dependence of the cross section. Using this information, we can indirectly infer the average strength of interaction, the size of target. (Optical potential fit may have physical relevance with the radius and diffuseness of the target)
- Total reaction cross section : 입자의 핵에의 흡수정도를 나타내고, 상호작용의 크기와 핵의 크기에 대한 정보를 준다.
- Inelastic scattering cross section: 각운동량에 민감하다. 핵의 들뜬상태의 에너지, 각운동량, 패리티등에 대한 정보를 준다. 산란된 입자이외에도 방출된 gamma 나 전자를 동시에 측정하는 경우 더 자세한 정보를 얻을 수 있음.
- rearrangement reaction의 미분 산란 단면적: transfer, fusion 등등. one nucleon transfer reaction can be used to extract information for a single particle state. Two nucleon transfer can be used to extract the correlation of two nucleons.
- 편극량 측정: spin dependence of cross section gives more information on the nuclear interaction. 입사 입자, 표적, 산란입자의 스핀에 대한 의존도를 측정할 수 있다.

1.3 Conservation laws in nuclear reaction

- Energy-Momentum Conservation
- (total) Angular momentum Conservation
- (approximate) Parity
- (electric) Charge
- (approximate) Isospin

1.4 Reaction Theory 기본 용어

이제 부터는 Direct Reaction 의 경우만 생각하자.

For process,

$$p(proj) + T(target) \rightarrow x + R(remnant), \quad \rightarrow T(p, x)R. \quad (1.2)$$

의 반응의 Q-value 는

$$Q = (m_p + m_T - m_x - m_R)c^2, \quad (1.3)$$

으로 최종 products에서 available한 운동에너지에 해당한다. 이때 가능한 최종 product가 여러가지인 경우를 생각할 수 있다.

- In standard nuclear reaction, the number of nucleons does not change and the target is at rest. Thus, in non-relativistic limit, we can rephrase, $Q = T_x + T_R - T_p$.
- One necessary condition for the reaction occur is $Q + T_p > 0$. Further constraint is $T_p \geq -Q \frac{m_p + m_T}{m_T}$.
- Mass excess Δ : difference between actual mass and atomic mass unit times atomic number. $m_A = Am_{amu} + \Delta$.
- In fact, Q-value can be determined by measuring ejectile energy T_x and angle θ_x from Q-equation:

$$Q = T_x(1 + \frac{m_x}{m_R}) - T_p(1 - \frac{m_p}{m_R}) - \frac{2}{m_R} \sqrt{m_p T_p m_x T_x} \cos \theta_x \quad (1.4)$$

1.5 Direct Reaction channel

예를 들어서, 3 개의 입자, a, b, c 로 이루어진 시스템을 생각해보자. Initially, b 와 c 가 bound state (bc) 를 만들수 있다고 하자. $a + (bc)$ 의 반응에서 일어날 수 있는 경우가 다음과 같다고 하자.

$$\begin{aligned} a + (bc) &\rightarrow a + b + c, & 0(\text{break up}), \\ &\rightarrow a + (bc), & 1(\text{elastic}), \\ &\rightarrow a + (bc)^*, & 2(\text{inelastic}), \\ &\rightarrow (ab) + c, & 3(\text{transfer}), \\ &\rightarrow (abc), & 4(\text{capture}) \end{aligned} \quad (1.5)$$

이와 같이 가능한 구분 가능한 최종 product들(1,2,3,4)을 channel 이라고 부르고, 이 중 particle 들의 구성만으로 구분하는 것을 partition 이라고 부른다.(0,1,3,4 는 모두 다른 partition이지만, 1과 2는 같은 partition이다.) Each partition further distinguished by state of excitation of each nucleus and each such pair of states is known as a "**reaction channel**". The initial partition is known as the incident, or entrance channel and various possible outcomes are the possible exit channels. In a particular reaction, if not enough energy for a particular exit channel then it is said to be closed.

보통 channel이라는 말을 reaction channel과 각 reaction channel의 partial wave form을 구분하지 않고 사용한다. 그러나, 여기서는 되도록, reaction channel과 partial wave channel 을 구분하여 사용하도록 하자. reaction channel은 각 channel의 energy, partition 그리고, spin states 를 나타내고, partial wave channel은 각 reaction channel을 partial wave expansion 하여 angular momentum 까지 정해지는 상태를 나타낸다.

- Direct reaction 계산에서는 각 channel α 에서의 wave function이 projectile wave function(ϕ_p), target wave function(ϕ_t) 그리고 이 둘의 relative wave function $\psi_\alpha(R)$ 으로 주어진다 고 가정하고, 이 때의 exit channel의 relative wave function을 구하면 differential cross section등을 얻을 수 있다.

$$(\text{channel wave function}) = [\phi_p \otimes \phi_t]_\alpha \psi_\alpha(R) \quad (1.6)$$

Target과 Projectile의 bound state wave function이 알려져 있다고 가정할 때, 필요한 것은 Schrodinger equation으로 부터 각 channel의 relative wave function을 구하는 것이다. 한편, 이러한 relative wave function들이 만족해야 하는 식은 각 channel 사이의 coupling을 가지는 coupled equation이 된다.

- cross section: DWBA에서는 bound state wave function과 distorted wave function이 이미 알려져 있다고 생각하고 T-matrix를 potential과 wave function에 대한 integral로 계산한다. 한편, CC 에서는 bound state wave function은 이미 알려져 있다고 가정하지만, distorted wave를 coupled equation을 풀어서 구하게 된다. 따라서, 이 경우에는 얻어진 wave function으로부터 T-matrix 또는 S-matrix 를 구하고, 이를 이용해서 cross section을 계산한다. 그러므로, multi-channel T-matrix또는 S-matrix와 cross section을 연결시키는 식이 필요하다.

따라서, 계산은

- (1) Obtain distorted wave for channels either by solving coupled equation.
- (2) Compute T-matrix either from Asymptotic form of distorted wave from coupled equation or by computing DWBA matrix elements
- (3) Compute Differential Cross section from T-matrix.

- 예를 들어서 x 라는 partition의 target 과 projectile 있을 때, 이 partition이 J_{tot}, M_{tot} 을 가질 때 wave function 은 아래와 같이 나타 낼 수 있다.

$$\begin{aligned}
\Psi_{x, J_{tot}}^{M_{tot}}(R_x, \xi_p, \xi_t) &= \sum_{\alpha} \left[\left[i^L Y_L(\hat{R}_x) \otimes \phi_{I_p}^{xp}(\xi_p) \right]_{J_p} \otimes \phi_{I_t}^{xt}(\xi_t) \right]_{J_{tot} M_{tot}} \frac{1}{R_x} \psi_{\alpha}^{J_{tot}}(R_x) \\
&= \sum_{\alpha} |xpt : (LI_p)J_p, I_t; J_{tot} M_{tot}\rangle \frac{\psi_{\alpha}^{J_{tot}}(R_x)}{R_x} \\
&= \sum_{\alpha} |\alpha; J_{tot} M_{tot}\rangle \frac{\psi_{\alpha}^{J_{tot}}(R_x)}{R_x} \quad (1.7)
\end{aligned}$$

여기서 $\alpha = (x, p, t, L, I_p, J_p, I_t)$ 으으로써, $|\alpha; JM\rangle$ 는 J_{tot}, M_{tot} 을 만드는 모든 channel 들을 나타내고, bound state wave function 즉 internal structure를 포함한다.

- 일반적으로 incident channel wave function 은 J, M 의 eigen state가 아닌 plane wave 이므로, 이 들을 각 J, M 의 eigen state α_i 로 분해하고, 동시에 반응후에 갈 수 있는 최종 channel α 의 합으로 나타내면,

$$\Psi_{x_i p_i t_i}^{\mu_{p_i} \mu_{t_i}}(R_x, \xi_p, \xi_t; \mathbf{k}_i) = \sum_{J_{tot} M_{tot}} \sum_{\alpha \alpha_i} |\alpha; J_{tot} M_{tot}\rangle \frac{\psi_{\alpha \alpha_i}^{J_{tot}}(R_x)}{R_x} A_{\mu_{p_i} \mu_{t_i}}^{J_{tot} M_{tot}}(\alpha_i, \mathbf{k}_i) \quad (1.8)$$

where, "incoming coefficient" is defined as

$$A_{\mu_{p_i} \mu_{t_i}}^{J_{tot} M_{tot}}(\alpha_i, \mathbf{k}_i) = \frac{4\pi}{k_i} \sum_{M_i m_i} Y_{L_i M_i}^*(\mathbf{k}_i) \langle L_i M_i, I_{p_i} \mu_{p_i} | J_{p_i} m_i \rangle \langle J_{p_i} m_i, I_{t_i} \mu_{t_i} | J_{tot} M_{tot} \rangle. \quad (1.9)$$

incoming coefficient 는 plane incoming wave를 잘 정의된 incoming channel 들로 변환시켜주고, 최종 wave function은 이 incoming channel들이 outgoing channel로 바뀐 결과들의 합이다. 따라서, 우리가 필요한 것은 하나의 잘 정의된 incoming channel로부터의 outgoing channel의 wave function $\psi_{\alpha \alpha_i}^{J_{tot}}(R_x)$ 를 구하는 것이다.

즉, 위 식에서 $\Psi_{x_i p_i t_i}^{\mu_{p_i} \mu_{t_i}}(R_x, \xi_p, \xi_t; \mathbf{k}_i)$ 는 incident reaction channel $|x_i p_i t_i, \mu_{p_i} \mu_{t_i}, \mathbf{k}_i\rangle$ 로부터 얻어지는 total system wave function 으로, 각 reaction channel로의 wave function의 합이다. 한편, $A_{\mu_{p_i} \mu_{t_i}}^{J_{tot} M_{tot}}(\alpha_i, \mathbf{k}_i)$ 는 incident channel을 partial wave decompose 시킬 때 나타나는 partial reaction channel의 계수이다. 한편, $|\alpha; J_{tot} M_{tot}\rangle$ 는 최종적으로 관찰되는 partial reaction channel을 나타내고, $\psi_{\alpha \alpha_i}^{J_{tot}}(R_x)$ 는 incident partial channel α_i 에서 final partial channel α 로의 전이 확률을 나타내는 파동함수가 된다.

Or, in combination,

$$\Psi_{x_i p_i t_i}^{\mu_{p_i} \mu_{t_i}} = \sum \frac{4\pi}{k_i} i^L Y_{L_i M_i}^*(\mathbf{k}_i) Y_L(\hat{R}_x) \frac{1}{R_x} \chi_{\alpha\alpha_i}^{J_{tot}}(R_x) (\text{C.G.'s})(\text{bound state waves}) \quad (1.10)$$

with ψ is normalized as Coulomb wave function,

$$\chi_{\alpha\alpha_i}(R_x) \rightarrow \frac{i}{2} [H_{\alpha}^{(-)} \delta_{\alpha\alpha_i} - S_{\alpha\alpha_i} H_{\alpha}^{(+)}] \quad (1.11)$$

주의할 것은 $1/k$ 는 incoming coefficient에 포함되어 있었다는 것이다.

- 어떤 channel α 에서 target과 projectile의 bound state 를 αp , αt 라고 하자. 마찬가지로 wave function은 $\phi_{\alpha p}(\xi_{\alpha p})$, $\phi_{\alpha t}(\xi_{\alpha t})$ 로 쓰고, ξ 는 bound state의 internal degree of freedom 을 나타낸다.

1.5.1 Wave functions

To start with, the [excitation energies, spins and parities](#) of all the states in each [mass partition](#) need to be specified, along with [the nuclear masses, charges and relative Q-values of the partitions](#). Each pair projectile and target excited states is then a distinct channel with its own [scattering wave function and boundary conditions](#). The initial projectile and target states will select one such channel as the ‘incoming channel’, with its boundary conditions specifying an incoming plane wave. All channels (including the incoming channel) will have outgoing spherical waves. Particular attention must be given to the consistent placement of i^L factors in these definitions.

The individual nuclear states are then specified in sufficient detail for the particular reaction mechanisms involved. It is not necessary to specify the full quantum mechanical states of all the nucleons in the nucleus, but rather, only the states of those changed in the reactions being considered. In particular, [one and two-nucleon wave functions](#) will have to be described, if those nucleons are to be transferred to other nuclei.

If a nuclear state consists of a particle of spin s bound outside a nucleus with possible core states ϕ_I , then the [bound state radial wave functions](#) $u_{lsjI}(r)$ will have to be found by [solving a coupled-channels set of equations for negative energy eigen-solutions](#)(즉, 여기서 u 는 core-valence particle wave function).

If the particle is not bound, in the other hand, then its continuum range of energies must be discretized into a finite collection of ‘[bin](#)’ states which can be scaled to unit normalisation.

If the nuclear state consists of two particles of intrinsic spins s_1 and s_2 outside a core, then it is usually specified by a shell-model or by a Sturmian-basis calculation in terms of the independent coordinates r_1 and r_2 . To calculate transfer rates, however, the [two-particle wave functions](#) need to be given in terms of the collective coordinates (usually $\mathbf{r} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and $\rho = r_1 - r_2$). In order to use the states in a reaction calculation, therefore, equations are given for the transformation from the independent coordinates.

1.5.2 Optical Model

1.5.3 Cross section

In case of multi-channel case, differential cross section for asymptotic wave

$$\psi_{\beta}(\mathbf{k}_{\alpha}) \rightarrow e^{i\mathbf{k}_{\alpha} \cdot \mathbf{r}_{\alpha}} \delta_{\alpha\beta} + f_{\beta\alpha}(\mathbf{k}_{\beta}, \mathbf{k}_{\alpha}) \frac{e^{i\mathbf{k}_{\beta} \cdot \mathbf{R}_{\beta}}}{R_{\beta}} \quad (1.12)$$

gives flux,

$$J_\alpha = \frac{\hbar k_\alpha}{\mu_\alpha}, \quad \hat{J}_{\beta\alpha} = \frac{\hbar k_\beta}{\mu_\beta} |f_{\beta\alpha}|^2, \quad (1.13)$$

with

$$\begin{aligned} f_{\beta\alpha}(\theta) &= -\frac{\mu_\beta}{2\pi\hbar^2} \langle e^{i\mathbf{k}_\beta \cdot \mathbf{r}_\beta} | V_\beta | \Psi_\alpha^{(+)} \rangle \\ &= -\frac{\mu_\beta}{2\pi\hbar^2} T_{\beta\alpha}(\mathbf{k}_\beta, \mathbf{k}_\alpha), \end{aligned} \quad (1.14)$$

and cross section

$$\begin{aligned} \frac{d\sigma_{\beta\alpha}}{d\Omega} &= \frac{\mu_\alpha}{\hbar k_\alpha} \frac{\hbar k_\beta}{\mu_\beta} |f_{\beta\alpha}|^2 = \left(\frac{v_\beta}{v_\alpha}\right) |f_{\beta\alpha}(\mathbf{k}_\beta, \mathbf{k}_\alpha)|^2, \\ &= \frac{\mu_\alpha \mu_\beta}{(2\pi\hbar^2)^2} \left(\frac{k_\beta}{k_\alpha}\right) |T_{\beta\alpha}(\mathbf{k}_\beta, \mathbf{k}_\alpha)|^2. \end{aligned} \quad (1.15)$$

단, 여기서, $(f, T)_{\beta\alpha}$ 는 아직 partial wave expansion을 하지 않은 형태이다.

In other words,

$$T = \langle \mathbf{k}_f; I_{p_f} M_{p_f} I_{t_f} M_{t_f} | V | \mathbf{k}_i; I_{p_i} M_{p_i} I_{t_i} M_{t_i} \rangle^{(+)} \quad (1.16)$$

In this form, V contains every interaction in the Hamiltonian, left state is a plane wave state and right state is a full solution of Hamiltonian. Each states includes bound state wave functions and wave function for relative motion. Bra-ket implies the integration over all coordinates.

Then,

$$\begin{aligned} \sigma_{xpt}(\theta) &= \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum_{\mu_p \mu_t \mu_{p_i} \mu_{t_i}} \frac{v_f}{v_i} |f_{\mu_p \mu_t \mu_{p_i} \mu_{t_i}}^{xpt}|^2 \\ &= \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum_{\mu_p \mu_t \mu_{p_i} \mu_{t_i}} |\tilde{f}_{\mu_p \mu_t \mu_{p_i} \mu_{t_i}}^{xpt}(\theta)|^2 \end{aligned} \quad (1.17)$$

with

$$\begin{aligned} \tilde{f}_{\mu_p \mu_t \mu_{p_i} \mu_{t_i}}^{xpt}(\theta) &= \delta_{\mu_p \mu_{p_i}} \delta_{\mu_t \mu_{t_i}} \delta_{xpt, x_i p_i t_i} f_c(\theta) \\ &+ \frac{4\pi}{k_i} \sum_{L_i M_i, LM} \sum_{J_{p_i} m_i, J_p m} \sum_{J_{tot} M_{tot}} C_{L_i M_i, I_{p_i} \mu_{p_i}}^{J_{p_i} m_i} C_{J_{p_i} m_i, I_{t_i} \mu_{t_i}}^{J_{tot} M_{tot}} C_{LM, I_p \mu_p}^{J_p m} C_{J_p m, I_t \mu_t}^{J_{tot} M_{tot}} \\ &\times Y_{LM}(\hat{k}) Y_{L_i M_i}^*(\hat{k}_i) \tilde{T}_{\alpha\alpha_i}^{J_{tot}\pi} e^{i\sigma_L(\eta_\alpha) + i\sigma_{L_i}(\eta_{\alpha_i})} \end{aligned} \quad (1.18)$$

where

$$\tilde{T}_{\alpha\alpha_i}^{J_{tot}\pi} = \frac{i}{2} [\delta_{\alpha\alpha_i} - \tilde{S}_{\alpha\alpha_i}^{J_{tot}\pi}] \quad (1.19)$$

with

$$\tilde{S}_{\alpha\alpha_i}^{J_{tot}\pi} = \sqrt{\frac{v_\alpha}{v_{\alpha_i}}} S_{\alpha\alpha_i} \quad (1.20)$$

Note that usual i^{L-L_i} factor is included into the coupling potential, and thus into the definition of T -matrix by the combination with bound state wave.

Or,

$$\tilde{f}_{m'M';mM}^{xpt}(\theta) = \delta_{\alpha\beta} f_c(\theta) + \sum_{L'} A_{m'M';mM}^{L'} P_{L'}^{m'+M'-m-M}(\cos \theta) \quad (1.21)$$

where

$$\begin{aligned} A_{m'M';mM}^{L'} &= \sum_{LJJ'T} C_{L0J_p m}^{Jm} C_{JmJ_t M}^{J_T M_T} C_{L'M_L' J_p' m'}^{J' M_{L'} + m} C_{J' M_{L'} + m' J_t' M'}^{J_T M_T} \\ &\times \frac{4\pi}{k_i} \sqrt{\frac{k'\mu}{\mu'k}} e^{i\sigma_L} e^{i\sigma_{L'}} \frac{i}{2} [\delta_{\alpha\alpha'} - S_{\alpha'\alpha}^{J_T}] \sqrt{\frac{2L+1}{4\pi}} Y_c(L', M_{L'}) \end{aligned} \quad (1.22)$$

where $Y_c(L, M)$ is the coefficient of $P_L^{|M|}(\cos \theta) e^{iM\phi}$ in $Y_{LM}(\theta, \phi)$

In the last expression, $T_{\alpha\alpha_i}$ is expressed in partial wave form. Thus, in practice, it implies

$$\begin{aligned} T_{\alpha\alpha_i} &= \langle k_\alpha, \alpha\phi_\alpha | V_\beta | k_{\alpha_i}, \alpha_i, \phi_\alpha \rangle^{(+)} \\ |k_{\alpha_i}, \alpha_i, \phi_\alpha \rangle^{(+)} &= \sum_{\alpha'} |\alpha', \phi_{\alpha'} \rangle \frac{\chi_{\alpha\alpha_i}^{(+)}(R_{\alpha'})}{R_{\alpha'}}, \\ |k_\alpha, \alpha\phi_\alpha \rangle &= |\alpha, \phi_\alpha \rangle j_\alpha(R_\alpha). \end{aligned} \quad (1.23)$$

here, the wave function $\chi_{\alpha\alpha_i}^{(+)}$ is a full solution of full Schrodinger equation. Also, note that the expression implies the integral over all relevant coordinates. But, usually, the T -matrix is inferred from the boundary condition of full wave function instead of the integral over potential.

In case of DWBA, ($\alpha \neq \alpha_i$), the T -matrix is computed by the integral over potential,

$$\begin{aligned} T_{\alpha\alpha_i}^{DW} &= (-),^{DW} \langle k_\alpha, \alpha\phi_\alpha | V_\beta | k_{\alpha_i}, \alpha_i\phi_{\alpha_i} \rangle^{(+),DW}, \\ |k_\alpha, \alpha, \phi_\alpha \rangle^{(\pm),DW} &= \sum_{\alpha'} |\alpha', \phi_{\alpha'} \rangle \frac{\chi_{\alpha'\alpha}^{(\pm),DW}(R_{\alpha'})}{R_{\alpha'}}. \end{aligned} \quad (1.24)$$

where $\chi_{\alpha'\alpha}^{(\pm),DW}$ is a solution of Schrodinger equation with optical potential. In many case, the optical potential is diagonal, $\chi_{\alpha'\alpha}^{(\pm),DW} = \delta_{\alpha'\alpha} \chi_\alpha^{(\pm),DW}$.

Additional terms

- Consider beam intensity ϕ particles/sec with beam area $A \text{ cm}^2$. After passing target with thickness dx , beam intensity will change as $\phi + d\phi$.

Usually, beam intensity of charged particle is written in terms of current (micro Ampere, μA).

For example, $1\mu A$ proton is

$$\phi = (1\mu A) \times \frac{10^{-6} \text{Coul/sec}}{\mu A} \times \frac{1}{1.602 \times 10^{-19} \text{Coul/proton}} = 6.24 \times 10^{12} \text{protons/sec} \quad (1.25)$$

Or, one can use (particle microamperes) = 6.24×10^{12} ions/sec.

- Assume change of intensity is from the blocking of target nuclei. Then the ratio of ϕ and $d\phi$ will be related with the ration of beam area and effective area covered by all target nuclei. Number of all target nuclei is $\rho dx A$. Let effective area of one nuclei as σ_{tot} .

$$\frac{-d\phi}{\phi} = \frac{\rho dx A \sigma_{tot}}{A} = \rho \sigma_{tot} dx \rightarrow \phi(x) = \phi_0 e^{-\rho \sigma_{tot} x}. \quad (1.26)$$

Usually, target density is written in terms of N atoms/cm² for a very thin target instead of ρ atoms/cm³.

- Number of product nuclei, dN_p , during time t passing dx can be computed as

$$dN_p = \phi(x)(1 - e^{-\rho \sigma_{tot} dx})t. \quad (1.27)$$

If the product is radioactive and decay, change of number of product in time is

$$\begin{aligned} \frac{dN_p}{dt} &= (\text{production rate}) - (\text{decay rate}) = \rho \sigma_{tot} dx \phi - \lambda N_p \\ \Rightarrow \frac{dN_p}{\rho \sigma_{tot} dx \phi - \lambda N_p} &= dt \end{aligned} \quad (1.28)$$

We may introduce A as the disintegration rate of product nuclei at the end of the irradiation.

$$A = \rho \sigma \Delta x \phi (1 - e^{-\lambda t}) \quad (1.29)$$

Number of product nuclei at the end of irradiation is A/λ , or

$$N = \frac{\rho \sigma \Delta x \phi}{\lambda} (1 - e^{-\lambda t}) \quad (1.30)$$

- Classically we can view total cross section as $\sigma_{tot} \simeq \pi r_0^2 (A_P + A_T)^2$. In semi-classically, to get a contribution from a particular orbital angular momentum, we may use impact parameter $l = bp$ and $\sigma_l = \pi \frac{(l+1)^2}{p^2} - \pi \frac{l^2}{p^2} = \frac{\pi}{p^2} (2l+1)$. This explains $(2l+1)$ factor of cross section in partial wave.

$$\sigma_{tot} = \frac{\pi}{p^2} \sum_l^{l_{max}} (2l+1) = \frac{\pi}{p^2} (l_{max} + 1)^2 \quad (1.31)$$

We may think $l_{max} = pR_T$ and $\sigma_{tot} \simeq \pi(R_T + \frac{1}{p})^2$.

In a similar way, one can estimate the radius of a nuclei from the cross section measurement by approximation,

$$\sigma_I = \pi(R_I^p + R_I^t)^2, \quad (1.32)$$

where R_I is the interaction radius.

- For reaction, we may introduce transmission coefficient T_l such that

$$\sigma_R = \frac{\pi}{p^2} \sum_l (2l+1) T_l \quad (1.33)$$

Classically, $T_l = 0$ if $l < l_{max}$ and $T_l = 1$ if $l \leq l_{max}$. At low energy, $T_l = \sqrt{E}$ and $\sigma \propto \frac{1}{\sqrt{E}} \propto \frac{1}{v}$.

Reaction cross section may be measured by sum of all exit channel cross section or by a beam attenuation.

Ambiguity of definition? Total cross section is the sum of elastic cross section and reaction cross section. Thus **reaction cross section** is the sum of all possible channels except elastic channel. Is this correct? One defines **interaction cross section** as the sum of all possible channels except elastic and inelastic channels. **absorption cross section** is defined for complex optical potential which is defined as disappearance of elastic channel flux. Is this correct?

- In case of zero energy limit, the total cross section becomes $2\pi R^2$, 2 times larger than the classical cross section. In case of classical black disk, we expect $\sigma_{sc} = 0$, $\sigma_r = \pi R^2$ classically. However, actually there is no shadow even in forward direction at low energy limit. It is because there is a large scattering at the edge of the disk and it is large enough to remove the shadow. In other words, $\sigma_{sc} = \pi R^2$. Thus, $\sigma_{tot} = 2\pi R^2$.
- Classical cross section near the barrier: With Coulomb interaction, when the projectile energy E and its closes approach $R_c = R_P + R_T$, we have a reaction barrier as equal to Coulomb barrier $B = \frac{Z_1 Z_2 e^2}{R_c}$. (Note that because of orbital angular momentum, $E - B \neq 0$. Also, in this case, nuclear interaction is partly included in the radius.) The kinetic energy at this point is $T = E - B$ gives $p = \sqrt{2mT} = \sqrt{2\mu E} \sqrt{1 - B/E}$. Classically, orbital angular momentum, $l = rp \rightarrow l_{max} = R\sqrt{2\mu E} \sqrt{1 - B/E}$. Thus, total cross section, when $E > B$,

$$\sigma_{tot} = \pi \bar{\lambda}^2 (l_{max} + 1)^2 = \pi R^2 \left(1 - \frac{B}{E}\right). \quad (1.34)$$

Thus, cross section near the barrier is closely approach to classical limit from below.

- More realistically, $V_{tot} = V_C + V_{nucl} + V_{cent}$ gives interaction barrier which is different from Coulomb barrier from V_C .
- In the elastic scattering, there are diffractive patterns. This can be understood classically as an interference of plane waves from two sides of a nucleus. This simple model gives diffractive maxima at

$$n\lambda = 2 \times 2R \times \sin\left(\frac{\theta}{2}\right) \quad (1.35)$$

- Schematic approach to transfer reaction. In case of (d,p), if the momentum of deuteron and proton are \mathbf{k}_d and \mathbf{k}_p , the momentum of captured neutron is $\mathbf{k}_n = \mathbf{k}_d - \mathbf{k}_p$ i.e. $k_n^2 = k_d^2 + k_p^2 - 2k_d k_p \cos \theta$. If the neutron is captured at impact parameter R , transferred orbital angular momentum will be $\hbar \Delta l = k_n R \hbar$. By squaring the relation, we get

$$\cos \theta = \frac{-(\frac{\Delta l}{R})^2 + k_d^2 + k_p^2}{2k_d k_p}, \quad (1.36)$$

by measuring θ and k_p , one can determine Δl . (or one may use $k_n R = \sqrt{l(l+1)}$) Though this is a rough classical view, we can expect the peak position in angular

distribution will be sensitive to the effective radius and orbital angular momentum of the transferred neutron.

- Schematic description of elastic scattering of optical potential. In Bertulani's book, by using simple approximation of black sphere and Born approximation with W.S. potential, one can understand why the elastic cross section oscillates over angles and decrease for larger angles. In black sphere approximation,

$$\frac{d\sigma}{d\Omega} \sim \frac{2R}{\pi k \theta^2 \sin^2 \theta} \cos^2(kR\theta + \frac{\pi}{4}), \quad kR\theta \ll 1, \quad (1.37)$$

implying the cross section decrease as increasing angle and oscillate over angle in terms of scale set by radius R . (Note this is only valid for small angle θ .)

In Born approximation of W.S. potential,

$$\frac{d\sigma}{d\Omega} \sim |4\pi a \sin(qR) e^{-\pi q a} + \dots|^2 \quad (1.38)$$

implying that the oscillation is set by the scale R , while the decrease of cross section is set by diffuseness a .

- mean free path: 단순히 constant potential을 가정할 경우, mean-free-path 또는 amplitude 의 decay 는 대략 $\lambda \sim \frac{k}{2mW}$ scale에 의해 결정 된다. Typical $W \sim 10$ MeV를 가정할 때 neutron의 경우 $\lambda \sim 5$ fm 정도가 된다. 반면, classical gas라면, $\lambda_{clas} \sim \frac{1}{\rho\sigma} \sim 0.4$ fm 정도가 되어 두 경우의 차이가 매우 크다. (다른 말로 하면 classical gas의 경우에 W 가 더 크다?) 이것은 QM 효과인 Pauli principle때문 일지도 모른다? Pauli principle은 density가 작을 수록 효과가 작아질 것이고, 핵의 표면 부근에서 density가 작으므로, surface부근에서 W 가 커질 것으로 예상할 수 있다...?
- Fermi gas model 은 입자 사이의 상호작용은 무시하지만, Pauli principle은 고려한다.
- astrophysical S-factor:

$$\sigma(E) = \frac{1}{E} e^{-2\pi\eta} S(E), \quad \eta = \frac{Z_1 Z_2 e^2}{\hbar v} \quad (1.39)$$

1.5.4 Fraunhofer and Fresnel diffraction

As like optics, quantum scattering also shows a diffraction patterns. But, here one have to consider both Coulomb and strong interaction.

Roughly speaking, at small $\eta \ll 1$ (weak Coulomb or high energy), Fraunhofer diffraction occurs and at $\eta \geq 1$ (strong Coulomb or low energy), Fresnel diffraction occurs.

Fraunhofer diffraction can be obtained in sharp-cutoff strong absorption model with approximation $\eta \simeq 0$.

$$\frac{d\sigma^{FH}}{d\Omega}(\theta) = |iR \frac{J_1(kR\theta)}{\theta}|^2 \quad (1.40)$$

On the other hand, Fresnel diffraction can be obtained in smooth-cutoff strong absorption model in $\eta \simeq 1$: ratio to Rutherford to be zero at $\theta \gg \theta_{gr}$, to be 1 at $\theta = 0$, to be 1/4 at grazing angle $\theta = \theta_{gr}$. In SAM, $S(l)$ is parameterized with parameters λ_{gr} , Δ , μ .

$$S_l^N = S^N(\lambda) = (1 + i\mu \frac{d}{d\lambda}) \frac{1}{1 + \exp(\frac{\lambda_{gr} - \lambda}{\Delta})} \quad (1.41)$$

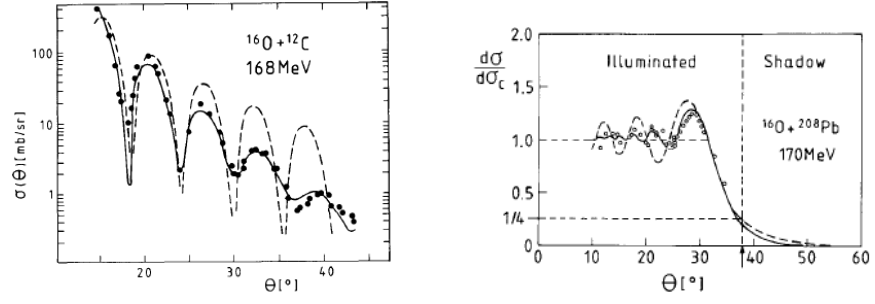


Figure 1.5: Fraunhofer /Fresnel diffraction

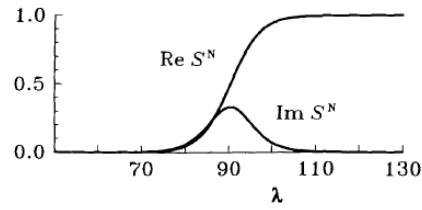


Figure 1.6: Strong absorption model is similar to black sphere but with smooth-cutoff as a function $S(l)$ which is absorption below grazing angular momentum is not perfect but still strong. This form can explain Fresnel diffraction pattern.

Chapter 2

Coupled Channel equations

2.1 Terms

Let us first make sure the terms to be used.

- **Partition** is specified by the projectile and target charge and mass.
- **Excited State pair** or **physical channels** are defined by projectile and target spin, parity and excitation energy in addition to partition.
- **channels** or **basis** are defined by orbital angular momentum and total angular momentum of the system in addition to physical channels.

Usually, we consider scattering of plane wave of projectile and target. The incident wave function can be considered as

$$\Psi_{\mathbf{k}}(\mathbf{r}, \xi_p, \xi_t) = e^{i\mathbf{k} \cdot \mathbf{r}} \phi_p(\xi_p) \phi_t(\xi_t). \quad (2.1)$$

In case of pure Coulomb scattering we may consider this wave function as

$$\Psi_{\mathbf{k}}(\mathbf{r}, \xi_p, \xi_t) = \psi_c(\mathbf{k}, \mathbf{r}) \phi_p(\xi_p) \phi_t(\xi_t). \quad (2.2)$$

with

$$\begin{aligned} \psi_c(\mathbf{k}, \mathbf{r}) &\sim e^{-\pi\eta/2} \Gamma(1+i\eta) e^{i\mathbf{k} \cdot \mathbf{r}} {}_1F_1(-i\eta, 1, i(kr - \mathbf{k} \cdot \mathbf{r})) \\ &= e^{i[kz + \eta \ln(k(r-z))]} + f_c(\theta) \frac{e^{i[kr - \eta \ln(2kr)]}}{r} \\ &= \sum_L (2L+1) i^L e^{i\sigma_L(\eta)} \frac{F_L(\eta, \rho)}{\rho} P_L(\cos \theta) \\ &= 4\pi \sum_{LM} i^L e^{i\sigma_L(\eta)} \frac{F_L(\eta, \rho)}{\rho} Y_{LM}(\hat{r}) Y_{LM}^*(\hat{k}) \end{aligned} \quad (2.3)$$

But, in case of multi-channel scattering, we have to consider all possible channels. Because the total angular momentum and parity will be conserved even after reaction, it would be convenient to combine spherical harmonics and projectile and target wave functions to define channel basis.

There are several ways to define basis with L , I_p , I_t and J_{tot} . By introducing additional J_p or S ,

$$\begin{aligned}
|\alpha\rangle &= \left[\left[i^L Y_L(\hat{R}_x) \otimes \phi_{I_p}^{xp}(\xi_p) \right]_{J_p} \otimes \phi_{I_t}^{xt}(\xi_t) \right]_{J_{tot} M_{tot}}, \text{ or} \\
&= \left[\left[Y_L(\hat{R}_x) \otimes \phi_{I_p}^{xp}(\xi_p) \right]_{J_p} \otimes \phi_{I_t}^{xt}(\xi_t) \right]_{J_{tot} M_{tot}}, \text{ or} \\
&= \left[i^L Y_L(\hat{R}_x) \otimes \left[\phi_{I_p}^{xp}(\xi_p) \otimes \phi_{I_t}^{xt}(\xi_t) \right]_S \right]_{J_{tot} M_{tot}}, \text{ or} \\
&= \left[Y_L(\hat{R}_x) \otimes \left[\phi_{I_p}^{xp}(\xi_p) \otimes \phi_{I_t}^{xt}(\xi_t) \right]_S \right]_{J_{tot} M_{tot}}, \text{ or} .
\end{aligned} \tag{2.4}$$

From now on, to denote whether i^L factor is included in the definition of $|\alpha\rangle$, let us denote $|i^\alpha, \alpha\rangle$ or $|\alpha\rangle$.

From the previous expansion we would expect, the scattering with nuclear potential to be

$$\Psi_{\mathbf{k}}^{(+)}(\mathbf{r}, \xi_p, \xi_t) = \sum_{\alpha_i} \frac{\psi_{\alpha_i}^{(+)}(\mathbf{r}, \xi_p, \xi_t)}{r} A_{\alpha_i}(\mathbf{k}) \tag{2.5}$$

where

$$A_{\alpha_i}(\mathbf{k}) = \frac{4\pi}{k_i} \sum_M e^{i\sigma_{L_i}} Y_{LM}^*(\hat{k}) \langle \alpha_i | \mathbf{k}, I_p, I_t \rangle \tag{2.6}$$

and $\langle \alpha_i | \mathbf{k}, I_p, I_t \rangle$ denote all combination of C.G. coefficients and coupling coefficients to make channel α_i . The wave function should be expanded in channel basis,

$$\frac{\psi_{\alpha_i}^{(+)}(\mathbf{r}, \xi_p, \xi_t)}{r} = \sum_{\alpha} |i^\alpha \alpha\rangle \frac{u_{\alpha, (\alpha_i)}(r)}{r} \tag{2.7}$$

In this convention, the i^L factor of plane wave limit is included in the channel basis. But if one use $|\alpha\rangle$ basis, we may define

$$\frac{\psi_{\alpha_i}^{(+)}(\mathbf{r}, \xi_p, \xi_t)}{r} = \sum_{\alpha} |\alpha\rangle \frac{u_{\alpha, (\alpha_i)}(r)}{r} \tag{2.8}$$

$$A_{\alpha_i}(\mathbf{k}) = \frac{4\pi}{k_i} \sum_M i^{L_i} e^{i\sigma_{L_i}} Y_{LM}^*(\hat{k}) \langle \alpha_i | \mathbf{k}, I_p, I_t \rangle \tag{2.9}$$

The asymptotic form of wave function $u_{\alpha, (\alpha_i)}$ defines scattering or collision matrix.

$$u_{\alpha, (\alpha_i)} \rightarrow \frac{i}{2} (I_{\alpha_i} \delta_{\alpha\alpha_i} - S_{\alpha, \alpha_i} O_{\alpha}) \tag{2.10}$$

or

$$u_{\alpha, (\alpha_i)} \rightarrow \sqrt{\frac{v_i}{v_{\alpha}}} \frac{i}{2} (I_{\alpha_i} \delta_{\alpha\alpha_i} - \tilde{S}_{\alpha, \alpha_i} O_{\alpha}) \tag{2.11}$$

The asymptotic form becomes

$$\frac{1}{r} \frac{i}{2} \left(H_{L'}^{(-)} \delta_{\alpha' \alpha} - S_{\alpha' \alpha} H_{L'}^{(+)} \right) \rightarrow \frac{1}{r} \sin \Theta_L \delta_{\alpha' \alpha} + \frac{e^{i\Theta_{L'}}}{r} \frac{(S_{\alpha' \alpha} - \delta_{\alpha' \alpha})}{2i}, \quad (2.12)$$

where

$$\Theta_L = kR - \frac{L\pi}{2} + \sigma_L - \eta \ln 2kR. \quad (2.13)$$

The first term becomes original Coulomb function which gives Coulomb scattering amplitude $f_C(\theta)$. the second term becomes

$$\frac{e^{i(\rho - \eta \ln 2\rho)}}{r} e^{i\sigma_{L'}} i^{-L'} \frac{(S_{\alpha' \alpha} - \delta_{\alpha' \alpha})}{2i} \quad (2.14)$$

Combining this expression for full wave function,

$$\Psi_{\mathbf{k}}^{(+)}(\mathbf{r}, \xi_p, \xi_t) = \psi_c(\mathbf{k}, \xi_p, \xi_t) + \sum_{\alpha} |i^{\alpha} \alpha\rangle \frac{e^{i(\rho - \eta \ln 2\rho)}}{r} \sum_{\alpha_i} A_{\alpha_i}(\mathbf{k}) e^{i\sigma_{L'}} i^{-L'} \frac{(S_{\alpha' \alpha} - \delta_{\alpha' \alpha})}{2i} \quad (2.15)$$

This implies the scattering amplitude $f(\theta)$ becomes in the first convention, multiplying $\phi_{I_p} \phi_{I_t}$ and integrate over internal degrees,

$$f(\theta) = 4\pi \sum_{\alpha, \alpha'} e^{i\sigma_L + i\sigma_{L'}} \frac{(S_{\alpha' \alpha} - \delta_{\alpha' \alpha})}{2ik_i} Y_{L'}(\hat{R}) Y_L^*(\hat{p}) \langle \alpha_i | \mathbf{k}, I_p, I_t \rangle \langle I'_p, I'_t | \alpha \rangle \quad (2.16)$$

where i^{-L} factor cancels with i^L factor in the channel basis. On the other hand, the other convention gives

$$f(\theta) = 4\pi \sum_{L, L'} e^{i\sigma_L + i\sigma_{L'}} i^{L-L'} \frac{(S_{\alpha' \alpha} - \delta_{\alpha' \alpha})}{2ik_i} Y_{L'}(\hat{R}) Y_L^*(\hat{p}) \langle \alpha_i | \mathbf{k}, I_p, I_t \rangle \langle I'_p, I'_t | \alpha \rangle \quad (2.17)$$

The differential cross section can be obtained by

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{v_f}{v_i} |f_c(\theta) \delta_{fi} + f_{fi}(\theta)|^2 \quad (2.18)$$

This gives for unpolarized elastic scattering ($v_i = v_f$)

$$\frac{d\sigma_{el}}{d\Omega} = \frac{1}{(2I_1 + 1)(2I_2 + 1)} \sum_{M_1 M_2} \sum_{M'_1 M'_2} |f_C(\Omega) \delta_{M'_1 M_1} \delta_{M'_2 M_2} + f_{cM'_1 M'_2}^{(cM_1 M_2)}(\Omega)|^2 \quad (2.19)$$

And Inelastic or reaction cross section $\tilde{f}_{fi} = f_c \delta_{fi} + \sqrt{\frac{v_f}{v_i}} f_{fi}$

$$\frac{d\sigma_{c \rightarrow c'}}{d\Omega} = \frac{1}{(2I_1 + 1)(2I_2 + 1)} \sum_{M_1 M_2} \sum_{M'_1 M'_2} |f_{c'M'_1 M'_2}^{(cM_1 M_2)}(\Omega)|^2 \quad (2.20)$$

The summation can be analytically done for reaction (Unpolarized cross section의 경우에만) as (Elastic scattering의 경우는 위의 식을 사용하는 것이 좋다.)

$$\frac{d\sigma_{c \rightarrow c'}}{d\Omega} = \frac{\pi}{k^2} \frac{1}{(2I_1 + 1)(2I_2 + 1)} \sum_{\lambda} B_{\lambda}(E) P_{\lambda}(\cos \theta) \quad (2.21)$$

where anisotropy coefficient (\propto , Legendre polynomial coefficient) are given by

$$B_\lambda(E) = \frac{1}{4\pi} \sum_{J\pi} \sum_{lL} \sum_{J'\pi'} \sum_{l'L'} (-1)^{I-I'} e^{i(\sigma_l + \sigma_{l'} - \sigma_L - \sigma_{L'})} Z(lJLJ', I\lambda) Z(l'JL'J', I'\lambda) S_{c'I'l', cIl}^{J\pi}(E) S_{c'I'l', cIl}^{J'\pi'}(E) \quad (2.22)$$

real coefficient Z is defined as

$$Z(lJLJ', I\lambda) = (-1)^{J+J'} [(2\lambda+1)(2J+1)(2J'+1)(2l+1)(2L+1)]^{1/2} \begin{pmatrix} l & L & \lambda \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{matrix} l & L \\ J' & J \end{matrix} \right\} \begin{pmatrix} \lambda \\ I \end{pmatrix} \quad (2.23)$$

(which have symmetry $Z(lJLJ', I\lambda) = Z(LJ'lJ, I\lambda)$. Different convention may have different symmetry and definition of Z .) Then, integrated cross section or reaction cross section is

$$\sigma_{c \rightarrow c'} = \frac{\pi}{k^2} \frac{1}{(2I_1+1)(2I_2+1)} \sum_{J\pi} (2J+1) \sum_{lL} \sum_{l'L'} |S_{c'I'l', cIl}^{J\pi}(E)|^2 \quad (2.24)$$

2.1.1 In one convention

Let us consider the cross section in (L,S) coupling scheme. Also, convention using \tilde{S} -matrix. Instead of computing scattering amplitude directly, it is better to compute $A_{m', M', m, M}^{L'}$ first because $A_{m', M', m, M}^{L'}$ is independent of angle.

The FRESKO manual writes (this is a Condon-Shortley phase convention)

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2L+1)}{4\pi} \frac{(L-m)!}{(L+m)!}} (-1)^m e^{im\phi} P_{lm}(\cos\theta). \quad (2.25)$$

However, in Varsharovich book Spherical Harmonics are defined without $(-1)^m$ factor. In fact, this $(-1)^m$ factor can be absorbed into the phase convention of $P_{lm}(\cos\theta)$ (In case of sci-py package, there is no $(-1)^m$ factor in the expression, however, it is included in the definition of Legendre polynomial. Thus, it is consistent with Condon-Shortley convention.) Thus, one can use either convention. However, one have to make sure about the phase convention of associated Legendre polynomial.

In any case, one can express spherical harmonics as

$$\begin{aligned} Y_{lm}(\theta, \phi = 0) &= Y_c(l, m) P_{|m|}(\cos\theta) \\ Y_c(l, m) &= \sqrt{\frac{(2L+1)}{4\pi} \frac{(L-|m|)!}{(L+|m|)!}} (-1)^m \quad \text{for } m > 0, \\ &= \sqrt{\frac{(2L+1)}{4\pi} \frac{(L-|m|)!}{(L+|m|)!}} \quad \text{for } m < 0. \end{aligned} \quad (2.26)$$

(Note that $(-1)^m$ factor will be moved to $m < 0$ case in other convention.)

Then, we can express, the scattering amplitude for polarization.

$$\tilde{f}_{m', M'; m, M}(\theta) = \delta_{xx'} \delta_{I_p, I'_p} \delta_{I_t, I'_t} f_c(\theta) + \sum_{L'} \tilde{A}_{m' M'; m M}^{L'} \times P_{L'}^{|m'+M'-m-M|}(\cos\theta) \quad (2.27)$$

and differential cross section

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{1}{(2I_p+1)(2I_t+1)} \sum_{m' M' m M} |\tilde{f}_{m' M'; m M}(\theta)|^2 \quad (2.28)$$

The expression

$$\tilde{A}_{m' M'; m M}^{L'} = \sum \quad (2.29)$$

2.2 Model space wave function

Because the total angular momentum is a conserved quantity, one only have to consider coupled channels which having the same total angular momentum.

$$|\Psi^J\rangle = \sum_{\alpha} |\phi_{\alpha}\rangle \psi_{\alpha}(R_{\alpha}) \quad (2.30)$$

where α includes quantum numbers of partitions, $|\alpha\rangle \equiv |xpt; (LI_p)J_p, I_t; J_{tot}M_{tot}\rangle$ where (xpt) represents partition of projectile and target, L is a orbital angular momentum between projectile and target, I_p is a spin of projectile, $\mathbf{J}_p = \mathbf{L} + \mathbf{I}_p$, I_t is a spin of target, $J_{tot}M_{tot}$ is a total angular momentum of system and its projection and R_{α} is a coordinates between projectile and target.

In other words, more explicit form of $|\phi_{\alpha}\rangle$ is ¹

$$|\phi_{\alpha}\rangle = \left[\left[i^L Y_L(\hat{R}_x) \otimes \phi_{I_p}^{xp}(\xi_p) \right]_{J_p} \otimes \phi_{I_t}^{xt}(\xi_t) \right]_{J_{tot}M_{tot}}, \quad (2.31)$$

where $\phi_{I_p}^{xp}(\xi_p)$ and $\phi_{I_t}^{xt}(\xi_t)$ are bound state wave functions of projectile and target with ξ_p and ξ_t as internal coordinates of projectile and target.

- For a system with specific total angular momentum, wave function becomes

$$\begin{aligned} \Psi_{x,J_{tot}}^{M_{tot}}(\mathbf{R}_x, \xi_p, \xi_t) &= \sum_{\alpha} \left[\left[i^L Y_L(\hat{R}_x) \otimes \phi_{I_p}^{xp}(\xi_p) \right]_{J_p} \otimes \phi_{I_t}^{xt}(\xi_t) \right]_{J_{tot}M_{tot}} \frac{1}{R_x} f_{\alpha}^{J_{tot}}(R_x) \\ &= \sum_{\alpha} |xpt : (LI_p)J_p, I_t; J_{tot}M_{tot}\rangle \frac{f_{\alpha}^{J_{tot}}(R_x)}{R_x} \\ &= \sum_{\alpha} |\alpha; J_{tot}M_{tot}\rangle \frac{f_{\alpha}^{J_{tot}}(R_x)}{R_x} \end{aligned} \quad (2.32)$$

where $\alpha = (x, p, t, L, I_p, J_p, I_t)$.

- For a wave function from initial plane wave with momentum \mathbf{k}_i and spin projections of μ_{p_i} and μ_{t_i} can be written as

$$\Psi_{x_i p_i t_i}^{\mu_{p_i} \mu_{t_i}}(R_x, \xi_p, \xi_t; \mathbf{k}_i) = \sum_{J_{tot}M_{tot}} \sum_{\alpha} |\alpha; J_{tot}M_{tot}\rangle \frac{f_{\alpha}^{J_{tot}}(R_x)}{R_x} A_{\mu_{p_i} \mu_{t_i}}^{J_{tot}M_{tot}}(\alpha_i, \mathbf{k}_i) \quad (2.33)$$

where, "incoming coefficient" is defined as

$$A_{\mu_{p_i} \mu_{t_i}}^{J_{tot}M_{tot}}(\alpha_i, \mathbf{k}_i) = \frac{4\pi}{k_i} \sum_{M_i m_i} Y_{L_i M_i}^*(\mathbf{k}_i) \langle L_i M_i, I_{p_i} \mu_{p_i} | J_{p_i} m_i \rangle \langle J_{p_i} m_i, I_{t_i} \mu_{t_i} | J_{tot} M_{tot} \rangle \quad (2.34)$$

with asymptotic normalization is

$$f_{\alpha\alpha_i}(R_x) \rightarrow \frac{i}{2} [H_{\alpha}^{(-)}(\eta_{\alpha}, k_{\alpha} R_x) \delta_{\alpha\alpha_i} - S_{\alpha\alpha_i}(k_{\alpha}) H_{\alpha}^{(+)}(\eta_{\alpha}, k_{\alpha} R_x)] \quad (2.35)$$

where H^{\pm} are Coulomb Hankel functions.

¹This expression follows the convention in Thompson's textbook.

2.3 Coupled channel equations

Let us consider a model space wave function $|\Psi^J\rangle$ which have definite total angular momentum. Then, coupled channel equation is only for those channels having the same total angular momentum.

One can group full Hamiltonian as

$$H = H_{xt}(\boldsymbol{\xi}_t) + H_{xp}(\boldsymbol{\xi}_p) + \hat{T}_x(\mathbf{R}_x) + \mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p). \quad (2.36)$$

where

$$\begin{aligned} H_{xp}(\boldsymbol{\xi}_p)\phi_{I_p}^{xp}(\boldsymbol{\xi}_p) &= \epsilon_{xp}\phi_{I_p}^{xp}(\boldsymbol{\xi}_p), \\ H_{xt}(\boldsymbol{\xi}_t)\phi_{I_t}^{xt}(\boldsymbol{\xi}_t) &= \epsilon_{xt}\phi_{I_t}^{xt}(\boldsymbol{\xi}_t), \end{aligned} \quad (2.37)$$

relative kinetic term and interaction between target and projectile becomes

$$\begin{aligned} \hat{T}_x(\mathbf{R}_x) &= -\frac{\hbar^2}{2\mu_x}\nabla_{\mathbf{R}_x}^2, \\ \mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p) &= \sum_{i \in p, j \in t} V_{ij}(\mathbf{r}_i - \mathbf{r}_j). \end{aligned} \quad (2.38)$$

with reduced mass $\mu_x = \frac{m_{xp}m_{xt}}{m_{xp}+m_{xt}}$.

Assume the Model space Wave function Ψ satisfies,

$$0 = [\mathcal{H} - E]|\Psi\rangle = [\mathcal{H} - E]|\phi_i\rangle\psi_i + [\mathcal{H} - E]\sum_{j \neq i} |\phi_j\rangle\psi_j \quad (2.39)$$

If we multiply $\langle\phi_i|$ on the left hand side, for each channel i ,

$$\langle\phi_i|E - \mathcal{H}|\phi_i\rangle\psi_i = -\sum_{j \neq i} \langle\phi_i|E - \mathcal{H}|\phi_j\rangle\psi_j. \quad (2.40)$$

In the left hand side, it would be natural to express $E - \mathcal{H}$ in channel i ,

$$\begin{aligned} \langle\phi_i|E - \mathcal{H}|\phi_i\rangle\psi_i &= \langle\phi_i|[E - H_i^{bd} - T_i - V_i]|\phi_i\rangle\psi_i = \langle\phi_i|[E_i - T_i - V_i]|\phi_i\rangle\psi_i \\ &= (E_i - T_i(R_i) - \langle\phi_i|V_i|\phi_i\rangle(R_i))\psi_i(R_i) \end{aligned} \quad (2.41)$$

where H_i^{bd} represent the Hamiltonian which gives projectile and target bound states in channel i , $V_i = \mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p)$ is a potential between projectile and target in channel i , and thus $\langle\phi_i|V_i|\phi_i\rangle(R_i)$ implies integration over projectile and target bound state internal coordinates ξ_p, ξ_t , which results in the function with only R_i radial dependence. (Note that the channel $|\phi_i\rangle$ also contains the orbital angular momentum \hat{R}_i dependence.)

Thus, the equation becomes

$$\boxed{(E_i - T_i(R_i) - U_i)\psi_i(R_i) = \langle\phi_i|\mathcal{V}_i - U_i|\phi_i\rangle\psi_i(R_i) - \sum_{j \neq i} \langle\phi_i|E - \mathcal{H}|\phi_j\rangle\psi_j(R_j)} \quad (2.42)$$

where we introduced auxiliary optical potential U_i for channel i .

For the right-hand side, we may use two different ways, in post form

$$-\langle\phi_i|E - \mathcal{H}|\phi_j\rangle\psi_j = \langle\phi_i|T_i - E_i + V_i|\phi_j\rangle\psi_j = (T_i - E_i)\langle\phi_i|\phi_j\rangle\psi_j + \langle\phi_i|V_i|\phi_j\rangle\psi_j \quad (2.43)$$

or in prior form

$$-\langle \phi_i | E - \mathcal{H} | \phi_j \rangle \psi_j = \langle \phi_i | T_j - E_j + V_j | \phi_j \rangle \psi_j = \langle \phi_i | \phi_j \rangle (T_j - E_j) \psi_j + \langle \phi_i | V_j | \phi_j \rangle \psi_j \quad (2.44)$$

For the same partition, $x' = x$, we may normalize $\hat{N}_{\alpha'\alpha} = \langle \phi_{\alpha'} | \phi_{\alpha} \rangle = \delta_{\alpha'\alpha}$. However, in general $\langle \phi_i | \phi_j \rangle$ are non-orthogonal and may not commute with $(T_i - E_i)$.

Thus, we get

$$(E_i - T_i(R_i) - \langle \phi_i | V_i | \phi_i \rangle(R_i)) \psi_i(R_i) = \sum_{j \neq i} [(T_i - E_i) \langle \phi_i | \phi_j \rangle \psi_j + \langle \phi_i | V_i | \phi_j \rangle \psi_j], \quad (2.45)$$

or

$$(E_i - T_i(R_i) - \langle \phi_i | V_i | \phi_i \rangle(R_i)) \psi_i(R_i) = \sum_{j \neq i} [\langle \phi_i | \phi_j \rangle (T_j - E_j) \psi_j + \langle \phi_i | V_j | \phi_j \rangle \psi_j] \quad (2.46)$$

If we introduce auxiliary optical potential U_i , we may rearrange in "post form"

$$\begin{aligned} (E_i - T_i - U_i) \psi_i &= \left(\sum_{j \neq i} \langle \phi_i | V_i | \phi_j \rangle \psi_j \right) + ([\langle \phi_i | V_i | \phi_i \rangle - U_i] \psi_i) \\ &\quad + \left(\sum_{j \neq i} (T_j - E_j) \langle \phi_i | \phi_j \rangle \psi_j \right) \end{aligned} \quad (2.47)$$

Or, in "prior form",

$$\begin{aligned} (E_i - T_i - U_i) \psi_i &= \left(\sum_{j \neq i} \langle \phi_i | V_j | \phi_j \rangle \psi_j \right) + ([\langle \phi_i | V_i | \phi_i \rangle(R_i) - U_i] \psi_i) \\ &\quad + \left(\sum_{j \neq i} \langle \phi_i | \phi_j \rangle (T_j - E_j) \psi_j \right), \end{aligned} \quad (2.48)$$

Let us denote

$$V^{prior}(R_i, R_j) = \langle \phi_i | V_j | \phi_j \rangle, \quad V^{post}(R_i, R_j) = \langle \phi_i | V_i | \phi_j \rangle, \quad (2.49)$$

If we define $U_i(R_i) = \langle \phi_i | V_i | \phi_i \rangle$, diagonal term in the right hand side would vanish.

For radial wave function $\psi_{\alpha} = f_{\alpha}/R$ in a prior form for rearrangement, while separating

inelastic scattering and rearrangements couplings,²

$$\begin{aligned}
& (E_{xpt} - T_{xL}(R_x) - U_x(R_x))f_\alpha(R_x) \\
&= \langle \phi_\alpha | V_x - U_x | \phi_\alpha \rangle (R_x) f_\alpha(R_x) + \left(\sum_{\alpha' \neq \alpha} V_{\alpha\alpha'}(R_x) f_{\alpha'}(R_x) \right) \\
&+ \left(\sum_{x \neq x', \alpha'} \int dR_{x'} V_{\alpha\alpha'}^{prior}(R_x, R_{x'}) f_{\alpha'}(R_{x'}) \right) \\
&+ \left(\sum_{\alpha \neq \alpha'} \int dR_{x'} N_{\alpha\alpha'}(T_{x'L'} - E_{x'p't'}) f_{\alpha'}(R_{x'}) \right) \tag{2.50}
\end{aligned}$$

where

$$\begin{aligned}
T_{xL}(R) &= -\frac{\hbar^2}{2\mu_x} \left(\frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right), \\
V_{\alpha\alpha'}(R_x) &= \langle \phi_\alpha | V_x(R_x, \xi) | \phi_{\alpha'} \rangle, \\
V_{\alpha\alpha'}^{prior}(R_x, R_{x'}) &= R_x \langle \phi_\alpha | V^{prior}(R_{x'}, \xi) | \phi_{\alpha'} \rangle \frac{1}{R_{x'}}, \\
N_{\alpha\alpha'}(R_x, R_{x'}) &= R_x \langle \phi_\alpha | \phi_{\alpha'} \rangle \frac{1}{R_{x'}}. \tag{2.51}
\end{aligned}$$

The actual form of couplings requires explicit form of potentials in a certain model. For example, inelastic channel coupling which have the same partition,

$$V_{\alpha\alpha'}(R_x) = \int d\xi_p d\xi_t \langle i^L [Y_L(\hat{R}_x) \otimes \phi_p(\xi_p)] \phi_t(\xi_t) | V_x(R_x, \xi_p, \xi_t) | i^{L'} [Y_{L'}(\hat{R}_x) \otimes \phi_{p'}(\xi_{p'})] \phi_{t'}(\xi_{t'}) \rangle \tag{2.52}$$

can be described as an collective model or single particle excitation models for the integration over internal coordinates.

Also, one may further multipole expand the couplings in terms of transferred angular momentums, $\Delta L, \Delta S, \Delta J$.

In the next chapter, let us consider more details on how to obtain couplings in various models.

²Compared to the coupled channel equation given in FRESKO document, (1) we have additional non-orthogonal terms (2) diagonal term in the right side can be ignored if $U_x(R_x) \equiv \langle \phi_\alpha | V_\alpha | \phi_\alpha \rangle$ (3) We have not expanded in multipole of couplings yet (4) $i^{L-L'}$ factor are hidden in the definition of $|\phi_\alpha\rangle$.

Chapter 3

Couplings, Interactions

3.1 Multipole expansion of interaction

In general, the reaction occurs through the interaction between target and projectile,

$$\mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p) = \sum_{i \in p, j \in t} V_{ij}(\mathbf{r}_i - \mathbf{r}_j) = \mathcal{V}_{xtp}(\mathbf{R}'_x, \boldsymbol{\xi}'_t, \boldsymbol{\xi}'_p). \quad (3.1)$$

where if there is a partition change, the interaction can be written either prior or post form.

In coupled reaction channel calculation, we need to compute (if there is no rearrangement) couplings

$$V_{\alpha\alpha'}^{prior}(R_x, R_{x'}) = R_x \langle \phi_\alpha | V^{prior}(\mathbf{R}_{x'}, \xi_p, \xi_t) | \phi_{\alpha'} \rangle \frac{1}{R_{x'}}. \quad (3.2)$$

where integration over internal coordinates and angular integration of \mathbf{R}_x is implied. (Note that there are \hat{R}_x and $\hat{R}_{x'}$ dependence are hidden in $|\alpha\rangle$ and $|\alpha'\rangle$).

By factoring out factor $i^{L-L'}$, we can write

$$\begin{aligned} V_{\alpha\alpha'}^{prior}(R_x, R_{x'}) &= \int d\Omega_{R'_x} d\xi_p d\xi_t \left\langle [Y_L(\hat{R}_x) \otimes \phi_p(\xi_p)]_{J_p} \otimes \phi_t(\xi_t) \right| \\ &\quad \times R_x V_x(\mathbf{R}_{x'}, \xi_p, \xi_t) \frac{1}{R_{x'}} \left| [Y_{L'}(\hat{R}_{x'}) \otimes \phi_{p'}(\xi_{p'})] \otimes \phi_{t'}(\xi_{t'}) \right\rangle \end{aligned} \quad (3.3)$$

Let us first do not consider rearrangements.

$$\begin{aligned} V_{\alpha\alpha'}(R_x) &= \int d^2\Omega_{R_x} d^3A_p d^3A_t d^3\xi_t \left\langle [Y_L(\hat{R}_x) \otimes \phi_p(\xi_p)]_{J_p} \otimes \phi_t(\xi_t) \right| \\ &\quad \times V_x(\mathbf{R}_x, \xi_p, \xi_t) \left| [Y_{L'}(\hat{R}_x) \otimes \phi_{p'}(\xi_p)] \otimes \phi_{t'}(\xi_t) \right\rangle \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} \mathcal{V}_x(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p) &= \sum_{i \in p, j \in t} V_{ij}(\mathbf{r}_i - \mathbf{r}_j) \\ &= \sum_{i \in p, j \in t} V_{ij}(\mathbf{R}_x + \boldsymbol{\xi}_{p,i} - \boldsymbol{\xi}_{t,j}; \mathbf{s}_i, \mathbf{s}_j, \mathbf{t}_i, \mathbf{t}_j) \\ &= \int d\mathbf{r}_t d\mathbf{r}_p \sum_{i \in p, j \in t} v(\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t; \mathbf{s}_i, \mathbf{s}_j, \mathbf{t}_i, \mathbf{t}_j) \delta^{(3)}(\mathbf{r}_p - \mathbf{r}_i) \delta^{(3)}(\mathbf{r}_t - \mathbf{r}_j) \end{aligned}$$

By tensor decompose of the potential, we may write¹

$$\mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p) = \sqrt{4\pi} \sum_{\lambda\mu} V_\lambda(R) Y_{\lambda\mu}(\hat{R}) T_{\lambda-\mu}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_p) \quad (3.6)$$

Then, the matrix elements for states $\langle\alpha'|\$ and $|\alpha\rangle$ can be expressed in terms of the product of form factor and reduced matrix elements, $V_\lambda(R)\langle\alpha||T_\lambda(\boldsymbol{\xi}_t, \boldsymbol{\xi}_p)||\alpha'\rangle$, and other all coefficients related with shuffling states and tensor decomposition of potential.

In FRESKO, the potential operators are decomposed into multipoles and the matrix elements related with orbital angular momentums are computed by FRESKO so that one only have to give inputs for reduced matrix elements and form factors of the coupling $V_\lambda(R)\langle\alpha||T_\lambda(\boldsymbol{\xi}_t, \boldsymbol{\xi}_p)||\alpha'\rangle$. In some special model, even the reduced matrix elements or form factors will be computed automatically.

This is explained in the manual of FRESKO and the textbook of I.J. Thompson. However, there is some convention difference between two.

3.2 Double folding model

Suppose we have a scalar folding potential $v(\mathbf{r}_i - \mathbf{r}_j)$, and full interaction between projectile and target becomes²

$$\begin{aligned} \mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p) &= \sum_{i \in p, j \in t} v(\mathbf{R} + \mathbf{r}_i - \mathbf{r}_j) \\ &= \int d\mathbf{r}_t d\mathbf{r}_p v(\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t) \sum_{i \in p, j \in t} \delta^{(3)}(\mathbf{r}_p - \mathbf{r}_i) \delta^{(3)}(\mathbf{r}_t - \mathbf{r}_j). \end{aligned} \quad (3.7)$$

We can decompose the channel state as

$$\left| [Y_{L'}(\hat{R}_x) \otimes \phi_{p'}(\boldsymbol{\xi}_{p'})] \otimes \phi_{t'}(\boldsymbol{\xi}_{t'}) \right\rangle = \sum_M C_{\alpha, M} Y_{L'}(\hat{R}) \phi_{I_p}(\boldsymbol{\xi}_p) \phi_{I_t}(\boldsymbol{\xi}_t) \quad (3.8)$$

where $C_{\alpha, M}$ is a collection of Wigner coefficients with projections $M = \{L_z, I_{pz}, I_{tz}, J_z\}$. If we only consider initial and final states as separate product $|Y_{Lm}\rangle \times |I_p M_p\rangle \times |I_t M_t\rangle$, we can simply set $C_{\alpha, M} = 1$. Then,

$$\begin{aligned} V_{\alpha, \alpha'}(R) &= \langle\alpha| \mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p) |\alpha'\rangle \\ &= \sum_{M, M'} C_{\alpha, M} C_{\alpha', M'} \int d\mathbf{r}_t d\mathbf{r}_p \langle Y_{Lm}(\hat{R}) | v(\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t) | Y_{L'm'}(\hat{R}) \rangle \\ &\quad \times \langle \phi_{I_p, I_{pz}} | \sum_i \delta^{(3)}(\mathbf{r}_p - \mathbf{r}_i) | \phi_{I'_p, I'_{pz}} \rangle \langle \phi_{I_t, I_{tz}} | \sum_j \delta^{(3)}(\mathbf{r}_t - \mathbf{r}_j) | \phi_{I'_t, I'_{tz}} \rangle \end{aligned} \quad (3.9)$$

Note that the integration over \mathbf{r}_i and \mathbf{r}_j is implied in the matrix element. For elastic and in-elastic excitation, the number of nucleon does not change in initial and final states.

However, this expression does not take into account anti-symmetrization of full wave function. Later, we will take into account Pauli principle between projectile and target as a exchange potential term approximately.

¹I suppose one would have $T_{\lambda-\mu}(R, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p)$ in general, instead of $V_\lambda(R) T_{\lambda-\mu}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_p)$.

²We assumed simple separation of wave function into projectile, target and orbital while ignoring anti-symmetrization of full wave function of the system.

3.2.1 density matrix

Let us define density matrix,

$$\rho_{II'}(\mathbf{r}) \equiv \langle \phi_{II_z} | \sum_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) | \phi_{I'I'_z} \rangle. \quad (3.10)$$

It will correspond to nucleon density of a nucleus if $I = I'$, or to transition density if $I \neq I'$. For example, in H.F. theory, one have for diagonal density as

$$\rho(\mathbf{r}) \simeq \sum_{\alpha} |\phi_{\alpha}(r)|^2 \quad (3.11)$$

where α are single particle states in Hartree-Fock space. We may want to multipole expand $\rho_{A'A}(\mathbf{r})$ by using

$$\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\delta(r_1 - r_2)}{r_1 r_2} \sum_{lm} Y_{lm}(\hat{r}_1) Y_{lm}^*(\hat{r}_2) \quad (3.12)$$

$$\begin{aligned} \rho_{A'A}(\mathbf{r}) &= \langle I_{A'} M_{A'} | \sum_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) | I_A M_A \rangle \\ &= \sum_{Lm} Y_{Lm}^*(\hat{r}) \frac{\langle I_{A'} M_{A'} | I_A M_A L m \rangle}{\sqrt{2I_{A'} + 1}} \langle A' | \sum_i r_i^{-2} \delta(r - r_i) Y_L(\hat{r}_i) | A \rangle \\ &= \sum_{Lm} Y_{Lm}^*(\hat{r}) \langle I_{A'} M_{A'} | I_A M_A L m \rangle C_L \rho_{A'A,L}(r). \end{aligned} \quad (3.13)$$

where we used the Wigner-Eckart theorem and defined multipole of density $\rho_L(r)$. ($\sqrt{2I+1}$ factor depends on the convention of reduced matrix element.) In other words,

$$C_L \rho_{A'A,L}(r) = \left(\frac{1}{\sqrt{2I_{A'} + 1}} \right) \langle A' | \sum_i r_i^{-2} \delta(r - r_i) Y_L(\hat{r}_i) | A \rangle \quad (3.14)$$

where normalization constant C_L is to be chosen for convenience. This transition density may be obtained microscopically by computing the matrix elements with wave function of A' and A . On the other hand, one may use macroscopic model by deforming ground state density.³

3.2.2 Double Folding using Fourier Transformation

Let us consider Fourier transformation,

$$\tilde{f}(\mathbf{k}) = \int d^3r e^{i\mathbf{k} \cdot \mathbf{r}} f(\mathbf{r}), \quad f(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{r}} \tilde{f}(\mathbf{k}). \quad (3.16)$$

³It is not clear how one can connect the deformation of a spherical density $\rho_0(r)$ for diagonal ground state, to the microscopic definition of a transition density. Just formally, if $\rho(\mathbf{r})$ is already known and if it is expanded, we should be able to define $\rho_{lm}(r)$,

$$\rho(\mathbf{r}) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\hat{r}) \quad (3.15)$$

We may consider partial wave expansion

$$f(\mathbf{r}) = \sum_{LM} C_L f_L(r) Y_{LM}(\hat{r}) \quad (3.17)$$

where $f_L(r)$ can be written as a integral of $f(\mathbf{r})$ with spherical harmonics. To use spherical symmetric case as $f(r) = f_0(r)$, it is convenient to define $C_{L=0} = \sqrt{4\pi}$, $C_{L \neq 0} = 1$. Then, the partial wave expansion gives

$$\begin{aligned} \tilde{f}(\mathbf{k}) &= \int d^3r \left(\sum_{lm} 4\pi j_l(kr) i^l Y_{lm}(\hat{k}) Y_{lm}^*(\hat{r}) \right) \left(\sum_{l'm'} C_{l'} f_{l'}(r) Y_{l'm'}(\hat{r}) \right) \\ &= \sum_{LM} C_L \tilde{f}_L(k) i^L Y_{LM}(\hat{k}) \end{aligned} \quad (3.18)$$

with

$$\tilde{f}_L(k) \equiv \int dr (4\pi r^2) j_L(kr) f_L(r) \quad (3.19)$$

The inverse relation becomes

$$f(\mathbf{r}) = \sum_{LM} C_L f_L(r) Y_{LM}(\hat{r}) \quad (3.20)$$

with

$$f_L(r) = \int \frac{dk k^2}{(2\pi)^3} (4\pi) j_L(kr) \tilde{f}_L(k) \quad (3.21)$$

($i^{\pm L}$ factor cancels each other.)

In terms of transition densities, Fourier transformation of density can be written as (Note that the complex conjugation of spherical harmonics and factor $i^{\pm L}$, depends on convention.)

$$\begin{aligned} \rho_{A'A}(\mathbf{r}) &= \sum_{Lm} \langle I_{A'} M_{A'} | I_A M_A L m \rangle Y_{Lm}^*(\hat{r}) C_L \rho_L(r) \\ \tilde{\rho}_{A'A}(\pm \mathbf{k}) &= \sum_{Lm} \langle I_{A'} M_{A'} | I_A M_A L m \rangle i^{\mp L} C_L \tilde{\rho}_L(k) Y_{Lm}^*(\hat{k}) \end{aligned} \quad (3.22)$$

with

$$\begin{aligned} \tilde{\rho}_L(k) &= 4\pi \int_0^\infty r^2 dr j_L(kr) \rho_L(r). \\ \rho_L(r) &= \int \frac{dk k^2}{(2\pi)^3} (4\pi) j_L(kr) \tilde{\rho}_L(k). \end{aligned} \quad (3.23)$$

We may use Fourier transformation for $v(\mathbf{r})$, to simplify multi-dimensional integrals into series of 1-d integrals,

$$\begin{aligned} V_{\alpha, \alpha'}(R) &= \sum_{M, M'} C_{\alpha, M} C_{\alpha', M'} \int d\mathbf{r}_t d\mathbf{r}_p \langle Y_{L'}(\hat{R}) | v(\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t) | Y_L(\hat{R}) \rangle \rho_{I_p I'_p}(\mathbf{r}_p) \rho_{I_t I'_t}(\mathbf{r}_t) \\ &= \sum_{M, M'} C_{\alpha, M} C_{\alpha', M'} \int d\mathbf{r}_t d\mathbf{r}_p \int \frac{d^3k}{(2\pi)^3} \langle Y_{L'}(\hat{R}) | \tilde{v}(\mathbf{k}) e^{-i\mathbf{k} \cdot (\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t)} | Y_L(\hat{R}) \rangle \rho_{I_p I'_p}(\mathbf{r}_p) \rho_{I_t I'_t}(\mathbf{r}_t) \\ &= \sum_{M, M'} C_{\alpha, M} C_{\alpha', M'} \int \frac{d^3k}{(2\pi)^3} \langle Y_{L'}(\hat{R}) | e^{-i\mathbf{k} \cdot \mathbf{R}} | Y_L(\hat{R}) \rangle \times \tilde{v}(\mathbf{k}) \tilde{\rho}_{I_p I'_p}(\mathbf{k}) \tilde{\rho}_{I_t I'_t}(-\mathbf{k}) \end{aligned} \quad (3.24)$$

In other words, we may use

$$U_F(\mathbf{R}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{R}} \tilde{v}(\mathbf{k}) \tilde{\rho}_1(\mathbf{k}) \tilde{\rho}_2(-\mathbf{k}) \quad (3.25)$$

and

$$\tilde{U}_F(\mathbf{k}) = \int d^3\mathbf{R} e^{i\mathbf{k}\cdot\mathbf{R}} U_F(\mathbf{R}) = \tilde{v}(\mathbf{k}) \tilde{\rho}_1(\mathbf{k}) \tilde{\rho}_2(-\mathbf{k}). \quad (3.26)$$

To get

$$V_{\alpha,\alpha'}(R) = \sum_{MM'} C_{\alpha,M} C_{\alpha',M'} \langle Y_{L'M'}(\hat{R}) | U_F(\mathbf{R}) | Y_{L'M'}(\hat{R}) \rangle \quad (3.27)$$

Then, by the partial wave expansion of plane wave, we get

$$\begin{aligned} V_{\alpha,\alpha'}(R) &= \sum_{\Delta L} \sum_{M,M'} \sum_{L_z} C_{\alpha,M} C_{\alpha',M'} \langle Y_{L'} | Y_{\Delta L}(\hat{R}) | Y_L \rangle \\ &\quad \times \sqrt{4\pi} i^{-\Delta L} \int \frac{d^3k}{(2\pi)^3} j_{\Delta L}(kR) \left(\tilde{v}(\mathbf{k}) \tilde{\rho}_{I_p I'_p}(-\mathbf{k}) \tilde{\rho}_{I_t I'_t}(\mathbf{k}) Y_{\Delta L}^*(\hat{k}) \right) \\ &= \sum_{\Delta L} \sum_{M,M'} \sum_{L_z} C_{\alpha,M} C_{\alpha',M'} \langle Y_{L'} | Y_{\Delta L}(\hat{R}) | Y_L \rangle V_{\alpha,\alpha'}^{\Delta L}(R) \end{aligned} \quad (3.28)$$

In the last line, we defined a coupling Form factor $V_{\alpha,\alpha'}^{\Delta L}(R)$ which includes all integration over internal degrees of freedom.

For scalar interaction $v(|\mathbf{r}|)$, (direct) folded potential is

$$\begin{aligned} U_F(\mathbf{R}; A_1 \rightarrow A'_1, A_2 \rightarrow A'_2) &= \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{R}} \tilde{v}(k) \tilde{\rho}_1(\mathbf{k}) \tilde{\rho}_2(-\mathbf{k}) \\ &= \sum_{LM} i^{-L} Y_{LM}^*(\hat{R}) \int \frac{d^3k}{(2\pi)^3} 4\pi j_L(kR) Y_{LM}(\hat{k}) \tilde{v}(k) \tilde{\rho}_1(\mathbf{k}) \tilde{\rho}_2(-\mathbf{k}) \\ &= \sum_{LM, L_1 m_1, L_2 m_2} \langle I'_1 M'_1 | I_1 M_1 L_1 m_1 \rangle \langle I'_2 M'_2 | I_2 M_2 L_2 m_2 \rangle C_{L_1} C_{L_2} i^{L_1 - L_2 - L} Y_{LM}^*(\hat{R}) \\ &\quad \times \left(\int \frac{dk k^2}{2\pi^2} j_L(kR) \tilde{v}(k) \tilde{\rho}_{L_1}(k) \tilde{\rho}_{L_2}(k) \right) \times \left(\int d\Omega_k Y_{LM}(\hat{k}) Y_{L_1 m_1}^*(\hat{k}) Y_{L_2 m_2}^*(\hat{k}) \right) \\ &= \sum_{LM, L_1 m_1, L_2 m_2} \langle I'_1 M'_1 | I_1 M_1 L_1 m_1 \rangle \langle I'_2 M'_2 | I_2 M_2 L_2 m_2 \rangle \langle LM | L_1 m_1 L_2 m_2 \rangle Y_{LM}^*(\hat{R}) \\ &\quad \times C_{L_1} C_{L_2} \sqrt{\frac{(2L_1+1)(2L_2+1)}{(4\pi)(2L+1)}} i^{L_1 - L_2 - L} \langle L_1 0 L_2 0 | L 0 \rangle \\ &\quad \times \left(\int \frac{dk k^2}{2\pi^2} j_L(kR) \tilde{v}(k) \tilde{\rho}_{L_1}(k) \tilde{\rho}_{L_2}(k) \right) \\ &= \sum_{LM, L_1 m_1, L_2 m_2} \langle I'_1 M'_1 | I_1 M_1 L_1 m_1 \rangle \langle I'_2 M'_2 | I_2 M_2 L_2 m_2 \rangle \\ &\quad \times \langle LM | L_1 m_1 L_2 m_2 \rangle Y_{LM}^*(\hat{R}) C_L U_{L_1 L_2}^L(R) \end{aligned} \quad (3.29)$$

where multipole of folding potential is defined. (One may define $C_{L=0} = (4\pi)^{1/2}$, $C_{L \neq 0} = 1$ for convenience so that scalar potential is the same as $L = 0$ component, $U(R) = U_0(R)$).

This is not the same with Clensch-Goran factors $C_{\alpha,m}$ defined earlier.) Thus,

$$\begin{aligned}
U_{L_1, L_2}^L(R) &= \int \frac{dk k^2}{(2\pi)^3} (4\pi) j_L(kR) \tilde{U}^L(k), \\
\tilde{U}^L(k) &= \frac{C_{L_1} C_{L_2}}{C_L} \sqrt{\frac{(2L_1+1)(2L_2+1)}{(4\pi)(2L+1)}} i^{L_1-L_2-L} \langle L_1 0 L_2 0 | L 0 \rangle \\
&\quad \times \tilde{v}(k) \tilde{\rho}_{L_1}(k) \tilde{\rho}_{L_2}(k)
\end{aligned} \tag{3.30}$$

consistency check

For $L_1 + L_2 = L$ is from the moments of potential and density,

$$\begin{aligned}
J_k(f^L) &= 4\pi \int f^L(R) r^{k+2} dr, \\
R_k(f^L) &= J_{k+2}(f^L) / J_k(f^L),
\end{aligned} \tag{3.31}$$

one can do some consistency check,

$$J_L(U^L) = \langle L 0 | L_1 0 L_2 0 \rangle \frac{(2L+1)!!}{(2L_1+1)!!(2L_2+1)!!} \frac{\hat{L}_1 \hat{L}_2}{\hat{L}} \frac{C_{L_1} C_{L_2}}{\sqrt{4\pi} C_L} J_{L_1}(\rho_1^{L_1}) J_{L_2}(\rho_2^{L_2}) J_0(V^0) \tag{3.32}$$

and

$$R_L(U^L) = (2L+3) \left(\frac{R_{L_1}(\rho_1^{L_1})}{(2L_1+3)} + \frac{R_{L_2}(\rho_2^{L_2})}{(2L_2+3)} + \frac{R_0(V^0)}{3} \right) \tag{3.33}$$

3.2.3 Spin/Isospin dependent folding potential

If the folding potential have additional dependence on spin or isospin,

$$v_{12} = v_0(r_{12}) + v_1(r_{12}) \tau_1 \cdot \tau_2 + v_2(r_{12}) \sigma_1 \cdot \sigma_2 + \dots \tag{3.34}$$

If the isospin does not change during the process we can simply replace, $\tau_1 \cdot \tau_2 \rightarrow \tau_1^z \tau_2^z$ and

$$\langle \phi_I | \sum_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) \tau_i^z | \phi_{I'} \rangle = \rho_{II'}^p(\mathbf{r}) - \rho_{II'}^n(\mathbf{r}). \tag{3.35}$$

In a very special case, $\rho_n = \frac{N}{Z} \rho_p = \frac{N}{A} \rho$, this corresponds to

$$\langle \phi_I | \sum_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) \tau_i^z | \phi_{I'} \rangle = \frac{Z - N}{A} \rho(\mathbf{r}) \tag{3.36}$$

For spin, we would get vector spin transition density

$$\langle \phi_I | \sum_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) \sigma_i | \phi_{I'} \rangle = \vec{\rho}_{II'}(\mathbf{r}) \tag{3.37}$$

For G-T transition, we will get,

$$\langle \phi_I | \sum_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) \sigma_i \tau_i^a | \phi_{I'} \rangle = \vec{\rho}_{II'}^a(\mathbf{r}) \tag{3.38}$$

Considering isospin dependence, ($v_{nn} = v_{pp} = v_0 + v_1$ and $v_{np} = v_0 - v_1$),

$$V = V_0 + V_1 \tau_1 \cdot \tau_2 \quad (3.39)$$

One have double folding potential

$$U_F = U_0 + U_1 \tau_1 \cdot \tau_2 \quad (3.40)$$

Thus,

$$\begin{aligned} U_{F_0} &= \int d^3 \mathbf{r}_1 \int d^3 \mathbf{r}_2 (\rho_{Pn}(r_1) + \rho_{Pp}(r_1)) (\rho_{Tn}(r_2) + \rho_{Tp}(r_2)) v_0(r_{12}), \\ U_{F_1} &= \int d^3 \mathbf{r}_1 \int d^3 \mathbf{r}_2 (\rho_{Pn}(r_1) - \rho_{Pp}(r_1)) (\rho_{Tn}(r_2) - \rho_{Tp}(r_2)) v_1(r_{12}). \end{aligned} \quad (3.41)$$

Thus, in particular case of $\rho_n = \rho_p \times (N/Z) = \rho \times (N/A)$,

$$U_{F_1} = \left(\frac{N_T - Z_T}{A_T} \right) \left(\frac{N_P - Z_P}{A_P} \right) \int d^3 \mathbf{r}_1 \int d^3 \mathbf{r}_2 \rho_P(r_1) \rho_T(r_2) v_1(r_{12}) \quad (3.42)$$

Also transition Form factor from transition density of projectile becomes

$$\begin{aligned} F_0 &= \int d^3 \mathbf{r}_1 \int d^3 \mathbf{r}_2 (\delta \rho_{Pn}(r_1) + \delta \rho_{Pp}(r_1)) (\rho_{Tn}(r_2) + \rho_{Tp}(r_2)) v_0(r_{12}), \\ F_1 &= \int d^3 \mathbf{r}_1 \int d^3 \mathbf{r}_2 (\delta \rho_{Pn}(r_1) - \delta \rho_{Pp}(r_1)) (\rho_{Tn}(r_2) - \rho_{Tp}(r_2)) v_1(r_{12}). \end{aligned} \quad (3.43)$$

These form factors can be used for in-elastic scattering cross section.

3.3 Exchange term of Double folding potential

Let us consider a general process $a_0 + A_0 \rightarrow a + A$. Let us re-summarize reaction theory.

The channel internal wave function is

$$\Psi_\alpha(x_\alpha) \equiv \Psi_{\alpha SM_S}(x_\alpha) = \sum_{M_a M_A} \langle J_a M_a J_A M_A | SM_S \rangle \Psi_{J_a M_a}(x_a) \Psi_{J_A M_A}(x_A) \quad (3.44)$$

Spin-angle part of total channel wave function is

$$\Phi_{\alpha LS}^{(JM_J)}(\hat{R}, x_\alpha) = \sum_{M_L M_S} \langle LM_L SM_S | JM_J \rangle i^L Y_{LM_L}(\hat{R}) \Psi_\alpha(x_\alpha) \quad (3.45)$$

Total scattering wave function is

$$\Psi_{JM_J}(R) = \frac{1}{R} \sum_{\alpha LS} \chi_{\alpha LS}^{(J)}(k_\alpha, R) \Phi_{\alpha LS}^{(JM_J)}(\hat{R}, x_\alpha). \quad (3.46)$$

The CC equation is

$$\begin{aligned} & \left\{ \frac{\hbar^2}{2\mu_\alpha} \left[\frac{d^2}{dR^2} + k_\alpha^2 - \frac{L(L+1)}{R^2} \right] - \langle \alpha(LS)J | V | \alpha(LS)J \rangle \right\} \chi_{\alpha LS}^{(J)}(k_\alpha, R) \\ &= \sum_{\alpha' L' S' \neq \alpha LS} \langle \alpha(LS)J | V | \alpha'(L'S')J \rangle \chi_{\alpha' L' S'}^{(J)}(k_{\alpha'}, R) \end{aligned} \quad (3.47)$$

$$\begin{aligned}
\langle \alpha'(L'S')J|V|\alpha(LS)J \rangle &= \int d\hat{R} dx_a dx_A [\Phi_{\alpha'L'S'}^{(JM_J)}(\hat{R}, x_{\alpha'})]^* V(R) \Phi_{\alpha LS}^{(JM_J)}(\hat{R}, x_{\alpha}) \\
&= \sum_{M_L M_{L'} M_S M_{S'}} \langle L' M_{L'} S' M_{S'} | J M_J \rangle \langle L M_L S M_S | J M_J \rangle \\
&\quad \times \int d\hat{R} [i^{L'} Y_{L' M_{L'}}(\hat{R})]^* [i^L Y_{L M_L}(\hat{R})] \langle \alpha'(J_{a'} J_{A'}) S' M_{S'} | V(\mathbf{R}) | \alpha(J_a J_A) S M_S \rangle.
\end{aligned}$$

Usually assumes diagonal potentials are the same local potential for all channels,

$$\langle \alpha(LS)J|V|\alpha(LS)J \rangle \simeq \langle \alpha_0(L_0 S_0)J|V|\alpha_0(L_0 S_0)J \rangle \quad (3.48)$$

With multipole expansion,

$$V(\mathbf{R}) = \sum_{\lambda\mu} C_{\lambda} V_{\lambda}(R) [i^{\lambda} Y_{\lambda\mu}(\hat{R})]^*. \quad (3.49)$$

with normalization convention $C_0 = \sqrt{4\pi}$ and $C_{\lambda} = 1$ for $\lambda \neq 0$.

$$\langle \alpha'(L'S')J|V|\alpha(LS)J \rangle = \frac{1}{\sqrt{4\pi}} \sum_{\lambda} i^{L'-L-\lambda} (-1)^{J-S} \hat{L} \hat{L}' \hat{S} \langle L_0 L' 0 | \lambda 0 \rangle W(LL'SS'; \lambda J) C_{\lambda} \langle \alpha' S' || V_{\lambda}(R) || \alpha S \rangle \quad (3.50)$$

where, reduced matrix convention is

$$\langle \alpha'(J_{a'} J_{A'}) S' M_{S'} | V_{\lambda\mu}(R) | \alpha(J_a J_A) S M_S \rangle = \langle S M_S \lambda \mu | S' M_{S'} \rangle \langle \alpha' S' || V_{\lambda}(R) || \alpha S \rangle \quad (3.51)$$

The folding model, including **knock-on exchange(the interchange of nucleons i and j that occurs when the dinuclear system is antisymmetrized)**, replace v_{ij} as

$$V = \sum_{i \in a, j \in A} v_{ij} \rightarrow v_{ij}(1 - P_{ij}) = v_D + v_{EX} P_{ij}^x, \quad (3.52)$$

$$v_D = v_{ij}^{(D)} = v_{ij}, \quad v_{EX} = v_{ij}^{(EX)} = -v_{ij} P_{ij}^{\sigma} P_{ij}^{\tau} \quad (3.53)$$

Then,

$$\begin{aligned}
&\langle \alpha'(J_{a'} J_{A'}) S' M_{S'} | V_{\lambda\mu}(R) | \alpha(J_a J_A) S M_S \rangle \\
&= \sum_{M_{a'} M_{A'} M_a M_A} \langle J_{a'} M_{a'} J_{A'} M_{A'} | S' M_{S'} \rangle \langle J_a M_a J_A M_A | S M_S \rangle \langle \alpha'(a' A') | V | \alpha(a A) \rangle
\end{aligned} \quad (3.54)$$

with

$$\begin{aligned}
\langle \alpha'(a' A') | V | \alpha(a A) \rangle &= \sum_{i \in a, j \in A} \sum_{i' \in a', j' \in A'} [\langle i' j' | v_D | i j \rangle + \langle i' j' | v_{EX} | j i \rangle] \\
&= V_{a'A', aA}^{(D)}(E_{\alpha}, \mathbf{R}) + V_{a'A', aA}^{(EX)}(E_{\alpha}, \mathbf{R}, \mathbf{R}')
\end{aligned} \quad (3.55)$$

In a H.F. picture of the system, in a plane wave approximation of the relative motion of each nucleus,

$$\begin{aligned}
\phi_{i \in A_1}(x_1) &= \varphi_i(r_1) \exp(ik_1 \cdot x_1), \\
\phi_{i \in A_1}(x_2) &= \varphi_i(r_1 + s) \exp(ik_1 \cdot x_2), \\
\phi_{j \in A_2}(x_1) &= \varphi_j(r_2 - s) \exp(ik_2 \cdot x_1), \\
\phi_{j \in A_2}(x_2) &= \varphi_j(r_2) \exp(ik_2 \cdot x_2)
\end{aligned} \quad (3.56)$$

where φ are intrinsic wave function of a nucleus. We may approximate all nucleons have the same wave number as $k_1 = K_1/A_1 = mv_1$, $k_2 = K_2/A_2 = mv_2$. The relative momentum of two nucleus may be written as $K = \mu(v_1 - v_2) = \frac{\mu}{m}(\frac{K_1}{A_1} - \frac{K_2}{A_2})$. Then, define density

$$\sum_{i \in A_1, j \in A'_1} \varphi_i^*(\mathbf{x}_1) \varphi_j(\mathbf{x}_2) = \rho_{a'a}(\mathbf{x}_1, \mathbf{x}_2) \quad (3.57)$$

Direct term becomes

$$\begin{aligned} \sum_{i'j', ij} \langle i'j' | v_D | ij \rangle &= \sum_{i'j', ij} \int \phi_{i'}^*(x_1) \phi_{j'}^*(x_2) v_D(s) \phi_i(x_1) \phi_j(x_2) d^3 r_a d^3 r_A \\ &= \sum_{i'j', ij} \int \phi_{i'}^*(x_1) \phi_i(x_1) \phi_{j'}^*(x_2) \phi_j(x_2) v_D(s) d^3 r_a d^3 r_A \\ &= \sum_{i'j', ij} \int \varphi_{i'}^*(r_a) \varphi_i(r_a) \varphi_{j'}^*(r_A) \varphi_j(r_A) v_D(s) d^3 r_a d^3 r_A \\ &= \int \rho_{a'a}(r_a, r_a) \rho_{A'A}(r_A, r_A) v_D(s) d^3 r_a d^3 r_A \end{aligned} \quad (3.58)$$

Exchange term,

$$\begin{aligned} \sum_{i'j', ij} \langle i'j' | v_{EX} | ji \rangle &= \sum_{i'j', ij} \int \phi_{i'}^*(x_1) \phi_{j'}^*(x_2) v_{EX}(s) \phi_j(x_1) \phi_i(x_2) d^3 r_a d^3 r_A \\ &= \sum_{i'j', ij} \int \phi_{i'}^*(x_1) \phi_i(x_2) \phi_{j'}^*(x_2) \phi_j(x_1) v_{EX}(s) d^3 r_a d^3 r_A \\ &= \sum_{i'j', ij} \int \varphi_{i'}^*(r_a) \varphi_i(r_a + s) e^{ik_1 \cdot (x_2 - x_1)} \varphi_{j'}^*(r_A) \varphi_j(r_A - s) e^{ik_2 \cdot (x_1 - x_2)} v_{EX}(s) d^3 r_a d^3 r_A \\ &= \int \rho_{a'a}(r_a, r_a + s) \rho_{A',A}(r_A, r_A - s) v_{EX}(s) e^{i(k_1 - k_2) \cdot (x_2 - x_1)} d^3 r_a d^3 r_A \\ &= \int \rho_{a'a}(r_a, r_a + s) \rho_{A',A}(r_A, r_A - s) v_{EX}(s) \exp(i \frac{m}{\mu} K \cdot s) d^3 r_a d^3 r_A \end{aligned} \quad (3.59)$$

In fact, the relative momentum $\mathbf{K}(R)$ should depends on the distance between nuclei as

$$\frac{K^2(R)}{2\mu} + V(E_\alpha, R) + V_C(R) = E_\alpha \quad (3.60)$$

The direct term is local if potential is local and can be written as

$$\begin{aligned} V_{a'A', aA}^{(D)}(E_\alpha, \mathbf{R}) &= \int \rho_{a'a}(\mathbf{r}_a) \rho_{A'A}(\mathbf{r}_A) v_D(\rho, E_\alpha, s) d^3 r_a d^3 r_A, \\ \mathbf{s} &= \mathbf{r}_A - \mathbf{r}_a + \mathbf{R} \end{aligned} \quad (3.61)$$

On the other hand, exchange term can be approximated as

$$V_{a'A', aA}^{(EX)}(E_\alpha, R) = \int \rho_{a'a}(\mathbf{r}_a, \mathbf{r}_a + \mathbf{s}) \rho_{A'A}(\mathbf{r}_A, \mathbf{r}_A - \mathbf{s}) v_{EX}(\rho, E_\alpha, s) \exp(i \frac{\mathbf{K}(R) \cdot \mathbf{s}}{M_\alpha}) d^3 r_a d^3 r_A \quad (3.62)$$

where $M_\alpha = aA/(a + A)$.

One may simplify the problem, by treating only the diagonal radial part of potential in a g.s. density for $V(R)$ in $K(R)$. Note that V^{EX} have to be calculated as self consistently.

Let us a bit generalize the F.T. method. Define $\mathbf{x} = \mathbf{r}_a - \mathbf{r}_A$, $\mathbf{y} = (\mathbf{r}_a + \mathbf{r}_A)/2$ and with change of variables,

$$\begin{aligned} & \int d^3\mathbf{r}_a d^3\mathbf{r}_A f(\mathbf{r}_a, \mathbf{r}_A, \mathbf{s} = \mathbf{R} + \mathbf{r}_A - \mathbf{r}_a, \mathbf{R}) \\ &= \int d^3\mathbf{x} d^3\mathbf{y} f(\mathbf{r}_a = \mathbf{y} + \frac{\mathbf{x}}{2}, \mathbf{r}_A = \mathbf{y} - \frac{\mathbf{x}}{2}, \mathbf{s} = \mathbf{R} - \mathbf{x}, \mathbf{R}), \\ &= \int d^3\mathbf{s} d^3\mathbf{y} f(\mathbf{r}_a = \mathbf{y} + \frac{\mathbf{R}}{2} - \frac{\mathbf{s}}{2}, \mathbf{r}_A = \mathbf{y} - \frac{\mathbf{R}}{2} + \frac{\mathbf{s}}{2}, \mathbf{s}, \mathbf{R}) \end{aligned} \quad (3.63)$$

In an approximation of density matrix expansion,

$$\begin{aligned} \rho(\mathbf{r}, \mathbf{r} + \mathbf{s}) &\simeq \rho(\mathbf{r} + \mathbf{s}/2) \hat{j}_1(\mathbf{k}_F(\mathbf{r} + \frac{\mathbf{s}}{2}) \cdot \mathbf{s}), \\ \hat{j}_1(x) &= 3(\sin(x) - x \cos(x))/x^3, \end{aligned} \quad (3.64)$$

with position dependent Fermi momentum $k_F(r)$,

$$k_F(r) = \left\{ \left[\frac{3}{2} \pi^2 \rho(\mathbf{r}) \right]^{3/2} + \frac{5C_S [\nabla \rho(\mathbf{r})]^2}{3\rho(\mathbf{r})^2} + \frac{5\nabla^2 \rho(\mathbf{r})}{36\rho(\mathbf{r})} \right\}^{1/2}, \quad C_S = \frac{1}{36} \frac{1}{4} \quad (3.65)$$

Density matrix can be formally written as

$$\begin{aligned} \rho(\mathbf{R} + \frac{\mathbf{s}}{2}, \mathbf{R} - \frac{\mathbf{s}}{2}) &= \sum_a \phi_a^*(\mathbf{R} + \frac{\mathbf{s}}{2}) \phi_a(\mathbf{R} - \frac{\mathbf{s}}{2}) \\ &= e^{\mathbf{s} \cdot (\nabla_1 - \nabla_2)/2} \sum_a \phi_a^*(\mathbf{R}_1) \phi_a(\mathbf{R}_2) |_{\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}} \end{aligned} \quad (3.66)$$

If time-reversed orbitals are filled pairwise, (in other words, $\phi_m^*(x) = \phi_{-m}(x)$) linear term vanishes,

$$(\nabla_1 - \nabla_2)[\phi_m^*(R_1) \phi_m(R_2) + \phi_m(R_1) \phi_m^*(R_2)] = 0. \quad (3.67)$$

Question: Because of using this property, it is only valid for even-even nucleus(?). Then, involving only even powers, angular integral of the square of a density matrix may be replaced by the integral of the square of the angle average of the density matrix,

$$\rho(\mathbf{R} + \frac{\mathbf{s}}{2}, \mathbf{R} - \frac{\mathbf{s}}{2}) = F\left(\left(\frac{\nabla_1 - \nabla_2}{2}\right)^2\right) \rho(\mathbf{R}_1, \mathbf{R}_2) \quad (3.68)$$

where

$$F(k^2) = \frac{\sinh(ks)}{ks}, k^2 > 0, \quad \frac{\sin(ks)}{ks}, k^2 < 0. \quad (3.69)$$

F is expanded around some average value $-k^2$,

$$\begin{aligned} F\left(\left(\frac{\nabla_1 - \nabla_2}{2}\right)^2\right) &= F(-k^2) + F'(-k^2) \left[\left(\frac{\nabla_1 - \nabla_2}{2}\right)^2 + k^2 \right] + \dots \\ &= j_0(ks) + \frac{s}{2k} j_1(ks) \left[\left(\frac{\nabla_1 - \nabla_2}{2}\right)^2 + k^2 \right] + \dots \end{aligned} \quad (3.70)$$

we can change the double folding integral into a form of

$$\begin{aligned}
& \rightarrow \int d^3s d^3y f_1(\mathbf{r}_a + \frac{\mathbf{s}}{2} = \mathbf{y} + \frac{\mathbf{R}}{2}, \mathbf{s}) f_2(\mathbf{r}_A - \frac{\mathbf{s}}{2} = \mathbf{y} - \frac{\mathbf{R}}{2}, \mathbf{s}) v(\mathbf{s}) \exp(\frac{i\mathbf{K}(R)\mathbf{s}}{M_\alpha}) \\
& = \int d^3s d^3y \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{-ik_1 \cdot (y + \frac{R}{2})} \tilde{f}_1(k_1, s) e^{-ik_2 \cdot (y - \frac{R}{2})} \tilde{f}_2(k_2, s) v(s) \exp(\frac{i\mathbf{K}(R)\mathbf{s}}{M_\alpha}) \\
& = \int d^3s \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot R} \tilde{f}_1(k, s) \tilde{f}_2(-k, s) v(s) \exp(\frac{i\mathbf{K}(R)\mathbf{s}}{M_\alpha}) \\
& \rightarrow \int ds s^2 (4\pi) j_0(\frac{K(R)s}{M_\alpha}) v(s) \int \frac{dk k^2}{2\pi^2} j_0(kR) \tilde{f}_1(k, s) \tilde{f}_2(-k, s) \tag{3.71}
\end{aligned}$$

By using this property, at least one of the integration can be simplified using F.T.

3.3.1 M3Y potential

Reference: PHYSICAL REVIEW C 69, 024610 (2004), PHYSICAL REVIEW C 84, 064602 (2011)

In double folding model, actually there can be two different kind of folding, direct term,

$$U_D(R) = g(E_p) \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p) v_D(\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t) \rho_t(\mathbf{r}_t), \tag{3.72}$$

where $g(E_p)$ is a correction for energy dependence and exchange term,

$$U_E(R) = g(E_p) \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p; \mathbf{r}_p + \mathbf{s}) v_E(\mathbf{s}) \rho_t(\mathbf{r}_t; \mathbf{r}_t - \mathbf{s}) \exp(i\mathbf{k}_{rel} \cdot \mathbf{s} / A_{rel}) \tag{3.73}$$

where $\mathbf{s} = \mathbf{R} + \mathbf{r}_p - \mathbf{r}_t$, $A_{rel} = A_p A_t / (A_p + A_t)$, $k_{rel}^2(R) = 2m_n A_{rel} [E_{cm} - U(R)] / \hbar^2$. The exchange term is usually approximated as a zero-range δ function terms. (Dao T. Khoa, W. von Oertzen, and H. G. Bohlen, Phys. Rev. C 49, 1652 (1994).) (The finite range version of exchange term have 3 Yukawa functions.)

The most frequently used nucleon-nucleon potentials in the theory of nuclear reactions are so-called density-dependant M3Y-Reid and M3Y-Paris potentials (here M is for Michigan, Y is for Yukawa) (Review: M.E. Brandan, G.R. Satchler, The interaction between light heavy-ions and what it tells us, Physics Reports 285 (1997) 143-243). These fitted to reproduce G-matrix calculation.

In general on the potential can be spin-isospin dependent, V_{ST} ,

$$v(r) = V_{00} + V_{01} \tau_1 \cdot \tau_2 + V_{10} \sigma_1 \cdot \sigma_2 + V_{11} \tau_1 \cdot \tau_2 \sigma_1 \cdot \sigma_2. \tag{3.74}$$

- The potential $v(r)$ can be characterized in terms of its moments

$$\begin{aligned}
J_v &= \int dr 4\pi r^2 v(r), \\
\langle r^2 \rangle_v &= \left(\int dr 4\pi r^4 v(r) \right) / J_v \tag{3.75}
\end{aligned}$$

For example, above $M3Y - Reid$ gives $J_{00} = -146 \text{ MeV} \cdot \text{fm}^3$ and $\langle r^2 \rangle_v = 7.26 \text{ fm}^2$.

- Exchange term correction in zero-range approximation for $S = T = 0$ ⁴

$$\begin{aligned}
v_{00}(r) &\rightarrow v_{00}(r) + F_{ex}(E)\delta(r), \\
F_{ex}(E) &\simeq J_{ex}(1 - \tau E/A), \\
\text{M3Y-Reid : } J_{ex} &= -276, \quad \tau = 0.005, \\
\text{M3Y-Paris : } J_{ex} &= -590, \quad \tau = 0.002.
\end{aligned} \tag{3.76}$$

This is not consistent with expression in PHYSICAL REVIEW C 69, 024610 (2004), where energy dependence is an overall factor.

- More generally, (according to the PHYSICAL REVIEW C 69, 024610 (2004))

$$\begin{aligned}
v_{00,D}(r) &= \sum_{i=1}^3 G_{D_i} \frac{e^{-\mu_i r}}{\mu_i r}, \\
v_{00,E}(r) &= \sum_{i=1}^3 G_{E_i} \frac{e^{-\mu_i r}}{\mu_i r}, \quad \text{or } G_{E\delta} \delta^3(r)
\end{aligned} \tag{3.77}$$

Energy dependence is approximated as

$$g(E_p) = 1 - kE_p. \tag{3.78}$$

- Parameters of bare M3Y is summarized as

Coefficients	Reid	Paris
$G_{D1}(\text{MeV})$	7999	11062
$G_{D2}(\text{MeV})$	-2134	-2537.5
$G_{D3}(\text{MeV})$	0	0
$G_{E1}(\text{MeV})$	4631.4	-1524.25
$G_{E2}(\text{MeV})$	-1787.1	-518.75
$G_{E3}(\text{MeV})$	-7.847	-7.847
$\mu_1(\text{fm}^{-1})$	4	4
μ_2	2.5	2.5
μ_3	0.707	0.707
$G_{E\delta}(\text{MeV fm}^3)$	-276	-592
$k(\text{MeV}^{-1})$	0.002	0.003

- Density dependence: But it is known that **one have to use density dependent NN potential for nuclear matter saturation**. Density dependence is introduced as

$$v^{DD}(r; \rho_{FA}, E) = f(\rho_{FA}, E) v_{NN}(r), \tag{3.79}$$

where

$$f(\rho_{FA}, E) = C(E) [1 + \alpha(E) \exp(-\beta(E) \rho_{FA}) - \gamma \rho_{FA}], \tag{3.80}$$

where the density is defined in a somewhat arbitrary as

$$\rho_{FA} = \rho_p(\mathbf{r}_p) + \rho_t(\mathbf{r}_t), \quad \text{or } \rho_p(\mathbf{r}_p + \mathbf{s}/2) + \rho_t(\mathbf{r}_t - \mathbf{s}/2). \tag{3.81}$$

This density dependence introduce a little bit of complication in folding potential.

⁴There are several different values of exchange term values. One may need to check the reference. Some case it is written as $J_{ex} = -262$ for Reid.

- In DDM3Y, $\gamma = 0$. In BDM3Y, $\alpha = 0$. CDM3Y is a combination of DDM3Y and BDM3Y. The parameters are fitted to reproduce correct saturation of nuclear matter. Parameters of DDM3Y are summarized as (different choice gives different nuclear matter incompressibility.)

DD label	interaction	C	α	$\beta(\text{fm}^{-3})$	$\gamma(\text{fm}^{-3})$
0	D independent	1	0	0	0
1	DDM3Y1-Paris	0.2963	3.7231	3.7384	0
2	CDM3Y1	0.3429	3.0232	3.5512	0.5
3	CDM3Y2	0.3346	3.0357	3.0685	1.0
4	CDM3Y3	0.2985	3.4528	2.6388	1.5
5	CDM3Y4-Paris	0.3052	3.2998	2.3180	2.0
6	CDM3Y5	0.2728	3.7367	1.8294	3.0
7	CDM3Y6	0.2658	3.8033	1.4099	4.0
8	BDM3Y1-Paris	1.2521	0.0	0.0	1.7452
9	DDM3Y-Reid	0.2845	3.6391	2.9605	0.0
10	BDM3Y-Reid	1.2253	0.0	0.0	1.5124

- On the other hand, iso-spin dependent part for Reid potential,

$$V_{01}(r) = -4886 \frac{e^{-4r}}{4r} + 1176 \frac{e^{-2.5r}}{2.5r} + 217\delta^3(r). \quad (3.82)$$

The transition density are sometimes approximated as a derivative WS form,

$$\beta_L^{(N)} R_W \frac{d}{dr} W(R) = \beta_L^{(N)} \left(\frac{W R_W}{a_W} \right) \frac{e^x}{(e^x + 1)^2} \quad (3.83)$$

while, $\beta_2^{(N)}$ may be determined from $\beta_2^{(c)}$ which is obtained from $B(E2)$ assuming uniform charge radius $R_c = 1.2A^{1/3}$ fm ,

$$\beta_2^{(N)} R_W = \beta_2^{(c)} R_c \quad (3.84)$$

3.3.2 Folding of DDM3Y

Here, only elastic part of folding potential will be considered. Details can be found in the Appendix of D.T. Khoa and G.R. Satchler, NPA668(2000)3-41.

Direct and exchange potential is

$$U_D(R) = \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p) v_D(\mathbf{R} + \mathbf{r}_p - \mathbf{r}_t) \rho_t(\mathbf{r}_t), \quad (3.85)$$

$$U_E(R) = \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p; \mathbf{r}_p + \mathbf{s}) v_E(\mathbf{s}) \rho_t(\mathbf{r}_t; \mathbf{r}_t - \mathbf{s}) \exp(i\mathbf{k}_{rel} \cdot \mathbf{s}/A_{rel}) \quad (3.86)$$

with density matrix expansion approximation of transition density.

Density/Energy dependence of M3Y potential is assumed,

$$Cg(E)[1 + \alpha e^{-\beta\rho_{FA}} - \gamma\rho_{FA}]v_{D,E}(s) \quad (3.87)$$

Direct potential

$$V_0^{(D)}(E, R) = \frac{Cg(E)}{2\pi^2} \int_0^\infty \left[\rho_0^a(k) \rho_0^A(k) + \alpha \bar{\rho}_0^a(k) \bar{\rho}_0^A(k) - \gamma (\tilde{\rho}_0^a(k) \bar{\rho}_0^A(k) + \bar{\rho}_0^a(k) \tilde{\rho}_0^A(k)) \right] v_D(k) j_0(kR) k^2 dk \quad (3.88)$$

where,

$$\begin{aligned} v_D(k) &= 4\pi \int_0^\infty v_D(r) j_0(kr) r^2 dr, \\ \rho_0^{a,A}(k) &= 4\pi \int_0^\infty \rho_0^{a,A}(r) j_0(kr) r^2 dr, \\ \bar{\rho}_0^{a,A}(k) &= 4\pi \int_0^\infty \rho_0^{a,A}(r) e^{-\beta \rho_0^{a,A}(r)} j_0(kr) r^2 dr, \\ \tilde{\rho}_0^{a,A}(k) &= 4\pi \int_0^\infty [\rho_0^{a,A}(r)]^2 j_0(kr) r^2 dr \end{aligned} \quad (3.89)$$

Exchange potential

$$V_0^{(E)}(E, R) = 4\pi Cg(E) \int_0^\infty G_0(R, s) j_0(K(R)s/M) v_{EX}(s) s^2 ds, \quad (3.90)$$

where

$$G_0(R, s) = \frac{1}{2\pi^2} \int_0^\infty \left[f_a(k, s) f_A(k, s) + \alpha \bar{f}_a(k, s) \bar{f}_A(k, s) - \gamma [f_a(k, s) \tilde{f}_A(k, s) + \tilde{f}_A(k, s) f_A(k, s)] \right] j_0(kR) k^2 dk \quad (3.91)$$

and

$$\begin{aligned} f_{a(A)}(k, s) &= 4\pi \int_0^\infty \rho_0^{a,A}(r) \hat{j}_1(k_{F_{a,A}}(r)s) j_0(kr) r^2 dr, \\ \bar{f}_{a(A)}(k, s) &= 4\pi \int_0^\infty \rho_0^{a,A}(r) e^{-\beta \rho_{a,A}(r)} \hat{j}_1(k_{F_{a,A}}(r)s) j_0(kr) r^2 dr, \\ \tilde{f}_{a(A)}(k, s) &= 4\pi \int_0^\infty [\rho_0^{a,A}(r)]^2 \hat{j}_1(k_{F_{a,A}}(r)s) j_0(kr) r^2 dr \end{aligned} \quad (3.92)$$

$$\frac{\hbar^2 K(R)^2}{2mM} = E_{cm} - V(E, R) - V_C(R) \quad (3.93)$$

where E_{cm} is a center of mass energy, while E is an incident laboratory energy per nucleon and $M = \frac{A_1 A_2}{A_1 + A_2}$. $V(E, R) = V_D(E, R) + V_E(E, R)$.

$$\begin{aligned} \hat{j}_1(x) &= \frac{3(\sin(x) - x \cos(x))}{x^3}, \\ k_{F_{a,A}}(r) &= \left\{ \left[\frac{3}{2} \pi^2 \rho(\mathbf{r}) \right]^{3/2} + \frac{5C_S [\nabla \rho(\mathbf{r})]^2}{3\rho(\mathbf{r})^2} + \frac{5\nabla^2 \rho(\mathbf{r})}{36\rho(\mathbf{r})} \right\}^{1/2}, \quad C_S = \frac{1}{36} \frac{1}{4} \end{aligned} \quad (3.94)$$

- From the assumed density dependence, we can write direct term

$$U_D(R) = C(E) U_{D1}(R) + C(E) \alpha(E) U_{D\alpha}(R) - C(E) \gamma U_{D\gamma}(R) \quad (3.95)$$

and exchange term

$$U_{EX}(R) = C(E) U_{EX1}(R) + C(E) \alpha(E) U_{EX\alpha}(R) - C(E) \gamma U_{EX\gamma}(R) \quad (3.96)$$

- U_{D1} can be done as previous F.T. method.

$$U_{D1}(R) = \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p) v_D(\mathbf{s}) \rho_t(\mathbf{r}_t) \quad (3.97)$$

if we are only dealing with spherical density and spherical potential, we need only spherical bessel transformation with $l = 0$.

Only considering $L = 0$ components of potential and density,

$$U_{D1}(R) = \int \frac{dk k^2}{(2\pi)^3} 4\pi j_0(kR) \tilde{U}_{D1}(k), \quad \tilde{U}_{D1}(k) = \tilde{v}_D(k) \tilde{\rho}_p(k) \tilde{\rho}_t(k) \quad (3.98)$$

$$\tilde{f}_D(k) = 4\pi \int dr r^2 j_0(kr) f(r), \quad f = (v, \rho_p, \rho_t) \quad (3.99)$$

- For $U_{D\alpha}$, it is convenient to choose $\rho_{FA} = \rho_p(\mathbf{r}_p) + \rho_t(\mathbf{r}_t)$,

$$U_{D\alpha} = \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p) e^{-\beta \rho_p(\mathbf{r}_p)} \rho_t(\mathbf{r}_t) e^{-\beta \rho_p(\mathbf{r}_t)} v(\mathbf{s}). \quad (3.100)$$

This can be done as like previous F.T. method, except that we need

$$U_{D\alpha}(R) = \int \frac{dk k^2}{(2\pi)^3} 4\pi j_0(kR) v_D(k) \tilde{\rho}_{p,D\alpha}(k) \tilde{\rho}_{t,D\alpha}(k) \quad (3.101)$$

with

$$\begin{aligned} \tilde{\rho}_{D\alpha}(\mathbf{k}) &\rightarrow \int d^3\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \rho(\mathbf{r}) e^{-\beta \rho(\mathbf{r})}, \\ &\rightarrow 4\pi \int dr r^2 j_0(kr) \rho(r) e^{-\beta \rho(r)} \end{aligned} \quad (3.102)$$

- For $U_{D\gamma}$, it is convenient to choose $\rho_{FA} = \rho_p(\mathbf{r}_p) + \rho_t(\mathbf{r}_t)$,

$$\begin{aligned} U_{D\gamma} &= \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p) (\rho_p(\mathbf{r}_p) + \rho_t(\mathbf{r}_t)) \rho_t(\mathbf{r}_t) v_D(\mathbf{s}) \\ &= \int \frac{dk k^2}{(2\pi)^3} 4\pi j_0(kR) (\tilde{\rho}_p(k) \tilde{\rho}_{t,D\gamma}(k) + \tilde{\rho}_t(k) \tilde{\rho}_{p,D\gamma}(k)) \tilde{v}_D(k) \end{aligned} \quad (3.103)$$

This also can be done as like previous F.T. method except that now it have sum of two terms and one $\rho(\mathbf{k})$ have to be replaced as

$$\begin{aligned} \tilde{\rho}_{D\gamma}(\mathbf{k}) &\rightarrow \int d^3\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \rho(\mathbf{r})^2, \\ &\rightarrow 4\pi \int dr r^2 j_0(kr) \rho(r)^2 \end{aligned} \quad (3.104)$$

- For U_{EX1} , it is convenient to choose $\rho_{FA} = \rho_p(\mathbf{r}_p + \mathbf{s}/2) + \rho_t(\mathbf{r}_t - \mathbf{s}/2)$,

$$\begin{aligned} U_{EX1} &\simeq \int d^3\mathbf{r}_p \int d^3\mathbf{r}_t \rho_p(\mathbf{r}_p + \mathbf{s}/2) \hat{j}_1(\mathbf{k}_F(\mathbf{r}_p + \frac{\mathbf{s}}{2}) \cdot \mathbf{s}) \rho_t(\mathbf{r}_t - \mathbf{s}/2) \hat{j}_1(\mathbf{k}_F(\mathbf{r}_t - \frac{\mathbf{s}}{2}) \cdot \mathbf{s}) \\ &\quad \times v_E(\mathbf{s}) \exp(i\mathbf{k}_{rel} \cdot \mathbf{s}/A_{rel}) \\ &= \int d^3\mathbf{y} \int d^3\mathbf{s} \rho_p(\mathbf{y} - \frac{\mathbf{R}}{2}) \hat{j}_1(\mathbf{k}_F(\mathbf{y} - \frac{\mathbf{R}}{2}) \cdot \mathbf{s}) \rho_t(\mathbf{y} + \frac{\mathbf{R}}{2}) \hat{j}_1(\mathbf{k}_F(\mathbf{y} + \frac{\mathbf{R}}{2}) \cdot \mathbf{s}) \\ &\quad \times v_E(\mathbf{s}) \exp(i\mathbf{k}_{rel} \cdot \mathbf{s}/A_{rel}) \end{aligned} \quad (3.105)$$

Then, consider F.T.

$$\rho_p(\mathbf{y} - \frac{\mathbf{R}}{2}) \hat{j}_1(\mathbf{k}_F(\mathbf{y} - \frac{\mathbf{R}}{2}) \cdot \mathbf{s}) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{y} - \frac{\mathbf{R}}{2})} \tilde{f}_1(\mathbf{k}, \mathbf{s}) \quad (3.106)$$

We can obtain

$$\tilde{f}(\mathbf{k}, s) = 4\pi \int dx x^2 j_0(kx) \rho(x) \hat{j}_1(k_F(x)s) \quad (3.107)$$

Then, above integral changes into

$$\begin{aligned} U_{EX1} &= \int d^3 \mathbf{s} \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{R}} \tilde{f}_1(\mathbf{k}, \mathbf{s}) \tilde{f}_2(-\mathbf{k}, \mathbf{s}) v_E(\mathbf{s}) e^{i\mathbf{k}_{rel} \cdot \mathbf{s} / A_{rel}} \\ &\rightarrow 4\pi \int ds s^2 j_0(\frac{k_{rel}(R)s}{A_{rel}}) v_E(s) \left(\int \frac{dk k^2}{2\pi^2} j_0(kR) \tilde{f}_1(k, s) \tilde{f}_2(k, s) \right) \end{aligned} \quad (3.108)$$

Thus, one needs to compute series of spherical Bessel transformation.

- For $U_{EX\alpha}$, it is convenient to choose $\rho_{FA} = \rho_p(\mathbf{r}_p + \mathbf{s}/2) + \rho_t(\mathbf{r}_t - \mathbf{s}/2)$,

$$\begin{aligned} U_{EX\alpha} &= \int d^3 \mathbf{s} \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{R}} \tilde{f}_1(\mathbf{k}, \mathbf{s}) \tilde{f}_2(-\mathbf{k}, \mathbf{s}) v_E(\mathbf{s}) e^{i\mathbf{k}_{rel} \cdot \mathbf{s} / A_{rel}} \\ &\rightarrow 4\pi \int ds s^2 j_0(\frac{k_{rel}(R)s}{A_{rel}}) v_E(s) \left(\int \frac{dk k^2}{2\pi^2} j_0(kR) \tilde{f}_1(k, s) \tilde{f}_2(k, s) \right) \end{aligned} \quad (3.109)$$

this time $\tilde{f}_1(\mathbf{k}, \mathbf{s})$ (also $\tilde{f}_2(\mathbf{k}, \mathbf{s})$ similarly) is defined as

$$\rho_p(\mathbf{y} - \frac{\mathbf{R}}{2}) e^{-\beta \rho_p(\mathbf{y} - \frac{\mathbf{R}}{2})} \hat{j}_1(\mathbf{k}_F(\mathbf{y} - \frac{\mathbf{R}}{2}) \cdot \mathbf{s}) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{y} - \frac{\mathbf{R}}{2})} \tilde{f}_1(\mathbf{k}, \mathbf{s}) \quad (3.110)$$

This implies

$$\tilde{f}(\mathbf{k}, \mathbf{s}) \rightarrow 4\pi \int dx x^2 j_0(kx) \rho(x) e^{-\beta \rho(x)} \hat{j}_1(k_F(x)s) \quad (3.111)$$

- For $U_{EX\gamma}$, it is convenient to choose $\rho_{FA} = \rho_p(\mathbf{r}_p + \mathbf{s}/2) + \rho_t(\mathbf{r}_t - \mathbf{s}/2)$,

$$\begin{aligned} U_{EX\gamma}(R) &= \int d^3 \mathbf{s} \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{R}} \left[\tilde{f}_1^m(\mathbf{k}, \mathbf{s}) \tilde{f}_2(-\mathbf{k}, \mathbf{s}) + \tilde{f}_1(\mathbf{k}, \mathbf{s}) \tilde{f}_2^m(-\mathbf{k}, \mathbf{s}) \right] v_E(\mathbf{s}) e^{i\mathbf{k}_{rel} \cdot \mathbf{s} / A_{rel}} \\ &\rightarrow 4\pi \int ds s^2 j_0(\frac{k_{rel}(R)s}{A_{rel}}) v_E(s) \left(\int \frac{dk k^2}{2\pi^2} j_0(kR) (\tilde{f}_p(k, s) \tilde{f}_t^m(k, s) + \tilde{f}_p^m(k, s) \tilde{f}_t(k, s)) \right) \end{aligned} \quad (3.112)$$

where

$$\begin{aligned} \rho_p(\mathbf{y} - \frac{\mathbf{R}}{2}) \hat{j}_1(\mathbf{k}_F(\mathbf{y} - \frac{\mathbf{R}}{2}) \cdot \mathbf{s}) &= \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{y} - \frac{\mathbf{R}}{2})} \tilde{f}_1(\mathbf{k}, \mathbf{s}) \\ \rho_p^2(\mathbf{y} - \frac{\mathbf{R}}{2}) \hat{j}_1(\mathbf{k}_F(\mathbf{y} - \frac{\mathbf{R}}{2}) \cdot \mathbf{s}) &= \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{y} - \frac{\mathbf{R}}{2})} \tilde{f}_1^m(\mathbf{k}, \mathbf{s}) \end{aligned} \quad (3.113)$$

In the same way,

$$\begin{aligned} \tilde{f}(k, s) &\rightarrow 4\pi \int dx x^2 j_0(kx) \rho_p(x) \hat{j}_1(\mathbf{k}_F(x) \cdot \mathbf{s}), \\ \tilde{f}^m(k, s) &\rightarrow 4\pi \int dx x^2 j_0(kx) \rho_p^2(x) \hat{j}_1(\mathbf{k}_F(x) \cdot \mathbf{s}) \end{aligned} \quad (3.114)$$

3.3.3 Parametrization of density, potential and F.T.

There are commonly used forms.

- 2-parameter Fermi form (WS form)

$$\rho_{2pF}(r) = \rho_0 \left[1 + \exp \left(\frac{r - R}{a} \right) \right]^{-1} \quad (3.115)$$

here ρ_0 is extracted by the constraint of the volume integral

$$4\pi \int dr r^2 \rho(r) = X. \quad (3.116)$$

(This can be approximated as

$$\rho_0 = \frac{3X}{4\pi R^3} \left[1 + \left(\frac{\pi a}{R} \right)^2 + 6 \left(\frac{a}{R} \right)^3 e^{-R/a} + \dots \right] \quad (3.117)$$

)

- 2-parameter Gaussian form

$$\rho_{2pG}(r) = \rho_0 \exp \left(-\frac{r^2}{R^2} \right) \quad (3.118)$$

where normalization

$$\rho_0 = \frac{X}{\pi^{3/2} R^3} \quad (3.119)$$

F.T.

$$\tilde{\rho}(k) = \rho_0 \pi^{3/2} R^3 e^{-R^2 k^2/4} \quad (3.120)$$

- 3-parameter Fermi/Gaussian form

$$\rho_{3pF}(r) = \left[1 + \omega \frac{r^2}{R^2} \right] \rho_{2pF}(r) \quad (3.121)$$

- Delta form

$$V_0 \delta^{(3)}(\mathbf{r}) \quad (3.122)$$

F.T.

$$\tilde{V}(k) = V_0. \quad (3.123)$$

- Yukawa form (and pure Coulomb)

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r} \quad (3.124)$$

F.T.

$$\tilde{V}(k) = \frac{4\pi V_0}{\mu} \frac{1}{k^2 + \mu^2} \quad (3.125)$$

- One global parametrization is available from Sao Paulo potential. They use 2-parameter Fermi function, with $R_0 = 1.31A^{1/3} - 0.84$ and $a = 0.56$ fm. Note that this is for matter distribution of Sao-Paulo folding potential. (In other words, it is not a point nucleon distribution.) For point nucleon distribution, $R_{0C} = 1.76Z^{1/3} - 0.96$, $R_{0p} = 1.81Z^{1/3} - 1.12$, $R_{0n} = 1.49N^{1/3} - 0.79$. with $a = 0.5$ fm.

3.4 Sao-Paulo potential

In Sao-Paulo potential, nuclear density is parameterized as

$$\begin{aligned} R_p &= 1.81Z^{1/3} - 1.12, & a_p &= 0.47 - 0.00083Z, \\ R_n &= 1.49N^{1/3} - 0.79, & a_n &= 0.47 + 0.00046N \end{aligned} \quad (3.126)$$

Alternatively, or

$$R_M = 1.31A^{1/3} - 0.84, \quad a_M = 0.56 \quad (3.127)$$

Then, the local equivalent(LE) SP potential in zero-range approach⁵ is

$$\begin{aligned} V_{SP}(R, E) &= V_F(R) \exp(-4v^2/c^2), \\ v^2(R, E) &= \frac{2}{\mu}[E - V_C(R) - V_{SP}(R, E)] \end{aligned} \quad (3.128)$$

where velocity $v^2(R, E)$ should be obtained in a self consistent way. While, the energy independent folding potential part is

$$V_F(R) = \int \rho_1(r_1)\rho_2(r_2)V_0\delta^{(3)}(R - r_1 + r_2)d^3r_1d^3r_2, \quad V_0 = -456 \text{ MeV.fm}^3 \quad (3.129)$$

3.5 DWBA matrix element

In DWBA approximation, initial and final state wave function becomes distorted waves and only potential for transition is included in the matrix element.

$$T^{DWBA} = {}^{(-)}\langle \mathbf{k}_f; I_{pf} M_{pf} I_{tf} M_{tf} | V_{fi} | \mathbf{k}_i; I_{pi} M_{pi} I_{ti} M_{ti} \rangle^{(+)} \quad (3.130)$$

where V_{fi} depends on the reaction and convention.

In partial wave form,

$$\begin{aligned} T_{\alpha\alpha_i}^{DW} &= {}^{(-),DW}\langle k_\alpha, \alpha\phi_\alpha | V_\beta | k_{\alpha_i}, \alpha_i\phi_{\alpha_i} \rangle^{(+),DW}, \\ |k_\alpha, \alpha, \phi_\alpha\rangle^{(\pm),DW} &= \left[\left[i^{L_\alpha} Y_{L_\alpha}(\hat{R}_\alpha) \otimes \phi_{I_p, \alpha}(\xi_{p, \alpha}) \right]_{J_p} \otimes \phi_{I_t, \alpha}(\xi_{t, \alpha}) \right]_{J_\alpha M_\alpha} \frac{\chi_\alpha^{(\pm),DW}(R_\alpha)}{R_\alpha} \end{aligned} \quad (3.131)$$

If there is no rearrangement(inelastic scattering), the potential $V_\beta(R_\alpha, \xi_p, \xi_t)$ and relative wave functions are all function of R_α . Thus, integral is over

$$\int d^3R_\alpha d^{3(p-1)}\xi_p d^{3(t-1)}\xi_t \quad (3.132)$$

In case of rearrangement, the initial DW is described in R_i but final DW is described in R_f . Thus, there should be relation(or Jacobian) connecting coordinates $(R_i, \xi_{p,i}, \xi_{t,i})$ and $(R_f, \xi_{p,f}, \xi_{t,f})$.

$$\int d^3R_i d^{3(p-1)}\xi_{p,i} d^{3(t-1)}\xi_{t,i} = J \int dR_i \int dR_f \left(\int \text{integral over internal degrees} \right) (R_i, R_f) \quad (3.133)$$

⁵There is a non-local version too.

The exact integration depends on the particular assumption on the structure model of bound nuclei and transition mechanism.

In other words, for reaction $a(=b+x)+A \rightarrow b+B(=A+x)$, integration is either

$$d^3\mathbf{R}_\alpha d^3\mathbf{r}_x [d^3\xi_b]_{b-1} [d^3\xi_A]_{A-1} \quad (3.134)$$

or

$$d^3\mathbf{R}_\beta d^3\mathbf{r}_x [d^3\xi_b]_{b-1} [d^3\xi_A]_{A-1}. \quad (3.135)$$

We can convert $d^3\mathbf{r}_x \rightarrow (factor)d^3R_\beta$ in the first choice or $d^3\mathbf{r}_x \rightarrow (factor)d^3R_\alpha$ for the second choice by using the relation between coordinates.

For example, in the figure, we can change r_1 as a function of R_a . Then,

$$\mathbf{r}_1 = \frac{m_B m_a}{m_x(m_a + m_A)} (\mathbf{r}_a - \frac{m_b}{m_a} \mathbf{r}_b), \quad (3.136)$$

and thus, we get

$$d\mathbf{r}_b d\mathbf{r}_1 = J d\mathbf{r}_b d\mathbf{r}_a, \quad J = \left(\frac{m_B m_a}{m_x(m_a + m_A)} \right)^3 \quad (3.137)$$

Or, equivalently,

$$\mathbf{r}_2 = \frac{m_A m_a}{m_x(m_a + m_A)} (\mathbf{r}_a - \frac{m_B}{m_A} \mathbf{r}_b), \quad (3.138)$$

$$d\mathbf{r}_a d\mathbf{r}_2 = J' d\mathbf{r}_a d\mathbf{r}_b, \quad J' = \left(\frac{m_B m_a}{m_x(m_a + m_A)} \right)^3 \quad (3.139)$$

For example, if there is no target spin dependent interaction in the optical potential⁶, there is no target spin projection change in distorted wave(but, projectile spin projection can change),

$$\begin{aligned} T^{DWBA} &= {}^{(-)}\langle \mathbf{k}_f; I_f M_f s_f \sigma_f | V_{fi} | \mathbf{k}_i; I_i M_i s_i \sigma_i \rangle^{(+)} \\ &= J \sum_{\sigma'_i \sigma'_f} \int d\mathbf{R}_f \int d\mathbf{R}_i \chi_{\sigma'_f \sigma_f}^{(-)*}(\mathbf{k}_f, \mathbf{R}_f) \langle I_f M_f s_f \sigma'_f | V_{fi} | I_i M_i s_i \sigma'_i \rangle (\mathbf{R}_f, \mathbf{R}_i) \chi_{\sigma'_i \sigma_i}^{(+)}(\mathbf{k}_i, \mathbf{R}_i) \end{aligned} \quad (3.140)$$

where, $\langle I_f M_f s_f \sigma'_f | V_{fi} | I_i M_i s_i \sigma'_i \rangle (\mathbf{R}_f, \mathbf{R}_i)$ implies integration over all internal coordinates with bound state wave functions. The spin index summation over σ' are introduced for the case that the distorting potential is spin-dependent. And Jacobi factor is from converting natural coordinate system into \mathbf{R}_i and \mathbf{R}_f . In case of post form, natural integral variable is \mathbf{R}_f and \mathbf{r}_f . (In prior form, they are \mathbf{R}_i and \mathbf{r}_i). Thus, Jacobi factor is to change \mathbf{r}_f into \mathbf{R}_i .

The potential for transfer reaction in DWBA is

$$\begin{aligned} V_{fi} &= V_{xA}(\mathbf{r}_1) + V_{bA}(\mathbf{R}) - U_{aA}(\mathbf{R}_\alpha), \quad \text{in prior form} \\ &= V_{xb}(\mathbf{r}_2) + V_{bA}(\mathbf{R}) - U_{bB}(\mathbf{R}_\beta), \quad \text{in post form} \end{aligned} \quad (3.141)$$

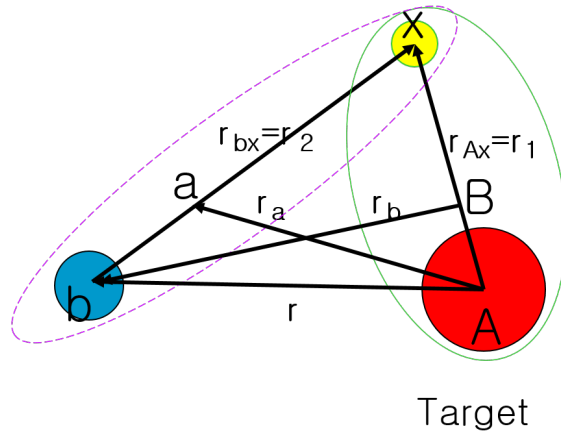


Figure 3.1: Direction of arrow are chosen so that from heavier to lighter particle. $\mathbf{R}_\alpha = \mathbf{r}_1 - \frac{m_b}{m_b+m_x}\mathbf{r}_2$, $\mathbf{R}_\beta = \frac{m_A}{m_A+m_x}\mathbf{r}_1 - \mathbf{r}_2$. $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2 = \frac{m_A\mathbf{R}_\alpha + m_B\mathbf{R}_\beta}{m_a+m_A}$.

where \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{R} are function of \mathbf{R}_α and \mathbf{R}_β . In zero-range approximation or neglecting remnant correction, the second and third terms for transfer reaction are ignored.

Suppose projectile stripping such that projectile $a = b + x$ stripped and bound state $B = A + x$ is formed(See Figure). In the post representation, natural coordinate will be R_b and r_1 .

And, distorted wave is expanded in partial waves

$$\begin{aligned}\chi_{m'm}^{(+)}(\mathbf{k}, \mathbf{r}) &= \frac{4\pi}{kr} \sum_{JLM} \langle LM, sm | J, M+m \rangle \langle LM+m-m', sm' | JM+m \rangle i^L \chi_{LJ}^{(+)}(kr) Y_{LM}^*(\hat{k}) Y_{L, M+m-m'}(\hat{r}), \\ \chi_{LJ}^{(+)}(kr) &\rightarrow \frac{i}{2} \exp(i\sigma_L) [H_L^{(-)}(kr) - \eta_L^J H_L^{(+)}(kr)]\end{aligned}\quad (3.142)$$

, η_L^J is a reflection coefficient or S-matrix. (주의: 여기서는 Coulomb phase factor를 distorted partial wave 에 포함 시켰음.)

Since the Hamiltonian is a scalar, we can write

$$\Delta J = I_i - I_f = L_f - L_i + s_f - s_i = \Delta L + \Delta s \quad (3.143)$$

and V can be multipole expanded in ranks of $\Delta J, \Delta L, \Delta s$.

Thus, the most important information we need to compute is $\langle I_f M_f s_f \sigma'_f | V_{fi} | I_f M_f s_f \sigma'_f \rangle(\mathbf{R}_f, \mathbf{R}_i)$ which is also a central part of coupled channel calculation. Details of computing this kernel will be discussed in next chapter.

If we ignore internal structures of b and A , only projectile and target s.p. wave function of x in projectile and target appears in the matrix elements $\langle I_f M_f s_f \sigma'_f | V_{fi} | I_f M_f s_f \sigma'_f \rangle(\mathbf{R}_f, \mathbf{R}_i)$. In other words, after partial wav expansion,

$$\begin{aligned}I(\mathbf{r}_1, \mathbf{r}_2) &= \langle I_f M_f s_f \sigma'_f | V_{fi} | I_f M_f s_f \sigma'_f \rangle(\mathbf{R}_f, \mathbf{R}_i), \\ &= \sum_{I_B I_A j l_1 n_1 I_x a_x} C_{I_B I_A j l_1 n_1 I_x a_x}^{(1)} C_{s_a s_b s l_2 n_2 I_x a_x}^{(2)} \\ &\quad \times (I_A M_A j m_j | I_B M_B) (s_b m_b s m_s | s_a m_a) (l_1 m_1 I_x M_x | j m_j) (l_2 m_2 I_x M_x | s m_s) \\ &\quad \times \phi_{l_1 m_1 n_1}(\mathbf{r}_1) W(\mathbf{r}_2) \phi_{l_2 m_2 n_2}(\mathbf{r}_2)\end{aligned}\quad (3.144)$$

⁶Usually, spin-orbit interaction is related with the projectile spin.

in post form with $\mathbf{I}_B - \mathbf{I}_A = \mathbf{j}$, $\mathbf{s}_b - \mathbf{s}_a = \mathbf{s}$, $\mathbf{s} + \mathbf{j} = \mathbf{l}$ and $C^{(1,2)}$ are c.f.p in target and projectile bound states.

3.6 Coupled equation sets

모든 가능한 channel을 고려하는 것은 불가능하므로, 적은수의 coupled model space를 생각해야한다. 여기서, model space wave function을 간단히 나타내자.

$$|\Psi\rangle = \sum_{\alpha} |\phi_{\alpha}\rangle \psi_{\alpha} \quad (3.145)$$

여기서, channel α 는 partition과 partial wave 를 모두 포함한 것이고, $|\phi_{\alpha}\rangle$ 에는 bound state 와 관련된 모든 양이 포함되고, ψ_{α} 에는 incoming coefficient를 비롯한 scattering wave의 모든 것이 포함되어 있다고 하자. 말하자면,

$$\begin{aligned} |\phi_{\alpha}\rangle &= \left[\left[i^L Y_L(\hat{R}_x) \otimes \phi_{I_p}^{xp}(\xi_p) \right]_{J_p} \otimes \phi_{I_t}^{xt}(\xi_t) \right]_{J_{tot} M_{tot}}, \\ \psi_{\alpha} &= \frac{\chi_{\alpha\alpha_i}^{J_{tot}}(R_x)}{R_x} \underbrace{A_{\mu_{p_i}\mu_{t_i}}^{J_{tot} M_{tot}}(\alpha_i, \mathbf{k}_i)}_{}, \end{aligned} \quad (3.146)$$

여기서 ψ_{α} 는 아직 분모의 R 을 빼내기 전이다. 또는 $i^L Y_L(\hat{R}_x)$ 까지 ψ_{α} 에 포함시켜서 생각하면, bound state $|\phi_{\alpha}\rangle$ 는 internal degree of freedom 만 포함하게 된다. Underbrace part 는 kinematic part로써, partial wave decomposition of initial wave function 을 나타낸다. (즉, $|\Psi\rangle$ 가 plane wave 인 경우 $|\mathbf{k}_i, \mu_{p_i}, \mu_{t_i}\rangle = \sum_{\alpha_i} |\Psi_{\alpha_i}\rangle A_{\mu_{p_i}\mu_{t_i}}^{J_{tot} M_{tot}}(\alpha_i, \mathbf{k}_i)$ 로 분해된다. 만약 $|\Psi\rangle$ 가 이미 잘 정의된 total angular momentum을 가지는 경우에는 이 부분은 필요가 없다. 아래식에서는 사실상 $|\Psi_{\alpha_i}\rangle$ 으로 생각하자.)

full Wave function Ψ 는 $[\mathcal{H} - E]\Psi = 0$ 을 만족한다. 원래의 many-body system full Hamiltonian \mathcal{H} 은 A-body 에 대한 operator 이지만, 각 channel에 따라 여러가지 다른 방식으로 regroup 할 수 있다.

$$H = H_{xt}(\xi_t) + H_{xp}(\xi_p) + \hat{T}_x(\mathbf{R}_x) + \mathcal{V}_{xtp}(\mathbf{R}_x, \xi_t, \xi_p). \quad (3.147)$$

여기서, $H_{t,p}$ 는 target과 projectile의 bound state를 기술하는 Hamiltonian이고,

$$\begin{aligned} H_{xp}(\xi_p) \phi_{I_p}^{xp}(\xi_p) &= \epsilon_{xp} \phi_{I_p}^{xp}(\mathbf{x}i_p), \\ H_{xt}(\xi_t) \phi_{I_t}^{xt}(\xi_t) &= \epsilon_{xt} \phi_{I_t}^{xt}(\xi_t), \end{aligned} \quad (3.148)$$

relative kinetic term and interaction between target and projectile becomes

$$\begin{aligned} \hat{T}_x(\mathbf{R}_x) &= -\frac{\hbar^2}{2\mu_x} \nabla_{R_x}^2, \\ \mathcal{V}_{xtp}(\mathbf{R}_x, \xi_t, \xi_p) &= \sum_{i \in p, j \in t} V_{ij}(\mathbf{r}_i - \mathbf{r}_j). \end{aligned} \quad (3.149)$$

with reduced mass $\mu_x = \frac{m_{xp}m_{xt}}{m_{xp}+m_{xt}}$.

Let us rewrite the full equation for the scattering wave.

$$0 = [\mathcal{H} - E]|\Psi\rangle = [\mathcal{H} - E]|\phi_i\rangle \psi_i + [\mathcal{H} - E] \sum_{j \neq i} |\phi_j\rangle \psi_j \quad (3.150)$$

If we multiply $\langle \phi_i |$ on the left hand side, (it is just a separation of terms and no orthogonality of wave functions are assumed.)

$$\langle \phi_i | E - \mathcal{H} | \phi_i \rangle \psi_i = - \sum_{j \neq i} \langle \phi_i | E - \mathcal{H} | \phi_j \rangle \psi_j. \quad (3.151)$$

The left hand side can be simplified by

$$\begin{aligned} \langle \phi_i | E - \mathcal{H} | \phi_i \rangle \psi_i &= \langle \phi_i | [E - H_i^{bd} - T_i - V_i] | \phi_i \rangle \psi_i = \langle \phi_i | [E_i - T_i - V_i] | \phi_i \rangle \psi_i \\ &= (E_i - T_i(R_i) - \langle \phi_i | V_i | \phi_i \rangle(R_i)) \psi_i(R_i) \end{aligned} \quad (3.152)$$

여기서, $V_i = \mathcal{V}_{xtp}(\mathbf{R}_x, \boldsymbol{\xi}_t, \boldsymbol{\xi}_p)$ 이고, $\langle \phi_i | V_i | \phi_i \rangle(R_i)$ 는 bound state internal coordinate ξ_p, ξ_t 에 대한 적분이 내포된다.

For the right-hand side, we may use two different ways,

$$-\langle \phi_i | E - \mathcal{H} | \phi_j \rangle \psi_j = \langle \phi_i | T_i - E_i + V_i | \phi_j \rangle \psi_j = (T_i - E_i) \langle \phi_i | \phi_j \rangle \psi_j + \langle \phi_i | V_i | \phi_j \rangle \psi_j \quad (3.153)$$

or

$$-\langle \phi_i | E - \mathcal{H} | \phi_j \rangle \psi_j = \langle \phi_i | T_j - E_j + V_j | \phi_j \rangle \psi_j = \langle \phi_i | \phi_j \rangle (T_j - E_j) \psi_j + \langle \phi_i | V_j | \phi_j \rangle \psi_j \quad (3.154)$$

같은 partition 안에서는 $x' = x$, $\hat{N}_{\alpha'\alpha} = \delta_{\alpha'\alpha}$ 으로 normalize 되어 있다고 생각한다. 여기서, $\langle \phi_i | \phi_j \rangle$ 는 non-orthogonal 이고, rearrangement channel의 경우에는 R_i 또는 R_j 에 대한 함수가 되기 때문에 $(T_i - E_i)$ 와 commute하지 않을 수 있다. 정리하면,

$$\begin{aligned} (E_i - T_i(R_i) - \langle \phi_i | V_i | \phi_i \rangle(R_i)) \psi_i(R_i) &= \sum_{j \neq i} [(T_i - E_i) \langle \phi_i | \phi_j \rangle \psi_j + \langle \phi_i | V_i | \phi_j \rangle \psi_j], \\ &= \sum_{j \neq i} [\langle \phi_i | \phi_j \rangle (T_j - E_j) \psi_j + \langle \phi_i | V_j | \phi_j \rangle \psi_j] \end{aligned} \quad (3.155)$$

Here, if we introduce auxiliary optical potential U_i ,

$$\begin{aligned} (E_i - T_i - U_i) \psi_i &= \left(\sum_{j \neq i} \langle \phi_i | V_i | \phi_j \rangle \psi_j \right) + ([\langle \phi_i | V_i | \phi_i \rangle - U_i] \psi_i) \\ &\quad + \left(\sum_{j \neq i} \langle \phi_i | \phi_j \rangle (T_j - E_j) \psi_j \right) \end{aligned} \quad (3.156)$$

Or,

$$\begin{aligned} (E_i - T_i - U_i) \psi_i &= \left(\sum_{j \neq i} \langle \phi_i | V_j | \phi_j \rangle \psi_j \right) + ([\langle \phi_i | V_i | \phi_i \rangle(R_i) - U_i] \psi_i) \\ &\quad + \left(\sum_{j \neq i} (T_j - E_j) \langle \phi_i | \phi_j \rangle \psi_j \right), \end{aligned} \quad (3.157)$$

보통의 경우, $\langle \phi_i | V_i | \phi_i \rangle \rightarrow U_i$ 로 근사시키기 때문에 오른쪽의 항 하나는 사라지게 된다. 그러나, U_i 를 auxiliary field로 쓰는 경우 오른쪽 항이 필요하게 된다.

아직 $1/R$ 을 분리하지 않았기 때문에, 여기서 $T_i = -\frac{\hbar^2 \nabla_i^2}{2\mu_i}$ 이다. (Text book 3.2.2. 의 식은 $1/R$ 을 분리한 다음이다.)⁷ 이때, $\langle \phi_i | \phi_j \rangle$ is non-orthogonality terms. And

$$V^{prior}(R_i, R_j) = \langle \phi_i | V_j | \phi_j \rangle, \quad V^{post}(R_i, R_j) = \langle \phi_i | V_i | \phi_j \rangle, \quad (3.159)$$

If we define $U_i(R_i) = \langle \phi_i | V_i | \phi_i \rangle$, diagonal term in the right hand side would vanish.

여기까지는 wave function이 각 channel에서 projectile 과 target의 곱이 된다는 것 이외에 어떠한 가정도 이루어 지지 않았다. 그러나, potential matrix element를 얻기 위해서는 bound state 의 structure에 대한 model이 필요하다. (이것을 bound state의 form factor라고 부른다.)

따라서, coupled channel equation을 만들기 위해서는 bound state wave function과 각 component 사이의 interaction 이 주어져야 하고, 이를 바탕으로 potential matrix element 를 계산해야 한다.(이 때 interaction의 형태도 모델에 따라 달라짐에 주의.) 이렇게 CC equation이 얻어진 다음에는 이를 풀어서 channel wave function을 구한다.

일반적으로, 1차 DWBA를 생각할 때는 non-orthogonality term $\hat{N}_{\alpha\alpha'}$ or remnant interaction may be ignored.

다시 쓰면, $p = (post \text{ or } prior)$,

$$(E_i - T_i - U_i)\psi_i = \left(\sum_{j \neq i} V^p(R_i, R_j) \psi_j \right) + (\langle \phi_i | V_i - U_i | \phi_i \rangle \psi_i) + \left(\sum_{j \neq i} K_{ij}(T_j - E_j) \psi_j \right) \quad (3.160)$$

한편, $j \neq i$ 중에는 partition이 같은 inelastic channel과 partition이 달라지는 rearrangement channel이 있다. inelastic channel의 경우에는 하나의 R_i 만으로 coupling potential을 나타낼 수 있어 post form과 prior form의 구분이 필요없고, non-orthogonality term도 zero가 된다. 따라서, 오른쪽 항을 inelastic scattering 에 의한 local coupling과 rearrangement에 의한 non-local coupling으로 나누자. 또한, partial wave 계산을 위해 $\psi_\alpha = f_\alpha/R$ 를 f_α 에 대한 식으로 바꾸어 주면, in a prior form for rearrangement,

$$\begin{aligned} & (E_{xpt} - T_{xL}(R_x) - U_x(R_x))f_\alpha(R_x) \\ &= \langle \phi_\alpha | V_x - U_x | \phi_\alpha \rangle (R_x) f_\alpha(R_x) + \left(\sum_{\alpha'} V_{\alpha\alpha'}(R_x) f_{\alpha'}(R_x) \right) \\ &+ \left(\sum_{x \neq x', \alpha'} \int dR_{x'} V_{\alpha\alpha'}^{prior}(R_x, R_{x'}) f_{\alpha'}(R_{x'}) \right) \\ &+ \left(\sum_{\alpha \neq \alpha'} \int dR_{x'} N_{\alpha\alpha'}(T_{x'L'} - E_{x'p't'}) f_{\alpha'}(R_{x'}) \right) \end{aligned} \quad (3.161)$$

⁷ $\sum_{j \neq i}$ part에도 optical potential을 도입하면, 오른쪽항을

$$\sum_j [\langle \phi_i | V_j - U_j | \phi_j \rangle] \psi_j + \sum_{j \neq i} \langle \phi_i | (T_j + U_j - E_j) | \phi_j \rangle \psi_j \quad (3.158)$$

로 쓸 수 있다. 즉, V_j 대신 $W_j = V_j - U_j$ 를 사용하는 것과 같게 된다.

where

$$\begin{aligned}
T_{xL}(R) &= -\frac{\hbar^2}{2\mu_x} \left(\frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right), \\
V_{\alpha\alpha'}(R_x) &= \langle \phi_\alpha | V_x(R_x, \xi) | \phi_{\alpha'} \rangle, \\
V_{\alpha\alpha'}^p &= R_x \langle \phi_\alpha | V^p | \phi_{\alpha'} \rangle \frac{1}{R_{x'}}, \\
N_{\alpha\alpha'} &= R_x \langle \phi_\alpha | \phi_{\alpha'} \rangle \frac{1}{R_{x'}}.
\end{aligned} \tag{3.162}$$

The actual form of couplings requires explicit form of potentials in a certain model. For example, inelastic channel coupling

$$V_{\alpha\alpha'}(R_x) = \langle i^L [Y_L(\hat{R}_x) \otimes \phi_p(\xi_p)] \phi_t(\xi_t) | V_x(R_x, \xi_p, \xi_t) | i^{L'} [Y_{L'}(\hat{R}_x) \otimes \phi_{p'}(\xi_{p'})] \phi_{t'}(\xi_{t'}) \rangle \tag{3.163}$$

can be described as an collective model or single particle excitation models. In case of

- 주의: 여기서 구한 식은 wave function을 bound state part와 relative wave function part로 단순히 분리했다.

$$\Psi_{m_p m_t}(\mathbf{k}) = \phi_{I_p m_p} \phi_{I_t m_t} \psi_{m_p m_t}(\mathbf{k}) \tag{3.164}$$

와 같이 표현하는 경우에 해당한다. 따라서, total angular momentum을 생각하는 경우 coupling을 고려해야하고, $\psi_{m_p m_t}(\mathbf{k})$ 는 partial wave로 쓸 때 $1/R$ 을 포함하는 형태이다.

- 실제로 Coupling potential을 계산하기 위해서는 bound state에 대한 structure, 즉, wave function을 알아야한다. 구체적인 식은 어떤 model을 사용하는가에 따라 달라진다. 예를 들어 collective rotational model을 사용할 경우에는 bound state의 internal degree of freedom을 알 필요가 없이, 전체적인 rotation만 생각하면 된다. 한편, one-nucleon transfer의 경우에는 bound state전체의 wave function을 알 필요가 없이 변화되는 부분만의 single particle wave function만 알면 된다.
- coupled equation을 풀 다음에는 boudnary condition으로부터 T-matrix를 구하여 cross section을 구한다. 또는 DWBA를 사용하여,

$$T_{ij}^{DWBA} = J \int d^3 R_i d^3 R_j \psi_i^{(-)*}(R_i) \langle \phi_i | W_i | \phi_j \rangle (R_i, R_j) \psi_j^{(+)}(R_j) \tag{3.165}$$

를 계산하는 것으로 구할 수 있다. 단 이 경우 $\psi_j^{(\pm)}(R_j)$ 는 coupling potential을 zero로 두었을 때의 relative wave function이고, J 는 Jacobian factor이다. 그리고, $W_i = V_i - U_i$.

3.6.1 Coupled equations in partial wave form

If we use the partial wave expansion of channel wave function, we get

$$\begin{aligned}
[E_{xpt} - T_{xL}(R_x) - U_x(R_x)] f_\alpha(R_x) &= \sum_{\alpha', \Gamma > 0} i^{L'-L} V_{\alpha; \alpha'}^\Gamma(R_{x'}) f_{\alpha'}(R_{x'}) \\
&+ \sum_{\alpha', x' \neq x} i^{L'-L} \int_0^{R_m} V_{\alpha; \alpha'}(R_x, R_{x'}) f_{\alpha'}(R_{x'}) dR_{x'} \tag{3.166}
\end{aligned}$$

where,⁸

$$T_{xL}(R) = \frac{-\hbar^2}{2\mu_x} \left(\frac{d^2}{dR^2} - \frac{L(L+1)}{R^2} \right). \quad (3.167)$$

and $f_\alpha(R_x)$ is after extracting $1/R_x$, (i.e. $\Psi = \sum_\alpha |\alpha\rangle \frac{f_\alpha(R_\alpha)}{R_\alpha}$).

여기서, $U_x(R_x)$ 는 diagonal optical potential 로 Coulomb과 Nuclear를 모두 포함한 것이다. 위 식에서는 non-orthogonality term 은 포함이 되지 않았다. (물론, potential 에 coupling 의 효과는 포함되어 있다.) $V_{\alpha;\alpha'}^\Gamma(R_{x'})$ 는 local coupling interaction of multipolarity Γ and $V_{\alpha;\alpha'}(R_x, R_{x'})$ 는 non-local couplings between partitions that arise from particle transfers.

For incoming channel α_0 , radial wave function $\psi_\alpha(R_x)$ satisfy the boundary conditions

$$\psi_\alpha(R_x) = \frac{i}{2} [\delta_{\alpha\alpha_0} H_{L\eta_\alpha}^{(-)}(K_\alpha(R_x)) - S_{\alpha_0\alpha} H_{L\eta_\alpha}^{(+)}(K_\alpha(R_x))]. \quad (3.168)$$

The asymptotic kinetic energy

$$E_{xpt} = E + Q_x - \epsilon_p - \epsilon_t \quad (3.169)$$

$$K_\alpha = \sqrt{\frac{2\mu_x}{\hbar^2 E_{xpt}}}, \quad \eta_\alpha = \frac{2\mu_x}{\hbar^2} \frac{Z_{xp} Z_{xt} e^2}{2K_\alpha} \quad (3.170)$$

Now the problem is how to obtain the optical potential, bound state wave function and coupling potential for various reactions and structures.

3.7 Optical Potential: Feshbach P

In case of multi-channel,

$$(E + i\epsilon - T_R - H_A - V)|\chi_{\mathbf{k}_0}^\epsilon\rangle = i\epsilon|\mathbf{k}_0, \phi_0\rangle, \quad \langle \mathbf{R}, \xi | \mathbf{k}_0, \phi_0 \rangle = e^{i\mathbf{k}_0 \cdot \mathbf{R}} \phi_0(\xi) \quad (3.171)$$

By expanding in channels,

$$\begin{aligned} \chi_{\mathbf{k}_0}^\epsilon(\mathbf{R}, \xi) &= \sum_i \chi_i^\epsilon(\mathbf{R}_i) \phi_i(\xi_i), \\ (E_i + i\epsilon - T_R - V_{ii})|\chi_i^\epsilon\rangle &= \sum_{j \neq i} V_{ij}|\chi_j^\epsilon\rangle + i\epsilon\delta_{i0}|\mathbf{k}_0\rangle, \end{aligned} \quad (3.172)$$

In case N=1 (2-channels),

$$\begin{aligned} (E_0 + i\epsilon - T_R - V_{00})|\chi_0^\epsilon\rangle &= V_{01}|\chi_1^\epsilon\rangle + i\epsilon|\mathbf{k}_0\rangle, \\ (E_1 + i\epsilon - T_R - V_{11})|\chi_1^\epsilon\rangle &= V_{10}|\chi_0^\epsilon\rangle. \end{aligned} \quad (3.173)$$

Then

$$|\chi_1^\epsilon\rangle = (E_1 + i\epsilon - T_R - V_{11})^{-1} V_{10} |\chi_0^\epsilon\rangle \quad (3.174)$$

$$(E_0 + i\epsilon - T_R - V_{00} - V_{01}(E_1 + i\epsilon - T_R - V_{11})^{-1} V_{10}) |\chi_0^\epsilon\rangle = i\epsilon|\mathbf{k}_0\rangle \quad (3.175)$$

⁸In some case, the monopole coupling $\Gamma = 0$ also have to be calculated from the given potentials. Thus, $U_x(R_x) = V_{\alpha,\alpha}^{\Gamma=0}(R_x)$ or right-hand side have to have $V_{\alpha,\alpha}^{\Gamma=0}(R_x)$ couplings.

Thus, we can consider effective potential

$$V_{opt} = V_{00} + V_{01} \frac{1}{E_1 + i\epsilon - T_R - V_{11}} V_{10} \quad (3.176)$$

The second term can only contribute when $E_1 > 0$ (i.e. above the threshold of excited state) (if $E_1 < 0$, no imaginary part contribute in the denominator). Also, the imaginary part of this optical potential is negative, implying loss of flux from incident channel.

DWBA derivation: We can rearrange the operator from $\frac{1}{A} - \frac{1}{B} = \frac{1}{A}(B - A)\frac{1}{B} = \frac{1}{B}(B - A)\frac{1}{A}$

$$\begin{aligned} \frac{1}{E_1 + i\epsilon - T_R - V_{11}} &= \frac{1}{E_1 + i\epsilon - T_R} + \frac{1}{E_1 + i\epsilon - T_R - V_{11}} V_{11} \frac{1}{E_1 + i\epsilon - T_R} \\ &= \frac{1}{E_1 + i\epsilon - T_R} + \frac{1}{E_1 + i\epsilon - T_R} V_{11} \frac{1}{E_1 + i\epsilon - T_R - V_{11}} \end{aligned} \quad (3.177)$$

Then,

$$\begin{aligned} |\chi_1^\epsilon\rangle &= \frac{1}{E_1 + i\epsilon - T_R} \left[1 + V_{11} \frac{1}{E_1 + i\epsilon - T_R - V_{11}} \right] V_{10} |\chi_0^\epsilon\rangle \\ &= \frac{1}{E_1 + i\epsilon - T_R} \Omega_1^{(-)\dagger} V_{10} |\chi_0^\epsilon\rangle \end{aligned} \quad (3.178)$$

with⁹

$$\Omega_1^{(-)} = 1 + \frac{1}{E_1 - i\epsilon - T_R - V_{11}} V_{11} \quad (3.179)$$

We want to find the expression for scattering amplitude from

$$\begin{aligned} \langle \mathbf{R} | \chi_1^{+\epsilon} \rangle &= \int d\mathbf{r}' \langle \mathbf{r}' | \frac{1}{E_1 + i\epsilon - T_R} | \mathbf{r}' \rangle \langle \mathbf{r}' | \Omega_1^{(-)\dagger} V_{10} | \chi_0^\epsilon \rangle \\ &\rightarrow -\frac{\mu}{2\pi\hbar^2} \frac{e^{ik_1 r}}{r} \int d\mathbf{r}' e^{-i\mathbf{k}_1 \cdot \mathbf{r}'} \langle \mathbf{r}' | \Omega_1^{(-)\dagger} V_{10} | \chi_0^\epsilon \rangle, \\ \langle \mathbf{R} | \chi_1^{+\epsilon} \rangle &\rightarrow f_{10}^{(+)} \frac{e^{ik_1 R}}{R} \end{aligned} \quad (3.180)$$

where $\mathbf{k}_1 = k_1 \hat{r}$. Thus, we get

$$f_{10}^{(+)} = -\frac{\mu}{2\pi\hbar^2} \langle \mathbf{k}_1 | \Omega_1^{(-)\dagger} V_{10} | \chi_0^\epsilon \rangle = -\frac{\mu}{2\pi\hbar^2} \langle \chi_{\mathbf{k}_1}^{(-)} | V_{10} | \chi_{\mathbf{k}_0}^{(+)} \rangle \quad (3.181)$$

Note here that

$$\begin{aligned} (E_1 + i\epsilon - T_R - V_{11}) | \chi_{\mathbf{k}_1}^{(+)} \rangle &= 0, \\ (E_0 + i\epsilon - T_R - V_{00}) | \chi_{\mathbf{k}_0}^{(+)} \rangle &= i\epsilon | \mathbf{k}_0 \rangle + V_{01} | \chi_{\mathbf{k}_1}^{(+)} \rangle \end{aligned} \quad (3.182)$$

If we ignore V_{01} term, we get DWBA.

More generally, introducing projector P and Q , we get effective interaction in model space P ,

$$\begin{aligned} (E + i\epsilon - H) | \Psi_{\mathbf{k}} \rangle &= i\epsilon | \mathbf{k} \rangle, \\ | \Psi_{\mathbf{k}} \rangle &= P | \Psi \rangle + Q | \Psi \rangle = | \Psi_P \rangle + | \Psi_Q \rangle \end{aligned} \quad (3.183)$$

⁹For the moment it is not clear whether this operator is Moeller operator. This will be the case when it acts on $|\mathbf{k}_1\rangle$ state.

with projection operator $P + Q = I$, $PQ = QP = 0$, $P^2 = P$, $Q^2 = Q$.

$$\begin{aligned}(E + i\epsilon - PHP)|\Psi_P\rangle &= -PHQ|\Psi_Q\rangle + i\epsilon P|\mathbf{k}\rangle, \\ (E + i\epsilon - QHQ)|\Psi_Q\rangle &= -QHP|\Psi_P\rangle\end{aligned}\quad (3.184)$$

Thus, we may get effective Hamiltonian such that

$$\begin{aligned}(E + i\epsilon - H_{eff})|\Psi_P\rangle &= i\epsilon P|\mathbf{k}\rangle, \\ H_{eff} &= PHP + PHQ \frac{1}{E + i\epsilon - QHQ} QHP\end{aligned}\quad (3.185)$$

This is called Generalized optical potential.

On the other hand, usual optical potential may be defined from the energy average of the generalized optical potential, which removes rapid fluctuation of energy dependence.

The optical potential for elastic channel α , (with V_α is a potential between two-body, $|a\rangle$ is a projectile and target wave,) From,

$$\frac{1}{x + i\eta} = \mathcal{P} \frac{1}{x} - i\pi\delta(x), \quad (3.186)$$

we get Hermitian(real) part

$$\text{Re}\mathcal{V}_\alpha = (a|V_\alpha|a) + (a|V_\alpha Q \frac{\mathcal{P}}{E - QHQ} QV_\alpha|a), \quad (3.187)$$

and imaginary part

$$\text{Im}\mathcal{V}_\alpha = -\pi(a|V_\alpha Q \delta(E - QHQ) QV_\alpha|a) \quad (3.188)$$

where $(a|V_\alpha|a)$ is a static part and the last are dynamic one depending on energy. We can consider an analytic continuation of complex z ,

$$\mathcal{V}_\alpha(z) = (a|V_\alpha|a) + (a|V_\alpha Q \frac{1}{z - QHQ} QV_\alpha|a), \quad (3.189)$$

with physical optical potential is at $z = E + i\eta$ with real E .

3.7.1 a model

Let us consider some model space in shell model. Suppose $A + 1$ system is described by shell model Hamiltonian

$$H_S = \sum_{i=0}^A [T(i) + V_S(i)] \quad (3.190)$$

where $i = 0$ is for continuum neutron and $i > 0$ are in bound nuclei. Let us consider model space,

- $P + Q_1 + Q_2 = 1$.
- $P = |0\rangle\langle 0| \cdot \int d^3k |S; \mathbf{k}_0^+\rangle\langle S; \mathbf{k}_0^+|$: bound nuclei+ scattering neutron
- $Q_1 = \sum_{\lambda=1}^N |S; \lambda\rangle\langle S; \lambda|$: "bound states embedded in the continuum" of $A+1$ nucleons. This corresponds to compound/resonance states.

- $Q_2 = \sum_{m=1}^M |m\rangle\langle m| \cdot \int d^3k_m |S; \mathbf{k}_m^+\rangle\langle S; \mathbf{k}_m^+|$: corresponds to excited state of bound A-nuclei + inelastic scattered neutron.
- let us denote the projected states of full reaction state $|a^+\rangle$ as

$$\begin{aligned} |0; \mathbf{k}^+\rangle &= |a^+\rangle = |0^+\rangle + |1\rangle + |2^+\rangle, \\ |0^+\rangle &= P|a^+\rangle, \quad |1\rangle = Q_1|a^+\rangle, \quad |2^+\rangle = Q_2|a^+\rangle \end{aligned} \quad (3.191)$$

- the full Hamiltonian is $H = H_S + V_{res}$,

$$(H_S + V_{res} - E)|a^+\rangle = 0. \quad (3.192)$$

By using the projection operators and orthogonality of those states, we get

$$\begin{aligned} 0 &= [H_{00} - E]|0^+\rangle + V_{01}|1\rangle + V_{02}|2^+\rangle, \\ 0 &= V_{10}|0^+\rangle + (H_{11} - E)|1\rangle + V_{12}|2^+\rangle, \\ 0 &= V_{20}|0^+\rangle + V_{21}|1\rangle + (H_{22} - E)|2^+\rangle, \end{aligned} \quad (3.193)$$

where $H_{00} = PHP = T(0) + V_S(0) + \epsilon_0 + V_{00}$, $V_{00} = PV_{res}P$, $H_{ii} = Q_i H Q_i$, $V_{ij} = Q_i V_{res} Q_j$.

- For simplicity, if we ignore all couplings to $|1\rangle$ (compound states), we get effective potential or GOP for direct channel coupling,

$$\mathcal{V}^{dir} = V_S(0) + \langle 0|V_{00}|0\rangle + \sum_{m=1}^M \int d^3k_m \frac{|V_{0m}(k_m)\rangle\langle V_{0m}(k_m)|}{E + i\eta - \epsilon_m - E_m} \quad (3.194)$$

where $|V_{0m}(k_m)\rangle = \langle 0|V_{res}|m\rangle|S; \mathbf{k}_m^+\rangle$. (matrix element between bound state waves times scattering wave)

- In this case, the direct couplings to other channel gives imaginary part of potential if $E > \epsilon_m$. In other words, optical potential is complex when energy is greater than in-elastic threshold.
- If we assume, the direct coupling between $|0^+\rangle \rightarrow |2^+\rangle$ is negligible, but there are coupling through compound states. (V_{02} are small but V_{01} , V_{12} are present.) In this case, the same procedure gives us effective potential in elastic channel, or compound GOP as

$$\mathcal{V}^{cpd} = V_{stat} + \sum_{\lambda=1}^N \frac{|V_{0\lambda}\rangle\langle V_{0\lambda}|}{E + i\eta - E_\lambda - \sum_{m=1}^M \int d^3k_m |V_{\lambda m}(\mathbf{k}_m)|^2 (E + i\eta - \epsilon_m - E_m)^{-1}} \quad (3.195)$$

- In other words, $V_{\lambda m}$ describes the decay channels of compound nuclei λ to in-elastic states m . Even though there is no direct channel to inelastic state from elastic, the compound state acts as a doorway state and optical potential contains those direct channel contributions.

- If $E < \epsilon_1$, the compound GOP has poles at energies $E = E_\lambda^{(pole)}$ which makes the denominator zero.

$$E_\lambda^{(pole)} - E_\lambda - \sum_{m=1}^M \int d^3 k_m \frac{|V_{\lambda m}(\mathbf{k})|^2}{E_\lambda^{(pole)} - \epsilon_m - E_m} = 0. \quad (3.196)$$

this means that the energy dependence of GOP for $E < \epsilon_1$ is very violent. On the other hand, if $E > \epsilon_1$, the GOP become complex. (continuously)

- Difference between direct part and compound part is that there is an integration over continuum momentum in direct part. Because of this integration, even though there is a pole from the denominator, the optical potential itself is a continuous function of energy.
- If one define energy shift and partial width as

$$\sum_{m=1}^M \int d^3 k_m \frac{|V_{\lambda m}(\mathbf{k})|^2}{E + i\eta - \epsilon_m - E_m} = \Delta_\lambda - i\frac{1}{2}\Gamma_\lambda \quad (3.197)$$

$$\begin{aligned} \Delta_\lambda(E) &= \sum_{m=1}^M \mathcal{P} \int d^3 k_m \frac{|V_{\lambda m}(\mathbf{k})|^2}{E - \epsilon_m - E_m}, \\ \Gamma_\lambda &= \frac{2\pi m_n}{\hbar^2} \sum_{m=1}^M k_m \int d\Omega_m |V_{\lambda m}(\mathbf{k})|^2 \end{aligned} \quad (3.198)$$

then dynamical part of GOP is

$$\sum_\lambda \frac{|V_{0\lambda}\rangle\langle V_{0\lambda}|}{E - E_\lambda - \Delta_\lambda + i\theta(E - \epsilon_1)\frac{1}{2}\Gamma_\lambda} \quad (3.199)$$

This form is a sum of resonances.

- Within approximation, including both direct and compound couplings, we may say

$$\begin{aligned} \mathcal{V}(E) &= \mathcal{V}^{pole} + \mathcal{V}^{cont}(E), \\ \mathcal{V}^{pole}(E) &= \sum_{\lambda(poles)} \frac{|\tilde{V}_{0\lambda}\rangle\langle\tilde{V}_{0\lambda}|}{E + i\eta - E_\lambda^{(poles)}}, \\ \mathcal{V}^{cont}(E) &= V_{stat} + \mathcal{V}^{dir} + \mathcal{V}^{cpd,cont} \\ &= V^{dir} + iW^{dir}\theta(E - \epsilon_1) + \sum_{\lambda(cont)} \frac{|V_{0\lambda}\rangle\langle V_{0\lambda}|}{E - E_\lambda - \Delta_\lambda + i\theta(E - \epsilon_1)\frac{1}{2}\Gamma_\lambda} \end{aligned} \quad (3.200)$$

where direct part only weakly depends on energy.

- the static part of the potential is local. (if total-anti-symmetrization is not considered. If total-anti-symmetrization is considered, static term should also become non-local.)

$$V_{stat}(\mathbf{r}_0, \mathbf{r}'_0) = \langle \mathbf{r}_0 | V_{stat} | \mathbf{r}'_0 \rangle = V_{fold}(\mathbf{r}_0) \delta(\mathbf{r}_0 - \mathbf{r}'_0), \quad (3.201)$$

On the other hand, dynamical part is non-local,

$$\langle \mathbf{r}_0 | V_{0\lambda} \rangle \langle V_{0\lambda} | \mathbf{r}'_0 \rangle = V_{\lambda 0}(\mathbf{r}_0) V_{\lambda 0}^*(\mathbf{r}'_0) \quad (3.202)$$

What we can learn from these example/model is that the optical potential and structure (including compound formation) are closely related quantity. The distinction of shape-elastic and compound elastic is from whether the static part of potential or compound part is dominant.

3.8 Appendix: Dispersion relation

Let us review shortly about the dispersion relation. Basically 'dispersion relation' is a relation between energy(E) or frequency ω to momentum p or wave vector k , $E(p)$ or $\omega(k)$. On the other hand, the relation between real part and imaginary part of a quantity is also referred as 'Dispersion relation'. It is more correct to call it Kramers-Kronig relation.

3.8.1 Kramers-Kronig relation

If a complex function $\chi(\omega)$ is analytic in upper half-plane and vanish as like $1/|\omega|$, we can have

$$0 = \oint_C \frac{\chi(\omega')}{\omega' - \omega} d\omega' = \mathcal{P} \int \frac{\chi(\omega')}{\omega' - \omega} d\omega' - i\pi\chi(\omega) \quad (3.203)$$

where contour C avoids a pole at ω in real axis. This implies for real and imaginary part of $\chi(\omega)$ satisfies

$$\chi(\omega) = \frac{1}{i\pi} \mathcal{P} \int \frac{\chi(\omega')}{\omega' - \omega} d\omega'. \quad (3.204)$$

3.8.2 In response

Consider a response $P(t)$ to a signal $F(t')$ with a response function $\chi(t - t')$.

$$P(t) = \int \chi(t - t') F(t') dt' \quad (3.205)$$

The causality implies $\chi(t - t') = 0$ for $t < t'$. This means $\chi(\omega)$ should be analytic in upper half plane of ω . (In other words, there is no pole in upper half-plane and one can close the contour in upper half-plane which results zero integral.) On the other hand, if frequency of signal is too large, the system can not respond to very large frequency. Thus, the response function should vanish for large ω . Thus, we can expect the response function satisfies the Kramers-Kronig relation.

In other words, following type of integral

$$f(t) = \int d\omega e^{-i\omega t} \frac{\tilde{f}(\omega)}{\omega - \omega' + i\eta} \propto \theta(t), \quad (3.206)$$

because there is no pole of ω in upper-half plane, and thus the contour integral can be closed in upper-half plane for $t < 0$.

3.8.3 In optical potential

Actually, the 'dispersion relation' of optical potential is not exactly Kramers-Kronig relation. But rather a consequence of Feshbach formalism. From the Feshbach formalism,

$$H_{eff} = PHP + \sum_Q PHQ \frac{1}{E + i\epsilon - QHQ} QHP \quad (3.207)$$

where the summation have to be replaced into integral for continuum states of Q . The imaginary part of optical potential arise from the $i\epsilon$ term. If we insert complete set of $P = |\psi_0\rangle\langle\psi_0|$, $Q = \sum_{n \neq 0} |\psi_n\rangle\langle\psi_n|$, $QHQ|\psi_n\rangle = E_n|\psi_n\rangle$, (well, following expression need to check again. Only rough interpretation is applicable.)

$$\langle\psi_0|H_{eff}|\psi_0\rangle = \langle\psi_0|H|\psi_0\rangle + \sum_n \frac{\langle\psi_0|H|\psi_n\rangle\langle\psi_n|H|\psi_0\rangle}{E - E_n + i\epsilon} \quad (3.208)$$

Let us define,

$$\begin{aligned} V(E) &= \langle\psi_0|H|\psi_0\rangle, \\ X(E_n) &= \langle\psi_0|H|E_n\rangle\langle E_n|H|\psi_0\rangle, \end{aligned} \quad (3.209)$$

then

$$\begin{aligned} &\rightarrow V(E) + \int dE' \frac{X(E')}{E - E' + i\epsilon} \\ &= V(E) + \mathcal{P} \int dE' \frac{X(E')}{E - E'} - i\pi X(E) \end{aligned} \quad (3.210)$$

Thus, we may identify real part of optical potential as

$$\begin{aligned} \text{Re}U(E) &= V(E) + \mathcal{P} \int dE' \frac{X(E')}{E - E'}, \\ \text{Im}U(E) &= -\pi X(E). \end{aligned} \quad (3.211)$$

In other words, we can write

$$\text{Re}U(E) = V(E) - \frac{1}{\pi} \mathcal{P} \int dE' \frac{\text{Im}U(E')}{E - E'} \quad (3.212)$$

(Can we identify the second term as $V_{EX}(E)$?)

We may separate optical potential into

$$\begin{aligned} V_{opt}(E) &= V_{stat} + V_{dyn}(E), \\ V_{dyn}(E) &= V_{pol}(E) + i\text{Im}V_{opt}(E). \\ V_{pol}(E) &= \frac{\mathcal{P}}{\pi} \int_{E_0}^{\infty} dE' \frac{\text{Im}V_{opt}(E')}{E' - E} \end{aligned} \quad (3.213)$$

Or, for complex z ,

$$V_{opt}(z) = V_{stat} + \frac{1}{\pi} \int_{z_0}^{\infty} dz' \frac{\text{Im} V_{opt}(z')}{z' - z - i\eta} \quad (3.214)$$

Thus, if the real potential V_{stat} and $\text{Im}V_{opt}(E)$ is given, one can compute real part of $V_{pol}(E)$ and get the full potential as

$$V_{opt}(E) = V_{stat} + V_{pol}(E) + i\text{Im}V_{opt}(E) \quad (3.215)$$

note that, it is about the depth of optical potential rather than spatial shape. Thus, one can consider $V_{opt}(E)$ is a depth of WS potential.

We may define time-dependent potential as

$$V_{opt}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt/\hbar} V_{opt}(E). \quad (3.216)$$

Above dispersion relation implies

$$V_{opt}(t) = V_{stat}\delta(t) + V_{dyn}(t)\theta(t) \quad (3.217)$$

Chapter 4

DWBA

4.1 DWBA expression from B.T.Kim's lecture

Be careful for conventions.

$$\begin{aligned}
 T_{M_B m_b; M_A m_a}(\mathbf{k}_b, \mathbf{k}_a) &= \langle \Psi_{M_B m_b}^{(-)} | W | \Psi_{M_A m_a}^{(+)} \rangle \\
 &= J \sum_{m'_a m'_b} \int \int d\mathbf{r}_b d\mathbf{r}_a \chi_{m'_b m_b}^{(-)*}(\mathbf{k}_b, \mathbf{r}_b) \langle I_B M_B, s_b m_b | W | I_A M_A, s_a m_a \rangle \chi_{m'_a m_a}^{(+)}(\mathbf{k}_a, \mathbf{r}_a)
 \end{aligned}$$

For stripping reaction $a = b + x$ and $B = A + x$ in mind, $J = (m_B m_a / (m_x (m_a + m_A)))^3$ and internal states are

$$\begin{aligned}
 \phi_{I_B M_B} &= \phi_{I_B M_B}(\zeta_A, \zeta_x, \mathbf{r}_1), \\
 \varphi_{s_b m_b} &= \varphi(\zeta_b) \\
 \phi_{I_A M_A} &= \phi_{I_A M_A}(\zeta_A), \\
 \varphi_{s_a m_a} &= \varphi_{s_a m_a}(\zeta_b, \zeta_x, \mathbf{r}_2)
 \end{aligned} \tag{4.2}$$

$$I(\mathbf{r}_1, \mathbf{r}_2) = \langle I_B M_B, s_b m_b | W | I_A M_A, s_a m_a \rangle = \int d\zeta_x \langle \phi_{I_B M_B} | \phi_{I_A M_A} \rangle \langle \varphi_{s_b m_b} | W(\mathbf{r}_2) | \varphi_{s_a m_a} \rangle$$

...

4.2 DWBA in Fresco

4.3 DWBA in other book...

Chapter 5

Couplings, Interactions

어떤 Reaction을 계산하기 위해서는 각 channel의 bound state들의 wave function 과, projectile-target을 기술하는 interaction 그리고, channel 사이의 coupling을 기술하는 potential이 필요하다.

5.1 Typical Form of optical potential

In many cases, the Optical potential is parameterized in the form of

$$U(\mathbf{R}) = V(R) + iW(R) + iW_s(R) + (V_{so}(R) + iW_{so}(R))2\mathbf{L} \cdot \mathbf{s} + V_C(R). \quad (5.1)$$

However, there can be additional spin-dependent or isospin-dependent interactions.

Optical potential cannot provide abrupt variations in the cross section like resonance.

- In many case, the form factor are parametrized as Woods-Saxon form

$$f_x(R, r_x, a_x) = \frac{1}{1 + \exp(\frac{R-R_x}{a_x})}, \quad R_x = r_x A^{\frac{1}{3}} \text{ or } r_x(A_1^{\frac{1}{3}} + A_2^{\frac{1}{3}}) \quad (5.2)$$

typically $r_0 = 1.25$ fm, $a = 0.65$ fm. (But for imaginary parts, $a_I = a_S = 0.47$ fm.)

- Volume potential:

$$V(R) + iW(R) = -V_r f_r(R, r_r, a_r) - iV_i f_i(R, r_i, a_i) \quad (5.3)$$

$$-V_0 \frac{1}{1 + \exp(\frac{R-R_v}{a_v})} - iW_0 \frac{1}{1 + \exp(\frac{R-R_w}{a_w})}, \quad R_{v,w} = r_{v,w}(A_p^{\frac{1}{3}} + A_t^{\frac{1}{3}}) \quad (5.4)$$

with $R_r = r_r(A_1^{\frac{1}{3}} + A_2^{\frac{1}{3}})$ or $R_r = r_r A_1^{\frac{1}{3}}$. Since the definition of r_r depends on the convention, one always have to check the convention. $V_r, V_i > 0$ for attractive potential.

Note that potential $-iW, W > 0$ provides absorption.

Optical potentials are energy dependent. Usually, real part of the potential is deeper than imaginary parts.

- Imaginary Surface potential: Central part can have additional surface imaginary potential.

$$iW_s(R) = iW_s \times 4a_s \frac{d}{dR} f_s(R, r_s, a_s) \quad (5.5)$$

$$4a_x \frac{d}{dR} f_x(R, r_x, a_x) = -4 \frac{\exp(\frac{R-R_x}{a_x})}{(1 + \exp(\frac{R-R_x}{a_x}))^2} \quad (5.6)$$

Be careful that one may also define the derivative with $\frac{d}{dR_x}$. $V_s > 0$ for attractive potential. (Because derivative part gives negative sign, $W > 0$ provide absorption.)

- The imaginary volume and surface is complementary. At low energy, there would be no available unoccupied states within nucleus. Thus, only surface provides absorption and the surface is more important than volume term. On the other hand, at high energy, volume part is more important.

$$V_s(R) + iW_s(R) = V_s \times 4a_v \frac{d}{dR} f(R, r_v, a_v) + iW_s \times 4a_w \frac{d}{dR} f(R, r_w, a_w),$$

$$4a_x \frac{d}{dR} f_x(R, r_x, a_x) = -4 \frac{\exp(\frac{R-R_x}{a_x})}{(1 + \exp(\frac{R-R_x}{a_x}))^2}, \quad R_{v,w} = r_{v,w}(A_p^{\frac{1}{3}} + A_t^{\frac{1}{3}}).$$

- Coulomb potential:

$$\begin{aligned} V_{Coul}(R) &= Z_p Z_e e^2 \left(\frac{3}{2} - \frac{R^2}{2R_{Coul}^2} \right) \frac{1}{R_{Coul}}, \quad R \leq R_{Coul} \\ &= Z_p Z_e e^2 \frac{1}{R}, \quad R \geq R_{Coul}, \\ R_{coul} &= r_C (A_p^{1/3} + A_t^{1/3}). \end{aligned} \quad (5.7)$$

with $R_{Coul} = r_{Coul} A^{1/3}$. The factor can be written as

$$Z_p Z_e e^2 = Z_p Z_e (1./137.036) * (197.32698) \simeq 1.43996 Z_p Z_e \quad \text{MeV.fm} \quad (5.8)$$

- Spin-orbit force: Always have to check the convention,

$$\begin{aligned} V_{so} &= \mathcal{F}_1^{so}(R) 2\mathbf{L} \cdot \mathbf{S}, \\ \mathcal{F}_1^{so}(R) &= \left(\frac{\hbar}{m_\pi c} \right)^2 \frac{1}{R} \frac{d}{dR} \frac{V_{so}}{1 + \exp(\frac{R-R_{so}}{a_{so}})}, \end{aligned} \quad (5.9)$$

where, (called Thomas Form factor.)

$$\frac{1}{R} \frac{d}{dR} \frac{V_{so}}{1 + \exp(\frac{R-R_{so}}{a_{so}})} = -\frac{V_{so}}{a_{so} R} \frac{\exp(\frac{R-R_{so}}{a_{so}})}{\left(1 + \exp(\frac{R-R_{so}}{a_{so}}) \right)^2} \quad (5.10)$$

Note that $2\mathbf{L} \cdot \mathbf{S} = J(J+1) - L(L+1) - S(S+1)$ is diagonal in J-basis, but not diagonal in S-basis. Also, $\left(\frac{\hbar}{m_\pi c} \right)^2 \simeq 2.00 \text{ fm}^2$ is conventional. (if used average pion

mass $\left(\frac{\hbar}{m_\pi c}\right)^2 \simeq 2.043 fm^2$, if use $m_\pi^\pm, \left(\frac{\hbar}{m_\pi c}\right)^2 \simeq 1.9988 fm^2$.) Since the derivative $V_{so} > 0$ for attractive potential. Usually, $V_{so} = 5 - 8$ MeV for nucleons.

spin orbit interaction provides asymmetry(polarization effect).

In fact, the relativistic Dirac equation naturally implies the spin-orbit interaction of form in the Schrodinger equation,

$$\left\{ -\frac{\hbar^2}{2m}[\Delta + k^2] + V_1(r) + \frac{1}{r} \frac{dV_2(r)}{dr} \vec{\sigma} \cdot \vec{L} \right\} f(\vec{r}) = 0 \quad (5.11)$$

- spin-spin force,

$$V_{ss} = \mathcal{F}_1^{ss}(R) \mathbf{I}_p \cdot \mathbf{I}_t. \quad (5.12)$$

The matrix element can be written as $\langle L(I_p, I_t) S; J_{tot} | \mathbf{I}_p \cdot \mathbf{I}_t | L(I_p, I_t) S; J_{tot} \rangle = \frac{1}{2} [S(S+1) - I_p(I_p+1) - I_t(I_t+1)]$ in S-basis. It is not diagonal in J-basis.

- VR^n **ambiguity**. If the depth and radius of real part of potential is varied so that VR^n is near constant, $n \simeq 2$, the cross section is insensitive to this variation.
- In general, optical potential is non-local.

$$\langle \mathbf{r} | V_{opt} | \mathbf{r}' \rangle = V(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + K(\mathbf{r}, \mathbf{r}') \quad (5.13)$$

Non-locality was parameterized in one study as

$$V(\mathbf{r}, \mathbf{r}') = U\left(\frac{|\mathbf{r} + \mathbf{r}'|}{2}\right) H\left(\frac{|\mathbf{r} - \mathbf{r}'|}{\rho}\right) \quad (5.14)$$

with U as W-S form and H as Gaussian. The study shows that there is an equivalent local potential which is energy dependent. (즉, 적어도 일부의 energy dependence는 non-locality를 무시하고 local potential을 사용하기 때문에 생긴다.)

- At least for the real part of optical potential may be obtained by using double folding model like M3Y potential.

Note that if the optical potentials are all function of \mathbf{R} , there will be only elastic scattering. To make a transition to other channels, the potential should contain additional coordinate dependence. For example, in rigid rotor model,

$$\begin{aligned} V(\mathbf{R}, \boldsymbol{\xi}) &= 4\pi \sum_{\lambda, \mu} V_\lambda(R) Y_\lambda^\mu(\hat{R}) Y_\lambda^{\mu,*}(\boldsymbol{\xi}), \\ V_\lambda(R) &= \frac{1}{2} \int_0^\pi V(\mathbf{R}, \boldsymbol{\xi}) P_\lambda(\cos \theta) \sin \theta d\theta \end{aligned} \quad (5.15)$$

(This expression is not correct, because there are internal coordinates $\boldsymbol{\xi}$ which have to be integrated with bound state wave functions.)

But, in general, the potential have to be decomposed into multipole operators acting on the full internal coordinates of the system.

5.2 Single Nucleon binding potential

In many cases, the single particle picture of nucleon in the nucleus is useful. The single particle wave function in a nucleus is written as

$$\phi_{lsj;b}^m(\mathbf{r}) = [Y_l(\hat{r}) \otimes X_s]_{jm} \frac{u_{lsj;b}(r)}{r} \quad (5.16)$$

but, the real nucleon in bound state would be different from the simple shell model wave function. Thus, one uses overlap functions and spectroscopic factors to describe single particle level.

5.3 Multipole expansion of interaction

일반적으로 projectile과 target사이의 Hamiltonian은 ξ_p, ξ_t, \mathbf{R} coordinate로 기술되며, Hamiltonian 자체는 scalar 이다. In other words, we may write

$$H(\xi_p, \xi_t, \mathbf{R}) = \sum_{i \in p, j \in t} V_{ij}(\mathbf{r}_{ij}) + \dots \quad (5.17)$$

여기서, orbital \mathbf{R} 에 대한 dependence를 spherical harmonics로 전개하면,

$$H(\xi_p, \xi_t, \mathbf{R}) = \sum_{\lambda} H_{\lambda}(\xi_p, \xi_t, \mathbf{R}), \quad (5.18)$$

where¹

$$\begin{aligned} H_{\lambda}(\xi_p, \xi_t, R) &= \sqrt{4\pi} \mathcal{F}_{\lambda}(R) \sum_m \mathcal{T}_{\lambda}^{*m}(\xi_p, \xi_t, R) Y_{\lambda}^m(\hat{R}) \\ &= \sqrt{4\pi} \mathcal{F}_{\lambda}(R) \mathcal{T}_{\lambda}(\xi_p, \xi_t, R) \cdot Y_{\lambda}(\hat{R}) \end{aligned} \quad (5.19)$$

² One can obtain Radial Form factor for reaction by integrating internal coordinates with wave functions, $\langle f | H | i \rangle$. orbital angular part 와 radial and structure part를 분리할 경우 계산할 필요가 있는 것은

$$V_{fi}^{\lambda}(R) \equiv \mathcal{F}_{\lambda}(R) \langle I_f || \mathcal{T}_{\lambda}(R, \xi) || I_i \rangle. \quad (5.20)$$

가 된다. 이때, ξ 는 projectile 과 target을 모두 포함한다. ³ 그러나, 실제로 coupled equation 을 계산하는데 필요한 것은 $V_{\alpha'\alpha}^{\lambda} = \langle \alpha' | H_{\lambda}(\xi, R) | \alpha \rangle$ 이다. 이것은 위에서 정의한 $V_{fi}^{\lambda}(R)$ 과는 정의가 다름에 주의(?).

Actual form of decomposition of potential depends on the convention and models. But important thing is to decompose the potential into a

$$(\text{Transition Potential}) \sim (\text{Orbital angular part}) \times (\text{Projectile part}) \times (\text{Target part}) \quad (5.23)$$

¹여기서 $\sqrt{4\pi}$ normalization은 $\mathcal{T}_0^0 = 1$ 로 쓸 때, $H_0(\xi_p, \xi_t, R) = \mathcal{F}_0(R)$ 이 되도록 정한 것이다.

²Actually obtaining \mathcal{F}_{λ} , \mathcal{T}_{λ} from original Hamiltonian is a work.

³특별히, angular part가 \mathbf{L} 인 경우 또는 $Y_{\lambda}(\hat{R})$ 로 표현되지 않는다면,

$$H_{\lambda}(\xi, R) = \mathcal{F}_{\lambda}(R) \mathcal{T}_{\lambda}(\xi, R) \cdot \mathcal{O}_{\lambda}(\hat{R}) \quad (5.21)$$

로 생각할 수 있을 것 같다. 예를들어, spin-orbit force라면, $\mathcal{O}_{\lambda=1} = 2\mathbf{L}$, $\mathcal{T}_{\lambda=1}(\xi) = \mathbf{s}$ 로 둘 수 있을듯. 주의할 것은 scalar product notation과 usual tensor product notation에는 차이가 있음.

$$A_J \cdot B_J \equiv \sum_m A_{Jm} B_{Jm}^* = \sum_m (-1)^{-m} A_{Jm} B_{J-m} = (-1)^{-J} \sqrt{2J+1} [A_J \otimes B_J]_{00} \quad (5.22)$$

5.4 FRESCO potentials

In Fresco, the potential part only describes the scalar part of each potentials, $\mathcal{F}_\lambda(R)$, for couplings

$$V_{fi}^\lambda(R) \equiv \mathcal{F}_\lambda(R) \langle I_f || \mathcal{T}_\lambda(R, \xi) || I_i \rangle. \quad (5.24)$$

Thus, actual calculation involves combining scalar part and operator matrix element part.

Central volume ($V_v(r) + iW_v(r)$) and surface potentials ($V_s(r) + iW_s(r)$) have simple structure and only $\mathcal{F}_{\lambda=0}(R)$ is required.

$$V_{fi}^0(R) = \mathcal{F}_{\lambda=0}(R) \langle I_f || 1 || I_i \rangle \quad (5.25)$$

Spin-orbit interactions are $\lambda = 1$,

$$V_{fi}^1(R) = \mathcal{F}_{\lambda=1}(R) \langle I_f || O_{\lambda=1} || I_i \rangle, \quad O_{\lambda=1} = 2\mathbf{l} \cdot \mathbf{s}_p, 2\mathbf{l} \cdot \mathbf{s}_t \quad (5.26)$$

Spin-Spin or isospin interactions are $\lambda = 0$, but requires matrix elements

$$V_{fi}^0(R) = \mathcal{F}_{\lambda=0}(R) \langle I_f || \mathbf{s}_p \cdot \mathbf{s}_t || I_i \rangle \quad (5.27)$$

Tensor interactions are $\lambda = 2$,

$$V_{fi}^2(R) = \mathcal{F}_{\lambda=2}(R) \langle I_f || O_{\lambda=2} || I_i \rangle, \quad O_{\lambda=2} = (\mathbf{s}_p \cdot \hat{R})^2 - \frac{2}{3}, (\mathbf{s}_t \cdot \hat{R})^2 - \frac{2}{3}, \dots \quad (5.28)$$

In general, any kind of coupling can be defined by supplying

$$V_{fi}^\lambda(R) \equiv \mathcal{F}_\lambda(R) \langle I_f || \mathcal{T}_\lambda(R, \xi) || I_i \rangle. \quad (5.29)$$

and scalar part can be defined in a specific form with small number of parameters or numerical table. The coupling part can be supplied in the potential part if the matrix element form can be simply written with some parameters, or can be supplied in the coupling part. It is the same for deformation part, too. If the analytic form of the scalar part is given, and the matrix element can be calculated in a simple way like rotational model, only small number of parameters are enough to define deformation.

5.5 Transition potential

Inelastic scattering이나 transfer reaction등과 같이 non-elastic reaction의 경우, 이러한 transition은 Hamiltonian에 의해 일어난다. The full Hamiltonian is a scalar which is combined with tensor operators, thus a sum of many multi-polarity λ . For a given multipolarity,⁴

$$\begin{aligned} H &= \sum_{\lambda} H_{intr}^{\lambda}(\xi, \mathbf{R}) \\ H_{intr}^{\lambda}(\xi, \mathbf{R}) &= \sqrt{4\pi} \mathcal{F}_{\lambda}(R) \sum_{m=-\lambda}^{\lambda} \mathcal{T}_{\lambda}^m(R, \xi)^* Y_{\lambda}^m(\hat{R}) = \sqrt{4\pi} \mathcal{F}_{\lambda}(R) \mathcal{T}_{\lambda}(R, \xi) \cdot Y_{\lambda}(\hat{R}) \end{aligned} \quad (5.31)$$

⁴여기서 two tensor scalar product,

$$A_{\lambda} \cdot B_{\lambda} \equiv \sum_{m=-\lambda}^{\lambda} A_{\lambda, m}^* B_{\lambda, m} = (-1)^{-\lambda} \sqrt{2\lambda + 1} [A_{\lambda} \otimes B_{\lambda}]^{(0)} \quad (5.30)$$

where λ is a orbital angular momentum transfer in R . 여기서, R 은 target과 projectile 사이의 distance 를 나타낸다. ξ includes internal coordinates of both target and projectile. Also, even writing the Hamiltonian in this form may not be an easy task. Target과 projectile의 internal structure change는 $\mathcal{T}_\lambda(R, \xi)$ 에 의해서 일어난다. 또한, multipolarity λ 는 transferred orbital angular momentum 이라고 생각할 수 있다.

이제 Hamiltonian의 matrix element를 생각해보자. projectile spin과 target spin 합이 $S = I_t + I_p$ 이고 relative angular momentum 이 L 인 , $|(LS)J\rangle$ 상태를 생각하자. Then⁵

$$\begin{aligned} & \langle (L_f S_f) J_{tot} | H^\lambda(R, \xi) | (L_i S_i) J_{tot} \rangle \\ &= \sqrt{4\pi} \mathcal{F}_\lambda(R) \sum_m \langle (L_f S_f) J_{tot} | \mathcal{T}_\lambda(R, \xi) \cdot Y_\lambda(\hat{R}) | (L_i S_i) J_{tot} \rangle \\ &= \sqrt{4\pi} \mathcal{F}_\lambda(R) (-1)^{\lambda+J_{tot}+L_i+S_f} \left\{ \begin{matrix} L_i & S_i & J_{tot} \\ S_f & L_f & \lambda \end{matrix} \right\} \langle L_f || Y_\lambda(R) || L_i \rangle \langle S_f || \mathcal{T}_\lambda(R, \xi) || S_i \rangle \end{aligned}$$

By using, ⁶

$$\langle L_f || Y_\lambda(R) || L_i \rangle = (4\pi)^{-1} \hat{\lambda} \hat{L}_i \langle L_i 0, \lambda 0 | L_f 0 \rangle, \quad (5.33)$$

and define coupling potential⁷

$$V_{fi}^\lambda(R) \equiv \mathcal{F}_\lambda(R) \langle I_f || \mathcal{T}_\lambda(R, \xi) || I_i \rangle, \quad (5.35)$$

$\mathcal{F}_\lambda(R)$ is called a form factor.

$$\begin{aligned} & \langle (L_f I_f) J_{tot} | H_{intr}^\lambda(\xi, R) | (L_i I_i) J_{tot} \rangle \\ &= V_{fi}^\lambda(R) (-1)^{\lambda+J_{tot}+L_i+I_f} \hat{\lambda} \hat{L}_i \langle L_i 0, \lambda 0 | L_f 0 \rangle \left\{ \begin{matrix} L_i & I_i & J_{tot} \\ I_f & L_f & \lambda \end{matrix} \right\} \end{aligned} \quad (5.36)$$

위의 분석은 어떤 coupled reaction model 의 경우에도 적용되는 일반적인 사항이다. 남은 문제는 $V_{fi}^\lambda(R)$ 의 계산을 구체적으로 어떻게 하는가 이고, 이것은 model dependent하다. 이제 몇가지 경우에 대해 구체적으로 살펴 보도록 하자.

5.6 General Spin transfer coupling

Let us consider a Hamiltonian which cause Δl , Δs_p , Δs_t . (Orbital angular momentum transfer, projectile spin transfer, target spin transfer). Suppose the Hamiltonian is already tensor decomposed, and denoted as (now Δl represent both transfer quantum number and operators)

$$[(O_{\Delta l} \otimes O_{\Delta s_p}) \otimes O_{\Delta s_t}]_{00} \rightarrow [(\Delta l \otimes \Delta s_p) \otimes \Delta s_t]_{00} \quad (5.37)$$

⁵여기서 두 텐서가 서로 교환가능이라고 가정한다. 최종 결과의 확인 필요.

⁶In normalization of W-E theorem of reduced matrix element, with $\hat{x} = \sqrt{2x+1}$,

$$\langle j_f m_f | \hat{T}_{jm} | j_i m_i \rangle = \frac{\langle j_m m_i, j_m | j_f m_f \rangle}{\hat{j}_f} \langle j_f || \hat{T}_j || j_i \rangle$$

⁷In case of rearrangement channels or single particle excitation, the coupling potential may be not in a simple form. For example, the coupling for transfer in prior form will require

$$V_{fi}(R_f, R_i) = \langle I_f | V_1(r_1) + V_2(r_2) - U(R_i) | I_i \rangle \quad (5.34)$$

which have to be further decomposed into multipoles and integrated.

Let us consider states $|(LI_p)J, I_t; J_{tot}M_{tot}\rangle$, we get matrix elements

$$\begin{aligned}
& \langle (LI_p)J, I_t; J_{tot}M_{tot} | [(\Delta l \otimes \Delta s_p) \otimes \Delta s_t]_{00} | (L'I'_p)J', I'_t; J_{tot}M_{tot} \rangle \\
&= \frac{1}{\sqrt{2J_{tot}+1}} \langle (LI_p)J, I_t; J_{tot} | [(\Delta l \otimes \Delta s_p) \otimes \Delta s_t]_{00} | (L'I'_p)J', I'_t; J_{tot} \rangle \quad (\text{W.E. theorem}) \\
&= \frac{1}{\sqrt{2J_{tot}+1}} \langle (LI_p)J | [(\Delta l \otimes \Delta s_p)_{\Delta s_t} | (L'I'_p)J'] \langle I_t | \Delta s_t | I'_t \rangle \\
&\quad \times \sqrt{(2J_{tot}+1)(2J_{tot}+1)} \begin{Bmatrix} J & J' & \Delta s_t \\ I_t & I'_t & \Delta s_t \\ J_{tot} & J_{tot} & 0 \end{Bmatrix} \\
&= \sqrt{2J_{tot}+1} \langle (LI_p)J | [(\Delta l \otimes \Delta s_p)_{\Delta s_t} | (L'I'_p)J'] \langle I_t | \Delta s_t | I'_t \rangle \\
&\quad \times \frac{(-1)^{J'+\Delta s_t+I_t+J_{tot}}}{\sqrt{(2\Delta s_t+1)(2J_{tot}+1)}} \begin{Bmatrix} J' & I'_t & J_{tot} \\ I_t & J & \Delta s_t \end{Bmatrix} \\
&= \frac{(-1)^{J'+\Delta s_t+I_t+J_{tot}}}{\sqrt{2\Delta s_t+1}} \begin{Bmatrix} J' & I'_t & J_{tot} \\ I_t & J & \Delta s_t \end{Bmatrix} \langle I_t | \Delta s_t | I'_t \rangle \\
&\quad \times \langle L | \Delta l | L' \rangle \langle I_p | \Delta s_p | I'_p \rangle \sqrt{(2J+1)(2J'+1)(2\Delta s_t+1)} \begin{Bmatrix} L & L' & \Delta l \\ I_p & I'_p & \Delta s_p \\ J & J' & \Delta s_t \end{Bmatrix} \\
&= (-1)^{J'+\Delta s_t+I_t+J_{tot}} \hat{J} \hat{J}' \begin{Bmatrix} J' & I'_t & J_{tot} \\ I_t & J & \Delta s_t \end{Bmatrix} \begin{Bmatrix} L' & I'_p & J' \\ \Delta l & \Delta s_p & \Delta s_t \\ L & I_p & J \end{Bmatrix} \langle L | \Delta l | L' \rangle \\
&\quad \times \langle I_p | \Delta s_p | I'_p \rangle \langle I_t | \Delta s_t | I'_t \rangle \tag{5.38}
\end{aligned}$$

where $\hat{J} = \sqrt{2J+1}$. Fresco compute the first line of matrix elements and only require input for $\langle I_p | \Delta s_p | I'_p \rangle \langle I_t | \Delta s_t | I'_t \rangle$ and potential Form factor.

If there is no target spin transfer, $\Delta s_t = 0$ and $T_{\Delta s_t=0} = 1$, we have $\langle I_t | \Delta s_t | I'_t \rangle = \delta_{I_t I'_t} \hat{I}_t$, and $J = J'$, $\Delta l = \Delta s_p$

$$\begin{aligned}
&\Rightarrow (-1)^{J+I_t+J_{tot}} \hat{J} \hat{J} \hat{I}_t \begin{Bmatrix} J & I_t & J_{tot} \\ I_t & J & 0 \end{Bmatrix} \begin{Bmatrix} L' & I'_p & J \\ \Delta l & \Delta s_p & 0 \\ L & I_p & J \end{Bmatrix} \langle L | \Delta l | L' \rangle \langle I_p | \Delta s_p | I'_p \rangle \\
&= (-1)^{J+I_t+J_{tot}} \hat{J} \hat{J} \hat{I}_t (-1)^{J+I_t+J_{tot}} \frac{1}{\hat{J} \hat{I}_t} (-1)^{L'+\Delta l+I_p+J} \frac{1}{\hat{\Delta l} \hat{J}} \begin{Bmatrix} L & L' & \Delta l \\ I'_p & I_p & J \end{Bmatrix} \\
&\quad \times \langle L | \Delta l | L' \rangle \langle I_p | \Delta l | I'_p \rangle \\
&= (-1)^{L'+\Delta l+I_p+J} \frac{1}{\hat{\Delta l}} \begin{Bmatrix} L & L' & \Delta l \\ I'_p & I_p & J \end{Bmatrix} \langle L | \Delta l | L' \rangle \langle I_p | \Delta l | I'_p \rangle \tag{5.39}
\end{aligned}$$

5.7 Inelastic couplings

- Multipolarity λ : transferred orbital angular momentum.
- Normal parity transition : If change parity between two states, when λ is odd.
- Inelastic transition can occur by electromagnetic and/or nuclear force. EM are almost entirely electric at low and medium energies and principally Coulomb.

5.8 Collective Model

Suppose Nuclear excitation occurs through the rotation or vibration of nuclei. Collective excitation can be parameterized as Radius in both case of rotation and vibration.

$$R(\theta, \phi) = R_0(1 + \sum_{\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \phi)) \quad (5.40)$$

We can think angles θ, ϕ as a angle of body-fixed frame of rotational case, or the angle of deformation in radius for vibrational case.

In case of vibrational model,

$$\begin{aligned} \alpha_{\lambda\mu} &= \frac{\beta_{\lambda}}{\sqrt{2\lambda+1}} (a_{\lambda\mu}^{\dagger} + (-1)^{\mu} a_{\lambda\mu}), \\ H_0 &= \hbar\omega_{\lambda} \sum_{\mu} \alpha_{\lambda\mu}^{\dagger} \alpha_{\lambda\mu} \end{aligned} \quad (5.41)$$

Or, for rotational case,

$$\begin{aligned} \alpha_{\lambda\mu} &= \sqrt{\frac{4\pi}{2\lambda+1}} \beta_{\lambda} Y_{\lambda\mu}(\theta_d, \phi_d), \text{ (for axial symmetry) }, \\ H_0 &= \frac{I(I+1)\hbar^2}{2\mathcal{J}} \end{aligned} \quad (5.42)$$

H_0 describes the energy of collective excited states.

In both cases, parameter β_{λ} ,

$$\beta_{\lambda} = \frac{4\pi}{3Z_T R_T^{\lambda}} \sqrt{\frac{B(E\lambda, \uparrow)}{e^2}}. \quad (5.43)$$

Then,

$$\begin{aligned} V(r, \hat{O}) &= V^{(N)}(r, \hat{O}) + V^{(C)}(r, \hat{O}) \\ V^{(N)}(r, \hat{O}) &= -\frac{V_0}{1 + \exp[(r - R_0 - R_T \hat{O})/a]}, \\ V^{(C)}(r, \hat{O}) &= \frac{3}{2\lambda+1} Z_P Z_T e^2 \frac{R_T^{\lambda}}{r^{\lambda+1}} \hat{O}, \end{aligned} \quad (5.44)$$

for Rotational coupling,

$$\hat{O} = \beta Y_{20}(\theta) \quad (5.45)$$

for Vibrational coupling

$$\hat{O} = \frac{\beta}{\sqrt{4\pi}} (a + a^{\dagger}) \quad (5.46)$$

For vibrational excitation, we can couple $|0\rangle$, $|n=1\rangle = a^{\dagger}|0\rangle = |2+\rangle$, $|n=2\rangle = \frac{(a^{\dagger})^2}{\sqrt{2}}|0\rangle = |0+, 2+, 4+\rangle$, $\epsilon_n = (n + \frac{1}{2})\hbar\omega$,

$$\langle n | \hat{H} | n' \rangle = \begin{pmatrix} 0 & F & 0 \\ F & \epsilon & \sqrt{2}F \\ 0 & \sqrt{2}F & 2\epsilon \end{pmatrix} \quad (5.47)$$

where $F = \frac{\beta}{\sqrt{4\pi}}$. For rotational excitation, $|0+\rangle = |Y_{00}\rangle$, $|2+\rangle = |Y_{20}\rangle$, $|4+\rangle = |Y_{40}\rangle$, with $\epsilon = E_{I=2}$, $\epsilon_I = \frac{I(I+1)}{6} E_{I=2}$

$$\langle I|H|I'\rangle = \begin{pmatrix} 0 & F & 0 \\ F & \epsilon + \frac{2\sqrt{5}}{7}F & \frac{6}{7}F \\ 0 & \frac{6}{7}F & \frac{10}{3}\epsilon + \frac{20\sqrt{5}}{77}F \end{pmatrix} \quad (5.48)$$

where the first terms in diagonal corresponds to rotational energy and the second terms to reorientation couplings. (Need to check above expressions)

Iso-centrifugal approximation: λ :independent of excitations

$$\frac{l(l+1)\hbar^2}{2\mu r^2} \rightarrow \frac{J(J+1)\hbar^2}{2\mu r^2} \quad (5.49)$$

In CCFull, pair transfer (2-nucleon transfer) coupling is approximated by macroscopic pair-transfer coupling which is similar to inelastic scattering. Basically, it is assumed that extra nucleons changes density of nucleus as (transition density)

$$\Delta\rho^p = \frac{\partial\rho}{\partial A}\Delta A, \quad \frac{\partial\rho}{\partial A} \sim \frac{0.4}{A^{2/3}} \frac{\partial\rho}{\partial r} \quad (5.50)$$

or roughly $\beta^p = 3A\beta^V$. Thus, to obtain pair transfer coupling, one needs ΔQ and strength such that F

$$V^p = F \frac{dV}{dr} \quad (5.51)$$

5.9 Coulomb deformation

Coulomb interaction의 경우에는 Potential의 Form 이 분명하게 알려져 있다고 볼수 있다. 하지만, 실제 적용에서는 charge distribution에 대한 tidal 효과가 고려 되어야한다. By the deformation of charge distribution, $\int \rho_q(\mathbf{r})d^3r = Z_p$, internal degree of freedom 과 relative motion으로 분리.

$$H_{intr} = eZ_t V_C(\mathbf{R}) = \sum_{\lambda\mu} \frac{4\pi Z_t e^2}{2\lambda+1} Y_{\lambda\mu}(\mathbf{R}) \int Y_{\lambda\mu}(\mathbf{r})^* \frac{r_{<}^\lambda}{r_{>}^{\lambda+1}} \rho_q(\mathbf{r}) d^3r \quad (5.52)$$

where $r_{<}$ and $r_{>}$ are either R or r depending on the position. Deformation can be considered as related with change of density or shape of density.

Coupling potential is obtained with nuclear matrix elements.

$$V_{fi}(\mathbf{R}) = \int d\xi \langle \phi_f(\xi) | V(\mathbf{R}, \xi) | \phi_i(\xi) \rangle \quad (5.53)$$

At long distance, $R \gg r$, Coulomb excitation coupling is (Note that the monopole part is separate, $\lambda \neq 0$.)

$$V_{fi}(\mathbf{R}) = \sum_{\lambda>0} \frac{4\pi}{2\lambda+1} \frac{Z_t e^2}{R^{\lambda+1}} \langle f; I_f M_f | \mathcal{M}(E\lambda, \mu) | i; I_i M_i \rangle Y_{\lambda\mu}(\mathbf{R}) \quad (5.54)$$

with

$$\langle f; I_f M_f | \mathcal{M}(E\lambda, \mu) | i; I_i M_i \rangle = \langle f; I_f M_f | \int Y_{\lambda\mu}(\mathbf{r})^* r^\lambda \rho_q(\mathbf{r}) d^3r | i; I_i M_i \rangle \quad (5.55)$$

따라서, reduced coupling potential은, $f(R, r) = \frac{r^\lambda}{r^{\lambda+1}}$,

$$V_{fi}^\lambda(R) = \langle I_f | \mathcal{F}_\lambda(R) \mathcal{T}_\lambda^\mu(\xi) | I_i \rangle = \frac{4\pi Z_t e^2}{2\lambda + 1} \langle I_f | Y_{\lambda\mu}(\mathbf{r}) f(R, r) \rho_q(\mathbf{r}) | I_i \rangle \quad (5.56)$$

We define Coulomb reduced matrix element

$$\langle I_f | E\lambda | I_i \rangle = \langle I_f | Y_{\lambda\mu}(\mathbf{r}) r^\lambda \rho_q(\mathbf{r}) | I_i \rangle, \quad (5.57)$$

Here, the integral over internal coordinates $\int d^3r$ is implied. We can immediately notice that if the charge distribution is spherical (in other words, $I_i = I_f = 0$) the diagonal matrix element $\langle 0 | E\lambda | 0 \rangle = 0$ for $\lambda \neq 0$. Also, the deformation can be included in the $\rho_q(\mathbf{r})$. The off-diagonal matrix element can be related to the reduced transition probability,

$$B(E\lambda, I_i \rightarrow I_f) = \frac{1}{2I_i + 1} |\langle I_f | E\lambda | I_i \rangle|^2. \quad (5.58)$$

and the diagonal matrix elements can be related to the multi-pole moment of charge distribution (quadrupole moment and so on).

In Coulomb excitation, which is characteristic its inverse power of $R^{\lambda+1}$, (R_c is a charge radius)

$$\begin{aligned} V_{fi}^\lambda(R) &\rightarrow_{R > R_c \gg r} \frac{\sqrt{4\pi} e^2 Z_t}{2\lambda + 1} \langle I_f | E\lambda | I_i \rangle \frac{1}{R^{\lambda+1}} \\ &\rightarrow_{R_c > R \gg r} \frac{\sqrt{4\pi} e^2 Z_t}{2\lambda + 1} \langle I_f | E\lambda | I_i \rangle \frac{R^\lambda}{R_c^{2\lambda+1}}. \end{aligned} \quad (5.59)$$

Relevant reduced matrix element $\langle I_f | E\lambda | I_i \rangle$ depends on the model. We consider two cases

- **Rotational model:** (The derivation of the following matrix elements will be given later in the explanation of rotor model.) Rotational model 의 경우, nuclei 의 internal structure는 body-fixed frame에서 $|\chi\rangle$ 로 일정하지만, 실험실에서는 회전한 효과가 합쳐져서 보이게 된다는 것이므로, 두 부분을 나누어 정리하면,

$$\langle I_f | E\lambda | I_i \rangle = \langle I_f | Y_{\lambda\mu}(\mathbf{r}) r^\lambda \rho_q(\mathbf{r}) | I_i \rangle = \sqrt{2I_i + 1} \langle I_i K \lambda 0 | I_f K \rangle \langle \chi | E\lambda | \chi \rangle \quad (5.60)$$

where, K is the rotational band and $\langle \chi | E\lambda | \chi \rangle = M_n(E\lambda)$ is the matrix element in intrinsic rest frame of the nucleus (In other words, $|I\rangle \rightarrow |\chi\rangle \otimes |IMK\rangle$). FRESKO requires this $M_n(E\lambda)$ as an input for Coulomb deformation. Note that $\langle \chi | E\lambda | \chi \rangle = M_n(E\lambda)$ is independent of K while $\langle I_f | E\lambda | I_i \rangle$ depends on the K .

For a deformed sphere of **constant** internal density with deformation length δ_λ nucleus, and mean radius R_c , we may get

$$\begin{aligned} \langle \chi | E\lambda | \chi \rangle &= \frac{3Z_p \delta_\lambda R_c^{\lambda-1}}{4\pi} \\ \langle I_f | E\lambda | I_i \rangle &= \frac{3Z_p \delta_\lambda R_c^{\lambda-1}}{4\pi} \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle = \langle \chi | E\lambda | \chi \rangle \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle \end{aligned} \quad (5.61)$$

Vibrational models also use similar form.

- **Experimentally** the reduced matrix $\langle I_f || E\lambda || I_i \rangle$ can be directly related to the reduced transition probability

$$B(E\lambda, I_i \rightarrow I_f) = \frac{1}{2I_i + 1} |\langle I_f || E\lambda || I_i \rangle|^2. \quad (5.62)$$

for off-diagonal matrix elements and Quadrupole moment by

$$Q_2 = \sqrt{\frac{16\pi}{5}} (2I + 1)^{-\frac{1}{2}} \langle II20 || II \rangle \langle I || E2 || I \rangle \quad (5.63)$$

for diagonal reduced matrix elements.

- In Fresco, sometimes the reduced matrix element $M(Ek)$ which has additional phase factor to $\langle I' || Ek || I \rangle$, to be a real number, is used.

$$\begin{aligned} M(Ek; I \rightarrow I') &= i^{I-I'+|I-I'|} \langle I' || Ek || I \rangle \\ &= \pm \sqrt{(2I + 1)B(Ek; I \rightarrow I')}, \\ &= M_n(Ek, I \rightarrow I') (-1)^{(I-I'+|I-I'|)/2} \sqrt{2I + 1} \langle IKk0 || I'K \rangle \end{aligned} \quad (5.64)$$

(Thus, $M(Ek; I \rightarrow I') = M(Ek; I' \rightarrow I)$.)

5.10 Inelastic couplings: simple rotor model

Hamiltonian 을 intrinsic motion과 rotational motion으로 나눌 수 있을 경우

$$H = H_{intr}(q, p) + H_{rot, \alpha}(P_\omega) \quad (5.65)$$

여기서, q, p 는 intrinsic coordinate, conjugate momentum이고, α 는 intrinsic eigen-state index 라고 할 수 있다. ($H_{intr}|\alpha\rangle = E_\alpha|\alpha\rangle$). 한편, body-fixed axis와 space-fixed axis의 angle ω 에 의해 그 orientation을 기술할 수 있는데, rotational Hamiltonian은 orientation ω 가 아니라, conjugate angular momenta P_ω 의 함수일 것이다. 이 Hamiltonian의 eigenstate는

$$\Psi_{\alpha, I} = \Phi_\alpha(q) \varphi_{\alpha, I}(\omega) \quad (5.66)$$

의 형태로, 주어진 intrinsic eigenstate에 대해, rotational level을 angular momentum quantum number I 로 나타낼 수 있을 것이다.

body-fixed axis/frame 에서 intrinsic excited states 를 $|IK\rangle$ 라고 하자. 이 때, I 와 K 에 의해 body-fixed frame에서의 state가 결정된다.(에너지라던지.) 한편, space-fixed frame 에서는 같은 state 가 다른 z-component 를 가질 수 있다. ($|IM\rangle$) 그런데, K 는 intrinsic 이므로, space-fixed axis가 어떻게 바뀌건 항상 같을 것이다. 따라서, 보존되는 양이고, 임의의 excited state in space-fixed frame은 $|IM, K\rangle$ 로 나타낼 수 있는데, 이때의 M 값은 space-fixed axis 와 body-fixed axis의 angle에 의존한다.

State of orientation of the body-fixed system is completely specified by the three angular momentum quantum numbers IKM (conjugate of orientation angles ω)

Then, such state will have rotational property,

$$|IKM\rangle_z = \sum_{M'} \mathcal{D}_{MM'}^I(\omega) |IKM'\rangle_{z'} \quad (5.67)$$

The state $|\omega\rangle$ with orientation ω w.r.t. coordinate system S has the orientation $\omega = 0$ w.r.t. S' ,

$$|\omega\rangle_S = |\omega = 0\rangle_{S'} \quad (5.68)$$

Then, wave function describing orientation of intrinsic system is

$$\Phi_{IKM}(\omega) = \langle \omega | IKM \rangle = \sum_{M'} D_{MM'}^I(\omega) \langle \omega = 0 | IKM' \rangle = D_{MK}^I(\omega) \langle \omega = 0 | IK, M = K \rangle \quad (5.69)$$

Thus, normalized wave function

$$\Phi_{IKM}(\omega) = \left(\frac{2I+1}{8\pi^2} \right)^{1/2} D_{MK}^I(\omega) \quad (5.70)$$

(In fact, $|IKM\rangle$ is not eigenstates of \mathcal{RT} . Thus, considering time-reversal one have to use both $|IKM\rangle$ and $|I(-K)M\rangle$) The \mathcal{D} functions can thus also be viewed as the wave functions describing the orientation of a dynamical system with specified angular momentum quantum numbers I , M , and K .

Bohr and Motelson, "There are no collective rotations about the symmetry axis" for axial symmetric system. (impossibility of distinguishing orientations of the intrinsic frame that differ only by a rotation about the symmetry axis. - absence of collective rotations for a spherical system.)

if the intrinsic Hamiltonian is invariant w.r.t. a rotation of 180 about an axis perpendicular to the symmetry axis. $\mathcal{R} = \mathcal{R}_2(\pi)$ invariance implies that the rotation \mathcal{R} is part of the intrinsic degree of freedom.

In collective model, the internal degree of freedom of bound state only changes collectively. Thus, it can be used only for inelastic scattering channel. Collective motion can be either rotation or vibrations of bound state wave function. In case of simple rotor model, all potential matrix elements and wave function of excited states are determined by deformation length.

Rotation of a deformed nuclear shape, deformational vibration of a nucleus and Coulomb excitation by deformation are described in a similar expression.

The deformation may change in the distance between projectile and target where optical potential is calculated or/and the change depending on the relative orientation of radius vector to the intrinsic orientation of the nucleus.

Spherical symmetric nuclei의 경우에는 quantum mechanical system에 대해 rotation을 생각하는 것이 어렵다. 어느 특정 방향으로의 rotation을 생각할 수 없고, 따라서, rotational motion을 생각할 때는 symmetry axis를 축으로 하는 rotation은 생각하지 않는다. deformed nuclei의 경우 intrinsic spin 의 방향은 symmetry axis와 평행할 것이고, symmetry axis를 축으로 도는 것은 생각할 수 없다. 대신, deformed nuclei의 경우에는 symmetry axis에 대해 수직인 축으로의 rotation을 생각할 수 있게 된다. 따라서 total angular momentum \mathbf{I} 는 rotation \mathbf{R} 과 intrinsic spin \mathbf{j} 의 합으로 생각할 수 있다, $\mathbf{I} = \mathbf{R} + \mathbf{j}$. 이 때, axial symmetric nuclei의 경우에는 \mathbf{R} 이 symmetry axis에 수직이므로, total angular momentum \mathbf{I} 의 symmetry axis에 대한 projection 과 intrinsic spin 의 symmetry axis에 대한 projection을 생각할 수 있고, 이것이 보존된다. (여기서, 헷갈리는 것은 Rotation이라는 것이 핵 전체가 rigid body 처럼 도는 것으로 생각해야 하는지, 아니면, 내부적으로 보이지 않는 핵자들의 flow가 있는 것으로, 내부적인 flow의 상태가 변하는 것으로 생각해야 하는지 하는 것이다. 아무래도, 외부적으로 보이지 않는 일종의 내부 flow의 변화로 rotational state를 이해해야 하는 것이 아닐까 생각된다.

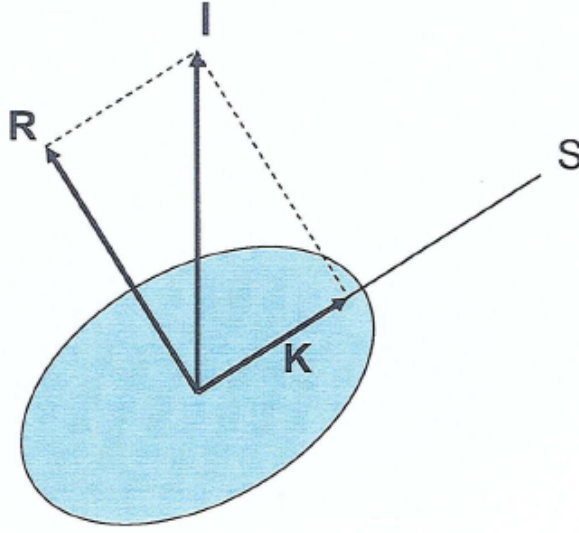


Figure 5.1: Simple Rotor with total angular momentum I of nucleus, a projection of I along symmetry axis, K .

Rigid body rotation 처럼 생각할 경우, 특정 R 위치에서의 optical potential은 시간에 따라 계속 변해야 하는 반면, internal flow로 생각할 경우에는 시간에 대한 변화없이 핵의 대칭축과 R vector의 사잇각에 의해서만 결정되고, 이것은 시간에 대해서 변하지 않는다. 다른 말로, 위 그림에서 Rotational excitation에 의해 그림의 핵이 돌아가는 속도가 바뀌는 것으로 생각하기 보다는 그림의 핵의 모양은 일정한 상태에서 R vector의 크기만 바뀌는 것으로 생각하는 것이다.)

In other words,

- In body fixed frame, the state of a nuclei can be written as $|jK\rangle$ where K is a projection of intrinsic spin on body fixed z' -axis. ($\mathbf{j} \cdot \hat{z}'|jK\rangle = K|jK\rangle$)
- Rotation about the body fixed symmetric axis has no meaning in quantum mechanics. Thus, only perpendicular rotation to body-fixed symmetry axis can be considered. Thus, $\hat{R} \cdot \hat{z}' = 0$.
- In laboratory frame, the total angular momentum(or total spin) of a nucleus and its projection on laboratory z -axis is a good quantum number. $I_z|IM\rangle = M|IM\rangle$. Also, it's projection on to body fixed z' axis is also a good quantum number, $I_{z'} = \mathbf{I} \cdot \hat{z}' = \mathbf{j} \cdot \hat{z}'$. Thus, the rotational state can be expressed as $|IMK\rangle$.
- Rotation can be described as an Euler angle of body fixed z' -axis from laboratory z -axis. This corresponds to rotation matrix.
- 여기서, IMK 라는 quantum number는 eigen-value로써, time-independent하다. 즉, momentum eigen state $|\mathbf{k}\rangle \sim e^{i\mathbf{k}\cdot\mathbf{r}}$ 로 time-dependence를 따로 생각하듯이, $|IMK\rangle$ 의 경우도 rotation에 대한 time-dependence는 따로 생각하고, wave function 의 representation 은 $\langle\omega|IMK\rangle$ 은 오직, intrinsic body-fixed axis와 laboratory frame의 z -axis 사이의 Euler angle ω 에만 의존한다. (angle이 시간에 따라 도는 식 $\theta(t)$ 로 생각하지 않는다.)

- Transitions occur among states with the same value of K .

Deformed rotating nuclei의 wave function 은 intrinsic part χ_K 와 rotational part, D , 로 분리해서 생각할 수 있고, 이 두 부분이 independent 라고 가정하면, wave function은 이 둘의 곱이 된다. Lab frame에서의 rotational state는 symmetry axis가 lab frame에서 어느 방향인지를 나타내는 것으로 나타내어 진다고 볼수 있다. Total angular momentum I 인 입자의 transformation property로부터 rotation motion을 기술하는 wave function은 Wigner D-matrix가 된다고 생각할 수 있다. (with rank I) 또한, spherical symmetry at laboratory system 에 의해 특정 방향에 대한 nuclei 의 에너지 차이를 생각할 수 없으므로, ψ_{IM} 은 M 에 대해 degenerate 해야하고, 또한 clock-wise motion과 counter clock-wise motion도 degenerate 해야하므로, K 와 $-K$ 에 대해서도 two-fold degeneracy를 가져야 한다. 따라서, Rotational Nuclei의 Wave function은

$$\Psi_{IMK} = \left(\frac{2I+1}{16\pi^2} \right)^{\frac{1}{2}} [D_{MK}^I \chi_K + (-1)^{I-K} D_{M-K}^I \chi_{-K}], \quad (5.71)$$

로 써지게 된다. even-even nuclei의 경우 intrinsic spin이 zero 이므로, $K = 0$ 이 되고, 이것은 I 가 even인 값(0,2,4..)만이 가능함을 의미한다. odd nuclei의 경우에는, K 의 값이 half-integer, 이러한 제한이 없다. 특히 $K = 0$ 인 경우는 simply

$$\Psi_{IM0} = \left(\frac{2I+1}{8\pi^2} \right)^{\frac{1}{2}} D_{M0}^I \chi_0, \quad (I = 0, 2, 4..) \quad (5.72)$$

이 때, $|IMK\rangle$ state는

$$I^2|IMK\rangle = I(I+1)\hbar^2|IMK\rangle, \quad I_z|IMK\rangle = M\hbar|IMK\rangle, \quad I_{z'}|IMK\rangle = K\hbar|IMK\rangle \quad (5.73)$$

를 만족한다.

Rotational energy는

$$H = \frac{\hbar^2}{2\tau} \mathbf{R}^2 = \frac{\hbar^2}{2\tau} (\mathbf{I} - \mathbf{j})^2 = \frac{\hbar^2}{2\tau} (\mathbf{I}^2 + \mathbf{j}^2 - 2\mathbf{I} \cdot \mathbf{j}) + T_{coup} \quad (5.74)$$

로 써지고, 여기서 T_{coup} 는 slow rotation에서는 무시가능하다. (\mathbf{R} 이 항상 \mathbf{j} 와 수직이고, $\mathbf{I} = \mathbf{R} + \mathbf{j}$ 라면, $-2\mathbf{I} \cdot \mathbf{j} = -2\mathbf{j}^2$ 이 될 것 이다. 따라서, $E \sim \mathbf{I}^2 - \mathbf{j}^2$. 정말??)

따라서, 간단히

$$E(I) = AI(I+1) + B \quad (5.75)$$

로 parametrize 할 수 있다. 단, Moment of inertia는 rotational band 에 따라 다른 값을 가질 수도 있고, 위 식은 단지 낮은 I 값에 대해서만 성립한다. 좀 더 일반적으로는

$$E(I) = AI(I+1) + B(I(I+1))^2 + \dots \quad (5.76)$$

를 생각할 수 있다.

한편, 다음의 세가지 경우에 대해 rotational energy를 생각해 보자.

(1) even-even nuclei 의 경우 $j = 0$, $K = 0$ 이고, band는 $I = 0+$ 로 부터 시작되며, I 는 even 값 $I = 2^+, 4^+, \dots$ 만 가진다.

$$E(I) = \frac{\hbar^2}{2\tau} I(I+1), \quad I = 0, 2, 4, \dots \quad (5.77)$$

(2) odd nuclei with $K \neq \frac{1}{2}$. 이 경우, band는 $I = K$ 부터 시작하고, $I = K + 1, K + 2, \dots$ 를 가질 수 있다. (모두 같은 parity를 갖는다?)

$$E_K(I) = \frac{\hbar^2}{2\tau} [I(I+1) - K(K+1)], \quad I = K, K+1, K+2, \dots \quad (5.78)$$

(3) Odd nuclei with $K = \frac{1}{2}$. 이 경우엔 특별하게, T_{coup} term에 의한 mixing 효과를 고려해야 한다.

$$E_{K=\frac{1}{2}}(I) = \frac{\hbar^2}{2\tau} [I(I+1) - a(-1)^{I+\frac{1}{2}}(I + \frac{1}{2})], \quad I = \frac{1}{2}, \frac{3}{2}, \dots \quad (5.79)$$

여기서, a 는 decoupling parameter.

(4) According to A. Bohr, ^{235}U levels (odd nuclei) can be parameterized as

$$E(I) = AI(I+1) + BI^2(I+1)^2 + \dots + (-1)^{I+K} \frac{(I+K)!}{(I-K)!} (A_{2K} + B_{2K}I(I+1) + \dots) \quad (5.80)$$

5.10.1 Deformation of potential

Nuclear Deformation을 생각하면, charge distribution의 change에 따라서 Coulomb interaction이 변형되고 matter distribution의 변화로 optical potential도 deform이 된다. 이러한 변형의 효과는 Coulomb excitation이나 Collective excitation (Rotation, vibration)에 의한 nuclear state 변화로 나타나게 된다.

Deformed nucleus 에 대해서 deformed surface of a nucleus 를 body-fixed reference frame 에서 정의하자.

$$\tilde{R}(\theta', \phi') = R_0 + \sum_{q=2}^{q_{max}} \sum_{\mu=-q}^q d_{q\mu} Y_{q\mu}(\theta', \phi') \quad (5.81)$$

여기서, parameter $d_{q\mu}$ 는 length dimension을 가지고, 특별히 d_{q0} 는 deformation length δ_q 라고 부른다. ($q = 0$ 는 R_0 에 포함되고, $q = 1$ 은 translation을 의미한다. Deformation에 대해 Volume이 변하지 않는다 등의 조건을 주면, $d_{q\mu} = (-1)^\mu d_{q-\mu}$ 를 만족해야한다. 특히 axially symmetric인 경우에는 $q = 2$ 에 대해 오직 d_{20} 만이 non-zero이다. 물론 triaxial symmetric인 경우에는 $d_{22} = d_{2-2}$ 도 non-zero.) fractional deformation β_q 는

$$\beta_q = \delta_q / R_0. \quad (5.82)$$

로 정의하고 $\beta_2 > 0$ is prolate, $\beta_2 < 0$ is oblate라고 부른다.

아주 단순한 rotational model에서 deformation of optical potential은

$$\begin{aligned} V(R, \theta', \phi') &= U(R - \tilde{R}(\theta', \phi') + R_0) \\ &\simeq U(R) - U'(R) \sum_{q\mu} d_{q\mu} Y_{q\mu}(\theta', \phi') \end{aligned} \quad (5.83)$$

Axially deformed rotator의 경우 q 는 multipolarity 로 생각할 수 있다.

$$V(R, \theta', \phi') = U(R) - U'(R) \sum_{\lambda=2}^q \delta_\lambda Y_{\lambda 0}(\theta', \phi') \quad (5.84)$$

여기서 angle은 laboratory vector \mathbf{R} 과 body-fixed coordinate에서의 axis \hat{z}' 의 angle이므로

$$Y_{\lambda 0}(\theta', \phi') = \frac{\hat{\lambda}}{\sqrt{4\pi}} P_{\lambda}(\cos \theta') = \frac{\hat{\lambda}}{\sqrt{4\pi}} \frac{4\pi}{2\lambda + 1} \sum_m Y_{\lambda m}(\hat{\xi}) Y_{\lambda m}^*(\hat{R}) \quad (5.85)$$

따라서,

$$V(R, \hat{\xi}, \hat{R}) = U(R) - U'(R) \sum_{\lambda} \delta_{\lambda} \frac{\sqrt{4\pi}}{\hat{\lambda}} \sum_m Y_{\lambda m}(\hat{\xi}) Y_{\lambda m}^*(\hat{R}) \quad (5.86)$$

This means that in the rotational model,

$$\begin{aligned} \mathcal{F}_{\lambda}(R) &= -U'(R), \\ T_{\lambda}(\xi) &= \frac{\delta_{\lambda}}{\hat{\lambda}} Y_{\lambda}^*(\xi). \end{aligned} \quad (5.87)$$

즉, reduced coupling potential은 (여기서는 δ_{λ} 는 deformed nucleus 의 intrinsic property 로 상수 취급. 하지만, 일반적으로는 $\hat{\delta}_{\lambda}$ 는 $V_{fi}^{\lambda}(R) = -\frac{1}{\sqrt{4\pi}} \frac{dU(R)}{dR} \langle I_f || \hat{\delta}_{\lambda}(\xi) || I_i \rangle$ 로 model dependent 한 양으로 계산해야 한다.) 위 식을 통해서 potential을 orbital angular momentum dependent part와 intrinsic part로 분리했다. 여기서, $\hat{\xi}$ 는 z-축과 body-fixed axis와의 angle 이므로 시간에 따라 변하는 것으로 생각할 수 있다.

$$V_{fi}^{\lambda}(R) = \mathcal{F}_{\lambda}(R) \langle I_f || \mathcal{T}(R, \xi) || I_i \rangle = -\frac{dU(R)}{dR} \frac{1}{\hat{\lambda}} \langle I_f || \delta_{\lambda} Y_{\lambda}(\xi) || I_i \rangle \quad (5.88)$$

한편, **Rotor model의 경우**, $Y_{\lambda m}(\xi) = \frac{\hat{\lambda}}{\sqrt{4\pi}} D_{m0}^{\lambda}(\omega)^*$ 이고, rotating state wave function 은 다음과 같이 생각할 수 있다. Let us consider the rotationally excited state of a rotor ϕ_{IM} from intrinsic state ϕ_K ,⁸

$$\phi_{IM} = \frac{\hat{I}}{\sqrt{8\pi^2}} D_{MK}^I(\omega)^* \phi_K \quad (5.89)$$

where D^I is a rotational matrix and ω is a Euler angles. From matrix element,

$$\begin{aligned} \langle I_f M_f | Y_{\lambda m}^*(\hat{\xi}) | I_i M_i \rangle &= \frac{\hat{I}_i \hat{I}_f}{8\pi^2} \int d\omega \langle \phi_K | D_{M_f K}^{I_f}(\omega) Y_{\lambda m}^*(\hat{\xi}) D_{M_i K}^{I_i}(\omega)^* | \phi_K \rangle \\ &= \frac{\hat{\lambda}}{\sqrt{4\pi}} \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle \hat{I}_f^{-1} \langle I_i M_i, \lambda m | I_f M_f \rangle, \end{aligned} \quad (5.90)$$

We get reduced matrix elements,

$$\langle I_f || Y_{\lambda}^*(\xi) || I_i \rangle = \frac{\hat{\lambda}}{\sqrt{4\pi}} \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle. \quad (5.91)$$

By using the integral of rotational matrix,

$$V_{fi}^{\lambda}(R) = -\frac{\delta_{\lambda}}{\sqrt{4\pi}} \frac{dU(R)}{dR} \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle. \quad (5.92)$$

⁸원래는 symmetry 를 고려한 식으로 계산해야한다.

따라서, $U(R)$ 이 주어진 경우, deformation length δ_λ 가 input 으로 필요하다.

주의할 것은 비록 Coulomb deformation과 비슷한 모양이긴 하지만, Coulomb은 charge distribution과 관련되고, 여기서는 nucleus 의 shape deformation 을 생각하므로, 이 둘이 완전히 같지는 않다는 점이다. 또한, Coulomb excitation의 경우, Radial form 이 $1/R^{\lambda+1}$ 인데 비해, Nuclear deformation에 의한 form factor는 $\frac{dU}{dR}$ form이다. 따라서, Coulomb excitation에 들어가는 $p1$ 은 deformation length가 아님.

5.11 Another attempt

The surface of collective nuclei which is deformed from spherical shape,

$$R = R_a(\Omega') = R_0 \left(1 + \sum_{\lambda\mu} a_{\lambda\mu} Y_{\lambda\mu}(\Omega') \right) \quad (5.93)$$

in body-fixed frame. (To be real R , $a_{\lambda\mu}^* = (-1)^\mu a_{\lambda-\mu}$.)⁹ In space fixed frame,

$$R = R_\alpha(\Omega) = R_0 \left(1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\Omega) \right) \quad (5.94)$$

The spherical Harmonics and coefficients between space-fixed frame and body-fixed frame are related via Euler angles and rotation matrix.

$$Y_{\lambda\mu}(\Omega') = \sum_{\mu'} D_{\mu'\mu}^\lambda(\omega) Y_{\lambda\mu'}(\Omega) \quad (5.95)$$

$$\alpha_{\lambda\mu} = \sum_{\mu'} D_{\mu\mu'}^\lambda(\omega) a_{\lambda\mu'} \quad (5.96)$$

Thus, rotation can be described by Euler angles. (In other words, we can see Euler angles as internal coordinates of rotational states.)

On the other hand, in case of vibration, we may think the Euler angle is fixed but the coefficient $a_{\lambda\mu}$ corresponds to vibration and considered in terms of internal coordinates.

With a (small) deformation of spherical potential,

$$V(\mathbf{r}, \alpha) = V_C(r) + V_N(r) + V_{coup}(\mathbf{r}, \alpha) \quad (5.97)$$

where non-spherical part of potential

$$V_{coup}(\mathbf{r}, \alpha) = V_{coup}^C(\mathbf{r}, \alpha) + V_{coup}^N(\mathbf{r}, \alpha) = \sum_{LK} v_L(r) \alpha_{LK} Y_{LK}(\Omega) \quad (5.98)$$

with $v_L(r) = v_L^C(r) + v_L^N(r)$ which can be approximated as derivative of spherical potential.

⁹If $\lambda = 1$, $R_a(\Omega') = R_0(1 + \sqrt{3/4\pi} \mathbf{a} \cdot \mathbf{r}')$ corresponds to overall translation in direction \mathbf{a} .

For spin coupled states, $|(I)JM\rangle$, one can obtain matrix elements of non-spherical potential as

$$\begin{aligned}
V_{bc}^J(r) &= \langle (l_b I_b) JM | V_{coup} | (l_c I_c) JM \rangle \\
&= \sum_L (-)^{l_c + I_b + J} \left\{ \begin{matrix} J & I_b & l_b \\ L & l_c & I_c \end{matrix} \right\} \langle I_b || \alpha_L || I_c \rangle \langle l_b || Y_L || l_c \rangle v_L(r) \\
&= \sum_L (-)^{l_c + I_b + J + l_b} \left(\frac{(2l_b + 1)(2L + 1)(2l_c + 1)}{4\pi} \right)^{1/2} \left(\begin{matrix} l_b & L & l_c \\ 0 & 0 & 0 \end{matrix} \right) \left\{ \begin{matrix} J & I_b & l_b \\ L & l_c & I_c \end{matrix} \right\} \\
&\quad \times \langle I_b || \alpha_L || I_c \rangle v_L(r)
\end{aligned} \tag{5.99}$$

Thus, one can compute the collective excitation by computing $\langle I_b || \alpha_L || I_c \rangle$. In case of rotation, $\alpha_{\lambda\mu}$ is related with $a_{\lambda\mu'}$ of body-fixed frame through rotation matrix. In case of vibration, we can treat the $\alpha_{\lambda\mu}$ as an operator.

5.11.1 rotational excitation

we have to specify quantum numbers of rotational states $|I_c\rangle$ and its energy levels in rotational model and then compute the matrix elements with α .

For permanently axially deformed nuclei,

$$a_{\lambda\mu} = \beta_\lambda \delta_{\mu 0} \tag{5.100}$$

and related with space fixed one

$$\alpha_{\lambda\mu} = \beta_\lambda D_{\mu 0}^\lambda(\omega) \tag{5.101}$$

Consider rotation of axially symmetric nucleus about a space-fixed axis with total spin I and spin projection K . (rotation corresponds to change of Euler angle. there is no rotation in body-fixed frame.) Projection on the body-fixed axis, K , is also a conserved quantum number. Thus, rotational state is described as $|IMK\rangle$.

Simplest case of structureless nucleus with moment of inertia \mathcal{J} rotating about an axis perpendicular to the symmetry axis. this implies $K = 0$ and $I = \text{even}$.

$$\begin{aligned}
\frac{\hbar^2}{2\mathcal{J}} I^2 |IM0\rangle &= \frac{\hbar^2}{2\mathcal{J}} I(I+1) |IM0\rangle \\
I_z |IM0\rangle &= M |IM0\rangle
\end{aligned} \tag{5.102}$$

In coordinate representation is a function of Euler angles ,

$$\langle \alpha, \beta, \gamma = 0 | IM0 \rangle = \left(\frac{2I+1}{4\pi} \right)^{1/2} D_{M0}^I(\alpha, \beta, 0) = (-)^M Y_{IM}(\beta, \alpha) \tag{5.103}$$

Note that we have to consider it as a wave function of rotational state in all Euler angles (α, β) instead of thinking the Euler angle is fixed in space as physical rotation. (Just like the quantum rotational state $|Lm\rangle$ has wave function $Y_{LM}(\theta, \phi)$ in space. One should not consider classical rotation as a function of time.)

$$\langle I_b || \alpha_L || I_c \rangle = \beta_L \langle I_b || D_{00}^L || I_c \rangle = \beta_L (-)^{I_b} \langle I_b 0 | D_{00}^L | I_c 0 \rangle \left(\begin{matrix} I_b & L & I_c \\ 0 & 0 & 0 \end{matrix} \right)^{-1} \tag{5.104}$$

where D_{00} is chosen because reduced matrix element does not depends on projection.

$$\langle I_b 0 | D_{00}^L | I_c 0 \rangle = \sqrt{\frac{4\pi}{2L+1}} \int d\Omega Y_{I_b 0}(\beta, \alpha) Y_{L 0}(\beta, \alpha) Y_{I_c 0}(\beta, \alpha) \tag{5.105}$$

5.11.2 Vibrational excitation

Because the wave function of potential matrix element is space-fixed ones, one have to work in the space-fixed system. Parameter $\alpha_{\lambda\mu}$ are intrepreted as elongation operators which is self-adjoint.

$$\alpha_{\lambda\mu}^\dagger = (-)^\mu \alpha_{\lambda,-\mu}, \quad \pi_{\lambda\mu}^\dagger = (-)^\mu \pi_{\lambda,-\mu} \quad (5.106)$$

where π is a cannnonically conjugate momentum satisfying,

$$\begin{aligned} [\alpha_{\lambda\mu}, \pi_{\lambda'\mu'}] &= i\hbar \delta_{\lambda\lambda'} \delta_{\mu\mu'} \\ [\alpha_{\lambda\mu}, \alpha_{\lambda'\mu'}] &= [\pi_{\lambda\mu}, \pi_{\lambda'\mu'}] = 0. \end{aligned} \quad (5.107)$$

the vibrational Hamiltonian is (In other words α corresponds to displacement x .)

$$h = \sum_{\lambda\mu} \left(\frac{1}{2B_\lambda} \pi_{\lambda\mu}^\dagger \pi_{\lambda\mu} + \frac{C_\lambda}{2} \alpha_{\lambda\mu}^\dagger \alpha_{\lambda\mu} \right) \quad (5.108)$$

Re-organize Hamiltonian in terms of creation and annihilation operators,

$$\begin{aligned} c_{\lambda\mu} &= \left(\frac{B_\lambda \omega_\lambda}{2\hbar} \right)^{1/2} \alpha_{\lambda\mu} + i(-)^\mu \left(\frac{1}{2\hbar B_\lambda \omega_\lambda} \right)^{1/2} \pi_{\lambda,-\mu} \\ c_{\lambda\mu}^\dagger &= (-)^\mu \left(\frac{B_\lambda \omega_\lambda}{2\hbar} \right)^{1/2} \alpha_{\lambda,-\mu} - i \left(\frac{1}{2\hbar B_\lambda \omega_\lambda} \right)^{1/2} \pi_{\lambda,\mu}, \quad \omega_\lambda = \left(\frac{C_\lambda}{B_\lambda} \right)^{1/2} \end{aligned} \quad (5.109)$$

or conversely

$$\alpha_{\lambda\mu} = \left(\frac{\hbar}{2B_\lambda \omega_\lambda} \right)^{1/2} \left(c_{\lambda\mu} + (-1)^\mu c_{\lambda,-\mu}^\dagger \right) \quad (5.110)$$

with

$$\begin{aligned} [c_{\lambda\mu}, c_{\lambda'\mu'}^\dagger] &= \delta_{\lambda\lambda'} \delta_{\mu\mu'} \\ [c_{\lambda\mu}, c_{\lambda'\mu'}] &= [c_{\lambda\mu}^\dagger, c_{\lambda'\mu'}^\dagger] = 0 \end{aligned} \quad (5.111)$$

$$h = \sum_{\lambda\mu} \hbar \omega_\lambda \left(c_{\lambda\mu}^\dagger c_{\lambda\mu} + \frac{1}{2} \right). \quad (5.112)$$

Thus, vibrational state can be specified with $|n; IM_I\rangle$. One can have one-phonon state from mode $\lambda = I$,

$$|n = 1; IM_I\rangle = c_{IM_I}^\dagger |0\rangle \quad (5.113)$$

or two-phonon state from modes λ_1 and λ_2 ,

$$|n = 2; IM_I\rangle = \frac{1}{\sqrt{1 + \delta_{\lambda_1 \lambda_2}}} \sum_{\mu_1 \mu_2} \langle \lambda_1 \mu_1 \lambda_2 \mu_2 | IM_I \rangle c_{\lambda_1 \mu_1}^\dagger c_{\lambda_2 \mu_2}^\dagger |0\rangle \quad (5.114)$$

and so on. With this in mind, the coupling matrix element between two vibration states is

$$\langle n_b; I_b || \alpha_L || n_c; I_c \rangle = \left(\frac{\hbar}{2B_L \omega_L} \right)^{1/2} (-)^{I_b} \begin{pmatrix} I_b & L & I_c \\ 0 & 0 & 0 \end{pmatrix}^{-1} \langle n_b; I_b 0 | (c_{L0} + c_{L0}^\dagger) | n_c; I_c 0 \rangle.$$

for example, between zero-phonon and one-phonon state,

$$\langle 0; 0 | \alpha_L | 1; I \rangle = \left(\frac{\hbar}{2B_I \omega_I} \right)^{1/2} (-)^I (2I+1)^{1/2} \delta_{LI}. \quad (5.115)$$

One can replace $(\frac{\hbar}{2B_I \omega_I})$ in the Coulomb part of the interaction with $B(EI)$ value for the one-phonon transition.

$$B(E1; 1 \rightarrow 0) = \left(\frac{3}{4\pi} Z_t e R_C^I \right)^2 \left(\frac{\hbar}{2B_I \omega_I} \right) \quad (5.116)$$

then,

$$\langle 0; 0 | \alpha_L | 1; I \rangle = (-)^I (2I+1)^{1/2} \delta_{LI} \frac{4\pi}{3Z_t e R_c^I} [B(EI; 1 \rightarrow 0)]^{1/2} \quad (5.117)$$

5.11.3 Coulomb polarization potential

(REF: Book of Froblich et al)

Consider one-step Coulomb excitation from ground state to $|IM\rangle$ states, $M = -I, \dots, I$. (In other words $P = |00\rangle\langle 00|$ and $Q = \sum_m |Im\rangle\langle Im|$.) In this simple model space, one can compute the GOP as

$$\mathcal{V} = V_0 + \langle 0 | V | 0 \rangle + \langle 0 | V Q \frac{1}{E + i\eta - T - \hbar - V} Q V | 0 \rangle \simeq V_0 + V_{pol} \quad (5.118)$$

here $V_0 = V_c + V_N$ is the real static Coulomb+nuclear potential. (In other words, here we are computing equivalent local potential with full coupled channel calculation.) Diagonal $\langle 0 | V | 0 \rangle$ is small. Coulomb polarization potential is complex non-local energy-dependent. (Previous polarization potential was meant to be real potential comes from dispersion relation.)

With many simplification/approximation, (treat the solution as plane wave), trivially equivalent local potential is

$$V_{pol}(r) = -\frac{2I+1}{16\pi E} V_I(r) [V_I(r) + 2ikF_I(r)] \quad (5.119)$$

where

$$V_I(r) = \frac{4\pi Z_p e}{R_C^I (2I+1)^{3/2}} [B(EI; 0 \rightarrow I)]^{1/2} \frac{r_{<}^I}{r_{>}^{I+1}} \quad (5.120)$$

$$F_I(r) = 2\pi^2 \int_0^\infty q dq j_I(qr) \tilde{V}_I(q) \quad (5.121)$$

Leading term in the Coulomb polarization potential is the $I = 2$ contribution, and at large r ,

$$\text{Re} V_{pol}(r) = -\frac{\pi}{25} \frac{B(E2; 0 \rightarrow 2)}{E} \frac{Z_p^2 e^2}{r^6} \quad (5.122)$$

$$\begin{aligned} \text{Im} V_{pol}(r) &= -\frac{2}{3} \frac{\omega_{pol}}{R_C^9} r^4, \quad \text{for } r < R_C, \\ &= -(1 - \frac{2}{7} \frac{R_C^2}{r^2} - \frac{1}{21} \frac{R_C^4}{r^4}) \omega_{pol} \frac{1}{r^5}, \quad \text{for } r > R_C, \end{aligned} \quad (5.123)$$

with

$$\omega_{pol} = \frac{4\pi}{75} \frac{2\mu}{\hbar^2 k} Z_p^2 e^2 B(E2; 0 \rightarrow 2) \quad (5.124)$$

5.11.4 Nuclear Moments

Let us define more clearly on the nuclear moments. Keep in mind that there can be electric nuclear moments (which is related with charge/magnetic distribution) and matter nuclear moments which are in general different.

- Nuclear Mean Squared Charge Radius: second radial moment of nuclear charge

$$\langle r^2 \rangle = \frac{\int_0^R \rho(\mathbf{r}) \mathbf{r}^2 d\mathbf{r}}{\int_0^R \rho(\mathbf{r}) d\mathbf{r}} \quad (5.125)$$

If the shape is axially deformed, we have charge deformation parameter β_2 as

$$R(\theta) = R_1[1 + \beta_2 Y_{20}(\theta)], \quad (5.126)$$

Liquid drop model gives

$$\begin{aligned} \langle r^2 \rangle &= \frac{3}{5} R^2 + \frac{3}{4\pi} R^2 \beta_2^2 \\ &\rightarrow \langle r^2 \rangle_{sph} + \frac{5}{4\pi} \langle r^2 \rangle_{sph} \beta_2^2 \end{aligned} \quad (5.127)$$

- magnetic moment operator of nuclei: sum of free nucleon magnetic moments with gyromagnetic factors

$$\hat{\boldsymbol{\mu}} = \sum_{i=1}^A g_l^i \mathbf{l}^i + \sum_{i=1}^A g_s^i \mathbf{s}^i \quad (5.128)$$

Magnetic dipole moment of nuclei μ_I (which is dependent on the spin state of nucleus) is

$$\mu_I = \langle I, m = I | \hat{\mu}^z | I, m = I \rangle \quad (5.129)$$

and nuclear spin \mathbf{I} is related to the gyromagnetic ratio

$$\boldsymbol{\mu} = g_I \mathbf{I} \mu_N \quad (5.130)$$

- Electric Quadrupole moment: electric quadrupole moment operator can be defined as

$$\hat{Q}_2^0 = \hat{Q}_z = \sqrt{\frac{16\pi}{5}} \sum_{i=1}^A e_i r_i^2 Y_{20}(\hat{r}_i) \quad (5.131)$$

Spectroscopic quadrupole moment of nuclear spin I is

$$\begin{aligned} Q_s(I) &= \langle I, m = I | \hat{Q}_2^0 | I, m = I \rangle = \hat{I}^{-1} \langle II, 20 | II \rangle \langle I || \mathbf{Q} || I \rangle \\ &= \sqrt{\frac{I(2I-1)}{(2I+1)(2I+3)(I+1)}} \langle I || \mathbf{Q} || I \rangle \end{aligned} \quad (5.132)$$

(Note spectroscopic quadrupole moment is zero if $I < 1$ even with deformed nucleus with spin is 0 or 1/2.)

- Intrinsic quadrupole moment Q_0 is related to the intrinsic deformation β_2 . The Q_s and Q_0 can be related only in a certain assumptions. Axially symmetric nuclei case,

$$Q_s(I) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0 \quad (5.133)$$

with K being the projection of nuclear spin on the deformation axis. Intrinsic quadrupole moment in a small deformation β_2 is related as

$$Q_0 = \frac{3}{\sqrt{5}\pi} ZR^2 \beta_2 (1 + 0.36\beta_2) \quad (5.134)$$

If matter distribution is considered we may replace $Z \rightarrow A$. (RIPL used 0.16 instead of 0.36 ? and $R = 1.2A^{1/3}$ fm.)

5.11.5 Fresco: deformation

FRESCO에서 collective excitation의 효과를 넣는 방법이 여러가지가 있다. Coulomb deformation의 경우와 nuclear deformation의 경우를 따로 생각해야 한다.

- &POT 에서 deformation length를 input으로 주는 경우. (TYPE=10,11)
- &POT 에서 &STEP 을 이용하여, matrix element를 주는 경우.(TYPE=12,13). (단, 이때에는 $I_i \rightarrow I_f$ 뿐 아니라 $I_f \rightarrow I_i$ 도 explicit하게 주어야 한다.)
- &COUPLING 에서 &INEL 을 이용하여 coupling을 define하는 경우.

Information of deformation is described in Potentials (Type=10,11, Shape=7,8,9)of FRESCO input.

Nuclear deformation에 대해 $P(k) = DEF(k)$ corresponds to deformation length in fm (δ_λ).

Coulomb deformation에 대해서는 $P(k) = Mn(Ek)$ corresponds to the intrinsic reduced matrix elements in $e.f m^k$ units. 이 때, $Mn(Ek)$ 는

$$Mn(Ek) = \frac{\langle I' || Ek || I \rangle}{\sqrt{2I+1} \langle IK, k0 | I'K \rangle} \quad (5.135)$$

로 정의 되는데, 이 때 $\langle I' || Ek || I \rangle$ 는 Alder and Winther의 definition을 따른다.

$$Mn(Ek) = \frac{3Z\beta_k R^k}{4\pi} \quad (5.136)$$

이것은 앞에 사용된 식과 일치하는 것으로 보인다. (RIPL 에 의하면, E.M. deformation parameters are extracted from measured $B(E\lambda)$ as

$$B(E\lambda) = \left(\frac{3R_0^\lambda Z}{4\pi} \beta_\lambda \right)^2, \quad R_0 = 0.12A^{\frac{1}{3}} [\sqrt{b}]. \quad (5.137)$$

)

주의할 것은 Type에 따라, SHAPE에 따라 $P(k)$ 의 의미가 달라짐.

Fresco manual에 typo가 있는 것으로 생각된다. Coulomb deformation 의 경우, TYPE=10,11 일 때의 입력값 $P(k)$ 는 $M_n(Ek)$ 가 맞는 것으로 보인다. 실제 포텐셜 계산에선, 필요한 factor 를 자동으로 곱하여 $M_n(Ek) \rightarrow M(Ek)$ 로 바꾸어서 계산하는 것으로 여겨진다.

5.11.6 Inelastic scattering: single particle excitation

we may consider excitation of a nucleus by a transition of a single valence particle state from $\phi_{l_i s j_i}$ to $\phi_{l_f s j_f}$, or $(I_c j_i)I_i$ to $(I_c j_f)I_f$ with core spin I_c .

Let us consider three interaction $V_{cv}(\mathbf{r})$, $V_{vt}(\mathbf{r}_{vt})$, $V_{ct}(\mathbf{r}_{ct})$. 이 경우엔 projectile($p = c + v$) 의 excitation 을 생각한다. For given \mathbf{R} , $\mathbf{r}_{vt} = \mathbf{R} + \frac{m_c}{m_p} \mathbf{r}$ and $\mathbf{r}_{ct} = \mathbf{R} - \frac{m_v}{m_p} \mathbf{r}$. $m_p = m_c + m_v$.

$$V_{fi}(R) = \langle (L_f J_f) \Lambda | V_{vt} + V_{ct} | (L_i J_i) \Lambda \rangle, \quad (5.138)$$

$\Lambda = L + J$ is a conserved total angular momentum. by the expansion, $z = \hat{R} \cdot \hat{r}$

$$V_{vt} + V_{ct} = \sum_{\lambda} (2\lambda + 1) \mathcal{F}_{\lambda}(R, r) P_{\lambda}(z) = 4\pi \sum_{\lambda} \mathcal{F}_{\lambda}(R, r) \sum_m Y_{\lambda m}(\hat{r})^* Y_{\lambda m}(\hat{R}) \quad (5.139)$$

where,

$$\mathcal{F}(R, r) = \frac{1}{2} \int_{-1}^{+1} [V_{vt}(r_{vt}) + V_{ct}(r_{ct})] P_{\lambda}(z) dz \quad (5.140)$$

Thus,

$$V_{fi}^{\lambda}(R) = \left(\int_0^{\infty} u_{l_f s j_f}(r) \mathcal{F}_{\lambda}(R, r) u_{l_i s j_i}(r) dr \right) \hat{\lambda} \hat{l}_i \langle l_i 0, \lambda 0 | l_f 0 \rangle. \quad (5.141)$$

5.12 single particle transfer

Stripping example, $p(= n + c) + c' \rightarrow n + t'(= n + c')$,

$$\begin{aligned} [H_p - \epsilon_p] \phi_p(\mathbf{r}) &= 0, & H_p &= T_r + V_p(\mathbf{r}), \\ [H_t - \epsilon_t] \phi_t(\mathbf{r}') &= 0, & H_t &= T_{r'} + V_p(\mathbf{r}') \end{aligned} \quad (5.142)$$

Full Hamiltonian can be written in two ways,

$$\begin{aligned} H &= T_{\mathbf{r}} + T_{\mathbf{R}} + V_p(\mathbf{r}) + V_t(\mathbf{r}') + U_{cc'}(\mathbf{R}_c) \\ &= H_{prior} = T_{\mathbf{R}} + U_i(R) + H_p(\mathbf{r}) + \mathcal{V}_i(\mathbf{R}, \mathbf{r}) \\ &= H_{post} = T_{\mathbf{R}'} + U_f(R') + H_t(\mathbf{r}') + \mathcal{V}_f(\mathbf{R}', \mathbf{r}') \end{aligned} \quad (5.143)$$

where

$$\begin{aligned} \mathcal{V}_i(\mathbf{R}, \mathbf{r}) &= V_t(\mathbf{r}') + U_{cc'}(\mathbf{R}_c) - U_i(R) \\ \mathcal{V}_f(\mathbf{R}', \mathbf{r}') &= V_p(\mathbf{r}) + U_{cc'}(\mathbf{R}_c) - U_f(R') \end{aligned} \quad (5.144)$$

Consider Model space, (only two channel)

$$\Psi = \left[\phi_p(\mathbf{r}) \otimes Y_L(\hat{R}) \right]_{\Lambda} \chi_i(R)/R + \left[\phi_t(\mathbf{r}') \otimes Y_{L'}(\hat{R}') \right]_{\Lambda} \chi_f(R)/R \quad (5.145)$$

We need to compute matrix element,

$$\langle \left[\phi_t(\mathbf{r}') \otimes Y_{L'}(\hat{R}') \right]_{\Lambda} | \mathcal{V}_x | \left[\phi_p(\mathbf{r}) \otimes Y_L(\hat{R}) \right]_{\Lambda} \rangle (R, R') \quad (5.146)$$

Details on the calculation

In general the matrix element becomes a non-local operator. We can simplify the integral by change of variable and angle integrations.

$$\mathbf{r} = p\mathbf{R}' + q\mathbf{R}, \quad \mathbf{r}' = p'\mathbf{R}' + q'\mathbf{R} \quad (5.147)$$

$$\begin{aligned} p &= -\omega, & q &= \nu_t \omega, & p' &= -\nu_p \omega, & q' &= \omega, \\ \nu_p &= m_c/m_p, & \nu_t &= m_{c'}/m_{t'}, & \omega &= (1 - \nu_p \nu_t)^{-1} \end{aligned} \quad (5.148)$$

Convert functions $\phi_t(\mathbf{r}')$ and $\phi_p(\mathbf{r})$ into a function of \mathbf{R} and \mathbf{R}' ,

Moshinsky solid-harmonic expansion: When $\mathbf{r} = a\mathbf{R} + b\mathbf{R}'$, we can expand the spherical harmonics as

$$\begin{aligned} Y_{lm}(\hat{r}) &= Y_{lm}(\widehat{a\mathbf{R} + b\mathbf{R}'}) \\ &= \sqrt{4\pi} \sum_{n\lambda} c(l, n) \frac{(aR)^{l-n} (bR')^n}{r^l} Y_{l-n, m-\lambda}(\hat{R}) Y_{n, \lambda}(\hat{R}') C_{l-n, m-\lambda, n, \lambda}^{lm}, \\ c(l, n) &= \sqrt{\frac{1}{2n+1} \binom{2l+1}{2n}} \end{aligned} \quad (5.149)$$

$$\begin{aligned} r^2 &= p^2 R'^2 + q^2 R^2 + 2pqRR'z, & z &= \hat{R} \cdot \hat{R}' \\ r'^2 &= p'^2 R'^2 + q'^2 R^2 + 2p'q'RR'z \end{aligned} \quad (5.150)$$

zero-range approximation

$$V_{l'L';0L}^L(R', R) = D_0 \frac{(-1)^{L'-l'}}{\hat{L}} \frac{\hat{l}' \hat{L} \hat{L}'}{\sqrt{4\pi}} \begin{pmatrix} l' & L & L' \\ 0 & 0 & 0 \end{pmatrix} \frac{u_{l'}(R')}{R'} \frac{q^2}{p} \delta(pR' + qR) \quad (5.151)$$

overlap

Here, we ignored the unchanged core-internal wave function. In fact, we need to consider overlap wave function which should be intrinsically many-body,

$$\Psi_{A;B}(r) = \langle \psi_A(r, \xi_B) | \psi_B(\xi_B) \rangle \quad (5.152)$$

If target or projectile is a bound state of core(spin I_c)+valence(j_v) with total JM , the bound state wave function in coupling order $|(ls)j, I_c; JM\rangle$, can be written as

$$\phi_{JM}(\xi) = \phi_{JM}(\xi_c, \mathbf{r}_v) = \sum_{ljI} A_{lsj_v}^{j_v I_c J} [\phi_{I_c}(\xi_c) \otimes \phi_{l_v s_v j_v}(\mathbf{r}_v)]_{JM} \quad (5.153)$$

where, bound state wave function can be a linear combination of several normalized single particle wave functions as

$$\phi_{l_v s_v j_v}(\mathbf{r}_v) = [Y_{l(\hat{r}_v)} \otimes \phi_s]_{j_v} \frac{u_{lsjI}(\mathbf{r}_v)}{r_v}. \quad (5.154)$$

And the $A_{lsj_v}^{j_v I_c J}$ is called coefficient of fractional parentage(=spectroscopic amplitude) for single particle state in bound state with JM . $|A_{lsj_v}^{j_v I_c J}|^2$ are called spectroscopic factor. Note

that u are normalized wave functions. If there is only one single particle states for given excited state, c.f.p= 1 for pure single particle state. In fact, we have to anti-symmetrize the total wave function. Thus, in that case spectroscopic amplitude have to contain factors related with anti-symmetrization.

한편, seperation energy 에 대한 정보로 부터, 우리는 asymptotic form of bound state wave function 이 Whittaker function에 비례할 것으로 볼 수 있다. 이 때의 비례 상수를 Asymptotic normalization constant 라고 부른다.

$$\phi_l(r) =_{r>R_n} C_l W_{-\eta, l+1/2}(-2k_p r) \simeq C_l e^{-k_p r} \quad (5.155)$$

만약, $\phi_l(r) \sim A u_l(r)$ 에서 spectroscopic amplitude A 와 normalized wave function 으로 쓴다면, normalized wave function 자체가 single particle ANC를 가지고, $(ANC) = (s.a.)(s.p.ANC)$ 로 생각할 수 있다.

The single particle wave functions can be determined by binding potential between core and valence. Note that in general, there will be non-zero off diagonal coupling potentials which connects between excited states. Thus, it is enough to specify binding potential and single particle state quantum number and c.f.p to obtain overlap function K_{ij} .

Transfer reaction 경우, $A(a, b)B$ with $a = b + x$, $B = A + x$, coupling potential을 생각해보자. DWBA formalism 에서 계산해야하는 coupling은 $\langle bB | \Delta V | aA \rangle$ 로 쓸 수 있고, ΔV 는 interaction 중 diagonal part가 빠진 potential이다. 즉, $\langle aA | \Delta V | aA \rangle = 0$ 또는 $\langle bB | \Delta V | bB \rangle = 0$. Cluster model에서의 모든 interaction은 아래와 같이 주어진다.

$$H_I = V_{xb} + V_{bA} + V_{xA} \quad (5.156)$$

이 때, entrance channel 을 기준으로 생각하면, $a = b + x$ 이므로, V_{xb} 는 bound state 의 internal 에 이미 포함되어 있다. 또한, aA 의 elastic scattering 은 optical potential U_{aA} 로 기술되므로, 이부분을 빼주어야한다. 따라서, entrance channel을 기준으로 할 때, (Prior Form)

$$\Delta V^{prior} = V_{xA} + [V_{bA} - U_{aA}] \quad (5.157)$$

라고 쓸 수 있는데, 마지막의 core-core interaction 과 optical potential의 차이를 Remnant potential 이라고 부른다. 만약 x 가 core b 나 A 에 비해 작은 경우라면, $V_{bA} - U_{aA} \simeq 0$ 로 approximation할 수 있고, 이 경우 transition coupling은 exit channel에서의 binding potential V_{xA} 가 된다.

마찬가지로, exit channel을 기준으로 하면, (Post form), V_{xA} 는 bound state B 에 이미 포함되므로,

$$\Delta V^{post} = V_{xb} + [V_{bA} - U_{bB}] \quad (5.158)$$

라고 쓸 수 있고, 이 경우 remnant potential을 무시하면, entrance channel에서의 binding potential V_{xa} 가 transition potential이 된다.¹⁰

In general, the coupling becomes non-local in R_i and R_f and requires finite range of non-locality. In zero-range approximation, $\mathcal{V}_o \phi(\mathbf{r}) \sim D_0 \delta(\mathbf{r})$, the calculation becomes simpler.¹¹

¹⁰**Q:** Suppose the inputs does not have optical potential between U_{t-p} , but only have V_{c-p} and V_{v-p} . In this case, FRESKO automatically computes U_{t-p} from V_{c-p} and V_{v-p} and subtract U_{t-p} for coupling potential? Or, only considers Coulomb interaction between target and projectile in this channel?

주의: FRESKO 의 경우, 책이나 매뉴얼의 식은 모두, s.p. excitation이나 transfer는 scalar potential에 의한 경우만 계산하는 것으로 보인다.

¹¹zero-range approximation assumes (1)Wave function is S-wave only (2) remnant term은 무시가능 (3) potential이 zero-range. 따라서, $\int d^3r V(\mathbf{r})\phi(\mathbf{r}) = \int d^3r D_0 \delta^3(\mathbf{r})$ 로 생각하여, D_0 를 얻을 수 있다.

Local energy approximation은 zero-range approximation에 finite effective radius에 의한 correction을 하는 것.¹² zero-range approximation parameter D_0 can be written as¹³

$$\begin{aligned} D_0 &= \sqrt{4\pi} \int_0^\infty r V_0(r) u_0(r) dr, \\ u_0(r) &\rightarrow \frac{2\mu_p}{\hbar^2} \frac{1}{\sqrt{4\pi}} D e^{-k_p r} \end{aligned} \quad (5.159)$$

Also, in low energy approximation,

$$D = \sqrt{4\pi} \int_0^\infty \frac{\sinh(k_p r)}{k_p} V_0(r) u_0(r) dr \quad (5.160)$$

Gives $D = (1 + k_p^2 \rho_{eff}^2) D_0$, $k_p^2 = 2\mu_p \epsilon_p / \hbar^2$, ρ_{eff} is a measure of mean radius of potential $V_0(r)$.

Asymptotic Normalization Coefficient is defined as the asymptotic coefficient C_l of the Whittaker function.

$$u_l(r) \rightarrow C_l W_{-\eta, l+\frac{1}{2}}(-2k_p r) \simeq C_l e^{-k_p r} \quad (5.161)$$

Thus, in case of S-wave zero-range approximation, $C_l = \frac{2\mu_p}{\hbar^2} \frac{1}{\sqrt{4\pi}} D$. ANC is related with the spectroscopic factor.

5.12.1 Cluster Wave function: two particle

Or, if it is a bound state of core and two particles, for coupling order $|\{L, (l, (s_1 s_2) S) j\} J_{12}, I_c; J\rangle$,

$$\begin{aligned} \phi_{JM}(\xi) &= \phi_{JM}(\xi_c, \mathbf{r}, \rho) \\ &= \sum_{LlSjJ_{12}I_c} A_{LjJ_{12}}^{J_{12}I_cJ} [\phi_{I_c}(\xi_c) \otimes [Y_L(r) \otimes [Y_l(\rho) \otimes [\phi_{s_1} \otimes \phi_{s_2}]_S]_j]_{J_{12}}]_{JM} \frac{u_{12}(r, \rho)}{r\rho} \end{aligned} \quad (5.162)$$

where coefficients A 's are called coefficients of fractional parentage. If the wave function of two particles are given in terms of single particle wave functions in each position, $\phi_{l_1, s_1, j_1}(r_1)$ and $\phi_{l_2, s_2, j_2}(r_2)$, they have to be converted by

$$\begin{aligned} \phi_{12}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_i c_i |(l_1(i), s_1) j_1(i), (l_2(i), s_2) j_2(i); J_{12} T)\rangle \\ &\rightarrow \sum_u c_i \sum_{LlSj} |L, (l, (s_1 s_2) S) j; J_{12}, T\rangle \phi_{L(lS)j}^{J_{12}T,i}(r, \rho) \end{aligned} \quad (5.163)$$

with

$$\begin{aligned} \phi_{L(lS)j}^{J_{12}T,i}(r, \rho) &= \langle L, (l, (s_1 s_2) S) j; J_{12} T | (l_1, s_1) j_1, (l_2, s_2) j_2; J_{12}, T \rangle \\ &\quad \times \langle [Y_L(\hat{r}) Y_l(\hat{\rho})]_\lambda | [\phi_{l_1 s_1 j_1}(\mathbf{r}_1) \phi_{l_2 s_2 j_2}(\mathbf{r}_2)]_{J_{12}, T} \rangle \end{aligned} \quad (5.164)$$

¹²Local energy approximation assumes (1) wave function은 s-wave only (2) potential is not zero-range but have small effective range.

¹³Considering s-wave bound state solution of delta function potential, which have the form $e^{-k_p r}$, we can interpret the D_0 as the asymptotic strength of wave function. From this, when the potential has some small range, it would corresponds to replace D_0 into modified asymptotic strength D of wave function.

5.12.2 Knockout reactions

For example, $^{14}\text{C}(p, \alpha)^{11}\text{B}$, 여기서 target의 일부가 떨어져 나가고 projectile로 대체되는 경우. 이때 가능한 mechanism은 두가지를 생각할 수 있다. 첫번째는 triton transfer

$$(^{14}\text{C} = ^{11}\text{B} + ^3\text{H}) + p \rightarrow ^{11}\text{B} + (\alpha = p + ^3\text{H}). \quad (5.165)$$

두번째는 일종의 $p - \alpha$ exchange,

$$(^{14}\text{C} = ^{10}\text{Be} + \alpha) + p \rightarrow (^{11}\text{B} = p + ^{10}\text{Be}) + \alpha \quad (5.166)$$

두번째 경우에는 exchange 효과를 고려하여 amplitude를 합해야한다. 단, transition potential, $\mathcal{V}_i = V_{p-\text{Be}} + U_{p\alpha} - U_{p\text{C}}$, 에서 보통은 첫번째 binding potential 이 중요하여 remnant interaction이 무시됨에 비해서, 이 경우는 $p - \alpha$ interaction 이 중요해진다.

5.12.3 Continuum: Break up

If the particle transfers to continuum, and the coupling with continuum states are important, it have to be specified by 'bin' wave functions in CDCC method in FRESKO. (Coupling between bound state and continuum 뿐 아니라 coupling between continuum 까지 고려하는 경우)

5.12.4 Capture

penetrability factor. direct capture by emission of photon. Or doorway resonances. compound nucleus by long resonance.

5.12.5 Charge exchange reactions

Charge exchange 의 경우 다양한 reaction mechanism 이 가능하다. 예를 들어 $^{12}\text{C}(p, n)^{12}\text{N}$ 의 경우

- heavy-particle transfer of ^{11}C from, $^{12}\text{C} = ^{11}\text{C} + n$ to $^{12}\text{N} = ^{11}\text{C} + p$
- two-step transfers via $^{11}\text{C} + d$ intermediate state. $^{12}\text{C} + p \rightarrow ^{11}\text{C} + d \rightarrow ^{12}\text{N} + n$
- direct conversion of a proton to neutron via isospin dependent interaction(like pion exchange)

마지막의 경우는 Lane potential

$$V(R) = V_0(R) + \frac{\mathbf{t} \cdot \mathbf{T}}{A} V_T(R) \quad (5.167)$$

로 표현된다.

Fermi transition : $H_F = V_F(R)\mathbf{t} \cdot \mathbf{T}$

Gamow-Teller transition: $H_{GT} = V_{GT}(R)(\mathbf{s} \cdot \mathbf{S})(\mathbf{t} \cdot \mathbf{T})$

Generalized multipole transitions

5.12.6 Photo-nuclear couplings

5.13 Structure Models

5.13.1 Shell Model

5.13.2 Cluster Models

5.13.3 Mean Field Models

Hartree theory

$$\begin{aligned}
[\hat{T}_{\mathbf{r}} + V_{mean}(\mathbf{r})]\phi_i(\mathbf{r}) &= \epsilon_i \phi_i(\mathbf{r}), \\
\rho(\mathbf{r}) &= \sum_{i=1}^F n_{occ}(i) |\phi_i(\mathbf{r})|^2, \quad \sum_i n_{occ}(i) = A, \\
V_{mean}(\mathbf{r}) &= \int d\mathbf{r}' V_{NN}(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')
\end{aligned} \tag{5.168}$$

Hartree-Fock theory use slater determinant of wave function and exchange term in the equation.

The density-dependent effective interaction is often used, $V_{NN}(\mathbf{r}; \rho)$.

Mean-field models are simpler for even-even nuclei, as then nucleons are in pairs of time-reversed orbits.(?)

Mean-field models may be used to predict bound-state wave functions for transfer or capture scattering calculations according to the orbital occupation numbers.

Mean-field are for ground state. For excited states, time-dependent analysis (Time-dependent Hartree-Fock) or linear perturbation analysis(Random Phase Approximation) is used.

5.13.4 Collective nuclear-matter description

Use plausible form of $\rho(\mathbf{r})$.

Laboratory density depends on the angular momentum $I\mu$.

Body-fixed densities are intrinsic and only depends on the K , which is the projection of intrinsic spin on the body-fixed z' axis. (Nuclear spin in Laboratory frame is a vector sum of intrinsic spin and rotation of the body-fixed axis($\mathbf{I} = \mathbf{s} + \mathbf{L}_r$)).

Mean-field, density-functional, collective rotation model gives nuclear wave functions initially in body-fixed frame. Thus, if we want density in lab frame, we need to rotate the wave function.

Shell model wave usually projected onto good angular-momentum states in laboratory frame.

We can think of 'matter' deformation in a similar way as 'potential' deformation.

$$\begin{aligned}
\rho(r, \theta', \phi') &= \rho_0(r - \tilde{R}(\theta', \phi') + R_0), \\
\tilde{R}(\theta', \phi') &= R_0 + \sum_{\lambda=0}^{\lambda_{max}} \sum_{\mu=-\lambda}^{\lambda} \delta_{\lambda\mu} Y_{\lambda\mu}(\theta', \phi')
\end{aligned} \tag{5.169}$$

$\delta_{\lambda\mu}$ 'matter deformation length', fractional matter deformation($\beta_{\lambda\mu} = \frac{\delta_{\lambda\mu}}{R_0}$) can be defined.

'intrinsic quadropole moment' for diagonal transitions ($I_i = I_f = I$) and $\lambda = 2$,

$$\begin{aligned} Q &= \sqrt{\frac{16\pi}{5}} \langle \Phi_{JJ} | \hat{Q}_{20} | \Phi_{JJ} \rangle \\ &= \sqrt{\frac{16\pi}{5}} \langle \Phi_{JJ} | \int d\mathbf{r}' r'^2 \hat{\rho}(\mathbf{r}') Y_{20}(\hat{r}') | \Phi_{JJ} \rangle \end{aligned} \quad (5.170)$$

$$Q_2 = \sqrt{\frac{16\pi}{5}} \frac{\langle II20 | II \rangle}{\sqrt{2I+1}} \langle I || M(E2) || I \rangle \quad (5.171)$$

여기서, $|\Phi_{JM}\rangle$ 은 deformed nucleus 의 lab frame에서의 wave function 인 반면 $\rho(\mathbf{r}')$ 은 body-fixed frame에서의 matter density 이다.

$$Q = \frac{3R_0 A}{\sqrt{\pi}} \delta_2 \quad (5.172)$$

example: If nuclus is a spherical core + unpaired single nucleon, ϕ_{lsj} , the quadropole moment is determined by the single particle. Then,

$$Q_{sp} = -\frac{2j-1}{2(j+1)} \langle r^2 \rangle, \quad \langle r^2 \rangle = \int r^2 |\phi_{lsj}(\mathbf{r})|^2 d\mathbf{r}. \quad (5.173)$$

5.14 Overlap functions

one-body overlap function for transfer, captures or single particle excitation involving nucleus B which is core A+ single particle, which is related with spectroscopic amplitude(or c.f.p.). 중요한것은 overlap function이 일반적으로 간단한 single particle wave function이 아닌, 복잡한 many-body wave function의 일부라는 것이다. 일반적인 shell model에서도 configuration mixing을 고려할 경우, overlap function은 단순한 single particle wave function으로 기술되지 않는다.

만약 antisymmetrization을 고려하지 않으면, A nuclei와 A-1 nuclei의 overlap은

$$\Psi_i^A = \sum_{flj} \phi_{i,f,l,j}(\mathbf{r}) \Psi_f^{A-1}, \quad \langle \Psi_f^{A-1} | \Psi_i^A \rangle(\mathbf{r}) = \sum_{l,j} \phi_{i,f,l,j}(\mathbf{r}). \quad (5.174)$$

와 같이 생각할 수 있다. 여기서, i와 f는 A와 A-1 nucleus 의 모든 양자수를 나타낸다. 즉, overlap function은 가능한 모든 configuration mixing 을 모두 고려하여야한다. 하지만, 많은 경우 overlap function이 오직 하나의 single particle wave로 근사 가능하다는 가정을 한다.

5.14.1 Non-antisymmetrized theory: Pauli principle이 무시되는 경우

Suppose $B = A + v$, core wave $\Phi_{I_A}^A(\xi_A)$ and composite wave function, $\Phi_{I_B}^B(\xi_B = \{\xi_A, \mathbf{r}\})$ (여기서, \mathbf{r} 은 single particle과 core 사이의 위치벡터), Overlap function,

$$\begin{aligned} \phi_{I_A:I_B}(\mathbf{r}) &= \langle \Phi_{I_A}^A(\xi_A) | \Phi_{I_B}^B(\xi_A, \mathbf{r}) \rangle, \\ \Phi_{I_B}^B(\xi_A, \mathbf{r}) &= \sum_{I_A} \phi_{I_A:I_B,nl,j}(\mathbf{r}) \Phi_{I_A}^A(\xi_A). \end{aligned} \quad (5.175)$$

여기서, I_A 와 I_B 는 처음과 나중의 모든 물리적인 값을 포함하는 것이고, $\phi_{I_A:I_B}$ 는 일반적으로 I_A 와 I_B 외에도 valence 입자의 n, l, j 에도 의존한다. $\phi_{I_A:I_B}(\mathbf{r})$ 은 복잡한 manybody

problem을 풀어야 얻을 수 있고, $\phi_{I_A:I_B}(\mathbf{r})$ 각각은 normalize가 되지 않았으며, 오직 전체 합만 normalize되어 있다.(또한 core 와의 anti-symmetrization이 고려되지 않았다.) 또한, 아직 overlap function을 partial wave decompose하지도 않았음.

$$\sum_{I_A} \int d\mathbf{r} |\phi_{I_A:I_B}(\mathbf{r})|^2 = 1 \quad (5.176)$$

그리고, 이때 overlap function의 square norm을 spectroscopic factor라고 부른다. (여기서, sum rule 값이 1인 것은 anti-symmetrization이 고려되지 않은 wave function normalization 때문.)

$$S^{I_A:I_B} = \int d^3r |\phi_{I_A:I_B}(\mathbf{r})|^2 \quad (5.177)$$

위 정의에서, overlap function은 nuclear interior에서만 중요하므로 spectroscopic factor는 사실상 nuclear interior 에 대한 정보를 나타낸다. (따라서, spectroscopic factor는 high energy 에서 더 정확히 얻을 수 있다?)

partial wave expand를 하면,

$$\Phi_{I_B}^B(\xi_B, \mathbf{r}) = \sum_{I_A} \sum_{lsj} \frac{u_{lsj}^{jI_A I_B}(r)}{r} \left[[Y_l(\hat{r}) \otimes \chi_s]_j \otimes \Phi_{I_A}^A(\xi_A) \right]_{I_B} \quad (5.178)$$

따라서, normalization condition gives

$$\sum_{I_A} \sum_{lsj} \int_0^\infty dr |u_{lsj}^{jI_A I_B}(r)|^2 = 1. \quad (5.179)$$

여기서, 각 $u_{lsj}^{jI_A I_B}(r)$ 은 unit normalization이 되어 있지 않음에 유의. 만약, radial function을 unit normalize 시킨다면, (v 는 unit normalized s.p. wave function)

$$u_{lsj}^{jI_A I_B}(r) = A_{lsj}^{iI_A I_B} v_{lsj}^{jI_A I_B}(r), \quad ||v_{lsj}^{jI_A I_B}|| = 1. \quad (5.180)$$

spectroscopic amplitude $A_{lsj}^{iI_A I_B}$ 와 spectroscopic factor는

$$S_{lsj}^{iI_A I_B} = |A_{lsj}^{iI_A I_B}|^2 \quad (5.181)$$

인 관계가 있게 된다. 따라서, sum rule도

$$\sum_{lsj I_A} |A_{lsj}^{iI_A I_B}|^2 = \sum_{lsj I_A} S_{lsj}^{iI_A I_B} = 1. \quad (5.182)$$

이 된다. (단, 이것은 anti-symmetrization을 생각하지 않은 것이다.)

한편, Asymptotic normalization coefficient는 overlap function의 radial wave function의 tail로 정의된다. Binding energy($k_B = \sqrt{2\mu B}$)를 아는 경우, overlap function의 tail은 Whitaker function이 될 것으로 예상할 수 있으므로,

$$\phi_{lsj}^{I_A:I_B}(r) \rightarrow C_{lsj}^{I_A:I_B} W_{-\eta, l+\frac{1}{2}}(-2k_B r) \rightarrow C_{lsj}^{I_A:I_B} e^{-k_B r}, \quad (5.183)$$

In many cases, the complex many-body overlap function is approximated as a single-particle overlap functions

$$\phi_{lsj}^{I_A;I_B}(r) \simeq \phi_{lsj}^{(sp)I_A;I_B}(r) = \left[S_{lsj}^{(sp)I_A;I_B} \right]^{\frac{1}{2}} \varphi_{(lsj)I_B}(r) \quad (5.184)$$

where $\varphi_{(lsj)I_B}(r)$ is normalized to one. Then, the asymptotic form of bound wave is

$$\varphi_{(lsj)I_B}(r) \rightarrow b_{lsj}^{I_B} W_{-\eta, l+\frac{1}{2}}(-2k_B r) \quad (5.185)$$

where, $b_{lsj}^{I_B}$ is single-particle ANC(즉, unit-normalized wave function 과 Whittaker function 의 ratio). Thus,

$$C_{lsj}^{I_A;I_B} = \left[S_{lsj}^{(sp)I_A;I_B} \right]^{\frac{1}{2}} b_{lsj}^{I_B} \rightarrow S_{lsj}^{(sp)I_A;I_B} = \left(\frac{C_{lsj}^{I_A;I_B}}{b_{lsj}^{I_B}} \right)^2. \quad (5.186)$$

Because ANC is a peripheral quantity, probing the region outside the nuclear radius r_N , it is best extracted using reactions at very low energies. This quantity should not depend on the choice of the neutron-target binding potential. The spectroscopic factor on contrary contains the nuclear interior information, therefore is commonly determined from high-energy reactions. The spectroscopic factor is sensitive to the assumptions made on the geometry of the neutron-target binding interaction.

Distinction between spectroscopic factor and ANC? Spectroscopic factor is norm integral of overlap function while ANC is a asymptotic tail of overlap function. Thus, they are not the same. The Coulomb breakup or peripheral reaction should be only sensitive to ANC not spectroscopic factor. When ANC is important, the procedure extracting spectroscopic factor from experiment may be questionable.

보통 Shell model 코드에서 spectroscopic factor 또는 spectroscopic amplitude를 주지만, spectroscopic amplitude의 sign에는 주의해야한다.

- (a) overall phase of $\Phi_{I_A}^A$ and $\Phi_{I_B}^B$ 는 어떻게 정해져 있는가?
- (b) 어떤 coupling scheme을 사용하고 있는가?
- (c) sign of normalized wave function은 어떻게 정해지는가?

FRESCO의 경우는 single particle wave function이 origin 부근에서 positive 이도록 정한다. Shell model wave function과 다른 경우 sign을 고쳐주어야 한다.

실제 실험과 관련된 spectroscopic factor는 isospin T_z 에 depend(즉 proton과 neutron을 구분) 하지만 이론적인 계산에서는 isospin symmetric인 경우로 계산하는 것이 편리하기도 하다. 따라서, 그런 경우는

$$S_{lsj}^{jI_A I_B}(t_z) = C^2 \tilde{S}_{lsj}^{jI_A I_B}(T), \quad C = \langle T_A T_{zA}, t t_z | T_B T_{zB} \rangle \quad (5.187)$$

와 같이 T_z 에 independent하고 T 에만 depend하는 $\tilde{S}_{lsj}^{jI_A I_B}(T)$ 를 이용하기도 한다. 여기서, 주의할 것은 creation operator로 생각할 경우, $S(t_z)$ 는 space에 대해서만 reduced matrix element 인데 비해서 $S(t_z) \sim \langle f, T' T_z' || a_k^\dagger || i, T T_z \rangle$ $S(T)$ 는 isospin에 대해서도 reduced matrix element 라는 것이다. (이를 나타내기 위해서, $S(T) \sim \langle f, T' || a_k^\dagger || i, T \rangle$ 로 나타내기도 한다.)

5.14.2 Antisymmetrized theory

앞에서는 transfered particle v 가 core A 안의 nucleon들과 구분이 된다고 가정하였다. 만약 identical nucleon이 여럿 있는 경우, anti-symmetrization을 고려해주어야 한다.

CFP

하나의 orbit에 여러개의 입자가 있는 경우를 생각하면, creation operator를 이용하여,

$$|k^n \omega J M\rangle = (-1)^n \sum_{\omega' J'} \frac{\langle k^n \omega J || a_k^\dagger || k^{n-1} \omega' J' \rangle}{n \sqrt{(2J+1)}} \left[Z^\dagger(k^{n-1} \omega' J) \otimes a_k^\dagger \right]_M^J | \rangle \quad (5.188)$$

로써, n-particle state를 n-1 particle state들과 creation operator를 이용하여 나타낼 수 있다. 이때, 계수의 합이 1이 되도록 normalize한 계수를 coefficient of fractional parentage라고 한다.

$$\langle j^n \omega J || j^{n-1} \omega' J' \rangle \equiv \frac{\langle k^n \omega J || a_k^\dagger || k^{n-1} \omega' J' \rangle}{\sqrt{(2J+1)}}, \quad \sum_{\omega' J'} |\langle j^n \omega J || j^{n-1} \omega' J' \rangle|^2 = 1 \quad (5.189)$$

여기서, $(cfp)^2$ 는 $|j^n \omega J\rangle$ 에서 nucleon하나를 제거하여 $|j^{n-1} \omega' J'\rangle$ 이 될 확률로 생각할 수 있다.

Spectroscopic factor

일반적인 many-body wave function을 생각하여 spectroscopic factor를 다음과 같이 정의 할 수도 있다.

$$S_{k, \omega, \omega'}^{J_A, J_{A-1}} \equiv \frac{|\langle \Psi^A \omega J || a_k^\dagger || \Psi^{A-1} \omega' J' \rangle|^2}{(2J+1)} = \frac{|\langle \Psi^{A-1} \omega' J' || \tilde{a}_k || \Psi^A \omega J \rangle|^2}{(2J+1)} \quad (5.190)$$

(여기서, reduced matrix element는 radial integral 을 포함한다.) 만약, overlap wave function을 순수한 single particle in j-shell 로 생각한다면,

$$S_{j, \omega, \omega'}^{J_A, J_{A-1}} = \frac{|\langle j^n \omega J || a_j^\dagger || j^{n-1} \omega' J' \rangle|^2}{(2J+1)} = n |\langle j^n \omega J || j^{n-1} \omega' J' \rangle|^2 \quad (5.191)$$

으로 얻을 수 있다. (따라서, $cfp = 1$ 인 경우, 즉 오직 하나의 j-shell만 available한 경우, $S = n$ 이 된다.) 그러나, 일반적으로 configuration mixing이 일어나기 때문에 위 식과 같이 써지지 않는다. 한가지 주의할 것은 위 식에서 J 는 더 무거운 핵을 기준으로 한다는 것이다. 때문에, nucleon removal의 경우,

$$\sigma_-(A \rightarrow A-1) = S \sigma_{sp}. \quad (5.192)$$

반면, nucleon capture의 경우,

$$\sigma_+(A \rightarrow A+1) = \frac{(2J_f + 1)}{(2J_i + 1)} S \sigma_{sp}, \quad (5.193)$$

because S value is defined for $A+1 \rightarrow A$ and have $2J_f + 1$ in denominator.

Spectroscopic factor의 sum-rule은 nucleon-removal의 경우

$$\sum_{f, -} S_{i, f, k} = \langle \hat{n}_k \rangle_i, \quad [A \rightarrow (A-1)]. \quad (5.194)$$

여기서, $S_{i, f, k}$ 는 A-nuclei i 로부터 A-1 nuclei f 로 k-state를 remove하는 경우의 spectroscopic factor로, configuration mixing이 있는 경우, 오른쪽항은 처음 A-nuclei 의 k-state의 occupation number 에 해당한다. 여기서, $f-$ 에 대한 summation이 있음에 유의.¹⁴

¹⁴가장 단순한 경우로 i 가 $J_A = j$ 로써, closed shell $J_B = 0$ 인 경우에는, $f, -$ 가 하나 밖에 없다. 또한 이 경우 spectroscopic factor는 $S = 1$ 이다. 또한, $\langle n_k \rangle_i = 1$ 이다. 따라서, sum rule이 만족된다.

비슷하게 nucleon-addition 의 경우,

$$\sum_{f+} S_{i,f,k} \frac{(2J_f + 1)}{(2J_i + 1)} = (2j + 1) - \langle n_k \rangle_i, \quad [A \rightarrow (A + 1)], \quad (5.195)$$

이 된다.¹⁵ (여기서, $S_{i,f,k} \frac{(2J_f+1)}{(2J_i+1)}$ 를 하나의 spectroscopic factor로 생각할 수도 있지 않을까? 또한, 이 경우 shell-model로 정의된 S와 실험에서 결정되는 S의 관계는?)

(?) Cluster의 transfer의 경우로 확장하여 생각하면, 만약, $a + A \rightarrow b + B$ 반응이 cluster v 의 transfer에 의해서 일어난다고 하면, n_A^B 를 the number of nucleons in B that are in the same orbital and identical to that of valence nucleons v transferred.

$$n_A^B = \binom{N_B}{v} = \frac{N_B!}{N_A!v!} \quad (5.196)$$

여기서, N_B 는 number of such valence nucleons in nucleus B, $N_A = N_B - v$ 는 such identical nucleons remaining in core nucleus A.¹⁶

$$\begin{aligned} \Phi_{I_B}^B(\xi_A, \mathbf{r}) &= \frac{1}{\sqrt{n_A^B}} \sum_{I_A, l_{sj}} A_{lsj}^{jI_A I_B} \frac{v_{lsj}^{jI_A I_B}(r)}{r} \left[[Y_l(\hat{r}) \otimes \chi_s]_j \otimes \Phi_{I_A}^A(\xi_A) \right]_{I_B}, \\ \sum_{l_{sj} I_A} |A_{lsj}^{jI_A I_B}|^2 &= \sum_{l_{sj} I_A} S_{lsj}^{jI_A I_B} = n_A^B \end{aligned} \quad (5.197)$$

Now, the spectroscopic factor S are now to be interpreted as a probability multiplied by the number of available valence nucleons, so that one-step cross sections are still proportional to these spectroscopic factors.

예: ^{16}O 은 shell model 에서 $|(0s_{1/2})^4(0p_{3/2})^8(0p_{1/2})^4\rangle$ 로 생각할 수 있다. (여기선 proton 과 neutron을 구분하지 않았음.) one-neutron removal을 생각하면,

$$|^{16}\text{O}\rangle = \sqrt{2}|\frac{1}{2}^- \otimes 0p_{1/2}\rangle + \sqrt{4}|\frac{3}{2}^- \otimes 0p_{3/2}\rangle + \sqrt{2}|\frac{1}{2}^+ \otimes 0s_{1/2}\rangle \quad (5.198)$$

로 생각할 수 있다. 여기서, 각 계수는 $\sqrt{C^2 S}$ 에 해당하고, 이것은 각 orbit에 있는 neutron의 갯수로부터 얻어진다.

5.15 Cluster overlaps

If the transferred particle v is a cluster of nucleons, in cluster models, we may compute the single cluster wave function as like single particle wave function. Usually, we would require specific l_{sj} for the cluster wave function. However, the number of nodes n have to be inferred by considering shell model counting.

Suppose, valence nucleon shell has $\mathcal{N}_i = 2n_i + l_i$ quanta. (i.e. p-shell $\mathcal{N} = 1$, sd-shell $\mathcal{N} = 2$.) If the transferred valence cluster is taken as composed of v such nucleons, then the total energy of the cluster nucleons will be $v\mathcal{N}_i\hbar\omega$. (즉, 대강, shell-model에서의 각각의 nucleon들의 energy의 합). If the internal cluster configuration has $\mathcal{N}_{int} = 2n_{int} + l_{int}$ quanta, then the cluster-core motion in partial wave L has N nodes,

$$2N + L + \mathcal{N}_{int} = v\mathcal{N}_i \quad (5.199)$$

¹⁵예를 들어, 가장 단순한 경우인, $J_A = 0$ 인 closed shell에 nucleon이 추가되어 $J_B = j$ 가 되는 경우, $\langle n_j \rangle_i = 0$ 이고, 하나뿐인 $f, +$ 에 대해 $S = 1$ 이다. 따라서, 위 sum rule이 만족됨을 알 수 있다.

¹⁶ $v = 1$ 인 간단한 경우, $n_A^B = N_B$ 가 된다. 즉, B-nuclei에서 orbit v 에 있는 핵자의 수가 된다.

(즉, 원래 shell model single particle들의 energy = (core-cluster energy)+(cluster internal energy).) 여기서, $N = 0, 1, \dots$ 는 origin을 제외한 값이다. (FRESCO 코드에서 사용하는 NN 은 origin을 포함하여 1부터 시작한다.)

5.16 One photon coupling

photon 을 내보내거나, 흡수하는 경우를 생각하자. 보통 radiative photon은 각각 react individually, To a first approximation, 하나의 photon과 핵이 반응하는 경우만 생각하도록 하자. photon 의 energy와 momentum은 $E_\gamma = \hbar\omega$ and $p_\gamma = \hbar k_\gamma = E_\gamma/c$ 로 쓰도록 하자.

Maxwell equation in Gaussian units,

$$\begin{aligned}\nabla \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{j}_q + \frac{1}{c} \frac{d\mathbf{E}}{dt}, \\ \nabla \times \mathbf{E} &= -\frac{1}{c} \frac{d\mathbf{H}}{dt} \\ \nabla \cdot \mathbf{H} &= 0 \\ \nabla \cdot \mathbf{E} &= 4\pi\rho_q.\end{aligned}\tag{5.200}$$

source terms \mathbf{j}_q and ρ_q 는 charge current and charge density. 여기서 모든 field와 source들은 position 과 시간의 함수이다, $\mathbf{X} = \mathbf{X}(\mathbf{r}, t)$.

5.16.1 vector potential

photon 의 vector potential 과 scalar potential을 다음과 같이 정의하자.

$$\begin{aligned}\mathbf{H} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\nabla\phi(t) - \frac{1}{c} \frac{d\mathbf{A}}{dt}\end{aligned}\tag{5.201}$$

이러한 정의를 Maxwell equation에 대입하고, $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ 를 이용하면, Maxwell equation을 \mathbf{A}, ϕ 에 대한 식으로 바꿀 수 있다.

$$\begin{aligned}\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j}_q + \frac{1}{c} \frac{d\mathbf{E}}{dt} &\rightarrow \nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{d\mathbf{A}}{dt^2} = -\frac{4\pi}{c} \mathbf{j}_q + \nabla(\nabla \cdot \mathbf{A}) + \frac{1}{c} \nabla \frac{d\phi}{dt} \\ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{d\mathbf{H}}{dt} &\rightarrow \nabla \times (-\nabla\phi - \frac{1}{c} \frac{d\mathbf{A}}{dt}) = -\frac{1}{c} \frac{d}{dt}(\nabla \times \mathbf{A}), \quad \text{trivial identity,} \\ \nabla \cdot \mathbf{H} = 0 &\rightarrow \nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad \text{trivial identity,} \\ \nabla \cdot \mathbf{E} = 4\pi\rho_q &\rightarrow \nabla^2 \phi = -4\pi\rho_q - \frac{1}{c} \frac{d}{dt}(\nabla \cdot \mathbf{A}).\end{aligned}\tag{5.202}$$

단, 위 식만으로는 $\mathbf{A}(\mathbf{r}, t), \phi(\mathbf{r}, t)$ 이 unique하게 결정되지 않는다. 따라서, gauge invariance 를 이용하여,

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \quad \phi' = \phi + \frac{\partial\chi}{\partial t}.\tag{5.203}$$

의 χ 를 적당히 정하여, Coulomb or transverse gauge

$$\nabla \cdot \mathbf{A} = 0\tag{5.204}$$

을 만족시키도록 조건을 줄 수 있고, 이러한 vector potential은 다음 식을 풀어 unique하게 결정할 수 있다.

$$\begin{aligned}\nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{d\mathbf{A}}{dt^2} &= -\frac{4\pi}{c} \mathbf{j}_q, \\ \nabla^2 \phi &= -4\pi \rho_q\end{aligned}\quad (5.205)$$

단 여기서, one-photon approximation을 생각하여(vector potential이 다시 photon의 scalar potential에 의해 결정되는 일이 없도록), 간단히 하였다. (즉, Coulomb interaction은 charge density에 의해 결정되고, photon field는 charge current에 의해 결정된다.)

photon과 source의 time-dependence를 다음과 같이 두면,

$$\begin{aligned}\mathbf{A}(t) &= \mathbf{A}e^{-i\omega t} + \mathbf{A}^*e^{i\omega t}, \\ \mathbf{j}_q(t) &= \mathbf{j}e^{i\omega t} + \mathbf{j}^*e^{-i\omega t}\end{aligned}\quad (5.206)$$

we get stationary equation

$$\nabla^2 \mathbf{A} + k_\gamma^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{j}. \quad (5.207)$$

핵의 charge current는 다음의 current operator의 matrix element로 생각할 수 있다.

$$\mathbf{j}(\mathbf{r}) = \langle \Phi_f | \hat{\mathbf{j}}_q(\mathbf{r}) | \Psi_i \rangle, \quad (5.208)$$

여기서 matrix element는 wave function $\Phi(\mathbf{r}_i), \Psi(\mathbf{r}_i)$ 의 모든 \mathbf{r}_i 에 대한 integral이 들어 있는 것으로, 일반적인 transition을 생각하여, initial state와 final state가 다를 수 있다. 엄밀하게는 current operator는 $\hat{\mathbf{j}}_{free} + \hat{\mathbf{j}}_2$ 으로, free current operator와 two-body current contribution을 포함하지만, (나아가 photon을 electric 과 magnetic component로 분리하면, magnetic current를 정의할 수도 있다.) 많은 경우 free current operator로 approximation을 한다.

따라서, 위 식을 rearrange하면,

$$\left[-\frac{\hbar c}{k_\gamma} \nabla^2 - E_\gamma \right] \mathbf{A}(\mathbf{r}) = \frac{4\pi\hbar}{k_\gamma} \langle \Phi | \hat{\mathbf{j}}_{free}(\mathbf{r}) | \Psi \rangle \quad (5.209)$$

로 쓸 수 있다. 이것은 nuclear transition (예를 들어 ${}^7\text{Be}(p, \gamma){}^8\text{B}$ 와 같은 photo-production or capture reaction)에 의해 생기는 photon \mathbf{A} 에 대한 식을 준다.

반면, Schrodinger equation에서 minimul substitution을 하여, photon과 입자의 반응을 나타내는 식을 얻으면

$$\begin{aligned}\hat{\mathbf{p}} &\rightarrow \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}, \\ \left[\frac{\hat{\mathbf{p}}^2}{2m} + V - E \right] \Psi &= 0 \\ \rightarrow \left[\frac{1}{2\mu} \left(\hat{\mathbf{p}}^2 - \frac{q}{c} (\mathbf{A} \cdot \hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot \mathbf{A}) + \frac{q^2}{c^2} |\mathbf{A}|^2 \right) + V - E \right] \Psi &= 0.\end{aligned}\quad (5.210)$$

가 얻어지고, 이것은 $\hat{\mathbf{p}} = \frac{\hbar}{i} \nabla$ 로부터, $|\mathbf{A}|^2$ 를 무시하면,

$$[\hat{T} + V - E] \Psi = \frac{1}{c} \int d\mathbf{r} \mathbf{A}(\mathbf{r}) \cdot \hat{\mathbf{j}}_{free}(\mathbf{r}) \Psi, \quad (5.211)$$

가 된다. 좀더 general한 경우, two-body current를 포함하여 다음과 같이 쓸 수 있다.

$$[\hat{T} + V - E] \Psi = \frac{1}{c} \int d\mathbf{r} \mathbf{A}(\mathbf{r}) \cdot \hat{\mathbf{j}}(\mathbf{r}) \Psi, \quad (5.212)$$

이로부터 주어진 nuclear transition에 의한 emitted photon과, photon absorption에 의한 nuclear wave function의 변화를 얻을 수 있다.

5.16.2 Photon Cross section

photo production과 photo absorption cross section을 구하기 위해서는 photon의 flux를 알아야 한다. 주어진 photon wave \mathbf{A} 에 의한 photon의 flux는 Poynting vector (photon energy flux) 를 계산하여 얻을 수 있다.

$\mathbf{A}(\mathbf{r}) = \mathbf{a}e^{i\mathbf{k} \cdot \mathbf{r}}$ 인 경우를 생각하면, physical vector potential과 electric and magnetic fields are

$$\begin{aligned}\mathbf{A}(t) &= 2\Re(\mathbf{A}e^{-i\omega t}) = 2\mathbf{a} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \\ \mathbf{E}(t) &= -\frac{1}{c} \frac{d\mathbf{A}}{dt} = 2k_\gamma \mathbf{a} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t), \\ \mathbf{H}(t) &= \nabla \times \mathbf{A}(t) = 2(\mathbf{k} \times \mathbf{a}) \sin(\mathbf{k} \cdot \mathbf{r} - \omega t), \\ \mathbf{S}_p(t) &= \frac{c}{4\pi} \mathbf{E}(t) \times \mathbf{H}(t)\end{aligned}\quad (5.213)$$

Thus,

$$|\mathbf{S}_p| \simeq \frac{c}{4\pi} |\mathbf{E}|^2 = \frac{c}{4\pi} |\mathbf{H}|^2 \quad (5.214)$$

and 시간에 대한 평균을 내면, $\langle |\sin(t)|^2 \rangle = \frac{1}{2}$ and $\langle |\mathbf{E}(t)|^2 \rangle = 2k_\gamma^2 |\mathbf{a}|^2$ 으로부터

$$\langle |\mathbf{S}_p(t)| \rangle = \frac{k_\gamma^2 c}{2\pi} |\mathbf{a}|^2. \quad (5.215)$$

이 얻어진다. 이것을 photon 하나의 energy로 나누면, outgoing photon number flux 가 얻어진다.

$$j_\gamma = \frac{k_\gamma}{2\pi\hbar} |\mathbf{a}|^2 \quad (5.216)$$

이것을 vector photon wave function $\mathbf{Z}(\mathbf{r})$ 을 도입하여 나타낼 수 있다.

$$j_\gamma = c|\mathbf{Z}(\mathbf{r})|^2, \quad \mathbf{Z}(\mathbf{r}) = \sqrt{\frac{k_\gamma}{2\pi\hbar c}} \mathbf{A}(\mathbf{r}) \quad (5.217)$$

이전의 식을 다시 쓰면, photo-production의 경우¹⁷

$$\left[-\frac{\hbar c}{k_\gamma} \nabla^2 - E_\gamma \right] \mathbf{Z}(\mathbf{r}) = 2\sqrt{\frac{\hbar}{\omega}} \langle \Phi_b | \hat{\mathbf{j}}(\mathbf{r}) | \Psi \rangle = V_{\gamma p} \Psi, \quad (5.218)$$

photo-disintegration의 경우

$$[\hat{T} - V - E] \Psi = \sqrt{\frac{\hbar}{\omega}} \int d^3\mathbf{r} \hat{\mathbf{j}}(\mathbf{r}) \Phi_b \cdot \mathbf{Z} = \mathbf{V}_{p\gamma} \cdot \mathbf{Z}, \quad (5.219)$$

로 쓸 수 있다. (주의, 두 경우에 factor 2 차이가 있음.¹⁸ 가 되어 symmetric 하기 위해서는 $\tilde{S}_{p\gamma} = \tilde{S}_{\gamma p}$, $V_{\gamma p} = 2V_{p\gamma}$ 가 되어야함을 알 수 있다. 즉, relativistic photon을 non-relativistic equation으로 취급하는데에서 오는 artifact.)

그러나, 이러한 current operator들은 derivative form이기 때문에 사용하기가 복잡하다. long wave length limit을 생각하면, 좀더 간단한 모양으로 쓸 수 있다.

¹⁷여기서, photon의 식에는 diagonal potential이 없음에 유의.

¹⁸factor 2 차이는 비록 위에서 non-relativistic 한 Schrodinger equation으로 photon의 wave function equation을 만들었지만, 여전히 massless photon의 경우 relativistic kinematics를 사용해야 하기 때문이다. Kinetic term의 경우, $-\frac{1}{2\mu} \nabla^2$ 대신, $-\frac{1}{k_\gamma} \nabla^2$ 가 사용되었는데, 이것은 kinetic term을 $-t_\alpha \nabla^2$ 라고 쓸 때, $t_\alpha = (E_\alpha - E_0)/k_\alpha^2 = E_0(\gamma - 1)/k^2$ for $k = m_0\gamma v/\hbar$ with $\gamma = (1 - v^2/c^2)^{-1/2}$ 으로 generalize할 수 있다. 여기서, non-relativistic

5.17 Appendix: Watanabe potential and JS adiabatic potential

Watanabe, Nucl.Phys. 8(1958)484.

J. D. Harvey and R. C. Johnson, Phys.Rev.C3(1971)636-645

Watanabe potential is basically a single folding potential for deuteron projectile.

$$V(R) = \int d^3\mathbf{r} |\chi(\mathbf{r})|^2 (V_1(\mathbf{r}_1) + V_2(\mathbf{r}_2)) \quad (5.222)$$

where, $\mathbf{r}_1 = \mathbf{R} + \frac{1}{2}\mathbf{r}$ and $\mathbf{r}_2 = \mathbf{R} - \frac{1}{2}\mathbf{r}$ are position vector from the target to each nucleon in the deuteron. (In general case, the relation can be $\mathbf{r}_1 = a\mathbf{R} + b\mathbf{r}$. Let us only consider same mass particles for the moment.)

On the other hand, JS adiabatic potential is a approximate potential which include the break up channel effects. Because of this, the adiabatic potential cannot be used for elastic scattering itself. Roughly speaking, if the adiabatic approximation is valid, $\epsilon_i \simeq \epsilon_0$ compared to incident energy E , we may ignore the energy difference in each breakup channels. In this case, the whole equation becomes in zero range approximation as

$$(T_R + U_{ct} + U_{nt} - (E - \epsilon_0))\Psi^{ad}(r=0, R) = 0. \quad (5.223)$$

Even though the Ψ^{sd} should be sum of all breakup channels, one may simply approximate it as $\chi^{ad}(R)\phi_0(0)$ with $\phi_0(0)$ as a zero-range np wave of deuteron. (Or, one may think the r is practically fixed during the reaction. Thus, we may treat r as a just a parameter and treat the radial wave function to have additional fixed parameter r , $\chi^{ad}(R; r)$. Then, for each fixed r , we will solve $\chi^{ad}(R; r)$. In zero-range approximation, we solve the equation at $r = 0$. In finite range case, r dependence are averaged over weight factor $V_{np}|\phi_0|^2$.) Simple JS adiabatic potential for transfer use zero range approximation for deuteron and adiabatic potential becomes simply

$$V_{ad}(R) = V_n(R) + V_p(R), \quad (5.224)$$

note that originally $V_n(\mathbf{r}_n)$ and $V_p(\mathbf{r}_p)$ potential had \mathbf{r} dependence. In other words, replace $U_n(r_n) \rightarrow U_n(R)$ as $r = 0$.

The finite range version of adiabatic potential (JT) is given as

$$\bar{V}^{(JT)}(R) = \frac{\langle \phi_0 | V_{np}(\mathbf{r}) (V_{nA}(\mathbf{r}_1) + V_{pA}(\mathbf{r}_2)) | \phi_0 \rangle}{\langle \phi_0 | V_{np}(\mathbf{r}) | \phi_0 \rangle} \quad (5.225)$$

note that this is different from Watanabe folding because of V_{np} contribution. **An explanation can be found Johnson's lecture, "Reaction Mechanims for Rare Isotope Beams",**

limit $t_\alpha \rightarrow \frac{\hbar^2}{2\mu_0}$ 이고, 반면, massless photon의 경우는 $t_\gamma \rightarrow \frac{\hbar c}{k_\gamma}$ 가 된다. 한편, T-matrix form은

$$\begin{aligned} T_{\alpha\alpha_i} &= -\frac{2\mu}{\hbar^2 k_\alpha} \int F_\alpha(R') \sum_{\alpha'} \langle \alpha | V(R', R) | \alpha' \rangle \psi_{\alpha'}(R) \\ &\rightarrow -\frac{1}{t_\alpha k_\alpha} \int \tilde{F}_\alpha \sum_\beta V_{\alpha\beta} \psi_\beta dR, \quad \text{relativistic} \end{aligned} \quad (5.220)$$

따라서, Symmetric S-matrix form becomes

$$\tilde{S}_{\alpha\alpha_i} = 2i \sqrt{\frac{v_\alpha}{v_{\alpha_i}}} T_{\alpha\alpha_i} = -\frac{2i}{\hbar \sqrt{v_\alpha v_{\alpha_i}}} \frac{\hbar v_\alpha}{t_\alpha k_\alpha} \int \tilde{F}_\alpha \sum_\beta V_{\alpha\beta} \psi_\beta \quad (5.221)$$

<https://doi.org/10.1063/1.2114701> . However, it explains in terms of sturmian basis such that orthogonality as $\langle \phi_i | V_{np} | \phi_j \rangle = -\delta_{ij}$ such that $|\phi_0\rangle = \phi_d(r)$?

In general, each potential contains both central complex potential and spin-orbit potential.

$$V_i(\mathbf{r}_i) = V(r_i) + iW(r_i) + (V_{so}(r_i) + iW_{so}(r_i))2\mathbf{l}_i \cdot \mathbf{s}_i \quad (5.226)$$

Then, after folding, the potential between deuteron and target also can have central, spin-orbit and tensor interactions.

$$V(\mathbf{R}) = V(R) + iW(R) + (V_{so}(R) + iW_{so}(R))\mathbf{L} \cdot (\sigma_1 + \sigma_2) + T(R)\hat{T}_{dA} \quad (5.227)$$

Also, central potential may have a contribution from each spin-orbit interaction.

In simple zero range adiabatic case, resulting adiabatic potentials are

$$V_{ad}^{(JS)}(R) = (V_1(R) + V_2(R)) + i(W_1(R) + W_2(R)) + \left(\frac{(V_{so,1}(R) + V_{so,2}(R))}{2} + i \frac{(W_{so,1}(R) + W_{so,2}(R))}{2} \right) \mathbf{L} \cdot \mathbf{S} \quad (5.228)$$

For a simplest case, where $V_i(\mathbf{r}_i)$ only contains the central part and deuteron only have S-waves, we can obtain a simple relation for $V(R)$. This can be done in two ways.

- Original Watanabe paper: Since $\mathbf{r} = 2\mathbf{r}_1 - 2\mathbf{R}$ for given \mathbf{R} and \mathbf{r}_1 , we may replace

$$\begin{aligned} r &= \sqrt{4r_1^2 + 4R^2 - 8r_1R \cos \theta} \rightarrow (-d \cos \theta) = \frac{rdr}{4r_1R}, \\ d^3\mathbf{r} &= 2^3 d^3\mathbf{r}_1 = 2^3 dr_1 r_1^2 (-d \cos \theta) d\phi, \\ &\rightarrow \frac{2^3}{4R} dr_1 r_1 d\phi r dr \end{aligned} \quad (5.229)$$

Thus, if deuteron wave function is S-wave,

$$\begin{aligned} \int d^3\mathbf{r} |\chi(r)|^2 \rho(r_1) &= \int \frac{2^3}{4R} dr_1 r_1 \rho(r_1) d\phi r dr |\chi(r)|^2 \\ &= \frac{4\pi}{R} \int_0^\infty dr_1 r_1 \rho(r_1) \int_{2|R-r_1|}^{2(R+r_1)} dr r |\chi(r)|^2. \end{aligned} \quad (5.230)$$

Thus, we can obtain $V(R)$ by computing double integration. For more complicate case, like non-central potential, non-equal mass particles, non-spherical wave function, we would need numerical integration over angles.

- Using angular integration: However, in a sense, it seems to be rather in-convenient to compute double integral. Alternative way to do the calculation is to first compute angle integral and then do a radial integration. If the potential only depends on the distance and wave function is pure S-wave, we may do the angular integration first.

Let us consider the effective coupling

$$X_{lL,l'L'}^\Lambda(R) = \langle (lL)\Lambda | V_1(\mathbf{r}_1) + V_2(\mathbf{r}_2) | (l'L')\Lambda \rangle \quad (5.231)$$

where $(lL)\Lambda$ implies the projectile wave function $\phi_l(\mathbf{r})$ and relative wave function between projectile and target. Since the potential is a function of $\mathbf{r}_1 = a\mathbf{R} + b\mathbf{r}$ and $\mathbf{r}_2 = p\mathbf{R} + q\mathbf{r}$, the righthand side implies the integration over projectile internal

coordinates \mathbf{r} . We may consider, with $u = \hat{R} \cdot \hat{r}$ is the cosine of angle between R and r , and then the potential itself can be considered as a function of (R, r, u) ,

$$\begin{aligned} d^3\mathbf{r} &= 2\pi r^2 dr du, \quad \mathbf{r}_i = \mathbf{r}_i(R, r, u), \\ V(R, r, u) &= V_1(\mathbf{r}_1) + V_2(\mathbf{r}_2) = \sum_K (2K+1) V_K(R, r) P_K(u) \end{aligned} \quad (5.232)$$

and

$$V_K(R, r) = \frac{1}{2} \int_{-1}^{+1} du [V_1(r_1) + V_2(r_2)] P_K(u) \quad (5.233)$$

If we only consider pure S-wave internal wave function, only $K = 0$ component will survive and we will get the same result as previous case.

If the bound state is only in S-wave, (However, if there are spin-orbit operators, the $V(\mathbf{R})$ itself have operators and cannot be simply expanded in terms of Legendre polynomials.)

Then, though the exact evaluation of $X_{lL,l'L'}^\Lambda(R)$ would involve detailed angular momentum decompositions, it will be proportional to the integral,

$$\int_0^\infty dr u_l^*(r) V_K(R, r) u_{l'}(r). \quad (5.234)$$

Similar method also works for adiabatic potential calculation. The front end code FRONT19 of TWOFNR have a subroutine FOLDER which compute general case of folding potential including deuteron S,D-waves and spin-orbit interaction of folding potential and returns general form of $V(\mathbf{R})$ including spin-orbit and tensor component.

For example, central potential is calculated as

$$\begin{aligned} V_c(R) &= \int dr [u^2(r) + w^2(r)] \int_{-1}^1 du V(R, r, u) \\ &\quad - \frac{3}{2} \int dr w^2(r) \int_{-1}^1 du V_{so}(R, r, u) - 3R \int dr \frac{1}{r} (\sqrt{2}uw - w^2) \int_{-1}^1 du V_{so}(R, r, u) \end{aligned}$$

Note: Since the FOLDER only uses given deuteron wave functions for REID, HULTHIN(?), AV18 case, it may be useful to add the possibility with empirical bound state wave function for FRESCO input.

Note: Since the FRONT19 provides global optical potentials and folding potentials with KD02, JLM, it may be useful to generalize it with arbitrary optical potential folding and provide FRESCO inputs.

Chapter 6

Inelastic Excitations

Here, let us look at the details of the computing kernels of inelastic excitation couplings.

6.1 Collective excitation

In collective model, we introduce angle dependence into the optical potential which enables transition. We introduce the angle dependence for surface vibration of spherically symmetric target

$$R_0 \rightarrow R(\theta, \phi) = R_0 \left(1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi) \right) \quad (6.1)$$

or for axially deformed nucleus

$$R_0 \rightarrow R(\theta') = R_0 \left(1 + \sum_{\lambda} \beta_{\lambda} Y_{\lambda 0}(\theta') \right) \quad (6.2)$$

where, θ' is an angle in body-fixed axis. **Note that (θ, ϕ) is an angle of distance R in the space-fixed frame, while (θ, ϕ) is an angle of distance R in the body-fixed frame since the surface shape $R(\theta')$ is defined in the body-fixed frame where symmetry axis is chosen as \hat{z}' .** We can convert the $Y_{\lambda 0}(\theta', 0)$ into a space-fixed angle by rotation, $Y_{\lambda 0}(\theta', 0) = \sum_{\mu} D_{\mu 0}^{\lambda}(\omega) Y_{\lambda, \mu}(\theta, \phi)$ where ω is an Euler angle between body fixed axis and space fixed axis.¹

Thus, changing optical potential

$$U(r - R_0) \rightarrow U(r - (R_0 + \delta R)) = U(r - \delta R - R_0) \quad (6.4)$$

¹We may rewrite it as θ' is an angle between $\hat{\xi} = \hat{z}'$ and \mathbf{R} , and $Y_{\lambda m}(\hat{\xi}) = \frac{\hat{\lambda}}{\sqrt{4\pi}} D_{m 0}^{\lambda}(\omega)^*$, (It is not clear what is the Euler angle here... also sign of m .)

$$\begin{aligned} Y_{\lambda 0}(\theta', \phi') &= \frac{\hat{\lambda}}{\sqrt{4\pi}} P_{\lambda}(\cos \theta') &= \frac{\hat{\lambda}}{\sqrt{4\pi}} \frac{4\pi}{2\lambda + 1} \sum_m Y_{\lambda m}^*(\hat{\xi}) Y_{\lambda m}(\hat{R}) \\ & &= \sum_m D_{m 0}^{\lambda}(\omega) Y_{\lambda m}(\hat{R}) \end{aligned} \quad (6.3)$$

For a simple verification, imagine the case all vectors are in the same plane. ($\phi = 0$) Then, only relative angles need to be considered and all becomes real number.

Note that Woods-Saxon potential already use the form of $r - R_0$ in the form factor. Thus, it is effectively the same as replacing $r \rightarrow r - \delta R$ and $R_0 \rightarrow R_0 + \delta R$.

Usually the vibration or deformation coupling is considered for central part of the potential (i.e. volume, surface and Coulomb but no spin-orbit). In case of vibration or small deformation and Coulomb deformation, power series expansion of δR is used. However, in case of rotation of deformed nuclei, sometimes the power series expansion form is not used.

6.1.1 Vibrational excitation

Assume the vibration of surface of spherically symmetric nucleus

$$R_0 \rightarrow R(\theta, \phi) = R_0 \left(1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi) \right) \quad (6.5)$$

we get by power series expansion for nuclear part

$$V(r - \delta R) = V(r) - \frac{dV(r)}{dr} \delta R(\theta, \phi) - \frac{1}{2} \frac{d^2 V(r)}{dr^2} (\delta R(\theta, \phi))^2 \quad (6.6)$$

$$\begin{aligned} \delta R(\theta, \phi) &= R_0 \left(\sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi) \right), \\ \delta R(\theta, \phi)^2 &= R_0^2 \sum_{\lambda\lambda_1\lambda_2} \frac{\hat{\lambda}_1 \hat{\lambda}_2}{\sqrt{4\pi\lambda}} \langle \lambda_1 0 \lambda_2 0 | \lambda 0 \rangle \sum_{\mu} Y_{\lambda\mu}(\theta, \phi) [\alpha_{\lambda_1} \otimes \alpha_{\lambda_2}]_{\lambda\mu} \end{aligned} \quad (6.7)$$

where, $\hat{\lambda} = \sqrt{2\lambda + 1}$.

In case of Coulomb potential, we have to think about the charge distribution in target.

$$\begin{aligned} V_{Coul} &= ZZ' e^2 \int d\mathbf{r}' \frac{\rho(r', \theta', \phi')}{|\mathbf{r} - \mathbf{r}'|} \\ &= 4\pi ZZ' e^2 \sum_{\lambda\mu} \int dr' d\Omega' r'^2 \frac{\rho(r', \theta', \phi')}{\hat{\lambda}^2} \frac{r_{<}^{\lambda}}{r_{>}^{\lambda+1}} Y_{\lambda\mu}(\theta, \phi) Y_{\lambda\mu}^*(\theta', \phi') \end{aligned} \quad (6.8)$$

Separate $\lambda = 0$ part and higher order part,

$$\begin{aligned} V_{Coul} &= ZZ' e^2 \int dr' d\Omega' \rho(r', \theta', \phi') \left[r' \theta(r' - r) + \frac{r'^2}{r} \theta(r - r') \right] \\ &\quad + 4\pi ZZ' e^2 \sum_{\lambda > 0, \mu} \int d^3 r' \frac{\rho(r', \theta', \phi')}{\hat{\lambda}^2} \left[\frac{r^{\lambda}}{r'^{\lambda+1}} \theta(r' - r) + \frac{r'^{\lambda}}{r^{\lambda+1}} \theta(r - r') \right] Y_{\lambda\mu}(\theta, \phi) Y_{\lambda\mu}^*(\theta', \phi') \end{aligned} \quad (6.9)$$

At long distance $R \gg R_n$, we may think

$$V_{Coul} = V_{Coul}(r) + \quad (6.10)$$

If we assume constant charge distribution within the charge radius $R_C(\theta', \phi')$, and

$$R_C(\theta', \phi') = R_C \left(1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta', \phi') \right) \quad (6.11)$$

we get

$$\begin{aligned}
V_{Coul}(r, \theta, \phi) = & \frac{ZZ'e^2}{2R_c} \left(3 - \frac{r^2}{R_c^2}\right) \theta(R_c - r) + \frac{ZZ'e^2}{r} \theta(r - R_c) \\
& + \sum_{\lambda\mu} \frac{3ZZ'e^2}{\hat{\lambda}^2} \left[\frac{r^\lambda}{R_c^{\lambda+1}} \theta(R_c - r) + \frac{R_c^\lambda}{r^{\lambda+1}} \theta(r - R_c) \right] (\alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi)) \\
& + \sum_{\lambda\mu} \frac{3ZZ'e^2}{2\hat{\lambda}^2} \left[\frac{(1-\lambda)r^\lambda}{R_c^{\lambda+1}} \theta(R_c - r) + \frac{(\lambda+2)R_c^\lambda}{r^{\lambda+1}} \theta(r - R_c) \right] \\
& \times \sum_{\lambda_1\lambda_2} \frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}\hat{\lambda}} \langle \lambda_1 0 \lambda_2 0 | \lambda 0 \rangle Y_{\lambda\mu}(\theta, \phi) [\alpha_{\lambda_1} \otimes \alpha_{\lambda_2}]_{\lambda\mu}
\end{aligned} \tag{6.12}$$

Thus, combining both nuclear part and Coulomb part, we can write

$$\begin{aligned}
V(r, \theta, \phi) &= V_{diag}(r) + V_{coupl}^{(v)}(r, \theta, \phi), \\
V_{coupl}^{(v)} &= \sum_{\lambda\mu} v_{cp;\lambda}^{(1)(v)} \alpha_{\lambda\mu} Y_{\lambda\mu} + \sum_{\lambda\mu\lambda_1\lambda_2} v_{cp;\lambda}^{(2)(v)} \frac{\hat{\lambda}_1\hat{\lambda}_2}{\sqrt{4\pi}\hat{\lambda}} \langle \lambda_1 0 \lambda_2 0 | \lambda 0 \rangle Y_{\lambda\mu}(\theta, \phi) [\alpha_{\lambda_1} \otimes \alpha_{\lambda_2}]_{\lambda\mu}
\end{aligned} \tag{6.13}$$

Explicit forms of $v_{cp;\lambda}^{(1)(v)}$ and $v_{cp;\lambda}^{(2)(v)}$ can be obtained from derivative of potential form factors.

Note that diagonal matrix element for $v_{cp;\lambda}^{(1)(v)}$ vanishes in vibrational model. (index (v) represent vibration) (**Why?**)

Actual computation of potential matrix element can be done by treating $\alpha_{\lambda\mu}$ as an operators to change phonon states of target.

6.1.2 Rotational excitation

Rotational coupling of deformed nuclei can be calculated in a similar way. Thus, replace $\alpha_{\lambda\mu}$ and $Y_{\lambda\mu}(\theta, \phi)$ in vibrational coupling by β_λ and $Y_{\lambda 0}(\theta') = \sum_\mu D_{\mu 0}^\lambda(\omega) Y_{\lambda\mu}(\theta, \phi)$ where rotation matrix and Euler angle between body-fixed and space-fixed coordinates. Same for Coulomb,

$$\alpha_{\lambda\mu} Y_{\lambda\mu} \rightarrow \beta_\lambda D_{\mu 0}^\lambda(\omega) Y_{\lambda\mu}$$

Alternatively, one can use numerical integration to get an multipole expansion. Since in general the potential will be a function of angles, it can be expanded as a series of Legendre Polynomial or Spherical Harmonics.

$$V(\mathbf{R}, \boldsymbol{\xi}) = \sqrt{4\pi} \sum_{\lambda\mu} \mathcal{F}_\lambda(R) Y_{\lambda\mu}(\theta', \phi') \tag{6.14}$$

For rotational model,

$$\begin{aligned}
V(\mathbf{R}, \boldsymbol{\xi}) &= \sqrt{4\pi} \sum_{\lambda} \mathcal{F}_\lambda(R) Y_{\lambda 0}(\theta', 0) = \sum_{\lambda} \mathcal{F}_\lambda(R) \hat{\lambda} P_\lambda(\cos \theta') \\
&= \sqrt{4\pi} \sum_{\lambda\mu} \mathcal{F}_\lambda(R) D_{\mu 0}^\lambda(\omega) Y_{\lambda\mu}(\theta, \phi) \\
&= \sqrt{4\pi} \sum_{\lambda\mu} \mathcal{F}_\lambda(R) \frac{\sqrt{4\pi}}{\hat{\lambda}} Y_{\lambda\mu}^*(\boldsymbol{\xi}) Y_{\lambda\mu}(\theta, \phi)
\end{aligned} \tag{6.15}$$

To get coefficient, we need numerical integration

$$\begin{aligned}
\mathcal{F}_\lambda(R) &= \frac{1}{2} \hat{\lambda} \int_{-1}^1 d \cos(\theta') V(\mathbf{R}, \boldsymbol{\xi}) P_\lambda(\cos \theta') \\
&= \frac{1}{\sqrt{4\pi}} \int d\Omega_\xi V(\mathbf{R}, \boldsymbol{\xi}) Y_{\lambda\mu}(\theta', 0) \\
&= \frac{\sqrt{4\pi}}{2} \int_{-1}^1 d \cos(\theta') V(\mathbf{R}, \boldsymbol{\xi}) Y_{\lambda\mu}(\theta', 0) \\
V(\mathbf{R}, \boldsymbol{\xi}) &= V(r(R, \cos \theta')), \quad r(R, u) = R - \sqrt{\frac{2\lambda+1}{4\pi}} P_\lambda(u) \delta_\lambda
\end{aligned} \tag{6.16}$$

where $\delta_\lambda = R_0 \beta_\lambda$.

Depending on the convention, $\sqrt{4\pi}$ factor and $\hat{\lambda}$ factor may be absorbed into $V_\lambda(r)$.

If we expand $\mathcal{F}_\lambda(R)$ for small δ_λ , we simply get

$$\mathcal{F}_{\lambda \neq 0}(R) \simeq -\frac{\delta_\lambda}{\sqrt{4\pi}} \frac{dV(R)}{dR} \tag{6.17}$$

In case of Coulomb potential, as like the vibration, (Note here ρ_q is normalized such that $\int dV \rho_q = Z_t$)

$$V_{Coul} = \sqrt{4\pi} \sum_{\lambda\mu} \left(\frac{\sqrt{4\pi} Z_p e^2}{2\lambda+1} \int dr' d\Omega' r'^2 \rho_q(r', \theta', \phi') \frac{r_{\leq}^\lambda}{r_{>}^{\lambda+1}} Y_{\lambda\mu}^*(\theta', \phi') \right) Y_{\lambda\mu}(\theta, \phi) \tag{6.18}$$

If we think $R \gg r'$,

$$\begin{aligned}
V_{Coul}(R) &= \sqrt{4\pi} \sum_{\lambda\mu} F_\lambda(R) \mathcal{T}_{\lambda\mu}(\xi)^* Y_{\lambda\mu}(\hat{R}) \\
&= \sqrt{4\pi} \sum_{\lambda\mu} \frac{\sqrt{4\pi} Z_p e^2}{2\lambda+1} \frac{1}{R^{\lambda+1}} \left(\int d^3\mathbf{r} \rho_q(\mathbf{r}) r^\lambda Y_{\lambda\mu}^*(\hat{r}) \right) Y_{\lambda\mu}(\theta, \phi)
\end{aligned} \tag{6.19}$$

Matrix element

The evaluation of matrix element requires $\langle f | D_{\mu 0}^\lambda(\omega) | i \rangle$, $\langle f | Y_{\lambda\mu}^*(\hat{\xi}) | i \rangle$, $\langle f | \rho_q(\mathbf{r}) r^\lambda Y_{\lambda\mu}(\hat{r}) | i \rangle$ with target collective states wave function and charge distribution. From the Wigner-Eckart theorem these can be computed from reduced matrix elements.

Details on the matrix elements for vibration and rotational model are **TO BE ADDED**.

6.2 Microscopic single particle excitation

6.3 Microscopic multi-particle excitation?

Chapter 7

Transfer reaction

7.1 Projectile stripping

7.1.1 Classical picture

Suppose (d, p) reaction with momentum k_d , k_p and k_n , where k_n is a momentum of stripped neutron. From momentum conservation $k_d = k_p + k_n$, $k_n^2 = k_d^2 + k_p^2 - 2k_p k_d \cos \theta$. If the transfer occurred at R , we can think orbital angular momentum transfer is $l_n = k_n R$. In other words, angle dependence of k_p gives information on the transferred orbital angular momentum.

7.2 Projectile pickup (or target stripping)

7.3 Knock out reaction

Chapter 8

R-matrix methods and Compound Nucleus Reaction

Thompson 책의 R-matrix 설명은 산만한 편이다. 대신, P. Descouvemont and D. Baye, Rep. Prog. Phys. **73**(2010)03601 이 잘 설명하는 편이고, 기본 reference로 삼도록 하자.

현상론적으로는 핵 내부를 black box로 취급하여, 모델을 계산하는 대신, 실험에서 결정된 cross section, S-matrix와의 matching을 통해 핵 내부의 성질을 기술하는 parameter를 fitting 하는 것이다.

Since the cross sections including non-elastic reactions can be described by the multichannel S-matrix and asymptotic wave functions in each reaction channels, the only information needed to get the cross section is R-matrix at the boundary regardless the dynamics in the internal region. Thus the phenomenological R-matrix is applicable to compound nucleus reaction as well as direct reactions. On the other hand, theoretical R-matrix is calculated from potential and thus is restricted to the model.

8.1 Very simple model

어떤 두 핵의 $C = a + A$ 인 Compound nuclei가 lifetime 이 Γ 인 state를 가진다고 하자. 그러면, 그 상태를 나타내는 wave function은 time dependence 만 생각할 때, 다음과 같은 형태를 가질 것으로 생각할 수 있다.

$$\psi(t) = \psi_0 \exp(-i\omega_r t) \exp(-\frac{t}{2\tau}), \quad |\psi(t)|^2 = |\psi_0|^2 \exp(-t/\tau) \quad (8.1)$$

이것을 stationary state $\phi(\omega)$ 의 combination으로 분해하면

$$\begin{aligned} \psi(t) &= \frac{1}{(2\pi)^{1/2}} \int_0^\infty \phi(\omega) \exp(-i\omega t) d\omega, \\ \phi(\omega) &= \frac{1}{(2\pi)^{1/2}} \frac{i\psi_0}{(\omega - \omega_r) + i\frac{1}{2\tau}}. \end{aligned} \quad (8.2)$$

라고 생각할 수 있다. $E = \hbar\omega$ 로부터, Probability of finding a compound nuclei in stationary state energy E will be proportional to

$$P(E = \hbar\omega) \propto |\phi(\omega)|^2 \propto \frac{1}{(E - E_r)^2 + (\frac{\Gamma}{2})^2} \quad (8.3)$$

Here $\Gamma = \hbar/\tau$ is a total decay rate of compound nuclear state. 따라서, energy E 인 $\alpha = a + A$ channel 이 compound nuclei를 만들 cross section은 이 확률에 비례할 것으로 예상할 수 있다.

$$\sigma_{\alpha,C}(E) = \frac{F(E)}{(E - E_r)^2 + (\frac{1}{2}\Gamma)^2} \quad (8.4)$$

$F(E)$ 의 형태에 대해서는 a, A, C 가 평형을 이루고 있는 상태를 가정하는 것으로 유추할 수 있다. 평형 상태에서 $a + A \rightarrow C$ 인 확률을 생각해보자. 이것은 a 의 모든 가능한 에너지 상태에서 A 를 만나 C 를 생성하는 확률을 생각하면 된다. A probability per unit time in volume V , for a particle a in velocity v_α to form a nuclei C will be $\frac{v_\alpha \sigma_{\alpha C}(E_\alpha)}{V}$. A probability of particle a in energy E_α will be

$$n(p)dp = \frac{V p^2 dp}{2\pi^2 \hbar^3}. \quad (8.5)$$

Thus,

$$\begin{aligned} P(\alpha \rightarrow C) &= \int \frac{\sigma_{\alpha C} v_\alpha}{V} \frac{V p_\alpha dp_\alpha}{2\pi^2 \hbar^3} \\ &= \int \frac{\sigma_{\alpha C} k_\alpha^2 dE}{2\pi^2 \hbar} \\ &\simeq \frac{k_\alpha^2 F(E_r)}{2\pi^2 \hbar} \int \frac{dE}{(E - E_r)^2 + \frac{\Gamma^2}{4}} = \frac{k_\alpha^2 F(E_r)}{\pi \hbar \Gamma} \end{aligned} \quad (8.6)$$

where it is assumed that $k_\alpha^2 F(E)$ is slowly varying. 한편 평형상태에서는 C 가 만들어지는 확률과 붕괴하는 확률이 균형을 이룰것이다. 단위 시간당 붕괴하는 전체 확률 Γ/\hbar 중, α 로 decay하는 부분의 partial width를 Γ_α 로 정의하면, 단위시간당 $C \rightarrow \alpha$ 의 확률은 Γ_α/\hbar 로 쓸 수 있다. 따라서, 이로부터

$$F(E_r) = \frac{\pi}{k_\alpha^2} \Gamma_\alpha \Gamma \quad (8.7)$$

라고 formal하게 쓸 수 있다. 즉, Compound nuclear cross section

$$\sigma_{\alpha,C}(E) = \frac{\pi}{k_\alpha^2} \frac{\Gamma_\alpha \Gamma}{(E - E_r)^2 + \Gamma^2/4} \quad (8.8)$$

Compound nuclei가 붕괴하는 다른 channel을 고려하면 $\Gamma = \sum_\beta \Gamma_\beta$, $\frac{1}{\tau} = \sum_\beta \frac{1}{\tau_\beta}$, $A(a, b)B$ 의 compound nuclear reaction cross section을

$$\sigma_{\alpha\beta} = \frac{\pi}{k_\alpha^2} \frac{\Gamma_\alpha \Gamma_\beta}{(E - E_r)^2 + \Gamma^2/4} \quad (8.9)$$

로 생각할 수 있다. (주의: 여기서 energy의 기준은 $a + A$ system이다. 만약 Compound nuclei의 ground state에 대해 $a + A$ 의 separation에너지가 S_α 라면, Compound nuclear g.s.를 기준으로 $E^* = E_r + S_\alpha$ 인 excited state에 해당한다.) (In fact Γ_α can have energy dependence usually in k_α . Thus one can define energy independent reduced width $\gamma_\alpha^2 \simeq \Gamma_\alpha/(2ka)$.) This reduced widths are closely related to the spectroscopic factor (and ANC) for bound states.

같은 내용을 좀더 formal하게 scattering wave function의 phase shift에 대한 argument로 증명할 수 도 있다. (Note 1 참조) ¹

¹Near resonance energy E_r , we may expand as

$$\cot \delta_l \simeq \cot \delta_l(E = E_R) - c(E - E_r) \simeq -c(E - E_r), \quad (8.10)$$

8.2 Compound Nucleus Reaction

- Compound Nucleus Reaction has longer lifetime than direct reaction. More than $10^{-18} \sim 10^{-16}$ secs compared to direct reaction 10^{-22} secs.
- Though not exact, there is the Bohr independence hypothesis or the amnesia assumption.
- Angular distribution from the Compound Nucleus is symmetric about 90 degrees.
- According to the hypothesis, the cross section depends on the probability of formation and decay.
- probability of decay can be thought as

$$\frac{T_l^\beta(E_\beta)}{\sum_{l_\gamma, E_\gamma} T_l^\gamma(E_\gamma)} \quad (8.13)$$

Cross section(or probability of formation) in reaction $a + A \rightarrow C$ can be written as

$$\sigma_{aA \rightarrow C} = \frac{\pi}{k^2} \sum_l (2l+1) T_l^{aA \rightarrow C}(k) \quad (8.14)$$

where transmission coefficient T_l which can be thought as $|1 - S_l|^2$ (absorption, note that it is not a t-matrix of elastic scattering.)

즉, 다른 말로 만약 모든 channel에 대한 transmission coefficient 를 T_α 를 계산할 수 있다면, Compound reaction cross section을

$$\sigma_{\alpha\beta} = \frac{\pi}{k_\alpha^2} \frac{g_\alpha T_\alpha T_\beta}{\sum_\gamma T_\gamma} \quad (8.15)$$

로 계산 할 수 있다. 여기서 g_α 는 statistical factor from angular momentum. 이런 방식이 Hauser-Feshbach 이론에 해당한다.

- Let Γ be the width of a compound nucleus and D be the difference between two levels. The levels are isolated in the limit $\frac{\Gamma}{D} \ll 1$, which is usually a low excitation energy of compound nucleus. Many levels are overlapping in the limit $\frac{\Gamma}{D} \gg 1$, which is usually a high excitation energy of a compound nucleus.

or

$$\begin{aligned} \delta(E) &= \delta_{bg}(E) + \delta_{res}(E), \\ \delta_{res}(E) &= \arctan\left(\frac{\Gamma/2}{E_r - E}\right) + n(E)\pi, \end{aligned} \quad (8.11)$$

then

$$\begin{aligned} f_l(k) &= \frac{1}{k \cot \delta_l - ik} \simeq -\frac{\Gamma/2}{k [(E - E_r) + i\frac{\Gamma}{2}]}, \quad \frac{d(\cot \delta_l)}{dE} \Big|_{E=E_r} = -c = -\frac{2}{\Gamma}, \\ s_l(k) &= e^{2i\delta_{bg}} \frac{k \cot \delta_l + ik}{k \cot \delta_l - ik} = e^{2i\delta_{bg}} \frac{E - E_r - i\Gamma/2}{E - E_r + i\Gamma/2}, \\ \sigma_l &= \frac{4\pi}{k^2} \frac{(2l+1)(\Gamma/2)^2}{(E - E_r)^2 + \Gamma^2/4} \end{aligned} \quad (8.12)$$

- For example, in the limit of $\frac{\Gamma}{D} \ll 1$, we may expect a sharp rise of cross section near the resonance energy. The reaction cross section for $a + A \rightarrow C \rightarrow b + B$ may be considered as transition from energy E of $a + A$ state through E_0 resonance state which is near E and be written as

$$\sigma(E) = \frac{\pi}{k^2} \frac{(2J_C + 1)}{(2J_a + 1)(2J_A + 1)} \frac{\Gamma_{aA}^C \Gamma_{bB}^C}{(E - E_0)^2 + (\frac{\Gamma}{2})^2} \quad (8.16)$$

Here note that $2J_C + 1$ is numerator since the total angular momentum of decay mode is fixed by J, M of CN. (In other words, J_B^z and J_b^z are not free.) This also consistent with

- In the limit of $\frac{\Gamma}{D} \gg 1$, for given energy E , there would be many overlapping resonances near E . Thus, it would be difficult to identify one resonance contribution and rather one have to average over contributions from all possible resonance states near E . Thus, one may consider averaged cross section over energy range

$$\sigma(E) = \left\langle \sum_{J_C} \int_{E-\Delta}^{E+\Delta} dE \rho_C(E) \pi D^2 \frac{(2J_C + 1)}{(2J_a + 1)(2J_A + 1)} \frac{\Gamma_{aA}^C \Gamma_{bB}^C}{(E - E_0)^2 + (\frac{\Gamma}{2})^2} \right\rangle \quad (8.17)$$

- Suppose the process $a + A \rightarrow C \rightarrow b + B$ and probability of the emitted particle b have energy e_b . Maximum possible energy of b is $e_b^{max} = E_C^* - S_b$, where E_C^* is a excited energy of compound nuclei C and S_b is the separation energy of b and B from C . The emitted particle energy can be smaller than e_b^{max} , in this case B is in an excited state E_B^* . Probability of emitting particle b in energy $e_b (< e_b^{max})$ is (? How ? Following equations are not clear..)

$$W(e_b) de_b = \frac{(2J_b + 1)\mu}{\pi^2 \hbar^3} e_b \sigma_{inv} \frac{\rho(E_B^*)}{\rho(E_C^*)} de_b \quad (8.18)$$

where σ_{inv} is a cross section of formation of C via $b + B$ and ρ are density of excited state levels. Fermi gas model gives

$$\rho(E^*) = C \exp(2\sqrt{aE^*}), \quad (8.19)$$

where level density parameter a is $A/12 - A/8$. Nuclear temperature T is given by $E^* = aT^2 - T$.

- 보통의 경우 individual resonance 는 오직 낮은 에너지 상태에서만 보이지만, isobaric analogue resonance는 예외적으로 높은 excitation energy에서 좁은 폭으로 나타나기도 한다. (background의 continuum level과 구분이 된다.) $T = \frac{1}{2}(N - Z)$ 인 핵에 proton 을 더하여 $(N, Z + 1)$ 인 상태를 만들었을 때, 보통 $T = T_0 - 1/2$ 인 상태들은 compound nuclei의 낮은 에너지 상태이고, $T = T_0 + 1/2$ 인 상태들은 excited state가 되는데, 이것은 $(N + 1, Z)$ 인 핵의 같은 $T = T_0 + 1/2$ 상태와 비슷하게 된다. (proton과 neutron이 바뀐 상태이고, Coulomb energy(Δ_C)에 대한 보정($E_c = \Delta_C - (m_n - m_p)c^2$) 이 필요하지 만.) 다른 말로 (N, Z) nuclei에 proton을 scattering 시킬때, $(N + 1, Z)$ 의 state들이 compound nuclei $(N, Z + 1)$ 의 high excited resonance의 형태로 나타날 수 있다. (예를 들어 ^{208}Pb 에 15 MeV proton을 입사시켜 만든 ^{209}Bi 의 high excited state(18.7 MeV) 는 ^{209}Pb 의 ground state의 IAR에 해당한다.)

8.3 Convention

이 장에서는 Rep. Prog. Phys. **73**(2010)03601 의 convention 을 따르도록 한다.
Coulomb functions asymptotic forms

$$\begin{aligned} I_l(\eta, x) &= G_l - iF_l \rightarrow e^{-i(x - \frac{l\pi}{2} - \eta \ln 2x + \sigma_l)}, \\ O_l(\eta, x) &= G_l + iF_l \rightarrow e^{i(x - \frac{l\pi}{2} - \eta \ln 2x + \sigma_l)}, \\ W_{-\eta_B, l + \frac{1}{2}}(x = 2\kappa_B r) &\rightarrow x^{-\eta_B} e^{-x/2}. \end{aligned} \quad (8.20)$$

Coulomb scattering wave function

$$\begin{aligned} \psi_C^{(+)}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} e^{-\pi\eta/2} \Gamma(1 + i\eta) e^{ikz} {}_1F_1(-i\eta, 1, ik(r - z)) \\ &\rightarrow \frac{1}{(2\pi)^{3/2}} \left(e^{i(kz + \eta \ln k(r - z))} + f_C(\Omega) \frac{e^{i(kr - \eta 2kr)}}{r} \right), \\ f_C(\Omega) &= -\frac{\eta}{2k \sin^2 \frac{\theta}{2}} e^{2i(\sigma_0 - \eta \ln \sin \frac{\theta}{2})}. \end{aligned} \quad (8.21)$$

$$\psi_C^{(+)}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{l=0}^{\infty} (2l+1) i^l e^{i\sigma_l} P_l(\cos \theta) \frac{F_l(\eta, kr)}{kr} \quad (8.22)$$

Scattering by a potential,

$$\begin{aligned} u_l(0) &= 0, \\ u_l(r) &\rightarrow C_l [I_l(\eta, kr) - U_l O_l(\eta, kr)] \end{aligned} \quad (8.23)$$

where $U_l = e^{2i\delta_l}$ is a S-matrix, collision matrix.

$$\begin{aligned} \Psi^{(+)}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} \sum_{l=0}^{\infty} (2l+1) i^l e^{i\sigma_l} P_l(\cos \theta) \left(\frac{i}{2} C_l^{-1} \frac{u_l(r)}{kr} \right) \\ &\rightarrow \psi_C^{(+)}(\mathbf{r}) + \frac{1}{(2\pi)^{3/2}} f(\Omega) \frac{e^{i(kr - \eta 2kr)}}{r} \end{aligned} \quad (8.24)$$

$$f(\Omega) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) e^{2i\sigma_l} (U_l - 1) P_l(\cos \theta) \quad (8.25)$$

$$\frac{d\sigma}{d\Omega} = |f_C(\Omega) + f(\Omega)|^2 \quad (8.26)$$

a channel c with two sub systems in internal states $\phi_{cI_1M_1}^{(1)}$ and $\phi_{cI_2M_2}^{(2)}$ is

$$|c\rangle = i^{l_c} \left| [\phi_{cI_1}^{(1)} \otimes \phi_{cI_2}^{(2)}]_{I_c} \otimes Y_{l_c}(\Omega_c) \right\rangle^{JM\pi} \quad (8.27)$$

(coupling의 순서가 Thompson 책과 다른데 상관 없나? i^{l_c} factor는 time-reversal 에 대해서, $K|JM\rangle = (-1)^{J-M}|J-M\rangle$ 을 만족하기 위한 것.)

$$\pi = \pi_{c1}\pi_{c2}(-1)^{l_c} \quad (8.28)$$

$$\begin{aligned}
H\Psi_{(c_0)}^{JM\pi} &= E\Psi_{(c_0)}^{JM\pi}, \\
\Psi_{(c_0)}^{JM\pi} &= \sum_c \mathcal{A}|c\rangle \frac{u_{c(c_0)}(r_c)}{r_c}, \\
u_{c(c_0)}(r_c) &\rightarrow C_{c_0} \frac{1}{\sqrt{v_c}} [\delta_{cc_0} I_c(k_c r_c) - U_{cc_0} O_c(k_c r_c)].
\end{aligned} \tag{8.29}$$

Be careful that because of $\sqrt{\frac{v_{c_0}}{v_c}}$ convention of asymptotic wave, collision matrix U_{cc_0} actually corresponds to \tilde{S}_{cc_0} in the FRESKO book. This makes the differential cross section does not need additional correction from the change of momentum. In other words, we can consider,

$$\begin{aligned}
u_{c(c_0)}(r_c) &\rightarrow C_{c_0} \frac{\sqrt{v_{c_0}}}{\sqrt{v_c}} [\delta_{cc_0} I_c(k_c r_c) - U_{cc_0} O_c(k_c r_c)] \\
&= C_{c_0} \left[\delta_{cc_0} I_c(k_c r_c) - \frac{\sqrt{v_{c_0}}}{\sqrt{v_c}} U_{cc_0} O_c(k_c r_c) \right]
\end{aligned} \tag{8.30}$$

Thus, the asymptotic form becomes

$$\frac{e^{ik_c r}}{r} \sum_c \frac{\sqrt{v_{c_0}}}{\sqrt{v_c}} \frac{(U_{cc_0} - \delta_{cc_0})}{2ik} Y_c(\theta) (\text{other factors}) \tag{8.31}$$

Thus, flux becomes

$$j_{out} = v_c \left| \frac{\sqrt{v_{c_0}}}{\sqrt{v_c}} \frac{(U_{cc_0} - \delta_{cc_0})}{2ik} Y_c(\theta) (\text{other factors}) \right|^2 \tag{8.32}$$

eigen-phase δ_n can be defined from diagonalization of collision matrix U . (Note that the phase convention i^{l_c} implies that collision matrix U is not just unitary, but also symmetric $U = U^T$.)

This gives coupled equations for channel wave function

$$\begin{aligned}
&[T_c + V_c(r) + E_c - E] u_{c(c_0)}(r) + \sum_{c'} \int_0^\infty W_{cc'}(r, r') u_{c'(c_0)}(r') dr' = 0, \\
T_c &= -\frac{\hbar^2}{2\mu_c} \left(\frac{d^2}{dr^2} - \frac{l_c(l_c + 1)}{r^2} \right), \\
V_c(r_c) &= \langle c | V | c \rangle, \quad \text{local or direct potential} \\
W_{cc'}(r, r') &= \langle c | \delta(r_c - r) V \mathcal{A} \delta(r_{c'} - r') | c' \rangle - V_c(r) \delta_{cc'} \delta(r - r'), \quad \text{non-local potential}
\end{aligned} \tag{8.33}$$

where V_c is a local potential in channel including Coulomb and nuclear interaction.

8.4 One-channel R-matrix method

We may consider Schrodinger equation in the internal region, $(0, a)$ as a solution of

$$(H_l - E)u_l = 0, \quad (r \leq 0), \tag{8.34}$$

with a boundary condition,

$$u_l(a) = u_l^{ext}(a), \quad u_l'(a) = u_l'^{ext}(a). \quad (8.35)$$

This is approximated by the inhomogeneous Bloch-Schrodinger equation,

$$(H_l + \mathcal{L}(B) - E)\phi_l^{int} = \mathcal{L}(B)\phi_l^{ext} \quad (8.36)$$

Where the Bloch operator is to make the Hamiltonian to be a hermitian in range $(0, a)$.

$$\mathcal{L}(B) = \frac{\hbar^2}{2\mu}\delta(r-a)\left(\frac{d}{dr} - \frac{B}{r}\right). \quad (8.37)$$

For example, Kinetic and Bloch term becomes symmetric(Hermitian),

$$\begin{aligned} & \langle \varphi_i | T_l + \mathcal{L}(B) | \varphi_j \rangle \\ &= \frac{\hbar^2}{2\mu} \int_0^a dr \varphi_i(r) \left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \delta(r-a)\left(\frac{d}{dr} - \frac{B}{r}\right) \right) \varphi_j(r) \\ &= \frac{\hbar^2}{2\mu} \int_0^a dr \left\{ \frac{d}{dr} \left[-\varphi_i \frac{d}{dr} \varphi_j \right] + \frac{d\varphi_i}{dr} \frac{d\varphi_j}{dr} + \varphi_i \frac{l(l+1)}{r^2} \varphi_j + \delta(r-a) \left(\varphi_i \frac{d}{dr} \varphi_j - \varphi_i \frac{B}{r} \varphi_j \right) \right\} \\ &= \frac{\hbar^2}{2\mu} \int_0^a dr \left[\frac{d\varphi_i}{dr} \frac{d\varphi_j}{dr} + \varphi_i \frac{l(l+1)}{r^2} \varphi_j - \delta(r-a) \varphi_i \frac{B}{r} \varphi_j \right] \end{aligned} \quad (8.38)$$

If we let the boundary condition,

$$\phi_l(a) = u_l^{ext}(a), \quad \frac{d}{dr}\phi_l(a) = \frac{d}{dr}u_l^{ext}(a) \quad (8.39)$$

, in other words $\mathcal{L}(B)\phi_l = \mathcal{L}(B)u_l^{ext}$, the solution of Bloch-Schrodinger equation is equivalent to the Schrodinger equation in the internal region while satisfying the continuity of wave function. And further, since the Bloch-Schrodinger Hamiltonian is Hermitian in the internal region, one can use finite square-integrable basis functions to solve Bloch-Schrodinger equation.

With a square integrable basis function in $(0, a)$, the equation becomes,

$$\sum_{j=1}^N C_{ij}(E, B) c_j = \frac{\hbar^2}{2\mu a} \varphi_i(a) (a u_l^{ext'}(a) - B u_l^{ext}(a)), \quad (8.40)$$

with symmetric matrix

$$C_{ij}(E, B) = \langle \varphi_i | T_l + \mathcal{L}(B) + V - E | \varphi_j \rangle \quad (8.41)$$

R-matrix is defined at energy E as (Note that traditional choice is $B = 0$.)

$$u_l(a) = R_l(E)(a u_l'(a) - B u_l(a)). \quad (8.42)$$

This R-matrix can be expressed as Green's function

$$\begin{aligned} & (H_l + \mathcal{L}(B) - E)G_l(r, r') = \delta(r - r'), \quad G_l(0, r) = 0, \\ & u_l^{int}(r) = \int_0^a dr' G_l(r, r') \mathcal{L}(B) u_l^{ext}(r'). \end{aligned} \quad (8.43)$$

Thus, we get R-matrix

$$R_l(E) = \frac{\hbar^2}{2\mu a} G_l(a, a). \quad (8.44)$$

Thus, if one get the approximate solution of Green's function of Bloch-Schrodinger equation, it is equivalent to get the internal wave function and R-matrix.

Then, the R-matrix is given in terms of basis functions,

$$R_l(E, B) = \frac{\hbar^2}{2\mu a} \sum_{ij=1}^N \varphi_i(a) (C^{-1})_{ij} \varphi_j(a) \quad (8.45)$$

from the R-matrix, one can get scattering matrix and also $u_l^{ext}(r)$.

$$\begin{aligned} U_l &= e^{2i\delta_l} = e^{2i\phi_l} \frac{1 - (L_l^* - B)R_l(E, B)}{1 - (L_l - B)R_l(E, B)}, \\ L_l &= ka \frac{O'_l(ka)}{O_l(ka)}, \\ \phi_l &= -\tan^{-1} \frac{F_l(ka)}{G_l(ka)} \end{aligned} \quad (8.46)$$

where, F_l, G_l are Coulomb functions and $O_l = G_l + iF_l$, $O'_l(z) = \frac{d}{dz} O_l(z)$. Sommefeld parameter η is implicit. Thus, we get internal wave function in R-matrix method,

$$u_l^{int}(r) = \frac{\hbar^2}{2\mu a} \frac{1}{R_l(E, B)} u_l^{ext}(a) \sum_{ij=1}^N \varphi_j(r) (C^{-1})_{ij} \varphi_i(a) \quad (8.47)$$

Thus, in calculable R-matrix, scattering problem becomes to finding inverse matrix C^{-1} . This may be formally expressed in terms of normalized eigen-vectors \mathbf{v}_{nl} of matrix $\mathbf{C}(0, B)$,

$$\mathbf{C}(0, B) \mathbf{v}_{nl} = E_{nl} \mathbf{v}_{nl}, \quad \mathbf{v}_{nl}^T \mathbf{v}_{n'l} = \delta_{nn'}. \quad (8.48)$$

Then, the spectral decomposition gives

$$[\mathbf{C}(E, B)]^{-1} = \sum_{n=1}^N \frac{\mathbf{v}_{nl} \mathbf{v}_{nl}^T}{E_{nl} - E} \quad (8.49)$$

and

$$R_l(E, B) = \sum_{n=1}^N \frac{\gamma_{nl}^2}{E_{nl} - E} \quad (8.50)$$

with

$$\begin{aligned} \gamma_{nl} &= \sqrt{\frac{\hbar^2}{2\mu a}} \phi_{nl}(a), \\ \phi_{nl}(r) &= \sum_{i=1}^N v_{nl,i} \varphi_i(r). \end{aligned} \quad (8.51)$$

Phenomenologically, if we fit E_{nl} and γ_{nl} of R-matrix with experimental data, we can obtain the energy dependence of wave function and phase shifts. In case of single pole approximation, $R_l(E) = \frac{\gamma_1^2}{E_1 - E}$, energy dependence of collision matrix is

$$\begin{aligned} U_l(E) &= e^{2i\phi_l(E)} \frac{\cot \delta_l(E) + i}{\cot \delta_l(E) - i}, \quad \tan \delta_l(E) = \frac{\gamma_1^2 P_l(E)}{E_1 - \gamma_1^2 S_l(E) - E}, \\ &\simeq e^{2i\phi_l} \frac{E_R - E + i\Gamma(E)/2}{E_R - E - i\Gamma(E)/2}, \quad E \rightarrow E_R. \end{aligned} \quad (8.52)$$

with

$$E_R = E_1 - \gamma_1^2 S_l(E_R), \quad \Gamma(E) = 2\gamma_1^2 P_l(E). \quad (8.53)$$

$\frac{2}{3}$, calculable R-matrix provides theoretical support to parametrize, R-matrix as (8.50). Once R-matrix is known, asymptotic form of wave function is completely determined and thus cross sections. Calculable R-matrix gives even more information about the internal region of wave function.

In other words, phenomenological R-matrix works for single channel(elastic scattering case),

- 1. Assume R-matrix is in the form of (8.50) and treat pole and reduced width as free parameter.
- 2. One can relate the R-matrix with phase shift from (8.46).
- 3. This can be related to the cross section.
- 4. Thus, one can fit the cross section by adjusting parameters in R-matrix.

In Calculable R-matrix, one can do more,

- 1. For given Hamiltonian, one can construct basis functions $\varphi_i(r)$.
- 2. Construct symmetric matrix C_{ij} matrix from (8.41).
- 3. Solve eigen-solution of (8.48). This gives E_{nl} and v_{nl} .
- 3. One can obtain R-matrix using (8.45) or (8.50).
- 4. Thus, one can do the same calculations as Phenomenological case without fitting.
- 5. Further, one can obtain internal wave function from (8.47).

In case of multichannel, phenomenological method is nearly the same, except one should introduce additional channel parameters in $R_{cc'}$ with γ_{cn} 's. Calculable case, one have to solve coupled channel equations and obtain coefficients for transition.

8.4.1 potential scattering and resonance scattering

From the collision matrix, we can obtain scattering amplitude f_l as (let us choose $B = 0$ for convenience)

$$\begin{aligned}
 f_l &= \frac{1}{2ik}(U_l - 1) = \frac{1}{2ik} \left(e^{2i\phi_l} \frac{1 - L_l^* R_l}{1 - L_l R_l} - 1 \right) \\
 &= \frac{1}{2ik} \left((e^{2i\phi_l} - 1) + e^{2i\phi_l} \left(\frac{1 - L_l^* R_l}{1 - L_l R_l} - 1 \right) \right) \\
 &= \frac{1}{2ik} \left((e^{2i\phi_l} - 1) + e^{2i\phi_l} \frac{(L_l - L_l^*) R_l}{1 - L_l R_l} \right)
 \end{aligned} \tag{8.54}$$

Here the first term is only from hard-sphere scattering at $r = a$ and the second term involves the details of the reaction in interior region. Thus, define potential scattering and resonance scattering amplitudes as

$$\begin{aligned}
 f_l &= f_{l,pot} + f_{l,res}, \\
 f_{l,pot} &= \frac{1}{2ik}(e^{2i\phi_l} - 1), \\
 f_{l,res} &= \frac{1}{2ik} e^{2i\phi_l} \frac{(L_l - L_l^*) R_l}{1 - L_l R_l} \\
 &= \frac{1}{k} e^{2i\phi_l} \frac{2P_l R_l}{1 - (S_l + iP_l) R_l}, \quad L_l = S_l + iP_l
 \end{aligned} \tag{8.55}$$

We can define resonance as phase shift pass $\pi/2$, or such that

$$1 - S_l(E_R) R_l(E_R) = 0. \tag{8.56}$$

near this energy we can expand, $S_l(E) R_l(E) \simeq (E - E_R) \left[\frac{d}{dE} S_l(E) R_l(E) \right]_{E=E_R} + \dots$, and define

$$\Gamma(E) = \frac{2P_l(E) R_l(E)}{[d(S_l R_l)/dE]_{E=E_R}} = 2\gamma^2 P_l(E), \quad \gamma^2 = R_l(E_R) / [d(S_l R_l)/dE]_{E=E_R}, \tag{8.57}$$

which gives

$$U_l^{BW}(E) \simeq e^{2i\phi_l} \frac{E_R - E + i\Gamma(E)/2}{E_R - E - i\Gamma(E)/2} \tag{8.58}$$

8.4.2 Gaussian basis

One of the basis used for the calculable R-matrix is Gaussian basis function

$$\varphi_i(r) = r^{l+1} \exp(-(r/b_i)^2), \quad b_i = b_1 x_0^{i-1}. \tag{8.59}$$

² One of the advantage of using Gaussian basis is that some matrix elements can be obtained analytically.

²This gives,

$$\frac{d}{dr} \varphi_i(r) = \left(\frac{l(l+1)}{r} - \frac{2r}{b_i^2} \right) \varphi_i(r) \tag{8.60}$$

Define integral function,

$$I_k(\nu) = \int_0^a r^k \exp(-\nu r^2) dr = \frac{1}{2\nu^{(k+1)/2}} \gamma\left(\frac{k+1}{2}, \nu a^2\right) \quad (8.61)$$

where γ is an incomplete gamma function.³ Then, integration in range $(0, a)$ gives

$$\langle \varphi_i | \varphi_j \rangle = I_{2l+2}(\nu_i + \nu_j), \quad \nu_i = 1/b_i^2 \quad (8.63)$$

Thus, for $B = 0$, in Gaussian basis

$$\begin{aligned} \langle \varphi_i | T_l + \mathcal{L}(0) | \varphi_j \rangle &= \frac{\hbar^2}{2\mu} \left[4\nu_i \nu_j I_{2l+4}(\nu_i + \nu_j) \right. \\ &\quad \left. - 2(l+1)(\nu_i + \nu_j) I_{2l+2}(\nu_i + \nu_j) + (l+1)(2l+1) I_{2l}(\nu_i + \nu_j) \right] \end{aligned} \quad (8.64)$$

potential term

$$\begin{aligned} \langle \varphi_i | \exp(-(r/r_0)^2) | \varphi_j \rangle &= I_{2l+2}(\nu_i + \nu_j + \frac{1}{r_0^2}), \\ \langle \varphi_i | r^k | \varphi_j \rangle &= I_{2l+2+k}(\nu_i + \nu_j), \\ \langle \varphi_i | \frac{1}{r} | \varphi_j \rangle &= I_{2l+1}(\nu_i + \nu_j). \end{aligned} \quad (8.65)$$

General matrix element requires numerical integration.

8.4.3 Lagrange function basis

In case of integral in range $(0, a)$ can be approximated by using Gaussian quadrature.

Gauss-Legendre quadrature weights and abscissas in $z \in [-1, 1]$ for degree N are defined as

$$w_i = \frac{2}{(1 - z_i^2)(P'_N(z_i))^2}, \quad P_N(z_i) = 0. \quad (8.66)$$

By change of variable, such that $x \in (0, 1)$, $z = (2x - 1)$ gives Gauss-Legendre quadratures in $x \in (0, 1)$

$$\lambda_i = \frac{1}{x_i(1 - x_i)(\frac{d}{dx} P_N(2x_i - 1))^2}, \quad P_N(2x_i - 1) = 0, \quad (8.67)$$

Then, Lagrange function defined as ($n = 1$ is chosen)

$$\varphi_i(r) = (-1)^{N+i} \left(\frac{r}{ax_i} \right)^n \sqrt{ax_i(1 - x_i)} \frac{P_N(2r/a - 1)}{r - ax_i}, \quad (8.68)$$

have properties at abscissas and radius,⁴

$$\varphi_i(ax_j) = \frac{1}{\sqrt{a\lambda_i}} \delta_{ij}, \quad \varphi_i(a) = (-1)^{N+i} \frac{1}{\sqrt{ax_i(1 - x_i)}}. \quad (8.69)$$

³In case of python, incomplete gamma function `gammainc(n, x)` is normalized as

$$\gamma(n, x) = \Gamma(n) \times \text{gammainc}(n, x) \quad (8.62)$$

⁴This can be proved from the definition. $\varphi_i(ax_j) = 0$ if $i \neq j$ because $P_N(2x_i - 1) = 0$. When $i = j$, $\lim_{x \rightarrow x_i} \frac{P_N(2x-1)}{x-x_i} = \frac{d}{dx} P'_N(2x_i - 1)$. However, it is not clear for the sign factor $(-1)^{N+i}$?.

This allows simple approximation for matrix elements,

$$\langle \varphi_i | \varphi_j \rangle = \int_0^a dr \varphi_i(r) \varphi_j(r) \simeq \sum_{k=1}^N a \lambda_k \varphi_i(ax_k) \varphi_j(ax_k) = \delta_{ij} \quad (8.70)$$

In a similar way, local potential

$$\langle \varphi_i | U | \varphi_j \rangle = \int_0^a dr \varphi_i(r) U(r) \varphi_j(r) \simeq U(ax_i) \delta_{ij}. \quad (8.71)$$

non-local potential,

$$\langle \varphi_i | W | \varphi_j \rangle \simeq a \sqrt{\lambda_i \lambda_j} W(ax_i, ax_j) \quad (8.72)$$

Thus, the potential matrix element does not need any integration.

The matrix element for kinetic term and Bloch term⁵ for $l = 0$ becomes for $i = j$,

$$\langle \varphi_i | T_0 + \mathcal{L}(0) | \varphi_j \rangle = \frac{(4N^2 + 4N + 3)x_i(1 - x_i) - 6x_i + 1}{3a^2 x_i^2 (1 - x_i)^2} \quad (8.74)$$

for $i \neq j$,

$$\langle \varphi_i | T_0 + \mathcal{L}(0) | \varphi_j \rangle = \frac{(-1)^{i+j}}{a^2 [x_i x_j (1 - x_i)(1 - x_j)]^{1/2}} \left[N^2 + N + 1 + \frac{x_i + x_j - 2x_i x_j}{(x_i - x_j)^2} - \frac{1}{1 - x_i} - \frac{1}{1 - x_j} \right] \quad (8.75)$$

for $l \neq 0$, it needs additional term

$$\langle \varphi_i | \frac{l(l+1)}{r^2} | \varphi_j \rangle = \delta_{ij} \frac{l(l+1)}{(ax_i)^2} \quad (8.76)$$

8.5 Phenomenology

현상론적으로는 R-matrix의 주어진 form의 parameter들을 fitting하여 실험 결과의 energy-dependence를 설명하는 것이다. (즉, fitting 방법의 하나.) R-matrix의 form은

$$\begin{aligned} R_l(E, B) &= \sum_{n=1}^N \frac{\gamma_{nl}^2}{E_n - E}, \quad \text{single channel,} \\ R_{cc'}(E) &= \sum_n \frac{\gamma_{nc} \gamma_{nc'}}{E_n - E}, \quad \text{multi-channel,} \end{aligned} \quad (8.77)$$

으로 주어지고, 여기서 energy pole E_n 과 reduced width γ_n^2 를 fitting parameter로 취급한다. (calculable R-matrix의 경우에는 이 값들이 포텐셜에 의해 계산되어지고 compound nucleus의 resonance와의 관계는 생각하지 않는다.) 단, 주의할 것은 실제 cross section이나 S-matrix에서 관찰되는 E_R , Γ_R 과 E_{nl} , γ_{nl} 은 같지 않다는 것이다.

⁵It is useful to have

$$\frac{d}{dr} \varphi_i(r) = \left(-\frac{ax_i}{r(r - ax_i)} + \frac{\frac{d}{dr} P_N(2r/a - 1)}{P_N(2r/a - 1)} \right) \varphi_i(r) \quad (8.73)$$

Full derivation of the expression is not done yet.

가장 쉬운 예로, isolated single resonance의 경우, elastic scattering의 결과는 Breit-Wigner form으로

$$\delta_l^{BW} \simeq \phi_l + \arctan \frac{\frac{1}{2}\Gamma_R}{E_R - E} \quad (8.78)$$

로 나타내거나, Γ_R 에 $P_l(E)/P_l(E_R)$ 을 곱하여 나타낼수 있고, fitting을 통해 E_R 과 Γ_R 을 결정하면, reduced width를 $\Gamma_R = 2\gamma_{obs}^2 P_l(E_R)$ 로 얻을 수 있다. 이 값을 통해 R-matrix를 만들 수 있다. 만들어진 R-matrix는 wave function에 대한 information을 준다.

8.5.1 single channel isolated resonance

R-matrix가 다음의 모양을 가진다고 할때,

$$R_l(E) = \frac{\gamma_1^2}{E_1 - E} \quad (8.79)$$

관련된 phase shift는

$$\delta_l \simeq \phi_l + \arctan \frac{\gamma_1^2 P_l(E)}{E_1 - \gamma_1^2 S_l(E) - E} \quad (8.80)$$

로 쓸 수 있다. 여기서 shift function에 대해 Thomas approximation, $S_l(E) \simeq S_l(E_1) + (E - E_1)S'_l(E)$, 을 쓰고 B-W form과 비교하면,

$$\begin{aligned} \gamma_1^2 &= \frac{\gamma_{obs}^2}{1 - \gamma_{obs}^2 S'_l(E_R)}, \\ E_1 &= E_R + \gamma_1^2 S_l(E_R) \end{aligned} \quad (8.81)$$

을 얻는다. 즉, 실험결과를 B-W form으로 fitting하여 (E_R, Γ_R) 을 얻고, γ_{obs} 를 얻으면, 이를 이용하여 (E_1, γ_1) 을 얻는것이다. 한편, shift와 penetration factor 는 Coulomb function으로 부터 구해진다.

$$L_l = ka \frac{O'_l(\eta, ka)}{O_l(\eta, ka)} = S_l + iP_l \quad (8.82)$$

8.6 Multichannel R matrix

Frankly speaking, the explanation in the text is not clear and difficult to understand. So, let's leave it for a moment.

In case of multi-channel R-matrix, one use diagonal real part of the potential to generate R-matrix basis functions. (Other parts of potential are treated as a matrix elements in the basis.) For a diagonal real potential, basis functions satisfy

$$[T_\alpha(R_\alpha) + U_\alpha(R_\alpha) - \epsilon_{n\alpha}] \omega_\alpha^n(R_\alpha) = 0, \quad \beta = \frac{d \ln \omega_\alpha^n(R_\alpha)}{dR_\alpha}. \quad (8.83)$$

(The same β for all n and α ?) Note this is not a solution with couplings with other channels.

Including the couplings to other channels, we solve

$$[T_\alpha(R_\alpha) + U_\alpha(R_\alpha) - \epsilon_{n\alpha}] g_\alpha^p(R_\alpha) + \sum_{\alpha' \neq \alpha} \hat{V}_{\alpha\alpha'} g_{\alpha'}^p(R'_\alpha) = e_p g_\alpha^p(R_\alpha) \quad (8.84)$$

equation to get eigen-values e_p and eigen functions $g_\alpha^p(R_\alpha)$ within the radius. Note that here index p is similar to n but for including off-diagonal couplings. (이 경우에 potential 과 에너지는 real인가 아니면 complex인가? 아마도 여전히 real) 여기서 ϵ_p 는 core excitation energy 에 해당 한다.

We may solve above equation by expanding $g_\alpha^p(R_\alpha)$ in terms of $\omega_\alpha^n(R_\alpha)$.

$$g_\alpha^p(R_\alpha) = \sum_{n=1}^N c_\alpha^{pn} \omega_\alpha^n(R_\alpha) \quad (8.85)$$

만약 M 개의 p state를 생각한다면,

- **Summary of R-matrix:** R matrix is used to parametrize the effects of Hamiltonian in the interior region $R \leq a$ with pole energies and reduced width amplitudes.

– R-matrix

$$\mathbf{R}_{\alpha'\alpha}(E) = \sum_p \frac{\gamma_{p\alpha} \gamma_{p\alpha'}}{e_p - E}, \quad (8.86)$$

where e_p are pole energies, $\gamma_{p\alpha}$ are reduced width amplitudes which are related with the internal wave function values at the boundary,

$$\gamma_{p\alpha} = \sqrt{\frac{t_\alpha}{a}} g_\alpha^p(a), \quad t_\alpha = \frac{\hbar^2}{2\mu_\alpha}. \quad (8.87)$$

– Relation between R-matrix and S-matrix

$$\begin{aligned} \mathbf{S}(E) &= \frac{t^{\frac{1}{2}} H^{(-)} - a \mathbf{R} t^{\frac{1}{2}} (H^{(-)'} - \beta H^{(-)})}{t^{\frac{1}{2}} H^{(+)} - a \mathbf{R} t^{\frac{1}{2}} (H^{(+)' } - \beta H^{(+)})}, \quad \text{most general,} \\ &= \frac{H^{(-)} - a \mathbf{R} (H^{(-)'} - \beta H^{(-)})}{H^{(+)} - a \mathbf{R} (H^{(+)' } - \beta H^{(+)})}, \quad \text{if } t_\alpha \text{ are all equal,} \\ &= \frac{H^{(-)} - a \mathbf{R} H^{(-)' }}{H^{(+)} - a \mathbf{R} H^{(+)' }}, \quad \text{if } \beta = 0, \end{aligned} \quad (8.88)$$

where only \mathbf{R} is a non-diagonal matrix and others are diagonal matrices($t_\alpha^{\frac{1}{2}}$ have elements $\sqrt{\hbar^2/2\mu_\alpha}$ and $H^{(\pm)}$ are asymptotic wave function values at $R = a$).

– the parameters can be fitted to the experimental cross section.

- Reformulation of R-matrix is useful for better understanding and some approximations. This can be done by introducing various (**reduced, partial, total**) **widths**, **shift functions**, **penetrabilities** and **level matrix**.
- Approximations leads to the models like Breit-Wigner , Hauser-Feshbach etc.

8.7 Short summary for multilevel multichannel R-matrix

External channel wave function,

$$\begin{aligned} \phi_c &= \frac{1}{\sqrt{v_c}} (y_c I_c + x_c O_c), \\ &= \frac{1}{\sqrt{v_c}} (y_c I_c - \sum_{c'} U_{cc'} y_{c'} O_c) \end{aligned} \quad (8.89)$$

where Collision matrix U is defined as,

$$x_c = - \sum_{c'} U_{cc'} y_{c'}. \quad (8.90)$$

In matrix form, with diagonal matrix Φ, v, I, O, y ,

$$\begin{aligned} \Phi &= v^{-\frac{1}{2}}(I - OU)y, \\ \Phi' &= v^{-\frac{1}{2}}(I' - O'U)y \end{aligned} \quad (8.91)$$

Internal channel wave function,

$$\phi_c = \left(\frac{m_c a_c}{\hbar^2} \right)^{\frac{1}{2}} \sum_{c'} R_{cc'} \left(\frac{\hbar^2}{m_{c'} a_{c'}} \right)^{\frac{1}{2}} [\rho_{c'} \phi'_{c'} - B_{c'} \phi_{c'}], \quad (8.92)$$

where

$$\rho_c = k_\alpha a_c, \quad \phi'_{c'} = \frac{d}{d\rho_{c'}} \phi_{c'}(\rho_{c'}) \quad (8.93)$$

and B_c represent boundary conditions in R-matrix theory.

In matrix form

$$\Phi = v^{-\frac{1}{2}}(\rho)^{\frac{1}{2}} R v^{\frac{1}{2}}(\rho)^{-\frac{1}{2}} [\rho \Phi' - B \Phi] \quad (8.94)$$

The R-matrix is parameterized as

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E}, \quad (8.95)$$

with eigenstate wave function X_{λ} , surface channel wave function ψ_c , and surface integral,

$$\gamma_{\lambda c} = \left(\frac{\hbar^2}{m_c a_c} \right)^{\frac{1}{2}} \int dS X_{\lambda}^* \psi_c. \quad (8.96)$$

From matching of internal/external wave function

$$\begin{aligned} v^{-\frac{1}{2}}(I - OU)y &= v^{-\frac{1}{2}}(\rho)^{\frac{1}{2}} R v^{\frac{1}{2}}(\rho)^{-\frac{1}{2}} \left[\rho v^{-\frac{1}{2}}(I' - O'U)y - B v^{-\frac{1}{2}}(I - OU)y \right] \\ (I - OU) &= (\rho)^{\frac{1}{2}} R (\rho)^{-\frac{1}{2}} [\rho(I' - O'U) - B(I - OU)] \\ &= \rho^{\frac{1}{2}} R \rho^{-\frac{1}{2}} [(\rho I' I^{-1} - B)I - (\rho O' O^{-1} - B)OU] \end{aligned} \quad (8.97)$$

Because I and O are conjugate, and using commutation of diagonal matrices,

$$(I - OU) = \rho^{\frac{1}{2}} R \rho^{-\frac{1}{2}} [L_0^* I - L_0 OU] \quad (8.98)$$

$$\rho^{\frac{1}{2}}(1 - RL_0^*)I\rho^{-\frac{1}{2}} = \rho^{\frac{1}{2}}(1 - RL_0)O\rho^{-\frac{1}{2}}U \quad (8.99)$$

The collision matrix can be obtained by logarithmic derivative matching of internal/external wave function,

$$U = \rho^{\frac{1}{2}} O^{-1} (1 - RL_0)^{-1} (1 - RL_0^*) I \rho^{-\frac{1}{2}}. \quad (8.100)$$

where ρ , I , O , and B are diagonal elements, ($\rho_{cc'} = \rho_c \delta_{cc'}$ for all channels.)

$$L_0 = \rho O' O^{-1} - B. \quad (8.101)$$

This can be further simplified by introducing

$$\begin{aligned} \mathcal{B} &= \rho I^{-1} O^{-1}, \\ \Omega &= I^{\frac{1}{2}} O^{-\frac{1}{2}}, \\ \mathcal{W} &= \mathcal{B}^{\frac{1}{2}} (1 - \mathbf{R} L_0)^{-1} (1 - \mathbf{R} L_0^*) \mathcal{B}^{-\frac{1}{2}}, \\ &= 1 + \mathcal{B}^{\frac{1}{2}} (1 - \mathbf{R} L_0)^{-1} \mathbf{R} \mathcal{B}^{\frac{1}{2}} w, \end{aligned} \quad (8.102)$$

where w is a Wronskian of O_c and I_c . Then we can obtain collision matrix as

$$U = \Omega W \Omega. \quad (8.103)$$

for positive energy, we can simplify

$$\begin{aligned} \mathcal{B}_{cc'} &= P_c \delta_{cc'}, \\ w_{cc'} &= 2i \delta_{cc'} \end{aligned} \quad (8.104)$$

with Penetrability P_c . This way simplifies the calculation of collision matrix from \mathbf{R} matrix. The calculation involves the inversion of $(1 - \mathbf{R} L_0)^{-1}$. In case of many channels, this inversion may be slow. On the other hand, if the number of levels are small but, number of channels are large, it is more convenient to use A -matrix inversion. Re-writing the expression of collision matrix by using explicit R -matrix expression in levels,

$$U_{cc'} = \Omega_c \Omega_{c'} \left[\delta_{cc'} + i \sum_{\lambda \lambda'} \Gamma_{\lambda c}^{\frac{1}{2}} \Gamma_{\lambda' c}^{\frac{1}{2}} A_{\lambda \lambda'} \right], \quad (8.105)$$

where (Derivation?)

$$(A^{-1})_{\lambda \lambda'} = (E_\lambda - E) \delta_{\lambda \lambda'} + \Delta_{\lambda \lambda'} - \frac{i \Gamma_{\lambda \lambda'}}{2}. \quad (8.106)$$

$$\begin{aligned} \Delta_{\lambda \lambda'} &= - \sum_c \gamma_{\lambda c} \gamma_{\lambda' c} (S_c - B_c), \\ \Gamma_{\lambda c} &= 2 P_c \gamma_{\lambda c}^2, \\ \Gamma_{\lambda \lambda'} &= \sum_c 2 P_c \gamma_{\lambda c} \gamma_{\lambda' c}. \end{aligned} \quad (8.107)$$

8.8 Relation to cross section

8.9 Compound nuclear reaction

Bohr's independence hypothesis 을 가정하면, compound nuclear reaction $a + A \rightarrow C^* \rightarrow b + B$ 은 다음과 같이 생각할 수 있다.

$$\sigma(a, b) = \sigma_{CN}(a, A \rightarrow C^*) P(C^* \rightarrow b + B) \quad (8.108)$$

여기서, $\sigma_{CN}(a, A \rightarrow C^*)$ 은 CN formation cross section이고, $P(C^* \rightarrow b + B)$ 는 CN이 b를 emit 하는 probability 이다.

CN formation cross section은 여러가지 가정을 이용하거나 optical potential로부터 계산이 가능하다.

Decay probability는 Decay width를 이용하여, 다음과 같이 나타낼 수 있다.

$$P(\beta) = \frac{\Gamma_\beta}{\Gamma}, \quad \Gamma = \sum_{\alpha} \Gamma_{\alpha} \quad (8.109)$$

많은 경우에 전체 width는 photon emission이나 neutron emission이 dominant이다. ($\Gamma \simeq \Gamma_n$.) 또한, emitted neutron의 energy가 작아 남은 핵의 level density가 클때는 emitting probability는 통계적인 경향을 따른다.

$$I(E_n) \propto E_n \exp(-E_n/\theta) dE, \quad \frac{1}{\theta} = \frac{dS}{dE}, \quad S = \ln \omega(E) \quad (8.110)$$

Chapter 9

Resonating Group Method(RGM) and Generator Coordinate Method(GCM)

Full many-body solution related with reaction is too difficult. However, one may have many-body solution/approximation to the anti-symmetrized projectile and target wave function. In RGM, one approximates the full anti-symmetrized wave function in terms of frozen projectile/target wave function, the ansatz wave function is

$$\Psi(\mathbf{r}_1 \dots \mathbf{r}_N) = \Psi_{CM}(\mathbf{R}_{cm}) \cdot \mathcal{A} [\psi(\mathbf{r}) \Phi_p(\xi_1 \dots \xi_{N_p}) \Phi_t(\xi_{N_p+1} \dots \xi_N)] \quad (9.1)$$

where $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_t$ and $\xi_i = \mathbf{r}_i - \mathbf{r}_{p,t}$. Here wave function is approximated as a anti-symmetrized production of anti-symmetric projectile and target wave function. One have to find the relative wave function $\psi(\mathbf{r})$. Important point is that the wave function is anti-symmetrized for any permutation between particles in projectile and target.

Re-writing the ansatz,

$$\begin{aligned} \Psi &= \int d^3 r'' \psi(\mathbf{r}'') \Phi_{r''}, \\ \Phi_{r''} &= \Psi_{CM}(\mathbf{R}_{cm}) \mathcal{A} [\delta(\mathbf{r}'' - \mathbf{r}) \Phi_p(\xi_1 \dots \xi_{N_p}) \Phi_t(\xi_{N_p+1} \dots \xi_N)] \end{aligned} \quad (9.2)$$

The relative wave $\psi(\mathbf{r})$ may be obtained by the variational condition,

$$\delta \left[\frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right] = 0 \quad (9.3)$$

with $H = H_0 + V - T_{CM}$. By using $\mathcal{A}^2 = \mathcal{A}$, one gets RGM equation,

RGM equation (derivation is not given) is

$$\left[-\frac{\hbar^2}{2\mu} \nabla_r^2 + V_D(\mathbf{r}) \right] \psi(\mathbf{r}) + \int d^3 r' V_E(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') = E \psi(\mathbf{r}) \quad (9.4)$$

where,

$$V_D(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') = \int d\xi_p d\xi_t d^3 r'' [\Phi_p(\xi_p) \Phi_t(\xi_t) \delta(\mathbf{r} - \mathbf{r}'')] \mathcal{V}(\mathbf{r}'', \xi_p, \xi_t) [\Phi_p(\xi_p) \Phi_t(\xi_t) \delta(\mathbf{r}' - \mathbf{r}'')] \quad (9.5)$$

with

$$\mathcal{V}(\mathbf{r}, \xi_p, \xi_t) = \sum_{i=1}^{N_p} \sum_{j=N_p+1}^N v(\mathbf{r}_i - \mathbf{r}_j) \quad (9.6)$$

and

$$V_E(\mathbf{r}, \mathbf{r}') = \int d\xi_p d\xi_t d^3\mathbf{r}'' [\Phi_p(\xi_p)\Phi_t(\xi_t)\delta(\mathbf{r} - \mathbf{r}'')] (H - E)(\mathcal{A} - 1) [\Phi_p(\xi_p)\Phi_t(\xi_t)\delta(\mathbf{r}' - \mathbf{r}'')](9.7)$$

Note that the calculation of V_E is very complicate because of \mathcal{A} . For a very simple form of approximations to bound state wave and interaction, it can be done. The non-local potential arise from the anti-symmetrization of projectile-target wave function. According to an example, for higher partial waves $l > 3$, the effects of V_E is small, but it plays very important role for lower l partial waves.

Chapter 10

Approximations

10.1 General

If we assume that the projectile is a sum of clusters c (core) and v (valence), the reaction with t (target) can be considered as three body wave function $\Psi(\mathbf{r}, \mathbf{R})$, (center of mass motion is already taken out),

$$\boxed{[\hat{T}_R + H_{int}(r) + U_{ct}(R_{ct}) + U_{vt}(R_{vt}) - E]\Psi(\mathbf{r}, \mathbf{R}) = 0.} \quad (10.1)$$

In a simple DWBA, one approximates,

$$\Psi(\mathbf{r}, \mathbf{R}) \simeq \chi(\mathbf{R})\phi_0(\mathbf{r}) \quad (10.2)$$

where ϕ_0 is a ground state wave function of projectile.

In CDCC, one approximates

$$\Psi(\mathbf{r}, \mathbf{R}) \simeq \sum_{n=0}^N \chi_n(\mathbf{R})\phi_n(\mathbf{r}) \quad (10.3)$$

including n of bound and continuum states of projectile.

In Time-Dependent (TD) approach, one approximate the projectile motion as a classical projectile, $\mathbf{R}(t)$ and thus considers time dependent equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, b, t) = (H_0 + U_{ct}(R_{ct}(t)) + U_{vt}(R_{vt}(t)) - V_{traj}(\mathbf{R}(t)))\Psi(\mathbf{r}, b, t) \quad (10.4)$$

where V_{traj} is the potential used to generate the trajectories and b is the impact parameter, which characterises each trajectory.

In Dynamical Eikonal Approximation(DEA), one assumes that the three-body wave function is not very different from plane wave,

$$\begin{aligned} \Psi(\mathbf{r}, \mathbf{R}) &\simeq e^{iKZ} \hat{\Psi}(\mathbf{r}, \mathbf{R}), \\ i\hbar v \frac{\partial}{\partial Z} \hat{\Psi}(\mathbf{r}, b, Z) &\simeq [H_0 - E_{n_0 l_0} + U_{ct}(\mathbf{R}_c) + U_{vt}(\mathbf{R}_v)] \hat{\Psi}(\mathbf{r}, b, Z). \end{aligned} \quad (10.5)$$

Usually, one introduce additional "adiabatic"(or "sudden") approximation in eikonal approximation. In adiabatic approximation, internal structure of projectile is fixed during the reaction. In other words, ignore \mathbf{r} dependence replacing $H_0 \simeq E_{n_0 l_0}$.

$$i\hbar v \frac{\partial}{\partial Z} \hat{\Psi}(\mathbf{r}, b, Z) \simeq [U_{ct}(\mathbf{R}_c) + U_{vt}(\mathbf{R}_v)] \hat{\Psi}(\mathbf{r}, b, Z) \quad (10.6)$$

10.2 Adiabatic methods

비록 reaction을 기술하기 위해서 asymptotic하게 wave function을 $\psi(\mathbf{r}, \mathbf{R}) \sim \phi(\mathbf{r})\chi(\mathbf{R})$ 으로 표시하지만, 실제로 이러한 separation은 \mathbf{r} 과 \mathbf{R} 이 완전히 independent하다는 가정에 기반하기 때문에 R 이 작을 때는 옳지 않을 것임을 예상할 수 있다. 그러나, 한편, separation을 하지 않은채 $\psi(\mathbf{r}, \mathbf{R})$ 자체를 구하는 것은 many-body problem으로 매우 어렵다. 그러나, 만약, bound state property (\mathbf{r} variation)는 \mathbf{R} variation에 비해 작다고 가정하면, \mathbf{r} 의 값을 고정시킨채로 \mathbf{R} dependence를 계산할 수 있고, 나중에 \mathbf{r} 에 대한 변화를 고려할 수 있을 것이다. 이처럼, slow varying part를 고정시킨채 계산하는 것이 adiabatic method로, $\chi(\mathbf{R})$ 대신, \mathbf{r} 은 dynamic variable이 아닌 일종의 parameter로 해석하고, 고정된 \mathbf{r} 에 대해, $\chi(\mathbf{R}, \mathbf{r})$ 을 계산하고 최종 결과는 parameter \mathbf{r} 에 대해 average를 시켜 구하는 것이 adiabatic(sudden) approximation 이다.

예를 들어, projectile(=c+v)+target(t) 의 경우, time-dependent equation 에서 $\Phi(\mathbf{R}, \mathbf{r}, t)$ 를 다음과 같이 도입하자.

$$\Psi(\mathbf{R}, \mathbf{r}, t) = \exp[-i(H_{int} - \epsilon_0)t/\hbar]\Phi(\mathbf{R}, \mathbf{r}, t) \quad (10.7)$$

그러면, time dependent Schrodinger equation은

$$\begin{aligned} H\Psi(\mathbf{R}, \mathbf{r}, t) &= i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{R}, \mathbf{r}, t), \\ \rightarrow (T_R + H_{int} + V_{ct} + V_{vt}) \exp[-i(H_{int} - \epsilon_0)t/\hbar] \Phi(\mathbf{R}, \mathbf{r}, t) \\ &= (H_{int} - \epsilon_0) \exp[-i(H_{int} - \epsilon_0)t/\hbar] \Phi(\mathbf{R}, \mathbf{r}, t) + \exp[-i(H_{int} - \epsilon_0)t/\hbar] i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{R}, \mathbf{r}, t), \\ \rightarrow (T_R + \epsilon_0 + V_{ct}(\mathbf{R}, \mathbf{r}(t)) + V_{vt}(\mathbf{R}, \mathbf{r}(t))) \Phi(\mathbf{R}, \mathbf{r}, t) &= i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{R}, \mathbf{r}, t) \end{aligned} \quad (10.8)$$

여기서, H_{int} 를 없애는 대신 time-dependent $\mathbf{r}(t)$ 를 다음과 같이 도입하였다.

$$\mathbf{r}(t) = \exp[i(H_{int} - \epsilon_0)t/\hbar] \mathbf{r} \exp[-i(H_{int} - \epsilon_0)t/\hbar] \quad (10.9)$$

여기서, internal degree의 variation이 작다고 가정하면(adiabatic approximation), $\mathbf{r}(t) \rightarrow \mathbf{r}(0) = \mathbf{r}$ 이고 $\Phi(\mathbf{R}, \mathbf{r}, t) = \exp(-iE_K t)\Phi_K(\mathbf{R}, \mathbf{r})$, 다음과 같은 stationary adiabatic equation 이 얻어진다.

$$[\hat{T}_R + \epsilon_0 + U_{ct}(\mathbf{r}, \mathbf{R}) + U_{vt}(\mathbf{r}, \mathbf{R})]\Phi_K(\mathbf{R}, \mathbf{r}) = E_K \Phi_K(\mathbf{R}, \mathbf{r}) \quad (10.10)$$

여기서, \mathbf{r} 은 일종의 parameter 역할을 한다. 이러한 approximation이 성립하기 위해서는

$$(H_{int} - \epsilon_0)t_{coll}/\hbar \ll 1, \quad (10.11)$$

즉, internal excitation energy 가 작고, 에너지가 높아 collision time이 짧아야한다는 것을 알 수 있다. 만약 더 나아가 \mathbf{R} 이 straight line이어야 한다는 조건을 더하면(eikonal approximation), Glauber model이 된다.

한편, adiabatic approximation은 breakup effect(or projectile excitation effect)를 포함하는 것으로 해석할 수 있다. Suppose three-body equation,

$$\boxed{[\hat{T}_R + H_{int}(\mathbf{r}) + U_{ct}(\mathbf{r}, \mathbf{R}) + U_{vt}(\mathbf{r}, \mathbf{R}) - E]\Psi(\mathbf{r}, \mathbf{R}) = 0} \quad (10.12)$$

with

$$\Psi(\mathbf{r}, \mathbf{R}) = \phi_0(\mathbf{r})\chi_0(\mathbf{R}) + \sum_{i>0} \phi_i(\mathbf{r})\chi_i(\mathbf{R}) \quad (10.13)$$

such that $H_{int} = T_r + V_{vc}(r)$, $H_{int}\phi_i = \epsilon_i\phi_i$. The equation naturally becomes a coupled channel equations,

$$[\hat{T}_R - [E - \epsilon_0] + U_{ct} + U_{vt}]\phi_0(\mathbf{r})\chi_0(\mathbf{R}) + \sum_i [\hat{T}_R - [E - \epsilon_i] + U_{ct} + U_{vt}]\phi_i(\mathbf{r})\chi_i(\mathbf{R}) = 0. \quad (10.14)$$

In adiabatic approximation, $|\epsilon_i - \epsilon_0| \ll E$, we may approximate $H_{int}\phi_i \simeq \epsilon_0\phi_i$. 그러면, 위식에서 \mathbf{r} 에 대한 derivative가 나타나지 않으므로, 식을 다음과 같이 simplify 할 수 있다. (FRESCO에서는 $|\epsilon_i - \epsilon_0| \ll E$ 를 이용한 CDCC 를 푸는 것으로 adiabatic approximation을 취급할 수 있다.)

$$[\hat{T}_R - [E - \epsilon_0] + U_{ct} + U_{vt}]\Psi^{ad}(\mathbf{r}, \mathbf{R}) = 0, \quad \Psi^{ad}(\mathbf{r}, \mathbf{R}) = \phi_0(\mathbf{r})\chi_0^{ad}(\mathbf{R}) + \sum_{i>0} \phi_i(\mathbf{r})\chi_i^{ad}(\mathbf{R}) \quad (10.15)$$

위 식에서 \mathbf{r} 은 parameter로써 취급할 수 있고, 각 r 에서 radial wave function을 구한 다음, r 에 대한 weighted average 하여 원하는 최종 elastic cross section 얻을 수 있다.

10.2.1 Johnson Soper approximation for transfer

In DWBA, the transfer reaction like (d,p) in post form is described by

$$T_{fi} = -\frac{2\mu_f}{\hbar^2 K_f} \langle \psi_f^{(-)} \phi_n | \mathcal{V}_f | \Psi(r, R) \rangle \quad (10.16)$$

where, $\mathcal{V}_f = V_{cv} + V_{ct} - U_f$, $\Psi(r, R)$ is an adiabatic three-body wave function, ϕ_n is a bound state wave function of $t+v$. For transfer reaction like (d,p), the dominating part of potential is $V_{np}(r)$ which has short range $\rho_{eff} \sim 1$ fm. In transfer reaction only $r \leq \rho_{eff}$ is necessary for the DWBA. Also, if the zero-range approximation is used, only wave function at $r = 0$ is necessary. Thus, by setting $r = 0$,

$$[\hat{T}_R - [E - \epsilon_0] + U_{ct} + U_{vt}]\Psi^{ad}(0, \mathbf{R}) = 0, \quad (10.17)$$

And this is equivalent to solve $\Psi^{ad}(0, \mathbf{R}) = \phi_0(0)\tilde{\chi}_0(R)$ for adiabatic potential,

$$(\hat{T}_R + U_{ad}(R) - (E - \epsilon_0))\tilde{\chi}(\mathbf{R}) = 0, \quad U_{ad}^{(JS)}(R) = U_{ct}(R) + U_{vt}(R). \quad (10.18)$$

If the finite range is considered, one can use averaging with weight $V_{np}(r)|u_{l_0}(r)|^2$

$$U_{ad}^{(JT)}(R) = \frac{\int d\mathbf{r} (V_n + V_p) V_{np} \phi_d(r)}{\int d\mathbf{r} V_{np} \phi_d(r)} \quad (10.19)$$

Another special case, which can be simplified is when $U_{ct} = 0$ like a neutron halo case. Then, without using zero-range approximation, one can have

$$[-\frac{\hbar^2 \nabla_R^2}{2\mu_{pt}} + U_{ct}(R_c) - (E - \epsilon_0)]\Psi_{K_0}^{sp}(\mathbf{r}, \mathbf{R}) = 0, \quad (10.20)$$

with $\mathbf{R} = \mathbf{R}_c + \frac{m_v}{m_p} \mathbf{r}$. This equation can be solved easily by shifting position,

$$\begin{aligned}\Psi_{K_0}^{sp}(\mathbf{r}, \mathbf{R}) &= \phi_0(\mathbf{r}) e^{i \frac{m_v}{m_p} K_0 \cdot \mathbf{r}} \chi_{K_0}(R_c), \\ \left[-\frac{\hbar^2 \nabla_{R_c}^2}{2\mu_{pt}} + U_{ct}(R_c) - (E - \epsilon_0) \right] \chi_{K_0}(R_c) &= 0,\end{aligned}\quad (10.21)$$

$\Phi_K(\mathbf{R}, \mathbf{r})$ 의 asymptotic form을 이용하면, $f(\theta, \phi, \mathbf{r})$ 을 얻을 수 있고, 이를 특정 state에 project하는 것으로 transition amplitude를 구할 수 있다.

$$f_{i0} = \int d\mathbf{r} \phi_i^*(\mathbf{r}) f(\theta, \phi, \mathbf{r}) \phi_0(\mathbf{r}). \quad (10.22)$$

비슷하게, adiabatic wave function을 각 channel의 합으로 생각하면,

$$\Phi(\mathbf{R}, \mathbf{r}) \simeq \sum_i \phi_i(\mathbf{r}) \chi_i(\mathbf{R}) \quad (10.23)$$

각 channel wave function $\chi_i(R)$ 이 만족시켜야 하는 식은

$$\begin{aligned}[\hat{T}_R + \epsilon_0 - E_K] \chi_i(\mathbf{R}) &= - \sum_{j \neq i} \int d\mathbf{r} \phi_i^*(\mathbf{r}) [U_{ct}(\mathbf{r}, \mathbf{R}) + U_{vt}(\mathbf{r}, \mathbf{R})] \phi_j(\mathbf{r}) \chi_j(\mathbf{R}) \\ &= - \sum_{j \neq i} V_{ij} \chi_j(\mathbf{R})\end{aligned}\quad (10.24)$$

마치 coupled channel equation에서 channel energy를 모두 같게 두는 것과 같아지게 된다. 일반적으로 Adiabatic wave function $\Phi(\mathbf{R}, \mathbf{r})$ 은 coupled equation의 solution이 된다.

다만, 경우에 따라, 식이 간단해 지게 되기도 한다. 예를 들어 $\mathbf{r} = 0$ 일때의 adiabatic wave function 만 필요한 경우에는 $V_{ij}(R) = \delta_{ij} [V_{ct}(\mathbf{R}) + V_{vt}(\mathbf{R})]$ 이 되어 un-coupled equation 으로 바뀐다.

10.3 Eikonal methods

만약 에너지가 충분히 높다면, 입자의 trajectory는 straight line에서 많이 벗어나지 않을 것이다. 이러한 가정에 따라, 입자의 wave function도 plane wave 와 비슷할 것으로 생각할 수 있다. structureless point particle의 wave function은

$$\Psi(\mathbf{R}) = e^{ikz} \phi(\mathbf{b}, z), \quad (10.25)$$

로 쓰고, $\phi(\mathbf{b}, z)$ 가 만족하는 식은 $[\hat{T}_R + V(\mathbf{R})]\Psi = E\Psi$ 로 부터

$$\begin{aligned}\hat{T}_R &= -\frac{\hbar^2}{2\mu} \left[\nabla_b^2 + \frac{\partial}{\partial z} \right], \\ 2ike^{ikz} \frac{\partial \phi}{\partial z} - \frac{2\mu}{\hbar^2} V(\mathbf{R}) e^{ikz} \phi + \underbrace{e^{ikz} \frac{\partial^2 \phi}{\partial z^2} + e^{ikz} \nabla_b^2 \phi}_{\text{ignore}} &= 0, \\ \rightarrow \frac{\partial \phi}{\partial z} &= -\frac{i}{\hbar v_p} V(\mathbf{R}) \phi, \quad \hbar k = \mu v_p, \\ \rightarrow \phi(\mathbf{b}, z) &= \exp \left[-\frac{i}{\hbar v_p} \int_{-\infty}^z V(\mathbf{b}, z') dz' \right]\end{aligned}\quad (10.26)$$

If we define eikonal phase $\chi(\mathbf{b}, z)$,

$$\chi(\mathbf{b}, z) = -\frac{i}{\hbar v_p} \int_{-\infty}^z V(\mathbf{b}, z') dz' \quad (10.27)$$

The solution becomes simply,

$$\Psi(\mathbf{R}) = e^{i(kz + \chi(\mathbf{b}, z))} \quad (10.28)$$

This is the eikonal wave function and it only requires one integration of potential. In case of long range Coulomb interaction, we approximate $\chi(\mathbf{b}, z) = \chi_c(\mathbf{b}, z) + \chi_n(\mathbf{b}, z)$ with $\chi_c(\mathbf{b}_c > R_{Coul}, z) = 2\eta_i \ln k_i b_c$. 그러면, scattering cross section은? 이미 eikonal wave function이 있으므로, 이를 이용하자. 일반적인 경우로 부터,

$$\begin{aligned} f(\theta) &= -\frac{\mu}{2\pi\hbar^2} \langle \mathbf{k}' | V | \mathbf{k} \rangle^{(+)} \rightarrow -\frac{\mu}{2\pi\hbar^2} \int d\mathbf{R} e^{-i\mathbf{k}' \cdot \mathbf{R}} V(\mathbf{R}) \Psi_{\mathbf{k}}(\mathbf{R}), \\ &= -\frac{\mu}{2\pi\hbar^2} \int d^2b e^{i\mathbf{q} \cdot \mathbf{b}} \int dz e^{i\mathbf{q} \cdot \hat{n} z} V(\mathbf{b}, z) \exp \left[-\frac{i}{\hbar v_p} \int_{-\infty}^z V(\mathbf{b}, z') dz' \right]. \end{aligned} \quad (10.29)$$

with $\mathbf{q} = \mathbf{k} - \mathbf{k}'$, $\mathbf{R} = z\hat{n} + \mathbf{b}$.

If $q \ll K$, $\mathbf{q} \cdot \hat{n} \simeq 0$, and $q = K \sin \frac{\theta}{2}$,¹

$$\begin{aligned} f(\theta) &= -\frac{iK_0}{2\pi} \int d^2b e^{i\mathbf{q} \cdot \mathbf{b}} [e^{i\chi(\mathbf{b}, \infty)} - 1], \\ &= \frac{K_0}{i} \int_0^\infty db b J_0(2K_0 b \sin(\frac{\theta}{2})) [e^{i\chi(\mathbf{b}, \infty)} - 1]. \end{aligned} \quad (10.31)$$

Be careful that J_0 is a Bessel function, not spherical Bessel function.

We may define eikonal S, with $\chi(\mathbf{b}) = \chi(\mathbf{b}, \infty)$,

$$S^{eik}(\mathbf{b}) = e^{i\chi(\mathbf{b})} \quad (10.32)$$

as a kind of S-matrix. In point particle, there is only one \mathbf{b} exists.

Or, we may relate this in terms of cross section. for a straight line path approximation, initial wave function is $\phi(b, z) = e^{ikz}$ and scattering wave function is $\psi(b, z \rightarrow \infty) = S^{eik}(b)e^{ikz}$. Then, the flux within range db around b becomes

$$\begin{aligned} j_{inc}(b) &= \frac{1}{2i\mu} (\phi^* \partial_z \phi - \phi \partial_z \phi^*) = \frac{k}{\mu}, \\ j_{out}(b) &= |S^{eik}(b)|^2 \frac{k}{\mu}. \end{aligned} \quad (10.33)$$

Thus, we may think the differential cross section and total cross section as

$$\frac{d\sigma}{d^2b} = |S^{eik}(b)|^2, \quad \sigma_{el} = \int d^2b |S^{eik}(b)|^2 \quad (10.34)$$

¹By the integral form of Bessel function,

$$J_0(z) = \frac{1}{\pi} \int_0^\pi d\phi e^{iz \cos \phi}. \quad (10.30)$$

If the projectile is composite, with internal wave function $\Phi(\mathbf{r})$, we use adiabatic approximation for the internal motion. (In other words, use 'freezing' approximation of internal motion and only consider eikonal phase change.) We may compute adiabatic eikonal wave function for each 'frozen' internal coordinate \mathbf{r} and average over internal wave function $|\Phi(\mathbf{r})|^2$. So that general transition amplitude in eikonal method is

$$f_{fi}(\theta) = -\frac{iK_0}{2\pi} \int d^2b e^{i\mathbf{q}\cdot\mathbf{b}} \langle \Phi_f(\mathbf{r}) | e^{i\chi(\mathbf{b}-\mathbf{b}_r)} - 1 | \Phi_i(\mathbf{r}) \rangle \quad (10.35)$$

with \mathbf{b} is projectile의 c.m.에 대한 impact parameter이고, \mathbf{b}_r 은 projection of internal coordinates.

we need to convolute the above expression for each pair. Let us consider projectile is composite, ($p = c + v$), and the impact parameters for each are $\mathbf{b} = \mathbf{b}_p$ and $\mathbf{b}_c, \mathbf{b}_v$ from the target. Suppose projectile internal wave function is $\Phi(\mathbf{r})$ with \mathbf{r} is a vector from core to valence particle.

10.4 Eikonal Knockout code : momdis

The reaction $A(b, c)X$ where projectile is a bound state of core and valence nucleon $b = c + n$ and one only measures residual core particles momentum and energy. (Thus, it can be $n + A$ scattering or $n' + A^*$ inelastic. One have to sum the cross section of these two cases.)

In MOMDIS code, this cross section is calculated by using Eikonal approximation. In other words, the wave function of projectile is approximated as

$$\Psi(\mathbf{r}, \mathbf{R}) = S_n(\mathbf{b}_n) S_c(\mathbf{b}_c) \phi_0(\mathbf{r}) \quad (10.36)$$

where \mathbf{r} is a relative coordinate between nucleon and core, \mathbf{R} is a center of mass of projectile. $\mathbf{b}_{n,c}$ is a impact parameter of nucleon/core. Eikonal phase change or profile function or S-matrix is calculated by

$$S(\mathbf{b}) = \exp \left[-\frac{i}{\hbar v} \int dz V(\mathbf{b} + z\hat{z}) \right] \quad (10.37)$$

And scattered wave function is

$$\Psi_{scat} = (S_n S_c - 1) \phi_0 \quad (10.38)$$

The elastic and diffractive scattering are calculated by taking overlaps of Ψ_{scat} with different final states.

- MOMDIS use $t - \rho - \rho$ approximation of Glauber theory for eikonal phase for nuclear interaction.

$$S(b) = \exp[i\chi(b)], \quad \text{with} \quad \chi_N(b) = \frac{1}{k_{NN}} \int_0^\infty dq q \rho_p(q) \rho_t(q) f_{NN}(q) J_0(qb) \quad (10.39)$$

with pre-defined $f_{NN}(q)$ NN-scattering amplitudes.

On the other hand, eikonal phase for Coulomb interaction is obtained with analytic form either with Rutherford scattering amplitude or modified by charge distribution.

- Elastic scattering case take overlap with ground state but with some arbitrary transverse momentum.

$$a(\mathbf{K}_\perp) = \int d^2\mathbf{R}_\perp e^{-i\mathbf{K}_\perp \cdot \mathbf{R}} \int d^3\mathbf{r} \phi_0^*(\mathbf{r})(S_c S_n - 1)\phi_0(\mathbf{r}) \quad (10.40)$$

And

$$\frac{d\sigma_{el}}{d^2\mathbf{K}_\perp} = \frac{|a(\mathbf{K}_\perp)|^2}{(2\pi)^2} \quad (10.41)$$

$$\begin{aligned} \sigma_{el} &= \int d^2\mathbf{R}_\perp \left| \int d^3\mathbf{r} \phi_0^*(\mathbf{r})(S_c S_n - 1)\phi_0(\mathbf{r}) \right|^2 \\ &= \frac{1}{2L_0 + 1} \sum_{M_0 M'_0} \int d^2\mathbf{R}_\perp \left| \int d^3\mathbf{r} \phi_{0,M'_0}^*(\mathbf{r})(S_c S_n - 1)\phi_{0,M_0}(\mathbf{r}) \right|^2 \end{aligned} \quad (10.42)$$

where the second line is for the case projectile has spin.

- For Diffractive breakup, the final state depends on the relative momentum \mathbf{k} of nucleon and core in their center of mass frame as well as on the transverse momentum \mathbf{K}_\perp of the center of mass.

$$\frac{d\sigma_{diff}}{d^2\mathbf{K}_\perp d^3k} = \frac{1}{(2\pi)^5} \frac{1}{2L_0 + 1} \sum_{M_0} \left| \int d^3r d^2\mathbf{R}_\perp e^{-i\mathbf{K}_\perp \cdot \mathbf{R}_\perp} \phi_{\mathbf{k}}^*(\mathbf{r}) S_c S_n \phi_{0,M_0}(\mathbf{r}) \right|^2 \quad (10.43)$$

where $\phi_{\mathbf{k}}(\mathbf{r})$ is a continuum nucleon-core wave function. After Integration

$$\frac{d\sigma_{diff}}{d^3k} = \frac{1}{(2\pi)^3} \frac{1}{2L_0 + 1} \sum_{M_0} \int d^2\mathbf{R}_\perp \left| \int d^3r \phi_{\mathbf{k}}^*(\mathbf{r}) S_c S_n \phi_{0,M_0}(\mathbf{r}) \right|^2 \quad (10.44)$$

if we consider only bound state

$$\sigma_{diff} = \frac{1}{2L_0 + 1} \sum_{M_0} \int d^2\mathbf{R}_\perp \left[\int d^3r \phi_{0,M_0}^*(\mathbf{r}) |S_c S_n|^2 \phi_{0,M_0}(\mathbf{r}) - \sum_{M'_0} \left| \int d^3r \phi_{0,M'_0}^*(\mathbf{r}) S_c S_n \phi_{0,M_0}(\mathbf{r}) \right|^2 \right] \quad (10.45)$$

A similar expression in MOMDIS manual is

$$\sigma_{el,bup} = \sum_{\mathbf{k}} |\langle \phi_{\mathbf{k}} | \hat{S}_b \hat{S}_c | \phi_0 \rangle|^2 \quad (10.46)$$

which can be re-written as

$$\sigma_{el,bup} = \langle \phi_0 | |\hat{S}_b|^2 |\hat{S}_c|^2 | \phi_0 \rangle - |\langle \phi_0 | \hat{S}_b \hat{S}_c | \phi_0 \rangle|^2 \quad (10.47)$$

from

$$\int d\mathbf{k} |\phi_{\mathbf{k}} \rangle \langle \phi_{\mathbf{k}}| + |\phi_0 \rangle \langle \phi_0| = 1. \quad (10.48)$$

- other absorption cross section are nucleon stripping (nucleon is absorbed by the target), core stripping, both nucleon and core stripping are written as

$$\frac{d\sigma_{n,str}}{d^3k_c} = \frac{1}{(2\pi)^3} \frac{1}{2L_0 + 1} \sum_{M_0} \int d^2b_n [1 - |S_n(b_n)|^2] \left| \int d^3r e^{-i\mathbf{k}_c \cdot \mathbf{r}} S_c(\mathbf{b}_c) \phi_{0,M_0}(\mathbf{r}) \right|^2 \quad (10.49)$$

$$\sigma_{n,str} = \frac{1}{(2\pi)^3} \frac{1}{2L_0 + 1} \sum_{M_0} \int d^2b_n [1 - |S_n(b_n)|^2] \int d^3r \phi_{0,M_0}^*(\mathbf{r}) |S_c(\mathbf{b}_c)|^2 \phi_{0,M_0}(\mathbf{r}) \quad (10.50)$$

$$\begin{aligned} \frac{d\sigma_{c,str}}{d^3k_c} &= \frac{d\sigma_{n \rightarrow c, str}}{d^3k_c} \\ \sigma_{c, str} &= \sigma_{n \rightarrow c, str} \end{aligned} \quad (10.51)$$

$$\sigma_{abs} = \frac{1}{2L_0 + 1} \sum_{M_0} \int d^2b_c [1 - |S_c(b_c)|^2] \int d^3r \phi_{0,M_0}^*(\mathbf{r}) [1 - |S_n(b_n)|^2] \phi_{0,M_0}(\mathbf{r}) \quad (10.52)$$

A similar expression in MOMDIS manual is

$$\sigma_{-v} = \sigma_{in, bup} = \frac{\pi}{k^2} \sum \langle \phi_0 | |S_c|^2 (1 - |S_v|^2) | \phi_0 \rangle \quad (10.53)$$

$$\sigma_{-c} = \frac{\pi}{k^2} \sum \langle \phi_0 | |S_v|^2 (1 - |S_c|^2) | \phi_0 \rangle \quad (10.54)$$

To compute stripping cross section in MOMDIS, one provide bound state wave function $\psi_{lm}(r)$ (assume no spin dependence of the reaction) , $S_n(b_n)$, $S_c(b_c)$.

Calculation of bound state wave requires binding potential and $nljm$ of nucleon state as input.

Calculation of S_n and S_c in $t - \rho - \rho$ approximation requires projectile and target density distribution as input.

Calculation of cross section, $\frac{d\sigma_{str}}{dk_z}$, $\frac{d\sigma_{str}}{d^2k_{\perp}}$, $\frac{d\sigma_{str}}{dk_y}$, $\frac{d\sigma_{str}}{k_{\perp} dk_{\perp} dk_z}$ are obtained by using integration of $\frac{d\sigma_{str}}{d^3k_c}$.

Chapter 11

Fusion and Fission

In fact, it seems like there is no microscopic theory of fusion. The description of fusion usually use imaginary part of optical potential (as an absorption or disappearance of flux from elastic channel) or strong absorption model (as the fusion occurs when the distance between two nuclei become smaller than a critical value).

Absorption cross section is defined as the loss of flux from the imaginary optical potential.

$$\sigma_A = \frac{\pi}{k^2} \sum_l (2l+1)(1 - |S_l|^2) = \frac{\pi}{k^2} \sum_l (2l+1)P_l(E) \quad (11.1)$$

where $P_l(E)$ is a reaction probability or transmission(penetration) probability.

$$P_l(E) = \frac{1}{|A|^2} \frac{4k}{E} \int dr W(r) |u_l(k, r)|^2, \quad (11.2)$$

where $|A|^2$ is a normalization factor for $u_l(k, r)$.

In case of coupled channel method with complex optical potential, there are additional reaction cross section other than absorption or fusion. Thus,

$$\sigma_R = \sigma_F + \sum_{\alpha \neq 0} \sigma_{\alpha}^{non-elastic} \quad (11.3)$$

And the fusion cross section can be written as a diagonal component of imaginary potential,

$$\begin{aligned} \sigma_F &= \sum_{\alpha} \sigma_F^{(\alpha)}, \\ \sigma_F^{(\alpha)} &= \frac{1}{|A|^2} \frac{k_0}{E} \langle \psi_{\alpha}^{(+)}(k_0) | W_{\alpha\alpha} | \psi_{\alpha}^{(+)}(k_0) \rangle. \end{aligned} \quad (11.4)$$

In case of strong absorption assumption, the penetration probability is calculated from the wave function at the critical radius. In this case, no imaginary potential is necessary and Penetration describes the loss of flux. CCFULL code basically use incoming wave boundary condition. (Without imaginary potential, usual S-matrix becomes unitary. Thus, first compute penetration probability and match the S-matrix for loss of flux.)

11.1 Incoming Wave Boundary condition: CCFULL case

CCFULL use no-Coriolis approximation or isocentrifugal approximation. (use J instead of L for centrifugal term)

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{J(J+1)\hbar^2}{2\mu r^2} + V_N^{(0)}(r) + \frac{Z_p Z_T e^2}{r} + \epsilon_n - E \right] \psi_n(r) + \sum_m V_{nm}(r) \psi_m(r) = 0 \quad (11.5)$$

where ψ_n is channel wave function, ϵ_n is a excitation energy of channel n, $V_N^{(0)}(r)$ is a real nuclear potential, V_{nm} is a coupling matrix element between different channel.

CCFULL solves $\psi_n(r)$ from the r_{min} (minimum position of Coulomb pocket) and use boundary condition,¹

$$\begin{aligned} \psi_n(r) &\rightarrow \left(\sqrt{\frac{k}{k_n(r)}} \right) T_n \exp \left(i \int_r^{r_{min}} k_n(r') dr' \right), \quad r \leq r_{min}, \\ &\rightarrow c \left(H_J^{(-)}(k_n r) \delta_{n,0} + R_n H_J^{(+)}(k_n r) \right), \quad r \geq r_{max}, \end{aligned} \quad (11.8)$$

where^{2 3}

$$\begin{aligned} k_n(r) &= \sqrt{\frac{2\mu}{\hbar^2} \left(E - \epsilon_n - \frac{J(J+1)\hbar^2}{2\mu r^2} - V_N^{(0)}(r) - \frac{Z_p Z_T e^2}{r} - V_{nn}(r) \right)}, \\ k_n &= \lim_{r \rightarrow \infty} k_n(r) \rightarrow \sqrt{\frac{2\mu(E - \epsilon_n)}{\hbar^2}} \end{aligned} \quad (11.9)$$

Numerical solution from r_{min} to r_{max} becomes,

$$\begin{aligned} \psi_m(r) &= \sum_n T_n \chi_{nm}(r), \\ \chi_{nm}(r) &\rightarrow C_{nm} H_J^{(-)}(k_m r) + D_{nm} H_J^{(+)}(k_m r), \quad r \rightarrow r_{max}. \end{aligned} \quad (11.10)$$

Matching with boundary condition fixes $T_n = (C^{-1})_{n0}$ (It must satisfy $\sum_n T_n C_{nm} = \delta_{m,0}$) and correspondingly determine S_n . Then, penetrability is

$$P_J(E) = \sum_n \frac{k_n(r_{min})}{k_0} |T_n|^2 \quad (11.11)$$

¹For a single channel case, the asymptotic form of IWBC becomes

$$\begin{aligned} u_l(r) &= \left(\sqrt{\frac{k}{k_l(r)}} \right) T_l \exp \left(i \int_r^{r_{min}} k_l(r') dr' \right), \\ \frac{d}{dr} u_l(r) &= u_l(r) \left[-ik_l - \frac{1}{2} k_l^{-1} k_l' \right], \\ \frac{d^2}{dr^2} u_l(r) &= u_l(r) \left[-\frac{1}{2} k_l k_l'' + \frac{3}{4} k_l^{-2} (k_l')^2 - k_l^2 \right], \\ \text{Thus, } \frac{-\hbar^2}{2\mu} \left(\frac{d}{dr^2} + k_l(r)^2 \right) u_l(r) &\rightarrow 0, \quad \text{if } k_l'(r) \rightarrow 0 \text{ for } r \leq r_{min}. \end{aligned} \quad (11.6)$$

Also, the radial flux is

$$\frac{1}{2ik_0} (u^* \frac{d}{dr} u - (\frac{d}{dr} u^*) u) = -\frac{k}{k_0} |T_l|^2 \quad (11.7)$$

²the square root factor in front of T_n should be checked.

³This is incoming wave boundary condition so that the incoming flux never returns back

즉, coupled channel method에서 fusion cross section은 각 channel에서의 fusion(= absorption) cross section을 모두 합한 것이다.

And,

$$\begin{aligned}\sigma_F(E) &= \sum_J \sigma_J(E) = \frac{\pi}{k_0^2} \sum_J (2J+1) P_J(E), \\ \langle l \rangle &= \frac{\sum_J J \sigma_J(E)}{\sum_J \sigma_J(E)}.\end{aligned}\quad (11.12)$$

Summation over J is truncated when angular momentum contribution is smaller than 10^{-4} of total cross section.

11.1.1 Semi-classical (above barrier)

Suppose projectile follows classical trajectory and there is a total absorption/fusion if impact parameter is smaller than grazing distance $b_g = l_g/k$.

$$\sigma^{cl} = \pi b_g^2. \quad (11.13)$$

Because

$$V_b + \frac{(kb_g)^2}{2\mu R_b^2} = E \quad (11.14)$$

we can re-express this cross section as

$$\sigma_{fus}^{cl}(E) = \pi R_b^2 \left(1 - \frac{V_b}{E}\right) \quad (11.15)$$

because this case cross section is purely absorption. (no elastic scattering.)

This simple model can be improved with WKB approximation. (In other words, fusion occurs by quantum tunneling.)

11.1.2 Parabolic Approximation (tunneling below barrier)

From strong absorption assumption, probability of fusion is probability to access r_{touch} . This probability can be calculated as penetrability of barrier.

The penetrability probability may be computed in WKB approximation with out solving numerical problem. Then,

$$P(E) = \exp \left[-2 \int_{r_{in}}^{r_{out}} dx' \sqrt{\frac{2\mu}{\hbar^2} (V(x') - E)} \right], \quad (11.16)$$

where $r_{in,out}$ is defined as $V(r_{in}) = V(r_{out}) = E$ (i.e. integration is within the barrier). Improved form is

$$P(E) = \frac{1}{1 + \exp \left[2 \int_{r_{in}}^{r_{out}} dx' \sqrt{\frac{2\mu}{\hbar^2} (V(x') - E)} \right]}, \quad (11.17)$$

Important quantities are barrier position and barrier height.

$$\left. \frac{dV}{dr} \right|_{R=R_B} = 0. \quad (11.18)$$

If we use $V_N = V_0 \exp[-(r - R_B)/a]$ and $V_c = \frac{Z_1 Z_2 e^2}{r}$, barrier height B and barrier position becomes

$$\begin{aligned} B &= V(R_B) = \frac{Z_1 Z_2 e^2}{R_B^2} \left(1 - \frac{a}{R_B}\right), \\ R_B &= \frac{1}{2} R_C \left(1 + \sqrt{1 - 4 \frac{a}{R_C}}\right), \quad R_C = \frac{Z_1 Z_2 e^2}{B}. \end{aligned} \quad (11.19)$$

If we approximate the barrier as parabolic,

$$V(r) = B + \frac{l(l+1)\hbar^2}{2\mu R_B^2} - \frac{1}{2} \mu \omega_B^2 (r - R_B)^2,$$

and sum over angular momentum (by using integral replacement, $\frac{\pi}{k^2} \sum_l (2l+1) P_l(E) \rightarrow \frac{\pi}{k^2} \int_0^\infty dl (2l+1) P_l(E)$) we get Wong's formula.⁴

$$\sigma_F = \frac{\hbar \omega_B}{2E} R_B^2 \ln \left(1 + \exp \left[\frac{2\pi}{\hbar \omega_B} (E - B) \right] \right) \quad (11.21)$$

Since this form is from the barrier position and energy for $l = 0$, we may use those at grazing angular momentum l_g for energy dependence, such that

$$\sigma_F = \frac{\hbar \omega_E}{2E} R_E^2 \ln \left(1 + \exp \left[\frac{2\pi}{\hbar \omega_E} (E - V_E) \right] \right), \quad \frac{l_g(l_g+1)\hbar^2}{2\mu R_E^2} = E - V_E, \quad (11.22)$$

Note that Wong's formula becomes a classical one in the limit of $E \gg V_b$.

Note that potential model reproduces the experimental data reasonably well for $E > V_b$. But, underpredict σ_{fus} for $E < V_b$. (즉 낮은 에너지에서 fusion이 이론값보다 더 잘 일어남.)

Definition of orbital angular momentum:

- grazing angular momentum: Defined by the barrier height. $V_B(l_g) = E$. $l \leq l_g$ is the condition to penetrate the barrier classically. If $l > l_g$, they are classically forbidden and only Coulomb interaction dominates.
- critical angular momentum: Defined by the Coulomb pocket. At $l = l_c$, the Coulomb pocket vanishes. Thus, for $l > l_c$, they cannot make resonance or fusion.

Fusion is possible for l is smaller than both l_g and l_c .

Thus, if $l_g < l_c$, $l < l_c$ can have fusion and $l \simeq l_g$ have direct reactions. $l > l_g$ are dominated by Coulomb.

If $l_g > l_c$ case, $l < l_c$ can have CN fusion and $l \simeq l_g$ will have direct nuclear reaction. However, for $l_c < l < l_g$, it cannot make fusion but it would be also not a direct reaction. Thus, it becomes Deep inelastic.

⁴In case $E - B \gg \hbar \omega_B / (2\pi)$, we get classical fusion cross section,

$$\sigma_{fus}(E) \simeq \pi R_B^2 \left(1 - \frac{B}{E}\right). \quad (11.20)$$

11.2 Simple fission Liquid drop model

Suppose the spherical compound nucleus is formed and immediately decay into two g.s. spherical nuclei which is in contact. In this case, the interaction between two daughter nuclei is dominated by Coulomb because the nuclear overlap is small and nuclear interaction is short ranged. Then, let us consider some energy supply for the fission as \tilde{B} . Then energy conservation between two cases is

$$\epsilon(A, Z) + \tilde{B} = \epsilon(A_1, Z_1) + \epsilon(A_2, Z_2) + V_C(R_C) \quad (11.23)$$

where ϵ is internal energy of a nucleus. The ground state energy can be estimated from liquid drop model. But note that compound nucleus is usually an excited state rather than ground state. If $\tilde{B} < 0$, spontaneous fission can occur. One can estimate \tilde{B} as a function of asymmetry ratio $\alpha = A_1/A$ by using mass formula and simple formula for R_C and Z_i values as a function of mass. This shows that

- For $A < 120$, it is preferred that $\alpha = 0$ or 1 is preferred. In other words, evaporation of light particle is preferred.
- For $120 < A < 300$, symmetric $\alpha = 0.5$ is preferred. So, fission is preferred.
- For $A > 300$, even the ground state of nucleus have $\tilde{B} < 0$. Thus, the formation of compound nucleus is difficult and fast fission to di-nuclear system occurs.

$\tilde{B}(\alpha)$ can be considered as driving potential of fission. One can also think \tilde{B} as a barrier for tunneling. Thus, \tilde{B} is important factor for the stability of compound nucleus.

Above is just a simple model and more detailed modification (like deformation of nucleus and stability against separation...) is necessary for actual case.

Chapter 12

FRESCO Examples

FRESCO use &PARTITION for the information of channels, &POT for the information of potentials, &OVERLAP for the information of overlap wave functions, and &Coupling for the construction of couplings between different channels. However, information for the deformation couplings are in the &POT and CFP of overlap are in &Coupling.

It is better to keep in mind the original coupled channel equations to be solved to understand the meaning of input values.

These examples are gathered from Book (B), Fresco Web page (T), and Moro's lecture (M).

12.1 Elastic Scattering :(M) $4\text{He}+^{58}\text{Ni}$

```
4He + 58Ni elastic scattering Ecm=10 MeV
NAMELIST
&FRESCO hcm=0.1 rmatch=25.0 jtmax=30
        thmin=1.00 thmax=-180.00 thinc=2.00
        smats=2 xstabl=1
        elab = 25.0 /
&PARTITION namep='ALPHA' massp=4 zp=2
            namet='58Ni' masst=58 zt=28 nex=1 /
&STATES jp=0.0 bandp=1 ep=0.0 cpot=1
        jt=0.0 bandt=1 et=0.0 /
&partition /
&POT kp=1 ap=4 at=58 rc=1.4 /
&POT kp=1 type=1
        p1=191.5 p2=1.37 p3=0.56 p4=23.5 p5=1.37 p6=0.56 /
&pot /
&overlap /
&coupling /
```

- This example from Moro's note and Moro's lecture at TALENT. Elastic scattering $4\text{He} + ^{58}\text{Ni}$. No excitation of projectile or target are considered. The Book's example for $p + ^{78}\text{Ni}$ is similar.
- $\text{thmax} > 0$ gives the elastic cross section as the ratio to Rutherford. ($\text{thmax} < 0$ for absolute value.)

- ap and at is only used for the estimation of radius parameter of potentials

$$R = r(ap^{\frac{1}{3}} + at^{\frac{1}{3}}) \quad (12.1)$$

In light projectile, usually ap=0 is taken. For heavy projectile, both are necessary.

12.1.1 test potential READ IN

FRESCO has options to get optical potential as a numeric table. However, it is said that the potential should start by defining TYPE=0 Coulomb potential. What if the numerical table of potential already included Coulomb interaction, $U = V_c + U_N$? To clarify, let us test. one can obtain numerical table of potentials using option 'TRENEG ≥ 3 ' in Fresco.

Be careful that the format of potential in 'fort.4' file is (**no list of r**).

```
comments
NPOINTS  RSTEP  RFIRST
V(1)  W(1)
V(2)  W(2)
...
```

(2) Replacing the nuclear interaction part U_N with numerical table gives the same result as expected. (3) However, if I do not provide TYPE=0, error occurs. Thus, only possible way to use read in if the potential $U = V_c + U_N$, is **to add V_{c0} to Fresco and subtract it from U** , $U - V_{c0} = V_c - V_{c0} + U_N$ where V_{c0} does not necessarily the same as original V_c .

It seems there should be a Coulomb interaction always between charged particles.

12.2 Inelastic scattering(Rotor Model): (B) $^{12}\text{C}(\alpha, \alpha)^{12}\text{C}(2^+)$

```
alpha+c12 -> alpha+c12* @ 100 MeV; nuc def
NAMELIST
&FRESCO hcm=0.05 rmatch=20.0
        jtmin=0.0 jtmax=40 absend= 0.01
        thmin=0.00 thmax=180.00 thinc=1.00
        iter=1 ips=0.0 iblock=0 chans=1 smats=2 xstabl=1
        elab(1)=100.0 /

&PARTITION namep='alpha' massp=4.0000 zp=2
            namet='12C'   masst=12.000 zt=6 qval=0.0 nex=2 /
&STATES jp=0. bandp=1 ep=0.0000 cpot=1
        jt=0.0 bandt=1 et=0.00 /
&STATES copyp=1          cpot=1
        jt=2.0 bandt=1 et=4.43 / ! target excited state
&partition /

&POT kp=1 ap=4.000 at=12.000 rc=1.2 / ! central coulomb
&POT kp=1 type=1          ! 12C-alpha central nuclear volume term
        p1=40.0 p2=1.2 p3=0.65 p4=10.0 p5=1.2 p6=0.500 /
&POT kp=1 type=11        p2=1.3 / ! deform of target in volume term
&POT kp=1 type=2          ! 12C-alpha central derivative term
        p1=0.00 p2=1.2 p3=0.65 p4=6.0 p5=1.2 p6=0.500 /
&POT kp=1 type=11        p2=1.3 / ! deform of target in derivative term
```

```
&pot /
&overlap /
&coupling /
```

- This example from the book. inelastic scattering $^{12}\text{C}(\alpha, \alpha')^{12}\text{C}^*(2^+)$ in collective rotational model.
- Inelastic scattering by rotational model의 경우, 파티션은 하나이지만, excited state pair는 $^{12}\text{C} + \alpha$ 와 $^{12}\text{C}(2^+; 4.438) + \alpha$ 로 2개가 된다. 한편, 이 때 state는 copy 또는 copyt를 이용하여 정의해야만 inelastic process가 계산된다.(테스트 결과 zero 가 나옴. 따라서, 반드시 copy or copyt 를 이용해야함.)
- Fresco cannot handle both target and projectile excitation at the same time in rotor model.
- Rotor model의 경우 deformation 의 정보를 &POT에서 주게 되는데, TYPE=10,11의 경우 Coulomb deformation의 경우에는 $P(k) = Mn(Ek)$ 를, Nuclear deformation의 경우는 deformation length $P(k) = DEF(k) = \delta_k$ 를 input으로 주면 된다. 단, 이 때 deformation 이 있는 모든 potential부분에 대해 TYPE = 10, 11을 밑에 추가해 주어야 한다. (예에서는 Coulomb deformation 없이 Nuclear Volume과 Surface potential에만 deformation이 적용되었다.)
- 즉, Nuclear deformation의 경우, **deformation length** δ_q which is defined as d_{q0} in the following expression,

$$\tilde{R}(\theta', \phi') = R_0 + \sum_{q=2, q_{max}} \sum_{\mu=-q, q} d_{q\mu} Y_{q\mu}(\theta', \phi') \quad (12.2)$$

Traditionally, fractional deformation $\beta_q = \delta_q / R_0$ are used. (Problem of β is that what value of R_0 should be used is not clear.)

Or sometimes, **reduced reformation length** are defined to represent the transition strength.

$$RDEF(k, I \rightarrow I') = \delta_k \times (-1)^{(I-I'+|I-I'|)/2} \sqrt{2I+1} \langle IKk0 | I'K \rangle \quad (12.3)$$

- Coulomb deformation의 경우, transition strength는 **Coulomb reduced matrix element**, $\langle I_f || E\lambda || I_i \rangle$ 를 이용하여 나타낼 수 있다.

$$\begin{aligned} \langle I_f || E\lambda || I_i \rangle &= \langle I_f || Y_{\lambda\mu}(\mathbf{r}) r^\lambda \rho_q(r) || I_i \rangle \\ &= M_n(E\lambda; I \rightarrow I') \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle, \end{aligned} \quad (12.4)$$

where **Coulomb intrinsic reduced matrix element** $M_n(E\lambda)$ is defined. In Fresco, Coulomb deformation input can be $p(k) = M_n(Ek)$.

rotor model을 사용하는 경우에는,

$$\langle I_f || E\lambda || I_i \rangle = \frac{3Z_p \delta_\lambda R_c^{\lambda-1}}{4\pi} \hat{I}_i \langle I_i K, \lambda 0 | I_f K \rangle \quad (12.5)$$

를 이용하여 얻을 수 있다. (이때, reduced matrix element는 Alder and Winther convention을 따른다. $\langle I || E\lambda || I' \rangle = \langle I' || E\lambda || I \rangle^*$.)

또는 STR value를 이용할 때는, $M_n(Ek)$ 대신, reduced matrix element를 바로 사용하지만, real-value를 얻기 위해서, 여기에 phase factor를 더 곱한 $STR = M(E\lambda; I \rightarrow I')$ 를 사용한다.

$$M(Ek) \quad (12.6)$$

- 주의할 것은 TYPE에 따라서 input value P(k) 의 의미가 달라진다는 것이다. 만약 deformation length를 주는 대신, 다른 model을 사용하고 싶은 경우에는 namelist &STEP 으로 coupling matrix를 input으로 줄 수 있다. TYPE=12,13의 경우에는 &STEP 을 통해 matrix element를 직접 입력한다. (Coupling from state 'ib' to state 'ia', multipolarity 'k' and strength 'str') 이 때의 STR은 Mn(EK)나 DEF(k)가 아니라 reduced matrix element M(Ek) for Coulomb transitions, reduced deformation length RDEF(k) for nuclear transitions.

Potential	TYPE	P(k)	STR	comment
Coulomb, rotational	10, 11	Mn(Ek)	-	unit $e.f m^k$
Coulomb, general	12,13	Mn(Ek)	$M(Ek) = (-1)^{\frac{I-I'+ I-I' }{2}} \langle I' Ek I \rangle ?$	
Nuclear, rotational	10, 11	$DEF(k) = R\beta_k$	-	
Nuclear, general	12,13	$DEF(k) = R\beta_k$	$RDEF(k) = (-1)^{\frac{I-I'+ I-I' }{2}} \langle I' \delta_k I \rangle ?$	

- Coulomb deformation의 정보는 실험값 $B(Ek; I \rightarrow I')$ 으로 부터 얻을 수 있다.
- Question: NNDC의 ^{12}C 의 quadropole quadrupole deformation parameter 는 $\beta_2 = 0.577$ 이다. input값 $\delta_2 = 1.3$ 과 consistent 한가? ¹

12.3 Inelastic scattering: (M) $^{16}O + ^{208}Pb$ (Projectile and Target Excitation)

```
160+208Pb 80 MeV with both deformation in proj and target
NAMELIST
&FRESCO hcm=0.010 rmatch=17.000 rintp=0.50
hn1=0.033 rn1=3.00 centre=0.00 jtmin=0.0 jtmax=100.0
absend=0.1000 dry=F jump(1:6:1)=0 0 0 0 0 0 jbord(1:6)=
0 0 0 0 0.0 0.0 thmin=80.00 thmax=180.00
thinc=2.50 cutl=0.00 cutr=0.00 cutc=0.00 ips=0.0100
iter=18 iblock=2 pralpha=F pade=1 nnu=24
erange=1.2000 dk=0.0200 chans=1 smats=2 veff=1 kfus=20
pel=1 exl=1 lab=1 lin=1 lex=1 elab=80.0000
nlab(1:3)=0 0 0 fatal=T nosol=F psiren=T
unitmass=1.000000 finec=137.03599 /

&PARTITION namep='16-0' massp=15.9949 zp=8
namet='PB-208' masst=207.9770 zt=82 qval=0.0000 pwf=T
nex=5 /

&STATES jp=0.0 bandp=1 ep=0.0000 cpot=1 jt=0.0
bandt=1 et=0.0000 fexch=F /
```

¹NNDC에 나오는 β_2 의 정의와 책의 (4.4.26),(4.4.3) 을 조합하고, $R_c = R_0$, $\langle I_i K, \lambda 0 | I_f K \rangle = 1$ 이라면, (4.4.9)와 같은 결과가 얻어지고, $R_c = 1.2A^{1/3}$ 을 대입하면, 약 1.585 가 얻어진다.

```

&STATES jp=3.0 bandp=-1 ep=6.1300 cpot=1 jt=0.0
      copyt=1 et=0.0000 fexch=F /
&STATES jp=0.0 copyp=1 ep=0.0000 cpot=1 jt=3.0
      bandt=-1 et=2.6100 fexch=F /
&STATES jp=0.0 ep=0.0000 cpot=1 jt=2.0 bandt=1
      et=4.0700 fexch=F /
&STATES jp=0.0 ep=0.0000 cpot=1 jt=5.0 bandt=-1
      et=3.2000 fexch=F /

&partition /

! Coulomb potential
&POT kp=1 itt=F ap=208.000 at=16.000 rc=1.200 /
&POT kp=1 type=10 itt=F p3=37.600 / ! Coulomb deformation of projectile
&POT kp=1 type=13 shape=10 itt=F p2=54.45 p3=815.0 p5=0.2380e+5 /
&STEP ib=1 ia=3 k=3 str=815.0 / ! Coulomb deformation of target
&STEP ib=3 ia=1 k=3 str=815.0 /
&STEP ib=1 ia=4 k=2 str=54.45 /
&STEP ib=4 ia=1 k=2 str=54.45 /
&STEP ib=1 ia=5 k=5 str=0.2380E+05 /
&STEP ib=5 ia=1 k=5 str=0.2380E+05 /
&step /

&POT kp=1 type=1 shape=1 itt=F p1=0.000 p2=1.179
      p3=0.658 p4=10.000 p5=1.000 p6=0.400 /
&POT kp=1 type=-1 itt=F p1=60.500 p2=1.179
      p3=0.658 /
&POT kp=1 type=10 shape=11 itt=F p3=2.150 / ! Nuclear deformation of projectile
&POT kp=1 type=13 shape=11 itt=F p2=0.400 p3=0.800 p5=0.468 /
&STEP ib=1 ia=3 k=3 str=0.8000 / ! Nuclear deformation of target
&STEP ib=3 ia=1 k=3 str=0.8000 /
&STEP ib=1 ia=4 k=2 str=0.4000 /
&STEP ib=4 ia=1 k=2 str=0.4000 /
&STEP ib=1 ia=5 k=5 str=0.4680 /
&STEP ib=5 ia=1 k=5 str=0.4680 /
&step /
&pot /
&overlap /
&coupling /

```

- This is from Moro's notes. Here considers both target and projectile excitation. In this example, the deformation of projectile(^{16}O) is included in a rotational model(thus only one parameter p2 is enough). On the other hand, the deformation of target is included as an explicit matrix elements.

- In collective model of inelastic scattering, the partition does not change. Thus, only

one partition ($^{16}O + ^{208}Pb$) is considered. There are five states

$$\begin{aligned}
st1 & \quad ^{16}O(0^+; g.s.) + ^{208}Pb(0^+; g.s.) \\
st2 & \quad ^{16}O^*(3^-) + ^{208}Pb(0^+; g.s.) \\
st3 & \quad ^{16}O(0^+; g.s.) + ^{208}Pb(3^-) \\
st4 & \quad ^{16}O(0^+; g.s.) + ^{208}Pb(2^+) \\
st5 & \quad ^{16}O(0^+; g.s.) + ^{208}Pb(5^-)
\end{aligned} \tag{12.7}$$

The first two states are fully coupled but other states are iterated.

- The diagonal couplings V_{11}, V_{22}, V_{55} would be non-deformed potential. For the coupling $V_{12}^{\lambda=3}$ and $V_{21}^{\lambda=3}$ can be obtained by the rotational model. The couplings involving state 1, $V_{1,j}^{\lambda}$ and $V_{j,1}^{\lambda}$ with $\lambda = 2, 3, 5, j = 3, 4, 5$ are included by using matrix elements. In this work, the couplings between excited states are ignored.
- It is important to use 'PWF=T' when there is a Coulomb deformation because there may remain Couplings by Coulomb multipoles at ' $R > R_{match}$ ' if charge is large. On the other hand, too large R_{match} raises numerical error. Thus one have to try to avoid large R_{match} values even with 'PWF=T'.
- Deformation of ^{16}O is considered in a rotational model(TYPE=10) with $M_n(E3) = 37.6$ and $\delta_3 = 2.150$. Coulomb deformation is inferred from $B(E3 \uparrow)$ and Nuclear deformation is inferred from $B(E3 \uparrow)$ assuming charge distribution and matter distribution is the same. For ^{16}O , NRV shows

$$\begin{aligned}
\langle R^2 \rangle^{\frac{1}{2}} &= 2.6991, \quad \beta_3 = -0.258, \\
B(E3 \uparrow) &= 0.00148 e^2 b^2, \quad \beta_3^0 = 0.729
\end{aligned} \tag{12.8}$$

If I treat (note that $1b = 100 fm^2$), for Coulomb deformation

$$M_n(E3) = \langle 3 || E3 || 0 \rangle = \sqrt{B(E3 \uparrow)} \simeq 38.47 \tag{12.9}$$

Thus, it seems to be consistent with used value 37.6 The nuclear deformation length can be inferred for uniformly charged sphere case,

$$M_n(Ek) = \frac{3Z\beta_k R^k}{4\pi} \rightarrow \delta_k = \frac{4\pi}{3Z} \frac{M_n(Ek)}{R^{k-1}} \tag{12.10}$$

by using values $R = 1.2 \times 16^{\frac{1}{3}}$, $M_n(E3) = 37.6$, we get $\delta_3 \simeq 2.153$ and $\beta_3^0 = 0.712$. Since there are large discrepancy with the NRV value of rms radius and $\beta_3 = -0.258$, I wonder which one is to be more physical.

- Coulomb deformation and Nuclear deformation of ^{208}Pb : Coulomb deformation can be inferred from $B(E\lambda \uparrow)$,

$$\begin{aligned}
B(E2 \uparrow) &= 0.287 e^2 b^2, \\
B(E3 \uparrow) &= 0.664225 \text{ or } 0.611 e^2 b^3
\end{aligned} \tag{12.11}$$

with

$$\text{str} = \langle ib || M(Ek) || ia \rangle = \sqrt{(2I_a + 1) B(E\lambda; ia \rightarrow ib)} \tag{12.12}$$

These gives

$$M(E3) = \pm \sqrt{(2*0+1)B(E3;0 \rightarrow 3)} = 781.7, \quad M(E2) = 53.57 \quad (12.13)$$

I found $B(E5)=4.47E+8 \text{ e}^2 fm^{10}$ from the web, which gives $M(E5)=2.1E+4$. Thus, it seems to be reasonably consistent with the numerical inputs.

For the nuclear deformation, input are

$$str = \langle ib || \delta_k || ia \rangle \quad (12.14)$$

this can be inferred by assuming the same deformation for both charge and matter. Then we can use

$$\delta_k = \frac{4\pi}{3Z} \frac{M_n(Ek)}{R^{k-1}} \quad (12.15)$$

with $R = 1.2 \times 208^{\frac{1}{3}}$, $Z = 82$, we get $\delta_3 \simeq 0.79$, $\delta_2 \simeq 0.38$, $\delta_5 \simeq 0.42$. These are consistent with the input values. Note that in this case $p2, p3$ are only informative purpose.

Or, we may use NRV value of β_2^0 and $R = 1.2 \times 208^{\frac{1}{3}}$ to get $\delta_3 \simeq 0.3846$ for quick estimation.

Q: Why not define diagonal coupling strength? They are not considered here. $\lambda = 0$ case are the original spherical potential?

12.3.1 Another example for 8Li excitation

```
8Li+208Pb quasielastic
NAMELIST
&FRESCO
  hcm=0.1 rmatch=100 rintp=0.5 hnl=0.033 rnl=3.0 centre=0.0
  jtmax=100 absend=0.001 dry=F
  jump(1:6:1)=0 0 0 0 0 0 jbord(1:6)=0 0 0 0 0 0
  thmin=2.0 thmax=-180.0 thinc=2.0
  ips=0.01 iblock=2 pralpha=F pade=1 nnu=24
  chans=1 smats=2 xstabl=1 pel=1 exl=1 lab=1 lex=1
  elab=34.404 nlab(1:3)=0 0 0
  fatal=F nosol=F psiren=T
  unitmass=1.0 finec=137.03599d0

&PARTITION
  namep='Li-8' massp=8 zp=3 namet='Pb-208' masst=207.977 zt=82
  qval=0.0 pwf=T nex=2 /
&STATES jp=2.0 bandp=1 ep=0.0 kkp=1 cpot=1
  jt=0.0 bandt=1 et=0.0 fexch=F /
  ! It seems excited state info is missing here
&PARTITION /

&POT kp=1 itt=F ap=8 at=208 rc=1.2 /
&POT kp=1 type=12 shape=10 itt=F
  p1=0.0 p2=10. p3=0. p4=0. p5=0. p6=0. /
&STEP ib=2 ia=1 k=2 str=12.24 /
```

```

&STEP  ib=1  ia=2  k=2  str=12.24  /
&STEP  ib=1  ia=1  k=2  str=-5.976  /
&STEP  ib=2  ia=2  k=2  str=5.477  /
&STEP  /

&POT  kp=1  type=1  itt=F  p1=60  p2=1.3  p3=0.65
      p4=150  p5=1.3  p6=0.60  /
&POT  kp=1  type=12  shape=10  itt=F
      p1=0.  p2=1.75  p3=0.  p4=0.  p5=0.  p6=0.  /
&STEP  ib=2  ia=1  k=2  str=2.1433  /
&STEP  ib=1  ia=2  k=2  str=2.1433  /
&STEP  ib=1  ia=1  k=2  str=-1.0458  /
&STEP  ib=2  ia=2  k=2  str=0.9585  /
&STEP  /
&POT /

&OVERLAP  /

&COUPLING /

```

- From the example of A. Moro's note
- ^8Li ground state 2^+ and 1st excited state 1^+ is considered. Only information given from the note is that $B(E2; 2^+ \rightarrow 1^+) = 30e^2 fm^4$ and deformation length $\delta_2 = 1.75$.
- Suppose the only information we have is $B(E2; 2^+ \rightarrow 1^+) = 30e^2 fm^4$ for Coulomb excitation. We may assume the intrinsic amplitude $M_n(E\lambda) = \langle \chi | M(E\lambda_0) | \chi \rangle$ which is not dependent on I values. Then, it is related to

$$\begin{aligned}
 STR(E\lambda; I_i \rightarrow I_f) &= (-1)^{(I_i - I_f + |I_i - I_f|)/2} \langle I_f || M(E\lambda) || I_i \rangle, \\
 \langle I_f || M(E\lambda) || I_i \rangle &= \sqrt{2I_i + 1} \langle I_i K \lambda 0 | I_f K \rangle \langle \chi | M(E\lambda_0) | \chi \rangle. \quad (12.16)
 \end{aligned}$$

From the property of C-G coefficient, $\langle I_f || M(E\lambda) || I_i \rangle = \langle I_i || M(E\lambda) || I_f \rangle$. And this matrix element can be related to the experiment by

$$B(E2; I_i \rightarrow I_f) = \frac{1}{\sqrt{2I_i + 1}} |\langle I_f || M(E\lambda) || I_i \rangle|^2. \quad (12.17)$$

for off-diagonal matrix or

$$Q_2(I) = \sqrt{\frac{16\pi}{5}} (2I + 1)^{-\frac{1}{2}} \langle II20 | II \rangle \langle I || M(E2) || I \rangle \quad (12.18)$$

for diagonal matrix elements.

- Note that the matrix elements are required for $1 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 1, 2 \rightarrow 2$. And in case of $2 \rightarrow 2$, because of additional factors $\langle 2 || M(E2) || 2 \rangle \neq \langle 2 || M_n(E2) || 2 \rangle$!
- Since in the problem, K cannot be 2 and K=0 gives zero amplitude $C_{20,20}^{10} = 0$, let us take K=1 (But why?),

$$C_{2120}^{11} = -\sqrt{\frac{3}{10}}, \quad C_{1120}^{21} = \frac{1}{\sqrt{2}}, \quad C_{2120}^{21} = -\frac{1}{\sqrt{14}}, \quad C_{1120}^{11} = \frac{1}{\sqrt{10}}. \quad (12.19)$$

This gives $M_n(E2) = 10$ and

$$\begin{aligned}
STR(E2; 2^+ \rightarrow 1^+) &= (-1)\sqrt{5}\langle 2K20|1K\rangle M_n(E2) \rightarrow 12.247, \\
STR(E2; 1^+ \rightarrow 2^+) &= \sqrt{3}\langle 1K20|2K\rangle M_n(E2) \rightarrow 12.247, \\
STR(E2; 2^+ \rightarrow 2^+) &= \sqrt{5}\langle 2K20|2K\rangle M_n(E2) \rightarrow -5.976, \\
STR(E2; 1^+ \rightarrow 1^+) &= \sqrt{3}\langle 1K20|1K\rangle M_n(E2) \rightarrow 5.477.
\end{aligned} \tag{12.20}$$

- we may do similar estimation by using $\delta_2 = 1.75$ as an intrinsic deformation length.

$$\begin{aligned}
RDEF(E2; 2^+ \rightarrow 1^+) &= (-1)\sqrt{5}\langle 2K20|1K\rangle \delta_2 \rightarrow 2.143, \\
RDEF(E2; 1^+ \rightarrow 2^+) &= \sqrt{3}\langle 1K20|2K\rangle \delta_2 \rightarrow 2.143, \\
RDEF(E2; 2^+ \rightarrow 2^+) &= \sqrt{5}\langle 2K20|2K\rangle \delta_2 \rightarrow -1.046, \\
RDEF(E2; 1^+ \rightarrow 1^+) &= \sqrt{3}\langle 1K20|1K\rangle \delta_2 \rightarrow 0.9585.
\end{aligned} \tag{12.21}$$

- Does this estimation, consistent with NRV value of $Q_2 = 0.0314 \text{ e b}$ for ${}^8\text{Li}$? How one can obtain δ_2 from experiment?
- In case of ${}^{16}\text{O} + {}^{208}\text{Pb}$, no diagonal coupling were defined. Here, the diagonal couplings are required because the ground state is not spherical. And thus, there is a non-zero coupling like $\langle 2||E2||2\rangle$.

Also another way to include deformation is to define couplings. Equivalent input is

```

!... same...
&POT kp=1 itt=F ap=8 at=208 rc=1.2 /
&POT kp=1 type=12 shape=10 itt=F p2=1 /
&step /

&POT kp=1 type=1 itt=F p1=60 p2=1.3 p3=0.65
      p4=150 p5=1.3 p6=0.60 /
&POT kp=1 type=12 shape=10 itt=F p2=1 /
&step /

&POT kp=2 itt=F ap=8 at=208 rc=1.2 /
&POT kp=2 type=1 itt=F p1=60 p2=1.3 p3=0.65 p4=150 p5=1.3 p6=0.6 /
&POT kp=2 type=12 shape=10 itt=F p2=1 /
&step /
&POT/

&OVERLAP /

&COUPLING icto=1 icfrom=1 kind=1 ip1=8 /
&INEL ib=2 ia=1 k=2 no=2 kp=1 a=12.24 /
&INEL ib=1 ia=2 k=2 no=2 kp=1 a=12.24 /
&INEL ib=1 ia=1 k=2 no=2 kp=1 a=-5.976 /
&INEL ib=2 ia=2 k=2 no=2 kp=1 a=5.477 /
&INEL ib=2 ia=1 k=2 no=3 kp=2 a=2.1433 /
&INEL ib=1 ia=2 k=2 no=3 kp=2 a=2.1433 /
&INEL ib=1 ia=1 k=2 no=3 kp=2 a=-1.0458 /
&INEL ib=2 ia=2 k=2 no=3 kp=2 a=0.9585 /
&INEL/

&coupling /

```

12.3.2 Inelastic scattering: (T) 11Be+197Au

- Simple 11Be + 197Au Coulex of 1/2- at the barrier
- Rotor model for excitation of $^{11}\text{Be}(0.5+; g.s.), ^{11}\text{Be}(0.5-; Ex = 0.32)$.
- The input values for the deformation are $p1 = M_n(E1) = 0.5899$ for Coulomb deformation and $p1 = \delta_1 = 0.6075$ for Nuclear deformation. [How can I understand these values? Is it related to \$\mu = -1.6816\$ nm from NRV?](#)

12.3.3 Inelastic scattering: (T) alpha+20Ne

- ALPHA + 20-NE (0-2-4) inelastic scattering excitation by BETA(2) ONLY.
- Rotor model for excitation of $^{20}\text{Ne}(0+; g.s.), ^{20}\text{Ne}(2+; Ex = 1.634), ^{20}\text{Ne}(4+; 4.248)$.
- No Coulomb deformation is considered. Use the $\beta_2 = 0.4$ value for nuclear deformation(?). But, use two different values of $\delta_2 = 1.325$ and 1.922 for real part and imaginary part of WS potential. [How can I understand these values? NRV values are \$B\(E2\) \uparrow = 0.0333 e^2b^2, \beta_2^0 = 0.721\$.](#)

12.3.4 Inelastic scattering: (T) 16O+58Ni

```

16O+58Ni  48.00 MeV, no transfer
0.040    -30.0    0.50    0.0375    1.000    -0.00
-5.00    0.010    0.0     0.0
  0.400.    -.001 F 0    1    0    0.
0  2.    180.    2.0
0.02    0    0 2 0 24 00
  1 0 0 0 1 0 0 0 0 0 0
16-0    15.9949 8.0          2T NI-58    57.935    28.0    0.00
0.0    1 0.0          1 0.0    1 0.0          F F
      1          1 2.0    +1 1.45
0 !&POT
  1 0 0 58.    16.0    1.2
 113    0.00    9.22 ! target Coulomb deform
      1 2 2 25.97
      2 1 2 25.97
     -2 2 2 -31.04
  1 1 1 00.00  1.179  0.658  10.00  1.000  0.400
 1-1 0 53.75  1.164  0.630 ! TYPE<0 for numerical addition
 113 10          0.400 !target Nuclear deform
      1 2 2 0.9925
      2 1 2 0.9925
     -2 2 2 -1.186
0
0
  0 1 1
48.00

```

- 16O+58Ni 48.00 MeV, no transfer
- consider inelastic excitation of $^{58}\text{Ni}(0+)$ and $^{58}\text{Ni}(2+; 1.45)$.

- Note that the TYPE<0 is used for the central part of nuclear interaction for numerical potential addition.
- deformation is given as a matrix inputs.
- Coulomb deformation matrix $M(E2; 0 \rightarrow 2) = 25.97 = M(E2; 2 \rightarrow 0)$ and $M(E2; 2 \rightarrow 2) = -31.04$
- Nuclear deformation $RDEF(E2; 0 \rightarrow 2) = 0.9925 = RDEF(E2; 2 \rightarrow 0)$ and $RDEF(E2; 2 \rightarrow 2) = -1.186$ Can I understand these input by NRV value $B(E2)=0.065 e^2 b^2 \beta_2^0 = 0.1768$ and $Q_2 = -0.10(b)$?

12.4 Inelastic scattering(Cluster Model): (M) $^{28}\text{Si}(^{19}\text{F}, ^{19}\text{F}^*)^{28}\text{Si}$ at 60 MeV

```

Test run: CC calculation 19F+28Si. Cluster form factors
NAMELIST
&FRESKO hcm=0.010 rmatch=25.000 rintp=0.50
        hnl=0.100 rnl=3.00 centre=0.00
        jtmin=0.0 jtmax=80.0 absend=0.0100 dry=F
        thmin=0.00 thmax=60.00 thinc=2.50
        ips=0.0000 iter=1 iblock=2 pralpha=F
        nnu=30 chans=1 listcc=2 smats=1 pel=1 exl=1
        lab=1 lin=1 lex=1 elab=60.0
        fatal=T nosol=F psiren=F unitmass=1.007335
        finec=137.03599 /

&PARTITION namep='19-F' massp=19 zp=9 namet='28-SI'
        masst=28 zt=14 qval=0.0000 pwf=T nex=2 /
&STATES jp=0.5 bandp=1 ep=0.0000 cpot=1 jt=0.0
        bandt=1 et=0.0000 fexch=F / ! 19F(1/2+)+28Si
&STATES jp=2.5 bandp=1 ep=0.2000 jt=0.0 et=0.0000
        fexch=F / !19F(5/2+)+28Si

&PARTITION namep='16-O' massp=16 zp=8 namet='31-P'
        masst=31 zt=15 qval=6.1990 pwf=T nex=1 /
&STATES jp=0.0 bandp=1 ep=0.0000 jt=0.5 bandt=1
        et=0.0000 fexch=F / ! 16O+31P
&partition /

&POT kp=1 itt=F ap=28.000 at=19.000 rc=1.200 / ! 19F-28Si central Coulomb
&POT kp=2 itt=F ap=31.000 at=16.000 rc=1.350 / !16O-31P central Coulomb
&POT kp=2 type=1 itt=F p1=31.200 p2=1.450 !16O-31P central nuclear
        p3=0.470 p4=15.100 p5=1.270 p6=0.310 p7=0.000 /
&POT kp=2 type=3 itt=F p1=0.750 p2=1.240 p3=0.370 / !16O-31P spin-orbit of projec

&POT kp=3 itt=F ap=19.000 at=0.000 rc=1.250 ! 16O-3H Coulomb
        ac=0.650 /
&POT kp=3 type=1 itt=F p1=115.000 p2=1.250 p3=0.650 /
&POT kp=3 type=3 itt=F p1=6.300 p2=1.250 p3=0.650 / ! 16O-3H spin-orbit of projec

&POT kp=4 itt=F ap=31.000 at=0.000 rc=1.250

```

```

        ac=0.650 / ! 31P=28Si+3H binding potential
&POT kp=4 type=1 itt=F p1=99.000 p2=1.250 p3=0.650 / !
&POT kp=4 type=3 itt=F p1=6.300 p2=1.250 p3=0.650 / !

&POT kp=5 itt=F ap=28.000 at=0.000 rc=1.200 / ! 160-28Si core-target interaction
&POT kp=5 type=1 itt=F p1=27.000 p2=2.425
        p3=0.460 p4=11.660 p5=2.145 p6=0.238 p7=0.000 /

&POT kp=6 itt=F ap=28.000 at=0.000 ! 3H-28Si valence target interaction
        rc=1.250 ac=0.650 /
&POT kp=6 type=1 itt=F p1=160.000 p2=1.070
        p3=0.720 p4=37.000 p5=1.350 p6=0.880 p7=0.000 /
&pot /

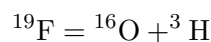
&OVERLAP kn1=1 ! 19F(1/2+)=160+3H(4S1/2) ! kn1 is index of overlap,
        kind=0 ! kind angular momentum scheme
        in=1 ! in=1 for projectile
        ic1=1 ic2=2 ! ic1 ic2 for partition number for core and composite
        nn=4 sn=0.5 l=0 j=0.5 ! 4S1/2
        kbpot=3 be=11.7300 ! potential index that binds fragment and core
        isc=1 nam=1 ampl=1.0000 /
&OVERLAP kn1=2 ic1=1 ic2=2 in=1 ! 19F(5/2+)=160+3H(3D1/2)
        nn=3 l=2 sn=0.5 j=2.5
        kbpot=3 be=11.5300
        isc=1 nam=1 ampl=1.0000 /
&overlap /

&COUPLING icto=1 ! index of partition being excited (19F)
        icfrom=2 ! index of partition containing core nucleus(160)
        kind=3 ! single particle excitation of projectile
        ip1=4 ! multipoles up to 4
        ip2=1 ! use nuclear potential only to construct coupling potential
        p1=6 ! potential index for valence-target interaction
        p2=5 ! potential index for core-target interaction
        jmax=0.00 rmax=0.0 /
&CFP in=1 / ! in=1 overlap for projectile <19F|160>
        ib=1 ! ib=1 index of state for projectile
        ia=1 ! ia=1 index of excitation state of core nucleus
        kn=1 ! kn=1 index of form factor of wave function
        ! which gives <19F|160>
        a=1.000 ! a= spectroscopic amplitude for the overlap
&CFP in=1 ib=2 ia=1 kn=2 a=1.000 / ! <19F*|160>
&cfp /

&coupling /

```

- This is from the example of Moro's note. $^{19}F(\frac{1}{2}^+)$ 와 $^{19}F(\frac{5}{2}^+; 0.2)$ 를



의 cluster model로 기술하여, inelastic scattering을 계산한다.

- 이 경우 비록 산란에서 나타나는 partition은 하나뿐이지만, core와 excited particle을 구분해 주기 위해서 새로운 partition이 필요하다. 즉, inelastic single particle exciation

계산은 transfer 계산과 동일하다. Partition 1 includes $^{19}F + ^{28}Si$ with g.s. and excited state of ^{19}F . The second partition은 $^{16}O + ^{31}P$ 이지만, 실제로는 overlap 계산에서만 쓰인다.

- iter=1 and iblock=2 : iblock=2 는 처음 2 개의 channel에 대해 full coupled equation 으로 푼다는 뜻이다. iter=1 은 iblock < number of states 인 경우에 나머지 state에 대해서는 iteration을 한다는 뜻이다.
- $^{19}F + ^{28}Si$ 사이의 interaction은 $kp = 1$ 은 Coulomb interaction만 주어졌다. 실제 계산에서 projectile-target 사이의 potential은 **cluster folding potential** 을 사용한다. (deuteron 의 경우 Watanabe potential.) 이것은 Coupled equation 의 오른쪽 항에서 $V_{\alpha,\alpha'}$ 의 diagonal part로 potential이 계산된다는 뜻. 따라서, 이 경우는 DWBA가 아닌, full coupled equation을 풀어야한다. elastic channel의 potential은 cluster interaction의 folding 으로 생각하여

$$U_{p-ct} = U_{cp-ct} + U_{v-ct} \quad (12.22)$$

으로 주어진다고 할 수 있다.(?) 이런 계산의 경우, U_{p-ct} 를 DWBA로 취급해서는 elastic channel이 제대로 계산이안될 것이다. 따라서, CRC로 계산해야 한다.

- 여기서 3H 를 valence , ^{16}O 를 cp (projectile core), ^{28}Si 를 ct (target core) 라고 부르자.

$$\begin{aligned} kp = 1 & \quad V_{cp+v,ct}^{coul}, \\ kp = 2 & \quad V_{cp,ct+v}^{coul} + V_{cp,ct+v}, \\ kp = 3 & \quad V_{cp,v}^{coul} + V_{cp,v}, \\ kp = 4 & \quad V_{ct,v}^{coul} + V_{ct,v}, \\ kp = 5 & \quad V_{cp,ct}^{coul} + V_{cp,ct}, \\ kp = 6 & \quad V_{v,ct}^{coul} + V_{v,ct}. \end{aligned} \quad (12.23)$$

이 중 실제로 사용되는 것은 $kp=3,5,6$ 뿐이다. $kp=2,4$ 는 필요 없음. $kp=3$ 은 projectile의 overlap function 계산에, $kp=5$ 와 6 은 projectile-target interaction을 만드는 데 사용된다.

- Overlap function: 필요한 overlap function은 $\langle ^{19}F(g.s) | ^{16}O \rangle$ 와 $\langle ^{19}F^*(ex) | ^{16}O \rangle$ 이다. 이것을 각각 potential $kp = 3$ 으로 계산한다.
- Coupling에 나타나는 parameter 들의 의미는 KIND 의 값에 따라 달라진다. Single particle excitation(KIND=3)의 경우 ICFROM은 core-particle의 partition, ICTO는 partition being excited 이다. ICFROM이 나타나기는 하지만, transfer와 달리 모든 coupling은 ICTO partition에서만 계산된다. ip1=q (mutipolarity of transition potential.) p1 은 fragment-target interaction , p2는 core-target interaction이다. 즉, $U_{p-ct} = U_{v-ct} + U_{cp-ct}$ 를 나타낸다.

ip2 = 1 means CRC equation의 오른쪽 항의 $U_{\alpha\alpha}$ (diagonal part) 와 $U_{\alpha\alpha'}$ (off-diagonal part)를 계산할 때 두 cluster potential을 folding 할때, nuclear potential부분만 이용하라는 뜻이다. 하지만, Coulomb potential은 CRC식의 왼쪽에 이미 포함되어 있으므로 ($kp=1$), Coulomb 효과가 무시되는 것은 아니다. 단지, transition을 일으킬 때, Coulomb 효과에 의한 transition은 고려하지 않는다는 것이다.

차후에 Transfer reaction $^{28}Si(^{19}F, ^{16}O)^{31}P$ 를 계산하기 위해서는 excitation 관련된 coupling 이외에 transfer와 관련된 coupling을 지정해 주어야 하고, 또한, ^{31}P 에 대한 overlap function 도 정의해 주어야 한다.

12.5 Breakup (CDCC) : (B) $^8B + ^{208}Pb$

```

8B+208Pb ; N+C q=0,1,2
CDCC
&CDCC
  hcm=0.01  rmatch=-60  rasym=1000  accrcy=0.001  absend=-50
  elab=656
  jbord=  0  200 300 1000 9000
  jump =  1  10  50  200
  thmax=20 thinc=0.05  cutr=-20  smats=2  xstabl=1
  nk=40  ncoul=0  reor=0  q=2 /
&NUCLEUS part='Proj'  name='8B'  charge=5  mass=8
      spin=1.5  parity=-1  be = 0.137  n=1  l=1  j=1.5 /
&NUCLEUS part='Core'  name='7Be'  charge=4  mass=7 /
&NUCLEUS part='Valence'  name='proton'  charge=1  mass=1  spin=0.5/
&NUCLEUS part='Target'  name='208Pb'  charge=82  mass=208  spin=0 /

&BIN spin=0.5  parity=+1  start=0.001  step=0.50  end=10.
  energy=F  l=0  j=0.5/
&BIN /

&POTENTIAL part='Proj'  a1=1  rc=2.65 /
&POTENTIAL part='Core'  a1=208  rc=1.3  v=114.2  vr0=1.286  a=0.853
      w=9.44  wr0=1.739  aw=0.809 /
&POTENTIAL part='Valence'  a1=208  rc=1.3  v=34.819  vr0=1.17  a=0.75
      w=15.340  wr0=1.32  aw=0.601/
&POTENTIAL part='Gs'  a1=1  rc=2.391  v=44.675  vr0=2.391  a=.48
      vso=4.898  rso0=2.391  aso=0.48 /

```

- Book example: There are two versions, short and long version, which are equivalent. In this example, $^8B = p + ^7Be(0+)$ break up is considered.
- Because of continuum, long range part and higher partial waves have to be included. Thus, `rmatch < 0`, `rasym`, `absend`, `jbord`, and `jump` are used.
- With the inclusion of so many partial waves, the strong repulsion at short distance can introduce numerical problem. Thus, `cutr` is used.
- Potential for V_{v-t} and V_{c-t} is named 'Core' and 'Valence'. Potential 'Proj' is a Coulomb part of the projectile-target interaction. Potential 'Gs' is for the binding and scattering wave of core-valence.
- In this example, relative wave between core-valence include s-wave only. Other partial waves (p,d,f are needed for convergence) are not considered here. `smallchan` and `smallcoup` drop off channels/couplings when they are weak.
- `&BIN` generates the states and overlaps between core and valence with total angular momentum and parity ('spin' and 'parity' is thus a total spin and parity of composed particle state) while varing energies from start to end with 'step' intervals. . 'l, j' represent angular momentums of valence particle relative to core. (In other words, `spin=(core spin)+j(=l+valence spin)`).

energy='T' case, bins are evenly spaced in energy. energy='F', bins are evenly spaced in k.

- In the example, total number of bins are $(end - start)/step \simeq 20$. 'nk' in the Namelist is the number of wave functions to be averaged in each bins.

The result is equivalent when we use the 'long' namelist format.

```
CDCC 8B+208Pb ; nuclear and coulomb s-waves
NAMELIST
&Fresco hcm= 0.01 rmatch= -60.000 rintp= 0.15 rsp= 0.0
  rasym= 1000.00 accrcy= 0.0010000
  jtmin= 0.0 jtmax= 9000.0 absend= -50.0000
  jump = 1 10 50 200
  jbord= 0.0 200.0 300.0 1000.0 9000.0
  thmin= 0.00 thmax= 20.00 thinc= 0.05 cutr=-20.00
  ips= 0.0000 it0= 0 iter= 0 iblock= 21 nnu= 24
  smallchan= 1.00E-12 smallcoup= 1.00E-12
  chans= 1 smats= 2 xstabl= 1 cdcc= 1
  elab= 656.0000 pel=1 exl=1 lab=1 lin=1 lex=1 /

&Partition namep='8B' massp= 8.0000 zp= 5 nex= 21 pwf=T
  namet='208Pb' masst=208.0000 zt= 82 qval= 0.1370/
&States jp= 1.5 ptyp=-1 ep= 0.0000 cpot= 1
  jt= 0.0 ptyt= 1 et= 0.0000/
&States jp= 0.5 ptyp= 1 ep= 0.1583 cpot= 1 copyt= 1/
&States jp= 0.5 ptyp= 1 ep= 0.2180 cpot= 1 copyt= 1/
..... ! more states
&States jp= 0.5 ptyp= 1 ep= 9.6583 cpot= 1 copyt= 1/
&Partition namep='7Be' massp= 7.0000 zp= 4 nex= -1 pwf=T
  namet='208Pb+p' masst=209.0000 zt= 83 qval= 0.0000/
&States jp= 0.0 ptyp= 1 ep= 0.0000 cpot= 2
  jt= 0.0 ptyt= 1 et= 0.0000/
&Partition / ! END OF DEFINING PARTITIONS

&Pot kp= 1 type= 0 shape= 0
  p(1:3)= 1.0000 0.0000 2.6500 /
&Pot kp= 2 type= 0 shape= 0
  p(1:3)= 208.0000 0.0000 1.3000 /
&Pot kp= 2 type= 1 shape= 0
  p(1:7)= 114.2000 1.2860 0.8530 9.4400 1.7390 0.8090
0.0000 /
&Pot kp= 3 type= 0 shape= 0
  p(1:3)= 208.0000 0.0000 1.3000 /
&Pot kp= 3 type= 1 shape= 0
  p(1:7)= 34.8190 1.1700 0.7500 15.3400 1.3200 0.6010
0.0000 /
&Pot kp= 4 type= 0 shape= 0
  p(1:3)= 1.0000 0.0000 2.3910 /
&Pot kp= 4 type= 1 shape= 0
  p(1:3)= 44.6750 2.3910 0.4800 /
&Pot kp= 4 type= 3 shape= 0
  p(1:3)= 4.8980 2.3910 0.4800 /
&Pot / ! END OF DEFINING POTENTIALS
```

```

&Overlap kn1= 1 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 1 l=1 lmax=0 sn=0.5 j= 1.5 nam=1 ampl= 1.0000
      kbpot= 4 be= 0.1370 isc= 1 ipc=0 / ! g.s. bound state
&Overlap kn1= 2 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 0 l=0 lmax=0 sn=0.5 j= 0.5 nam=1 ampl= 1.0000
      kbpot= 4 be= -0.0182 isc=12 ipc=2 nk= 20 er= -0.0344 /
&Overlap kn1= 3 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 0 l=0 lmax=0 sn=0.5 j= 0.5 nam=1 ampl= 1.0000
      kbpot= 4 be= -0.0771 isc=12 ipc=2 nk= 20 er= -0.0834 /
&Overlap kn1= 4 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 0 l=0 lmax=0 sn=0.5 j= 0.5 nam=1 ampl= 1.0000
      kbpot= 4 be= -0.1850 isc=12 ipc=2 nk= 20 er= -0.1324 /
..... ! more overlaps
&Overlap kn1= 21 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 0 l=0 lmax=0 sn=0.5 j= 0.5 nam=1 ampl= 1.0000
      kbpot= 4 be= -9.5173 isc=12 ipc=2 nk= 20 er= -0.9655 /
&Overlap / ! END OF DEFINING OVERLAPS

&Coupling icto= 1 icfrom= 2 kind=3 ip1= 2 ip2= 0 ip3= 0
      p1= 3.0000 p2= 2.0000 /
&Cfp in= 1 ib= 1 ia= 1 kn= 1 a= 1.0000 /
&Cfp in= 1 ib= 2 ia= 1 kn= 2 a= 1.0000 /
&Cfp in= 1 ib= 3 ia= 1 kn= 3 a= 1.0000 /
.... ! more cfps
&Cfp in= 1 ib= 20 ia= 1 kn= 20 a= 1.0000 /
&Cfp in= 1 ib= 21 ia= 1 kn= 21 a= 1.0000 /
! ***** END OF CDCC INPUTS *****

!***** This is a part of code which determine the Bin energies
      if(energy) then ! equal energy steps
          bstart(ib) = start + (ie-1)*step
          bend(ib) = bstart(ib) + step
          xk = sqrt(c1*abs(bstart(ib)))
          kstep = sqrt(c1*abs(bend(ib))) - xk
          bhat(ib) = bstart(ib)+step/2
      else ! equal momentum steps
          xk = kstart + (ie-1)*kstep
          bstart(ib) = xk**2/c1
          bend(ib) = (xk + kstep)**2/c1
          bhat(ib) = ( (xk+0.5*kstep)**2 + kstep**2/12.)/c1 * bound
      endif
      if(isc.ge.10 .and. bound>0) ! bound is a sign of (start)
          x      bhat(ib) = 0.6*((xk+kstep)**5-xk**5)
          x      /((((xk+kstep)**3-xk**3)*c1) * bound
          bmid(ib) = abs(bstart(ib)+bend(ib))*0.5 * bound
      if(.not.hat) bhat(ib) = bmid(ib)
      if(quasi>-100) bhat(ib) = quasi

```

- iter= 0 and iblock= 21 makes that the equations are fully coupled.
- total number of continuum bin states are 20. for each bin, the 'ep' represent channel energy of the break up channel. This corresponds to $ep = \hat{\epsilon}_i - \epsilon_0$ with $\epsilon_0 = -2.225$ MeV. $\hat{\epsilon}_i$ is a n-p scattering energy.

- In overlap, for continuum wave, suppose overlap wave function is computed by averaging wave functions computed in range $e1 > 0$ and $e2 > 0$ in 20 steps. 'be' and 'er' in the &OVERLAP represent bin's mid-point energy and width, $(e1+e2)/2$ and $e1-e2$. Bins are constructed in this way. (1) number of states $N = (\text{end-start})/\text{step}$ (2) if 'energy=T', $e_n = e_0 + n\Delta$, $\Delta = \text{step}$. Then, $\hat{e}_i = \text{'be'}$. (3) if 'energy=F', the equidistance in momentum space is assumed. initial and final momentum is computed $k_n = \sqrt{2\mu * \text{end}}$, $k_0 = \sqrt{2\mu * \text{start}}$. (Note the code use $2\mu = 0.0478436 * (\mu_A)$, $\mu_A = (A_p A_t)/(A_p + A_t)$) Then $\Delta = (k_n - k_0)/(\text{number of states})$ and $k_i = k_0 + i * \Delta$. (3) $e_i = \frac{\hbar^2}{2\mu} k_i^2$. (4) Then, for each Bin overlap, $be_i = (e_i + e_{i+1})/2$, $er_i = e_{i+1} - e_i$. (5) average energy of bin, \hat{e}_i , is calculated. (6) Then, the state energy $ep = \hat{e}_i - \epsilon_0$ with $\epsilon_0 < 0$.

According to the paper, M.Yahiro et.al., Progress of Theoretical Physics, Vol. 67, No.5, May 1982, the Bin state energy are obtained with

$$\begin{aligned}\hat{e}_i &= \frac{\hbar^2}{2\mu_p} \hat{k}_i^2, \\ \hat{k}_i^2 &= \left(\frac{k_i + k_{i-1}}{2} \right)^2 + \frac{\Delta k_i^2}{12}, \\ E &= \epsilon_0 + E_0 = \hat{e}_i + E_i\end{aligned}\tag{12.24}$$

where ϵ_0 is a binding energy of projectile(deuteron), E_0 is a entrance channel kinetic energy, \hat{e}_i is a bin state energy of projectile(np), E_i is a corresponding break up channel energy.

However, it seems to be that other definition of \hat{e}_i is also available in FRESKO.

- amplitude $\sqrt{nam} * \text{ampl}$
- isc determines the weight for averaging. (isc=2 for non-resonant bins and isc=4 for resonant bins.)
- kind=3 for single particle excitations of the projectile. (kind=4 for target.) The break up is treated as a single particle excitation between partition 1 and 2. In this case, ip1 is the maximum multipolarity of coupling potential, ip2=0 means both Coulomb and nuclear interaction is used for coupling. ip3=0 includes all couplings. ip=1, there are no reorientation couplings for all but the monopole, ip3=2, only reorientation couplings are included, ip3=3, it includes only couplings to and from the ground state. p1=3 is a core-target potential and p2=2 is valence-target potential.
- Because the results are in each energy bin, the total angular distribution requires summation over all bins. This can be done with utility 'sumbins' with 'fort.16'
- for the energy distribution of break up, use utility 'sumxen' with 'fort.13'.

12.5.1 CDCC : (T) 11Be+4He

```
&BIN spin=0.5 parity=+1 step=0.5 end=10. energy=F l=0 j=0.5/
&BIN spin=0.5 parity=-1 step=0.5 end=10. energy=F l=1 j=0.5/
&BIN spin=1.5 parity=-1 step=0.5 end=10. energy=F l=1 j=1.5/
&BIN spin=1.5 parity=+1 step=1.0 end=10. energy=F l=2 j=1.5/
```

```
&BIN spin=2.5 parity=+1 step=1.0 end=10. energy=F l=2 j=2.5/
&BIN spin=2.5 parity=-1 step=2.0 end=10. energy=F l=3 j=2.5/
&BIN spin=3.5 parity=-1 step=2.0 end=10. energy=F l=3 j=3.5/
```

- CDCC of ^{11}Be breakup as $^{11}\text{Be}(0.5+; g.s) = ^{10}\text{Be}(0+) + n(2s_{1/2})$.
- Continuum Bins are constructed for $l = 0, 1, 2, 3$ up to $E_{max} = 10$.

12.5.2 1-DWDCC: (T) 8B + 58Ni

- 1-DWDCC 8B(1.5-) + 58Ni(0+) Coulex p3/2 to s (esb), at 26 MeV of 1+13 chs, Rc=1.3
- Coulomb excitation and breakup of 8B=7Be+p is calculated in DWBA.
- 1 bound 8B(1.5-) = 7Be(0+) + p (1p3/2-) and 13 Bins of 8B(0.5+) = 7Be(0+)+ p(l=0, j=0.5, sn=0.5)
- single particle excitation coupling is calculated only for $Q = 1$ coupling.

12.6 Adiabatic CDCC:(T) $d + ^{208}\text{Pb}$

```
1-ADCC d + 208Pb Q=0,1,2 to p-waves at 50 MeV of 1+10*3 chs to 20 MeV, coul only
0.050 -30.0 0.40 0.025 1.5 -0.00
300.0 0.001
0. 40. -1.001 F 1 4 00.
1 20.0 000. 20.0 -1.6 -10.
0.000 0 099 0 24 0 0 8 0.000 -0.1000 0.002 !iblock=99
1 1 0 0 4 0 0 0 0 0 0 0 0 0 0
Deuteron 2. 1. 11 208Pb 208. 82. 0.0000
0.0 +1 0.0 1 0.0 1 0.0 F F
1.0 -1 0.000 1 1 F F ! note be=0.
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
1.0 -1 0.000 1 1 F F
Proton 1.0000 1.0 -1 209Pb 209. 82. 0.0
0.0 +1 0.0 3 0.5 +1 0.00
0 !&POT
1 0 0 208. 0. 1.3
1 1 0
2 1 0 165.53 0.4 0.60 1.0
3 0 0 208. 0. 1.3
3 1 0 50.0 1.20 0.75 15.0 1.20 0.75
5 0 0 208. 00.0 1.3
0 !&OVERLAP
1 1 2 1 0 1 0 0.0 0.0 2 0 2.226 1 0 0
```

```

23      1 2 1 0      1 1      0.0      1.0      2 0 -1.000      0 2 0-40 -1.980
25      1 2 1 0      1 1      0.0      1.0      2 0 -3.000      0 2 0-40 -2.000
27      1 2 1 0      1 1      0.0      1.0      2 0 -5.000      0 2 0-40 -2.000
29      1 2 1 0      1 1      0.0      1.0      2 0 -7.000      0 2 0-40 -2.000
31      1 2 1 0      1 1      0.0      1.0      2 0 -9.000      0 2 0-40 -2.000
33      1 2 1 0      1 1      0.0      1.0      2 0 -11.00      0 2 0-40 -2.000
35      1 2 1 0      1 1      0.0      1.0      2 0 -13.00      0 2 0-40 -2.000
37      1 2 1 0      1 1      0.0      1.0      2 0 -15.00      0 2 0-40 -2.000
39      1 2 1 0      1 1      0.0      1.0      2 0 -17.00      0 2 0-40 -2.000
41      1 2 1 0      1 1      0.0      1.0      2 0 -19.00      0 2 0-40 -2.000
0  !&COUPLING single particle excitation kind=3
    1      2      3 2 0 0      5.0      3.0
        1      1      1      1      1.00
        1      2      1      23      1.00
        1      3      1      25      1.00
        1      4      1      27      1.00
        1      5      1      29      1.00
        1      6      1      31      1.00
        1      7      1      33      1.00
        1      8      1      35      1.00
        1      9      1      37      1.00
        1     10      1      39      1.00
       -1     11      1      41      1.00
    0      1      1
50.0

```

- Adiabatic approximation for the deuteron break up effects. Note that the energy of excited states are the same as bound state deuteron while the BIN wave functions are computed with $be=-1, -3, -5...$
- $kp = 2$ is V_{n+Pb} , $kp = 3$ is V_{p+Pb} $kp = 5$ is V_{n+Pb}^C ? Couplings are computed by $(kp=5)+(kp=3)-(kp=1)$. Why use $kp=5$ for fragment-target interaction?

12.6.1 Adiabatic, R-matrix solution: (T) d+208Pb

```

1-ADCC d + 208Pb Q=0,1,2 to p-waves at 50 MeV of 1+10*3 chs to 20 MeV, coul only
0.050      -30.0 0.40      0.025      1.5      -0.00
300.0      0.001
    0. 40. -1.001 F      1      4 00.
1  20.0 000.      20.0      -1.6      -10.
0.000 0      0-1 0 24      0      0 8 0.000 -0.1000 0.002 ! iblock=-1
    10      5 F 0 20. ! card 4.5 for R-matrix solution
!...below are the same as previous example

!-----
!---above input corresponds to following output
R-MATRIX SOLUTIONS with <= 10 basis states/channel , and at least
5 states.
    buttle=0=complex+sh, pralpha=F, pcon= 0, meigs = 0,
    Rmatr = 20.0000, ebeta = 0.0000 0.0000, weak = 0.00000

```

- This is the same as previous ADCC calculation except the additional input for R-matrix solution.
- IBLOCK=-1 and the following line is only difference.

12.7 1 nucleon Transfer reaction: (B) $^{14}\text{N}(^{17}\text{F}, ^{18}\text{Ne})^{13}\text{C}$

```

n14(f17,ne18)c13 @ 170 MeV;
NAMELIST
&FRESKO hcm=0.03 rmatch=40
    rintp=0.20 hnl=0.1 rnl=5.00 centre=0.0 ! for non-local coupling
    jtmin=0.0 jtmax=120 absend=-1.0
    thmin=0.00 thmax=40.00 thinc=0.10
    iter=1 nnu=36
    chans=1 smats=2 xstabl=1
    elab=170.0 /

&PARTITION namep='f17' massp=17. zp=9
    namet='n14' masst=14. zt=7 nex=1 /
&STATES jp=2.5 bandp=1 ep=0.0 cpot=1 jt=1.0 bandt=1 et=0.0000/

&PARTITION namep='ne18' massp=18. zp=10
    namet='c13' masst=13. zt=6 qval=3.6286 nex=1/
&STATES jp=0. bandp=1 ep=0.0 cpot=2 jt=0.5 bandt=1 et=0.0000/
&partition /

&POT kp=1 ap=17.000 at=14.000 rc=1.3 / ! entrance channel pot
&POT kp=1 type=1 p1=37.2 p2=1.2 p3=0.6 p4=21.6 p5=1.2 p6=0.69 /

&POT kp=2 ap=18.000 at=13.000 rc=1.3 / ! exit channel pot
&POT kp=2 type=1 p1=37.2 p2=1.2 p3=0.6 p4=21.6 p5=1.2 p6=0.69 /

&POT kp=3 at=17 rc=1.2 / ! 18Ne=17F+p binding pot
&POT kp=3 type=1 p1=50.00 p2=1.2 p3=0.65 /
&POT kp=3 type=3 p1=6.00 p2=1.2 p3=0.65 /

&POT kp=4 at=13 rc=1.2 / ! 14N=p+13C binding pot
&POT kp=4 type=1 p1=50.00 p2=1.2 p3=0.65 /
&POT kp=4 type=3 p1=6.00 p2=1.2 p3=0.65 /

&POT kp=5 ap=17.000 at=14.000 rc=1.3 / !core-core pot? same as kp=1?
&POT kp=5 type=1 p1=37.2 p2=1.2 p3=0.6 p4=21.6 p5=1.2 p6=0.69 /
&pot /

&Overlap kn1=1 ic1=1 ic2=2 in=1 kind=0
    nn=1 l=2 sn=0.5 j=2.5 kbpot=3 be=3.922 isc=1 ipc=0 /
    ! < 17F|18Ne>
&Overlap kn1=2 ic1=2 ic2=1 in=2 kind=3
    nn=1 l=1 sn=0.5 ia=1 ib=1 j=1.0
    kbpot=4 be=7.5506 isc=1 ipc=0 /
    ! < 14 N| 13 C> ! kind=3 for non-zero spin core
&overlap /

```

```

&Coupling icto=-2 icfrom=1 kind=7 ip1=0 ip2=-1 ip3=5 /
&CFP in=1 ib=1 ia=1 kn=1 a=1.00 /
&CFP in=2 ib=1 ia=1 kn=2 a=1.00 /
&CFP /
&coupling /

```

- Example from text book. Considering the $^{14}\text{N}(^{17}\text{F}, ^{18}\text{Ne})^{13}\text{C}$ as a proton transfer. $^{14}\text{N}(1^+) = p(1p_{\frac{1}{2}, \frac{3}{2}}) + ^{13}\text{C}(\frac{1}{2}^-)$ and $^{18}\text{Ne}(0^+) = p(1d_{\frac{5}{2}}) + ^{17}\text{F}(\frac{5}{2}^+)$.
- optical potential in entrance/exit channel U_1 and U_2 , binding potential V_3 and V_4 , core-core optical potential U_5 are defined.
- positive Q-value seems to be wrong. The correct Q-value of the reaction is $Q = -3.626$ MeV.
- Because non-local kernel $V_{fi}(R', R)$ is required for transfer calculation, **rintp**, **hnl**, **rnl**, centre values are defined. **rintp** is a step in R , **hnl** is a step in non-local range **rnl** = $R' - R$ from **centre**.
- For non-local kernel, angular integration is done with Gaussian quadrature number **nnu**.
- The projectile overlap $\langle ^{18}\text{Ne} | ^{17}\text{F} \rangle$ for $^{18}\text{Ne}(0^+) = p(1d_{\frac{5}{2}}) + ^{17}\text{F}(\frac{5}{2}^+)$ is done using **KIND=0** $|(ln, sn)j\rangle$ coupling scheme. In this case, $ln = 2$, $sn = 0.5$ and $j = 2.5$ corresponds to single particle state $d_{\frac{5}{2}}$. Core spin 5/2 and Composite spin 0 is given in the partition.

Q: Shouldn't use **KIND=3** for non-zero core-spin? spin is ignored? Is it because the composite spin is zero?

- In case of target overlap, $\langle ^{14}\text{N} | ^{13}\text{C} \rangle$ for $^{14}\text{N}(1^+) = p(1p_{\frac{1}{2}, \frac{3}{2}}) + ^{13}\text{C}(\frac{1}{2}^-)$.

Because the core has non-zero spin, overlap requires **KIND=3** coupling scheme, $|(l_n, s_n)j_n, I_A; I_B\rangle$ where I_A and I_B is defined in core state number **ia** and composite state number **ib**. thus, only needs to specify ln, sn, jn ?

Q: Why $ic1 = 1, ic2 = 2$ for **kn=1** but $ic1 = 2, ic2 = 1$ for **kn=2**? Or it does not make difference?

Q: Why $j = 1.0$? Shouldn't it be $j = 1/2$ or $3/2$? or in this case, **j** means maximum value and thus, only $j = 1/2$ is included?

Note that, there can be a summation over $l < l_n < l_{max}$, j_n and all core states $I_A(\text{core})$ coupled to form I_B . (for deformed potential). j is the maximum j_n in the summation if $l_{max} > 1$. **Q:** Does it mean in general **kind=3** implies $|I_B; s_n\rangle = \sum_{l_n, j_n, I_A} |(l_n, s_n)j_n, I_A; I_B\rangle$? **It looks like l and j for **KIND=3** actually means l_{max} and j_{max} .**

- In the coupling, **Kind=7** is a finite range transfer. In this case, $ip1 = 0, 1$ is for post or prior. $ip2 = 0, 1, -1$ for remnant term. $ip3$ is a core-core optical potential index.

In the case of post form, coupling is from

$$\begin{aligned}
\text{prior} : & V_{p+^{17}\text{F}} + [V_{^{17}\text{F}+^{13}\text{C}} - U_{^{17}\text{F}+^{14}\text{N}}], \\
\text{post} : & V_{p+^{13}\text{C}} + [V_{^{17}\text{F}+^{13}\text{C}} - U_{^{18}\text{Ne}+^{13}\text{C}}].
\end{aligned} \tag{12.25}$$

The choice of initial and final partition and the choice of prior and post form determines the dominant part of coupling potential and channel optical potential. Then, $ip3$ would be used to specify the core-core potential part $V_{17F+13C}$.

If $ip3 = 0$, it would use optical potential of projectile core partition. Does it mean $ip3 = 0$, use $U_{17F+14N}(R_{17F+13C})$ as a core-core potential for remnant term?

In this example, $ip3 = 5$ is the same as $kp = 1$ and thus corresponds to $U_{17F+14N}$ as a core-core potential. Even in this choice with prior form, there will be a remnant term because of the coordinate difference.

12.8 Transfer: (T) $^{136}\text{Xe}(t,\alpha)^{135}\text{I}$ in ZR approximation

```

ZR (t,Alpha) reaction on 136-Xenon at sub-barrier energies
0.1000 25.0 0.50 0.2 5.00 0.9
0. 10. -.1 F F
0 66.0 146. 5.
0.00 1 0 0 24 0 .0
1 0 0 0 4 1 0 0 0 0
TRITON 3.016 1.0 1 XE-136 135.9072 54. -9.879
0.5 +1 0.0 0 1 0.0 +1 0.0
ALPHA 4.0026 2.0 1 I-135 134.910 53.0 0.000
0.0 +1 0.0 0 2 3.5 1 0.00
! &POT
1 0 0 135.90 1.25 0.65
1 1 0 150.0 1.28 0.68 10.0 1.42 0.88
2 0 0 134.91 1.25 0.65
2 1 0 225.0 1.32 0.585 33.0 1.32 0.585
4 0 0 135.0 0.0 1.25 0.65
4 1 0 60.000 1.250 0.650
4 3 0 8.00 1.25 0.650
0 !&OVERLAP for <Xe-136|I-135>
2 1 2 2 0 1 4 0.5 3.5 4 0 9.936 1 0 0 1 1.0000
0 !&COUNPLING KIND=5 Zero Range Transfer
2 1 5 0-1 0 -600. ! p1=D_0=-600
-2 1 1 2 1.000
0 1 1
5.89

```

- The example is from the web(or the test directory of FRESCO).
- A proton transfer between $t(\frac{1}{2}^+) + ^{136}\text{Xe}(0^+)$ and $\alpha(0^+) + ^{135}\text{I}(\frac{7}{2}^+)$ in a model $\alpha = t(\frac{1}{2}^+) + p(1s_{1/2})$ and $^{136}\text{Xe}(0) = ^{135}\text{I}(\frac{7}{2}^+) + p(1g_{7/2})$.
- In finite range model the $CFP = 1.4142$ for $\langle \alpha | t \rangle$ is used. [Can I understand the origin of this CFP ?](#)
- In ZR approximation(KIND=5), $p1 = D_0 = -600$ is used without defining overlap function $\langle \alpha | t \rangle$. [Can I understand the origin of this \$D_0\$?](#)
- Zero range approximation means that the product of $V_{t+p}\phi_{t+p}$ can be approximated as $D_0\delta(r)$. Ie. Approximated the t and p interaction is zero range. And Thus, there

is no need to specify, V_{t+p} or ϕ_{t+p} . Only need, $I + p$ binding potential, here V_4 , and wave function. And no need of core-core potential. But, it have U_1 and U_2 of optical potential in entrance/exit channel.

12.9 Transfer CCBA: (T) $^{28}\text{Si}(^{19}\text{F}, ^{16}\text{O})^{31}\text{P}$

```

Test run: CCBA calculation 28Si(19F,16O) 31P, cluster form factors.
0.04      25.0    0.50    0.100    3.
  0. 80.    .01    F F
1 0.0    60.0    2.5
0.00      1 2 0 30
  1 2 0 0 1 0 0 0 0
19-F      19.0    9.0          2 28-SI    28.0    14.0    -00000000
0.5    +1 0.0          0          0.0    +1 0.0
2.5    +1 0.2
16-0      16.0    8.0          1 31-P    31.0    15.0    6.199
0.0    +1 0.0          0          0.5    +1 0.0
! &POT
  1 0 0 28.0      19.0    1.2    ! V1= 19F-2Si Coul ?
  2 0 0 31.0      16.0    1.35   ! U2= 160-31P optical ?
  2 1 0 31.20    1.45    0.47    15.10    1.270    0.310
  2 3 0 0.75    1.24    0.37    ! V3 = 31P binding pot
  3 0 0 19.0     0.0      1.25    0.65
  3 1 0 115.00   1.250    0.650
  3 3 0 6.30     1.25     0.650
  4 0 0 31.0     0.0      1.25     0.65 !V4= 19F binding pot
  4 1 0 99.000   1.250    0.650
  4 3 0 6.30     1.25     0.650
  5 0 0 28.0     0.0      1.2          !U5= core-target
  5 1 0 27.00    2.42452  0.460    11.66    2.14456  0.238
  6 0 0 28.0     0.0      1.25     0.65 !U6 = frag-target
  6 1 0 160.00   1.07     0.72     37.0     1.35     0.88
0 !&OVERLAP
  1    1 2 1 0    4 0    0.5    0.5    3 0    11.73    1 0 0 1    1.0000
  2    1 2 2 0    4 0    0.5    0.5    4 0    17.899    1 0 0 1    1.0000
  3    1 2 1 0    3 2    0.5    2.5    3 0    11.53    1 0 0 1    1.0000
0 !&COUPLING
  2    1    7 0 0 0          ! Finite Range transfer(Kind=7)
    1    1    1    1    1.00
    1    2    1    3    1.00
   -2    1    1    2    1.00
  1    2    3 4 1 0    6.0    5.0 ! S.P. excitation of projectile (KIND=3)

  0    1    1
60.0

```

- This is an extension of previous in-elastic scattering calculation.
- CPOT value seems to be wrong. V_1 is entrance channel coulomb and U_2 is exit channel optical potential. No optical potential in entrance channel is defined. But, probably corresponding diagonal coupling is calculated by using cluster folding.

- V_3 is a $^{31}P = ^{28}Si + ^3H$ binding potential. V_4 is a ^{19}F binding potential. U_5 is a core-core potential. However, U_6 is a $^{28}Si + ^3H$ optical potential. Thus, one use different potential between $^{28}Si + ^3H$ for entrance channel and bound state in exit channel.
- The in-elastic coupling part and potentials are the same.
- $kn=1$ and $kn=3$ are the overlap function for $\langle ^{19}F(\frac{1}{2}^+, \frac{5}{2}^+) | ^{16}O \rangle$ and $kn=2$ is a overlap function for $\langle ^{31}P | ^{28}Si \rangle$.
- Finite Range transfer (Kind=7) is added in Post(IP1=0) form without remnant (IP2=0).

12.10 Two Neutron Transfer: (T) 9Li(t,p)11Li

```

9Li(t,p) with s^2 triton & Yabana 11Li. Sim+seq
NAMELIST
&FRESCO hcm= 0.100 rmatch= 25.000 rintp= 0.50
  hnl= 0.300 rnl= 12.00 centre= 0.00
  hnn= 0.500 rnn= 20.00 rmin= 0.50
  jtmin= 0.0 jtmax= 30.0 absend= 0.0100 dry=F
  thmin= 0.00 thmax= 90.00 thinc= 2.00
  it0=0 iter= 2 fatal=T iblock= 0
  iso='□' nnu= 24 maxl= 0 minl= 0 mtmin= 0 epc= 0.0000
  chans= 1 smats= 1 xstabl= 1
  unitmass= 1.000000 finec=137.03599
  inh=0 pel=1 exl= 1 lab=0 lin=0 lex= 0
  elab(1)= 39.8000 /

&PARTITION namep='Triton' massp= 3.0160 zp= 1 nex= 1 pwf=T
  namet='9Li' masst= 9.0000 zt= 3 qval= 0.0000/
&STATES jp= 0.5 ptyp= 1 ep= 0.0000 cpot= 1
  jt= 0.0 ptyt= 1 et= 0.0000/
&PARTITION namep='Deuteron' massp= 2.0141 zp= 1 nex=-1 pwf=T
  namet='10Li' masst= 10.0000 zt= 3 qval= -6.1570/
&STATES jp= 1.0 ptyp= 1 ep= 0.0000 cpot= 2
  jt= 0.5 ptyt=-1 et= 0.0000/
&PARTITION namep='Proton' massp= 1.0078 zp= 1 nex= 1 pwf=T
  namet='11Li□' masst= 11.0000 zt= 3 qval= -8.2800/
&STATES jp= 0.5 ptyp= 1 ep= 0.0000 cpot= 3
  jt= 0.0 ptyt= 1 et= 0.0000/
&partition / ! END OF DEFINING PARTITIONS

```

- Two neutron transfer reaction in 2nd DWBA including simultaneous and sequential transfer channels between 3-partitions:

$$\begin{aligned}
 t(\frac{1}{2}^+) + ^9Li(0^+) &\rightarrow p(\frac{1}{2}^+) + ^{11}Li(0^+), \\
 t(\frac{1}{2}^+) + ^9Li(0^+) &\rightarrow d(1^+) + ^{10}Li(\frac{1}{2}^-) \rightarrow p(\frac{1}{2}^+) + ^{11}Li(0^+). \quad (12.26)
 \end{aligned}$$

One can consider the simultaneous transfer as a di-neutron transfer in DWBA. However, in this example, the simultaneous transfer is considered as a two single neutron transfer at once.

- Simultaneous transfer is modeled as a structureless di-neutron transfer with structure $^{11}\text{Li} = ^9\text{Li} + (nn)$ and $t = p + (nn)$,

$$|^{11}\text{Li} = ^9\text{Li} \otimes (nn)\rangle, \quad |t = p \otimes (nn)\rangle \quad (12.27)$$

Q-value for the simultaneous transfer is $Q \simeq -6.2$ MeV

However, instead, in this example, the triton and ^{11}Li are modeled as two single particle neutrons. Thus, even the simultaneous transfer is computed by using the two neutron form factors.

- Sequential transfer is modeled as one neutron transfer at a time, $^{10}\text{Li} = ^9\text{Li} + n$, $^{11}\text{Li} = ^9\text{Li} + n + n$. Then, required form factors are

$$\begin{aligned} &|^{10}\text{Li} = ^9\text{Li} \otimes n(1p_{1/2})\rangle, \quad |^{11}\text{Li} = ^9\text{Li} \otimes n(1p_{1/2})^2\rangle, \quad |^{11}\text{Li} = ^{10}\text{Li} \otimes n(1p_{1/2})\rangle \\ &|d = p \otimes n(1s_{1/2})\rangle, \quad |t = p \otimes n(1s_{1/2})^2\rangle, \quad |t = d \otimes n(1s_{1/2})\rangle \end{aligned} \quad (12.28)$$

Note that the s.p. wave functions in two form factors ($|t = p \otimes n\rangle$ and $|t = d \otimes n\rangle$) are not necessary to be the same. (In case of Li, all three neutron form factors in $|^{10}\text{Li} = ^9\text{Li} \otimes n(1p_{1/2})\rangle, |^{11}\text{Li} = ^9\text{Li} \otimes n(1p_{1/2})^2\rangle, |^{11}\text{Li} = ^{10}\text{Li} \otimes n(1p_{1/2})\rangle$ are treated differently.)

Q-value for the intermediate partition is $Q \simeq -8.3$ MeV. In this example, the intermediate deuteron and ^{10}Li are treated to have physical binding energies.

```
! kp=1 t+9Li optical
&pot kp= 1 type= 0
  p(1:3)= 9.000 0.0000 1.2500 /
&pot kp= 1 type= 1
  p(1:7)= 146.000 1.2400 0.6780 25.0000 1.4500 0.8410 0.0000 /
! kp=2 d+10Li optical
&pot kp= 2 type= 0
  p(1:3)= 10.000 0.0000 1.2500 /
&pot kp= 2 type= 1
  p(1:7)= 85.800 1.1700 0.7600 1.1170 1.3250 0.7310 0.0000 /
&pot kp= 2 type= 2
  p(1:7)= 0.000 0.0000 0.0000 11.8630 1.3250 0.7310 0.0000 /
! kp=3 p+11Li optical
&pot kp= 3 type= 0
  p(1:3)= 11.000 0.0000 1.2500 /
&pot kp= 3 type= 1
  p(1:7)= 44.200 1.1700 0.7500 6.1000 1.3200 0.5400 0.0000 /
&pot kp= 3 type= 2
  p(1:7)= 0.000 0.0000 0.0000 2.4000 1.3200 0.5400 0.0000 /
! kp=4 d=n+p binding pot, p0=1 so type=0 is no need.
&pot kp= 4 type= 1
  p(1:7)= 100.000 0.4000 0.6000 0.0000 0.0000 0.0000 1.0000 /
! kp=5 t=d+n binding pot p0=1 so type=0 is no need.
&pot kp= 5 type= 1
  p(1:7)= 100.000 0.9500 0.6500 0.0000 0.0000 0.0000 1.0000 /
! kp=7 Li9+n pot
&pot kp= 7 type= 0
  p(1:3)= 1.000 0.0000 1.2000 /
```

```

&pot kp= 7 type= 1
      p(1:3)= 80.000 2.6700 0.6000 /
&pot kp= 7 type= 3
      p(1:3)= 10.000 2.6700 0.6000 /
&pot / ! END OF DEFINING POTENTIALS

```

- Potentials $V_1 = U_{t+^9Li}$, $V_2 = U_{d+^{10}Li}$, $V_3 = U_{p+^{11}Li}$ are optical potentials in each channel. $V_4 = V_{d=n+p}$, $V_5 = V_{t=d+n}$, $V_7 = V_{n+^9Li}$ are binding potentials.
- But note that V_5 is not an interaction between d and n , but the p and n within triton.

```

! overlap function for t=d+n
&OVERLAP kn1= 1 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 1 l=0 sn= 0.5 j= 0.5
      kbpot= 5 krpot= 0 be= 6.2570 isc= 1 ipc=0 / !<t|d>
! overlap for d=p+n
&OVERLAP kn1= 2 kn2= 0 ic1=2 ic2=3 in= 1
      kind=0 nn= 1 l=0 sn= 0.5 j= 0.5
      kbpot= 4 krpot= 0 be= 2.2245 isc= 1 ipc=0 / !<d|p>

```

- $kn=1$ overlap $\langle t = d + n(1s_{1/2}) | d \rangle$ with $S_n = 6.257$ of 3H . shouldn't we use coupling $KIND=3$ for core spin? Without the spin-orbit interaction the overlap function and there is only one way to combine $1 \otimes \frac{1}{2} = \frac{1}{2}$. Thus, $KIND=0$ or 3 will give the same result.
- $kn=2$ overlap $\langle d = p + n(1s_{1/2}) | p \rangle$ with $S_n = 2.225$. Note that $\phi_{kn=1}$ and $\phi_{kn=2}$ are both the neutron wave overlap for $1s_{1/2}$ state, but they have different binding energies.

```

! overlap for t=p+n+n
&OVERLAP kn1= 3 kn2= 0 ic1=1 ic2=2 in= 1
      kind=0 nn= 1 l=0 sn= 0.5 j= 0.5
      kbpot= 5 krpot= 0 be= 4.2400 isc= 1 ipc=0 / !<t|d>
&OVERLAP kn1= 10 kn2= 90 ic1=1 ic2=3 in= 1
      kind=9 ch1='□' nn= 1 l=0 lmax=1 sn= 0.0 ia= 1 j= 0.0 ib= 1
      kbpot= 1 krpot= 0 be= 0.1000 isc= 0 ipc=2
      nfl=-21 nam= 1 ampl= 1.0000 / !<t|p>
&twont
      tnt(1:4, 1)= 3 3 0 0 coef( 1)= 1.0000 /

```

- $kn=3(KIND=0)$ is the neutron s.p. overlap in $|t = p + n + n\rangle$ with $S_n = 4.24$. $kn=10-90$ are for the two neutron overlap in $|t = p + n + n\rangle$ with $\phi_{kn=3}$ s.p. overlap functions. The binding energy have to be matched to $S_{2n} = 8.481$ of triton. Thus, each neutron has binding energy of $S_{2n}/2$.

Note that $\phi_{kn=1}$ and $\phi_{kn=2}$ is used for sequential transfer but $\phi_{kn=3}$ is used for the construction of two neutron state of triton $|t = p \otimes n(1s_{1/2})^2\rangle$.

- $kn=10-90$ are two-neutron overlaps ($KIND=9$) using $\phi_{kn=3}$ s.p. overlaps. The particle state is constructed as

$$\begin{aligned}
\varphi(\mathbf{r}_1, \mathbf{r}_2) &= \sum_i c_i |l_1(i), s_1) j_1(i), (l_2(i), s_2) j_2(i); J_{12} T\rangle \\
&\rightarrow \sum_{i=1} c_i \sum_{LjSj} |L, (l, (s_1 s_2) S) j; J_{12} T\rangle \phi_{L(lS)j}^{J_{12} T, i}(r, \rho)
\end{aligned} \tag{12.29}$$

Further, the triton is constructed (KIND=9) as

$$|(L_{nn}, (l, s_{12})j)J_{12}, J_{core}; J_{com}\rangle |t_{12}, T_{core}; T_{com}\rangle \quad (12.30)$$

- Note that the meaning of each input parameters change when KIND=9.

NN	→ NPAIRS : number of pair products to be summed(?)
L	→ l_{min} , minimum orbital angular momentum l
LMAX	→ l_{max} , maximum orbital angular momentum l
SN	→ S_{min} , minimum sum S_{12} of the two nucleons' intrinsic spins ($S_{max} = 1.0$ always)
IA,IB	give core and composite states. ($J_{core}, T_{core}, J_{com}, T_{com}$)
JN, J	→ J_{12}
KBPOT	→ T, total isospin of two nucleons (used to enforce $l + S_{12} + T = odd.$)
KRPOT	→KNZR, the KN index to a single-particle state $u_{12}(r)$ of KIND 0 or 1, giving the N-N relative motion in the other participating nucleus (usually in the light ion). If KNZR>0, then just the overlap $u(R) = \langle u_{12}(r) U(r, R)\rangle$ is produced, suitable for zero-range two-nucleon transfer calculations.
BE	→ EPS, the threshold percentage to define components with square norms sufficiently small to be omitted in the final two-nucleon state.
ISC	≤ 0 : use Gaussian quadrature grid for the NN-distance RMIN to RNN
IPC	summary of $U(r, R)$ form factor for each r
NFL	file writing an reading

- &TWONT specifies hwat kind of ϕ should be used and the coefficient c_i values from shell model calculations.

```

! overlap for 10Li=9Li+n
&OVERLAP kn1= 6 kn2= 0 ic1=1 ic2=2 in= 2
      kind=0 nn= 1 l=1 sn= 0.5 j= 0.5
      kbpot= 7 krpot= 0 be= 0.1000 isc= 1 ipc=0 nfl= 0
      nam= 1 ampl= 1.0000 / !<9Li|10Li>
! overlap for 11Li=10Li+n
&OVERLAP kn1= 7 kn2= 0 ic1=2 ic2=3 in= 2
      kind=0 nn= 1 l=1 sn= 0.5 j= 0.5
      kbpot= 7 krpot= 0 be= 0.1000 isc= 1 ipc=0 nfl= 0
      nam= 1 ampl= 1.0000 / !<11Li|10Li>
! overlap for 11Li=9Li+n+n
&OVERLAP kn1= 90 kn2=170 ic1=1 ic2=3 in= 2
      kind=6 ch1='_' nn= 1 l=0 lmax=1 sn= 0.0 ia= 1 j= 0.0 ib= 1
      kbpot= 1 krpot= 0 be= -0.1000 isc= 0 ipc=2 nfl=-22
      nam= 1 ampl= 1.0000 / !<9Li|11Li>
&twont
      tnt(1:4, 1)= 6 6 0 0 coef( 1)= 1.0000 /

&overlap / ! END OF DEFINING OVERLAPS

```

- In case of overlap $kn=6 \langle 10Li(0.5-) = 9Li(0+)+p(1p_{1/2})|9Li(0+)\rangle$ and $kn=7 \langle 11Li(0+) = 10Li(0.5-) + p(1p_{1/2})|10Li(0.5-)\rangle$, use unphysical $S_n = 0.1$. It is because the ^{10}Li is an unbound state with $S_n = -0.026$. Also, note that $S_n = 0.3954$, $S_{2n} = 0.3692$ for ^{11}Li .

Thus, in the calculation, un-realistic shallow bound state wave function is used for ^{10}Li overlap functions.

Also, $kn=6$, $kn=7$ is used for single particle transfer calculation. And $kn=6$ is also used for two nucleon transfer.

- $kn=90-170$ is constructed from two $\langle 10Li|9Li \rangle$ single neutron overlap functions.

```
!simultaneous transfer (p,t)
&COUPLING icto= 2 icfrom= 1 kind=7 ip1= 1 ip2= 1 ip3= 0
  p1= -163.2000 p2= 0.0000 jmax= 0.0 rmax= 0.00/
  &cfp in= 1 ib= 1 ia= 1 kn= 1 a= 1.176 /
  &cfp in=-2 ib= 1 ia= 1 kn= 6 a= 1.000 /
! sequential transfer (p,d)
&COUPLING icto= 3 icfrom= 2 kind=6 ip1= 0 ip2= 1 ip3= 0
  p1= -125.7000 p2= 0.0000 jmax= 0.0 rmax= 0.00/
  &cfp in= 1 ib= 1 ia= 1 kn= 2 a= 1.000 /
  &cfp in=-2 ib= 1 ia= 1 kn= 7 a= 1.000 /
! sequential transfer (d,t)
&COUPLING icto= 3 icfrom= 1 kind=7 ip1= 1 ip2= 1 ip3= 0 /
  &cfp in= 1 ib= 1 ia= 1 kn= 10 a= 1.000 /
  &cfp in=-2 ib= 1 ia= 1 kn= 90 a= 1.000 /
&COUPLING icto= 0 icfrom= 1 kind=1 ip1= 0 ip2= 0 ip3= 0 /
```

- The couplings between partition are calculated as a finite range transfers.
- Can I understand the value CFP=1.176 for triton?
- The coupling between partition 2 and 3 is written as $kind = 6$ (ZR or LEA) transfer. However, it could be an typo and actually $kind = 7$.
- The coupling between partition , V_{21} and V_{31} are calculated in PRIOR FORM. But the V_{32} is computed in POST form, if $kind = 7$.

12.10.1 Another two neutron transfer example

```
two n transfer Zn64(p,t)Zn62 at 24 MeV Q=-12.50
NAMELIST
&FRESCO hcm=0.05 rmatch=20.0 rintp=0.2 hnl=0.1 rnl=16.0
  centre=-1.2 hnn=0.3 rnn=10.0 rmin=0.30
  jtmin=0.0 jtmax=20 absend=-1.0
  thmin=0.0 thmax=180.0 thinc=1.0
  iter=2 nnu=36
  chans=1 xstabl=1 smats=2
  pel=2 elab=24.0 /

&PARTITION namep='t3' massp=3.016 zp=1
  namet='Zn62' masst=62.0 zt=30 nex=1/
&STATES jp=0.5 bandp=1 ep=0.0 cpot=1 jt=0.0 bandt=1 et=0.0/

&PARTITION namep='p1' massp=1.0078 zp=1
  namet='Zn64' masst=64.0 zt=30 qval=12.497 nex=1 /
&STATES jp=0.5 bandp=1 ep=0.0 cpot=2 jt=0.0 bandt=1 et=0.0 /

&PARTITION namep='d2' massp=2.0141 zp=1
  namet='Zn63' masst=63.0 zt=30 qval= 4.232 nex=-4 / !qval?
&STATES jp=1.0 bandp=1 ep=0.0 cpot=9 jt=3.5 bandt=-1 et=0.0 / ! jt? et?
```

```

&STATES copyyp=1          cpot=9 jt=1.5 bandt=-1 et=0.0 /
&STATES copyyp=1          cpot=9 jt=2.5 bandt=-1 et=0.0 /
&STATES copyyp=1          cpot=9 jt=0.5 bandt=-1 et=0.0 /
&PARTITION /

```

```

&POT kp=1 at=62.0 rc=1.422 / ! t+Zn62
&POT kp=1 type=1 p1=149.921 p2=1.082 p3=0.759 p4=11.927 p5=1.275 p6=1.195 /
&POT kp=1 type=2 p4=28.008 p5=1.134 p6=0.857 /
&POT kp=1 type=3 p1=1.903 p2=0.503 p3=0.148 /

&POT kp=2 at=64.0 rc=1.270 / ! p+Zn64
&POT kp=2 type=1 p1=49.708 p2=1.194 p3=0.690 p4=1.607 p5=1.225 p6=0.690 /
&POT kp=2 type=2 p4= 7.209 p5=1.225 p6=0.690 /
&POT kp=2 type=3 p1= 5.9 p2=1.04 p3=0.630 /

&POT kp=4 at=1.0 rc=1.28 / ! triton p+n
&POT kp=4 type=1 p1=100. p2=0.95 p3=0.65 p4=0. p5=0. p6=0. /

&POT kp=5 at= 62.0 rc=1.270 / ! p
&POT kp=5 type=1 p1=49.708 p2=1.194 p3=0.690 p4=1.607 p5=1.225 p6=0.690 /
&POT kp=5 type=2 p4= 7.209 p5=1.225 p6=0.690 /
&POT kp=5 type=3 p1= 5.9 p2=1.04 p3=0.630 /

&POT kp=6 at=62.0 rc=1.250 / ! n-binding
&POT kp=6 type=1 p1=50.0 p2=1.25 p3=0.65 /
&POT kp=6 type=3 p1= 6.5 p2=1.25 p3=0.65 /

&POT kp=9 at=63.0 rc=1.3 / ! d-pot
&POT kp=9 type=1 p1=90.868 p2=1.170 p3=0.737 p4=0.335 p5=1.325 p6=0.808 /
&POT kp=9 type=2 p4=12.291 p5=1.325 p6=0.808 /
&POT kp=9 type=3 p1=6.854 p2=1.070 p3=0.660 /

&POT kp=11 type=1 p(1:7)= 100.0 0.4 0.6 0. 0. 0. 1.0 / !d=n+p
&POT kp=12 type=1 p(1:7)= 100.0 0.95 0.65 0. 0. 0. 1.0 / ! t=d+n
&POT /

```

```

&OVERLAP kn1=1 ic1=1 ic2=3 in=1 kind=0
nn=1 l=0 sn=0.5 j=0.5
kbpot= 12 be=6.2570 isc=1 ipc=0 / !<d|t>

&OVERLAP kn1=2 ic1=2 ic2=3 in=1 kind=0
nn=1 l=0 sn=0.5 j=0.5
kbpot= 11 be=2.2245 isc=1 ipc=0 / !<p|d>

&OVERLAP kn1=3 ic1=1 ic2=3 in=1 kind=0
nn=1 l=0 sn=0.5 j=0.5
kbpot=4 be=4.2409 isc=1 ipc=0 /

&OVERLAP kn1=30 kn2=60 ic1=1 ic2=2 in=1 kind=6
nn=1 l=0 lmax=0 sn=0.0 ia=1 j=0.0 ib=1
kbpot=1 be=0.1 isc=0 ipc=1 / !proj
&twont tnt(1,1)=3 tnt(2,1)=3 coef(1)=1.0 /

```

Triton Overlap part is almost the same as previous example, except, (1) KIND=6 is used here instead of KIND=9 in previous example.

```
&OVERLAP kn1=12 ic1=3 ic2=1 in=2 kind=0
nn=1 l=3 sn=0.5 j=3.5
kbpot=6 be=10.489 isc=1 ipc=0 / ! 1f7/2
&OVERLAP kn1=14 ic1=3 ic2=1 in=2 kind=0
nn=2 l=1 sn=0.5 j=1.5
kbpot=6 be=10.489 isc=1 ipc=0 / ! 2p3/2
&OVERLAP kn1=13 ic1=3 ic2=1 in=2 kind=0
nn=1 l=3 sn=0.5 j=2.5
kbpot=6 be=10.489 isc=1 ipc=0 / ! 1f5/2
&OVERLAP kn1=15 ic1=3 ic2=1 in=2 kind=0
nn=2 l=1 sn=0.5 j=0.5
kbpot=6 be=10.489 isc=1 ipc=0 / ! 2p1/2

&OVERLAP kn1=200 kn2=270 ic1=2 ic2=1 in=2 kind=6
nn=4 l=0 lmax=0 sn=0.0 ia=1 j=0.0 ib=1
kbpot=1 be=0.1 isc=0 ipc=2 /
&twont tnt(1,1)=12 tnt(2,1)=12 coef(1)=0.2006
tnt(1,2)=13 tnt(2,2)=13 coef(2)=0.8135
tnt(1,3)=14 tnt(2,3)=14 coef(3)=0.6708
tnt(1,4)=15 tnt(2,4)=15 coef(4)=0.4138 /
&OVERLAP /
```

Here the ^{64}Zn is modeled as a combination of several configuration. However, unlike the case of previous example, the binding energy of all neutron are set as $be=10.489$ which is half of two-neutron separation energy of ^{64}Zn . And the same overlap os also used to model ^{63}Zn which have $S_n = 9.116$.

Note $nn=4, j=0$ (i.e. only two neutrons forming 0^+) are considered.

The coefficients for two-neutron configurations are from shell model. Question is how one can obtain such information.

```
! simultaneous transfer
&COUPLING icto=2 icfrom=1 kind=7 ip1=0 ip2=-1 ip3=5 /
&CFP in=1 ib=1 ia=1 kn=30 a=1.0 /
&CFP in=-2 ib=1 ia=1 kn=200 a=1.0 /
! sequential (d,t)
&COUPLING icto=3 icfrom=1 kind=7 ip1=1 ip2=-1 ip3=0 p1=-163.2 / ! (t3,d)
&CFP in=1 ib=1 ia=1 kn=1 a=1.176 /
&CFP in=2 ib=1 ia=1 kn=12 a=0.2006 /
&CFP in=2 ib=3 ia=1 kn=13 a=0.8135 /
&CFP in=2 ib=2 ia=1 kn=14 a=0.6708 /
&CFP in=2 ib=4 ia=1 kn=15 a=0.4138 /
&CFP /
! sequential (p,d)
&COUPLING icto=2 icfrom=3 kind=7 ip1=0 ip2=-1 ip3=0 p1=-122.5 / ! (d,p)
&CFP in=1 ib=1 ia=1 kn=2 a=1.0 /
&CFP in=2 ib=1 ia=1 kn=12 a=1.4142 / ! sqrt(2)
&CFP in=2 ib=1 ia=3 kn=13 a=1.4142 /
&CFP in=2 ib=1 ia=2 kn=14 a=1.4142 /
&CFP in=2 ib=1 ia=4 kn=15 a=1.4142 /
&CFP /
```

- In this example, triton and ${}^{64}\text{Li}$ is treated as two neutron s.p. states.
- In this example, the Q-value for the intermediate state ${}^{63}\text{Zn}$ is not physical value. (deuteron binding energy is physical value) Instead, it's binding energy is inferred from the $S_{2n} = 20.978$ of ${}^{64}\text{Zn}$ and set as $S_{2n}/2 = 10.489$ MeV. This is because the ${}^{64}\text{Zn}$ is treated as a bound state of two single particle neutron states. From triton's $S_n = 6.257$, this gives $Q = 10.489 - 6.257 = 4.232$.
- The ${}^{64}\text{Zn}$ is modeled in several mixed configuration,

$$\begin{aligned} |{}^{64}\text{Zn}(0^+)\rangle &= a_1|{}^{62}\text{Zn} \otimes n(1f_{7/2})^2\rangle + a_2|{}^{62}\text{Zn} \otimes n(2p_{3/2})^2\rangle \\ &+ a_3|{}^{62}\text{Zn} \otimes n(1f_{5/2})^2\rangle + a_4|{}^{62}\text{Zn} \otimes n(2p_{1/2})^2\rangle \end{aligned} \quad (12.31)$$

At the same time, this implies that the ${}^{63}\text{Zn}$ can have several configuration,

$$\begin{aligned} |{}^{63}\text{Zn}(7/2^-)\rangle &= a_1|{}^{62}\text{Zn} \otimes n(1f_{7/2})\rangle \\ |{}^{63}\text{Zn}(3/2^-)\rangle &= a_2|{}^{62}\text{Zn} \otimes n(2p_{3/2})\rangle \\ |{}^{63}\text{Zn}(5/2^-)\rangle &= a_3|{}^{62}\text{Zn} \otimes n(1f_{5/2})\rangle \\ |{}^{63}\text{Zn}(1/2^-)\rangle &= a_4|{}^{62}\text{Zn} \otimes n(2p_{1/2})\rangle \end{aligned} \quad (12.32)$$

- For the two neutron configuration, shell model inputs of *&Twont* is necessary. And also *&CFP*.
- triton CFP for $< p|t > 1.1736$ could be from ab-initio calculations.
- the CFP for ${}^{63}\text{Zn} = {}^{62}\text{Zn} + n$ is the same as the two-neutron TNT values of ${}^{64}\text{Zn}$. For the CFP of ${}^{64}\text{Zn} = {}^{63}\text{Zn} + n$ is $\sqrt{2}$, which may be expected from number of valence neutrons above ${}^{62}\text{Zn}$.
- According to the thesis of KGL, the spectroscopic amplitudes are from NUSHELLX results in (*.tna) files. However, it is noted that the required conversion is done using A.Brown's code. (How?)

<i>s.p.</i>	<i>tna</i>	<i>FRESCO</i>
$1f_{7/2}$	0.18238	$\rightarrow 0.2006$
$2p_{3/2}$	0.73959	$\rightarrow 0.8135$
$1f_{5/2}$	0.60986	$\rightarrow 0.6708$
$2p_{1/2}$	0.37620	$\rightarrow 0.4138$

In this example, the conversion seems to be a overall factor. $(tna)*1.1 \simeq (tna)/0.909 \simeq (cfp)$. Can I understand this?

- For the sequential p-d-t process, the choice of the two prior forms are used, since the interactions for the smaller particle systems are better known. But this introduce non-orthogonality terms. If these states which are not orthogonal are included, the possibility of overcounting the intermediate states is present, and therefore a simple sum of the two-step amplitude can not be done.

12.11 CRC : (T) 16O+208Pb 80 MeV multi-channel path

16O+208Pb 80 MeV, 170 + 15N + 12C channels, S(212Po) = 1.30 BARE in TR.

NAMELIST

```
&FRESCO hcm= 0.025 rmatch= 30.000 rintp= 0.50
  hnl= 0.025 rnl= 6.00 centre= -0.00
  jtmin= 0.0 jtmax= 100.0 absend= 0.1000 dry=F rela='_' nearfa= 1
  thmin= 80.00 thmax=180.00 thinc= 2.50 koords=0 kqmax=0 pp=0
  cutl= 0.00 cutr=-10.00 cutc= 0.00
  ips= 0.0100 it0=0 iter= 30 fatal=T iblock= 3 pade=1 nosol=F psiren=T
  iso='_' nnu= 24 maxl= 0 minl= 0 mtmin= 0 epc= 0.0000
  erange= 1.2000 dk= 0.0200
  chans= 1 listcc= 0 treneg= 0 cdetr= 0 smats= 2 xstabl= 0 nlp1= 0
  waves= 0 lampl= 0 veff= 1 kfus=20 wdisk= 0 bpm= 0 melfil= 0
  unitmass= 1.000000 finec=137.03599
  inh=0 pel=1 exl= 1 lab=0 lin=0 lex= 0
  elab(1)= 80.0000 /

&PARTITION namep='16-0_0000' masssp= 15.9949 zp= 8 nex= 5 pwf=T
  namet='PB-208_00' masst=207.9770 zt= 82 qval= 0.0000/
&STATES jp= 0.0 ptyp= 1 ep= 0.0000 cpot= 1
  jt= 0.0 ptyt= 1 et= 0.0000/
&STATES jp= 3.0 ptyp=-1 ep= 6.1300 cpot= 1
  copyt= 1/
&STATES copyp= 1 cpot= 1
  jt= 3.0 ptyt=-1 et= 2.6100/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 1
  jt= 2.0 ptyt= 1 et= 4.0700/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 1
  jt= 5.0 ptyt=-1 et= 3.2000/
&PARTITION namep='17-0_0000' masssp= 16.9990 zp= 8 nex= 6 pwf=T
  namet='PB-207_00' masst=206.9760 zt= 82 qval= -3.2330/
&STATES jp= 2.5 ptyp= 1 ep= 0.0000 cpot= 2
  jt= 0.5 ptyt=-1 et= 0.0000/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 2
  jt= 2.5 ptyt=-1 et= 0.5710/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 2
  jt= 1.5 ptyt=-1 et= 0.8980/
&STATES jp= 0.5 ptyp= 1 ep= 0.8710 cpot= 2
  copyt= 1/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 2
  copyt= 2/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 2
  copyt= 3/
&PARTITION namep='15-N_0000' masssp= 14.9998 zp= 7 nex= 2 pwf=T
  namet='BI-209_00' masst=208.9800 zt= 83 qval= -8.3239/
&STATES jp= 0.5 ptyp=-1 ep= 0.0000 cpot= 3
  jt= 3.5 ptyt=-1 et= 0.8966/
&STATES jp= 0.0 ptyp= 0 ep= 0.0000 cpot= 3
  jt= 1.5 ptyt=-1 et= 3.1180/
&PARTITION namep='12-C_0000' masssp= 12.0000 zp= 6 nex= 1 pwf=T
  namet='P0-212_00' masst=212.0000 zt= 84 qval=-20.0000/
&STATES jp= 0.0 ptyp= 1 ep= 0.0000 cpot= 8
```



```

        jt= 0.0 ptyt= 1 et= 0.0000/
&partition /      ! END OF DEFINING PARTITIONS

&pot kp= 1 type= 0
p(1:3)= 208.000 16.0000 1.2000 /
&pot kp= 1 type=10
p(1:3)= 0.000 0.0000 37.6000 /
&pot kp= 1 type=13
p(1:7)= 0.000 54.45 815.0
        0.000 0.2380E+05 0.000 0.000 /
&step ib= 1 ia= 3 k=3 str= 815.0 /
&step ib= 3 ia= 1 k=3 str= 815.0 /
&step ib= 1 ia= 4 k=2 str= 54.45 /
&step ib= 4 ia= 1 k=2 str= 54.45 /
&step ib= 1 ia= 5 k=5 str= 0.2380E+05 /
&step ib=-5 ia= 1 k=5 str= 0.2380E+05 /
&pot kp= 1 type= 1 itt=F shape= 1
p(1:7)= 0.000 1.1790 0.6580 10.0000 1.0000 0.4000 0.0000 /
&pot kp= 1 type=-1
p(1:3)= 60.500 1.1790 0.6580 /! TYPE=-1 for numerical addition
&pot kp= 1 type=10 itt=F shape=11
p(1:3)= 0.000 0.0000 2.1500 /
&pot kp= 1 type=13 itt=F shape=11
p(1:7)= 0.000 0.4000 0.8000 0.0000 0.4680 0.0000 0.0000 /
&step ib= 1 ia= 3 k=3 str= 0.8000 /
&step ib= 3 ia= 1 k=3 str= 0.8000 /
&step ib= 1 ia= 4 k=2 str= 0.4000 /
&step ib= 4 ia= 1 k=2 str= 0.4000 /
&step ib= 1 ia= 5 k=5 str= 0.4680 /
&step ib=-5 ia= 1 k=5 str= 0.4680 /
&pot kp=20 type= 1 itt=F shape= 1
p(1:7)= 0.000 1.1790 0.6580 10.0000 1.0000 0.4000 0.0000 /
&pot kp= 2 type= 0
p(1:3)= 207.000 17.0000 1.2000 /
&pot kp= 2 type= 1
p(1:7)= 78.280 1.2150 0.6500 0.0000 1.1620 0.6230 0.0000 /
&pot kp= 2 type= 1 itt=F shape= 1
p(1:7)= 0.000 1.1790 0.6580 10.0000 1.0000 0.4000 0.0000 /
&pot kp= 3 type= 0
p(1:3)= 209.000 15.0000 1.2000 /
&pot kp= 3 type= 1
p(1:7)= 78.280 1.2150 0.6500 0.0000 1.1620 0.6230 0.0000 /
&pot kp= 3 type= 1 itt=F shape= 1
p(1:7)= 0.000 1.1790 0.6580 10.0000 1.0000 0.4000 0.0000 /
&pot kp= 4 type= 0
p(1:3)= 16.000 0.0000 1.2000 /
&pot kp= 4 type= 1
p(1:3)= 60.000 1.2000 0.6000 /
&pot kp= 4 type= 3
p(1:3)= 6.000 1.2000 0.6000 /
&pot kp= 5 type= 0
p(1:3)= 15.000 0.0000 1.2000 /
&pot kp= 5 type= 1
p(1:3)= 60.000 1.2000 0.6000 /

```

```

&pot kp= 5 type= 3
p(1:3)= 6.000 1.2000 0.6000 /
&pot kp= 6 type= 0
p(1:3)= 207.000 0.0000 1.2500 /
&pot kp= 6 type= 1
p(1:3)= 60.000 1.2500 0.6500 /
&pot kp= 6 type= 3
p(1:3)= 7.000 1.2500 0.6500 /
&pot kp= 7 type= 0
p(1:3)= 208.000 0.0000 1.2500 /
&pot kp= 7 type= 1
p(1:3)= 60.000 1.2800 0.7600 /
&pot kp= 7 type= 3
p(1:3)= 5.000 1.2800 0.7600 /
&pot kp= 8 type= 0
p(1:3)= 212.000 12.0000 1.2000 /
&pot kp= 8 type= 1
p(1:7)= 78.280 1.2150 0.6500 0.0000 1.1620 0.6230 0.0000 /
&pot kp= 8 type= 1 itt=F shape= 1
p(1:7)= 0.000 1.1790 0.6580 10.0000 1.0000 0.4000 0.0000 /
&pot kp=10 type= 0
p(1:3)= 12.000 4.0000 1.2500 /
&pot kp=10 type= 1
p(1:3)= 60.000 1.2800 0.7600 /
&pot kp=11 type= 0
p(1:3)= 208.000 0.0000 1.2500 /
&pot kp=11 type= 1
p(1:3)= 110.900 1.4640 0.5000 /
&pot / ! END OF DEFINING POTENTIALS

&OVERLAP kn1= 1 kn2= 0 ic1=1 ic2=2 in= 1
kind=0 ch1='□' nn= 1 l=2 lmax=0 sn= 0.5 ia= 1 j= 2.5 ib= 1
kbpot= 4 krpot= 0 be= 4.1425 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 3 kn2= 0 ic1=1 ic2=2 in= 1
kind=0 ch1='□' nn= 2 l=0 lmax=0 sn= 0.5 ia= 1 j= 0.5 ib= 2
kbpot= 4 krpot= 0 be= 3.2715 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 9 kn2= 0 ic1=1 ic2=2 in= 2
kind=0 nn= 3 l=1 sn= 0.5 j= 0.5
kbpot= 6 krpot= 0 be= 7.3755 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 10 kn2= 0 ic1=1 ic2=2 in= 2
kind=0 nn= 2 l=3 sn= 0.5 j= 2.5
kbpot= 6 krpot= 0 be= 7.9465 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 11 kn2= 0 ic1=1 ic2=2 in= 2
kind=0 nn= 3 l=1 sn= 0.5 j= 1.5
kbpot= 6 krpot= 0 be= 8.2735 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 2 kn2= 0 ic1=1 ic2=3 in= 1
kind=0 ch1='□' nn= 1 l=1 lmax=0 sn= 0.5 ia= 1 j= 0.5 ib= 1
kbpot= 5 krpot= 0 be= 12.1259 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /

```

```

&OVERLAP kn1= 5 kn2= 0 ic1=1 ic2=3 in= 2
      kind=0 nn= 2 l=3 sn= 0.5 j= 3.5
      kbpot= 7 krpot= 0 be= 2.9054 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 7 kn2= 0 ic1=1 ic2=3 in= 2
      kind=0 nn= 3 l=1 sn= 0.5 j= 1.5
      kbpot= 7 krpot= 0 be= 0.6840 isc= 1 ipc=0 nfl= 0 nam= 1 ampl=
0.0000 /
&OVERLAP kn1= 4 kn2= 0 ic1=1 ic2=4 in= 1
      kind=0 nn= 5 l=0 sn= 0.0 j= 0.0
      kbpot=10 krpot= 0 be= 7.1610 isc= 1 ipc=0 nfl= 0 nam= 0 ampl=
0.0000 /
&OVERLAP kn1= 12 kn2= 0 ic1=1 ic2=4 in= 2
      kind=0 nn=10 l=0 sn= 0.0 j= 0.0
      kbpot=11 krpot= 0 be= 1.0000 isc= 1 ipc=0 nfl= 0 nam= 0 ampl=
0.0000 /
&overlap /      ! END OF DEFINING OVERLAPS

&COUPLING icto= 2 icfrom= 1 kind=7 ip1= 1 ip2= 1 ip3= 0 /
&cfp in= 1 ib= 1 ia= 1 kn= 1 a= 1.050 /
&cfp in= 1 ib= 4 ia= 1 kn= 3 a= 1.015 /
&cfp in= 2 ib= 1 ia= 1 kn= 9 a= 1.550 /
&cfp in= 2 ib= 1 ia= 2 kn= 10 a= 2.790 /
&cfp in=-2 ib= 1 ia= 3 kn= 11 a= 1.812 /
&COUPLING icto= 3 icfrom= 1 kind=7 ip1= 1 ip2= 1 ip3= 0 /
&cfp in= 1 ib= 1 ia= 1 kn= 2 a= 1.414 /
&cfp in= 2 ib= 1 ia= 1 kn= 5 a= 1.050 /
&cfp in=-2 ib= 2 ia= 1 kn= 7 a= 1.000 /
&COUPLING icto= 4 icfrom= 1 kind=7 ip1= 1 ip2= 1 ip3= 0 /
&cfp in= 1 ib= 1 ia= 1 kn= 4 a= 0.5000 /
&cfp in=-2 ib= 1 ia= 1 kn= 12 a= 1.300 /
&COUPLING icto= 0 icfrom= 1 kind=1 ip1= 0 ip2= 0 ip3= 0 /

```

- CCBA(iter= 30, iblock= 3) for 16O+208Pb multiple channel inelastic+transfer.

- States are

$$\begin{array}{ll}
1 & 16O(0+) + 208Pb(0+) \quad c_{pot} = 1 \\
2 & 16O(3-) + 208Pb(0+) \\
3 & 16O(0+) + 208Pb(3-) \\
4 & 16O(0+) + 208Pb(2+) \\
5 & 16O(0+) + 208Pb(5-) \\
6 & 17O(2.5+) + 207Pb(0.5-) \quad c_{pot} = 2 \\
7 & 17O(2.5+) + 207Pb(2.5-) \\
8 & 17O(2.5+) + 207Pb(1.5-) \\
9 & 17O(0.5+) + 207Pb(0.5-) \\
10 & 17O(0.5+) + 207Pb(2.5-) \\
11 & 17O(0.5+) + 207Pb(1.5-) \\
12 & 15N(0.5-) + 209Bi(3.5-) \quad c_{pot} = 3 \\
13 & 15N(0.5-) + 209Bi(1.5-) \\
14 & 12C(0+) + 212Po(0+) \quad c_{pot} = 8
\end{array} \tag{12.33}$$

- Only first 3-states are fully solved.
- The excitation of 16O and 208Pb are considered in collective excitation. 16O is considered as rotor but 208Pb deformation is included as an input matrix elements.
- Coulomb deformation of 16O is given by $p3 = M_n(E3) = 37.6$. While nuclear deformation of 16O is given as $p3 = \delta_3 = 2.15$
- deformation of 208Pb are given in terms of matrix elements $M(Ek)$ and $RDEF(Ek)$. [Can I understand these values?](#)
- Transfer to other channel $V_{part1,part2}$, $V_{p1,p3}$ and $V_{p1,p4}$ are calculated as a finite range transfer.
- note that the non-trivial CFP coefficients in the couplings. [Can I understand these values from shell model?](#)

12.12 Capture : (B) $^{14}C(n, \gamma)^{15}C$

```

14C(n,g)15C E1 only
NAMELIST
&FRESCO hcm= 0.100 rmatch=100
          jtmin=0 jtmax=4.5 absend=-1
          thmin=0 thmax=0 iter=1
          elab(1)= 0.005 4.005 nlab=50/

&PARTITION namep='neutron' massp=1.0087 zp=0 nex=1
          namet='14C' masst=14.0032 zt=6 /
&STATES jp=0.5 ptyp=1 ep=0 cpot=1 jt=0.0 ptyt=1 et=0/

```

```

&PARTITION namep='Gamma' massp=0 zp=0 nex=1
            namet='15C' masst=15.0106 zt=6 qval=1.218 /
&STATES jp=1.0 ptyp=1 ep= 0 cpot=3 jt=0.5 ptyt=1 et=0/
&partition / ! for photon, cpot must be non-existing

&Pot kp=1 type= 0 shape= 0 p(1:3)= 14.0000 0.0000 1.3000/
&Pot kp=1 type= 1 shape= 0 p(1:3)= 57.0000 1.7 0.7000/
&Pot kp=1 type= 3 shape= 0 p(1:3)= 0.0000 1.7 0.5000/
&Pot kp=2 type= 0 shape= 0 p(1:3)= 14.0000 0.0000 1.2000/
&Pot kp=2 type= 1 shape= 0 p(1:3)= 55.7700 1.2230 0.5000/
&Pot kp=2 type= 3 shape= 0 p(1:3)= 5.0000 1.2230 0.5000/
&pot /

&OVERLAP kn1=1 ic1=1 ic2=2 in=-2 kind=0
          nn=2 l=0 sn=0.5 j=0.5 kbpot=2 be=1.218 isc=1 /
&overlap /

&COUPLING icto=2 icfrom=1 kind=2 ip1=-1 ip2= 1/
&cfp in=2 ib=1 ia=1 kn=1 a=1.000 /

```

- From textbook. E1 dominated capture.
- for capture reaction, in the second partition, projectile have to be 'Gamma' and 'cpot' have to refer non-existing potential.
- $\langle {}^{15}\text{C}(2s_{\frac{1}{2}}) | {}^{14}\text{C} \rangle$ overlap is defined. $IN < 0$ is for relativistic correction.
- The coupling kind=2 is for EM one-photon coupling. ICTO is the gamma partition and photon must be on the projectile side. $ip1 = \lambda > 0$ include all multipolarity up to λ . $ip1 < 0$ case, only λ is included. $ip2=0,1,2$ for electric and magnetic convection current, electric only, magnetic only.
- 'fort.39' and column 5 is the capture cross section as a function of neutron energy.

12.13 Exchange transfer: ${}^7\text{Li}({}^8\text{Li}, {}^7\text{Li}){}^8\text{Li}$

```

DWBA calculation of Exchange trnsafer 7Li(8Li,7Li)8Li, 11 MeV
NAMELIST
&FRESCO
  hcm=0.10 rmatch=20.0 rintp=0.2
  hnl=0.10 rnl=5 centre=0.0
  jtmin=0.0 jtmax=10.0 absend=0.001
  thmin=1.0 thmax=180.0 thinc=1.0
  ips=0.05 it0=1 iter=1 nnu=24
  chans=1 smats=1 xstabl=1
  elab=11.0 /
  ! iter=1 thus, DWBA calculation

&PARTITION namep='8Li' massp=8.0 zp=3.0
            namet='7Li' masst=7.0 zt=3.0 qval=0.0 nex=1 /
&STATES jp=2.0 bandp=+1 ep=0.0 cpot=1
          jt=1.5 bandt=-1 et=0.0 /

```

```

&PARTITION namep='7Li' massp=7.0 zp=3.0
              namet='8Li' masst=8.0 zt=3.0 qval=0.0 nex=1 /
&STATES cpot=1 copy=-1/
&PARTITION /

! Entrance and Exit Channel potential 7Li-8Li
&POT kp=1 ap=7.0 at=8.0 rc=1.3 /
&POT kp=1 type=1 p1=175.0 p2=0.64 p3=0.8
                  p4=16.9 p5=1.09 p6=0.8 /
! binding pot for p3/2( Davids-Typel binding pot)
&POT kp=2 ap=1.0 at=0.0 rc=2.5 /
&POT kp=2 type=1 p1=43.19 p2=2.5 p3=0.65 /
! binding pot for p1/2(Esbensen-Bertsch binding pot)
&POT kp=4 ap=1.0 at=0.0 rc=2.391 /
&POT kp=4 type=1 p1=42.14 p2=2.391 p3=0.52 /
&POT kp=4 type=3 p1=4.9 p2=2.391 p3=0.52 /
! Pohhast vol dominated 7Li-7Li pot
! core-core 7Li-7Li pot at 11 MeV
&POT kp=10 ap=7.0 at=7.0 rc=1.3 /
&POT kp=10 type=1 p1=4.66 p2=0.173 p3=1.617
                  p4=5.60 p5=1.589 p6=0.397 /
&POT /

! p3/2 form factor in projectile, target
&OVERLAP kn1=1 ic1=1 ic2=2 in=1
              kind=0 nn=1 l=1 sn=0.5 j=1.5
              kbpot=2 be=2.03229 isc=1 /
&OVERLAP kn1=2 ic1=2 ic2=1 in=2
              kind=0 nn=1 l=1 sn=0.5 j=1.5
              kbpot=2 be=2.03229 isc=1 /
! p1/2 form factor in projectile target
&OVERLAP kn1=3 ic1=1 ic2=2 in=1
              kind=0 nn=1 l=1 sn=0.5 j=0.5
              kbpot=4 be=2.03229 isc=1 /
&OVERLAP kn1=4 ic1=2 ic2=1 in=2
              kind=0 nn=1 l=1 sn=0.5 j=0.5
              kbpot=4 be=2.03229 isc=1 /
&OVERLAP /

# FR-Transfer coupling with complex remnant
&COUPLING icto=2 icfrom=1 kind=7 ip1=0 ip2=-1 ip3=10 /
&CFP in=1 ib=1 ia=1 kn=1 a=0.868 /
&CFP in=2 ib=1 ia=1 kn=2 a=0.868 /
&CFP in=1 ib=1 ia=1 kn=3 a=0.0737/
&CFP in=2 ib=1 ia=1 kn=4 a=0.0737 /
&CFP /
&COUPLING /

```

- This example is from the thesis of D. Howell (Google search result). Thus, it may have some problem and requires some careful study for the possible error in the code.
- Since the transferred channel and elastic scattering channel cannot be distinguished, it is necessary to specify that the second partition is an exchange of first partition by

using 'copyp=-1'.

- The finite range transfer coupling is calculated with spectroscopic amplitudes. Note that the spectroscopic amplitudes must be given twice for the transition.

12.14 Sfresco: (B) p+112Cd elastic scattering

For fitting with SFRESCO, requires input and output file of FRESKO and an search file for SFRESCO and an input commands for MINUIT. Thus, it goes 'MINUIT input file → SEARCH file → FRESKO input file'.

```
p + 112Cd elastic ! input file for FRESKO 'B7-p-cd.frin'
NAMELIST
&FRESKO hcm= 0.100 rmatch= 20.000 jtmin= 0.0 jtmax= 200.0
        thmin= 0.00 thmax=180.00 thinc= 2.00 xstabl= 1
        elab(1)= 27.9 /
&PARTITION namep='Protonuu' massp= 1.0000 zp= 1 nex= 1
            namet='112Cduuu' masst=112.0000 zt= 48 qval= 0/
&STATES jp= 0.5 ptyp= 1 ep= 0.0000 cpot= 1 jt= 0.0 ptyt= 1 et= 0.0000/
&partition /
&pot kp= 1 type= 0 p(1:3)= 112.000 0.0000 1.2000 /
&pot kp= 1 type= 1 p(1:6)= 52.500 1.1700 0.7500 3.5000 1.3200
0.6100 /
&pot kp= 1 type= 2 p(1:6)= 0.000 0.0000 0.0000 8.5000 1.3200
0.6100 /
&pot kp= 1 type= 3 p(1:3)= 6.200 1.0100 0.7500 /
&pot /
&overlap /
&coupling /

'B7-p-cd.frin' 'B7-p-cd.fROUT' ! input and output file name of FRESKO
4 1 ! number of variable and number of data
&variable kind=1 name='r0' kp=1 pline=2 col=2 potential=1. step=0.01/
! kind=1 : potential variable
! pline=2 : 2nd line of NAMELIST for potential index kp
! col=2 : p(col=2) variable from initial value 'potential=1' with 'step=0.01'
&variable kind=1 name='V' kp=1 pline=2 col=1 potential=50.00 step=0.1/
&variable kind=1 name='W' kp=1 pline=2 col=4 potential=5.0 step=0.1/
&variable kind=1 name='WD' kp=1 pline=3 col=4 potential=10. step=0.1/
&data iscale=0 idir=1 lab=F abserr=T/
!iscale=0 : absolute data in fm^2/sr.
!lab='F' data is in cm frame
!abserr=T error is absolute value not ratio.
22. 0.548 0.044
26. 0.475 0.024
30. 0.481 0.014
38. 0.447 0.009
50. 0.144 0.004
66. 0.499 0.010
70. 0.248 0.005
86. 0.463 0.014
90. 0.485 0.015
106. 0.087 0.003
```

110.	0.135	0.004
130.	0.161	0.005

&

- data type=0: angular distribution for fixed energy type=1 : excitation and angular cross-section double istribution type=2 : excitation cross section for fixed angle type=3 : excitation function for the total, reaction, fusion or inelastic cross section type=4 : an excitation phase shift for fixed partial wave type=5: a desired factor for bound state search type=6 a specific experimental constraint on som search parameter.

```
B8-p-cd.search ! input file name of search
q ! query status of search variables
plot B9-p-cd-init.plot ! write data and theory curve
min ! call MINUIT interactively
migrad ! in MINUIT, MINGRAD search
end ! return to SFRESCO
plot B9-p-cd-fit.plot
```

12.15 Deuteron overlap using Reid potential

```
82Ge(d,p)83Ge @ Ed=8 MeV;
NAMELIST
&FRESCO .../

&PARTITION namep='d' massp=2.014 zp=1
            namet='82Ge' masst= 82 zt=32 nex=1 pwf=T/
&STATES jp=1.0 bandp=1 ep=0.0 cpot=1 jt=0.0 bandt=1 et=0.0 /
&PARTITION namep='p' massp=1.0078 zp=1
            namet='83Ge' masst= 83 zt=32
            qval=1.47 nex=1 pwf=T/
&STATES jp=0.5 bandp=1 ep=0.0 cpot=2 jt=2.5 bandt=1 et=0.0 /
&partition /

! n+p binding pot
&POT kp=4 type=0 at=1 rc=1.2 / ! np binding pot
&POT kp=4 type=1 shape=5 p1=1.0/ ! Reid T=0 central
!&POT kp=4 type=2 shape=5 p1=1.0/ ! (?surface) Reid T=0 spin-orbit *r
&POT kp=4 type=3 shape=5 p1=1.0/ ! (spin-orbit for proj?) Reid T=0 spin-orbit
&POT kp=4 type=4 shape=5 p1=1.0/ ! (spin-orbit for targ?) Reid T=0 spin-orbit
&POT kp=4 type=7 shape=5 p1=1.0/ ! Reid 12*tensor
!&POT kp=4 type=8 shape=5 p1=1.0/ ! spin-spin
&pot /

! test for deuteron bound state with Reid Pot
! Does it give the spectroscopic factor ?
&OVERLAP kn1=2 kn2=3 ic1=2 ic2=1 in=1
            kind=3
            nn=1 l=0 lmax=2 sn=0.5 ia=1 ib=1 j=1.5
            kbpot=4 be=2.225 isc=0/
&overlap /
```



```

--> output
The following 2 components are in a group labelled 2 for 1.0+
0 2: 2 1 1 3 * 1 0 0.5 1 0.5 1 4 0 2.2062 0 0 0 0.0000 BE
2.2062; 0. 1.006 0.2305 0.9672 4.010 -124.9 -128.7 0.8745
0 3: 2 1 1 3 1 2 0.5 1 1.5 1 4 0 2.2062 0 0 0 0.0000 BE
2.2062; 0. 1.006 0.2305 0.2538 2.422 -7.6 -8.3 0.0228
Overall rms norm & radius = 1.0000 3.9275, & Overall D0 & D = -132.48
-137.01

```

- Original Reid68 is defined for each partial waves, instead of operator form. $T = 0$ and $T = 1$ is separate and for each coupled and uncoupled partial waves,

$$\begin{aligned}
 V(\text{uncoupled}) &= V_c^T(r) \\
 V(\text{coupled}) &= V_c^T(r) + V_{ts}^T(r)S_{12} + V_{ls}^T(r)\vec{L} \cdot (\vec{s}_1 + \vec{s}_2)
 \end{aligned} \quad (12.34)$$

- The tensor operator $S_{12} = 3\sigma_1 \cdot \hat{r}\sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2$. On the other hand, Satchler's convention which is used in the FRESKO is

$$S_2 \cdot R_2(a, b) = (s \cdot b)(s \cdot b) - \frac{1}{2}is \cdot (a \times b) - \frac{1}{3}s^2a \cdot b \quad (12.35)$$

This difference introduces factor 12 in the tensor operator.

- Reid93 SC potential have six operators. However, it is not clear how the FRESKO input is related with Reid potential.
- According to the Reid68 form, I expect the deuteron needs TYPE=0,1,3,4,7 . HOWEVER, it is not clear yet.
- TYPE=2,8 is for Reid93 ? unclear
- it seems to be important to use exact atomic mass in the &Partition.
- if necessary one can read external wave function for overlap function.

12.16 Runtime Error

FRESKO 결과는 항상 테스트 받아야한다. 중요한 stability test는

- rmatch
- jtmax
- rnl
- hcm
- cutl
- 'fort.3' input echo.

12.17 Charge Exchange

...

Chapter 13

Appendix

13.1 Center of mass correction

I am not sure whether the following arguments makes sense or not.

In may cases, the single particle wave function for the reaction is made so that it can gives experimental single nucleon separation energy by adjusting WS potential with the given (n, l, j) quantum numbers. This (n, l, j) quantum numbers would corresponds to those of single particle shell model. However, it is not certain how to connect the single particle shell model wave function which is defined from the distance from center of mass of nuclei and the relative wave function which is a function of relative distance between $A - 1$ nuclei and a nucleon.

Let us assume that the whole system is spherically symmetric and only the distance from the center is a matter. Then, in the center of mass frame, A -th one nucleon and center of mass of $(A - 1)$ nucleon will satisfy,

$$\mathbf{r}_A + (A - 1)\mathbf{R}_{A-1} = 0, \quad \mathbf{R}_{A-1} = \frac{\sum_{i=1}^{A-1} \mathbf{r}_i}{A - 1} \quad (13.1)$$

Then for the relative motion between \mathbf{r}_A and \mathbf{R}_{A-1} , we have reduced mass and relative distance as is

$$\mu = \frac{A - 1}{A} m_N, \quad \mathbf{r}_{rel} = \mathbf{r}_A - \mathbf{R}_{A-1} = \frac{A}{A - 1} \mathbf{r}_A \quad (13.2)$$

Let us suppose we have a solution to the equation for $A - th$ single particle ,

$$\left(-\frac{\nabla_A^2}{2m} + V(r_A) \right) \phi_\alpha(r_A) = E_\alpha \phi_\alpha(r_A), \quad \alpha = (n, l, j) \quad (13.3)$$

We would like to connect this solutions with the solution of equation for relative motion,

$$\left(-\frac{\nabla_{rel}^2}{2\mu} + V_{rel}(r_{rel}) \right) \psi_{\alpha'}(r_{rel}) = E_{\alpha'} \psi_{\alpha'}(r_{rel}). \quad (13.4)$$

If we consider relative motion, even though the potential may be written in terms of relative distance if the actual interaction is described by central potential , the potential will be just a function of distance from center by the above relation. Thus, the solution of relative motion will be a scaled version of original problem.

If we define the relative potential as

$$V_{rel}(r_{rel}) = \frac{A-1}{A} V(r_A = \frac{A-1}{A} r_{rel}), \quad (13.5)$$

then, rescaling $r_{rel} \rightarrow r_A$ gives,

$$\left(-\frac{\nabla_{rel}^2}{2\mu} + V_{rel}(r_{rel}) \right) = \frac{A-1}{A} \left(-\frac{\nabla_A^2}{2m} + V(r_A) \right) \quad (13.6)$$

Thus,

$$\frac{A-1}{A} \left(-\frac{\nabla_A^2}{2m} + V(r_A) \right) \psi_{\alpha'}(r_{rel}) = E_{\alpha'} \psi_{\alpha'}(r_{rel}) \quad (13.7)$$

Thus, we may identify,

$$E_{\alpha} = \frac{A}{A-1} E_{\alpha'}^{rel}, \quad \psi_{\alpha'}(r_{rel} = \frac{A}{A-1} r_A) = \left(\frac{A}{A-1} \right)^{3/2} \phi_{\alpha}(r_A) \quad (13.8)$$

because the rescaling does not change n, l, j , quantum numbers $\alpha' = \alpha$ and the normalization is fixed from the normalization condition for $\int |\phi(r_A)|^2 = \int |\psi(r_{rel})|^2 = 1$. Thus, we can obtain the single particle wave function which is a function of distance from center can be obtained from the solution for relative motion.

However, I am not quite sure whether this can be applicable for non-S wave or deformed case.

Chapter 14

FRESCO INPUT CARDS

FRESCO는 old fortran input form과 NAMELIST form을 함께 사용한다. ('fr2nl' 이라는 프로그램이나, FRESCO 자체에는 old fortran input을 NAMELIST format으로 바꾸어 주는 기능이 있다.)

NAMELIST FORMAT은 두번째 줄에서 'NAMELIST' 로 시작한다. CDCC FORMAT의 경우, 두번째 줄에서 'CDCC'로 시작한다. CDCC FORMAT의 경우는 오직 inelastic breakup 만 가능하므로, CDCC를 이용하여 필요한 Channel states와 overlap function들을 만든뒤에는 NAMELIST format을 이용하여 더 자세한 조정을 한다.

14.1 Definition Cards or &FRESCO

14.1.1 Card 0

10A8 HEADNG(10)

14.1.2 Card 1

10F8.4 HCM,RMATCH,RINTP,HNL,RNL,CENTRE

HCM RMATCH	Wave function is calculated at intervals of HCM up to abs(RMATCH) If RMATCH < 0, read Card 1a for coupled Coulomb Wave functions, implying the integration of the equations should be done up to rmatch, numerically, but these should be matched with coupled-channel Coulomb functions up to rasym. This is necessary, in case of CDCC to include long range of Coulomb interaction.
RINTP HNL RNL CENTRE	Non-local kernels $K'_{fi}(R_f, D_{fi})$ calculated at R_f intervals of RINTP, and for a non-local (D_{fi}) range of RNL centred at CENTRE in steps of HNL. Thus, RNL represent non-locality of kernel. RMATCH and RINTP are rounded to multiples of HCM, and HNL is rounded either to a multiple or a sub-multiple of HCM.

Non-local Kernel을

$$S(R_f) = \int_0^{R_{match}} K_{fi}(R_f, R_i) u(R_i) dR_i \quad (14.1)$$

(R_f, R_i) 대신, $D_{fi} = R_f - R_i < |Centre \pm RNL|$ 로 나타낸다.

14.1.3 Card 1a : In case of Rmatch < 0

2F8.6, 2F8.2 RASYM, ACCUR, SWITCH, AJSWITCH

RASYM	Use coupled Coulomb wave functions from CRCWFN out to asymptotic radius RASYM from inner radius abs(RMATCH) for those partitions in which PWF is TRUE. If RASYM < 0, then determine the outer radius in order that classical Coulomb trajectories reach an angle abs(RASYM) degrees.
ACCUR	accuracy parameter controlling the piecewise step length. Default is 0.01: smaller values give greater accuracy
SWITCH	radius at which to switch from Airy functions to sines and cosines in piecewise method. Default is 1000 fm.
AJSWITCH	default is 0.0 Normally the Coupled Coulomb wfns are matched to zero and the Numerov integration is omitted, when the Coulomb distance of closest approach is more than 4.5 fm outside abs(RMATCH) (or the -CUTR distance if CUTR negative). This is only allowed when $J_{total} \geq \text{AJSWITCH}$.

14.1.4 Card 2

2F4.0, F8.4, L2, I2, 1X, A1, I2 4(I4, F4.0)
JTMIN, JTMAX, ABSEND, DRY, JSETS, RELA, NEARFA, (JUMP(i), JBORD(i), i=2,5)

JTMIN JTMAX	range of Partial waves in the calculation for initial and final total angular momentum. If $j_{tmin} < 0$, only include elastic channels in range $J < \text{abs}(j_{tmin})$, transfer or excited states are ignored in the calculation.
ABSEND	controls convergence. If in the interval $\max(0, j_{tmin}) < J < j_{tmax}$, the absorption in the elastic channel is smaller than absend mb for three consecutive $J\pi$ sets, the calculation stops. If absend < 0, it takes the full J-intervals.
DRY JSETS RELA NEARFA	enables the calculation of positive parity(='P') or negative parity(='M' or 'N') only. =0," or 'F', no restriction = 'T' for relativistic kinematics for the incident projectile. <i>Not yet implemented</i>
JUMP JBORD	Number of angular momentum intervals, $jump(i), i = 2, 5$ and the steps $jbord(i), i = 2, 5$. $jump(1) = 1, jbord(1) = j_{tmin}$ The omitted J values are provided by interpolation on the scattering amplitudes $A(m', M' : mM; L)$ prior to calculating cross sections.

14.1.5 Card 3

2I1, F6.4, F8.4, F7.4, I1, 3F8.4 KQMAX, PP, THMIN, THMAX, THINC, KOORDS, CUTL, CUTR, CUTC

KQMAX PP	Give cross sections (and tensor analysing powers up to rank $K = KQMAX$) Calculate analysing powers/polarisations for projectile (PP=0 or blank), target (PP=1), ejectile (PP=2) or residual nucleus (PP=3). PP=4 gives projectile (PP=0) analysing powers, along with Kyy results.
THMIN THMAX THINC	angular range for cross section calculation. (thmin,thmin) in degrees in steps of thinc. $THMAX > 0$ gives elastic channels in ratio to Rutherford $THMAX < 0$ gives elastic channel in absolute value [mb/sr]
KOORDS	determines the coordinate systems used for the analysing powers: = 0 : Madison coordinates (default) = 1 : Madison + Transverse = 2 : Madison + Transverse + Recoil = 3 : Madison + Transverse + Recoil + Hooton-Johnson
CUTL CUTR CUTC	= radial points per L (angular momentum of partial wave) of lower radial cutoff. Define lower radial cutoff in the coupled channel equations for radial equations. this is useful for scattering below the Coulomb barrier Default = -1.6 CUTL is L-dependent cutoff When CUTL>0, use $\ell = J_{total}$ (total angular momentum of CC set), When CUTL<0, use $\ell = L_{in}$ (orbital angular momentum of incoming partial wave). Using CUTL<0 gives more accurate analysing powers. = lower radial cutoff. CUTR is the same for all partial waves the code use $\max(CUTL * L * hcm, CUTR)$ If cutr<0, the lower cutoff is put at the distance inside the Coulomb turning point. (For example, cutr= -20 fm puts the cutoff 20 fm inside the Coulomb turning point.) CUTC removes off-diagonal couplings inside the given radius. = lower radial cutoff for off-diagonal couplings.

14.1.6 Card 4

F6.4, I2, I4, I2, I2, A1, I3, 2I4 I2, F6.4
IPS, ITO, ITER, IBLOCK, PADE, ISO, NNU, MAXL, MINL, MTMIN, EPC,

2F8.4 2I2, 2e8.1, 4x, i4
ERANGE, DK, INH, plane, smallchan, smallcoup, initwf

ITO	Solve the coupled channels equations by at least ITO iterations, and up to ITER iterations.
ITER	ITO=ITER=0 solves only elastic channels(along with IBLOCK channels) ITO=ITER=1 or 2 gives 1 or 2-step DWBA Normally, a run is terminated if more than ITER steps are required for convergence. Setting ITER < 0 allows continuation even after convergence has failed after abs(ITER) iterations.
IPS	Iterations are normally also stopped if the successive differences are smaller than the errors estimated for the numerical integration of the coupled equations.

IBLOCK	<p>Setting $IPS < 0$ uses $\text{abs}(IPS)$, without this extra check. Stop sooner if the absolute differences between successive S-matrix elements (scaled by $(2J+1)/(2.JTMAX+1)$) are less than IPS percent. (Excited state pairs with IGNORE set on Card 7 are not counted against IPS). IBLOCK is the number of pairs of excitation levels (starting from partition 1, excitation 1) that are coupled exactly by blocking together. If $IBLOCK < 1$, then read Card 4.5 (see below) for R-matrix solution of the coupled equations.</p>
PADE ISO NNU	<p>$= 0$ for no Pade acceleration, $= 1$ for Pade acceleration by the epsilon algorithm. (not implemented in this version) NNU is the number of Gaussian integration points in the angular integration used for the non-local transfer kernels. NNU should be a multiple of 6; $NNU = 18$ is the minimum, and 24 or 36 give acceptable accuracy for all the reactions tried so far.</p>
MAXL,MINL MTMIN EPC ERANGE, DK INH PLANE SMALLCHAN SMALLCOUP INITWF	<p>MAXL,MINL are the maximum and minimum L values for the non-local kernels. If zero, MAXL has the default value $JTMAX+6$, and if $MINL < 0$ it takes the default value $JTMIN -6$. MTMIN is the lowest L-transfer for calculating transfer form factors using the m-dependent expressions for spherical harmonics. Putting $MTMIN = 0$ gives default value $MTMIN = 6$ (use $MTMIN < 0$ to avoid invoking default, if all transfers are to use this method). $=$ percentage cutoff accuracy in the NNU angular integration. If zero, the default is $(30/NNU)^2\%$. – <i>ignored</i> $= 0$: zero-range transfer forms in intervals of HCM exactly $= 1$: stored in steps of $HCM * (\text{proj. core})/(\text{proj. composite mass})$ $= 2$: stored in steps of $HCM * (\text{targ. core})/(\text{targ. composite mass})$ So $INH=2$ corrects for longitudinal recoil during transfers with zero-range projectiles. – <i>ignored</i> SMALLCHAN = fraction of unitarity to define a 'small channel'. A channel that is 'small' for $NSMALL=2$ times is dropped permanently. $=$ if all nonelastic channels are weaker than the fraction SMALLCOUP of unitarity, then permanently change from coupled-channels to DWBA. $=$file number from which to read fixed channel wave functions during all iterations. The channels whose wfs are changed must have the same IT index, that counts cards 7 as excited state pairs. This file as the same format as the fort.17 produced when WDISK is nonzero. The $INITWF > 0$ for formatted wf file, and < 0 for unformatted (the same sign convention as for WDISK).</p>

14.1.7 Card 4.5: If $IBLOCK < 0$, then read:

2i4, I1, L1, I2, i2, f6.4, 3f8.4,

NRBASES, NRBMIN, BUTTLE, PRALPHA, PCON, MEIGS, RMATR, EBETA(2), WEAK

NRBASES	= target number of radial basis states in each channel. (Use 2*NRBASES for the elastic and first-inelastic channel for more accuracy).
NRBMIN	= minimum number of radial basis states.
BUTTLE	= 4 for none, 0,2 for complex, 1,3 for real (2,3 without energy shift) Buttle correction (default 0)
PRALPHA	= print basis-state eigenvalues to files fort.60,61,62,63
PCON	= trace variable for calculation of radial basis states (same meaning as IPC on Cards 13 for single-particle bound states).
MEIGS	= maximum number of bound states to find, if ENLAB < 0 on Card 19.
RMATR	= R-matrix matching radius (default is RMATCH on Card 1). Warning: RMATR will be changed to an even multiple of HCM.
EBETA(i)	= energy $\hbar^2 k^2 / 2m$, where $k = f'/f$, the logarithmic derivative for all radial basis states at $r=RMATR$, with k having the same sign as EBETA(i). Use i=1,2 for positive,negative parity coupled channels sets (respectively). If R-matrix solutions are selected (by IBLOCK<0) then all channels are 'blocked' together and solved in a full CRC procedure.
WEAK	If WEAK>0, then non-elastic columns of the R-matrix are set to zero, when penetrabilities < WEAK.

14.1.8 Card 5

14I2

CHANS, LISTCC, TRENEG, CDETR, SMATS, XSTABL, NLPL, WAVES, LAMPL, VEFF, KFUS, WDISK, BPM, MEL, CDCC, NFUS

A value of 0 gives no trace, increasing values give progressively more printed output.

*Decrement*ed variables are decreased by 1 on each use.

CHANS	≥ 1 : Print the sets of coupled partial waves for each J,parity. Decrement.
LISTCC	= 1 : Print coupling coefficients between these channels. Decrement. = 1,2,... Print progressively more detail of couplings.
TRENEG	≥ 1 : Print all potentials as they are calculated from Cards 10 ≥ 3 : Print all potentials as they are calculated from Cards 9
CDETR	≥ 1 : Print information on the solving of the coupled equations. (decrement).
SMATS	≥ 1 : Print absorbtion & reaction cross sections for successive partitions and excitations. ≥ 2 : Print elastic S-matrix elements (S_{el}). Also 'punch' these elastic S_{el} on output file 7, in format (2F15.10,L,J,JTOT) for S_{el} complex, L, J and JTOT. See WDISK below for description of these quantum numbers. ≥ 3 : Print all S-matrix elements for the 'grazing partial waves' defined by $0.05 < Re(S_{el}) < 0.95$ ≥ 4 : Always print all the S-matrix elements. ≥ 5 : Print all S-matrix elements at each iteration of the coupled equations (or, if PADE > 0, the Pade approximant)

	≥ 6 : Print all actual S-matrix elements at each iteration (these may be divergent before Pade acceleration).
XSTABL	$\neq 0$: If XSTABL is non-zero, in file 16 punch output cross sections for all excitation levels in all partitions. A header card in FORMAT(5I6) gives partition IC, level pair IA, number of tensor ranks of analysing powers $1 < KQ1PR < XSTABL$, number of angles NANGL, and NEARF. NEARF=1 for total cross section, 2 for far-side component, and 3 for near-side component. Then follow NANGL print operations in FORMAT(1P,6E12.4), repeating the FORMAT for each operation if KQ1PR is large, of THETA, elastic xs (mb), T_{10} , iT_{11} , T_{20} , T_{21} , T_{22} , iT_{30} , iT_{31} etc.

NLPL > 0 : print a ‘contour plot’ of the non-local kernels $K_{fi}(R_f, D_{fi})$. This is useful to determine if the parameters on card 1 are adequate. Decrementd.

WAVES

± 1 or ± 3 : print out wave function solutions of the coupled equations at the end of the iterations. (If WAVES < 0 : print out the RATIO of the w/f to its asymptotic form $((G - iF) - S.(G + iF)).i/2$)

2 or 3 : print out the source terms at *each* iteration of the coupled equations.

LAMPL

$\neq 0$: Print out (on Fortran file 36) the coefficients $A(m'M' : mM; L)$ for the Legendre coefficients in the scattering amplitude for the partition number abs(LAMPL), and print out the $f(m'M' : mM; \theta)$ for each angle θ .

< 0 : only print out on file 37 the amplitudes f , not the A ’s, for partition abs(LAMPL). The phase convention here is that there is no Coulomb phase shift for $L = 0$ in the Coulomb scattering amplitude : factors such as $\exp i(\sigma_L - \sigma_0)$ appear in the A ’s.

VEFF

$\neq 0$: Calculate the ‘coupled channels effective potential’ found by averaging the ‘trivially equivalent potential’ over all the J, π sets, with weights of the elastic wave functions times the reaction cross section, in each set.

< 0 : Add this effective potential to optical potential of the elastic channel before printing.

$= -2$ or $+2$: Exclude partial waves with elastic S -matrix element $S_\ell < 0.1$ from the averaging sum.

The results show the real and imaginary parts for successive values of $J - L$, for the projectile only.

KFUS,NFUS

> 0 : Calculate ‘core fusion’ using the imaginary and scalar parts of potential number KFUS (i.e. cards 10 with TYPE = 1 or 2, and KP = KFUS), also for the first NFUS inelastic channels

WDISK

$= 1$: Print elastic wave functions on output file 17, FORMATTED

$= 2$: Print all wave functions on output file 17, FORMATTED

$= -1$: Print elastic wave functions on output file 17, UNFORMATTED

$= -2$: Print all wave functions on output file 17, UNFORMATTED

The following data formats are used when WDISK > 0 :

card A: (I4,2F8.4,F8.1,I3)

NR,H,ENLAB,JTOTAL,PARITY,MP,MT,ZP,ZT :
 number radial points, step size, lab. energy, J,pi,
 projectile and target masses and charges
 card B: (2I4,2F6.1,I4,F6.1,2F15.10,f12.8)
 IT,L,J,JTOT,LIN,JIN,SMAT (complex), ETA

where

IT = index to excited state pair, counts cards 7.
 L = partial wave
 J = L + projectile spin
 JTOT = total spin = $J_{total} = J + \text{target spin}$
 LIN = incoming partial wave
 JIN = incoming J value.
 SMAT = S matrix element for this partial wave.

card C: (6E12.4) (psi(I),I=1,NR) wave function
 card C is repeated until NR complex values given
 NB: the first point psi(1) = 0 always, as at r=0

Cards B & C are repeated for each channel, until IT < 0.
 When WDISK < 0, successive records contain the two real values of psi(I),
 starting IN THIS CASE, from I=2 (i.e. $r = h$).

BPM

≥ 1 : Calculate fusion cross sections in the Barrier Penetration Model using first the bare potential, and then the bare potential + the 'weighted equivalent potential' calculated when VEFF $\neq 0$.
 ≥ 2 : Print out L-distributions of the fusion cross section.

MEL – *ignored*

CDCC

$\neq 0$: Print out the $f(m'M' : mM; \theta)$ for each angle θ on file 57 for partition PEL,
 after the following information:

card A: (F10.4,3F8.4) ENLAB,Bproj,H2SM,e^2 lab energy,projectile binding energy
 hbar^2/2.m, e^2
 card B: (4f8.4) massp,masst,masst,masst masses projectile, target, core, valence
 card C: (4f8.4) Zp,Zt,Zc,Zv charges
 card D: (4A8) namep,namet,namec,namev names
 card E: (4f8.1) Jp,Jt,Jc,Jv g.s. spins
 card F: (4i8) Pp,Pt,Pc,Pv g.s. parities
 card G: (4I4) NBINS,NKMAX,NEXB,NNJMAX no. CDCC bins, max NK,
 no. excited states, max(2*Jex+1)
 card H: (I4,2f8.4) NANGL,THMIN,THINC (cm angular range from Card 3)
 for each of the NBINS bins:
 card I: (i2,2f4.1,3f8.4,2i4)
 l,j,Emid,kmin,kmax,NK,KN,ISC
 l,j: quantum numbers (s==Jv)
 Emid: centre of bin with respect to continuum threshold
 kmin,kmax,NK: Min,max and number of k values in bin integral
 KN: original KN index for bin state
 ISC: normalisation used for bin

```

    for each IK=1,NK
      card J: (10f8.4) delta(IK): nuclear phase shift used in bin integral (radians)

for each excited state pair in the entrance partition: IA=1,NEXB::
  card K: (f4.1,i4,f8.4,i4) Jex,Parity,Eex,IBIN:
    Jex :          spin of this projectile excited state
    Parity:        parity of this projectile state
    Eex:           excitation energy of this state above g.s.
    IBIN:          (first) bin defined for this excited state
  for each IANG=1,NANGL: read complex numbers:
    card L: (6E12.4): ((FAM(MEX,MP,IANG,IA),MEX=1,2*Jex(IA)+1),MP=1,2*Jp+1)

```

The phase convention again is that there is no Coulomb phase shift for $L = 0$ in the Coulomb scattering amplitude : factors such as $\exp i(\sigma_L - \sigma_0)$ appear in the A 's.

Summary of bin normalisation factors for different ISC values:

ISC

$$\begin{aligned}
 &= 2: \exp(-i\delta(k)) \\
 &= 4: \sin(\delta(k)) \exp(-i\delta(k)) \\
 &= 12: k \exp(-i\delta(k)) \\
 &= 14: k \sin(\delta(k)) \exp(-i\delta(k))
 \end{aligned}$$

14.2 Potentials

14.2.1 Card 9: Coulomb potential

Card 9: I3,I2,3X,2F8.4, 2F8.4
 KP, 0, A#1,A#2, ROC,AC (i.e. would be Card 10, but TYPE=0)

KP	potential index: all potentials starts from Card9 with TYPE=0 and Card10 will be added to the potential.
A#1,A#2	All the radii are multiplied by CC,(i.e. $R = r_0 \times CC$) which is updated by a TYPE=0 Card 9 to $CC = A\#1^{1/3} + A\#2^{1/3}$ and by TYPE>0 Cards 10, if $P_0 > 0$, to $CC = P_0^{1/3}$
ROC	Note that A#1, A#2 are only used for the conversion of r_0 to R .
AC	= radius (when multiplied by CC) of charge distribution = diffuseness of charge distribution

14.2.2 Card 10

Card 10: I3,I2, A1,I2, 3F8.4, 3F8.4, F8.4
 KP,TYPE,IT,SHAPE, P1,P2,P3, P4,P5,P6, P0
 ---REAL--- -IMAGINARY- A#
 V,Vr0,Va, W,Wr0,Wa, A#

TYPE indicate the type of spin tensor or excitation coupling.

Deformation case: TYPE=10-13 corresponds to the deformation. Card 11 은 TYPE과 SHAPE에 따라 필요해지고, 이때, P(k)와 STR의 의미도 달라짐에 유의.

TYPE=10 or 11 이고 SHAPE=10-13 인 경우에는 Card 11 이 필요 없이, P(k) 만으로 바로 앞에 정의된 potential을 deformation 시킨다. 단, 이 때, Coulomb deformation이라면, 앞의 potential은 TYPE=0 이어야하고, Nuclear deformation의 경우에는 TYPE=1,2 이어야 한다.

TYPE=10 or 11 이고, SHAPE=7,8,9 인 경우 Card 11을 읽어들인다. (그리고 나서 P(k)를 곱한다?)

TYPE=12,13 인 경우 Card 11을 읽어들인다. 이 때는 P(k)는 information only이고 실제로 곱해지지 않는다.(?)

14.2.3 Card 11: For TYPEs 12-13 or TYPE=10-13 with SHAPE=7,8,9 (&STEP)

Card 11 4X,3I4,F8.4 IB,IA,k, STR

14.2.4 Card 12: Blank card for indication of end of potential