

CONCLUSION

The experimental data available at present are sufficient to permit one to attempt a theoretical treatment of the intensities of α lines belonging to a single rotational band. A problem of equal importance is to obtain the theoretical relation between the empirical constants in the formulas and the geometrical parameters of the nucleus.

A careful study of the α spectra of Pu^{240} , Am^{241} , and U^{233} has made it possible for the first time to carry out a comparison of the existing formulas with experiment.

Formula (4), proposed by L. D. Landau, satisfactorily describes the intensity of α decay to various levels, at least for even-even nuclei. For odd nuclei the picture is not clear.

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Nuclear Saturation and Two-Body Forces: Self-Consistent Solutions and the Effects of the Exclusion Principle*

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The equilibrium properties of uniform nuclear matter have been studied in previous papers by using a self-consistency method to determine two-body reaction matrices and the average effective one-particle potential which they generate. In this paper the self-consistency method is simply illustrated by using some explicit examples. An investigation is also made of the previously neglected effects of the exclusion principle on transitions to intermediate states. A variational expression is used for the reaction matrix, utilizing as a trial function the wave function which is exact if the exclusion effect is neglected. Two-body potentials are used which simulate the actual forces, i.e., square wells with a repulsive core of $0.35 \hbar/\mu c$, range of $1.15 \hbar/\mu c$, and depth 98.3 Mev, acting on s -states only. It is found that for effective mass values of $0.5 M$ and $0.6 M$, the reaction matrix is appreciably altered, particularly

for low values of relative momentum. The requirements of self-consistency, however, almost entirely compensate for the change in the reaction matrices so that at normal density in the effective mass approximation the final result for the average binding energy is *precisely* the same as that obtained when exclusion effects are neglected. The reasons for this simple result are discussed.

Further approximations are discussed which are suitable for more exact computations and which allow inclusion of the exclusion effects, the departures from the effective mass approximation, and the effects of "propagation off the energy shell." Approximation methods for a finite nucleus are discussed and a simplified Hartree-Fock method using "pseudo-potentials" (in the sense of Fermi) is described.

I. INTRODUCTION

IN previous papers an approximation method for treating quantum mechanical systems of many particles has been developed and applied to the determination of the equilibrium properties of nuclear matter, such as the binding energy, equilibrium density, surface energy, etc.¹⁻⁵ We shall here develop in some detail the procedure used to obtain approximate solutions to a nuclear system of infinite extent by the method of self-consistency. In addition, it will be shown, by an explicit calculation of the average nuclear binding energy using a specific two-body nuclear interaction

potential, how an approximation previously made in neglecting the effects of the exclusion principle in the intermediate states may now be removed. It is found that the average properties of the nuclear matter are altered only very slightly by the exclusion effects, although there exist appreciable alterations in the details of the reaction matrix and hence in some details of the velocity-dependent potential.

We also show how the effects of propagation off the energy shell in excited states can be included approximately in a simplified form of the theory suitable for computation. The effects of the finite size of the nuclear matter have already been studied in previous papers.^{2,3} Some comments are made on an alternative method for investigating these effects.

II. FORMULATION OF THE SELF-CONSISTENCY PROBLEM

In earlier work³⁻⁵ it was shown that an excellent approximation to the binding energy of extended nuclear matter can be obtained by solving the following

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¹ Brueckner, Levinson, and Mahmoud, *Phys. Rev.* **95**, 217 (1954).

² K. A. Brueckner, *Phys. Rev.* **96**, 508 (1954).

³ K. A. Brueckner, *Phys. Rev.* **97**, 1353 (1955).

⁴ K. A. Brueckner and C. A. Levinson, *Phys. Rev.* **97**, 1344 (1955).

⁵ K. A. Brueckner, *Phys. Rev.* **100**, 36 (1955).

equation for the reaction matrix⁶

$$K_{ij,kl} = v_{ij,kl} + \sum_{mn} v_{ij,mn} \frac{1}{E_k + E_l - E_m - E_n} K_{mn,kl}, \quad (1)$$

where the matrix elements are taken with respect to the independent-particle eigenstates of a uniform medium, i.e., plane waves. The sums over m and n are over all empty states. The energy E_k is not the unperturbed kinetic energy, but includes the interaction energy of a particle in the state k with the rest of the system, i.e.,

$$E_k = (k^2/2M) + V(k), \quad (2)$$

where

$$V(k) = \sum_i (K_{ki,ki} - K_{ki,ik}).$$

We assume here that the potential has the same form in excited states as well, and neglect the effects of propagation off the energy shell discussed in reference 5. We shall return to these in Sec. V.

Equations (1) and (2) form a coupled set; their solution is most simply approached using a method of interpolation and iteration which has in previous discussions^{3,4} been called a "self-consistency" method. Finally, the energy of the system is

$$E = \sum_i k_i^2/2M + \frac{1}{2} \sum_{ij} (K_{ij,ij} - K_{ij,ji}), \quad (3)$$

where the sums are over all filled states.

In previous studies of the coupled system of equations, use was made of the similarity of Eq. (1) for the reaction matrix to an equation of scattering theory for the reaction matrix. For scattering, the equation for the reaction matrix (which we call R to avoid confusion) is

$$R_{ij,kl} = v_{ij,kl} + \sum_{mn} v_{ij,mn} \frac{1}{\epsilon_k + \epsilon_l - \epsilon_m - \epsilon_n} R_{mn,kl}, \quad (4)$$

where the energies are the kinetic energies alone, i.e., $\epsilon_k = k^2/2M$, and the sum over mn is over all states except

⁶ This equation does not give the exact energy shift due to a two-body potential v , which is $\Delta E = K'_{ii,ij}$, where K' is defined by the integral equation [see Reifman, DeWitt, and Newton, Phys. Rev. **101**, 877 (1956)]

$$K'_{ij,kl} = v_{ij,kl} + \sum_{mn} v_{ij,mn} \frac{1 - P_{kl}}{E_k + E_l + \Delta E - E_m - E_n} K'_{mn,kl},$$

with P_{kl} a projection operator on the state kl . K' differs from K of Eq. (1) by the appearance of ΔE in the energy denominator; although ΔE (the interaction energy of a single pair) is very small compared with the energies E_k , it cannot in general be neglected if the excitation energy can take on values of the order ΔE . In the case of a degenerate Fermi gas, however, the exclusion principle requires that the excitation energy be of the order of the Fermi energy except for states very near the Fermi momentum, so that in this case ΔE can be neglected. Alternatively ΔE can be expanded as a perturbation which can be shown to vanish for large A as $1/A$ for a degenerate Fermi gas of A particles (see reference 4). Thus in the many-body fermion problem it is possible to discuss the K matrix rather than the more complicated K' matrix if the effects of the exclusion are properly taken into account. The authors are indebted to Professor R. G. Newton and Professor N. Fukuda for pointing out the difference between the K and K' matrices.

the energy conserving state kl . The solution to this integral equation is most easily obtained by solving the related Schrödinger equation. If we ask for only the energy-conserving part of R (which we call T), then we have the familiar result for central forces

$$T_{ij} = -\frac{4\pi}{M} \sum_l \left| \frac{\mathbf{k}_i - \mathbf{k}_j}{2} \right|^{-1} (2l+1) \tan \delta_l P_l(\cos \theta), \quad (5)$$

where θ is the angle between $\mathbf{k}_i - \mathbf{k}_j$ and $\mathbf{k}_k - \mathbf{k}_l$ and δ_l is the phase shift induced in the l th partial wave by the interaction v .

Equation (1) for K differs from Eq. (4) for R in two respects: first, the energy differences are not solely the result of changes in kinetic energy and second, transitions to occupied states are forbidden by the exclusion principle. The first effect can be taken approximately into account by introducing the effective-mass approximation.³ We shall discuss this in more detail in the next section. The effect of the exclusion principle is more difficult to correct for; in Sec. IV we shall develop a method for taking it into account.

III. SOLUTION IN THE EFFECTIVE-MASS APPROXIMATION

We shall outline in this section the solution to the problem in the effective-mass approximation in more detail than was done in previous work; we do this to clarify the application of the methods. We shall also obtain solutions using factorable potentials⁷ where again the application of the methods is particularly simple and instructive.

The basic procedure of the effective-mass approximation lies in an approximate reduction of the exact equation for the many-body reaction matrix [Eq. (1)] to a corresponding equation for a two-body reaction matrix which can be solved by using methods developed in the study of the relatively simple two-body system. The exact reaction matrix $K_{ij,kl}$ depends explicitly on the matrix elements $v_{ij,kl}$ of the interaction potential for the interacting pair, and depends implicitly on all of the remaining particles through the connection of the energy of each interacting particle with the average potentials $V(k)$ of Eq. (2). For a given state of the rest of the medium, $V(k)$ depends explicitly on the state of a single particle, although it of course is altered if the state of the rest of the system is changed. Thus it is correct if we fix our attention on only one particle at a time, to assume that $V(k)$ depends explicitly on the variable k but only parametrically on the state of the remaining particles. The form which has been taken for $V(k)$ is

$$V(k) = V(0) + bk^2 + ck^4 + \dots, \quad (6)$$

where $V(0)$, b , and c are parameters which change if the state of the system is altered, but are fixed for a

⁷ The use of factorable potentials was suggested to us by Professor K. M. Watson.

given state. In this expression for $V(k)$, the odd powers of k are absent if the medium is isotropic.

It is possible to obtain rather simply a semiquantitative result for the K matrix if only the constant and the quadratic terms in $V(k)$ are retained. In this approximation we can write, from Eq. (2),

$$\begin{aligned} E_k &= (k^2/2M) + V(0) + bk^2 \\ &= (k^2/2M^*) + V(0), \end{aligned} \quad (7)$$

where the effective mass M^* is given by

$$1/M^* = (1/M) + 2b. \quad (8)$$

When the energy terms in Eq. (1) are replaced by those modified by the velocity-dependent potentials, we have an approximate equation for the two-body reaction matrix, which, except for the modified mass and for the intermediate states excluded by the exclusion principle, is identical with the exact equation for the reaction matrix for scattering, i.e., Eq. (4). Thus, if we neglect the exclusion effects for the present, the diagonal element of the reaction matrix in this approximation is obtained from Eq. (5):

$$\langle \mathbf{k} | K | \mathbf{k} \rangle = -\frac{4\pi}{M^*} \sum_l k^{-1}(2l+1) \tan \delta_l(M^*, k), \quad (9)$$

where $\delta_l(M^*, k)$ is now the phase shift induced by the potential when the particles are moving with the effective mass M^* . The potential in the medium, $V(k)$, can be obtained from the relationship

$$V(k) = \sum_i (K_{ik, ik} - K_{ik, ki}), \quad (10)$$

and the predicted value of M^* can be tested for self-consistency with the input values. When the self-consistent result is obtained for M^* , the reaction matrices for this value then provide a solution to the coupled equations (1) and (2).

It is easy to solve the problem in some special cases. First we consider the s -wave scattering from a two-body potential with parameters which approximate to average values for the singlet and triplet wells. We also include a repulsive core of typically assumed range. This interaction potential is

$$\begin{aligned} v(r) &= \infty & \text{for } r < r_c \\ &= -v_0 & \text{for } R > r > r_c \\ &= 0 & \text{for } r > R, \end{aligned} \quad (11)$$

where $r_c = 0.35\mu^{-1}$, $R = 1.15\mu^{-1}$, and the depth is 1.22 times the depth which gives binding at zero energy, i.e.,

$$(R - r_c)(Mv_0)^{1/2} = (1.22)^{1/2}\pi/2,$$

or

$$v_0 = 0.703\mu = 98.3 \text{ Mev}, \quad (12)$$

where μ is the π -meson rest mass (we use the system of units for which $\hbar = c = 1$).

To proceed, we obtain the phase shifts as a function

of k and the effective mass M^* by solving the Schrödinger equation which is equivalent to solving Eq. (1). For s -wave scattering, the phase shift is given by the equation

$$k_0 \cot[k_0(R - r_c)] = k \cot(kR + \delta), \quad (13)$$

where

$$k_0 = [M^*v_0 + k^2]^{1/2}.$$

The potential $V(k)$ defined by Eq. (10) is then given by¹

$$V(k) = -\frac{24}{\pi M^*} \int_0^{\frac{1}{2}(k_F - k)} x dx \tan(x) P_k(x), \quad (14)$$

where

$$\begin{aligned} P_k(x) &= 2 & \text{for } 0 < x < \frac{1}{2}(k_F - k) \\ &= [\frac{1}{4}k_F^2 - (\frac{1}{2}k - x)^2]/kx \end{aligned} \quad (15)$$

for

$$\frac{1}{2}(k_F - k) < x < \frac{1}{2}(k_F + k).$$

To determine the momentum dependence of $V(k)$, it is simplest to evaluate $V(k)$ at $k=0$ and $k=k_F$ (Fermi momentum). At these two values,

$$V(0) = -\frac{48}{\pi M^*} \int_0^{\frac{1}{2}k_F} x dx \tan \delta(x), \quad (16)$$

$$V(k_F) = -\frac{24}{\pi M^*} \int_0^{k_F} x dx \left(1 - \frac{x}{k_F}\right) \tan \delta(x).$$

Then, using Eqs. (7) and (8), we find

$$\sigma = \frac{M^*}{M} = \frac{k_F^2/2M}{(k_F^2/2M) + V(k_F) - V(0)}. \quad (17)$$

To solve this equation for σ , the simplest procedure is to interpolate between values of $\tan \delta(k)$ computed for values of M^* close to the final value. For example, for $k_F = 1.50 \mu$, we find, from Eq. (16)

$$\begin{aligned} V(0) &= -43.2 \text{ Mev} \\ V(k_F) &= -29.3 \text{ Mev} \end{aligned} \quad \text{for } \sigma_1 = 0.500, \quad (18)$$

$$\begin{aligned} V(0) &= -71.9 \text{ Mev} \\ V(k_F) &= -46.7 \text{ Mev} \end{aligned} \quad \text{for } \sigma_2 = 0.600.$$

Assuming a linear dependence of $V(0)$ and $V(k_F)$ upon σ over this narrow interval (this is very close to correct), we may write

$$V(0) = -\left[43.2 + \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}(28.7)\right] \text{ Mev}, \quad (19)$$

$$V(k_F) = -\left[29.3 + \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}(17.4)\right] \text{ Mev}.$$

These, combined with Eq. (17), give a quadratic equation for σ , which has as solution $\sigma = 0.548$. The reaction matrix K at this value of σ is plotted in Fig. 2,

from which we obtain $V(0) = -56.9$ Mev and $V(k_F) = -37.3$ Mev. These are the self-consistent solutions. The average interaction energy is^{1,3}

$$V_{av} = \frac{1}{2}\{V(0) + \frac{3}{5}[V(k_F) - V(0)]\} = -22.6 \text{ Mev}, \quad (20)$$

and the binding energy per particle

$$E/A = \frac{3}{5}(k_F^2/2M) + V_{av} = -7.9 \text{ Mev}. \quad (21)$$

In practice, it is necessary to carry out this procedure as a function of k_F or the density ($k_F = 1.52 \mu/\eta$, where the density parameter η corresponds to a nuclear radius $R = \eta 1.40 \times 10^{-13} A^{1/3}$ cm) and thus determine the curve of binding energy per particle *vs* density. It is also necessary to correct this result for the effects of the k^4 and higher terms in $V(k)$. This problem has been considered in reference 3 where it was shown that the approximate inclusion of the k^4 term increases the interaction energy by about 20%. We shall not discuss this detailed problem further here.

As a second illustration, we consider a factorable potential which allows an exact solution for the reaction matrix. It is known from the work of Yamaguchi⁸ that it is possible to represent accurately the scattering from zero to 100 Mev by using such potentials. We shall not consider the problem here with tensor force included; we instead take only averaged central singlet and triplet potentials which fit the scattering but, of course, not the quadrupole moment. The integral equation to be solved is

$$(\mathbf{k}' | K | \mathbf{k}) = (\mathbf{k}' | v | \mathbf{k}) + \int (\mathbf{k}' | v | \mathbf{k}'') \times \frac{1}{(k^2/M^*) - (k''^2/M^*)} (\mathbf{k}'' | K | \mathbf{k}) \frac{d\mathbf{k}''}{(2\pi)^3}. \quad (22)$$

If v is assumed to have the form

$$(\mathbf{k}' | v | \mathbf{k}) = f(\mathbf{k})f(\mathbf{k}'), \quad (23)$$

then it is easily shown that the solution for K is

$$(\mathbf{k}' | K | \mathbf{k}) = f(\mathbf{k})f(\mathbf{k}') / \left(1 - M^* \int \frac{f^2(\mathbf{k}'')}{k^2 - k''^2} \frac{d\mathbf{k}''}{(2\pi)^3}\right). \quad (24)$$

For s -waves we shall take here a simple form for $f^2(k)$ since we are not interested in numerical accuracy. A suitable choice is

$$f^2(k) = \lambda/(\beta^2 + k^2). \quad (25)$$

To give reasonable depth and range, we take $\beta = \mu$ (π -meson mass), and $\lambda = -4\pi\mu s/M$. For a bound state at zero energy, $s = 1$; we take $s = 1.22$ as a reasonable average for singlet and triplet states.

⁸ Y. Yamaguchi, Phys. Rev. **95**, 1629 (1954); Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. **95**, 1635 (1954).

The integration for K then gives for the diagonal element

$$(\mathbf{k} | K | \mathbf{k}) = -\frac{4\pi\mu s/M}{\alpha^2 + k^2}, \quad (26)$$

where $\alpha^2 = \mu^2(1 - \sigma s)$. It is also possible to carry out the integrations for $V(k)$ explicitly; we find

$$V(0) = -\frac{24}{\pi} \frac{\mu s k_F}{M} \left[1 - \frac{2\alpha}{k_F} \tan^{-1}\left(\frac{k_F}{2\alpha}\right)\right],$$

$$V(k_F) = -\frac{24}{\pi} \frac{\mu s k_F}{M} \times \left[\frac{1}{2} - \frac{\alpha}{k_F} \tan^{-1}\left(\frac{k_F}{\alpha}\right) + \frac{\alpha}{2k_F^2} \ln\left(\frac{\alpha^2 + k_F^2}{\alpha^2}\right)\right]. \quad (27)$$

Thus we must solve a transcendental equation to determine the self-consistent solution for the effective mass $M^* = \sigma M$. At normal density of the nuclear matter, the solution is $\sigma = 0.480$; at this value we obtain

$$V(0) = -76.7 \text{ Mev}, \quad V(k_F) = -50.3 \text{ Mev}, \quad (28)$$

and the binding energy per particle is found to be

$$E/A = \frac{3}{5}(k_F^2/2M) + \frac{1}{2}\{V(0) + \frac{3}{5}[V(k_F) - V(0)]\} = (14.7 - 30.4) \text{ Mev} = -15.7 \text{ Mev}. \quad (29)$$

This method can easily be extended to other forms of factorable potentials.

IV. CORRECTION FOR THE EXCLUSION PRINCIPLE

In the previous sections, as in the earlier work on the saturation problem, we have neglected the effects of the exclusion principle in intermediate states. This neglect has previously been justified by two arguments.³ The first of these makes use of the fact that, because of the great depth of the potentials (due in part to the presence of repulsive cores), the interacting particles will usually be at momenta considerably greater than the Fermi momentum and thus not be appreciably affected by the exclusion of states of low momentum. For example, for the square wells with repulsive cores considered in the last section, the well depth is about 100 Mev compared with 15 Mev for the mean kinetic energy. The second argument arises from a partial cancellation of the intermediate state exclusion effects when the expectation value of the total interaction energy is taken with respect to properly antisymmetrized states. To see this effect in detail, consider the second-order term in the reaction matrix $K_{ij,ij}$ which is

$$\sum_{mn} |V_{ij,mn}|^2 \frac{1}{E_i + E_j - E_m - E_n}. \quad (30)$$

Here the sum over mn runs only over unfilled states;

we thus write

$$\Sigma = \left[\sum_{mn} - \sum_{m(\text{all})} - \sum_{m(\text{filled})} \right] \left[\sum_{n} - \sum_{n(\text{all})} - \sum_{n(\text{filled})} \right], \quad (31)$$

where in each factor the sums run over all and over only the filled states, respectively. To compute the total interaction energy, we evaluate the sum over all filled states,

$$\sum_{ij} K_{ij, ij}, \quad (32)$$

to which the second-order contribution is

$$\sum_{k_i < k_F} \sum_{k_j < k_F} \sum_{k_m > k_F} \sum_{k_n > k_F} |v_{ij, mn}|^2 \frac{1}{E_i + E_j - E_m - E_n}. \quad (33)$$

For the case in which the total momentum of the interacting pair vanishes, the allowed regions of the relative momenta \mathbf{k}_{ij} and \mathbf{k}_{mn} are the interior and the exterior of a sphere of radius k_F , respectively. For this case the above expression becomes

$$\int_0^{k_F} k_{ij}^2 dk_{ij} \left[\int_0^\infty k_{mn}^2 dk_{mn} - \int_0^{k_F} k_{mn}^2 dk_{mn} \right] \frac{1}{k_{ij}^2 - k_{mn}^2} \times f(|\mathbf{k}_{ij}|, |\mathbf{k}_{mn}|), \quad (34)$$

where $f(|\mathbf{k}_{ij}|, |\mathbf{k}_{mn}|)$ is a symmetric function of the arguments. The second term would vanish because of the antisymmetry of the energy denominator in k_{ij} and k_{mn} . Thus, we make no error in the second order term in neglecting the exclusion principle when the total momentum of the interacting pair is vanishingly small in comparison to the relative momenta of the pair in the intermediate states. The cancellation can also be expected to persist to some extent even when this condition is not satisfied. The effect is, of course, not so simple in the actual solution of the integral equation for $K_{ij, ij}$; we expect, however, that some of this cancellation of effects will still occur. On the basis of these two arguments, it was concluded in the earlier studies that the exclusion effects could be safely neglected, the error introduced being small. We shall now proceed to make a quantitative study of the exclusion effects which will bear out the qualitative arguments we have just made.

The answer to these questions is, of course, immediately obvious if a solution is obtained to the integral equation for the reaction matrix K . This is not in general a simple procedure, however, and we shall first use an approximate method to obtain a semiquantitative determination of the exclusion effects. We make use of the fact that in the absence of the exclusion effects, it is relatively simple to obtain K by solving the related differential equation. To make this more explicit, we note that the integral equation for K , i.e., Eq. (1), can be written

$$K_{ij, kl} = \sum_{mn} v_{ij, mn} \Omega_{mn, kl}, \quad (35)$$

where the wave matrix Ω satisfies the integral equation⁹

$$\Omega_{mn, kl} = 1_{mn, kl} + \sum_{rs} \frac{1}{e_{kl, mn}} v_{mn, rs} \Omega_{rs, kl}, \quad (36)$$

writing $E_k + E_l - E_m - E_n = e_{kl, mn}$. Neglecting exclusion effects we can thus determine Ω in the usual manner by solving the matrix equation

$$(E_k + E_l - E_m - E_n) \Omega_{mn, kl} = \sum_{rs} v_{mn, rs} \Omega_{rs, kl}, \quad (37)$$

which on transformation from momentum to coordinate space, is the usual Schrödinger equation for the wave function. This simple method for determining K cannot be used in the presence of the exclusion principle since $1/e$ is no longer the Green's function for the operator e . To circumvent this difficulty, we make use of a variational equation for K in which the wave function determined neglecting the exclusion effects is used as a trial function. Since the corrections to K turn out to be rather small, this procedure should be quite accurate.

We use the Schwinger variation principle for K .¹⁰ We start from the equation defining K in terms of v and Ω ,

$$K_{ij, kl} = (\varphi_{ij}, v \Omega \varphi_{kl}) = (\varphi_{ij}, v \psi_{kl}), \quad (38)$$

where φ_{ij} and φ_{kl} are free-particle wave functions and ψ_{kl} is related to φ_{kl} by the wave matrix

$$\psi_{kl} = \sum_{mn} \varphi_{mn} \Omega_{mn, kl}. \quad (39)$$

It then follows that

$$K_{ij, kl} = \frac{(\varphi_{ij}, v \psi_{kl})^2}{(\varphi_{ij}, v \psi_{kl})} = \frac{(\varphi_{ij}, v \psi_{kl})^2}{(\psi_{ij}, v \psi_{kl}) - [\psi_{ij}, v(1/e)v\psi_{kl}]}, \quad (40)$$

where the last expression follows from the integral equation satisfied by ψ_{ij} . In this form, K is unaltered by first order errors in ψ ; this expression also has the useful property of being independent of the normalization of ψ . We now proceed to determine the corrections due to the exclusion effects.

For an example, we take the square well potential considered in Sec. III. It will prove simplest first to take the repulsive core as having a finite height v_1 and then let $v_1 \rightarrow \infty$. The wave function ψ in coordinate

⁹ For a more detailed discussion of the following see, for example, C. M. Möller, Kgl. Danske Videnskab. Selskab, Mat-fys. Medd. **23**, 1 (1945).

¹⁰ B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

space is, in the effective mass approximation, for large v_1

$$\psi_{kl}(r) = \frac{e^{-\alpha r_c} \sinh \alpha r \sin \delta_c}{k_0 r} \quad \text{for } r < r_c \quad (41)$$

$$= \frac{\sin k_0(r - r_c + \delta_c)}{k_0 r} \quad \text{for } r_c < r < R,$$

where

$$\alpha = (M^* v_1)^{\frac{1}{2}}, \quad (42)$$

$$k_0 = (M^* v_0 + k_{kl}^2)^{\frac{1}{2}}.$$

The phase shift δ_c , which vanishes as $v_1 \rightarrow \infty$, is determined by matching boundary conditions at $r = r_c$, and is given for large v_1 by

$$\delta_c = k_0 / \alpha. \quad (43)$$

The second term in the denominator for K of Eq. (40) may be written

$$[\psi_{ij}, v(1/e) v \psi_{kl}]$$

$$= \frac{V^2}{(2\pi)^6} \int (v \psi_{ij} | \mathbf{k}_m \mathbf{k}_n) d\mathbf{k}_m d\mathbf{k}_n \frac{1}{e_{kl, mn}} (\mathbf{k}_m \mathbf{k}_n | v \psi_{kl}), \quad (44)$$

where V is the normalization volume. The Fourier transform of $v\psi$ which is required is

$$(\mathbf{k}_m \mathbf{k}_n | v \psi_{kl})$$

$$= \int \frac{e^{-i\mathbf{k}_m \cdot \mathbf{r}_1}}{V^{\frac{1}{2}}} \frac{e^{-i\mathbf{k}_n \cdot \mathbf{r}_2}}{V^{\frac{1}{2}}} v(|\mathbf{r}_1 - \mathbf{r}_2|) \psi_{kl}(|\mathbf{r}_1 - \mathbf{r}_2|)$$

$$\times \frac{e^{i(\mathbf{k}_k + \mathbf{k}_l) \cdot (\mathbf{r}_1 + \mathbf{r}_2)}}{V} d\mathbf{r}_1 d\mathbf{r}_2$$

$$= \delta_{\mathbf{k}_m + \mathbf{k}_n, \mathbf{k}_k + \mathbf{k}_l} \frac{4\pi}{V}$$

$$\times \left\{ v_1 \int_0^{r_c} \frac{e^{-\alpha r} \sinh \alpha r \sin \delta_c \sin k_{mn} r}{k_0 r} \frac{\sin k_{mn} r}{k_{mn} r} r^2 dr \right.$$

$$\left. - v_0 \int_{r_c}^R \frac{\sin k_0(r - r_c + \delta_c) \sin k_{mn} r}{k_0 r} \frac{\sin k_{mn} r}{k_{mn} r} r^2 dr \right\}. \quad (45)$$

Dropping the Kronecker delta function this is, as $v_1 \rightarrow \infty$,

$$(\mathbf{k}_m \mathbf{k}_n | v \psi_{kl}) = \frac{4\pi}{V} A_{kl}(k_{mn}),$$

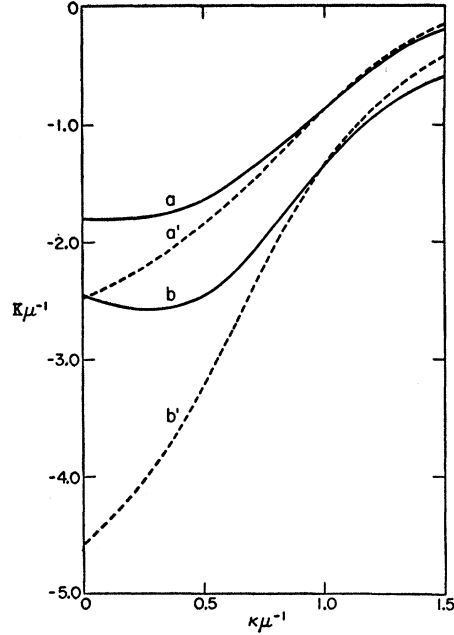


FIG. 1. Reaction matrices as functions of the relative momentum k . Solid curves are with exclusion effects and dotted curves are without. Curves a and a' are for $\sigma=0.500$, and b and b' for $\sigma=0.600$.

where

$$A_{kl}(k_{mn}) = \frac{1}{k_0^2 - k_{mn}^2} \left[\frac{k_{kl}^2 - k_{mn}^2}{M^* k_{mn}} \sin k_{mn} r_c \right.$$

$$- \frac{v_0}{k_0} \sin k_0(R - r_c) \cos k_{mn} R$$

$$\left. + \frac{v_0}{k_{mn}} \cos k_0(R - r_c) \sin k_{mn} R \right]. \quad (46)$$

We also need another integral,

$$(\psi_{ij}, v \psi_{ij})_{v_1 \rightarrow \infty} = - \frac{4\pi v_0}{k_0^2} \int_{r_c}^R \sin^2 k_0(r - r_c) dr$$

$$= - \frac{2\pi v_0}{k_0^2} (R - r_c) \left[1 - \frac{\sin 2k_0(R - r_c)}{2k_0(R - r_c)} \right]. \quad (47)$$

Introducing these results into the variational expression for K of Eq. (40), we find

$$K_{ij, ij} = \frac{(4\pi/V) k_0^2 A_{ij}^2(k_{ij})}{-\frac{1}{2} v_0 (R - r_c) \left[1 - \frac{\sin 2k_0(R - r_c)}{2k_0(R - r_c)} \right] - \frac{2M^* k_0^2}{\pi} \int_0^\infty \frac{A_{ij}^2(k') k'^2 dk'}{k_{ij}^2 - k'^2} \int \frac{d\Omega_{k'}}{4\pi}}. \quad (48)$$

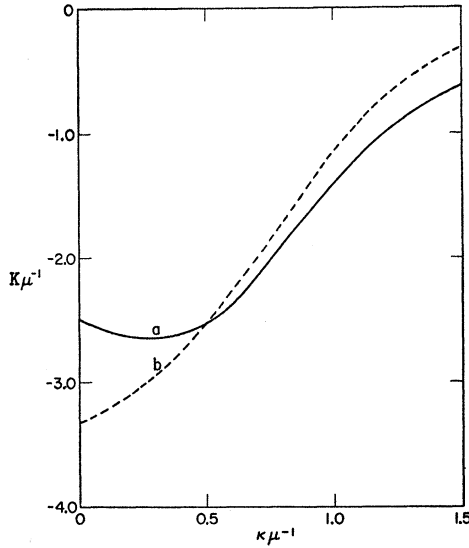


FIG. 2. Reaction matrices as functions of the relative momentum k . Curve a corresponds to the self-consistent solution (effective mass $M^*=0.606M$) with exclusion effects included, and curve b corresponds to the self-consistent solution ($M^*=0.548M$) neglecting exclusion effects.

The exclusion principle requires that

$$\begin{aligned} |\mathbf{k}' + \frac{1}{2}\mathbf{P}| &\geq k_F, \\ |-\mathbf{k}' + \frac{1}{2}\mathbf{P}| &\geq k_F, \end{aligned} \quad (49)$$

where $\mathbf{P} = \mathbf{k}_i + \mathbf{k}_j$. In neglect of this requirement, it is easily shown that $K_{ij,ij}$ is given by the usual expression [Eq. (9)]:

$$K_{ij,ij} = -\frac{4\pi}{M^*} (k_{ij})^{-1} \tan \delta(M^*, k_{ij}). \quad (50)$$

Carrying out first the integration over the angles of k' and taking into account the region in momentum space forbidden by the exclusion principle, we obtain

$$\int \frac{d\Omega_{k'}}{4\pi} = f_P(k'),$$

where

$$\begin{aligned} f_P(k') &= 0 \text{ for } k' < (k_F^2 - \frac{1}{4}P^2)^{\frac{1}{2}}, \\ &= [k'^2 + \frac{1}{4}P^2 - k_F^2] / Pk' \text{ for } \\ &\quad (k_F^2 - \frac{1}{4}P^2)^{\frac{1}{2}} < k' < k_F + \frac{1}{2}P, \\ &= 1 \text{ for } k' > k_F + \frac{1}{2}P. \end{aligned} \quad (51)$$

Thus, the final result for the reaction matrix K will depend explicitly upon the center-of-mass momentum $P = |\mathbf{k}_i + \mathbf{k}_j|$; we will suppress this dependence by replacing P by its average value compatible with a given value of the relative momentum k_{ij} . An elementary computation shows that, to a very good approximation,

$$\frac{1}{4}(P^2)_{Av} = \frac{3}{5}k_F^2(1 - k_{ij}/k_F). \quad (52)$$

One further comment is necessary with regard to the effects of the exclusion principle; when k_{ij} is close to the Fermi momentum k_F , one finds $\frac{1}{4}(P^2)_{Av} \rightarrow 0$. At this point the integral over k_i becomes logarithmically

infinite, being of the form

$$\int_{k_F}^{\infty} \frac{g(k')dk'}{k_F^2 - k'^2} \quad (53)$$

This singularity is, of course, without physical content; it is the result of the breakdown of these approximation methods in describing states very near the Fermi momentum. An explicit evaluation of the effect of this singularity on the energy shows that it is extremely small and consequently can be safely neglected. Thus, we shall not investigate here the problem of making a more accurate treatment of these states.

The result of the evaluation of K given by Eq. (48) are shown in Fig. 1 for two values of the effective mass. It is apparent that, as expected, the alteration of the reaction matrix is most pronounced for low momentum values, since it is for these that on the average the exclusion effect is most important. To obtain a solution to the self-consistency problem, we again, as in the last section, use an interpolation procedure. At the two values of the effective mass ratio $\sigma = 0.500$ and 0.600 , the integrals for $V(0)$ and $V(k_F)$ give

$$\begin{aligned} V(0) &= -38.0 \text{ Mev} \\ V(k_F) &= -24.6 \text{ Mev} \end{aligned} \quad \left\{ \begin{array}{l} \text{for } \sigma_1 = 0.500, \end{array} \right. \quad (54)$$

$$\begin{aligned} V(0) &= -53.3 \text{ Mev} \\ V(k_F) &= -38.1 \text{ Mev} \end{aligned} \quad \left\{ \begin{array}{l} \text{for } \sigma_2 = 0.600. \end{array} \right.$$

It is to be noted that the singularity in the reaction matrix K has no effect on $V(k_F)$ since the integrand contains a factor which vanishes at k_F [see Eq. (16)]. Solving the equation for the effective mass leads to the self-consistent result that $\sigma = 0.606$. The reaction matrix K at this self-consistent value of σ is plotted in Fig. 2 together with the corresponding reaction matrix in neglect of the exclusion effects (Sec. III). We then obtain

$$V(0) = -54.6 \text{ Mev}, \quad V(k_F) = -38.9 \text{ Mev}. \quad (55)$$

The average interaction energy (V_{Av}) and the binding energy per particle at the normal nuclear density then turn out to be -22.6 Mev and -7.9 Mev, respectively. These values are compared in Table I with the values previously obtained neglecting exclusion effects; it is apparent that the result has been altered very little by the exclusion effects.

The surprisingly small alteration of the interaction energy is due partly to the previously mentioned partial cancellation of the exclusion effects which can be expected to occur when the expectation values of the sum of the expansion terms of the reaction matrix in powers of the interaction potential is taken with respect to the antisymmetrized wave functions for the system. In our study, the cancellation is due to the increase in magnitude of the reaction matrix for large momentum values which tend to compensate for the decrease for low momentum values. Even more im-

portant, however, is the shift in the self-consistent value of the effective mass. The exclusion effect reduces the variation of the reaction matrix K with momentum, as is apparent from Fig. 1. Thus, for a given value of M^* , the predicted velocity dependence of the potential is less than the previously obtained result. This upsets the self-consistency of the solution, which can be re-established only if the effective mass is increased. (The increase in the effective mass can be understood on the basis that the exclusion effects act as a weak repulsive potential). This alteration of M^* also increases the over-all magnitude of the reaction matrix so that the final self-consistent result, as Table I shows, is very close to that obtained when the exclusion effects are neglected. It must be emphasized, however, that although the average effect is very small, there are appreciable alterations in detail, not only in the velocity dependence of the potential, but also in the reaction matrix for momenta very close to k_F .

V. IMPROVED APPROXIMATION TO THE K MATRIX

In the last two sections, some simple procedures were outlined for obtaining approximate solutions to the equations for the K matrix and the average potential. In this section we shall outline further procedures suitable for numerical computations with fast computing techniques. We return to the exact system of equations for the K matrices and for the average potential.

The energies given by Eqs. (1) and (2), E_k and E_l , in the ground state are given in terms of the diagonal elements of the K matrix as defined by Eq. (2); the excited energies E_m and E_n are, however, modified by being the energies appropriate to virtual states, i.e., they are for particles propagating off the energy shell. As shown in reference (5), they are given by

$$E_m(kl, nm) = \frac{km^2}{2M} + \sum_s (K_{ms, ms}^* - K_{ms, sm}^*), \quad (56)$$

where K^* now depends on the state of excitation of the system compared with the ground state. Since particles in states kl have made transitions to the states nm , K^* in this state is given by

$$K_{m's', ms}^* = v_{m's', ms} + \sum_{uv} \frac{v_{m's', uv}}{E_k + E_l + E_s - E_n - E_u - E_v} K_{uv, ms}^*. \quad (57)$$

As has been previously pointed out, in many circumstances the dependence of $E_m(kl, mn)$ on k, l, n is weak. This is true if the kinetic energy is large compared with the potential energy as is the case for strong interactions, or if the Born approximation is roughly valid, since then K^* does not depend on k, l, n . The effect can also be expected to be small since it is a high order correction to the ground state energies, appearing only in fourth order in a perturbation expansion. For these reasons we shall content ourselves with an approxima-

TABLE I. The self-consistent effective mass and energies (Mev) per particle at normal nuclear density with and without exclusion effects.

	M^*/M	$V(0)$	$V(k_F)$	V_{Av}	Bind. energy
Without exclusion effects	0.548	-56.9	-37.3	-22.6	-7.9
With exclusion effects	0.606	-54.6	-38.9	-22.6	-7.9

tion which reproduces the general effects of going off the energy shell but shall not try to do so in an exact way. The approximation we propose is to replace the exact equation for K^* , i.e., Eq. (57), by the approximate equation, assumed valid for any state,

$$K_{m's', ms}^* = v_{m's', ms} + \sum_{uv} v_{m's', uv} \times \frac{1}{E(k_F, k_m) + E(k_F, k_s) - E_u - E_v} K_{uv, ms}^*, \quad (58)$$

where we take $E(k_F, k_m)$ to be

$$E(k_F, k_m) = \begin{cases} E(k_m) & \text{if } |\mathbf{k}_m| < k_F \\ E(k_F) & \text{if } |\mathbf{k}_m| > k_F, \end{cases} \quad (59)$$

and k_F is the Fermi momentum. This form for K is continuous with the result valid on the energy shell; it reproduces in a semiquantitative way the restriction that the unexcited particles occupied states below the Fermi momentum. The error made by this approximation can be accurately determined by eventually treating the difference between the exact and approximate off-the-energy-shell K 's as a perturbation.

Equation (58) for the K matrix together with Eq. (2) for the potential are now suitable for computation; it is, however, useful to introduce a further approximation, the validity of which is based on the smallness of certain corrections to the energy. We expect, on the basis of the studies of NS III (reference 3) and of this paper, that the effective-mass approximation neglecting exclusion effects is a quite good first approximation to the K matrix. This suggests that we introduce certain simplification in the equation for K . First, in the reduced-mass approximation, a typical energy of excitation is

$$E_m + E_n - E_u - E_v = 2[E(\frac{1}{2}(K_m - K_n)) - E(\frac{1}{2}(K_u - K_v))]. \quad (60)$$

This result does not hold if the energy depends on the state other than through a constant and a quadratic term; it is approximately satisfied, however, if in the quartic and higher terms in k , we neglect the center-of-mass momentum compared with the relative momentum. The validity of this approximation derives from the fact that the terms of higher order in k than quadratic are important only for considerable excitation where the relative momentum takes on quite large values. At the same time, the total momentum is restricted to values of the order of the mean Fermi

momentum corresponding to kinetic energy of 15 or 20 Mev. Thus we shall assume that Eq. (60) is valid even if $E(k)$ has a general momentum dependence. We also, as in Sec. V, treat approximately the exclusion effects in introducing the function $f_k(k')$ defined by Eq. (51). This eliminates correctly transitions to the filled states, but averages over the dependence of the exclusion effect on the total momentum. The final result of these approximations is to replace Eqs. (1) for K by the approximate equation (we call the approximate reaction matrix K_a)

$$(\mathbf{k}'|K_a|\mathbf{k}) = (\mathbf{k}'|v|\mathbf{k}) + \sum_{\mathbf{k}''} (\mathbf{k}'|v|\mathbf{k}'') \frac{f_{k,k_F}(k'')}{2[E(k,k_F) - E(k'')]} (\mathbf{k}''|K_a|\mathbf{k}), \quad (61)$$

where in f_{k,k_F} and $E(k,k_F)$ we take the momentum equal to k if k is below the Fermi momentum and equal to k_F if k is above the Fermi momentum. The energy is finally

$$E(k) = \frac{k^2}{2M} + \sum_i \left\{ \left(\frac{k-k_i}{2} |K_a| \frac{k-k_i}{2} \right) - \text{exchange} \right\}, \quad (62)$$

valid for any state.

These approximations treat to a moderate accuracy the effects of the exclusion principle, the departure of $E(k)$ from the reduced mass approximation, and the effects of "propagation off the energy shell." Since these effects collectively alter the K matrix by only the order of (10% to 20%), and since their effect is reproduced in a semiquantitative way, the final result is estimated to be accurate to within a few percent. This can be further checked by finally treating the difference between the exact expressions and the approximation as a perturbation on the approximate solution.

These equations are now being studied at Los Alamos Scientific Laboratory and at the AEC computing facility at New York University. The results of the studies will be reported in a separate communication.

VI. APPROXIMATION METHOD FOR FINITE SYSTEMS

In the discussions of this paper and in most of the previous studies, the nuclear medium was considered to be of infinite extent. This approximation was useful primarily in that surface effects could be neglected and in that plane waves could be used as independent-particle states. In a previous paper⁴ a method was outlined for dealing with a finite system, which is restated in the following in somewhat greater detail than before.¹¹

A. Assume a potential $W^{(0)}$ ("0" for the zeroth approximation) in which the nucleons move independently. This potential will in general be nonlocal in coordinate space; it will also depend on the energy-momentum relation of the particle. When a particle is excited to a virtual state, for example, the potential it sees must be such as to confine it to the nuclear volume.

¹¹ Alternate methods for dealing with finite systems have been discussed by R. J. Eden and by H. Bethe (to be published).

These effects can easily be approximated in the first input $W^{(0)}$, and will automatically follow in higher iterates from the approximation of Sec. V.

B. Determine eigenstates $\varphi_i^{(0)}$ and energy eigenvalues $E_i^{(0)}$ for this potential.

C. In this representation, determine the reaction matrices defined by the equation:

$$K_{ij,kl}^{(0)} = v_{ij,kl} + \sum_{mn} v_{ij,mn} \times \frac{1}{E_i^{(0)} + E_j^{(0)} - E_m^{(0)} - E_n^{(0)}} K_{mn,kl}^{(0)}. \quad (63)$$

D. Finally determine from

$$V_{ii}^{(1)} = \sum_j (K_{ij,ij}^{(0)} - K_{ij,ji}^{(0)}) \quad (64)$$

the first iterate for W :

$$W^{(1)} = \sum_i (\varphi_i^{(0)}, v_{ii}^{(0)}, \varphi_i^{(0)}). \quad (65)$$

The self-consistency procedure then is to adjust the input iterate $W^{(n)}$ until it is equal to the output $W^{(n+1)}$.

It is apparent that this procedure is extremely difficult to carry out in practice, the complication arising from the need to construct a set of single-particle wave functions and eigenvalues for each $W^{(n)}$. It seems probable, however, that the procedure may be very much shortened by the following approximation. At the first state, determine the reaction matrices $K_{ij,kl}$ in a uniform medium. From these it is then possible to construct two-body interaction "pseudo-potentials" given by the equation

$$(\mathbf{r}_1' \mathbf{r}_2' | v_s | \mathbf{r}_1 \mathbf{r}_2) = \sum_{ij,kl} \int \varphi_i^*(\mathbf{r}_1') \varphi_j^*(\mathbf{r}_2') K_{ij,kl} \varphi_k(\mathbf{r}_1) \varphi_l(\mathbf{r}_2) \times d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_1' d\mathbf{r}_2'. \quad (66)$$

These are then potentials defined in the sense of Fermi,¹² who first introduced similar concepts in the theory of proton scattering in molecular hydrogen. The second step is to follow the usual Hartree-Fock procedure for a finite system by replacing the usual local two-body potentials by the nonlocal "pseudo-potentials." This procedure differs from that outlined above in that the alteration of the reaction matrix in going from a uniform medium of infinite extent to a finite system is neglected. This neglect can be partially corrected for by solving the reaction matrices as functions of density (the medium is still assumed uniform in density). For this case, the "pseudo-potential" v_s would also depend on the density; in the case of the finite (and varying density) medium, it would be reasonable to evaluate v_s at the local value of the density.¹³

The procedure outlined here together with the approximation of Sec. V provides a relatively simple method for studying finite systems; its applications to specific nuclear problems will be discussed in other papers.

¹² E. Fermi, *Ricerca sci.* 7, 13 (1936).

¹³ A similar idea has been independently proposed by T. H. R. Skyrme (to be published).