Appendix C
Kinematics

In the following, expressions are presented that describe the kinematics of a binary interaction $a + A \rightarrow b + B$, where species a, A, and B are particles with rest mass. For more detailed discussions of the kinematics in nuclear physics, the reader is referred to Marmier and Sheldon (1969) and references therein. Kinematics calculations can be conveniently performed by using readily available computer codes.

The kinematics of a nuclear reaction or of elastic scattering is determined by the conservation of total energy and linear momentum. Consider Fig. C.1 (left panel), showing a collision between a projectile a and a stationary target nucleus A in the laboratory. After the collision, the recoil nucleus B moves into a direction specified by the laboratory angle ϕ, while species b moves into a direction given by laboratory angle θ. If species b is a photon, then the collision represents a radiative capture process. If species a is identical to b, and species A is identical to B (that is, their state of excition), then the collision represents elastic scattering. First, expressions are given that relate quantities appropriate for the laboratory coordinate system only. Afterward, formulas for the transformation of quantities between laboratory and center-of-mass coordinate systems are presented.

C.1
Relationship of Kinematic Quantities in the Laboratory Coordinate System

Consider first a collision involving only particles with rest mass. The target nucleus A is assumed to be stationary in the laboratory system. Conservation of energy and linear momentum yields the three equations

$$m_a c^2 + E_a + m_A c^2 = m_b c^2 + E_b + m_B c^2 + E_B \tag{C.1}$$

$$\sqrt{2m_a E_a} = \sqrt{2m_B E_B}\cos\phi + \sqrt{2m_b E_b}\cos\theta \tag{C.2}$$

$$0 = \sqrt{2m_B E_B}\sin\phi - \sqrt{2m_b E_b}\sin\theta \tag{C.3}$$

non - relativistic

where E and m denote the kinetic energy and the rest mass, respectively. The linear momenta are given by $p = \sqrt{2mE}$. The second and third expression

Nuclear Physics of Stars. Christian Iliadis

ISBN: 978-3-527-40602-9

describes the total linear momentum parallel and perpendicular, respectively, to the incident beam direction. It is usually difficult to detect species B if it represents a heavy recoil nucleus. By eliminating E_B and ϕ and by using the definition of the reaction Q-value, $\boxed{Q = (m_a + m_A - m_b - m_B)c^2}$ (see Eq. (1.4)), one finds

$$\boxed{Q = E_b\left(1 + \frac{m_b}{m_B}\right) - E_a\left(1 - \frac{m_a}{m_B}\right) - \frac{2}{m_B}\sqrt{m_a m_b E_a E_b}\cos\theta} \tag{C.4}$$

This expression is sometimes used to determine an unknown Q-value by measuring E_a, E_b, and θ if the masses m_a, m_b, and m_B are known. Frequently, one is interested in the energy E_b of the emitted particle as a function of the bombarding energy E_a and the angle θ. From Eq. (C.4) one obtains

$$\boxed{\sqrt{E_b} = r \pm \sqrt{r^2 + s}} \tag{C.5}$$

where

$$r = \frac{\sqrt{m_a m_b E_a}}{m_b + m_B}\cos\theta \qquad \text{and} \qquad s = \frac{E_a(m_B - m_a) + m_B Q}{m_b + m_B} \tag{C.6}$$

We assumed above that in low-energy nuclear reactions the speeds of the particles are sufficiently small to disregard relativistic effects. For very accurate work, one can take the relativistic correction into account if each mass m in the above expressions is replaced by $m + E/(2c^2)$. Only real and positive solutions of E_b in Eqs. (C.5) and (C.6) are physically allowed. A number of different cases can be distinguished. If the reaction is exothermic ($Q > 0$) and if the projectile mass is smaller than the mass of the residual nucleus ($m_a < m_B$), then $s > 0$ and there will only be one positive solution for E_b. Because of the $\cos\theta$ dependence of r, E_b has a minimum at $\theta = 180°$. For very small projectile energies, for example, in reactions involving thermal neutrons, we find $r \to 0$ and hence

$$E_b(E_a \approx 0) \approx s \approx Q m_B/(m_B + m_b) \tag{C.7}$$

This implies that the kinetic energy of the emitted particle b has the same value for all angles. The situation is more complex if the reaction is endothermic ($Q < 0$). For very small projectile energies, $E_a \approx 0$, one has again $r \to 0$, but s becomes negative so that no positive value of E_b exists. Hence, for each angle θ there will be a minimum energy below which the reaction cannot proceed. The value of this minimum energy is smallest at $\theta = 0°$ and is referred to as the *threshold energy*, given by

$$E_a^{\text{min}}(\theta = 0°) = E_a^{\text{thresh}} = -Q\frac{m_b + m_B}{m_b + m_B - m_a} \tag{C.8}$$

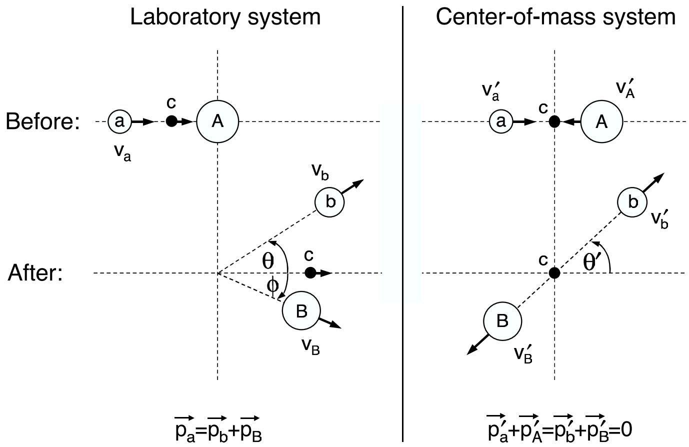

Fig. C.1 Kinematic properties of a reaction $A(a,b)B$ in the laboratory coordinate system (left) and the center-of-mass coordinate system (right). The target nucleus A is assumed to be stationary in the laboratory ($v_A = 0$). Unprimed and primed quantities are used in the laboratory and center-of-mass frame, respectively. The location of the center of mass is labeled "c."

At the threshold energy, the particles are emitted only in the direction $\theta = 0°$ with an energy of

$$E_b(E_a = E_a^{\text{thresh}}) = E_a^{\text{thresh}} \frac{m_a m_b}{(m_b + m_B)^2} \tag{C.9}$$

If one increases the bombarding energy beyond the threshold energy, then the particles b can be emitted at angles greater than $\theta = 0°$. It is also interesting to note that for endothermic reactions Eqs. (C.5) and (C.6) yield two positive solutions for $\theta < 90°$. In other words, two particle groups of different discrete energies are emitted in the forward direction. For bombarding energies exceeding

$$E_a = -Q \frac{m_B}{m_B - m_a} \tag{C.10}$$

there exists only a single positive solution for Eqs. (C.5) and (C.6).

Consider now a radiative capture process $a + A \to B + \gamma$. In this case we have to replace in Eqs. (C.1)–(C.3) the total energy, $m_b c^2 + E_b$, and linear momentum, $\sqrt{2m_b E_b}$, of species b by E_γ and E_γ/c, respectively. Eliminating again E_B and ϕ and solving for the energy of the emitted photon yields

$$E_\gamma = Q + \frac{m_A}{m_B} E_a + E_\gamma \frac{v_B}{c} \cos\theta - \frac{E_\gamma^2}{2m_B c^2} = Q + \frac{m_A}{m_B} E_a + \Delta E_{\text{Dopp}} - \Delta E_{\text{rec}} \tag{C.11}$$

The photon energy is given by a sum of four terms: (i) the value of $Q = (m_a + m_A - m_B)c^2 = E_B + E_\gamma - E_a$; (ii) the bombarding energy in the center-of-mass system (see below); (iii) the *Doppler shift* since the photon is emitted by a recoil nucleus B moving at a speed of $v_B = v_a(m_a/m_B)$; and (iv) the *recoil shift* which is caused by the energy shift of the recoiling nucleus. The last two contributions represent relatively small corrections and are numerically given by

$$\Delta E_{\text{Dopp}} = 4.63367 \times 10^{-2} \frac{\sqrt{M_a E_a}}{M_B} E_\gamma \cos\theta \qquad \text{(MeV)} \tag{C.12}$$

$$\Delta E_{\text{rec}} = 5.36772 \times 10^{-4} \frac{E_\gamma^2}{M_B} \qquad \text{(MeV)} \tag{C.13}$$

where all energies are in units of MeV and the rest masses are in units of u. The calculation of the photon energy from Eq. (C.11) is not strictly valid since E_γ also occurs on the right-hand side. If an answer with a precision of a few keV or so is sufficient, then one may replace the masses with (integer) mass numbers and use the approximation $E_\gamma \approx Q + E_a(m_A/m_B)$ on the right-hand side of Eq. (C.11). For accurate work, however, the masses of a, A, and B in Eqs. (C.1)–(C.3) should be replaced by the factors $m_i + E_i/(2c^2)$. The exact relativistic expression for the photon energy is then given by

$$E_\gamma = \frac{Q(m_a c^2 + m_A c^2 + m_B c^2)/2 + m_A c^2 E_a}{m_a c^2 + m_A c^2 + E_a - \cos\theta\sqrt{E_a(2m_a c^2 + E_a)}} \tag{C.14}$$

The relationship between the photon emission angle θ and the recoil angle ϕ can be obtained from the ratio of Eqs. (C.2) and (C.3),

$$\phi = \arctan\left(\frac{\sin\theta}{E_\gamma^{-1}\sqrt{2m_a c^2 E_a} - \cos\theta}\right) \tag{C.15}$$

The maximum angle of ϕ is obtained when the photon is emitted perpendicular to the incident beam direction ($\theta = 90°$),

$$\phi_{\text{max}} = \arctan\left(\frac{E_\gamma}{\sqrt{2m_a c^2 E_a}}\right) \tag{C.16}$$

Hence, the recoil nucleus B is emitted in the forward direction into a cone of half-angle ϕ_{max}.

A few comments are in order. If the reaction $A + a \to B + b$ or $A + a \to B + \gamma$ populates an excited state in nucleus B, then the Q-value in the above expressions must account for the energy of the excited state,

$$Q = Q_0 - E_x \tag{C.17}$$

where Q_0 is the Q-value for the ground state of B. Several excited levels may be populated in a given reaction. For a fixed angle θ, each of these states will give rise to a different value for the energy of the reaction products (for example, E_b or E_γ), where the largest observed energy corresponds to the population of the ground state. From a measurement of E_b or E_γ we may thus deduce an unknown excitation energy E_x by using Eqs. (C.5), (C.11), or (C.14). Note that for a radiative capture reaction the maximum emission angle $\phi_{\max}$ of B is given by Eq. (C.16), with E_γ denoting the photon energy for the ground state transition, even if the primary decay proceeds to an excited level (since subsequent de-excitation photons may also be emitted at $\theta = 90°$). The above expressions disregard the beam energy loss in the target and assume that the reaction is induced with a bombarding energy of E_a in the laboratory. If the reaction excites a narrow resonance, then the interaction is induced at the resonance energy E_r rather than at the actual incident beam energy. In this case, E_a in the above expressions represents E_r. Finally, for the case of radiative capture reactions it is assumed that the γ-ray emission occurs on a sufficiently short time scale for recoil energy losses in the target to be negligible, that is, the emitted photon experiences the full Doppler energy shift. If the photon is emitted after the recoil nucleus experienced an energy loss in the target, then the Doppler shift is attenuated. It is sometimes possible to deduce the mean lifetime of a nuclear level by measuring the attenuated Doppler shift (see, for example, Bertone et al. 2001).

C.2
Transformation Between Laboratory and Center-of-Mass Coordinate System

In experimental nuclear physics, all observations take place in a reference frame that is at rest in the laboratory. It is referred to as the *laboratory coordinate system*. From the theoretical point of view, however, the motion of the center of mass is of no consequence for the properties of a nuclear reaction. It is then often more convenient to use a moving coordinate frame in which the center of mass of the two colliding nuclei is at rest. It is called the *center-of-mass coordinate system*. Most kinematic quantities in Chapters 3 and 5 are given in the center-of-mass system. However, in Chapter 4 these quantities are frequently presented in the laboratory system, as is customary in the nuclear physics literature, since this is where the quantities are directly observed. We will only consider here the nonrelativistic transformation of kinematic quantities between these two reference frames. For the relativistic case, see Marmier and Sheldon (1969) and references therein.

The kinematic properties of a nuclear reaction $A(a,b)B$ in the laboratory and center-of-mass frames are shown in Fig. C.1. Unprimed and primed quantities

' represent CM frame

will be used in this section for the former and the latter coordinate system, respectively. It is assumed that the target nucleus is stationary in the laboratory ($v_A = 0$). In the center-of-mass frame, the total linear momentum is always equal to zero and, therefore, the nuclei b and B will recede in opposite directions. In other words, there is only one scattering angle θ'.

We will first consider the situation before the collision. The velocity $\vec{v}_c$ of the center-of-mass is given by the relations

$$(m_a + m_A)\vec{v}_c = m_a\vec{v}_a + m_A \cdot 0 \qquad \text{or} \qquad \vec{v}_c = \frac{m_a}{m_a + m_A}\vec{v}_a \tag{C.18}$$

and hence the projectile and target have velocities in the center-of-mass frame of

$$\vec{v}'_a = \vec{v}_a - \vec{v}_c = \left(1 - \frac{m_a}{m_a + m_A}\right)\vec{v}_a = \frac{m_A}{m_a + m_A}\vec{v}_a \tag{C.19}$$

$$\vec{v}'_A = \vec{v}_A - \vec{v}_c = -\vec{v}_c = -\frac{m_a}{m_a + m_A}\vec{v}_a \tag{C.20}$$

Since the total linear momentum of $a + A$ is zero in the center-of-mass frame, we find for the ratio of speeds

$$m_a\vec{v}'_a = m_A\vec{v}'_A \qquad \text{or} \qquad \frac{v'_a}{v'_A} = \frac{m_A}{m_a} \tag{C.21}$$

The kinetic energies of the two particles in the center-of-mass system are given by (see Eqs. (C.19) and (C.20))

CM energy before reaction

$$E'_a = \frac{1}{2}m_a(v'_a)^2 = \frac{1}{2}m_a v_a^2\left(\frac{m_A}{m_a + m_A}\right)^2 = E_a\frac{m_A^2}{(m_a + m_A)^2} \tag{C.22}$$

$$E'_A = \frac{1}{2}m_A(v'_A)^2 = \frac{1}{2}m_A v_a^2\left(\frac{m_a}{m_a + m_A}\right)^2 = E_a\frac{m_A m_a}{(m_a + m_A)^2} \tag{C.23}$$

and the total kinetic energy in the center-of-mass system before the collision is related to the laboratory bombarding energy by

$$E'_i = E'_a + E'_A = E_a\frac{m_A^2 + m_A m_a}{(m_a + m_A)^2} = E_a\frac{m_A}{m_a + m_A} \tag{C.24}$$

The laboratory bombarding energy, E_a, can be expressed as the sum of total kinetic energy in the center-of-mass system before the collision, E'_i, and the kinetic energy of the center-of-mass motion, E_c, as can be seen from (see Eqs. (C.18) and (C.24))

$$\begin{aligned} E_a &= \frac{1}{2}m_a v_a^2 = \frac{1}{2}\frac{m_A m_a}{m_a + m_A}v_a^2 + \frac{1}{2}\frac{m_a^2}{m_a + m_A}\frac{m_a + m_A}{m_a + m_A}v_a^2 \\ &= E_a\frac{m_A}{m_a + m_A} + \frac{1}{2}(m_a + m_A)v_c^2 = E'_i + E_c \end{aligned} \tag{C.25}$$

Furthermore, we find from Eq. (C.24)

$$E'_i = \frac{1}{2}\frac{m_a m_A}{m_a + m_A} v_a^2 = \frac{1}{2} m_{aA} v_a^2 \tag{C.26}$$

and thus the total center-of-mass kinetic energy can be expressed in terms of the laboratory bombarding velocity, v_a, and the *reduced mass* of particles a and A, defined as $m_{aA} \equiv m_a m_A/(m_a + m_A)$. Obviously, the above expressions apply equally to a radiative capture reaction, $A(a,\gamma)B$, or to elastic scattering, $A(a,a)A$.

We will now consider the situation after the collision. The total linear momentum in the center-of-mass system remains zero. For a reaction $A(a,b)B$, the two residual particles b and B separate in opposite directions with equal but opposite linear momenta,

$$m_b v'_b = m_B v'_B \tag{C.27}$$

The kinetic energies in the center-of-mass system are given by

$$E'_b = \frac{1}{2} m_b (v'_b)^2 \tag{C.28}$$

$$E'_B = \frac{1}{2} m_B (v'_B)^2 = \frac{1}{2} m_b (v'_b)^2 m_B \frac{m_b}{m_B^2} = \frac{m_b}{m_B} E'_b \tag{C.29}$$

The total kinetic energy in the center-of-mass system after the collision is then

$$E'_f = E'_b + E'_B = E'_b + \frac{m_b}{m_B} E'_b = E'_b \left(1 + \frac{m_b}{m_B}\right) \tag{C.30}$$

The kinetic energies in the center-of-mass system after the collision can be expressed in terms of the laboratory bombarding energy by using $E'_i + Q = E'_f$ (see Eq. (1.5)). The total kinetic energy is given by (see Eq. (C.24))

$$E'_f = E'_i + Q = E_a \frac{m_A}{m_a + m_A} + Q = Q + E_a \left(1 - \frac{m_a}{m_a + m_A}\right) \tag{C.31}$$

After some algebra one obtains for the kinetic energies of the particles

CM energy
after reaction

$$E'_b = \frac{m_B}{m_b + m_B}\left[Q + E_a\left(1 - \frac{m_a}{m_b + m_B}\right)\right] \tag{C.32}$$

$$E'_B = \frac{m_b}{m_b + m_B}\left[Q + E_a\left(1 - \frac{m_a}{m_b + m_B}\right)\right] \tag{C.33}$$

Finally, we will present the transformation equations for the angles and solid angles in the laboratory and center-of-mass systems. After the collision, we have for a reaction $A(a,b)B$ (see Eq. (C.19))

$$\vec{v}'_b = \vec{v}_b - \vec{v}_c \tag{C.34}$$

or, in terms of the components parallel with and perpendicular to the beam direction

$$v_b' \cos\theta' = v_b \cos\theta - v_c \tag{C.35}$$

$$v_b' \sin\theta' = v_b \sin\theta - 0 \tag{C.36}$$

From these expressions one can derive either of the following two relationships:

$$\boxed{\tan\theta = \frac{v_b' \sin\theta'}{v_b' \cos\theta' + v_c} = \frac{\sin\theta'}{\cos\theta' + v_c/v_b'} = \frac{\sin\theta'}{\cos\theta' + \gamma}} \tag{C.37}$$

$$\boxed{\cos\theta = \frac{\gamma + \cos\theta'}{\sqrt{1 + \gamma^2 + 2\gamma\cos\theta'}}} \tag{C.38}$$

The parameter γ is defined by the ratio of velocities of the center of mass and of particle b in the center-of-mass system,

$$\boxed{\begin{aligned}\gamma \equiv \frac{v_c}{v_b'} &= \sqrt{\frac{m_a m_b E_a}{m_B(m_b + m_B)Q + m_B(m_B + m_b - m_a)E_a}} \\ &\approx \sqrt{\frac{m_a m_b}{m_A m_B}\frac{E_a}{(1 + m_a/m_A)Q + E_a}}\end{aligned}} \tag{C.39}$$

where the approximation is obtained by setting $m_a + m_A \approx m_b + m_B$. For a very heavy target nucleus, one finds $\gamma \approx 0$ and hence the angle of the emitted particle b has about the same value in the laboratory and center-of-mass systems ($\theta \approx \theta'$). For elastic scattering, $m_a = m_b$, $m_A = m_B$, $Q = 0$, and thus one finds $\gamma = m_a/m_A$. For a radiative capture reaction, $A(a,\gamma)B$, the laboratory and center-of-mass angle of the emitted photon are related by (given here without proof)

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \tag{C.40}$$

where the relativistic parameter β is defined as

$$\beta \equiv \frac{\sqrt{E_a(E_a + 2m_a c^2)}}{m_A c^2 + m_a c^2 + E_a} \tag{C.41}$$

The definition of the differential cross section implies that the same number of reaction products are emitted into the solid angle element $d\Omega$ in the direction θ (laboratory system) as are emitted into $d\Omega'$ in the corresponding direction θ' (center-of-mass system). Thus

$$\left(\frac{d\sigma}{d\Omega}\right)_\theta d\Omega = \left(\frac{d\sigma}{d\Omega}\right)'_{\theta'} d\Omega' \tag{C.42}$$

We assume that the cross section depends on θ or θ', but not on the azimuthal angle. Hence

$$\frac{(d\sigma/d\Omega)'_{\theta'}}{(d\sigma/d\Omega)_{\theta}} = \frac{d\Omega}{d\Omega'} = \frac{d(\cos\theta)}{d(\cos\theta')} \tag{C.43}$$

From Eq. (C.38) we find for a reaction $A(a,b)B$

$$\frac{d(\cos\theta)}{d(\cos\theta')} = \frac{1+\gamma\cos\theta'}{(1+\gamma^2+2\gamma\cos\theta')^{3/2}} = \frac{\sqrt{1-\gamma^2\sin^2\theta}}{\left(\gamma\cos\theta+\sqrt{1-\gamma^2\sin^2\theta}\right)^2} \tag{C.44}$$

For a radiative capture reaction, $A(a,\gamma)B$, one obtains from Eq. (C.40)

$$\frac{d(\cos\theta)}{d(\cos\theta')} = \frac{1-\beta^2}{(1+\beta\cos\theta')^2} \tag{C.45}$$

for the relationship of the solid angles of the emitted photon.