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MULTIPOLE EXPANSION OF A TWO-BODY INTERACTION IN HELICITY FORMALISM AND ITS APPLICATIONS TO NUCLEAR STRUCTURE AND NUCLEAR REACTION CALCULATIONS

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Abstract: For a multipole expansion in a helicity formalism, the calculation of matrix elements of a two-body interaction involves a geometry quite independent of the spins. The total angular momentum quantum numbers j are used instead of the orbital ones; the number of different form factors increases with the value of the spin but a useful factorization can be obtained in the calculations.

1. Introduction

The helicity formalism [1,2]) introduced in the theory of collisions between particles with spin uses eigenvalues of the longitudinal spin components to label the S-matrix. The relations between matrix elements are simplified and, consequently the independent amplitudes are obtained more easily. Furthermore, these amplitudes are given by simpler expressions because all the geometrical coefficients can be written explicitly, as long as the spins involved are relatively small. The main advantage of this formalism is its ability to provide a clearly relativistic description of the spin, but it also leads to some important simplifications when used in the calculation of two-body reaction cross sections at non-relativistic energies [3]).

The helicity formalism can be generalized to simplify some calculations in nuclear structure and nuclear reaction theory which require the evaluation of a matrix element of a two-body interaction by means of a multipole expansion. Such an expansion can be avoided in nuclear structure theory when harmonic oscillator wave functions are employed and relative coordinates are separated out, but this procedure involves an increasing complication of Moshinsky coefficients when the angular momenta are large and cannot be extended easily to reaction calculations.

If no spins are involved, the interaction may be expanded around the origin as follows:

$$V(|\boldsymbol{r}_1-\boldsymbol{r}_2|) = \sum_L (2L+1)V_L(r_1, r_2)P_L(\cos\theta). \tag{1}$$

Since the total interaction $V(|\boldsymbol{r}_1-\boldsymbol{r}_2|)$ does not depend upon the choice of the origin, there are relations between the different multipoles $V_L(r_1, r_2)$ which can be obtained as follows: If $W(q)$ is the Fourier transform of V,

$$W(q) = \int \exp(i\boldsymbol{q}\cdot\boldsymbol{r})V(r)\mathrm{d}^3r, \tag{2}$$

all the multipoles are obtained from this unique function $W(q)$ by

$$V_L(r_1, r_2) = \frac{2}{\pi}\int_0^\infty qW(q)j_L(qr_1)j_L(qr_2)\mathrm{d}q, \tag{3}$$

and the recurrence relations between the Bessel functions can be used to obtain direct relations between multipoles.

An interaction between particles with spins is usually an operator in the space of the spins with a form factor as described by formula (1). The present generalization of the helicity formalism projects the spin of particle 1 on the direction $\boldsymbol{r}_1$, the spin of particle 2 on the direction $\boldsymbol{r}_2$ and replaces the operators in spin space by their matrix elements between states of a given helicity. The multipole expansions of the interaction between different helicity states have to be defined separately. The relationship between multipoles are more numerous and complicated because the definition of the helicity states also depends upon the choice of coordinates. The helicity multipoles of commonly used interactions are given in terms of the multipoles of the form factor.

The helicity formalism uses the elements of the rotation matrices instead of Legendre polynomials and of their derivatives, but this does not lead to practical difficulties because the magnetic quantum numbers involved are small and the rotation matrix elements can be easily computed from the Legendre polynomials [1,2]). Messiah's [4]) notation for the rotations and for all the geometrical coefficients will be used here. Other authors [1,2,5]) use the inverse rotation; the difference between both definitions is a complex conjugation and a permutation of the two magnetic quantum numbers of the matrix elements.

2. Description of a spin $\frac{1}{2}$ particle bound state

Let us consider in detail the bound state of a spin $\frac{1}{2}$ particle in the helicity formalism in order to show what general procedure will be followed. The usual decomposition of a wave function $|jm\rangle$ with an orbital angular momentum l, a total angular momentum j and its projection m on the axis of quantization is

$$|jm\rangle = f_{lj}(r)\sum_{\mu,\sigma}\langle l\tfrac{1}{2}\mu\sigma|jm\rangle Y_l^\mu(\theta,\varphi)|\sigma\rangle, \tag{4}$$

where $|\sigma\rangle$ is the spin eigenfunction with a projection σ along the axis of quantization, and $f_{lj}(r)$ is a radial function normalized according to

$$\int_0^\infty r^2 f_{lj}^2(r)\mathrm{d}r = 1. \tag{5}$$

The angles θ and φ describe the direction of $\boldsymbol{r}$ in the reference frame. The spherical harmonic can be replaced by a matrix element of the rotation operator which takes the axis of quantization into the direction of $\boldsymbol{r}$, i.e.

$$Y_l^\mu(\theta, \varphi) = \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} R_{\mu,0}^{(l)*}(\varphi, \theta, \psi). \tag{6}$$

The angle ψ is related to some origin; when rotated around $\boldsymbol{r}$, the spherical harmonic does not depend upon it, but this angle can be introduced if the wave function is divided by $(2\pi)^{\frac{1}{2}}$ to maintain its normalization. Meanwhile, $|\sigma\rangle$ can be expressed in terms of the two helicity functions $|\lambda\rangle$ which have a projection $\lambda = \pm\frac{1}{2}$ along $\boldsymbol{r}$

$$|\sigma\rangle = \sum_\lambda R_{\sigma,\lambda}^{(\frac{1}{2})*}(\varphi, \theta, \psi)|\lambda\rangle. \tag{7}$$

If the expressions (6) and (7) are substituted into (4), and the Clebsch-Gordan series for the product of the two rotation matrix elements is used, together with the orthogonality relation between Clebsch-Gordan coefficients, the state $|jm\rangle$ becomes

$$|jm\rangle = \left(\frac{2l+1}{8\pi^2}\right)^{\frac{1}{2}} f_{lj}(r) \sum_\lambda \langle l\tfrac{1}{2}0\lambda|j\lambda\rangle R_{m,\lambda}^{(j)*}(\varphi, \theta, \psi)|\lambda\rangle. \tag{8}$$

The Clebsch-Gordan coefficient which remains must be replaced by its explicit value

$$\begin{aligned} \langle l\tfrac{1}{2}0\lambda|j\lambda\rangle &= (-)^{l+j-\frac{1}{2}}\left\{\frac{2j+1}{2(2l+1)}\right\}^{\frac{1}{2}} \qquad \text{if} \quad \lambda = \tfrac{1}{2} \\ &= \left\{\frac{2j+1}{2(2l+1)}\right\}^{\frac{1}{2}} \qquad \text{if} \quad \lambda = -\tfrac{1}{2}. \end{aligned} \tag{9}$$

The helicity description of the state $|jm\rangle$ is then

$$|jm\rangle = \frac{1}{4\pi}(2j+1)^{\frac{1}{2}} \sum_\lambda \varphi_\lambda^j(r) R_{m,\lambda}^{(j)*}(\varphi, \theta, \psi)|\lambda\rangle, \tag{10}$$

with

$$\varphi_{\frac{1}{2}}^j(r) = (-)^{l+j-\frac{1}{2}} f_{lj}(r), \qquad \varphi_{-\frac{1}{2}}^j(r) = f_{lj}(r). \tag{11}$$

All references to the orbital angular momentum have disappeared from the description of the bound state but there are now two radial functions $\varphi_{\frac{1}{2}}^j(r)$ and $\varphi_{-\frac{1}{2}}^j(r)$ which are equal to within a sign for a state of well-defined parity. The value of j and the difference in sign give the parity of the state; there is only one possible value of l if j and the parity are specified. Such a description has been used by Bohr and Mottelson [6]) for the computation of the matrix elements of a zero-range interaction.

It should be noted that a helicity state as defined here has no direct physical significance. In the usual helicity formalism, the spins are projected along the momentum of the particle whose direction coincides with that of $\boldsymbol{r}$ at infinity and a pure helicity state can be defined. Here, no superposition of an even- and an odd-parity state can be described by a pure helicity state. This is so because their orbital angular momenta are different and thus the two radial functions have different behaviour around the origin. Accordingly, some form factor of the intrinsic $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction, which are given in the appendix, cannot be rigorously defined in the framework used here because they have a finite contribution even though they should be multiplied by a vanishing Clebsch-Gordan coefficient in the general formalism.

3. Description of a two-body interaction

Any two-body interaction which is invariant under rotations can be written

$$V(1,2) = \sum_{J,\lambda_1,\lambda_2,\lambda'_1,\lambda'_2} |\lambda'_1\rangle|\lambda'_2\rangle V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2)\langle\lambda_1|\langle\lambda_2| \times \sum_\mu (-)^\mu R^{(J)}_{\mu,\lambda_1-\lambda'_1}(\varphi_1,\theta_1,\psi_1)R^{(J)}_{-\mu,\lambda_2-\lambda'_2}(\varphi_2,\theta_2,\psi_2). \quad (12)$$

In the reference frame, φ_1, θ_1, ψ_1 describes a coordinate system with its z-axis along $\boldsymbol{r}_1$ and φ_2, θ_2, ψ_2 another one with its z-axis along $\boldsymbol{r}_2$. The scalar product form of the leading subscripts of the rotation matrix elements comes from the independence of the total expression with respect to the frame of reference. Under a rotation $\psi \to \psi+\Delta\psi$ the helicity state becomes

$$|\lambda\rangle_\psi = \exp(i\lambda\Delta\psi)|\lambda\rangle_{\psi+\Delta\psi}, \quad (13)$$

and the total expression (12) is independent of ψ_1 or ψ_2. If the axis of quantization is taken along $\boldsymbol{r}_1$ or $\boldsymbol{r}_2$ the angular part reduces to only one rotation matrix element.

Some symmetry properties are also required, in general, of the two-body force. These are parity conservation, time-reversal invariance and, if the interaction occurs between two identical particles, invariance under their permutation. The $V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2)$ are functions only of r_1 and r_2 for a local interaction (scalar, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ or tensor). They can also include derivative terms or expressions involving the total angular momenta of the states considered for a non-local interaction, such as the relative $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction. Such velocity dependent operators are parity invariants but may change sign under the time-reversal operation.

In order to study the consequences of the different invariances it is simplest to choose the axis of quantization along $\boldsymbol{r}_1$ together with a frame of reference for particle 2 given by the Euler angles $(0, \theta, 0)$; in this way, the angular dependence can be expressed with the reduced rotation matrix elements $r^{(J)}_{m,m'}(\theta)$ which depend only upon the invariant angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The properties

$$V_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = \sum_J V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2)(-)^{\lambda'_1-\lambda_1}r^{(J)}_{\lambda'_1-\lambda_1,\lambda_2-\lambda'_2}(\theta), \quad (14)$$

can be deduced from those of the helicity states, and then, the properties of each $V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2)$ can be obtained.

The action of the parity operator P on a helicity state introduced here is the same as for standard helicity [1]) because the vector $\boldsymbol{r}$ and the impulsion behave similarly.

$$P|\lambda\rangle = \eta(-)^{s-\lambda}\exp(i\pi J_y)|-\lambda\rangle, \tag{15}$$

where η is the intrinsic parity of the particle and s its spin. The rotation $\exp(i\pi J_y)$ has no effect as the interaction is invariant under this operation. For two nucleons ($\eta = 1; s = \frac{1}{2}$) and provided that $V(1, 2)$ is expressed with operators which are parity invariant

$$V_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = (-)^{\lambda_1+\lambda_2+\lambda'_1+\lambda'_2}V_{-\lambda'_1-\lambda'_2,-\lambda_1-\lambda_2}(1,2). \tag{16}$$

For each multipole the condition is:

$$V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = V^J_{-\lambda'_1-\lambda'_2,-\lambda_1-\lambda_2}(1,2). \tag{17}$$

Time-reversal invariance implies that the interaction commutes with an antiunitary operator T which gives for a helicity state

$$T|\lambda\rangle = (-)^{s-\lambda}|-\lambda\rangle, \tag{18}$$

because it does not change the vector $\boldsymbol{r}$.

In the case where $V(1, 2)$ is expressed with time-reversal invariant operators

$$V_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = (-)^{\lambda_1+\lambda_2+\lambda'_1+\lambda'_2}V_{-\lambda_1-\lambda_2,-\lambda'_1-\lambda'_2}(1,2), \tag{19}$$

$$V^J_{\lambda'_2\lambda'_2,\lambda_1\lambda_2}(1,2) = V^J_{-\lambda_1-\lambda_2,-\lambda'_1-\lambda'_2}(1,2). \tag{20}$$

However, this is not always the case for a non-local interaction, such as an intrinsic $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction, a part of which can be odd under time reversal. For this part the relations (19) and (20) have to be replaced by

$$V_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = -(-)^{\lambda_1+\lambda_2+\lambda'_1+\lambda'_2}V_{-\lambda_1-\lambda_2,-\lambda'_1-\lambda'_2}(1,2), \tag{19'}$$

$$V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = -V^J_{-\lambda_1-\lambda_2,-\lambda'_1-\lambda'_2}(1,2). \tag{20'}$$

When the two nucleons are identical

$$V_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = V_{\lambda'_2\lambda'_1,\lambda_2\lambda_1}(2,1), \tag{21}$$

and the corresponding relation between multipoles is

$$V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2) = V^J_{\lambda'_2\lambda'_1,\lambda_2\lambda_1}(2,1). \tag{22}$$

For a nucleon-nucleon interaction there are, *a priori*, 16 helicity multipoles for each value of J. Each of them will yield a two-body form factor. If one requires only the conservation of parity, eight of them are independent. Under time-reversal invariance, ten of them are even and six odd. For identical particles, four of them are symmetric under the permutation of particles 1 and 2, and there are six pairs of multipoles which transform into one another. Six of the eight parity-invariant multipoles are even and two odd under time reversal. Four parity-invariant multipoles are symmetric under particle interchange and two pairs transform into one another under the permutation of the two nucleons. Under both time reversal and particle interchange, there are four even symmetric multipoles, three even pairs and three odd ones. If all three invariances are imposed, there are four multipoles symmetric even for time reversal under the permutation of the two nucleons and two pairs which transform into one another, one of which is even for time reversal and the other odd.

To summarize these results, and for applications, it is convenient to consider all the multipoles $V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2)$ for a given J in a matrix form as a sum of two-body form factors multiplying elementary matrices $M_1 \otimes M_2$, where M_1 and M_2 can be one of the following:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, \tag{23}$$

spanning the spin space of particle 1 or 2. Among the four matrices (23), the first two satisfy the parity condition (17) and $M_1 \otimes M_2$ satisfies the condition (17) only when M_1 and M_2 both satisfy it or both fail to satisfy it. If parity conservation applies the two-body form factors can be separated into two groups, one an even part

$$a^J(1,2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + b^J(1,2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$
$$+ b'^J(2,1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + c^J(1,2)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \tag{24}$$

and the other an odd part

$$d^J(1,2)\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + e^J(1,2)\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$
$$+ e'^J(2,1)\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + f^J(1,2)\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}. \tag{25}$$

If the two particles are identical,

$$\begin{aligned} &a^J(1,2) = a^J(2,1), \qquad b^J(1,2) = b'^J(1,2), \qquad c^J(1,2) = c^J(2,1), \\ &d^J(1,2) = d^J(2,1), \qquad e^J(1,2) = e'^J(1,2), \qquad f^J(1,2) = f^J(2,1). \end{aligned} \tag{26}$$

If time-reversal invariance applies, a, c, d, e and f must be expressed in terms of time-reversal invariant operators, and b must change sign; consequently, b vanishes for a local interaction.

In the expansion (14) the magnetic quantum numbers of the reduced rotation matrix element $r^{(J)}_{\lambda'_1-\lambda_1,\lambda'_2-\lambda_2}(\theta)$ must be less than or equal to J. Thus, for $J=0$, some two-body form factors vanish:

$$\begin{aligned} &a^0(1,2)\neq 0, \qquad b^0(1,2)=b'^0(1,2)=c^0(1,2)=0,\\ &d^0(1,2)\neq 0, \qquad e^0(1,2)=e'^0(1,2)=f^0(1,2)=0. \end{aligned} \tag{27}$$

For the scalar interaction (1), the helicity representation involves only the form factor

$$a^J(1,2)=(2J+1)V_J(r_1,r_2), \tag{28}$$

and all the other two-body form factors vanish.

For an interaction containing $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, the following expressions are obtained:

$$\begin{aligned} c^J(1,2) &= -(2J+1)V_J(r_1,r_2),\\ d^J(1,2) &= JV_{J-1}(r_1,r_2)+(J+1)V_{J+1}(r_1,r_2),\\ e^J(1,2) &= \{J(J+1)\}^{\frac{1}{2}}\{V_{J-1}(r_1,r_2)-V_{J+1}(r_1,r_2)\},\\ f^J(1,2) &= (J+1)V_{J-1}(r_1,r_2)+JV_{J+1}(r_1,r_2),\\ a^J(1,2) &= b^J(1,2)=0. \end{aligned} \tag{29}$$

The expressions for the tensor interaction and the relative $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction are given in the appendix. The expressions are more complicated, especially for the $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction which includes a term $b(1,2)$.

For local potentials only, the helicity formalism requires five form factors, but the usual works define only three of them: however, there are two form factors for the even part of the interaction and three in the odd part. It will be shown that these two parts come in separately.

4. Matrix element between bound states

After integration over angles, the matrix element of the interaction (12) between four bound states described by their helicity functions (10), uncoupled and unsymmetrized, is given by

$$\begin{aligned} &\langle j'_1 m'_1|\langle j'_2 m'_2|V(1,2)|j_1 m_1\rangle|j_2 m_2\rangle\\ &=\sum_{J,\mu}(-)^{j_1-m_1+j'_2-m'_2}(2J+1)\begin{pmatrix} j'_1 & J & j_1\\ m'_1 & \mu & -m_1\end{pmatrix}\begin{pmatrix} j'_2 & J & j_2\\ m'_2 & -\mu & -m_2\end{pmatrix} f^J_{j'_1j'_2,j_1j_2}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} f^J_{j'_1j'_2,j_1j_2} &= \sum_{\lambda'_1,\lambda'_2,\lambda_1,\lambda_2}(-)^{j_1-\lambda_1+j'_2-\lambda_2}\frac{1}{4(2J+1)}\\ &\times\{(2j'_1+1)(2j'_2+1)(2j_1+1)(2j_2+1)\}^{\frac{1}{2}}\begin{pmatrix} j'_1 & J & j_1\\ \lambda'_1 & \lambda_1-\lambda'_1 & -\lambda_1\end{pmatrix}\begin{pmatrix} j'_2 & J & j_2\\ \lambda'_2 & \lambda_2-\lambda'_2 & -\lambda_2\end{pmatrix}\\ &\times\iint V^J_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(1,2)\varphi^{j'_1*}_{\lambda'_1}(r_1)\varphi^{j'_2*}_{\lambda'_2}(r_2)\varphi^{j_1}_{\lambda_1}(r_1)\varphi^{j_2}_{\lambda_2}(r_2)r_1^2r_2^2\,\mathrm{d}r_1\,\mathrm{d}r_2. \end{aligned} \tag{31}$$

With the particle-particle coupling the antisymmetrized matrix element is

$$\langle j_1' j_2'; JM|V(1,2)|j_1 j_2; JM\rangle = \sum_{m'_1, m'_2, m_1, m_2} \langle j_1 j_2 m_1 m_2|JM\rangle\langle j_1' j_2' m_1' m_2'|JM\rangle$$

$$\times\{\langle j_1' m_1'|\langle j_2' m_2'|V(1,2)(|j_1 m_1\rangle|j_2 m_2\rangle - |j_2 m_2\rangle|j_1 m_1\rangle)\}$$

$$= \sum_{J'} (-)^{J+j_1+j_2}(2J'+1)\begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & J' \end{Bmatrix} f^{J'}_{j'_1 j'_2,\, j_1 j_2}$$

$$- \sum_{J'} (2J'+1)\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \end{Bmatrix} f^{J'}_{j'_1 j'_2,\, j_2 j_1}. \tag{32}$$

The particle-hole antisymmetrized matrix element is

$$\langle j_1' j_1^{-1}; JM|V(1,2)|j_2 j_2'^{-1}; JM\rangle = \sum_{m'_1, m'_2, m_1, m_2} (-)^{j_1-m_1+j'_2-m'_2}$$

$$\times\langle j_1' j_1 m_1' - m_1|JM\rangle\langle j_2 j_2' m_2 - m_2'|JM\rangle$$

$$\times\{\langle j_1' m_1'|\langle j_2' m_2'|V(1,2)(|j_1 m_1\rangle|j_2 m_2\rangle - |j_2 m_2\rangle|j_1 m_1\rangle)\}$$

$$= f^{J}_{j'_1 j'_2,\, j_1 j_2} - \sum_{J'} (-)^{j_1+j_2+J+J'}\begin{Bmatrix} j_1 & j_1' & J \\ j_2 & j_2' & J' \end{Bmatrix} f^{J'}_{j'_1 j'_2,\, j_2 j_1}. \tag{33}$$

The matrix element of eq. (31) is the direct particle-hole term and it involves only multipoles with the J-value to which the particle and the hole are coupled. The exchange term of the particle-hole matrix element and both the direct and the exchange term in the particle-particle coupling are expressed as a sum of f^J multiplied by a 6-j recoupling coefficient.

The usual interactions have an exchange part and can be written

$$V = V_0 + \tfrac{1}{2}(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)V_1, \tag{34}$$

where $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are the isospins. The isospin formalism introduces in the formulae a geometry which turns out to be very simple. For a particle-particle matrix element, the direct term has to be calculated with $V_0+(-)^{1+T}V_1$ and the exchange with $V_1+(-)^{1+T}V_0$; for a particle-hole matrix element the direct term has to be calculated with $V_0+\{1+(-)^T\}V_1$ and the exchange with $V_1+\{1+(-)^T\}V_0$. All these considerations on the matrix elements depend on the phase convention. The one used here is defined by the Clebsch-Gordan coefficients explicitly written in formulae (32) and (33).

In order to evaluate the matrix element (31) let us perform successively the integration and the summation on the helicities of particle 1 to obtain a one-body form factor

$$F^{Jj_1j'_1}_{\lambda'_2\lambda_2}(2) = \sum_{\lambda'_1, \lambda_1} (-)^{j_1-\lambda_1}\tfrac{1}{2}\left\{\frac{(2j_1+1)(2j_1'+1)}{2J+1}\right\}^{\frac{1}{2}}\begin{pmatrix} j_1' & J & j_1 \\ \lambda_1' & \lambda_1-\lambda_1' & -\lambda_1 \end{pmatrix}$$

$$\times\int V^{J}_{\lambda'_1\lambda'_2,\, \lambda_1\lambda_2}(1,2)\varphi^{j'_1}_{\lambda'_1}(r_1)^*\varphi^{j_1}_{\lambda_1}(r_1)r_1^2\,\mathrm{d}r_1, \tag{35}$$

and then the same operation for particle 2

$$f^J_{j'_1j'_2,\,j_1j_2} = \sum_{\lambda'_2,\,\lambda_2} (-)^{j'_2-\lambda_2\frac{1}{2}} \left\{\frac{(2j'_2+1)(2j_2+1)}{2J+1}\right\}^{\frac{1}{2}} \begin{pmatrix} j'_2 & J & j_2 \\ \lambda'_2 & \lambda_2-\lambda'_2 & -\lambda_2 \end{pmatrix} \times \int F^{Jj'_1j_1}_{\lambda'_2\lambda_2}(2)\varphi^{j'_2}_{\lambda'_2}(r_2)\varphi^{j_2}_{\lambda_2}(r_2)r_2^2\mathrm{d}r_2\,. \quad (36)$$

The one-body form factors (35) can be written in a matrix form using the elementary matrices (23)

$$F^{Jj_1j'_1}(2) = A^{Jj_1j'_1}(2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + B^{Jj_1j'_1}(2)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + C^{Jj_1j'_1}(2)\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + D^{Jj_1j'_1}(2)\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}. \quad (37)$$

The relation (11) between the two components $\varphi^j_{\pm\frac{1}{2}}$ must be taken into account to obtain A, B, C and D for an interaction which conserves parity. In the summation over the helicities, the two terms of each elementary matrix (23) are equal or opposite. Consequently, there are two kinds of matrix elements.

(i) The "natural parity" matrix elements, when the product of the parities of the states j_1 and j'_1 is equal to $(-)^J$. The even part of the interaction only contributes and the one-body form factors C and D vanish. The "natural parity" or "even" one-body form factors are

$$A^{Jj_1j'_1}(2) = G^J_{j_1j'_1}\int [a^J(1,2)+\varepsilon_{j_1}\alpha^J_{j_1j'_1}b'^J(2,1)]\varphi^*_{l'_1j'_1}(r_1)\varphi_{l_1j_1}(r_1)r_1^2\mathrm{d}r_1\,,$$

$$B^{Jj_1j'_1}(2) = G^J_{j_1j'_1}\int [b^J(1,2)+\varepsilon_{j_1}\alpha^J_{j_1j'_1}c^J(1,2)]\varphi^*_{l'_1j'_1}(r_1)\varphi_{l_1j_1}(r_1)r_1^2\mathrm{d}r_1\,. \quad (38)$$

(ii) The "unnatural parity" matrix elements, corresponding to the case where the product of the parities of the states j_1 and j'_1 is different from $(-)^J$. The odd part of the interaction only contributes, yielding the "unnatural parity" of "odd" one-body form factors

$$C^{Jj_1j'_1}(2) = G^J_{j_1j'_1}\int [d^J(1,2)+\varepsilon_{j_1}\alpha^J_{j_1j'_1}e'^J(2,1)]\varphi_{l'_1j'_1}(r_1)\varphi_{l_1j_1}(r_1)r_1^2\mathrm{d}r_1\,,$$

$$D^{Jj_1j'_1}(2) = G^J_{j_1j'_1}\int [e^J(1,2)+\varepsilon_{j_1}\alpha^J_{j_1j'_1}f^J(1,2)]\varphi_{l'_1j'_1}(r_1)\varphi_{l_1j_1}(r_1)r_1^2\mathrm{d}r_1\,. \quad (39)$$

The evaluation of the matrix element (36) is straightforward. If parity is conserved, it vanishes unless the product of the parities of the states j_2 and j'_2 is equal to the product of those of j_1 and j'_1. For a natural parity matrix element:

$$f^J_{j'_1j'_2,\,j_1j_2} = (-)^{j'_2-j_2}G^J_{j_2j'_2}\int [A^{Jj_1j'_1}(2)+\varepsilon_{j_2}\alpha^J_{j_2j'_2}B^{Jj_1j'_1}(2)] \times \varphi_{l'_2j'_2}(r_2)\varphi_{l_2j_2}(r_2)r_2^2\mathrm{d}r_2. \quad (40)$$

For an unnatural parity matrix element, A and B have to be replaced by C and D, respectively.

In formulae (38)–(40), the usual wave functions $\varphi_{lj}(r)$ have been used to avoid confusion. Special notations have been introduced for the geometrical coefficients to emphasize their simplicity; they are

$$\varepsilon_j = (-)^{l-j+\frac{1}{2}} \quad (41)$$

(the notation used here is incomplete – instead of $|jm\rangle$, we should have used $|\varepsilon jm\rangle$ where ε indicates the parity of the state);

$$\alpha^J_{j,\,j'} = \frac{(j+\frac{1}{2})+(-)^{j+j'+J}(j'+\frac{1}{2})}{\{J(J+1)\}^{\frac{1}{2}}}, \quad (42)$$

which comes from the following relation among 3-j coefficients for j and j' half-integers

$$\begin{pmatrix} j' & J & j \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} = -\alpha^J_{jj'}\begin{pmatrix} j' & J & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}; \quad (43)$$

$$G^J_{jj'} = (-)^{j+\frac{1}{2}}\left\{\frac{(2j+1)(2j'+1)}{(2J+1)}\right\}^{\frac{1}{2}}\begin{pmatrix} j' & J & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad (44)$$

which is a special 3-j coefficient for which Racah's formula is not needed. Its presence was to be expected as a generalization of the well-known relation [7])

$$\begin{pmatrix} j' & J & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = -\{(2l+1)(2l'+1)\}^{\frac{1}{2}}\begin{pmatrix} l & l' & J \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l & l' & J \\ j' & j & \frac{1}{2} \end{Bmatrix}, \quad (45)$$

which can be used to simplify the matrix elements for an interaction independent of the spins. The coefficients $G^J_{jj'}$ can easily be obtained with the following expression:

$$G^J_{jj'} = (-)^{\mathrm{In}\{(J+j'-j+2)/2\}}(2J+1)^{-\frac{1}{2}}\frac{g(j+j'+J+1)}{g(j+j'-J)g(j+J-j')g(j'+J-j)}, \quad (46)$$

where $\mathrm{In}\{p\}$ is the integer part of p and

$$g(n) = \frac{(n!)^{\frac{1}{2}}}{n!!} = \left\{\frac{2\times 4\times \dots \times(n-1)}{3\times 5\times \dots \times n}\right\}^{\frac{1}{2}} \quad \text{for } n \text{ odd}$$

$$= \left\{\frac{2\times 4\times \dots \times n}{3\times 5\times \dots \times(n-1)}\right\}^{\frac{1}{2}} \quad \text{for } n \text{ even.} \quad (47)$$

The notation in formula (12) has been chosen to give an expression symmetric in the two particles. Formulae (30) and (31) are written in a manner which uses as fundamental matrix element the direct term of a particle-hole matrix element. The notation for the elementary matrices (23), was designated to give similar formulae for (38) and (39). The sign $(-)^{j_2-j'_2}$ in formula (40) is not present if the phase of the particle-hole state is $(-)^{\frac{1}{2}-m_n}$ instead of $(-)^{j_n-m_n}$.

In the usual coupling with the quantum numbers (L, S, J), A is the one-body form factor for $S = 0$, B for $S = 1$ and $L = J$; C and D are a mixture of $S = 1$ and $L = J \pm 1$.

5. DWBA and coupled-channel calculations

Distorted Wave Born approximation calculations [8]) involve a variety of problems and it would be difficult to go into all of them here. The simplest DWBA calculation which uses the nucleon-nucleon interaction considered here involves the transfer of a definite value of angular momentum to the nucleus and neglects the antisymmetrization of the incident or outgoing particle with the nucleons of the nucleus. We shall first consider the simplest case, i.e. the inelastic scattering of a nucleon on a spin-zero target with a residual nucleus which can be described by a single particle-hole state; we shall indicate some generalizations later.

The expansion of the distorted wave is usually written as

$$X_\sigma^{(+)}(\boldsymbol{k}, \boldsymbol{r}) = \frac{4\pi}{kr} \sum_{j, l, m, \mu, \mu', \sigma'} i^l X_{lj}(kr) \langle l\tfrac{1}{2}\mu\sigma | jm\rangle \langle l\tfrac{1}{2}\mu'\sigma' | jm\rangle \times Y_l^{\mu *}(\theta_k, \varphi_k) Y_l^{\mu'}(\theta_r, \varphi_r) |\sigma'\rangle, \tag{48}$$

where σ is the spin projection of the ingoing plane wave on an arbitrary axis and σ' its projection at the point $\boldsymbol{r}$ on the same axis. If we choose this arbitrary axis along $\boldsymbol{k}$, we introduce the usual helicity λ instead of σ; then the transformation (7) used for bound states can be performed. If φ_r, θ_r and ψ_r are Euler angles between a frame with its z-axis along $\boldsymbol{k}$ and a frame along $\boldsymbol{r}$, the wave function (48) may be written as

$$X_\lambda^{(+)}(\boldsymbol{k}, \boldsymbol{r}) = \frac{1}{2k} (2\pi)^{-\frac{1}{2}} \sum_{j, \lambda'} (2j+1) X^j_{\lambda\lambda'}(kr) R^{(j)*}_{\lambda, \lambda'}(\varphi_r, \theta_r, \psi_r) |\lambda'\rangle, \tag{49}$$

where the helicity functions $X^j_{\lambda\lambda'}$ are

$$X^j_{\lambda\lambda'} = \frac{i^{j-\frac{1}{2}}}{r} \{X_{l=j-\frac{1}{2}, j}(kr) + i(-)^{\lambda-\lambda'} X_{l=j+\frac{1}{2}, j}(kr)\}. \tag{50}$$

They depend upon the usual helicity λ and do not have a well-defined parity. With the same transformation the ingoing wave is

$$X_\lambda^{(-)*}(\boldsymbol{k}, \boldsymbol{r}) = \frac{1}{2k} (2\pi)^{-\frac{1}{2}} \sum_{j, \lambda'} (2j+1) \tilde{X}^j_{\lambda\lambda'}(kr) R^{(j)}_{\lambda, \lambda'}(\varphi_r, \theta_r, \psi_r) |\lambda'\rangle, \tag{51}$$

with

$$\tilde{X}^{j}_{\lambda\lambda'}(kr) = \frac{i^{\frac{1}{2}-j}}{r}\{X_{l=j-\frac{1}{2},j}(kr) - i(-)^{\lambda-\lambda'}X_{l=j+\frac{1}{2},j}(kr)\}. \tag{52}$$

The definition (52) differs from (50) by a complex conjugation of the coefficients and not by a complex conjugation of the radial wave functions. The helicity forms (49) and (51) of the distorted waves are of very limited use because parity conservation splits them into an even and an odd part.

The reaction is described by the helicity amplitudes

$$f_{\sigma_f\mu_f;\sigma_i}(\boldsymbol{k}_i, \boldsymbol{k}_f) = -\frac{m}{2\pi\hbar^2}\left(\frac{v_f}{v_i}\right)^{\frac{1}{2}}\langle X^{(-)}_{\sigma_f}(\boldsymbol{k}_f, \boldsymbol{r})\Psi^{I_f}_{\mu_f}|V|X^{(+)}_{\sigma_i}(\boldsymbol{k}_i, \boldsymbol{r})\Psi^{I_i}\rangle \tag{53}$$

for an incoming particle in the direction $\boldsymbol{k}_i$ with the helicity σ_i and an outcoming particle in the direction $\boldsymbol{k}_f$ with the helicity σ_f, the residual nucleus having a helicity μ_f. Here Ψ^{I_i} and $\Psi^{I_f}_{\mu_f}$ are the eigenfunctions of the initial and the final nucleus (we consider a target without spin in the initial state and described by a particle j_p and a hole j_h in the final state), m the reduced mass of the particle and v_i and v_f its velocity in the initial and the final state. The normalization has been chosen in such a way that

$$\frac{d\sigma}{d\Omega}(\boldsymbol{k}_i, \boldsymbol{k}_f) = \tfrac{1}{2}\sum_{\sigma_i,\sigma_f,\mu_f}|f_{\sigma_f\mu_f;\sigma_i}(\boldsymbol{k}_i, \boldsymbol{k}_f)|^2 \tag{54}$$

for an unpolarized incident beam. In this case, the polarizations of the outcoming particle and of the residual nucleus, and their correlations are described by the matrix

$$F_{\sigma_f\mu_f;\sigma'_f\mu'_f} = \left[2\frac{d\sigma}{d\Omega}(\boldsymbol{k}_i, \boldsymbol{k}_f)\right]^{-1}\sum_{\sigma_i} f_{\sigma_f\mu_f;\sigma_i}(\boldsymbol{k}_i, \boldsymbol{k}_f) f^{*}_{\sigma'_f\mu'_f;\sigma_i}(\boldsymbol{k}_i, \boldsymbol{k}_f). \tag{55}$$

The amplitude (53) is easily calculated if the axis of quantization is chosen along $\boldsymbol{k}_f$, φ and ψ being zero and θ the angle between $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$

$$f_{\sigma_f\mu_f;\sigma_i}(\theta) = -\frac{m}{\hbar^2}\left(\frac{v_f}{v_i}\right)^{\frac{1}{2}}\frac{1}{k_f k_i}\sum_{j_i,j_f,m_i}(-)^{j_i-\sigma_i}\{(2j_i+1)(2j_f+1)(2I_f+1)\}^{\frac{1}{2}}$$
$$\times\begin{pmatrix} j_i & j_f & I_f \\ m_i & -\sigma_f & \mu_f \end{pmatrix} r^{(j_i)}_{m_i,\sigma_i}(\theta) f^{I_f}_{j_p(j_f\sigma_f),\, j_h(j_i\sigma_i)}, \tag{56}$$

$f^{J}_{j'_1(j_f\sigma_f),\, j_1(j_i\sigma_i)}$ is given by the expression (31) where $\varphi^{j'_2}_{\lambda'_2}$ and $\varphi^{j_2}_{\lambda_2}$ have to be replaced by $\tilde{X}^{j_f}_{\sigma_f\lambda'_2}$ and $X^{j_i}_{\sigma_i\lambda_2}$, respectively.

If a normal parity state has been excited, the integration over the particle-hole variable gives the two one-body form factors A and B (38). The second integration

(40) becomes

$$f^J_{j_\mathrm{p}(j_\mathrm{f}\sigma_\mathrm{f}),\, j_\mathrm{h}(j_\mathrm{i}\sigma_\mathrm{i})} = (-)^{j_\mathrm{f}-j_\mathrm{i}}\tfrac{1}{2}G^J_{j_\mathrm{i}j_\mathrm{f}} \times \Big[\int A^{Jj_\mathrm{p}j_\mathrm{h}}(2)\{\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}-\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}-\frac{1}{2}}-(-)^{j_\mathrm{f}+J+j_\mathrm{i}}\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}\frac{1}{2}}\}r_2^2\mathrm{d}r_2 - \alpha^J_{j_\mathrm{i}j_\mathrm{f}}\int B^{Jj_\mathrm{p}j_\mathrm{h}}(2)\{\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}-\frac{1}{2}}-(-)^{j_\mathrm{f}+J+j_\mathrm{i}}\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}-\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}\frac{1}{2}}\}r_2^2\mathrm{d}r_2\Big]. \tag{57}$$

For an unnatural parity transition, the two one-body form factors are C and D as given by the formulae (41). For the second integration

$$f^J_{j_\mathrm{p}(j_\mathrm{f}\sigma_\mathrm{f}),\, j_\mathrm{h}(j_\mathrm{i}\sigma_\mathrm{i})} = (-)^{j_\mathrm{f}-j_\mathrm{i}}\tfrac{1}{2}G^J_{j_\mathrm{i}j_\mathrm{f}} \times \Big[\int C^{Jj_\mathrm{p}j_\mathrm{h}}(2)\{\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}-\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}-\frac{1}{2}}+(-)^{j_\mathrm{f}+J+j_\mathrm{i}}\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}\frac{1}{2}}\}r_2^2\mathrm{d}r_2 - \alpha^J_{j_\mathrm{i}j_\mathrm{f}}\int D^{Jj_\mathrm{p}j_\mathrm{h}}(2)\{\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}-\frac{1}{2}}+(-)^{j_\mathrm{f}+J+j_\mathrm{i}}\tilde{X}^{j_\mathrm{f}}_{\sigma_\mathrm{f}-\frac{1}{2}}X^{j_\mathrm{i}}_{\sigma_\mathrm{i}\frac{1}{2}}\}r_2^2\mathrm{d}r_2\Big]. \tag{58}$$

In both cases the change of sign of σ_i and σ_f multiplies the matrix element by $-\eta(-)^{j_\mathrm{f}+J+j_\mathrm{i}}$, where η is the intrinsic parity of the residual nucleus (1 for natural parity states, -1 for unnatural ones). Taking into account the change of sign of the helicity, the 3-j coefficient and the reduced rotation matrix element in the expression (56) for the amplitudes, one can verify that the usual parity condition

$$f_{\sigma_\mathrm{f}\mu_\mathrm{f};\,\sigma_\mathrm{i}}(\theta) = \eta(-)^{\sigma_\mathrm{f}-\mu_\mathrm{f}-\sigma_\mathrm{i}}f_{-\sigma_\mathrm{f}-\mu_\mathrm{f};\,-\sigma_\mathrm{i}}(\theta) \tag{59}$$

is obtained. The integrations in eqs. (57) and (58) need only be performed for $\sigma_\mathrm{i} = \sigma_\mathrm{f}$ and $\sigma_\mathrm{i} = -\sigma_\mathrm{f}$; the factors of A and B (or C and D) are interchanged in going from one case to the other and four integrals are needed for each set of values $(j_\mathrm{i}, j_\mathrm{f})$.

However, the helicity functions (50) and (52) are not directly obtained. For this reason and for further factorization, it is useful to replace them in the integrals (57) or (58) by their expression in terms of the usual wave functions $X_{lj}(kr)$. With the notation

$$F_{\pm,\pm} = \int F(2)X_{l_\mathrm{f}=j_\mathrm{f}\pm\frac{1}{2},\,j_\mathrm{f}}(k_\mathrm{f}r_2)X_{l_\mathrm{i}=j_\mathrm{i}\pm\frac{1}{2},\,j_\mathrm{i}}(k_\mathrm{i}r_2), \tag{60}$$

where the first sign is related to the ingoing wave and the second one to the outgoing wave, the following expressions are obtained:

For a natural-parity transition, if $j_\mathrm{i}+j_\mathrm{f}+J$ is even

$$f^J_{j_\mathrm{p}(j_\mathrm{f}\frac{1}{2});\,j_\mathrm{h}(j_\mathrm{i}\pm\frac{1}{2})} = i^{j_\mathrm{i}-j_\mathrm{f}+1}G^J_{j_\mathrm{i}j_\mathrm{f}}\{(A+\alpha^J_{j_\mathrm{i}j_\mathrm{f}}B)_{+,-}\mp(A-\alpha^J_{j_\mathrm{i}j_\mathrm{f}}B)_{-,+}\}. \tag{61}$$

For a natural-parity transition, if $j_\mathrm{i}+j_\mathrm{f}+J$ is odd

$$f^J_{j_\mathrm{p}(j_\mathrm{f}\frac{1}{2});\,j_\mathrm{h}(j_\mathrm{i}\pm\frac{1}{2})} = i^{j_\mathrm{i}-j_\mathrm{f}}G^J_{j_\mathrm{i}j_\mathrm{f}}\{(A-\alpha^J_{j_\mathrm{i}j_\mathrm{f}}B)_{-,-}\pm(A+\alpha^J_{j_\mathrm{i}j_\mathrm{f}}B)_{+,+}\}. \tag{62}$$

For an unnatural-parity transition, if j_i+j_f+J is even

$$f^J_{j_p(j_f\frac{1}{2});\,j_h(j_i\pm\frac{1}{2})} = i^{j_i-j_f} G^J_{j_ij_f}\{(C-\alpha^J_{j_ij_f}D)_{-,\,-} \pm (C+\alpha^J_{j_ij_f}D)_{+,\,+}\}. \tag{63}$$

For an unnatural-parity transition, if j_i+j_f+J is odd

$$f^J_{j_p(j_f\frac{1}{2});\,j_h(j_i\pm\frac{1}{2})} = i^{j_i-j_f+1} G^J_{j_ij_f}\{(C+\alpha^J_{j_ij_f}D)_{+,\,-} \mp (C-\alpha^J_{j_ij_f}D)_{-,\,+}\}, \tag{64}$$

where $\pm$ refers to the sign of σ_i.

A partial factorization can be seen in formulae (61)–(64); for all the matrix elements for which $j_f = j_i+J$ the one-body form factors are present with the combination $A\pm J\{J(J+1)\}^{-\frac{1}{2}}B$ or $C\pm J\{J(J+1)\}^{-\frac{1}{2}}B$; for $j_f = j_i+J-2$, they are $A\pm(J-2)\cdot\{J(J+1)\}^{-\frac{1}{2}}B$ or $C\pm(J-2)\{J(J+1)\}^{-\frac{1}{2}}D$ and so on; similarly, for the (J) matrix elements $(j_i = p;\ j_f = p+J-1)$, $(j_i = p+1; j_f = p+J-2)$, ..., $(j_i=p+J-1; j_f=p)$ the one-body form factors contribute by $A\pm(2p+J)\{J(J+1)\}^{-\frac{1}{2}}B$ or $C\pm(2p+J)\{J(J+1)\}^{-\frac{1}{2}}D$. If these circumstances are used, the time of integrations is reduced almost by a factor of two.

The problem described so far considers a very simple description of the nucleus. These results however can be generalized to other DWBA calculations involving a nucleon-nucleon interaction.

Any inelastic scattering on a spin-zero nucleus neglecting exchange terms needs only a more elaborate calculation of the one-body form factors, there is still only one multipole involved. The calculation of the one-body form factors depends upon the model and can be performed in any case for which there is some description of the spins (for example, a crude vibrational model is excluded).

If the spins of the initial and residual nucleus are non-zero, and neglecting antisymmetrization, all the multipoles with $|I_i-I_f| \leqq J \leqq I_i+I_f$ are present; the one-body form factors are alternatively of the type (A, B) or (C, D) for different J-values. The contributions of different multipoles must be added, giving for the amplitude

$$f_{\sigma_f\mu_f;\,\sigma_i\mu_i}(\theta) = \sum_{J,\,\mu'_i,\,\mu} (-)^{I_f+\mu-\mu_i}(2J+1)^{\frac{1}{2}} \begin{pmatrix} I_f & I_i & J \\ -\mu_f & \mu'_i & \mu \end{pmatrix} r^{(I_i)}_{-\mu'_i,\,-\mu}(\theta) f^J_{\sigma_f\mu;\,\sigma_i}(\theta), \tag{65}$$

with $f^J_{\sigma_f\mu;\,\sigma_i}(\theta)$ given by formula (56). The reduced rotation matrix elements in the expressions (56) and (65) can be summed

$$f_{\sigma_i\mu_f;\,\sigma_i\mu_i}(\theta) = \frac{m}{\hbar^2}\left(\frac{v_f}{v_i}\right)^{\frac{1}{2}} \frac{1}{k_i k_f} \sum_{J,\,J',\,M,\,M'} (2J+1)(2J'+1)\{(2j_i+1)(2j_f+1)\}^{\frac{1}{2}}$$
$$\times \begin{pmatrix} j_f & I_f & J' \\ \sigma_f & -\mu_f & -M \end{pmatrix} \begin{pmatrix} j_i & I_i & J' \\ -\sigma_i & \mu_i & M' \end{pmatrix} \begin{Bmatrix} j_i & j_f & J \\ I_f & I_i & J' \end{Bmatrix} r^{(J')}_{M,\,M'}(\theta) f^J_{I_f(j_f\sigma_f),\,I_i(j_i\sigma_i)}. \tag{66}$$

In this expression the direct term of the particle-particle matrix element (32) can be recognized besides the 3-j coefficients coupling the initial and final helicities to a total angular momentum J. However, the first expression (65) avoids the use of the 6-j coefficient by introducing a second reduced matrix element of rotation $r^{(I_i)}_{m,m'}(\theta)$; if I_i is small, all these matrix elements can be calculated easily; if I_f is smaller than I_i a different formula can be used by exchanging the initial and the final state. The other reduced rotation matrix element has a magnetic quantum number $\pm\frac{1}{2}$ and can easily be deduced [1]) from the Legendre polynomials and their associated functions.

In some other problems, such as knock-on reactions, only the exchange part of the matrix element is taken into account. In the expression (56), f^{I_f} must be replaced by

$$-\sum_J (-)^{J+I_f+j_h+j_i}(2J+1)\begin{Bmatrix} j_i & j_p & J \\ j_h & j_f & I_f \end{Bmatrix} f^J_{j_p(j_f\sigma_f);(j_i\sigma_i)j_h}. \tag{67}$$

Each matrix element must be calculated from the two-body form factors; for each value of j_i and for J ranging from $|j_i-j_p|$ to j_i+j_p one-body form factors of all kinds are obtained with the two parities of the partial wave j_i; each of them has to be integrated with the wave functions of some parity for j_f ranging from the greatest of the two values $|j_h-J|$ and $|j_i-I_f|$ to the lowest of the two values j_h+J and j_i+I_f.

A complete calculation would involve both the direct and the exchange terms.

The helicity formalism cannot be used directly in a coupled channel calculation because the part of the Hamiltonian which describes the relative motion of the two fragments depends upon the quantum numbers of the particle relative to the target nucleus. The set of coupled equations for each total angular momentum J of the system and each parity must be written between channels defined by the spin of the nucleus I, the orbital angular momentum l and the total angular momentum j of the particle; the coupling [9]) between two channels n and m is

$$\sum_{\mu_n,\mu_m,m_n,m_m} \langle j_n I_n \mu_n m_n|JM\rangle\langle j_m I_m \mu_m m_m|JM\rangle\langle l_n j_n \mu_n|\langle I_n m_n|V|l_m j_m \mu_m\rangle|I_m m_m\rangle$$
$$= \sum_{J'} (-)^{J+I_m+j_n}(2J'+1)i^{l_m-l_n}\begin{Bmatrix} I_m & I_n & J' \\ j_n & j_m & J \end{Bmatrix} G^J_{j_m j_n}\left\{\begin{pmatrix} A^{J'}(r) \\ C^{J'}(r)\end{pmatrix} + \varepsilon_{j_m}\alpha^{J'}_{j_n j_m}\begin{pmatrix} B^{J'}(r) \\ D^{J'}(r)\end{pmatrix}\right\}. \tag{68}$$

If the product of the intrinsic parities of the target in the channels n and m is $(-)^{J'}$, the one-body form factors are A and B; otherwise they are C and D. If $(-)^{l_n+l_m}$ is not equal to the product of the intrinsic parities, the coupling vanishes.

Formula (68) shows that the use of a spin-dependent nucleon-nucleon interaction does not introduce an important complication of the problem; the summation over J' includes all values instead of being restricted to those of a given parity; a coupling can exist between a channel for which the nucleus has zero spin and a channel with an unnatural-parity excited state of the target.

6. Generalization for higher spins

The method followed here could be applied to an interaction between particles of spin greater than $\frac{1}{2}$ and between two particles of different spins. For example, for spin one, the elementary matrices corresponding to (23) are

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}, \tag{69}$$

and the formula given for a scalar, a $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ and a tensor interaction holds between two particles of spin 1 and between particles of spin 1 and $\frac{1}{2}$ (for the relative $\boldsymbol{L}\cdot \boldsymbol{S}$ interaction, the value $\frac{1}{2}$ of the spin has been used to derive the expressions given in the appendix). But, for spin-one particles there are five more elementary matrices, three of which are of even kind and two odd, and there can be more general interactions which introduce them.

For an arbitrary spin s, $(2s+1)^2$ elementary matrices are needed to span the spin-space and this number is equal to the number of different sets of (L, S, J) for a given J. They can be separated into two groups with respect to parity and the number of matrices of each group is equal to this of (L, S, J) with $L+J$ even or $L+J$ odd (if s is half-integer the two groups have the same number of matrices; if s is an integer the even group includes one matrix more than the odd one). Another useful classification is their tensorial character λ which ranges from zero to $2s$ and is equal to the S of (L, S, J); there are $2\lambda+1$ matrices for the tensor λ; if λ is even, $\lambda+1$ of them are even and λ odd; if λ is odd, $\lambda+1$ of them are odd and λ even. Within a group defined by the parity and its tensor character, the correspondence to the (L, S, J) coupling mixes all the values of L.

The formulae (35) and (36) can be used to perform successively the integration over the coordinates of the two particles. From now on, we shall assume that the first integration has been done, giving some one-body form factors.

The main complication for a spin greater than $\frac{1}{2}$ arises because the knowledge of the total angular momentum and of the parity is not sufficient to specify the angular momentum. The bound states of spin-one particles can be described by their helicity wave functions (10) as for spin $\frac{1}{2}$, but with

$$\begin{aligned}
&\varphi_1^j = -\varphi_{-1}^j = -f_{lj}, && \varphi_0^j = 0 && \text{if} \quad l = j,\\
&\varphi_1^j = \varphi_{-1}^j = \left(\frac{j}{2j+1}\right)^{\frac{1}{2}} f_{lj}, && \varphi_0^j = -\left\{\frac{2(j+1)}{2j+1}\right\}^{\frac{1}{2}} f_{lj} && \text{if} \quad l = j+1,\\
&\varphi_1^j = \varphi_{-1}^j = \left(\frac{j+1}{2j+1}\right)^{\frac{1}{2}} f_{lj}, && \varphi_0^j = \left(\frac{2j}{2j+1}\right)^{\frac{1}{2}} f_{lj} && \text{if} \quad l = j-1.
\end{aligned} \tag{70}$$

The most general interaction includes five one-body form factors for the even part

$$F_{\text{even}} = \begin{vmatrix} a+e & \frac{1}{\sqrt{2}}(b+c) & d \\ \frac{1}{\sqrt{2}}(b-c) & a-2e & \frac{1}{\sqrt{2}}(b-c) \\ d & \frac{1}{\sqrt{2}}(b+c) & a+e \end{vmatrix}, \tag{71}$$

where a is scalar, b vector, c, d and e tensor. There are four one-body form factors for the odd part

$$F_{\text{odd}} = \begin{vmatrix} a' & \frac{1}{\sqrt{2}}(b'+c') & d' \\ \frac{1}{\sqrt{2}}(c'-b') & 0 & \frac{1}{\sqrt{2}}(b'-c') \\ -d' & \frac{-1}{\sqrt{2}}(b'+c') & -a' \end{vmatrix}, \tag{72}$$

where a' and b' are vector, c' and d' tensor. The matrices (69) have been used to write (71) and (72) for their vector part.

If $j+j'+J$ is odd the recurrence relations between 3-j coefficients allow some factorizations in the formula (36) as

$$\tfrac{1}{4}G^J_{jj'}\{jj'\}^{-\frac{1}{2}}[a'+\{J(J+1)\}^{-\frac{1}{2}}\{j(c'+b')+j'(c'-b')\}+d'\beta^J_{jj'}]$$
$$\text{for } l=j+1, \qquad l'=j'+1,$$
$$\tfrac{1}{4}G^J_{jj'}\left\{\frac{2j'+1}{j'(j'+1)(j+1)}\right\}^{\frac{1}{2}}[a+e+\{J(J+1)\}^{-\frac{1}{2}}(j'+1)(b+c)-d\beta^J_{jj'}]$$
$$\text{for } l=j+1, \qquad l'=j', \tag{73}$$

where

$$\beta^J_{jj'} = \frac{j'(j'+1)-j(j+1)}{\{(J-1)(J)(J+1)(J+2)\}^{\frac{1}{2}}}, \tag{74}$$

$$G^J_{jj'} = 4(-)^j\left\{\frac{j(j+1)j'(j'+1)}{2J+1}\right\}\begin{pmatrix} j' & J & j \\ 1 & 0 & -1\end{pmatrix}, \tag{75}$$

is still given by the expression (46).

If $j+j'+J$ is even, the 3-j coefficients cannot be reduced so easily. However, all of them can be obtained with the following relations:

$$\begin{pmatrix} j' & J & j \\ -1 & 2 & -1\end{pmatrix} = -\left\{\frac{j(j+1)}{(J-1)(J+2)}\right\}^{\frac{1}{2}}\begin{pmatrix} j' & J & j \\ -1 & 1 & 0\end{pmatrix} - \left\{\frac{j'(j'+1)}{(J-1)(J+2)}\right\}^{\frac{1}{2}}\begin{pmatrix} j' & J & j \\ 0 & 1 & -1\end{pmatrix},$$
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 1 & -1 & 0\end{pmatrix} = 2\,\frac{j_3(j_3+1)-j_1(j_1+1)-j_2(j_2+1)}{\{j_1(j_1+1)j_2(j_2+1)\}^{\frac{1}{2}}}\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0\end{pmatrix},$$
$$\begin{pmatrix} j' & J & j \\ 0 & 0 & 0\end{pmatrix} = (-)^{j-1}(2J+1)^{\frac{1}{2}}G^J_{jj'}. \tag{76}$$

The same formula (46) provides the starting point of the recurrence relations for the values of the 3-j coefficients involved for any spin.

In DWBA calculations the main complication introduced by spin greater than $\frac{1}{2}$ is the multiplicity of radial functions for a given j-value. In the most general problem their number is equal to that of the even form factors. For spin one, there are five of them: one of parity $(-)^j$ and four of parity $(-)^{j+1}$, two of them with an ingoing wave plus an outgoing one and the other two with only an outgoing part. Five independent helicity functions can be defined of which two are equal if there is no tensor potential. They can be used in expressions like (56) and (57), but, going back to the usual notations, a factorization is obtained showing that the only integrations needed are those indicated in formulae (73); the amplitudes for given helicities are sums of these integrals multiplied by simple Clebsch-Gordan coefficients.

Another possible generalization of this formalism could be the consideration of one-body form factors between particles of different spins (for example, for pick-up and stripping reactions; a phenomenological form factor could be defined between an helicity state of the deuteron and an helicity state of the nucleon; the elementary matrices in the spin space are rectangular and the multipoles would be labelled by half-integer values of J).

7. Conclusion

In any problem involving a multipole expansion of a two-body interaction, the angular momentum couplings are similar for all values of the spins. The particle-particle and particle-hole matrix elements involve the same 6-j coefficients needed for spinless particles, but the orbital angular momentum is replaced by the total angular momentum. For higher spins, there are more wave functions with the same j and more form factors. The helicity formalism demands a special study of each case to write explicitly the simple 3-j coefficients relative to the spin; however, the framework is the same in all the cases.

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Appendix

The multipole expansion in the helicity formalism of the tensor interaction

$$V(|\boldsymbol{r}_1-\boldsymbol{r}_2|)[3\{\boldsymbol{\sigma}_1\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\}\{\boldsymbol{\sigma}_2\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\}-(\boldsymbol{r}_1-\boldsymbol{r}_2)^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2] \tag{77}$$

is

$$a^J(1,2) = b^J(1,2) = 0,$$
$$c^J(1,2) = (r_1^2+r_2^2)(2J+1)V_J - r_1 r_2\{(2J+3)V_{J-1}+(2J-1)V_{J+1}\},$$

$$d^J(1,2) = 2(r_1^2+r_2^2)\{JV_{J-1}+(J+1)V_{J+1}\}-r_1 r_2\left\{\frac{J(J-1)}{2J-1}V_{J-2}\right.$$
$$\left.+(2J+1)\frac{14J^2+14J-10}{(2J-1)(2J+3)}V_J+\frac{(J+1)(J+2)}{2J+3}V_{J+2}\right\},$$
$$e^J(1,2) = (2r_1^2-r_2^2)\{J(J+1)\}^{\frac{1}{2}}\{V_{J+1}-V_{J-1}\}+r_1 r_2\{J(J+1)\}^{\frac{1}{2}}$$
$$\times\left\{\frac{J-1}{2J-1}V_{J-2}+\frac{2J+1}{(2J-1)(2J+3)}V_J-\frac{J+2}{2J+3}V_{J+2}\right\},$$
$$f^J(1,2) = -(r_1^2+r_2^2)\{(J+1)V_{J-1}+JV_{J+1}\}-r_1 r_2\left\{\frac{(J-1)(J+1)}{2J-1}V_{J-2}\right.$$
$$\left.-(2J+1)\frac{10J^2+10J-9}{(2J-1)(2J+3)}V_J+\frac{J(J+2)}{2J+3}V_{J+2}\right\}. \quad (78)$$

For the relative $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction

$$V(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\left\{\boldsymbol{L}_1+\boldsymbol{L}_2+i\boldsymbol{r}_1\wedge\boldsymbol{r}_2\left(\frac{1}{r_2}\frac{\partial}{\partial r_2}-\frac{1}{r_1}\frac{\partial}{\partial r_1}\right)+\boldsymbol{r}_1\wedge(\boldsymbol{r}_2\wedge\boldsymbol{L}_2)\frac{1}{r_2^2}\right.$$
$$\left.+\boldsymbol{r}_2\wedge(\boldsymbol{r}_1\wedge\boldsymbol{L}_1)\frac{1}{r_1^2}\right\}\{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2\}, \quad (79)$$

the multipoles invariant for time reversal are

$$a^J(1,2) = -2(2J+1)V_J-\tfrac{1}{2}\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}\right)\left\{J(J+3)V_{J-1}-(J+1)(J-2)V_{J+1}\right\}$$
$$+\tfrac{1}{2}\left(\frac{r_1}{r_2}(\alpha^J_{j_2j'_2})^2+\frac{r_2}{r_1}(\alpha^J_{j_1j'_1})^2\right\}\{V_{J-1}-V_{J+1}\},$$
$$c^J(1,2) = (2J+1)V_J-\tfrac{1}{2}\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}\right)\{(J+1)V_{J-1}+JV_{J+1}\},$$
$$d^J(1,2) = -\frac{J(J+1)}{2}\{2-(\alpha^J_{j_1j'_1})^2-(\alpha^J_{j_2j'_2})^2\}\{V_{J-1}-V_{J+1}\},$$
$$e^J(1,2) = -\tfrac{1}{2}\{J(J+1)\}^{\frac{1}{2}}\left[\left\{(J+2)V_{J-1}-(2J+1)\frac{r_2}{r_1}V_J+(J-1)V_{J+1}\right\}\right.$$
$$\left.+(\alpha^J_{j_1j'_1})^2\left\{(J+1)V_{J-1}-(2J+1)\frac{r_2}{r_1}V_J+JV_{J+1}\right\}\right],$$
$$f^J(1,2) = -(J+1)V_{J-1}-JV_{J+1}+\frac{2J+1}{2}\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}\right)V_J. \quad (80)$$

For this interaction there is also a multipole b which is the sum of two parts

(i) a derivative term

$$b_1^J(1,2) = \tfrac{1}{2}\{J(J+1)\}^{\frac{1}{2}} \left[(V_{J-1} - V_{J+1}) \Delta_{12} \left(r_2 \frac{\partial}{\partial r_1} - r_1 \frac{\partial}{\partial r_2} \right) + \left(r_2 \frac{\partial}{\partial r_1} - r_1 \frac{\partial}{\partial r_2} \right) \Delta_{12} (V_{J-1} - V_{J+1}) \right], \quad (81)$$

where $\Delta_{12} = r_1^2 r_2^2$ if the normalization (5) is used and $\Delta_{12} = 1$ if the wave functions are multiplied by r as is usual for unbound functions. The expansion (81) can be written directly with the derivatives of the wave functions which are on the right, in the following form:

$$b_1^J(1,2) = \{J(J+1)\}^{\frac{1}{2}} \left\{ (V_{J-1} - V_{J+1}) \Delta_{12} \left(r_2 \frac{\partial}{\partial r_1} - r_1 \frac{\partial}{\partial r_2} \right) + \tfrac{1}{2} \left(\frac{r_2}{r_1} - \frac{r_1}{r_2} \right) \times (\gamma_1 V_{J-1} + \gamma_2 V_{J+1}) \right\}, \quad (82)$$

where $\gamma_1 = J+1$ and $\gamma_2 = J$ if $\Delta_{12} = r_1^2 r_2^2$ and $\gamma_1 = J-1$ and $\gamma_2 = J+2$ if $\Delta_{12} = 1$.

(ii) A term odd for the permutation of j_1 and j_2 with j_1' and j_2'

$$b_2^J(1,2) = \frac{(j_1+\frac{1}{2})^2 - (j_1'+\frac{1}{2})^2}{2\{J(J+1)\}^{\frac{1}{2}}} \left\{ -(2J+1)V_J + \frac{r_2}{r_1} [(J+1)V_{J-1} + JV_{J+1}] \right\} + \frac{(j_2+\frac{1}{2}) - (-)^{j_2+j'_2+J}(j_2'+\frac{1}{2})}{2\alpha^J_{j_2 j'_2}} \left\{ (2J+1)V_J + \frac{r_1}{r_2} [JV_{J-1} + (J+1)V_{J+1}] \right\}. \quad (83)$$

If $j_2 = j_2'$ and J is even $\alpha^J_{j_2 j'_2} = 0$ and the second part of $b_2^J(1,2)$ becomes infinite; however, the calculation of the one-body form factors, $b^J(1,2)$ is multiplied by $\alpha^J_{j_2 j'_2}$ and the contribution is finite.

A review of formulae (80), (82) and (83) gives for the one-body form factors the following radial dependences:

$$A(1) = A_1(r_1) + A_2(r_1) \frac{\partial}{\partial r_1} + A_3(r_1)(\alpha^J_{j_1 j'_1})^2 + A_4(r_1)\{(j_1+\tfrac{1}{2})^2 - (j_1'+\tfrac{1}{2})^2\},$$

$$B(1) = B_1(r_1) + B_2(r_1) \frac{\partial}{\partial r_1} + B_3(r_1) \frac{(j_1+\frac{1}{2}) - (-)^{j_1+j'_1+J}(j_1'+\frac{1}{2})}{\alpha^J_{j_1 j'_1}},$$

$$C(1) = C_1(r_1) + C_2(r_1)(\alpha^J_{j_1 j'_1})^2,$$

$$D(1) = D(r_1). \quad (84)$$

The calculation of the relative $\boldsymbol{L}\cdot\boldsymbol{S}$ interaction is very simple for the unnatural-parity matrix elements. For the natural-parity ones, the two-one body form factors involve seven radial functions.

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