Chapter 5. Inelastic Scattering and Transfer Reactions - DWBA

Contents

This lecture is summarized in the books by G. R. Satchler, entitled "Direct Nuclear Reactions" (Oxford, 1983) and by H. Feshbach, "Theoretical Nuclear Physics - Nuclear Reactions" (Wiley, 1992), and in the paper by T. Tamura, (*Phys. Reports*, **14**, 59 (1974)) entitled "Compact Reformulation of Distorted-wave and Coupled-channel Born Approximations for Transfer Reactions between Nuclei".

1 Definitions of variables

Partition: We will only discuss reactions where there are two particles involved; such a pair of nuclei a and A, say, will be called a *partition* and will be denoted by α. It also sometimes refer to a particular internal state of that partition (i.e., a particular state of each of the nuclei a and A). We will use α, α' to different states of the same partition.

Channel: We will use the term *channel* to refer to a particular internal state of a partition in a particular state of relative motion. The term will also be used flexibly; sometimes just denote the energy of relative motion, sometimes energy and momentum, sometimes energy and angular momentum, and so on.

Internal states: For any partition $\alpha = a + A$, the wavefunctions and Hamiltonian of internal states will be written as

$$\begin{aligned} \psi_\alpha(x_\alpha) &\equiv \psi_a(x_a)\psi_A(x_A) \\ H_\alpha &\equiv H_a + H_A \end{aligned}$$

so that

$$(\epsilon_\alpha - H_\alpha)\psi_\alpha = 0$$

Channel Coordinates: The relative, or channel, coordinate $\vec{r}_\alpha$ is the vector joining the positions of the centers of mass of the two nuclei a and A.

$$\vec{r}_\alpha = \vec{r}_a - \vec{r}_A$$

The corresponding operator for the kinetic energy of relative motion is

$$K_\alpha = -\frac{\hbar^2}{2\mu_\alpha}\nabla_\alpha^2$$

where the subscript α on ∇ denotes differentiation with respect to $\vec{r}_\alpha$, and μ_α is the reduced mass for the α partition. Then the wavenumber k_α for the relative motion in this channel is obtained from the corresponding kinetic energy,

$$E_\alpha = \frac{\hbar^2 k_\alpha^2}{2\mu_\alpha} = E - \epsilon_\alpha$$

where E is the total energy of the system. Finally the momentum of relative motion is written as $\vec{p}_\alpha = \hbar\vec{k}_\alpha$.

Interaction Potential: The interaction potential between a and A will be written $V_\alpha \equiv V_\alpha(x_\alpha, \vec{r}_\alpha)$. If V_α is assumed to be the result of two-body forces, which depend on the spins and isospins of the two nucleons as well as their separations $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, only, it has the form

$$V_\alpha = \sum_{i=1}^{a}\sum_{j=a+1}^{a+A} v_{ij}(x_\alpha)$$

Total Hamiltonian : The total Hamiltonian for the system is then

$$H = H_\alpha + K_\alpha + V_\alpha$$

For other partitions,

$$\begin{aligned} H &= H_\beta + K_\beta + V_\beta \\ &= H_\gamma + K_\gamma + V_\gamma, \quad \text{etc.} \end{aligned}$$

Total wavefunction : The total wavefunction Ψ for the system obey the complete Schrödinger equation

$$(E - H)\Psi = 0$$

This Ψ may be expanded in terms of any complete set of internal states ψ_α

$$\Psi = \sum_\alpha \xi_\alpha(\vec{r}_\alpha)\psi_\alpha(x_\alpha)$$

where the sum runs over all internal states of some particular partition α. Note that the sum over α includes states, called *continuum*, in which either a or A or both are unbound. The continuum states can be used to construct wavefunctions for a different pair of nuclei. (The rearrangement collisions are possible.) They also describe exchange scattering in which one or more pairs of nucleons are interchanged between a and A.

Then the coefficients $\xi_\alpha(\vec{r}_\alpha)$ of this expansion describes the relative motion of the two nuclei a and A. If we invert this equation by using the orthonormality of ψ_α,

$$\xi_\alpha(\vec{r}_\alpha) = (\psi_\alpha|\Psi) \equiv \int \psi^*_\alpha(x_\alpha)\Psi dx_\alpha$$

Model wavefunctions : In practice, we do not use the full, infinite expansion, but we simply take the entrance channel plus a small subset of terms in which a and A are each bound and are coupled strongly to the entrance channel, and also include terms from two or more partitions to explain rearrangement collisions. We may regard this as a model for Ψ, to be used in conjunction with effective interactions,

$$\Psi_{model} = (\sum_\alpha)' u_\alpha(\vec{r}_\alpha)\psi_\alpha(x_\alpha) + (\sum_\beta)' u_\beta(\vec{r}_\beta)\psi_\beta(x_\beta) + ..$$

Unfortunately, wavefunctions from different partitions are not orthogonal. It must be taken into account when a mixed representation is used.

2 Transition amplitudes

We must impose boundary conditions in order to specify fully the wavefunction Ψ. The physical situation is one in which the incident beam is in the α channel, say; the two nuclei a and A are in their ground states and moving in the plane-wave state with relative momentum $\vec{k}_\alpha$. Thus Ψ contains incoming plane waves only in the ground state α channel but will have outgoing spherical waves in this and all other channels which are open at that energy. We denote this particular Ψ by $\Psi^{(+)}_\alpha(\vec{k}_\alpha)$. We may write

$$\Psi^{(+)}_\alpha = \sum_\beta \xi_\beta(\vec{r}_\beta)\psi_\beta(x_\beta)$$

The coefficients $\xi_\alpha(\vec{r}_\alpha)$, or the projection of $\Psi^{(+)}_\alpha$ onto the β channels, can be given by

$$\xi_\beta(\vec{r}_\beta) = (\psi_\beta|\Psi^{(+)}_\alpha) \equiv \int \psi^*_\beta(x_\beta)\Psi^{(+)}_\alpha dx_\beta$$

Since the function $\xi_\beta(\vec{r}_\beta)$ describes the relative motion in the β channel and tells us about any $\alpha \to \beta$ transition, it must have the asymptotic form

$$\xi_\beta(\vec{r}_\beta) \sim \exp[i\vec{k}_\alpha \cdot \vec{r}_\alpha]\delta_{\alpha\beta} + f_{\beta\alpha}(\hat{\vec{r}}_\beta, \vec{k}_\alpha)\frac{1}{r_\beta}\exp[ik_\beta r_\beta]$$

as $r \to \infty$. Here $\hat{\vec{r}}_\beta$ is the unit vector along $\vec{r}_\beta$ and stands for the polar angles of the direction of $\vec{r}_\beta$. Asymptotically, the relative momentum $\vec{k}_\beta$ has the same direction as $\vec{r}_\beta$, that is $\hat{\vec{k}}_\beta = \hat{\vec{r}}_\beta$, so we may equally write $f_{\beta\alpha}(\hat{\vec{k}}_\beta, \vec{k}_\alpha)$.

The above equation defines the scattering amplitude $f_{\beta\alpha}$ as the amplitude of the outgoing spherical wave in channel β induced by a plane wave of unit amplitude in the α channel. If the β and α partitions are the same, $f_{\beta\alpha}$ refers to elastic ($f_{\alpha\alpha}$) or inelastic ($f_{\acute{\alpha}\alpha}$). If $\beta \neq \alpha$, it describes a rearrangement collision.

The differential cross section for a reaction $A(a,b)B$, corresponding to a transition $\alpha \to \beta$, is given by

$$\frac{d\sigma_{\beta\alpha}}{d\Omega} = (\frac{v_\beta}{v_\alpha})|f_{\beta\alpha}(\vec{k}_\beta, \vec{k}_\alpha)|^2$$

For manipulative purposes, it is frequently more convenient to use a *transition amplitude* $T_{\beta\alpha}$, defined by renormalizing the scattering amplitude,

$$\begin{aligned} T_{\beta\alpha} &= -\frac{2\pi\hbar^2}{\mu_\beta} f_{\beta\alpha} \\ \frac{d\sigma_{\beta\alpha}}{d\Omega} &= \frac{\mu_\alpha\mu_\beta}{(2\pi\hbar^2)^2}(\frac{k_\beta}{k_\alpha})|T_{\beta\alpha}(\vec{k}_\beta, \vec{k}_\alpha)|^2 \end{aligned}$$

This seems to ignored any Coulomb fields that may be present. However, there is no loss of generality because in practice these Coulomb fields are screened asymptotically. Of course, any Coulomb effects on the scattering are included in the amplitudes f, and any Coulomb potentials are included in the interactions V_α and so forth.

3 Distorted waves

The Schrödinger equation for $\Psi_\alpha^{(+)}$ may be written as

$$(E - H_\beta - K_\beta)\Psi_\alpha^{(+)} = V_\beta \Psi_\alpha^{(+)}$$

Multiplying from the left by $\psi_\beta^*(x_\beta)$ and integrating over the x_β, we get an equation for ξ_β:

$$\begin{aligned} (E_\beta - K_\beta)\xi_\beta(\vec{r}_\beta) &= (\psi_\beta|V_\beta|\Psi_\alpha^{(+)}) \\ &\equiv \int \psi_\beta^*(x_\beta)V_\beta(\vec{r}_\beta, x_\beta)\Psi_\alpha^{(+)} dx_\beta \end{aligned}$$

This equation may be solved formally by standard Green function Techniques. (See Satchler Sec. 2.8.).

We introduce on both sides of the above equation an arbitrary auxiliary potential $U_\beta(r_\beta)$ which depend only on the channel radius r_β and that, therefore, cannot change the internal states of the β partition. In general, $U_\beta(r_\beta)$ may be complex. Then we have

$$(E_\beta - K_\beta - U_\beta(r_\beta))\xi_\beta(\vec{r}_\beta) = (\psi_\beta|W_\beta|\Psi_\alpha^{(+)})$$

where now

$$W_\beta = V_\beta(x_\beta, \vec{r}_\beta) - U_\beta(r_\beta)$$

is the *residual interaction.*

The motivation for introducing it is to give us an opportunity to include a large part of the average effects of the interaction V_β so that the effects of the inhomogeneous term on the right side of the equation may be minimized. The residual interaction W_β may be treated as a perturbation. In particular, some of the effects of the short-ranged nuclear forces may be included through the use of a complex optical potential. In addition, some approximation to the average Coulomb potential between b and B will usually be included in U_β, say, the two finite-size spherical charge distributions.

The formal solution of the equation may be expressed in terms of the solutions of the homogeneous equation

$$(E_\beta - K_\beta - U_\beta(r_\beta))\chi_\beta^{(+)}(\vec{k}_\beta, \vec{r}_\beta) \quad = \quad 0$$

These $\chi_\beta^{(+)}(\vec{k}_\beta, \vec{r}_\beta)$ are known as *distorted waves* and describes the elastic scattering of b on B due to the potential U_β by itself. Asymptotically, they have the form of an incident plane wave plus outgoing spherical waves

$$\chi_\beta^{(+)}(\vec{k}_\beta, \vec{r}_\beta) = e^{i\vec{k}_\beta \cdot \vec{r}_\beta} + f_\beta^{(0)}(\theta)\frac{e^{ik_\beta r_\beta}}{r_\beta}$$

as we have seen in Lecture V. We also need the function $\chi_\beta^{(-)}(\vec{k}_\beta, \vec{r}_\beta)$, which has an asymptotic form similar to the above equation but with incoming spherical waves. It is the time-reversal state of $\chi_\beta^{(+)}(\vec{k}_\beta, \vec{r}_\beta)$,

$$\chi_\beta^{(-)}(\vec{k}_\beta, \vec{r}_\beta) = \chi_\beta^{(+)*}(-\vec{k}_\beta, \vec{r}_\beta)$$

The $\chi_\beta^{(-)}(\vec{k}_\beta, \vec{r}_\beta)$ is a solution of the Schrödinger equation

$$(E_\beta - K_\beta - U_\beta^*(r_\beta))\chi_\beta^{(-)}(\vec{k}_\beta, \vec{r}_\beta) \quad = \quad 0$$

The formal solution (See Satchler Sec. 2.8.), $\xi_\beta(\vec{r}_\beta)$ of the original equation then gives

$$\xi_\beta(\vec{r}_\beta) \quad = \quad \chi_\alpha^{(+)}(\vec{k}_\beta, \vec{r}_\beta)\delta_{\alpha\beta} + \int G_\beta^{(+)}(\vec{r}_\beta, \vec{r}'_\beta)(\psi_\beta|W_\beta|\Psi_\alpha^{(+)})d\vec{r}'_\beta.$$

Here $G_\beta^{(+)}(\vec{r}_\beta, \vec{r}'_\beta)$ is the outgoing-wave Green function for propagation in the potential U_β. The distorted wave χ_α only appears in the equation when β stands for the entrance channel. Note that the potential U_β cannot induce transitions between different channels.

The above equation leads to the transition amplitude for the $\alpha \rightarrow \beta$ transition,

$$T_{\beta\alpha}(\vec{k}_\beta, \vec{k}_\alpha) \quad = \quad T_{\beta\alpha}^{(0)}(\vec{k}_\beta, \vec{k}_\alpha)\delta_{\alpha\beta} + < \chi_\alpha^{(-)}(\vec{k}_\beta)\psi_\beta|W_\beta|\Psi_\alpha^{(+)}(\vec{k}_\alpha) > .$$

where $T_{\beta\alpha}^{(0)}(\vec{k}_\beta, \vec{k}_\alpha)$ is the elastic transition amplitude due to U_β alone

$$T_{\beta\alpha}^{(0)}(\vec{k}_\beta, \vec{k}_\alpha) = -\frac{2\pi\hbar^2}{\mu_\beta} f_\beta^{(0)}(\theta)$$

where θ is the scattering angle between $\vec{k}_\beta$ and $\vec{k}_\alpha$.

4 Distorted Wave Born Approximation - DWBA

When we generalize the Lippman-Schwinger equation (See Lecture IV.) for scattering of systems with internal structure, we may obtain

$$\begin{aligned}\Psi_\alpha^{(+)} &= [1 + \frac{1}{E - H + i\epsilon} V_\alpha] e^{i\vec{k}_\alpha \cdot \vec{r}_\alpha} \psi_\alpha \\ &= [1 + G^{(+)} V_\alpha] e^{i\vec{k}_\alpha \cdot \vec{r}_\alpha} \psi_\alpha\end{aligned}$$

If we utilize the operator relation

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(B - A)\frac{1}{B} + \frac{1}{B}(B - A)\frac{1}{B}(B - A)\frac{1}{B} + ..$$

and the binomial expansion of $(1 + x)^{-1}$, we may obtain the operator expression for G when the Hamiltonian has the form $H = H_0 + V$,

$$\begin{aligned}G &= [1 + \frac{1}{E - H_0} V + \frac{1}{E - H_0} V \frac{1}{E - H_0} V + ..] \frac{1}{E - H_0} \\ &= [1 + G_0 V + G_0 V G_0 V + ..] G_0\end{aligned}$$

if the free propagator $G_0 = (E - H_0)^{-1}$. The first term of the series is called the *Born Approximation* or *Plane waves Born Approximation; PWBA* for the former

$$T_{\beta\alpha}^{PW}(\text{post}) = < e^{i\vec{k}_\beta \cdot \vec{r}_\beta} \psi_\beta | V_\beta | e^{i\vec{k}_\alpha \cdot \vec{r}_\alpha} \psi_\alpha >$$

We can also reformulate in a similar way for an auxiliary potential U_α, and obtain

$$\begin{aligned}\Psi_\alpha^{(+)} &= [1 + \frac{1}{E - H + i\epsilon} W_\alpha] \chi_\alpha^{(+)} \psi_\alpha \\ &= [1 + G^{(+)} W_\alpha] \chi_\alpha^{(+)} \psi_\alpha\end{aligned}$$

If we take V in the above propagator equation to be the residual interaction in channel α, $V = W_\alpha = V_\alpha - U_\alpha$, then we have

$$\Psi_\alpha^{(+)} = [1 + \tilde{G}^{(+)} W_\alpha + \tilde{G}^{(+)} W_\alpha \tilde{G}^{(+)} W_\alpha + ..] \chi_\alpha^{(+)} \psi_\alpha$$

Here $\tilde{G}^{(+)}$ is the distorted-waves propagator for the auxiliary potential U_α:

$$\tilde{G}^{(+)} = (E - H_\alpha - K_\alpha - U_\alpha + i\epsilon)^{-1}$$

The first term of the series is called the *Distorted waves Born Approximation; DWBA*

$$T_{\beta\alpha}^{DW}(\text{post}) = < \chi_\beta^{(-)}(\vec{k}_\beta) \psi_\beta | W_\beta | \chi_\alpha^{(+)}(\vec{k}_\alpha) \psi_\alpha >$$

where the residual interaction is $W_\beta = V_\beta - U_\beta$ (post). Similar Born and distorted-waves Born approximation may be obtained from the equivalent prior interaction forms for the transition amplitude:

$$\begin{aligned}T_{\beta\alpha}^{PW}(\text{prior}) &= < e^{i\vec{k}_\beta \cdot \vec{r}_\beta} \psi_\beta | V_\alpha | e^{i\vec{k}_\alpha \cdot \vec{r}_\alpha} \psi_\alpha > \\ T_{\beta\alpha}^{DW}(\text{prior}) &= < \chi_\beta^{(-)}(\vec{k}_\beta) \psi_\beta | W_\alpha | \chi_\alpha^{(+)}(\vec{k}_\alpha) \psi_\alpha >\end{aligned}$$

where the residual interaction is $W_\alpha = V_\alpha - U_\alpha$ (prior). We can show that provided no further approximations are made, these prior forms are exactly equal to the corresponding post forms. (See Satchler Sec.2.8.8.)

In the coordinate representation, the transition amplitude in the post form may be written as

$$\begin{aligned} T^{DW}_{\beta\alpha}(\vec{k}_\beta,\vec{k}_\alpha) &= \int\int d\vec{r}_\beta d\vec{r}_\alpha \chi^{(-)*}_\beta(\vec{k}_\beta,\vec{r}_\beta)(\psi_\beta|W_\beta|\psi_\alpha)\chi^{(+)}_\alpha(\vec{k}_\alpha,\vec{r}_\alpha) \\ &= \int\int d\vec{r}_\beta d\vec{r}_\alpha \chi^{(-)*}_\beta(\vec{k}_\beta,\vec{r}_\beta) I_{\beta\alpha}(\vec{r}_\beta,\vec{r}_\alpha)\chi^{(+)}_\alpha(\vec{k}_\alpha,\vec{r}_\alpha) \end{aligned}$$

The "nuclear matrix element", $I_{\beta\alpha}$, is separated explicitly from the distorted waves. This factor is defined to include the Jacobian $J_{\beta\alpha}$ for the transformation. This kernel plays the role of an effective interaction causing transitions between the elastic scattering states χ_α and χ_β, and contains all the information on nuclear structure, on angular momentum and parity selection rules, and even on the type of reaction being considered (whether inelastic scattering, knock-on, transfer, etc.). The radial part of this function is often called the *form factors*.

After the kernel $I_{\beta\alpha}$ has been calculated, the amplitude remains a six-dimensional integral over $\vec{r}_\alpha$ and $\vec{r}_\beta$. We can reduce this to a sum of two-dimensional integrations by making partial wave and multipole expansions.

4.1 Total wave functions

We now consider the reaction $A(a,b)B$ with nuclear spins and their z-components explicitly. When we include the effects of having spin-spin or spin-orbit couplings in the distorting potentials U, the distorted waves of relative motion could be written as matrices in the projections of the nuclear spins, and thus the total wave function in the incident channel can be generalized as

$$\Psi^{(+)}_{M_A m_a} = \chi^{(+)}_{m_a}\psi_{M_A m_a} \rightarrow \sum_{m'_a} \chi^{(+)}_{m_a m'_a}\psi_{M_A m'_a}$$

where the internal states become

$$\psi_{M_A m_a} = \phi_{I_A M_A}(x_A)\varphi_{s_a m_a}(x_a)$$

and the distorted waves may be written as, using the partial wave expansion,

$$\begin{aligned} \chi^{(+)}_{m_a m'_a}(\vec{k}_a,\vec{r}_a) &= \frac{4\pi}{k_a r_a} \sum_{\ell_a j_a m_{\ell_a} m'_{\ell_a}} (\ell_a m_{\ell_a} s_a m_a | j_a m_{j_a})(\ell_a m'_{\ell_a} s_a m'_a | j_a m_{j_a}) \\ &\qquad \times\ i^{\ell_a}\chi_{\ell_a j_a}(k_a,r_a)\ Y_{\ell_a \tilde{m}_{\ell_a}}(\hat{k}_a)\ Y_{\ell_a m'_{\ell_a}}(\hat{r}_a) \end{aligned}$$

where

$$\chi_{\ell_a j_a}(k_a, r_a\rightarrow\infty) \longrightarrow e^{i\sigma_{\ell_a}}[F_{\ell_a} + C_{\ell_a j_a}(G_{\ell_a} + iF_{\ell_a})]$$

In the exit channel, a ket vector becomes

$$\begin{aligned} |\Psi^{(-)}_{M_B m_b}> &= \mathcal{T}(-)^{I_B+M_B+s_b+m_b}\ |\Psi^{(+)}_{-M_B,-m_b}> \\ &= \mathcal{T}(-)^{I_B+M_B+s_b+m_b} \sum_{m'_b}\chi^{(+)}_{-m_b,-m'_b}(\vec{k}_b,\vec{r}_b)\phi_{I_B,-M_B}\varphi_{s_b,-m'_b} \end{aligned}$$

where $\mathcal{T}$ is the time-reversal operator (See Sec. VII-A1.). We have

$$\begin{aligned} \mathcal{T}\phi_{I_B,-M_B} &= (-)^{I_B+M_B}\phi_{I_B M_B} \\ \mathcal{T}\varphi_{s_b,-m'_b} &= (-)^{s_b+m'_b}\varphi_{s_b m'_b} \end{aligned}$$

$$\chi^{(+)}_{-m_b,-m'_b}(\vec{k}_b,\vec{r}_b) = \frac{4\pi}{k_b r_b} \sum_{\ell_b j_b m_{\ell_b} m'_{\ell_b}} (\ell_b m_{\ell_b} s_b, -m_b|j_b m_{j_b})(\ell_b m'_{\ell_b} s_b, -m'_b|j_b m_{j_b}) \times i^{\ell_b} \chi_{\ell_b j_b}(k_b,r_b)\, Y^{(T)}_{\ell_b \tilde{m}_{\ell_b}}(\hat{k}_b)\, Y_{\ell_b m'_{\ell_b}}(\hat{r}_b)$$

$$\mathcal{T}\chi^{(+)}_{-m_b,-m'_b}(\vec{k}_b,\vec{r}_b) = \frac{4\pi}{k_b r_b} \sum_{\ell_b j_b m_{\ell_b} m'_{\ell_b}} (\ell_b m_{\ell_b} s_b, -m_b|j_b m_{j_b})(\ell_b m'_{\ell_b} s_b, -m'_b|j_b m_{j_b}) \times (-i)^{\ell_b} \chi^*_{\ell_b j_b}(k_b,r_b)\, Y_{\ell_b m_{\ell_b}}(-\hat{k}_b)\, (-)^{\ell_b - m'_{\ell_b}} Y_{\ell_b,-m'_{\ell_b}}(\hat{r}_b)$$

We thus obtain

$$|\Psi^{(-)}_{M_B m_b}> = \sum \frac{4\pi}{k_b r_b}(-i)^{\ell_b}(-)^{m'_b - m_b}(\ell_b m_{\ell_b} s_b, -m_b|j_b m_{j_b})(\ell_b m'_{\ell_b} s_b, -m'_b|j_b m_{j_b}) \times \chi^*_{\ell_b j_b}(k_b,r_b)\, Y_{\ell_b m_{\ell_b}}(-\hat{k}_b)\, Y_{\ell_b \tilde{m}'_{\ell_b}}(\hat{r}_b)|\phi_{I_B M_B}\varphi_{s_b,m'_b}>$$

For a bra vector,

$$<\Psi^{(-)}_{M_B m_b}| = \sum_{m'_b} <\phi_{I_B M_B}\varphi_{s_b m'_b}|\chi^{(-)*}_{m'_b m_b}(\vec{k}_b,\vec{r}_b)$$

We thus have a relation, as seen in the previous section

$$\chi^{(-)*}_{m'_b m_b}(\vec{k}_b,\vec{r}_b) = (-)^{m'_b - m_b}\ \chi^{(+)}_{-m'_b,-m_b}(-\vec{k}_b,\vec{r}_b)$$

4.2 DWBA transition amplitudes

Using the ket and bra vectors in the preceding section, the transition amplitude of the DWBA can be written as

$$\begin{aligned} T_{M_B m_b; M_A m_a} &= <\Psi^{(-)}_{M_B m_b}|W|\Psi^{(+)}_{M_A m_a}> \\ &= J \sum_{m'_a m'_b} \int\!\!\int d\vec{r}_b d\vec{r}_a \chi^{(-)*}_{m'_b m_b}(\vec{k}_b,\vec{r}_b) < I_B M_B, s_b m_b|W|I_A M_A, s_a m_a > \chi^{(+)}_{m'_a m_a}(\vec{k}_a,\vec{r}_a) \end{aligned}$$

where J is the Jacobian needed in transforming the *natural* coordinate system to that described by r_a and r_b.

In a stripping (pick-up) reaction which is seen from the point of view of the *lighter (heavier)* system a and/or b (A and/or B), one can write $a = b + x$ and $B = A + x$, if the transferred particle is denoted as x. We then choose the coordinate system explicitly as is given in Fig.1. Throughout this lecture the heavier (lighter) system will be called system 1 (system 2) and thus the coordinate r_{xA} (r_{xb}) will be called as $r_1(r_2)$. We will first work with the post representation.

$$\vec{r}_{xA} = \vec{r}_1, \quad \vec{r}_{xb} = \vec{r}_2, \quad \vec{r}_{bB} = \vec{r}_b, \quad \vec{r}_{aA} = \vec{r}_a$$

They are related as

$$\begin{aligned} \vec{r}_2 &= \frac{m_B m_a}{m_x(m_a + m_A)}(\vec{r}_a - \frac{m_A}{m_B}\vec{r}_b) \\ \vec{r}_1 &= \frac{m_B m_a}{m_x(m_a + m_A)}(\vec{r}_a - \frac{m_b}{m_a}\vec{r}_b) \\ d\vec{r}_1 &= J d\vec{r}_a, \quad J = (m_B m_a/m_x(m_a + m_A))^3 \end{aligned}$$

where m denotes the mass, and J the Jacobian.

The interaction potential W in the post form becomes

$$W_\beta = W_{bB} = W_{bx} + W_{bA} \approx W_{bx} = W(r_2)$$

where W_{bA} may be put in the U_β.

The internal states may be written as

$$\begin{aligned}
\phi_{I_B M_B} &= \phi_{I_B M_B}(\vec{\zeta}_A, \vec{\zeta}_x, \vec{r}_1) \\
\varphi_{s_b m_b} &= \varphi_{s_b m_b}(\vec{\zeta}_b) \\
\phi_{I_A M_A} &= \phi_{I_A M_A}(\vec{\zeta}_A) \\
\varphi_{s_a m_a} &= \varphi_{s_a m_a}(\vec{\zeta}_b, \vec{\zeta}_x, \vec{r}_2)
\end{aligned}$$

where ζ denotes the internal coordinates of each nucleus.

We now evaluate the nuclear matrix elements for the stripping reaction in the post form explicitly; then the kernel $I(\vec{r}_1, \vec{r}_2)$ becomes

$$\begin{aligned}
I(\vec{r}_1, \vec{r}_2) &= < I_B M_B, s_b m_b | W | I_A M_A, s_a m_a > \\
&= < \phi_{I_B M_B} \varphi_{s_b m_b} | W | \phi_{I_A M_A} \varphi_{s_a m_a} > \\
&= \int d\vec{\zeta}_x < \phi_{I_B M_B} | \phi_{I_A M_A} >< \varphi_{s_b m_b} | W | \varphi_{s_a m_a} >
\end{aligned}$$

$$\begin{aligned}
I_{BA} &= < \phi_{I_B M_B} | \phi_{I_A M_A} > \\
&= \int d\vec{\zeta}_A \phi^*_{I_B M_B}(\vec{\zeta}_A, \vec{\zeta}_x, \vec{r}_1) \phi_{I_A M_A}(\vec{\zeta}_A) \\
I_{ba} &= < \varphi_{s_b m_b} | W | \varphi_{s_a m_a} > \\
&= \int d\vec{\zeta}_b \varphi^*_{s_b m_b}(\vec{\zeta}_b) W(r_2) \varphi_{s_a m_a}(\vec{\zeta}_b, \vec{\zeta}_x, \vec{r}_2)
\end{aligned}$$

We have

$$\begin{aligned}
\phi^*_{I_B M_B}(\vec{\zeta}_A, \vec{\zeta}_x, \vec{r}_1) &= \sum_{I_A I_x \ell_1 j, M_A m_j} C^{(1)}_{I_B I_A j \ell_1 n_1 I_x \alpha_x} \left[\phi_{\ell_1 \tilde{n_1}}(\vec{r}_1) \phi_{I_x \tilde{\alpha_x}}(\vec{\zeta}_x) \right]_{j m_j} \\
&\qquad \times \phi^*_{I_A M_A}(\vec{\zeta}_A) (I_A M_A j m_j | I_B M_B) \\
&= \sum_{I_A I_x \ell_1 j, m's} C^{(1)}_{I_B I_A j \ell_1 n_1 I_x \alpha_x} \sum_{m_1 M_x} (\ell_1 m_1 I_x M_x | j m_j) \\
&\qquad \times \ \phi_{\ell_1 \tilde{m_1} n_1}(\vec{r}_1) \phi_{I_x \tilde{M_x} \alpha_x}(\vec{\zeta}_x) \phi^*_{I_A M_A}(\vec{\zeta}_A) (I_A M_A j m_j | I_B M_B)
\end{aligned}$$

where $\phi_{I_x \tilde{\alpha_x}}(\vec{\zeta}_x)$ stands for the internal wave function of x, and α_x labels any additional quantum numbers needed. $\phi_{\ell_1 m_1 n_1}(\vec{r}_1)$ accounts for the motion of x relative to the core A. If the transferred particle is a nucleon, the $\phi_{\ell_1 m_1 n_1}(\vec{r}_1)$ is nothing but the orbital part of the wave function $u_{\ell_1 n_1}(r_1)$. The coefficient $C^{(1)}$ denotes a coefficient of fractional parentage (cfp). Integrating over ζ_A gives

$$\begin{aligned}
I_{BA} &= \sum_{I_A I_x \ell_1 j, m's} C^{(1)}_{I_B I_A j \ell_1 n_1 I_x \alpha_x} (\ell_1 m_1 I_x M_x | j m_j) \\
&\qquad \times \ \phi_{\ell_1 \tilde{m_1} n_1}(\vec{r}_1) \phi_{I_x \tilde{M_x} \alpha_x}(\vec{\zeta}_x) (I_A M_A j m_j | I_B M_B)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi_{s_a m_a}(\vec{\zeta}_b, \vec{\zeta}_x, \vec{r}_2) &= \sum_{s_b I_x \ell_2 s, m_b m_s} C^{(2)}_{s_a s_b s \ell_2 n_2 I_x \alpha_x} \left[\phi_{\ell_2 n_2}(\vec{r}_2)\phi_{I_x\alpha_x}(\vec{\zeta}_x)\right]_{s m_s} \\
&\qquad \times \varphi_{s_b m_b}(\vec{\zeta}_b)(s_b m_b s m_s|s_a m_a) \\
&= \sum_{s_b I_x \ell_2 s, m_b m_s} C^{(2)}_{s_a s_b s \ell_2 n_2 I_x \alpha_x} \sum_{m_2 M_x} (\ell_2 m_2 I_x M_x|s m_s) \\
&\qquad \times\ \phi_{\ell_2 m_2 n_2}(\vec{r}_2)\phi_{I_x M_x \alpha_x}(\vec{\zeta}_x)\varphi_{s_b m_b}(\vec{\zeta}_b)(s_b m_b s m_s|s_a m_a)
\end{aligned}$$

Integrating over ζ_b gives

$$\begin{aligned}
I_{ba} &= \sum_{s_b I_x \ell_2 s, m' s} C^{(2)}_{s_a s_b s \ell_2 n_2 I_x \alpha_x}(\ell_2 m_2 I_x M_x|s m_s) \\
&\qquad \times\ W(r_2)\phi_{\ell_2 m_2 n_2}(\vec{r}_2)\phi_{I_x M_x \alpha_x}(\vec{\zeta}_x)(s_b m_b s m_s|s_a m_a)
\end{aligned}$$

where $C^{(2)}$ is the corresponding cfp for the projectile system ($a = b + x$).

Finally, integrating over ζ_x gives the kernel $I(\vec{r}_1, \vec{r}_2)$ as

$$\begin{aligned}
I(\vec{r}_1, \vec{r}_2) &= < I_B M_B, s_b m_b|W|I_A M_A, s_a m_a > \\
&= \sum_{j s \ell_1 \ell_2 n_1 n_2 I_x \alpha_x m_1 m_2 M_x} C^{(1)}C^{(2)}(I_A M_A j m_j|I_B M_B)(s_b m_b s m_s|s_a m_a) \\
&\quad \times (\ell_1 m_1 I_x M_x|j m_j)(\ell_2 m_2 I_x M_x|s m_s)\phi_{\ell_1 \tilde{m_1} n_1}(\vec{r}_1)W(r_2)\phi_{\ell_2 m_2 n_2}(\vec{r}_2)
\end{aligned}$$

We here define the angular momentum transfers as

$$\vec{I}_B - \vec{I}_A = \vec{j}, \quad \vec{s}_b - \vec{s}_a = \vec{s}, \quad \vec{s} + \vec{j} = \vec{\ell}$$

We rearrange the Clebsch-Gordan coefficients

$$\begin{aligned}
CG_1 &= (I_A M_A j m_j|I_B M_B) \\
&= (-)^{I_A - M_A}(\hat{I}_B/\hat{j})(I_A M_A I_B, -M_B|j, -m_j) \\
CG_2 &= (s_b m_b s m_s|s_a m_a) \\
&= (-)^{s_b - m_b}(\hat{s}_a/\hat{s})(s_a m_a s_b, -m_b|s, m_s) \\
CG_3 &= (\ell_1 m_1 I_x M_x|j m_j)(\ell_2 m_2 I_x M_x|s m_s) \\
&= (-)^{\ell_2 - m_2}(-)^{\ell_2 + s - I_x}(\hat{s}/\hat{I}_x)(\ell_1 m_1 I_x M_x|j m_j)(\ell_2, -m_2 s m_s|I_x M_x) \\
&= (-)^{\ell_2 - m_2}(-)^{\ell_2 + s - I_x}(\hat{s}/\hat{I}_x)\sum_{\ell} \hat{\ell}\hat{I}_x(\ell_1 m_1 \ell_2, -m_2|\ell m_\ell) \\
&\qquad \times\ (\ell m_\ell s m_s|j m_j)W(\ell_1 \ell_2 j s|\ell I_x) \\
&= (-)^{\ell_2 - m_2}(-)^{\ell_2 + s - I_x}\sum_{\ell} \hat{s}\hat{\ell}(-)^{\ell_1 + \ell_2 - \ell}(\ell_1, -m_1 \ell_2 m_2|\ell, -m_\ell)(-)^{s + m_s} \\
&\qquad \times\ (\hat{j}/\hat{\ell})(j, -m_j s m_s|\ell, -m_\ell)W(\ell_1 \ell_2 j s; \ell I_x)
\end{aligned}$$

where $\hat{I} = \sqrt{2I+1}$. Now we have

$$\begin{aligned}
A &= \left[\phi_{\ell_1 \tilde{n}_1}(\vec{r}_1)\phi_{\ell_2 n_2}(\vec{r}_2)\right]_{\ell \tilde{m}_\ell} \\
&= u^*_{\ell_1 n_1}(r_1)u_{\ell_2 n_2}(r_2)(\ell_1, -m_1 \ell_2 m_2|\ell \tilde{m}_\ell)Y_{\ell_1 m_1}(\hat{r}_1)Y_{\ell_2 m_2}(\hat{r}_2) \\
&= u^*_{\ell_1 n_1}(r_1)u_{\ell_2 n_2}(r_2)(-)^{\ell - m_\ell}(\ell_1, -m_1 \ell_2 m_2|\ell, -m_\ell)(-)^{\ell_1 + m_1}Y^*_{\ell_1 m_1}(\hat{r}_1)Y_{\ell_2 m_2}(\hat{r}_2) \\
&= \phi_{\ell_1 \tilde{n}_1}(\vec{r}_1)\phi_{\ell_2 n_2}(\vec{r}_2)(-)^{\ell_1 + \ell - m_\ell + m_1}(\ell_1, -m_1 \ell_2 m_2|\ell, -m_\ell)
\end{aligned}$$

Combining together yields the kernel I

$$
\begin{aligned}
I &= \sum_{j\ell s}\{\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}\}(-)^{I_A - M_A + s_b - m_b + s + m_s} \\
&\quad \times (j, -m_j s m_s|\ell, -m_\ell)\ \hat{I}_B \hat{s}_a (I_A M_A I_B, -M_B|j, -m_j)(s_a m_a s_b, -m_b|s m_s)
\end{aligned}
$$

where

$$
\begin{aligned}
d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} &= \sum_{I_x \alpha_x} C^{(1)}_{I_B I_A j \ell_1 n_1 I_x \alpha_x} C^{(2)}_{s_a s_b s \ell_2 n_2 I_x \alpha_x} (-)^{\ell_2 + s - I_x}\ W(\ell_1 \ell_2 j s; \ell I_x) \\
f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell} &= \left[\phi_{\ell_1 \tilde{n}_1}(\vec{r}_1)\phi_{\ell_2 n_2}(\vec{r}_2)\right]_{\ell \tilde{m}_\ell} W(r_2)
\end{aligned}
$$

We can evaluate it for the pick-up reaction in the prior form, and obtain (See Tamura Sec. 2.1.)

$$
\begin{aligned}
I &= \sum_{j\ell s}\{\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}\}(-)^{I_A - M_A + s_b - m_b + s + m_s} \\
&\quad \times (j, m_j s, -m_s|\ell, -m_\ell)\epsilon^{js}_{AaBb} \hat{I}_A \hat{s}_b (I_A M_A I_B, -M_B|j, m_j)(s_a m_a s_b, -m_b|s, -m_s)
\end{aligned}
$$

where

$$
\epsilon^{js}_{AaBb} = (-)^{I_B + j - I_A + s_a + s - s_b}
$$

We can express these two different reactions in a fashion,

$$
\begin{aligned}
I &= \sum_{j\ell s}\{\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2)\}(-)^{I_A - M_A + s_b - m_b + s + m_s} \\
&\quad \times (j, -m_j s m_s|\ell, -m_\ell)\epsilon^{js}_{AaBb}\ \hat{I}_> \hat{s}_> (I_A M_A I_B, -M_B|j, -m_j)(s_a m_a s_b, -m_b|s m_s)
\end{aligned}
$$

$$
\begin{aligned}
d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} &= \sum_{I_x \alpha_x} C^{(1)}_{I_> I_< j \ell_1 n_1 I_x \alpha_x} C^{(2)}_{s_> s_< s \ell_2 n_2 I_x \alpha_x} (-)^{\ell_2 + s - I_x}\ W(\ell_1 \ell_2 j s; \ell I_x) \\
f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell} &= \left[\phi_{\ell_1 \tilde{n}_1}(\vec{r}_1)\phi_{\ell_2 n_2}(\vec{r}_2)\right]_{\ell \tilde{m}_\ell} W(r_2)
\end{aligned}
$$

The following definitions are introduced;
Stripping reaction ($a > b,\ A < B$):
$s_> = s_a,\ \ s_< = s_b,\ \ I_> = I_B,\ \ I_< = I_A,\ \ \epsilon^{js}_{AaBb} = 1$
Pick-up reaction ($a < b,\ A > B$):
$s_> = s_b,\ \ s_< = s_a,\ \ I_> = I_A,\ \ I_< = I_B,\ \ \epsilon^{js}_{AaBb} = (-)^{I_B + j - I_A + s_a + s - s_b}$

The transition amplitude of the DWBA can then be written as

$$
\begin{aligned}
T &= T_{M_B m_b; M_A m_a} = < \Psi^{(-)}_{M_B m_b}|W|\Psi^{(+)}_{M_A m_a} > \\
&= J \sum_{m'_a m'_b} \int\int d\vec{r}_b d\vec{r}_a \chi^{(-)*}_{m'_b m_b}(\vec{k}_b, \vec{r}_b) < I_B M_B, s_b m_b|W|I_A M_A, s_a m_a > \chi^{(+)}_{m'_a m_a}(\vec{k}_a, \vec{r}_a) \\
&= J \sum_{m'_a m'_b} \frac{(4\pi)^2}{k_a k_b} \int\int d\vec{r}_b d\vec{r}_a \sum_{j\ell s}\{\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s} f_{\ell \tilde{m}_\ell}\} r_a^{-1} r_b^{-1} \epsilon^{js}_{AaBb} (-)^{I_A - M_A + s_b - m_b + s + m_s} \hat{I}_> \hat{s}_> \\
&\quad \times (I_A M_A I_B, -M_B|j, -m_j)(s_a m'_a s_b, -m'_b|s m_s)(j, -m_j s m_s|\ell, -m_\ell) Y_{\ell_b \tilde{m}_{\ell_b}}(-\hat{k}_b) Y_{\ell_a m_{\ell_a}}(\hat{k}_a) \\
&\quad \times (-)^{-m_b} (\ell_b m_{\ell_b} s_b, -m_b|j_b m_{j_b})(\ell_a m_{\ell_a} s_a m_a|j_a m_{j_a})(-)^{m'_b}(\ell_b m'_{\ell_b} s_b, -m'_b|j_b m_{j_b}) \\
&\quad \times (\ell_a m'_{\ell_a} s_a m'_a|j_a m_{j_a}) i^{\ell_a + \ell_b} \chi_{\ell_b j_b}(k_b, r_b)\ \chi_{\ell_a j_a}(k_a, r_a)(\ell_a m'_{\ell_a} \ell_b m'_{\ell_b}|\ell m_\ell)[Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell m_\ell}
\end{aligned}
$$

From the Racah algebra, we have

$$
\begin{aligned}
VC \equiv & \sum_{m'_a m'_b m'_{\ell_a} m'_{\ell_b} m_\ell m_s} (-)^{s+m_s} (\ell_a m'_{\ell_a} s_a m'_a | j_a m_{j_a})(\ell_b m'_{\ell_b} s_b, -m'_b | j_b m_{j_b})(\ell_a m'_{\ell_a} \ell_b m'_{\ell_b} | \ell m_\ell) \\
& \times (s_a m'_a s_b, -m'_b | s m_s)(j, -m_j s m_s | \ell, -m_\ell) \\
= & \frac{\hat{\ell}}{\hat{j}} \hat{\ell} \hat{s} \hat{j}_a \hat{j}_b \left\{ \begin{array}{ccc} \ell_a & s_a & j_a \\ \ell_b & s_b & j_b \\ \ell & s & j \end{array} \right\} (j_a m_{j_a} j_b m_{j_b} | j m_j)
\end{aligned}
$$

where we use a relation

$$
(j, -m_j s m_s | \ell, -m_\ell) = (-)^{s+m_s} \frac{\hat{\ell}}{\hat{j}} (\ell m_\ell s m_s | j m_j)
$$

If we choose $\vec{k}_a \parallel z$-axis and the polar angle of $\vec{k}_b$ is called θ, then

$$
\begin{aligned}
Y_{\ell_a m_{\ell_a}}(0) &= i^{-\ell_a} \frac{\hat{\ell}_a}{\sqrt{4\pi}} \delta_{m_{\ell_a},0} \\
Y_{\ell_b \tilde{m}_{\ell_b}}(-\hat{k}_b) &= i^{-\ell_b} (-)^{\ell_b + m_{\ell_b}} \frac{\hat{\ell}_b}{\sqrt{4\pi}} G_{\ell_b, -m_{\ell_b}} P_{\ell_b m_{\ell_b}}(\theta) \\
G_{\ell - m} &= (-)^{(m-|m|)/2} \left[\frac{(\ell - |m|)!}{(\ell + |m|)!} \right]^{1/2}
\end{aligned}
$$

Therefore, we finally obtain

$$
\begin{aligned}
T &= T_{M_B m_b; M_A m_a} \\
&= J \frac{4\pi}{k_a k_b} \hat{I}_> \hat{s}_> \epsilon^{js}_{AaBb} (-)^{I_A - M_A + s_b - m_b} \\
& \times \sum_{j\ell s} (I_A M_A I_B, -M_B | j, -m_j)(j_a m_{j_a} j_b m_{j_b} | j m_j)(\ell_b m_{\ell_b} s_b, -m_b | j_b m_{j_b})(\ell_a 0 s_a m_a | j_a m_{j_a}) \\
& \times \frac{\hat{\ell}^2 \hat{s} \hat{\ell}_a \hat{\ell}_b \hat{j}_a \hat{j}_b}{\hat{j}} \left\{ \begin{array}{ccc} \ell_a & s_a & j_a \\ \ell_b & s_b & j_b \\ \ell & s & j \end{array} \right\} (-)^{\ell_b + m_{\ell_b}} G_{\ell_b, -m_{\ell_b}} P_{\ell_b m_{\ell_b}}(\theta) I^{j\ell s}_{\ell_b j_b, \ell_a j_a}
\end{aligned}
$$

where the dynamical factor $I^{j\ell s}_{\ell_b j_b, \ell_a j_a}$ becomes

$$
\begin{aligned}
I^{j\ell s}_{\ell_b j_b, \ell_a j_a} &= \frac{1}{2\ell + 1} \sum_{m_\ell} J \int \int d\vec{r}_b d\vec{r}_a \\
& \times \{ \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) \} (r_a r_b)^{-1} \chi_{\ell_b j_b}(k_b, r_b)\ \chi_{\ell_a j_a}(k_a, r_a) [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell m_\ell}
\end{aligned}
$$

The factor $(2\ell + 1)^{-1} \sum_{m_\ell}$ can be replaced by unity, if it is remembered that summand is independent of the value of m_ℓ.

In order to obtain the final transition amplitudes, we need to change coordinates $(\vec{r}_1, \vec{r}_2) \rightarrow (\vec{r}_a, \vec{r}_b)$ using the relations earlier, which will be done in the form factor calculations.

4.3 DWBA differential cross sections

DWBA differential cross sections can be written as

$$\frac{d\sigma^{DWBA}}{d\Omega} = \frac{\mu_a\mu_b}{(2\pi\hbar^2)^2}\frac{k_b}{k_a}\frac{1}{(2I_A+1)(2s_a+1)}\sum_{M_AM_Bm_am_b}|T_{M_Bm_b;M_Am_a}(\theta)|^2$$

Summing over M_A, M_B gives

$$\begin{aligned} A &= \sum_{M_AM_B}[(-)^{I_A-M_A}]^2(I_AM_AI_B,-M_B|j,-m_j)(I_AM_AI_B,-M_B|j',-m_j') \times \text{ something} \\ &= \delta_{jj'}\delta_{m_jm_j'} \times \text{ something} \end{aligned}$$

These delta functions makes $\sum_j$ take out from $|...|^2$. Thus we have

$$\begin{aligned} \frac{d\sigma^{DWBA}}{d\Omega} &= \frac{\mu_a\mu_b}{(2\pi\hbar^2)^2}\frac{k_b}{k_a}(\frac{\hat{I}_>\hat{s}_>}{\hat{I}_A\hat{s}_a})^2(\frac{4\pi}{k_ak_b})^2\sum_{jm_jm_am_b} \\ &\times |\sum_{\ell s}(j_am_{j_a}j_bm_{j_b}|jm_j)(\ell_bm_{\ell_b}s_b,-m_b|j_bm_{j_b})(\ell_a0s_am_a|j_am_{j_a})\epsilon^{js}_{AaBb}(-)^{s_b-m_b} \\ &\times \frac{\hat{\ell}^2\hat{s}\hat{\ell}_a\hat{\ell}_b\hat{j}_a\hat{j}_b}{\hat{j}}\begin{Bmatrix} \ell_a & s_a & j_a \\ \ell_b & s_b & j_b \\ \ell & s & j \end{Bmatrix} I^{j\ell s}_{\ell_bj_b,\ell_aj_a}(-)^{\ell_b+m_{\ell_b}}G_{\ell_b,-m_{\ell_b}}P_{\ell_bm_{\ell_b}}(\theta)|^2 \end{aligned}$$

4.4 DWBA without spin-orbit interaction

Without spin-orbit interaction we may have

$$m_{\ell_a} = m'_{\ell_a}, \quad m_{\ell_b} = m'_{\ell_b}, \quad m_a = m'_a, \quad m_b = m'_b$$

Hence the CG coefficients in the distorted waves become

$$\sum_{m_{\ell_a}m'_{\ell_a}}(\ell_am_{\ell_a}s_am_a|j_am_{j_a})(\ell_am'_{\ell_a}s_am'_a|j_am_{j_a}) = 1$$
$$\sum_{m_{\ell_b}m'_{\ell_b}}(\ell_bm_{\ell_b}s_b,-m_b|j_bm_{j_b})(\ell_bm'_{\ell_b}s_b,-m'_b|j_bm_{j_b}) = 1$$

The distorted waves may be written as

$$\begin{aligned} \chi^{(+)}_{m_am'_a}(\vec{k}_a,\vec{r}_a) &= \frac{4\pi}{k_ar_a}\sum_{\ell_a}i^{\ell_a}\chi_{\ell_a}(k_a,r_a)\ Y_{\ell_a\tilde{m}_{\ell_a}}(\hat{k}_a)Y_{\ell_am'_{\ell_a}}(\hat{r}_a) \\ \chi^{(-)*}_{m_bm'_b}(\vec{k}_b,\vec{r}_b) &= \frac{4\pi}{k_br_b}\sum_{\ell_b}i^{\ell_b}\chi_{\ell_a}(k_a,r_a)\ Y_{\ell_b\tilde{m}_{\ell_b}}(-\hat{k}_b)Y_{\ell_bm'_{\ell_b}}(\hat{r}_b) \end{aligned}$$

Remembering that $m_{\ell_a} = 0$, we obtain the transition amplitude

$$\begin{aligned} T &= T_{M_Bm_b;M_Am_a} \\ &= J\frac{4\pi}{k_ak_b}\hat{I}_>\hat{s}_>\epsilon^{js}_{AaBb}\sum_{j\ell s}(-)^{I_A-M_A+s_b-m_b+s+m_s}(I_AM_AI_B,-M_B|jm_j)(s_am_as_b,-m_b|sm_s) \\ &\quad \times (jm_jsm_s|\ell,-m_\ell)\hat{\ell}_a\hat{\ell}_b(\ell_a0\ell_bm_{\ell_b}|\ell m_\ell)(-)^{\ell_b+m_{\ell_b}}G_{\ell_b,-m_{\ell_b}}P_{\ell_bm_{\ell_b}}(\theta)I^{j\ell s}_{\ell_b,\ell_a} \end{aligned}$$

where the dynamical factor $I^{j\ell s}_{\ell_b,\ell_a}$ is the same as $I^{j\ell s}_{\ell_bj_b,\ell_aj_a}$ in Sec. VII-6 except that j_a and j_b dependence of χ is suppressed.

To obtain the differential cross section we use again

$$
\begin{aligned}
A &= \sum_{M_A M_B} [(-)^{I_A - M_A}]^2 (I_A M_A I_B, -M_B | j, m_j)(I_A M_A I_B, -M_B | j' m_j') \ \times \ \text{something} \\
&= \delta_{jj'} \delta_{m_j m_j'} \ \times \ \text{something} \\
B &= \sum_{m_a m_b} [(-)^{s_a - m_b}]^2 (s_a m_a s_b, -m_b | s m_s)(s_a m_a s_b, -m_b | s' m_s') \ \times \ \text{something} \\
&= \delta_{ss'} \delta_{m_s m_s'} \ \times \ \text{something} \\
C &= \sum_{m_j m_s} [(-)^{s + m_s}]^2 (j m_j s m_s | \ell, -m_\ell)(j m_j s m_s | \ell', -m_\ell') \ \times \ \text{something} \\
&= \delta_{\ell\ell'} \delta_{m_\ell m_\ell'} \ \times \ \text{something}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{d\sigma^{DWBA}}{d\Omega} &= \frac{\mu_a \mu_b}{(2\pi\hbar^2)^2} \frac{k_b}{k_a} (\frac{\hat{I}_> \hat{s}_>}{\hat{I}_A \hat{s}_a})^2 (\frac{4\pi}{k_a k_b})^2 \\
&\times \sum_{j\ell s m_{\ell_b}} | \sum_{\ell_a \ell_b} \hat{\ell}_a \hat{\ell}_b (\ell_a 0 \ell_b m_{\ell_b} | \ell m_\ell) I^{j\ell s}_{\ell_b, \ell_a} (-)^{\ell_b + m_{\ell_b}} G_{\ell_b, -m_{\ell_b}} P_{\ell_b m_{\ell_b}}(\theta) |^2
\end{aligned}
$$

Note that $(\ell_a 0 \ell_b m_{\ell_b} | \ell m_\ell)$ gives $m_{\ell_b} = m_\ell$.

5 Form factors for stripping reactions

5.1 Exact Finite Range (EFR) form factors

In this section we shall discuss in some detail how to carry out the six-dimensional integral that appeared in the dynamical kernel $I^{j\ell s}_{\ell_b j_b,\ell_a j_a}$. for stripping reactions. If this integral is carried out exactly without introducing any approximation, what one has is called an *exact-finite-range* (EFR) calculation. The next section introduce a few approximate ways of evaluating this integral, the re-coil (NR) and zero-range (ZR) approximations. In other words, to carry-out EFR calculations means to take into account exactly the finite-range and recoil effects in the calculations.

In carrying out this integral, the first thing we have to do is to transform a function of $(\vec{r}_1, \vec{r}_2)$ to a function of $(\vec{r}_a, \vec{r}_b)$. As seen in Fig.1, two coordinates are related as

$$\begin{aligned}
\vec{r}_1 &= \frac{m_B m_a}{m_x(m_a+m_A)}(\vec{r}_a - \frac{m_b}{m_a}\vec{r}_b) \equiv s_1\vec{r}_a + t_1\vec{r}_b \\
\vec{r}_2 &= \frac{m_B m_a}{m_x(m_a+m_A)}(\vec{r}_a - \frac{m_A}{m_B}\vec{r}_b) \equiv s_2\vec{r}_a + t_2\vec{r}_b
\end{aligned}$$

where

$$\begin{aligned}
s_1 &= \frac{m_B m_a}{m_x(m_a+m_A)}, \qquad t_1 = \frac{-m_B m_b}{m_x(m_a+m_A)} \\
s_2 &= \frac{m_A m_a}{m_x(m_a+m_A)}, \qquad t_2 = \frac{-m_B m_a}{m_x(m_a+m_A)}
\end{aligned}$$

and the Jacobian is

$$d\vec{r}_1 = J d\vec{r}_a, \quad J = s_1^3 = (m_B m_a / m_x(m_a+m_A))^3$$

where m denotes the mass.

Now we write $f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}$ as

$$f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) = \sum_{\ell_a \ell_b} F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a)[Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell}$$

The new function can be obtained by an inverse transformation as

$$\begin{aligned}
F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a) &= \frac{1}{\hat{\ell}^2}\sum_{m_\ell}\int\int f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2)\ [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} d\hat{r}_a d\hat{r}_b \\
&= \frac{1}{\hat{\ell}^2}\sum_{m_\ell}\int\int r_1^{-\ell_1}\phi_{\ell_1 n_1}(r_1) r_2^{-\ell_2}\phi_{\ell_2 n_2}(r_2) W(r_2)\ [\mathcal{Y}_{\ell_1}(\vec{r}_1)\mathcal{Y}_{\ell_2}(\vec{r}_2)]_{\ell \tilde{m}_\ell} \\
&\qquad \times\ [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} d\hat{r}_a d\hat{r}_b
\end{aligned}$$

where $\mathcal{Y}_{\ell m}(\vec{r}) \equiv r^\ell Y_{\ell m}(\hat{r})$ is called the *solid harmonics* (See Sec. VII-A2.). We then expand radial functions, $r_1^{-\ell_1}\phi_{\ell_1 n_1}(r_1) r_2^{-\ell_2}\phi_{\ell_2 n_2}(r_2) W(r_2)$, in a series of Legendre polynomials (See Sec. VII-A3.),

$$\begin{aligned}
A &\equiv r_1^{-\ell_1}\phi_{\ell_1 n_1}(r_1) r_2^{-\ell_2}\phi_{\ell_2 n_2}(r_2) W(r_2) \\
&= \sum_k \frac{2k+1}{2} G^{\ell_1 n_1 \ell_2 n_2}_k(r_a, r_b) P_k(\mu) \\
&= 2\pi \sum_{kq} G^{\ell_1 n_1 \ell_2 n_2}_k(r_a, r_b) Y_{kq}(\hat{r}_a) Y_{kq}(\hat{r}_b)
\end{aligned}$$

where

$$G_k^{\ell_1 n_1 \ell_2 n_2}(r_a, r_b) = \int_{-1}^{1} r_1^{-\ell_1} \phi_{\ell_1 n_1}(r_1) r_2^{-\ell_2} \phi_{\ell_2 n_2}(r_2) W(r_2) P_k(\mu) d\mu$$

and the parameter μ denotes the cosine of angle between 2 vectors $\hat{r}_a$ and $\hat{r}_b$.

From the properties of solid harmonics, we can express solid harmonics $Y_{\ell_1 m_1}(\vec{r}_1)$ and $Y_{\ell_2 m_2}(\vec{r}_2)$in terms of $\vec{r}_a$ and $\vec{r}_b$.

$$\begin{aligned}
Y_{\ell_1 m_{\ell_1}}(\vec{r}_1) &= \sum_{\lambda_1 \lambda_1'} \sqrt{4\pi} D_{\ell_1 \lambda_1 \lambda_1'} (s_1 r_a)^{\lambda_1} (t_1 r_b)^{\lambda_1'} \ [Y_{\lambda_1}(\hat{r}_a) Y_{\lambda_1'}(\hat{r}_b)]_{\ell_1 m_{\ell_1}} \\
D_{\ell_1 \lambda_1 \lambda_1'} &= \delta_{\lambda_1 + \lambda_1', \ell_1} [\frac{(2\ell_1 + 1)!}{(2\lambda_1 + 1)!(2\lambda_1' + 1)!}]^{1/2} \\
Y_{\ell_2 m_{\ell_2}}(\vec{r}_2) &= \sum_{\lambda_2 \lambda_2'} \sqrt{4\pi} D_{\ell_2 \lambda_2 \lambda_2'} (s_2 r_a)^{\lambda_2} (t_2 r_b)^{\lambda_2'} \ [Y_{\lambda_2}(\hat{r}_a) Y_{\lambda_2'}(\hat{r}_b)]_{\ell_2 m_{\ell_2}} \\
D_{\ell_2 \lambda_2 \lambda_2'} &= \delta_{\lambda_2 + \lambda_2', \ell_2} [\frac{(2\ell_2 + 1)!}{(2\lambda_2 + 1)!(2\lambda_2' + 1)!}]^{1/2}
\end{aligned}$$

After putting them these relations to F, one can perform angular integrations and obtain the function $F_{\ell \ell_a \ell_b}^{\ell_1 n_1 \ell_2 n_2}(r_b, r_a)$

Calculation of $F_{\ell \ell_a \ell_b}^{\ell_1 n_1 \ell_2 n_2}(r_b, r_a)$.

Let us take a break to obtain the explicit form of $F_{\ell \ell_a \ell_b}^{\ell_1 n_1 \ell_2 n_2}(r_b, r_a)$.

$$F_{\ell \ell_a \ell_b}^{\ell_1 n_1 \ell_2 n_2}(r_b, r_a) = \frac{1}{\hat{\ell}^2} \sum_{m_\ell} \int \int A [Y_{\ell_1}(\vec{r}_1) Y_{\ell_2}(\vec{r}_2)]_{\ell \tilde{m}_\ell} \ [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} d\hat{r}_a d\hat{r}_b$$

where

$$\begin{aligned}
A &\equiv r_1^{-\ell_1} \phi_{\ell_1 n_1}(r_1) r_2^{-\ell_2} \phi_{\ell_2 n_2}(r_2) W(r_2) \\
&= 2\pi \sum_{kq} G_k^{\ell_1 n_1 \ell_2 n_2}(r_a, r_b) Y_{kq}(\hat{r}_a) Y_{kq}(\hat{r}_b)
\end{aligned}$$

We now collect all spherical harmonics in F,

$$\begin{aligned}
B &\equiv [Y_{\ell_1}(\vec{r}_1) Y_{\ell_2}(\vec{r}_2)]_{\ell \tilde{m}_\ell} \ [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} Y_{kq}(\hat{r}_a) Y_{kq}(\hat{r}_b) \\
&= \sum_{m's} (\ell_1 m_1 \ell_2 m_2 | \ell \tilde{m}_\ell) Y_{\ell_1 m_1}(\vec{r}_1) Y_{\ell_2 m_2}(\vec{r}_2) (\ell_a m_a \ell_b m_b | \ell \tilde{m}_\ell) Y_{\ell_a m_a}(\hat{r}_a) Y_{\ell_b m_b}(\hat{r}_b) Y_{kq}(\hat{r}_a) Y_{kq}(\hat{r}_b) \\
&= \sum [C'] (\ell_1 m_1 \ell_2 m_2 | \ell, -m_\ell) (\lambda_1 m_{\lambda_1} \lambda_1' m_{\lambda_1'} | \ell_1 m_1) \ (\lambda_2 m_{\lambda_2} \lambda_2' m_{\lambda_2'} | \ell_2 m_2) (\ell_a m_a \ell_b m_b | \ell, -m_\ell) \\
&\quad \times Y_{\lambda_1 m_{\lambda_1}}(\hat{r}_a) \ Y_{\lambda_2 m_{\lambda_2}}(\hat{r}_a) \ Y_{\ell_a m_a}(\hat{r}_a) \ Y_{kq}(\hat{r}_a) \ Y_{\lambda_1' m_{\lambda_1'}}(\hat{r}_b) \ Y_{\lambda_2' m_{\lambda_2'}}(\hat{r}_b) \ Y_{\ell_b m_b}(\hat{r}_b) \ Y_{kq}(\hat{r}_b)
\end{aligned}$$

Now we have

$$\begin{aligned}
Y_{\lambda_1 m_{\lambda_1}}(\hat{r}_a) Y_{\lambda_2 m_{\lambda_2}}(\hat{r}_a) &= \sum_{\Lambda_a, m_{\Lambda_a}} \frac{1}{\sqrt{4\pi}} \frac{\hat{\lambda}_1 \hat{\lambda}_2}{\Lambda_a} (\lambda_1 0 \lambda_2 0 | \Lambda_a 0)(\lambda_1 m_{\lambda_1} \lambda_2 m_{\lambda_2} | \Lambda_a m_{\Lambda_a}) Y_{\Lambda_a m_{\Lambda_a}} \\
Y_{\lambda_1' m_{\lambda_1'}}(\hat{r}_b) Y_{\lambda_2' m_{\lambda_2'}}(\hat{r}_b) &= \sum_{\Lambda_b, m_{\Lambda_b}} \frac{1}{\sqrt{4\pi}} \frac{\hat{\lambda}_1' \hat{\lambda}_2'}{\Lambda_b} (\lambda_1' 0 \lambda_2' 0 | \Lambda_b 0)(\lambda_1' m_{\lambda_1'} \lambda_2' m_{\lambda_2'} | \Lambda_b m_{\Lambda_b}) Y_{\Lambda_b m_{\Lambda_b}}
\end{aligned}$$

Performing integrations over angles $\hat{r}_a$ and $\hat{r}_b$, respectively, give

$$\int d\hat{r}_a\ Y^*_{\ell_a,-m_a} Y_{\Lambda_a m_{\Lambda_a}} Y_{kq}(\hat{r}_a) = \frac{1}{\sqrt{4\pi}} \frac{\hat{\Lambda}_a \hat{k}}{\hat{\ell}_a} (\Lambda_a 0 k 0|\ell_a 0)\ (\Lambda_a m_{\Lambda_a} kq|\ell_a, -m_a)$$

$$\int d\hat{r}_b\ Y^*_{\ell_b,-m_b} Y_{\Lambda_b m_{\Lambda_b}} Y_{kq}(\hat{r}_b) = \frac{1}{\sqrt{4\pi}} \frac{\hat{\Lambda}_b \hat{k}}{\hat{\ell}_b} (\Lambda_b 0 k 0|\ell_b 0)\ (\Lambda_b m_{\Lambda_b} kq|\ell_b, -m_b)$$

We have here used the Condon-Shortly phase such that

$$Y_{\ell m} = (-)^m Y^*_{\ell,-m} = (-)^m i^{-\ell} Y^{*(CS)}_{\ell,-m}$$

Therefore, the general form of $F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a)$ becomes

$$F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a) = \sum [C][CG's] \times\ G^{\ell_1 n_1 \ell_2 n_2}_k (r_a, r_b)$$

where the summation is a collection of

$$[m_\ell, kq, \lambda_1 \lambda'_1 \lambda_2 \lambda'_2, m_1 m_2 m_a m_b, m_{\lambda_1} m_{\lambda'_1} m_{\lambda_2} m_{\lambda'_2}, \Lambda_a \Lambda_b, m_{\Lambda_a} m_{\Lambda_a}]$$

and the factor $[C]$ is

$$\begin{aligned}
[C] &= \frac{2\pi}{\hat{\ell}^2}(4\pi) D_{\ell_1 \lambda_1 \lambda'_1} D_{\ell_2 \lambda_2 \lambda'_2} r_a^{\lambda_1+\lambda_2} r_b^{\lambda'_1+\lambda'_2} s_1^{\lambda_1} t_1^{\lambda'_1} s_2^{\lambda_2} t_2^{\lambda'_2}\ \frac{1}{4\pi} \frac{\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}'_1 \hat{\lambda}'_2 \hat{k}^2}{\hat{\ell}_a \hat{\ell}_b} i^{\ell_1+\ell_2-\ell_a-\ell_b} (-)^{m_a+m_b} \\
&= \frac{1}{2} i^{\ell_1+\ell_2-\ell_a-\ell_b} (-)^{m_\ell} s_1^{\lambda_1} t_1^{\lambda'_1} s_2^{\lambda_2} t_2^{\lambda'_2} D_{\ell_1 \lambda_1 \lambda'_1} D_{\ell_2 \lambda_2 \lambda'_2}\ \frac{\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}'_1 \hat{\lambda}'_2 \hat{k}^2}{\hat{\ell}_a \hat{\ell}_b \hat{\ell}^2} r_a^{\lambda_1+\lambda_2} r_b^{\lambda'_1+\lambda'_2}
\end{aligned}$$

The Clebsch-Gordan coefficients are

$$\begin{aligned}
CG_1 &= (\lambda_1 0 \lambda_2 0|\Lambda_a 0)(\lambda'_1 0 \lambda'_2 0|\Lambda_b 0)(\Lambda_a 0 k 0|\ell_a 0) \\
&\quad \times (\Lambda_b 0 k 0|\ell_b 0) \\
CG_2 &= \sum_{m_1 m_2, m_{\lambda_1} m_{\lambda'_1} m_{\lambda_2} m_{\lambda'_2}} (\ell_1 m_1 \ell_2 m_2|\ell, -m_\ell) \\
&\quad \times (\lambda_1 m_{\lambda_1} \lambda'_1 m_{\lambda'_1}|\ell_1 m_1)(\lambda_2 m_{\lambda_2} \lambda'_2 m_{\lambda'_2}|\ell_2 m_2)\ (\Lambda_a m_{\Lambda_a} kq|\ell_a, -m_a)(\Lambda_b m_{\Lambda_b} kq|\ell_b, -m_b) \\
&= \hat{\ell}_1 \hat{\ell}_2 \hat{\Lambda}_a \Lambda_b (\Lambda_a m_{\Lambda_a} \Lambda_b m_{\Lambda_b}|\ell, -m_\ell) \begin{Bmatrix} \lambda_1 & \lambda_2 & \Lambda_a \\ \lambda'_1 & \lambda'_2 & \Lambda_b \\ \ell_1 & \ell_2 & \ell \end{Bmatrix} \\
CG_3 &= \sum_{m_\ell} (-)^{m_\ell} (\Lambda_a m_{\Lambda_a} \Lambda_b m_{\Lambda_b}|\ell, -m_\ell)\ (\ell_a m_a \ell_b m_b|\ell, -m_\ell) \\
&= (-)^{\Lambda_a - m_{\Lambda_a} + \ell_a - m_a} \frac{\hat{\ell}^2}{\hat{\Lambda}_b \hat{\ell}_b} \sum_{m_\ell} (-)^{m_\ell}\ (\Lambda_a m_{\Lambda_a} \ell m_\ell|\Lambda_b, -m_{\Lambda_b})(\ell_a m_a \ell m_\ell|\ell_b, -m_b) \\
&= (-)^{\Lambda_a - m_{\Lambda_a} + \Lambda_b - m_a} \hat{\ell}^2 \sum_{k'q'} W(\Lambda_a \Lambda_b \ell_a \ell_b; \ell k')\ (\Lambda_a m_{\Lambda_a} \ell_a m_a|k'q')(\Lambda_b m_{\Lambda_b} \ell_b, -m_b|k'q') \\
CG_4 &= \sum_{m_a m_b, m_{\Lambda_a} m_{\Lambda_a}} \sum_q [CG_3]\ (\Lambda_a m_{\Lambda_a} kq|\ell_a, -m_a)(\Lambda_b m_{\Lambda_b} kq|\ell_b, -m_b) \\
&= \sum_{k'q'q} (-)^{m_a - m_{\Lambda_a} - m_{\Lambda_a} - m_{\Lambda_b}} (-)^{\Lambda_b + \ell_b} \frac{\hat{\ell}^2 \hat{\ell}_a \hat{\ell}_b}{\hat{k}^2} \\
&\quad \times \sum_{m_a m_{\Lambda_a}} (\Lambda_a m_{\Lambda_a} \ell_a m_a|k, -q)(\Lambda_a m_{\Lambda_a} \ell_a m_a|k'q') \\
&\quad \times \sum_{m_b m_{\Lambda_b}} (\Lambda_b m_{\Lambda_b} \ell_b m_b|k, -q)(\Lambda_b m_{\Lambda_b} \ell_b m_b|k'q')\ W(\Lambda_a \Lambda_b \ell_a \ell_b; \ell k') \\
&= (-)^{k+\ell} \frac{\hat{\ell}^2 \hat{\ell}_a \hat{\ell}_b}{\hat{k}^2} W(\ell_a \Lambda_a \ell_b \Lambda_b; k\ell)
\end{aligned}$$

Combining together gives

$$\begin{aligned} F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a) &= \frac{1}{2} \sum_{\lambda_1 \lambda'_1 \lambda_2 \lambda'_2, \Lambda_a \Lambda_b, k} i^{\ell_1+\ell_2-\ell_a-\ell_b} (-)^{k+\ell} \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}'_1 \hat{\lambda}'_2 \hat{\ell}_1 \hat{\ell}_2 \hat{\Lambda}_a \hat{\Lambda}_b D_{\ell_1 \lambda_1 \lambda'_1} D_{\ell_2 \lambda_2 \lambda'_2} \\ & \times\ s_1^{\lambda_1} t_1^{\lambda'_1} s_2^{\lambda_2} t_2^{\lambda'_2}\ (\lambda_1 0 \lambda_2 0 | \Lambda_a 0)(\lambda'_1 0 \lambda'_2 0 | \Lambda_b 0)(\Lambda_a 0 k 0 | \ell_a 0)(\Lambda_b 0 k 0 | \ell_b 0) \\ & \times\ W(\ell_a \Lambda_a \ell_b \Lambda_b; k\ell) \left\{ \begin{array}{ccc} \lambda_1 & \lambda_2 & \Lambda_a \\ \lambda'_1 & \lambda'_2 & \Lambda_b \\ \ell_1 & \ell_2 & \ell \end{array} \right\} r_a^{\lambda_1+\lambda_2} r_b^{\lambda'_1+\lambda'_2} G_k^{\ell_1 n_1 \ell_2 n_2}(r_a, r_b) \end{aligned}$$

Q.E.D.

We introduce another function in the dynamical factor $I^{j\ell s}_{\ell_b j_b, \ell_a j_a}$

$$\begin{aligned} [D] &\equiv \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) \\ &= \sum_{\ell_a \ell_b} \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a)\ [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} \\ &= \sum_{\ell_a \ell_b} F^{j\ell s}_{\ell_a \ell_b}(r_b, r_a) [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} \end{aligned}$$

where $F^{j\ell s}_{\ell_a \ell_b}(r_b, r_a)$ may simply be called the *EFR form factor*,

$$F^{j\ell s}_{\ell_a \ell_b}(r_b, r_a) \equiv \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} F^{\ell_1 n_1 \ell_2 n_2}_{\ell \ell_a \ell_b}(r_b, r_a)$$

The explicit form is then given by

$$\begin{aligned} F^{j\ell s}_{\ell_a \ell_b}(r_b, r_a) &= \frac{1}{2} \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} \sum_{\lambda' s \Lambda' s, k} i^{\ell_1+\ell_2-\ell_a-\ell_b} (-)^{k+\ell} \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}'_1 \hat{\lambda}'_2 \hat{\ell}_1 \hat{\ell}_2 D_{\ell_1 \lambda_1 \lambda'_1} D_{\ell_2 \lambda_2 \lambda'_2} \\ & \times\ s_1^{\lambda_1} t_1^{\lambda'_1} s_2^{\lambda_2} t_2^{\lambda'_2} \hat{\Lambda}_1 \hat{\Lambda}_2 (\lambda_1 0 \lambda_2 0 | \Lambda_a 0)(\lambda'_1 0 \lambda'_2 0 | \Lambda_b 0)(\Lambda_a 0 k 0 | \ell_a 0)(\Lambda_b 0 k 0 | \ell_b 0 \\ & \times\ W(\ell_a \Lambda_a \ell_b \Lambda_b; k\ell) \left\{ \begin{array}{ccc} \lambda_1 & \lambda_2 & \Lambda_a \\ \lambda'_1 & \lambda'_2 & \Lambda_b \\ \ell_1 & \ell_2 & \ell \end{array} \right\} r_a^{\lambda_1+\lambda_2} r_b^{\lambda'_1+\lambda'_2} G_k^{\ell_1 n_1 \ell_2 n_2}(r_a, r_b) \end{aligned}$$

where

$$G_k^{\ell_1 n_1 \ell_2 n_2}(r_a, r_b) = \int_{-1}^{1} r_1^{-\ell_1} \phi_{\ell_1 n_1}(r_1) r_2^{-\ell_2} \phi_{\ell_2 n_2}(r_2) W(r_2) P_k(\mu) d\mu$$

Then the dynamical factor in the transition amplitude becomes

$$\begin{aligned} I^{j\ell s}_{\ell_b j_b, \ell_a j_a} &= J \int\int d\vec{r}_b d\vec{r}_a \{ \sum_{\ell_a \ell_b} F^{j\ell s}_{\ell_a \ell_b}(r_b, r_a) [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell \tilde{m}_\ell} \} (r_a r_b)^{-1} \\ &\times\ \chi_{\ell_b j_b}(k_b, r_b)\ \chi_{\ell_a j_a}(k_a, r_a) [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell m_\ell} \\ &= J \int\int dr_b dr_a r_a r_b\ \chi_{\ell_b j_b}(k_b, r_b) F^{j\ell s}_{\ell_a \ell_b}(r_b, r_a)\ \chi_{\ell_a j_a}(k_a, r_a) \end{aligned}$$

5.2 No-recoil (NR) approximation

When the mass of a transferred particle is small compared to that of a and B, i.e., $m_x/m_a << 1$ and $m_x/m_B << 1$, the coordinate relation between $(\vec{r}_1, \vec{r}_2)$ and $(\vec{r}_a, \vec{r}_b)$ becomes, as seen in Fig.1,

$$\begin{aligned}
\vec{r}_a &= \vec{r} + \frac{m_x}{m_a}\vec{r}_2 \approx \vec{r} \\
\vec{r}_b &= \vec{r} - \frac{m_x}{m_B}\vec{r}_1 = \frac{m_A}{m_B}\vec{r} - \frac{m_x}{m_B}\vec{r}_2 \approx \frac{m_A}{m_B}\vec{r} \\
\vec{r}_1 &= \vec{r} + \vec{r}_2 \equiv \vec{r} + \vec{r}' \\
\vec{r}_2 &\equiv \vec{r}'
\end{aligned}$$

and the Jacobian is

$$d\vec{r}_1 = J_{NR} d\vec{r}_a, \quad J_{NR} = 1$$

This approximation, so-called the *No-recoil approximation*, may be justified in heavy-ion reactions. The intuitive meaning of "no-recoil" can be found in Sec. 4.4 of Tamura's paper.

In the PWBA, the relative motion can be described in terms of plane waves as

$$\exp(ik_a \cdot r_a - ik_b \cdot r_b) = \exp[i(k_a - \frac{m_A}{m_B}k_b) \cdot r] \exp[i(\frac{m_x}{m_a}k_a - \frac{m_x}{m_B}(-k_b)) \cdot r']$$

$\frac{m_x}{m_a}k_a$ and $\frac{m_x}{m_B}(-k_b)$ are linear momenta that the particle x carries before and after the reaction, respectively. Therefore the momentum

$$\Delta k = \frac{m_x}{m_a}k_a - \frac{m_x}{m_B}(-k_b)$$

is nothing but the change of the linear momentum in the process of reaction, i.e., its *recoil* momentum. Since the approximation to set $\frac{m_x}{m_a} = \frac{m_x}{m_B} = 0$ makes $\Delta k = 0$, it is called the NR approximation.

(a) Dynamical factor $I^{j\ell s}_{\ell_b j_b, \ell_a j_a}$(NR)

Let us start with definition of $I^{j\ell s}_{\ell_b j_b, \ell_a j_a}$

$$\begin{aligned}
I^{j\ell s}_{\ell_b j_b, \ell_a j_a} &= \frac{1}{2\ell+1}\sum_{m_\ell} J \int\int d\vec{r}_b d\vec{r}_a \{ \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\tilde{\ell m_\ell}}(\vec{r}_1, \vec{r}_2)\} \\
&\times \; (r_a r_b)^{-1} \chi_{\ell_b j_b}(k_b, r_b) \; \chi_{\ell_a j_a}(k_a, r_a) [Y_{\ell_a}(\hat{r}_a) Y_{\ell_b}(\hat{r}_b)]_{\ell m_\ell}
\end{aligned}$$

Now

$$\begin{aligned}
J d\vec{r}_b &\rightarrow d\vec{r}', \qquad (r_a r_b)^{-1} \;\rightarrow\; \frac{m_B}{m_A} r^{-2} \\
J (r_a r_b)^{-1} d\vec{r}_a d\vec{r}_b &\rightarrow \frac{m_B}{m_A} r^{-2} d\vec{r} d\vec{r}' \\
[Y_{\ell_a}(\hat{r}) Y_{\ell_b}(\hat{r})]_{\ell m_\ell} &= i^{\ell_a + \ell_b - \ell} \frac{1}{\sqrt{4\pi}} \frac{\hat{\ell}_a \hat{\ell}_b}{\hat{\ell}} (\ell_a 0 \ell_b 0 | \ell 0) Y_{\ell m_\ell}(\hat{r})
\end{aligned}$$

Then we have

$$\begin{aligned}
I^{j\ell s}_{\ell_b j_b, \ell_a j_a} &= \frac{1}{2\ell+1}\sum_{m_\ell} \int d\vec{r} \; [A] \; i^{\ell_a + \ell_b - \ell} \frac{1}{\sqrt{4\pi}} (\ell_a 0 \ell_b 0 | \ell 0) \\
&\times \frac{\hat{\ell}_a \hat{\ell}_b}{\hat{\ell}} \{ \frac{m_B}{m_A r^2} \chi_{\ell_b j_b}(k_b, \frac{m_A}{m_B} r) \; \chi_{\ell_a j_a}(k_a, r) \} Y_{\ell m_\ell}(\hat{r})
\end{aligned}$$

The factor, $[A]$, defines the no-recoil form factor as

$$
\begin{aligned}
[A] &\equiv \int d\vec{r}' \{ \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) \} \\
&= \sqrt{4\pi} F^{jls}_{(NR)}(r) Y_{\ell \tilde{m}_\ell}(\hat{r})
\end{aligned}
$$

Performing the angular integration gives

$$
I^{j\ell s}_{\ell_b j_b, \ell_a j_a}(\mathrm{NR}) = \frac{m_B}{m_A} \frac{\hat{\ell}_a \hat{\ell}_b}{\hat{\ell}} (\ell_a 0 \ell_b 0 | \ell 0)\ i^{\ell_a + \ell_b - \ell} \int dr \chi_{\ell_b j_b}(k_b, \frac{m_A}{m_B} r)\ F^{jls}_{(NR)}(r) \chi_{\ell_a j_a}(k_a, r)
$$

Note that $(\ell_a 0 \ell_b 0 | \ell 0)$ gives the normal parity selection rule, i.e., ($\ell_a + \ell_b + \ell =$ even), which is not necessary for EFR calculation.

(b) No-recoil Form Factors

The form factor for NR approximation was defined as

$$
\begin{aligned}
[A] &\equiv \int d\vec{r}' \{ \sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) \} \\
&= \sqrt{4\pi} F^{jls}_{(NR)}(r) Y_{\ell \tilde{m}_\ell}(\hat{r})
\end{aligned}
$$

In parallel to the EFR form factor, we write

$$
\begin{aligned}
f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) &= \sum_{k_1 k_2} F^{\ell_1 n_1 \ell_2 n_2}_{\ell k_1 k_2}(r, r') [Y_{k_1}(\hat{r}) Y_{k_2}(\hat{r}')]_{\tilde{\ell} m_\ell} \\
\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} f^{\ell_1 n_1 \ell_2 n_2}_{\ell \tilde{m}_\ell}(\vec{r}_1, \vec{r}_2) &= \sum_{k_1 k_2} F^{j\ell s}_{k_1 k_2}(r, r')\ [Y_{k_1}(\hat{r}) Y_{k_2}(\hat{r}')]_{\tilde{\ell} m_\ell}
\end{aligned}
$$

Take integrations over $\vec{r}'$ on both sides of the above equation. The right hand side defines the NR form factor and the left hand side becomes

$$
\begin{aligned}
[B] &= \int d\vec{r}' \sum_{k_1 k_2} F^{j\ell s}_{k_1 k_2}(r, r')\ [Y_{k_1}(\hat{r}) Y_{k_2}(\hat{r}')]_{\tilde{\ell} m_\ell} \\
&= \int d\vec{r}' \sum_{k_1 k_2} F^{j\ell s}_{k_1 k_2}(r, r') (k_1 q_1 k_2 q_2 | \ell \tilde{m}_\ell) Y_{k_1 q_1}(\hat{r}) Y_{k_2 q_2}(\hat{r}')
\end{aligned}
$$

The angular integration over $\hat{r}'$ gives

$$
\int d\hat{r}' Y_{k_2 q_2}(\hat{r}') = \sqrt{4\pi} \int d\hat{r}' Y^*_{00} Y_{k_2 q_2}(\hat{r}') = \sqrt{4\pi} \delta_{k_2,0} \delta_{q_2,0}
$$

Therefore, LHS becomes

$$
[B] = \sqrt{4\pi} [\int dr' r'^2 F^{j\ell s}_{\ell 0}(r, r')] Y_{\ell \tilde{m}_\ell}(\hat{r})
$$

By comparing both sides, we have

$$
F^{jls}_{(NR)}(r) = \int dr' r'^2 F^{j\ell s}_{\ell 0}(r, r')
$$

Let us evaluate the $F^{j\ell s}_{\ell 0}(r, r')$ from the EFR form factor we obtained in the preceding section. Make replacements

$$
\vec{r}_a \to \vec{r}, \quad \vec{r}_b \to \vec{r}', \quad \ell_a \to \ell, \quad \ell_b \to 0
$$

and also put

$$s_1 = t_1 = t_2 = 1, \quad s_2 = 0$$

Since $\vec{r}_2 = \vec{r}'$, the solid harmonics $Y_{\ell_2 m_2}(\vec{r}_2) = Y_{\ell_2 m_2}(\vec{r}')$ is not needed to expand. Thus we just take

$$\lambda_2 = 0, \lambda_2' = \ell_2$$

The vector coupling coefficients in the EFR form factor become

$$\begin{aligned}
(\lambda_1 000|\Lambda_a 0) &= \delta_{\lambda_1,\Lambda_a} \\
(\Lambda_a 0 k 0|\ell_a 0) &= (\lambda_1 0 k 0|\ell 0) = (-)^{\lambda_1}\frac{\hat{\ell}}{\hat{k}}(\lambda_1 0 \ell 0|k0) \\
(\Lambda_b 0 k 0|00) &= (-)^{\lambda_b}\frac{1}{\hat{k}}\delta_{\Lambda_b,k} \\
(\lambda_1' 0 \lambda_2' 0|\Lambda_b 0) &= (\lambda_1' 0 \ell_2 0|k0) \\
W(\ell_a \Lambda_a \ell_b \Lambda_b; k\ell) &= W(\ell\lambda_1 0 \Lambda_b; k\ell) = \frac{1}{\hat{\ell}\hat{k}}\delta_{\Lambda_b,k} \\
\left\{\begin{array}{ccc} \lambda_1 & \lambda_2 & \Lambda_a \\ \lambda_1' & \lambda_2' & \Lambda_b \\ \ell_1 & \ell_2 & \ell \end{array}\right\} &= \left\{\begin{array}{ccc} \lambda_1 & 0 & \lambda_1 \\ \lambda_1' & \ell_2 & \Lambda_b \\ \ell_1 & \ell_2 & \ell \end{array}\right\} \\
&= (-)^{\lambda_1+\ell_2-\ell_1-k}\frac{1}{\hat{\ell}_2\hat{\lambda}_1}W(\ell_1\ell\lambda_1' k;\ell_2\lambda_1)
\end{aligned}$$

The NR form factor is explicitly then given by

$$\begin{aligned}
F^{j\ell s}_{\ell 0}(r,r') &= \frac{1}{2}\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2}\sum_{\lambda\lambda' k} i^{\ell_1+\ell_2-\ell}(-)^k \hat{\lambda}_1\hat{\lambda}_1'\hat{\ell}_1\hat{\ell}_2 D_{\ell_1\lambda_1\lambda_1'} \\
&\quad \times\ (-)^{\lambda_1}(\lambda_1 0 \ell 0|k0)(\lambda_1' 0 \ell_2 0|k0)\ W(\ell_1\ell_2\lambda_1 k;\ell\lambda_1')\ r^{\lambda_1} r'^{\lambda_1'+\ell_2} G_k^{\ell_1 n_1 \ell_2 n_2}(r,r')
\end{aligned}$$

and

$$\begin{aligned}
F^{jls}_{(NR)}(r) &= \int dr' r'^2 F^{j\ell s}_{\ell 0}(r,r') \\
&= \frac{1}{2}\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2}\sum_{\lambda\lambda' k} i^{\ell_1+\ell_2-\ell}(-)^k \hat{\lambda}_1\hat{\lambda}_1'\hat{\ell}_1\hat{\ell}_2 D_{\ell_1\lambda_1\lambda_1'}(-)^{\lambda_1}(\lambda_1 0 \ell 0|k0)(\lambda_1' 0 \ell_2 0|k0) \\
&\quad \times\ W(\ell_1\ell_2\lambda_1 k;\ell\lambda_1')\ r^{\lambda_1}\int dr' r'^{\lambda_1'+\ell_2+2} G_k^{\ell_1 n_1 \ell_2 n_2}(r,r')
\end{aligned}$$

Now defining μ to denote the cosine of the angle between the vectors $\vec{r}$ and $\vec{r}'$, we get

$$\begin{aligned}
[C] &\equiv r'^{\lambda_1'+\ell_2} G_k^{\ell_1 n_1 \ell_2 n_2}(r,r') \\
&= r'^{\lambda_1'}\int_{-1}^{1}\phi_{\ell_1 n_1}(r_1)\phi_{\ell_2 n_2}(r_2)W(r_2)P_k(\mu)d\mu \\
&= r'^{\lambda_1'}[\phi_{\ell_2 n_2}(r')W(r')]\int_{-1}^{1} r_1^{-\ell_1}\phi_{\ell_1 n_1}(r_1)P_k(\mu)d\mu \\
&= r'^{\lambda_1'}[\phi_{\ell_2 n_2}(r')W(r')]G_k^{\ell_1 n_1}(r,r')
\end{aligned}$$

where

$$G_k^{\ell_1 n_1}(r,r') \equiv \int_{-1}^{1} r_1^{-\ell_1}\phi_{\ell_1 n_1}(|\vec{r}+\vec{r}'|)P_k(\mu)d\mu$$

Finally we have

$$
\begin{aligned}
F^{jls}_{(NR)}(r) &= \frac{1}{2}\sum_{\ell_1 n_1 \ell_2 n_2} d^{j\ell s}_{\ell_1 n_1 \ell_2 n_2} \sum_{\lambda_1 \lambda_1' k} i^{\ell_1+\ell_2-\ell}(-)^k \hat{\lambda}_1 \hat{\lambda}_1' \hat{\ell}_1 \hat{\ell}_2 D_{\ell_1 \lambda_1 \lambda_1'} (-)^{\lambda_1} (\lambda_1 0 \ell 0|k0)(\lambda_1' 0 \ell_2 0|k0) \\
&\qquad \times\ W(\ell_1 \ell_2 \lambda_1 k; \ell \lambda_1') \left[r^{\lambda_1} \int dr' r'^{\lambda_1'+2} [\phi_{\ell_2 n_2}(r') W(r')] G^{\ell_1 n_1}_k (r, r') \right]
\end{aligned}
$$

5.3 Zero-range (ZR) approximation

When the interaction is zero-range, it is called *Zero-range (ZR) approximation.* Thus $\vec{r}_a \approx \vec{r}_b \approx r$. The essence of the ZR approximation is to set

$$
\phi_{\ell_2 n_2}(r') W(r') = \frac{D_0}{\sqrt{4\pi}} \delta(r')/r'^2 = \bar{D}_0 \delta(r')/r'^2
$$

where D_0 is the strength factor defined by Satchler. In order to justify this equation, we set $\ell_2 = 0, \lambda_1' = 0$, and thus $\lambda_1 = \ell_1$. Thus the vector coupling coefficients in the NR form factor become

$$
\begin{aligned}
(\lambda_1' 0 \ell_2 0|k0) &= (\lambda_1' 000|k0) = \delta_{\lambda_1',k} \\
(\lambda_1 0 \ell 0|k0) &= (\lambda_1 0 \ell 0|00) = \frac{1}{\hat{e}ll}\delta_{\lambda_1,\ell} \\
W(\ell_1 \ell_2 \lambda_1 k; \ell \lambda_1') &= W(\ell_1 0 \lambda_1 k; \ell 0) = \frac{1}{\hat{\ell}}
\end{aligned}
$$

The integration becomes

$$
\begin{aligned}
[A] &= r^{\lambda_1} \int dr' r'^{\lambda_1'+2} [\phi_{\ell_2 n_2}(r') W(r')] G^{\ell_1 n_1}_k (r, r') \\
&= r^{\ell_1} \int dr' \frac{D_0}{\sqrt{4\pi}} \delta(r') G^{\ell_1 n_1}_k (r, r') \\
&= \frac{D_0}{\sqrt{4\pi}} r^{\ell_1} G^{\ell_1 n_1}_k (r, 0) \\
&= \frac{D_0}{\sqrt{4\pi}} r^{\ell_1} \int_{-1}^{1} r_1^{-\ell_1} \phi_{\ell_1 n_1}(|\vec{r} + \vec{r}'|_{r'=0}) P_k(\mu) d\mu \\
&= \frac{D_0}{\sqrt{4\pi}} r^{\ell_1} r^{-\ell_1} \phi_{\ell_1 n_1}(r) \int_{-1}^{1} P_k(\mu) d\mu \\
&= \frac{2D_0}{\sqrt{4\pi}} \phi_{\ell_1 n_1}(r) \delta_{k,0}
\end{aligned}
$$

Usually in the ZR calculations, the form factor is defined,

$$
F^{jls}_{(ZR)}(r) = \sum_{\ell_1 n_1} (-)^{\ell} d^{j\ell s}_{\ell_1 n_1 00} \frac{D_0}{\sqrt{4\pi}} \phi_{\ell_1 n_1}(r)
$$

Further in the ZR calculations it is customary to omit the factors $D_0/\sqrt{4\pi}$ and $(-)^{\ell} d^{j\ell s}_{\ell_1 n_1 00}$ from the definition of the form factor. Thus, the ZR form factor is simply given as

$$
F^{jls}_{(ZR)}(r) = \phi_{\ell_1 n_1}(r)
$$

With this form factor, the dynamical factor for ZR approximation becomes

$$
I^{jls}_{\ell_b j_b, \ell_a j_a}(\mathrm{ZR}) = \frac{D_0}{\sqrt{4\pi}} \frac{m_B}{m_A} \frac{\hat{\ell}_a \hat{\ell}_b}{\hat{\ell}} (\ell_a 0 \ell_b 0|\ell 0)\ i^{\ell_a+\ell_b-\ell} \int dr \chi_{\ell_b j_b}(k_b, \frac{m_B}{m_A} r)\ F^{jls}_{(ZR)}(r) \chi_{\ell_a j_a}(k_a, r)
$$

This is almost exactly the same form as that of NR approximation, only difference is that $I(ZR)$ has an extra factor $D_0/\sqrt{4\pi}$.

6 Form factors for pick-up reactions

We gave an expression for the DWBA amplitude (Sec. VII-6) which is valid for stripping and pick-up processes, with assumption that the dynamical factor $I^{j\ell s}_{\ell_b j_b,\ell_a j_a}$ is defined in such a way as to be valid for both processes. In fact, however, we have given so far explicit form to this dynamical factor having in mind only stripping reactions.

In defining the dynamical factor to be used for pick-up reactions, it is convenient to introduce first the coordinate system defined by Fig.2 rather than Fig.1. With this new coordinate system, we have

$$s_1 = \frac{-m_A m_a}{m_x(m_a+m_A)}, \qquad t_1 = \frac{m_A m_b}{m_x(m_a+m_A)}$$
$$s_2 = \frac{-m_A m_b}{m_x(m_a+m_A)}, \qquad t_2 = \frac{m_B m_b}{m_x(m_a+m_A)}$$

and the Jacobian is

$$d\vec{r}_1 \;=\; J d\vec{r}_a, \quad J_{EFR} = s_2^3 = (m_A m_b/m_x(m_a+m_A))^3$$

Once these replacements are made, together with obvious redefinition of the bound state wavefunctions and prior-form interaction, all the expressions for EFR calculation (Sec. VII-9) given above for stripping reaction can be used also for pick-up reactions just as they stand.

For NR approximation, the coordinates are replaced by

$$\vec{r}_a = \frac{m_B}{m_A}\vec{r}, \quad \vec{r}_b = \vec{r}, \quad J_{EFR} r_a^{-1} r_b^{-1} dr_a dr_b = \frac{m_A}{m_B}\frac{1}{r^2} dr dr'$$

Since the evaluation of the form factor $F^{jls}_{(NR)}(r)$ can be made in exactly the same way as it is for stripping reactions (except again for appropriate redefinition of the bound state wavefunctions and prior-form interaction), the dynamical factor for pick-up reaction in NR approximation can be given

$$\begin{aligned} I^{j\ell s}_{\ell_b j_b,\ell_a j_a}(\text{NR-P}) \;&=\; \frac{m_B}{m_A}\frac{\hat{\ell}_a\hat{\ell}_b}{\hat{\ell}}(\ell_a 0 \ell_b 0|\ell 0)\; i^{\ell_a+\ell_b-\ell}\int dr \chi_{\ell_b j_b}(k_b,r)\; F^{jls}_{(NR)}(r)\chi_{\ell_a j_a}(k_a,\frac{m_B}{m_A}r) \\ &=\; (\frac{m_A}{m_B})^2\frac{\hat{\ell}_a\hat{\ell}_b}{\hat{\ell}}(\ell_a 0 \ell_b 0|\ell 0)\; i^{\ell_a+\ell_b-\ell}\int dr \chi_{\ell_b j_b}(k_b,\frac{m_A}{m_B}r)\; F^{jls}_{(NR)}(\frac{m_A}{m_B}r)\chi_{\ell_a j_a}(k_a,r) \end{aligned}$$

To choose which of the two versions in this equation to use is simply a matter of taste, although the choice of the second expression seems to have made customarily.

The expression for $I^{j\ell s}_{\ell_b j_b,\ell_a j_a}$(ZR-Pick-up) to be used for pick-up reactions is obtained by rewriting the expression for $I^{j\ell s}_{\ell_b j_b,\ell_a j_a}$(ZR) given in the Sec. VII-11 in exactly the same way as $I^{j\ell s}_{\ell_b j_b,\ell_a j_a}$(NR) (Sec. VII-10) was rewritten to give $I^{j\ell s}_{\ell_b j_b,\ell_a j_a}$(NR-Pick-up).

7 (Homework Set #15) DWBA calculation

Reproduce the results of P. D. Bond et. al. (Phys. Lett. 47B, 232 (1973)) by using the DWBA programs, SATURN and MARS.

A Time reversal operator

A.1 Definition

It is defined by the correspondence principle as

Classically		Quantum Mechanically
If $t \to -t$,		$\mathcal{T}$
$\vec{r} \to \vec{r}$	$\Leftrightarrow$	$\mathcal{T}\hat{r}\mathcal{T}^{-1} = \hat{r}$
$\vec{p} \to -\vec{p}$	$\Leftrightarrow$	$\mathcal{T}\hat{p}\mathcal{T}^{-1} = -\hat{p}$
$\vec{L} \to -\vec{L}$	$\Leftrightarrow$	$\mathcal{T}\hat{L}\mathcal{T}^{-1} = -\hat{L}$
$\vec{S} \to -\vec{S}$	$\Leftrightarrow$	$\mathcal{T}\hat{S}\mathcal{T}^{-1} = -\hat{S}$

A.2 Properties

(a) $\mathcal{T}$ is not unitary.

Let's take $[p_x, x] = -i\hbar$, for an example.

$$\begin{aligned} \mathcal{T}[p_x, x]\mathcal{T}^{-1} &= -i\hbar\mathcal{T}\mathcal{T}^{-1} = -i\hbar \\ [\mathcal{T}p_x\mathcal{T}^{-1}, \mathcal{T}x\mathcal{T}^{-1}] &= [-p_x, x] = i\hbar \end{aligned}$$

Since the equality of two equations does not hold, the time reversal operator is not unitary.

(b) $\mathcal{T}$ is antiunitary.

Antiunitary: The transformation $|\bar{A}> = \mathcal{T}|A>$, $|\bar{B}> = \mathcal{T}|B>$ is said to be *antiunitary* if the following equations hold

$$\begin{aligned} <\bar{B}|\bar{A}>^* &= <B|A> \\ \mathcal{T}(c_1|A> + c_2|B>) &= c_1^*\mathcal{T}|A> + c_2^*\mathcal{T}|B> . \end{aligned}$$

An antiunitary operator can be written as $\mathcal{T} = \mathcal{U}\mathcal{K}$,
where $\mathcal{K}$ is the complex conjugation operator and $\mathcal{U}$ is the unitary operator.

Proof.

$$\begin{aligned} \mathcal{T}(c_1|A> + c_2|B>) &= \mathcal{U}\mathcal{K}(c_1|A> + c_2|B>) \\ &= c_1^*\mathcal{U}\mathcal{K}|A> + c_2^*\mathcal{U}\mathcal{K}|B> \\ &= c_1^*\mathcal{T}|A> + c_2^*\mathcal{T}|B> . \end{aligned}$$

$$\begin{aligned} <\bar{B}|\bar{A}>^* &= <\mathcal{U}\mathcal{K}B|\bar{A}>^* = <\mathcal{K}B|\mathcal{U}^{-1}|\bar{A}>^* \\ &= <\mathcal{K}B|\mathcal{K}\mathcal{K}\mathcal{U}^{-1}|\bar{A}>^* = <B|\mathcal{K}\mathcal{U}^{-1}|A> \\ &= <B|\mathcal{K}\mathcal{U}^{-1}\mathcal{U}\mathcal{K}|\bar{A}> = <B|A> \end{aligned}$$

(c) $<\bar{B}|\bar{T}|\bar{A}>^* = <B|T|A>$, where $\bar{T} = \mathcal{T}T\mathcal{T}$.

$$\begin{aligned} <\bar{B}|\bar{T}|\bar{A}>^* &= <\mathcal{K}B|\mathcal{U}^{-1}\mathcal{U}\mathcal{K}T\mathcal{K}^{-1}\mathcal{U}^{-1}|\bar{A}>^* \\ &= <\mathcal{K}B|\mathcal{K}\mathcal{K}\mathcal{U}^{-1}\mathcal{U}\mathcal{K}T\mathcal{K}^{-1}\mathcal{U}^{-1}|\bar{A}>^* \\ &= <B|\mathcal{K}\mathcal{U}^{-1}\mathcal{U}\mathcal{K}T\mathcal{K}^{-1}\mathcal{U}^{-1}\mathcal{U}\mathcal{K}|A> \\ &= <B|T|A> \end{aligned}$$

(d) Operation of $\mathcal{T}$ on a single particle state

(1) Spatial wave function ($\mathcal{U}=1, \mathcal{T}=\mathcal{K}$.)

$$\mathcal{T}\phi_A(\vec{r}) = \phi_{\bar{A}}(\vec{r}) = \phi^*_A(\vec{r})$$

Proof:

$$\begin{aligned}
\phi_A(\vec{r}) &= <\vec{r}|A> \\
\mathcal{T}|\vec{r}> &= |\vec{r}>
\end{aligned}$$

From the antiunitary property, we have

$$\begin{aligned}
<\bar{\vec{r}}|\bar{A}>^* &= <\vec{r}|A> = \phi_A(\vec{r}) \\
&= <\vec{r}|\bar{A}>^* = \phi^*_{\bar{A}}(\vec{r})
\end{aligned}$$

Thus, the equality holds.

Time development

$$\begin{aligned}
\phi_A(\vec{r},t) &= e^{-iHt}\phi_A(\vec{r},0) \\
\phi_{\bar{A}}(\vec{r},t) &= e^{-iHt}\phi_{\bar{A}}(\vec{r},0) = e^{-iHt}\phi^*_A(\vec{r},0) \\
&= (e^{iHt}\phi_A(\vec{r},0))^* = \phi^*_A(\vec{r},-t)
\end{aligned}$$

(2) Angular Momentum Representation

Phase convention;

$$\begin{aligned}
i^\ell Y_{\ell m}(\hat{r}) &= <\hat{r}|\ell m> \quad \text{Wigner phase convention (Tamura)} \\
Y_{\ell m}(\hat{r}) &= <\hat{r}|\ell m> \quad \text{Condon-Shortley (CS)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}[i^\ell Y_{\ell m}(\hat{r})] &= i^{-\ell}Y^*_{\ell m} = (-)^{-\ell} i^\ell Y_{\ell,-m}(-)^m \\
&= (-)^{-\ell+m}[i^\ell Y_{\ell,-m}] = (-)^{\ell+m}[i^\ell Y_{\ell,-m}] \\
\mathcal{T}|\ell m> &= (-)^{\ell+m}|\ell,-m> \\
\mathcal{R}^{-1}(\pi)|\ell m> &= (-)^{\ell+m}|\ell,-m>
\end{aligned}$$

Thus, we have

$$\mathcal{T} = \mathcal{R}^{-1}(\pi), \qquad \mathcal{R}(\pi)\mathcal{T} = 1$$

(3) Spin-Angular Momentum ($\mathcal{K}=1, \mathcal{T}=\mathcal{U}$)

$$\begin{aligned}
\mathcal{T}\hat{S}\mathcal{T}^{-1} &= -\hat{S} \\
\mathcal{U}\hat{s}\mathcal{U}^{-1} &= -\hat{s} \implies \mathcal{U} = e^{i\pi\hat{S}_y} \\
\mathcal{U}\hat{\ell}\mathcal{U}^{-1} &= -\hat{\ell} \implies \mathcal{U} = e^{i\pi\hat{L}_y} \\
\mathcal{U}\hat{j}\mathcal{U}^{-1} &= -\hat{j} \implies \mathcal{U} = e^{i\pi\hat{J}_y} \\
\mathcal{T}|sm> &= (-)^{s+m}|s,-m> \\
\mathcal{T}|\ell m> &= (-)^{\ell+m}|\ell,-m> \\
\mathcal{T}|jm> &= (-)^{j+m}|j,-m>
\end{aligned}$$

$$
\begin{aligned}
|jm> &= \sum_{m_\ell m_s} <\ell m_\ell s m_s|jm> |\ell m_\ell> |s m_s> \\
\mathcal{T}|jm> &= \sum_{m_\ell m_s} <\ell m_\ell s m_s|jm> \\
&\qquad \times (-)^{\ell+m_\ell+s+m_s}|\ell,-m_\ell> |s,-m_s> \\
&= \sum_{m_\ell m_s} <\ell,-m_\ell s,-m_s|j,-m> (-)^{\ell+s-j} \\
&\qquad \times (-)^{-\ell-s-m}|\ell,-m_\ell> |s,-m_s> \\
&= (-)^{j+m}|j,-m>
\end{aligned}
$$

(e) Operation of $\mathcal{T}$ on many-particle bound state

Operate $\mathcal{T}$ on $|IM>$, where $I = \sum(\hat{\ell}_i + \hat{s}_i)$

$$
\mathcal{T}|IM>= (-)^{I+M}|I,-M>
$$

(f) Operation of $\mathcal{T}$ on momentum eigenstates

$$
\begin{aligned}
\hat{p}|\vec{p}> &= \vec{p}|\vec{p}>, \qquad <\vec{r}|\vec{p}>= e^{i\vec{p}\cdot\vec{r}} \\
\mathcal{T}|\vec{p}> &= |-\vec{p}> \\
\hat{p}\mathcal{T}|\vec{p}> &= -\vec{p}\mathcal{T}|\vec{p}>
\end{aligned}
$$

Since

$$
\begin{aligned}
|\vec{p}> &= \int d\vec{r}|\vec{r}><\vec{r}|\vec{p}>= \int d\vec{r}\ e^{i\vec{p}\cdot\vec{r}}|\vec{r}> \\
\mathcal{T}|\vec{p}> &= \int d\vec{r}\ e^{-i\vec{p}\cdot\vec{r}}|\vec{r}>= \int d\vec{r}|\vec{r}><\vec{r}|-\vec{p}> \\
&= |-\vec{p}> \\
\mathcal{T}\hat{p}|\vec{p}> &= \mathcal{T}\hat{p}\mathcal{T}^{-1}\mathcal{T}|\vec{p}>= -\hat{p}\mathcal{T}|\vec{p}> \\
&= \mathcal{T}\vec{p}|\vec{p}>= \vec{p}\mathcal{T}|\vec{p}>
\end{aligned}
$$

(g) Collision Process

Define $U^{(+)}, U^{(-)}$ as

$$
\begin{aligned}
U^{(+)} &= U(0,-\infty), \qquad U^{(+)}|\phi>= |\psi^{(+)}> \\
&= 1 + \frac{1}{E_0 - H + i\epsilon}V \\
U^{(-)} &= U(0,\infty), \qquad U^{(-)}|\phi>= |\psi^{(-)}> \\
&= 1 + \frac{1}{E_0 - H - i\epsilon}V
\end{aligned}
$$

where

$$
\begin{aligned}
H &= H_0 + V \\
H_0|\phi> &= E_0|\phi> \\
H|\psi^{(+)}> &= E_0|\psi^{(+)}>
\end{aligned}
$$

Operation of $\mathcal{T}$ on $|\psi^{(+)}>, |\psi^{(-)}>$ gives

$$
\mathcal{T}|\psi_A^{(+)}>= |\psi_{\bar{A}}^{(-)}>, \quad \mathcal{T}|\psi_A^{(-)}>= |\psi_{\bar{A}}^{(+)}>
$$

since

$$\begin{aligned}
\mathcal{T}|\psi_A^{(+)}> &= \mathcal{T}U^{(+)}|\phi_A>= \mathcal{T}(1+\frac{1}{E_0-H+i\epsilon}V)|\phi_A> \\
&= \mathcal{T}(1+\frac{1}{E_0-H+i\epsilon}V)\mathcal{T}^{-1}\mathcal{T}|\phi_A> \\
&= (1+\frac{1}{E_0-H-i\epsilon}V)|\phi_{\bar{A}}>=|\psi_{\bar{A}}^{(-)}>
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}|\psi_A^{(-)}> &= \mathcal{T}U^{(-)}|\phi_A>= \mathcal{T}(1+\frac{1}{E_0-H-i\epsilon}V)|\phi_A> \\
&= \mathcal{T}(1+\frac{1}{E_0-H-i\epsilon}V)\mathcal{T}^{-1}\mathcal{T}|\phi_A> \\
&= (1+\frac{1}{E_0-H_i\epsilon}V)|\phi_{\bar{A}}>=|\psi_{\bar{A}}^{(+)}>
\end{aligned}$$

(h) Reciprocity Relation

Define the $\hat{S}$ operator as

$$\hat{S}=U(\infty,0)U(0,-\infty)=U^{(-)\dagger}U^{(+)}$$

then, we have the so-called *reciprocity relation*.

$$<B|\hat{S}|A>=<\bar{A}|\hat{S}|\bar{B}>$$

Proof:

$$\begin{aligned}
<B|\hat{S}|A> &= <\phi_B|\hat{S}|\phi_A> \\
&= <\phi_B|U^{(-)\dagger}U^{(+)}|\phi_A> \\
&= <\psi_B^{(-)}|\psi_A^{(+)}> \\
&= <\bar{\psi}_B^{(-)}|\bar{\psi}_A^{(+)}>^* \\
&= <\psi_{\bar{B}}^{(+)}|\psi_{\bar{A}}^{(-)}>^* \\
&= <\psi_{\bar{A}}^{(-)}|\psi_{\bar{B}}^{(+)}> \\
&= <\bar{A}|\hat{S}|\bar{B}>
\end{aligned}$$

(i) Operation of $\mathcal{T}$ on scattering eigenstates

$$\begin{aligned}
&\mathcal{T}\Psi^{(+)}_{A=(\vec{k},I_AM_As_am_a)} \\
&\quad = (-)^{I_A+M_A+s_a+m_a}\Psi^{(-)}_{A'=(-\vec{k},I_A,-M_As_a,-m_a)} \\
&\quad = \Psi^{(-)}_{\bar{A}}
\end{aligned}$$

$$\begin{aligned}
\Psi_A^{(+)} &= \sum \chi^{(+)}_{I'_AM'_As'_am'_a,I_AM_As_am_a}(\vec{k},\vec{r})|I'_AM'_As'_am'_a> \\
\Psi_A^{(-)} &= \sum \chi^{(-)}_{I'_AM'_As'_am'_a,I_AM_As_am_a}(-\vec{k},\vec{r})|I'_AM'_As'_am'_a>
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\chi^{(-)*}_{Q'Q}(\vec{k},\vec{r}) \\
&\quad = (-)^{I'_A-M'_A+s'_a-m'_a-I_A+M_A-s_a+m_a}\chi^{(+)}_{-Q',-Q}(-\vec{k},\vec{r})
\end{aligned}$$

where $Q=(I_AM_As_am_a)$.

B Solid Harmonics

The Schrödinger equation for the central forces is

$$\hat{H}\varphi = [\frac{\hat{p}^2}{2\mu} + V(r)]\varphi = [\frac{\hat{p}_r^2}{2\mu} + \frac{\hat{\mathbf{L}}^2}{2\mu r^2} + V(r)]\varphi = E\varphi$$

The solution can be set

$$\varphi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi)$$

Then $R(r)$ satisfies

$$[-\frac{\hbar^2}{2\mu}(\frac{1}{r}\frac{d^2}{dr^2}r) + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} + V(r)]R(r) = ER(r)$$

Furthermore, if we set

$$R(r) \equiv \frac{u(r)}{r}$$

then, $u(r)$ satisfies

$$[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} + V(r)]u(r) = Eu(r)$$

and is normalized

$$\int u^*(r)u(r)dr = 1$$

If very near the origin, V can be neglected in comparison with the centrifugal term ($\sim r^{-2}$), that is, $V = Cr^\alpha,\ \alpha \geq -1$, $u(r)$ satisfies

$$\frac{d^2u(r)}{dr^2} - \frac{\ell(\ell+1)}{r^2}u(r) = 0$$

and its solution would be

$$u(r \to 0) = Ar^{\ell+1} + Br^{-\ell}$$

The second term will be discarded from the boundary condition $u(0) = 0$. Thus for any but S-state, u must be $\sim r^{\ell+1}$ at the origin and $\varphi(r,\theta,\phi)$ must behave as r^ℓ.

Assuming that V vanishes at ∞ (bound), the equation reduces

$$\frac{d^2u(r)}{dr^2} + \frac{2\mu E}{\hbar^2}u(r) = 0$$

and its solution would be

$$u(r \to \infty) = e^{-\rho}, \quad \text{where}, \quad \rho = \sqrt{-2\mu E/\hbar^2}\ r$$

Thus we can put

$$u(\rho) = \rho^{\ell+1}e^{-\rho}\omega(\rho)$$

where $\omega(\rho)$ satisfies

$$\frac{d^2\omega(\rho)}{d\rho^2} + 2(\frac{\ell+1}{\rho} - 1)\frac{d\omega(\rho)}{d\rho} + [\frac{V}{E} - \frac{2(\ell+1)}{\rho}]\omega(\rho) = 0$$

Therefore, we have

$$\begin{aligned}
\varphi(r,\theta,\phi) &= r^\ell Y_{\ell m}(\theta,\phi)e^{-\rho}\omega(\rho)\\
&= \mathcal{Y}_{\ell m}(\vec{r})e^{-\rho}\omega(\rho)
\end{aligned}$$

and we define the solid harmonics

$$\mathcal{Y}_{\ell m}(\vec{r}) \equiv r^\ell Y_{\ell m}(\theta,\phi)$$

is a solution of the Laplace equation.

We need $\mathcal{Y}_{\ell m}(\vec{r})$ in terms of $\vec{r}_a$ and $\vec{r}_b$. This is the translation of multipole fields. Consider $\vec{r} = \vec{r}_a + \vec{r}_b$. We see from this that

$$\begin{aligned}
\nabla_a^2\ r^\ell Y_{\ell m}(\theta,\phi) &= 0\\
\nabla_b^2\ r^\ell Y_{\ell m}(\theta,\phi) &= 0
\end{aligned}$$

So $r^\ell Y_{\ell m}(\theta,\phi)$ should be expanded in terms of multipole fields in $\vec{r}_a$ and $\vec{r}_b$. Furthermore, the multipole fields are basis for the representation of the rotation group. Combining these two facts we immediately see that

$$\begin{aligned}
&\mathcal{Y}_{\ell m}(\vec{r}) = r^\ell Y_{\ell m}(\theta,\phi)\\
&\quad = \sum_{\ell_a,\ell_b=0}^{\ell} \delta_{\ell_a+\ell_b,\ell}\ G(\ell_a\ell_b\ell)\ [\mathcal{Y}_{\ell_a m_a}(\vec{r}_a)\otimes \mathcal{Y}_{\ell_b m_b}(\vec{r}_b)]_{\ell m}
\end{aligned}$$

where $G(\ell_a\ell_b\ell)$ is a coefficient to be determined. Assuming now that $\theta_a = \theta_b = 0$, we see that $\theta = 0$ and $r = r_a + r_b$, so that the above equation becomes

$$\begin{aligned}
LHS &= [(2\ell+1)4\pi]^{1/2}(r_a+r_b)^\ell\\
&= [(2\ell+1)4\pi]^{1/2}\sum_{\ell_a,\ell_b=0}^{\ell}\delta_{\ell_a+\ell_b,\ell}\ \frac{\ell!}{\ell_a!\ell_b!}r_a^{\ell_a}r_b^{\ell_b}\\
RHS &= \sum_{\ell_a,\ell_b=0}^{\ell}\delta_{\ell_a+\ell_b,\ell}\ G(\ell_a\ell_b\ell)\ (\ell_a0\ell_b0\ell0)\\
&\quad\times\ [(2\ell_a+1)(2\ell_b+1)]^{1/2}r_a^{\ell_a}r_b^{\ell_b}
\end{aligned}$$

Equating both sides gives

$$\sqrt{4\pi} = G(\ell_a\ell_b\ell)\ [\frac{(2\ell_a+1)!(2\ell_b+1)!}{(2\ell+1)!}]^{1/2}$$

where we use a relation

$$(\ell_a0\ell_b0\ell0) = [\frac{(2\ell+1)(2\ell_a)!(2\ell_b)!}{(2\ell+1)!}]^{1/2}\frac{\ell!}{\ell_a!\ell_b!}$$

We obtain

$$G(\ell_a\ell_b\ell) = \sqrt{4\pi}\ [\frac{(2\ell+1)!}{(2\ell_a+1)!(2\ell_b+1)!}]^{1/2}$$

and

$$\begin{aligned}
\mathcal{Y}_{\ell m}(\vec{r}) &= \sqrt{4\pi}\sum_{\ell_a,\ell_b=0}^{\ell} D_{\ell\ell_a\ell_b}\\
&\qquad\times\ r_a^{\ell_a}r_b^{\ell_b}\ [Y_{\ell_a m_a}(\hat{r}_a)\otimes Y_{\ell_b m_b}(\hat{r}_b)]_{\ell m}\\
D_{\ell\ell_a\ell_b} &\equiv \delta_{\ell_a+\ell_b,\ell}\ [\frac{(2\ell+1)!}{(2\ell_a+1)!(2\ell_b+1)!}]^{1/2}
\end{aligned}$$

C Multipole expansions

C.1 Scalar function of r_{12}

We wish to express a scalar function $f(r_{12})$ in terms of function of r_1 and r_2, where

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2, \quad r_{12}^2 = r_1^2 + r_2^2 - 2r_1 r_2 \mu, \quad \mu = \cos\theta$$

where θ is the angle between $\vec{r}_1$ and $\vec{r}_2$, as shown in Fig.3.

It can simply be done by Legendre expansion,

$$\begin{aligned} f(r_{12}) &= \sum_\ell f_\ell(r_1, r_2)(2\ell+1)P_\ell(\cos\theta) \\ f_\ell(r_1, r_2) &= \frac{1}{2}\int_{-1}^{1} f(r_{12})P_\ell(\cos\theta)d\mu \end{aligned}$$

We now express $f(r_{12})$ in the form of

$$\begin{aligned} f(r_{12}) &= \sum_\ell f'_\ell(r_1, r_2)(-)^\ell [Y_\ell Y_\ell]_{00} \\ f'_\ell(r_1, r_2) &= \sqrt{16\pi^3}\int_{-1}^{1} f(r_{12})Y_{\ell 0}(\theta, 0)d\mu, \qquad \mu \equiv \cos\theta \end{aligned}$$

Proof:

$$\begin{aligned} f(r_{12}) &= \sum_\ell f_\ell(r_1, r_2)(2\ell+1)P_\ell(\cos\theta) \\ &= \sum_\ell f_\ell(r_1, r_2)(2\ell+1)\frac{4\pi}{(2\ell+1)}\sum_m Y_{\ell m}(\hat{r}_1)Y^*_{\ell m}(\hat{r}_2) \\ &= \sum_\ell f_\ell(r_1, r_2)4\pi(-)^\ell \hat{\ell}(\ell m \ell, -m|00)[Y_\ell Y_\ell]_{00} \\ &= \sum_\ell f_\ell(r_1, r_2)4\pi(-)^\ell \hat{\ell}[Y_\ell Y_\ell]_{00} \end{aligned}$$

Thus we have

$$\begin{aligned} f'_\ell(r_1, r_2) &= f_\ell(r_1, r_2)4\pi\hat{\ell} \\ &= 4\pi\hat{\ell}\frac{1}{2}\int_{-1}^{1} f(r_{12})P_\ell(\cos\theta)d\mu \\ &= 2\pi\hat{\ell}\int_{-1}^{1} f(r_{12})P_\ell(\mu)d\mu \\ &= 2\pi\hat{\ell}\int_{-1}^{1} f(r_{12})\sqrt{4\pi}\hat{\ell}^{-1}Y_{\ell 0}(\theta, 0)d\mu \\ &= \sqrt{16\pi^3}\int_{-1}^{1} f(r_{12})Y_{\ell 0}(\theta, 0)d\mu, \qquad \text{QED} \end{aligned}$$

C.2 Vector function of $\vec{r}_{12}$

A vector function $f(r_{12})Y_{\ell m}(\hat{r}_{12})$ can be expanded

$$f(r_{12})Y_{\ell m}(\hat{r}_{12}) = \sqrt{4\pi}\sum_{\ell_1 \ell_2} f_{\ell\ell_1\ell_2}(r_1, r_2)[Y_{\ell_1}Y_{\ell_2}]_{\ell m}$$

$$
\begin{aligned}
f_{\ell\ell_1\ell_2} &= \frac{1}{\sqrt{4\pi}}\sum_m \int f(r_{12})Y_{\ell m}(\hat{r}_{12})[Y_{\ell_1}Y_{\ell_2}]^*_{\ell m}d\hat{r}_1 d\hat{r}_2 \\
&= \frac{1}{\sqrt{4\pi}}\frac{1}{2\ell+1}\sum_m \int\int f(r_{12})Y_{\ell m}(\hat{r}_{12}) \\
&\times \sum_{m_1 m_2}(\ell_1 m_1 \ell_2 m_2|\ell m)Y_{\ell_1 m_1}(\hat{r}_1)Y^*_{\ell_2 m_2}(\hat{r}_2)d\hat{r}_1 d\hat{r}_2
\end{aligned}
$$

Now we choose $\hat{r}_1 \parallel \hat{z}$, and thus $m_1 = 0$ and $m_2 = m$. Then we have

$$
\begin{aligned}
Y_{\ell_1 m_1}(\hat{r}_1) &= Y_{\ell_1 0}(\cos\theta) = \frac{\hat{\ell}_1}{\sqrt{4\pi}}, \\
Y_{\ell_2 m_2}(\hat{r}_2) &= Y_{\ell_2 m}(\theta,0), \qquad Y_{\ell m}(\hat{r}_{12}) = Y_{\ell m}(\theta',0) \\
\int d\hat{r}_1 &\Rightarrow 4\pi, \qquad \int d\hat{r}_2 \Rightarrow 2\pi\int d\mu
\end{aligned}
$$

where θ is the angle between z-axis ($\vec{r}_1$) and $\vec{r}_2$ while θ' the angle between z-axis and $\vec{r}_{12}$ as shown in Fig.4. Then we have

$$
\begin{aligned}
f_{\ell\ell_1\ell_2}(r_1,r_2) &= \frac{1}{2\ell+1}\sum_m \int f(r_{12})Y_{\ell m}(\theta',0)\ \frac{1}{\sqrt{4\pi}}\frac{\hat{\ell}_1}{\sqrt{4\pi}}(\ell_1 0 \ell_2 m|\ell m)4\pi Y^*_{\ell_2 m}(\theta,0)2\pi d\mu \\
&= \frac{2\pi}{2\ell+1}\sum_m \hat{\ell}_1(\ell_1 0 \ell_2 m|\ell m)\int f(r_{12})Y_{\ell m}(\theta',0)Y^*_{\ell_2 m}(\theta,0)d\mu
\end{aligned}
$$

where

$$
\begin{aligned}
\mu &\equiv \cos\theta, \quad (\hat{r}_1(\parallel \hat{k})\cdot\hat{r}_2), \\
\mu' &\equiv \cos\theta' = \frac{r_2\mu - r_1}{r_{12}}, \quad (\hat{r}_1(\parallel \hat{k})\cdot\hat{r}_{12}) \\
r_{12}^2 &= r_1^2 + r_2^2 - 2r_1 r_2, \mu \\
r_2^2 &= r_{12}^2 + r_1^2 + 2r_1 r_{12}\mu'
\end{aligned}
$$

5. DWBA_fig

Coordinate System (Stripping)

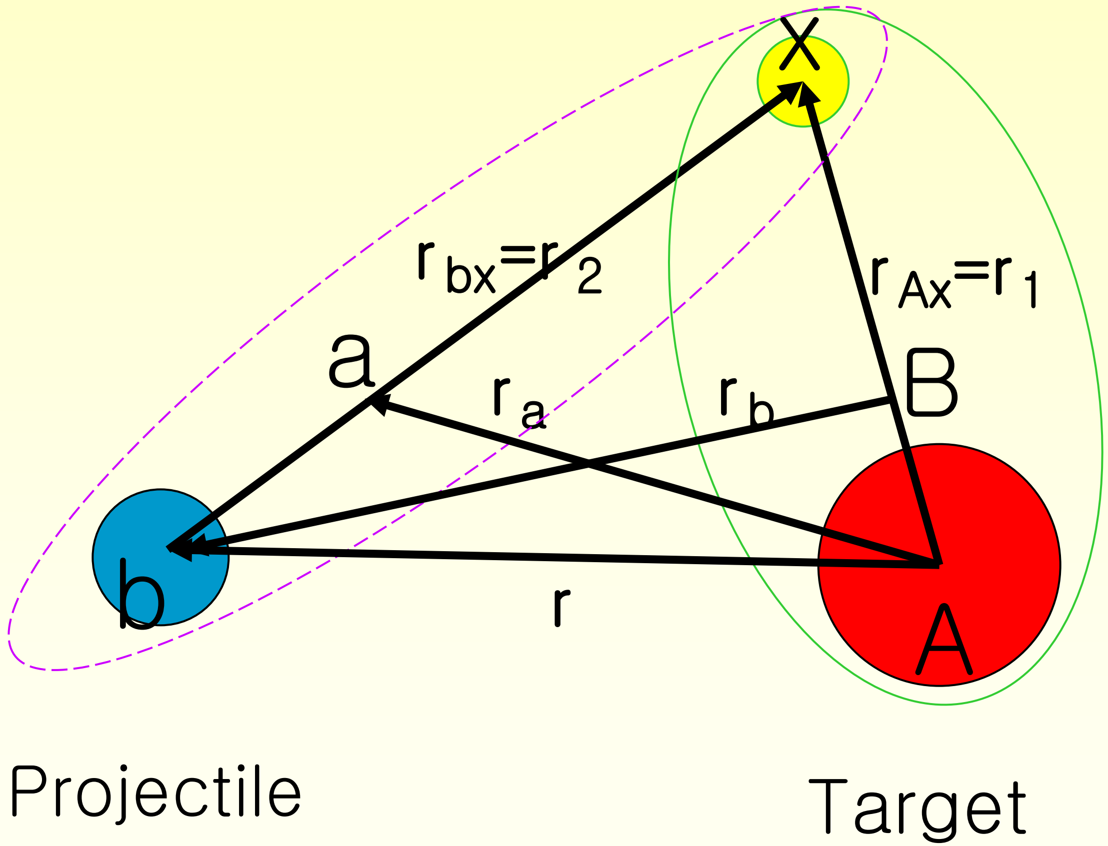


Figure 1

Coordinate System (Pick-up)

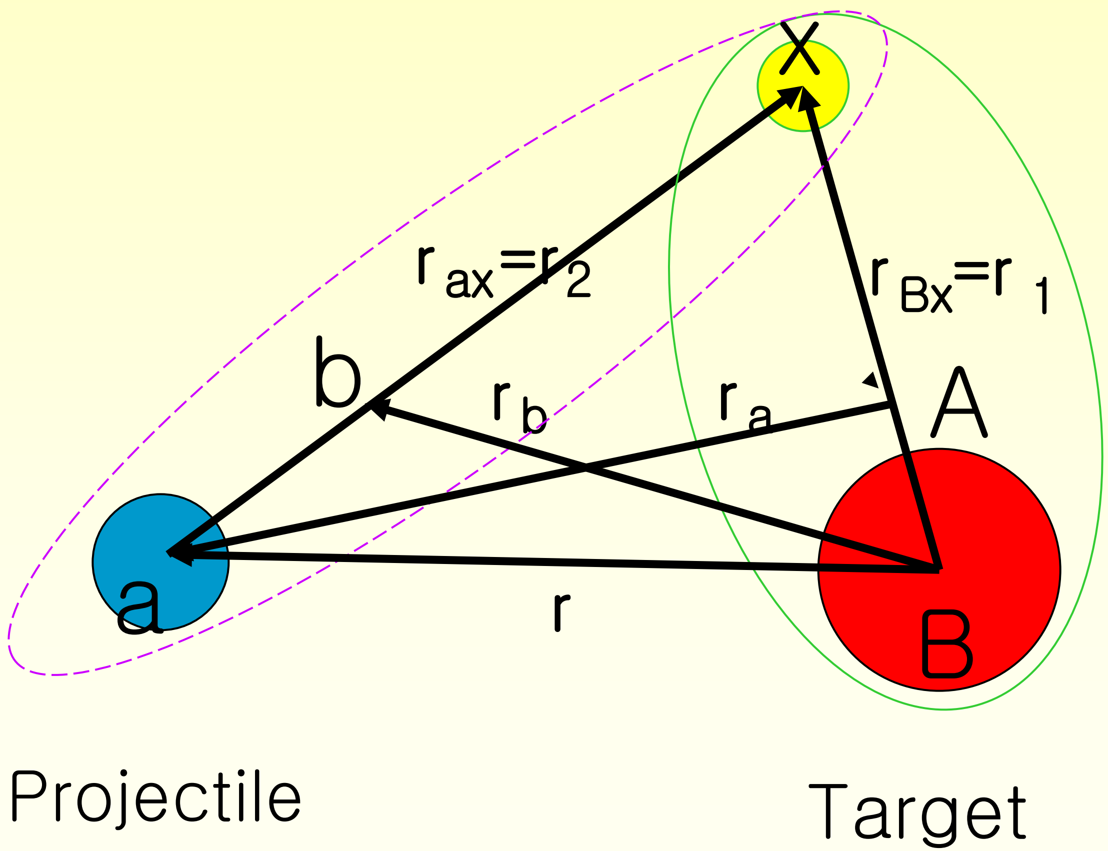
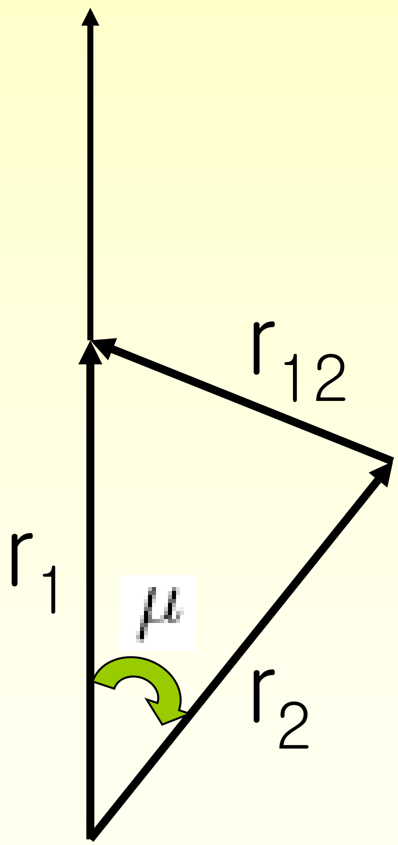


Figure 2

Multipole Expansion of Scalar Function

(1) A scalar function $f(r_{12})$ in terms of function of r_1 and r_2.



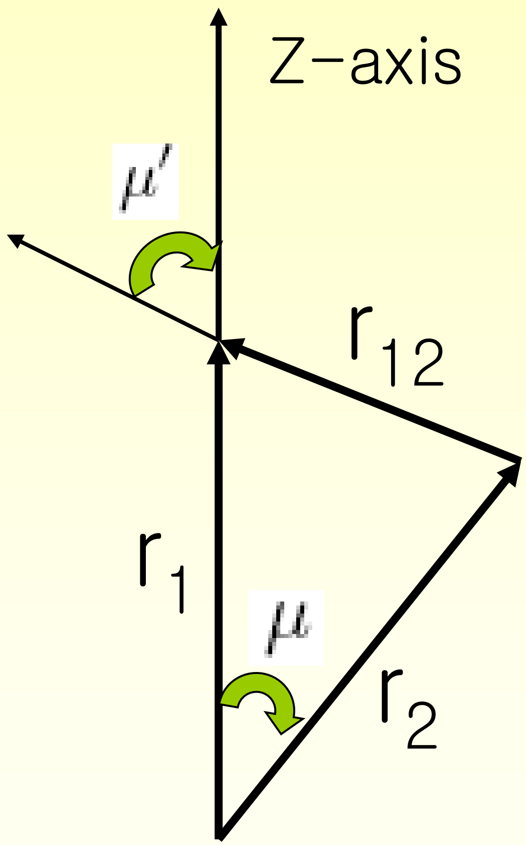
$$
\begin{aligned}
\vec{r}_{12} &= \vec{r}_1 - \vec{r}_2 \\
r_{12}^2 &= r_1^2 + r_2^2 - 2r_1 r_2 \mu, \\
\mu &= \cos\theta
\end{aligned}
$$

$$
\begin{aligned}
f(r_{12}) &= \sum_{\ell} f_\ell(r_1, r_2)(2\ell+1)P_\ell(\cos\theta) \\
&= \sum_{\ell} f'_\ell(r_1, r_2)(-)^\ell [Y_\ell Y_\ell]_{00} \\
f_\ell(r_1, r_2) &= \frac{1}{2}\int_{-1}^{1} f(r_{12})P_\ell(\cos\theta)d\mu \\
f'_\ell(r_1, r_2) &= \sqrt{16\pi^3}\int_{-1}^{1} f(r_{12})Y_{\ell 0}(\theta, 0)d\mu
\end{aligned}
$$

Figure 3

Multipole Expansion of Vector Function

(2) A vector function $f(r_{12})Y_{\ell m}(\hat{r}_{12})$ in terms of function of r_1 and r_2.



$$
\begin{aligned}
\vec{r}_{12} &= \vec{r}_1 - \vec{r}_2 \\
r_{12}^2 &= r_1^2 + r_2^2 - 2r_1 r_2 \mu, \\
\mu &= \cos\theta = \hat{r}_1 \cdot \hat{r}_2 \\
\mu' &= \cos\theta' = \hat{r}_1 \cdot \hat{r}_{12} \\
&= \frac{r_1 - r_2\mu}{r_{12}} \\
f(r_{12})Y_{\ell m}(\hat{r}_{12}) &= \sqrt{4\pi}\sum_{\ell_1\ell_2} f_{\ell\ell_1\ell_2}(r_1, r_2)[Y_{\ell_1}Y_{\ell_2}]_{\ell m} \\
f_{\ell\ell_1\ell_2}(r_1, r_2) &= \frac{2\pi}{2\ell+1}\sum_m \hat{\ell}_1(\ell_1 0 \ell_2 m|\ell m) \\
&\quad \times \int f(r_{12})Y_{\ell m}(\mu', 0)Y^*_{\ell_2 m}(\mu, 0)d\mu
\end{aligned}
$$

Figure 4