Homework 4: Dynamic optimization

This homework is due at 5pm on Friday March 24.
See homework #1 for guidelines on submitting your work.

Consider the decision problem of Harold Zurcher, bus mechanic. Zurcher's job is to make decisions about the maintenance and replacement of bus engines. Zurcher oversees a fleet of 100 buses, and during each period he must choose whether to maintain (a = 0) or replace (a = 1) an engine. Each bus (i) is inspected once per quarter (t), and we keep track of a bus engine's age (s) in integer values. Think of Zurcher solving 100 parallel infinite-horizon dynamic optimization problems (1 problem for each bus). We will observe Zurcher for 6 years (24 periods) with an exogenous collection of initial s values for the buses. The problem would be largely unchanged if we observed Zurcher maintaining a single bus for 2400 quarters rather than 100 buses for 24 quarters, so I omit i subscripts wherever possible below.

When Zurcher performs maintenance on a bus engine, he expends an effort cost of $\theta_0 s + \varepsilon_0$. In the next period the bus's age becomes $s' = s + 1$ if $s < 5$, and $s' = 5$ if $s = 5$. Whenever a bus engine is replaced, Zurcher pays the effort cost of $\theta_1 + \varepsilon_1$, and that bus's age is set to $s = 0$. (The new engine has age $s = 1$ during the next period.) This is an "optimal stopping problem" in that Zurcher is choosing whether to "stop" maintenance on an engine by replacing it. The vector $\theta = (\theta_0, \theta_1)$ contains parameters to be estimated, and ε_a is a taste shock that follows Type-I extreme value distribution, and it is independent across buses, actions, and time. Zurcher's objective is to maximize his discounted expected utility; he discounts future periods at rate $\beta = 0.95$.

Let $V_a(s,\varepsilon)$ represent Zurcher's choice-specific value function for action a while in state s. Let $V(s'|s,a)$ be next period's expected value when Zurcher enters that period in state s' following action a in the previous period. Once the next period is reached, the previous period's action is not relevant, so in some places below I replace $V(s'|s,a)$ with $V(s')$ or $V(s)$. Incorporating the notation above, Zurcher's value from a = 0 in state s is:

$$V_0(s,\varepsilon_0) = -\theta_0 s + \varepsilon_0 + \beta V(s'|s,a=0).$$

When in state s, Zurcher's expected value from a = 1 is:

$$V_1(s,\varepsilon_1) = -\theta_1 + \varepsilon_1 + \beta V(s'|s,a=1).$$

Due to the functional form of ε, Zurcher's choice probabilities in each period (for each bus) are multinomial logit (MNL) probabilities. Conditional on entering the period at state s, Zurcher chooses a = 0 with probability:

$$\Pr(a=0|s) = \frac{\exp[-\theta_0 s+\beta V(s'|s,a=0)]}{\exp[-\theta_0 s+\beta V(s'|s,a=0)]+\exp[-\theta_1+\beta V(s'|s,a=1)]}.$$

$\Pr(a = 1|s)$ is defined similarly. Before realizing a period's ε values, Zurcher's expected value at the start of the period has the logit inclusive value form:

$$V(s) = \log\{\exp[-\theta_0 s + \beta V(s'|s,0)] + \exp[-\theta_1 + \beta V(s'|s,1)]\}.$$

The value function $V(s'|s,a)$ is defined over a very limited support (i.e. possible combinations of s and a), and if we knew these values we could estimate θ with a simple MNL likelihood routine using data on bus maintenance choices. For buses i = 1 to N and periods t = 1 to T, the log likelihood function would be:

$$\mathcal{L} = \sum_i \sum_t [1\{a_{it} = 0\} \log(\Pr(a_{it} = 0|s_{it}) + 1\{a_{it} = 1\} \log(\Pr(a_{it} = 1|s_{it})] \,.$$

V depends on θ, however, so this simple approach is not possible. We must solve for $V(s)$ at each s conditional on θ. Once we have these function values, we can estimate θ with the likelihood function above.

You will use an iterative numerical procedure to solve for V conditional on θ. The steps of this procedure are:

1. Guess at a value of θ.
2. Set initial values of V at each s. (Perhaps V = 0 for all s.) Let the index r = 0 and denote the initial value function V_r.
3. Using $V_r(s'|s, a)$ and θ, write $V_{r+1}(s)$ using the logit inclusive value expression:

   $$V_{r+1}(s) = \log\{\exp[-\theta_0 s + \beta V_r(s'|s,0)] + \exp[-\theta_1 + \beta V_r(s'|s,1)]$$

   for each s.
4. Replace V_r with V_{r+1}, and repeat step (3) until the Vs converge at each s.
5. Use $V(s|\theta)$ to evaluate the likelihood function $\mathcal{L}(\theta)$. Revise the guess at θ and return to step (2). Continue until θ converges.

This is a "nested fixed point" (NFXP) procedure for estimating θ.

For this assignment, you will use the data in bus_data_hw4.csv to estimate θ for two assumptions on β. First, assume that β = 0, as if Zucher is completely myopic. This is equivalent to fixing V = 0 for future actions. (You should be able to estimate θ in Stata or (e.g.) Matlab when β = 0.) Second, assume that β = 0.95 and estimate the dynamic model using the nested fixed point algorithm described above. You should submit all programs plus a write-up that includes:

1. Summary statistics on the data, including $\Pr(a|s)$ values.
2. Estimates of θ for β = 0. Provide standard errors.
3. Estimates of θ for β = 0.95. Provide standard errors.