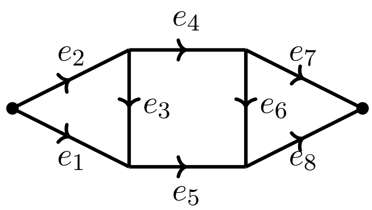

$$= I = \frac{1}{E_1+E_2-q^0}\frac{1}{E_1+E_3+E_4-q^0}\times$$
$$\times\left[\frac{1}{E_4+E_5-q^0}+\frac{1}{E_1+E_3+E_6+E_7-q^0}\right]\frac{1}{E_5+E_6+E_7-q^0}\frac{1}{E_7+E_8-q^0} \tag{1}$$

We now consider the IR subset given by the following three cuts $c_1=\{e_7,e_8\}$, $c_2=\{e_5,e_6,e_7\}$, $c_3=\{e_1,e_3,e_6,e_7\}$. The forest formula is

$$R[I] = I - \sum_{\substack{C\subseteq\{c_1,c_2,c_3\}\\ C\neq\emptyset}} Z_C[I] \tag{2}$$
$$= I - Z_{123}[I] - Z_{12}[I] - Z_{23}[I] - Z_{13}[I] - Z_1[I] - Z_2[I] - Z_3[I], \tag{3}$$

with

$$Z_C[I] = T_C\left[I - \sum_{\substack{C'\subseteq\{c_1,c_2,c_3\}\\ \text{s.t.} C\subset C'}} Z_{C'}[I]\right]. \tag{4}$$

The triple counter-term reads:

$$Z_{123}[I] = T_{123}[I] = \frac{1}{E_2-E_3-E_6-E_7}\frac{1}{E_4-E_6-E_7}\frac{1}{E_1+E_3+E_6+E_7-q^0}\frac{1}{E_5+E_6+E_7-q^0}\frac{1}{E_7+E_8-q^0}. \tag{5}$$

Notice that one of the two cross-free families of the orientation has been set to zero because it does not have all three surfaces. $T_{123}[I]$ is the only term that will lead to three distinct cuts after the application of Local Unitarity. Then, we consider

$$Z_{12}[I] = T_{12}[I - T_{123}[I]], \tag{6}$$

where

$$\begin{aligned}T_{12}[I] =& \frac{1}{E_1+E_2-q^0}\frac{1}{E_1+E_3+E_4-q^0}\times\\ &\times\left[\frac{1}{E_4-E_6-E_7}+\frac{1}{E_1+E_3+E_6+E_7-q^0}\right]\frac{1}{E_5+E_6+E_7-q^0}\frac{1}{E_7+E_8-q^0}\\ =& \frac{1}{E_1+E_2-q^0}\frac{1}{E_4-E_6-E_7}\frac{1}{E_1+E_3+E_6+E_7-q^0}\frac{1}{E_5+E_6+E_7-q^0}\frac{1}{E_7+E_8-q^0}\end{aligned} \tag{7}$$

$$T_{12}[T_{123}[I]] = T_{123}[I] = \frac{1}{E_2-E_3-E_6-E_7}\frac{1}{E_4-E_6-E_7}\frac{1}{E_1+E_3+E_6+E_7-q^0}\frac{1}{E_5+E_6+E_7-q^0}\frac{1}{E_7+E_8-q^0} \tag{8}$$

Importantly, the surface $E_1+E_3+E_4-q^0$ has disappeared in the approximation $T_{12}[I]$, as it identifies two connected components with the cut c_2. Combining the terms we get

$$Z_{12}[I] = -\frac{1}{E_2-E_3-E_6-E_7}\frac{1}{E_4-E_6-E_7}\frac{1}{E_1+E_2-q^0}\frac{1}{E_5+E_6+E_7-q^0}\frac{1}{E_7+E_8-q^0}, \tag{9}$$

which does not have a singularity on c_3, as we expected. We now look at

$$Z_{13}[I] = T_{13}[I - T_{123}[I]]. \tag{10}$$

The graph in between c_1 and c_3 has a propagator, which will need to be approximated, but there is an ambiguity due to the lack of energy conservation. We choose to approximate it with c_1; this choice must be kept consistent both when we compute $T_{13}[I]$ and $T_{13}[T_{123}[I]]$. We find:

$$T_{13}[I] = T_{13}[T_{123}[I]] = \frac{1}{E_2 - E_3 - E_6 - E_7} \frac{1}{E_4 - E_6 - E_7} \frac{1}{E_1 + E_3 + E_6 + E_7 - q^0} \frac{1}{E_5 + E_6 - E_8} \frac{1}{E_7 + E_8 - q^0}, \tag{11}$$

so that $Z_{13}[I] = 0$. It is easy to check that the same holds for $Z_{23}[I]$, which also equals to zero.

Finally, we get to triple nesting of approximation operators. We start with

$$Z_1[I] = T_1[I - Z_{12}[I] - Z_{13}[I] - Z_{123}[I]] = T_1[I - Z_{12}[I] - Z_{123}[I]] = I - Z_{12}[I] - Z_{123}[I], \tag{12}$$

where in the last step we used the fact that in this case T_1 does not approximate anything (all thresholds are "loop level" from the point of view of c_1). Thus

$$Z_1[I] = -\frac{1}{(E_4 - E_5 - E_7)(E_1 + E_2 - q^0)(E_1 + E_3 + E_4 - q^0)(E_4 + E_5 - q^0)(E_7 + E_8 - q^0)}, \tag{13}$$

which, as expected, is only singular at c_1. Continuing, we have

$$Z_2[I] = T_2[I - Z_{12}[I] - Z_{23}[I] - Z_{123}[I]] = T_2[I - Z_{12}[I] - Z_{123}[I]], \tag{14}$$

with

$$\begin{aligned} T_2[I] =& \frac{1}{E_1 + E_2 - q^0} \frac{1}{E_1 + E_3 + E_4 - q^0} \times \\ & \times \left[\frac{1}{E_4 - E_6 - E_7} + \frac{1}{E_1 + E_3 + E_6 + E_7 - q^0} \right] \frac{1}{E_5 + E_6 + E_7 - q^0} \frac{1}{E_8 - E_5 - E_6} \end{aligned} \tag{15}$$

$$T_2[Z_{12}[I]] = -\frac{1}{E_2 - E_3 - E_6 - E_7} \frac{1}{E_4 - E_6 - E_7} \frac{1}{E_1 + E_2 - q^0} \frac{1}{E_5 + E_6 + E_7 - q^0} \frac{1}{E_8 - E_5 - E_6} \tag{16}$$

$$T_2[Z_{13}[I]] = \frac{1}{E_2 - E_3 - E_6 - E_7} \frac{1}{E_4 - E_6 - E_7} \frac{1}{E_1 + E_3 + E_6 + E_7 - q^0} \frac{1}{E_5 + E_6 + E_7 - q^0} \frac{1}{E_8 - E_5 - E_6}. \tag{17}$$

Combining everything, we get

$$Z_2[I] = 0. \tag{18}$$

This is expected: the cut c_2 gives a real-virtual interference diagram, which necessarily is singular at c_1. It is thus not possible to "isolate" c_2. An analogous argument holds for c_3; indeed:

$$Z_3[I] = T_3[I - Z_{13}[I] - Z_{23}[I] - Z_{123}[I]] = T_3[I - Z_{123}[I]] = 0, \tag{19}$$

since $T_3[I] = T_3[Z_{123}[I]]$. Finally, we may easily check that

$$R[I] = I - Z_{123}[I] - Z_{12}[I] - Z_1[I] = 0, \tag{20}$$

showing that the algebraic subtraction correctly isolates singular behaviours.