1 (b) Assume the measurements for $X_1 \cdots X_i$ are $n_1 \cdots n_i$.

$$L(R) = P(n_1, n_2, \cdots, n_N | R) = \prod_{i=1}^{N} P(n_i | R) = \prod_{i=1}^{N} \frac{R^{n_i} e^{-R}}{n_i!}$$

Maximizing $L(R)$ is minimizing $B(R) = -2\ln L(R)$:

$$B(R) := -2\ln L(R) = 2\sum_{i=1}^{N}\left[R - n_i \ln R + \ln(n_i!)\right]$$

$$\frac{\partial B(R)}{\partial R} = 2\sum_{i=1}^{N}\left(1 - \frac{n_i}{R}\right) = 0; \quad \frac{\partial^2 B}{\partial R^2} = \frac{2}{R^2}\sum_{i=1}^{N} n_i$$

$$\frac{\partial B(R)}{\partial R} = 0 \Rightarrow \hat{R} = \frac{1}{N}\sum_{i=1}^{N} n_i = R_{ML}.$$

$$Var[\hat{R}] = \frac{2}{\left.\frac{\partial^2 B}{\partial R^2}\right|_{\hat{R}}} = \frac{\hat{R}^2}{\sum_{i=1}^{N} n_i} = \frac{1}{N^2}\sum_{i=1}^{N} n_i = \frac{\hat{R}}{N}$$

$\hat{R}$ is an unbiased estimator for R, since it is the expected value of R.
$\hat{R}$ is a minimum variance statistic.

2 (a) $F(\lambda) = C + A\,G(\lambda)$

$$L(C,A) = \prod_{i=1}^{N} P(F_i | C, A) = \prod_{i=1}^{N}\left[C + AG_i(\lambda)\right]$$

$$-2\ln L = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}(F_i - C - AG_i)^2 + 2\sum_{i=1}^{N}\ln\sigma_i + N\ln 2\pi$$

$$0 = \frac{\partial(2\ln L)}{\partial C} = \sum_{i=1}^{N}\frac{1}{\sigma_i^2}\frac{\partial}{\partial C}(F_i - C - AG_i)^2 = -2\sum_{i=1}^{N}\frac{F_i - AG_i}{\sigma_i^2} + 2C\sum_{i=1}^{N}\frac{1}{\sigma_i^2}$$

$$\Rightarrow \boxed{\hat{C} = \sum_{i=1}^{N}\frac{F_i - AG_i}{\sigma_i^2} \Big/ \sum_{i=1}^{N}\frac{1}{\sigma_i^2}}$$

$$\boxed{Var[\hat{C}]} = \frac{2}{\left.\frac{\partial^2}{\partial C^2}(-2\ln L)\right|_{C=\hat{C}}} = \frac{2}{\left.\frac{\partial}{\partial C}\left(-\sum_{i=1}^{N}\frac{2}{\sigma_i^2}(F_i - C - AG_i)\right)\right|_{C=\hat{C}}} = \boxed{\frac{1}{\sum_{i=1}^{N}\frac{1}{\sigma_i^2}}}$$

$$0 = \frac{\partial(-2\ln L)}{\partial A} = \frac{\partial}{\partial A}\left(\sum_{i=1}^{N}\frac{1}{\sigma_i^2}(F_i - C - AG_i)^2\right) = -2\sum_{i=1}^{N}\frac{G_i}{\sigma_i^2}(F_i - C) + 2\sum_{i=1}^{N}\frac{AG_i^2}{\sigma_i^2}$$

$$\Rightarrow \boxed{\hat{A} = \sum_{i=1}^{N}\frac{G_i}{\sigma_i^2}(F_i - C) \Big/ \sum_{i=1}^{N}\frac{G_i^2}{\sigma_i^2}}$$

$$\boxed{Var[\hat{A}]} = \frac{2}{\frac{\partial^2}{\partial A^2}(-2\ln L)\Big|_{A=\hat{A}}} = \frac{1}{\sum_{i=1}^{N}\frac{G_i}{\sigma_i^2}\frac{\partial}{\partial A}(F_i - C - AG_i)} \boxed{= \frac{1}{\sum_{i=1}^{N}\frac{G_i^2}{\sigma_i^2}}}$$

2(c) $$H = \begin{bmatrix} \frac{\partial^2\chi^2}{\partial C^2} & \frac{\partial^2\chi^2}{\partial C\partial A} \\ \frac{\partial^2\chi^2}{\partial A\partial C} & \frac{\partial^2\chi^2}{\partial A^2} \end{bmatrix} \qquad \chi^2 = \sum_{i=1}^{N}\left(\frac{F_i - C - AG_i}{\sigma_i}\right)^2$$

$$\frac{\partial\chi^2}{\partial C} = \sum_{i=1}^{N}\frac{-2(F_i - AG_i) + 2C}{\sigma_i^2}; \quad \frac{\partial\chi^2}{\partial C^2} = \sum_{i=1}^{N}\frac{2}{\sigma_i^2}; \quad \frac{\partial\chi^2}{\partial A\partial C} = \sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2}$$

$$\frac{\partial\chi^2}{\partial A} = \sum_{i=1}^{N}\frac{-2(F_i - C)G_i + 2G_i^2 A}{\sigma_i^2}; \quad \frac{\partial\chi^2}{\partial A^2} = \sum_{i=1}^{N}\frac{2G_i^2}{\sigma_i^2}; \quad \frac{\partial\chi^2}{\partial C\partial A} = \sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2}$$

$$\Rightarrow H = \begin{bmatrix} \sum_{i=1}^{N}\frac{2}{\sigma_i^2} & \sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2} \\ \sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{2G_i^2}{\sigma_i^2} \end{bmatrix} \qquad H^{-1} = \frac{1}{H_{11}\cdot H_{22} - H_{12}\cdot H_{21}}\begin{bmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11} \end{bmatrix}$$

$$H^{-1} = \frac{1}{\sum_{i=1}^{N}\frac{2}{\sigma_i^2}\sum_{i=1}^{N}\frac{2G_i^2}{\sigma_i^2} - \left(\sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2}\right)^2}\begin{bmatrix} \sum_{i=1}^{N}\frac{2G_i^2}{\sigma_i^2} & -\sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2} \\ -\sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2} & \sum_{i=1}^{N}\frac{2}{\sigma_i^2} \end{bmatrix}$$

$$Var[C] = \frac{2}{\frac{\partial^2\chi^2}{\partial C^2}} = \frac{1}{\sum_{i=1}^{N}\frac{1}{\sigma_i^2}}; \quad Var[A] = \frac{2}{\frac{\partial^2\chi^2}{\partial A^2}} = \frac{1}{\sum_{i=1}^{N}\frac{G_i^2}{\sigma_i^2}}$$

$$Cov(C,A) = [H^{-1}]_{12} = \frac{-\sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2}}{\sum_{i=1}^{N}\frac{2}{\sigma_i^2}\sum_{i=1}^{N}\frac{2G_i^2}{\sigma_i^2} - \left(\sum_{i=1}^{N}\frac{2G_i}{\sigma_i^2}\right)^2}$$

3(a) $K_x = K_c \cdot q$

$$K_c(1+q) = \frac{2\pi a \sin i}{P} \Rightarrow a = \frac{K_c(1+q)P}{2\pi \sin i}$$

$$\left(\frac{2\pi}{P}\right)^2 = \frac{G(M_x+M_c)}{a^3} = \frac{G}{a^3} M_x (1+q)$$

$$\Rightarrow M_x = \frac{(2\pi)^2 a^3}{P^2(1+q)G} = \frac{\cancel{(2\pi)^2}}{\cancel{P^2}\cancel{(1+q)}G} \cdot \frac{K_c^3 (1+q)^{\cancel{3}\,2} P^{\cancel{3}}}{(2\pi)^{\cancel{3}}(\sin i)^3} = \frac{K_c^3 (1+q)^2 P}{2\pi G (\sin i)^3}$$

$$\Rightarrow F_x = M_x \frac{(\sin i)^3}{(1+q)^2} = \frac{K_c^3 P}{2\pi G}.$$

since $(\sin i)^3 \le 1$, $(1+q)^2 \geqslant 1$, $\frac{(\sin i)^3}{(1+q)^2} \le 1$.

$\therefore F_x \le M_x \Rightarrow F_x$ is a lower limit on M_x.

$$(V_{rot}\sin i)^3 = (0.462)^3 \frac{M_c}{\cancel{M_X}} \cdot \frac{\cancel{M_X}(\sin i)^3}{K_c^3 P} \cdot 2\pi G$$

$$M_c = \frac{(V_{rot}\sin i)^3 K_c^3 P}{(0.462)^3(\sin i)^3 2\pi G} = \frac{(V_{rot}\sin i)^3}{(0.462)^3(\sin i)^3} F_X$$

$$M_X = M_c \cdot \frac{K_c}{K_X} = \frac{(V_{rot}\sin i)^3 F_X}{(0.462)^3(\sin i)^3} \cdot \frac{K_c}{\frac{2\pi a \sin i}{P} - K_c}.$$