3(b)

1st Moment: $m_1 = \langle x \rangle$

Uniform: $\langle x \rangle = \int_{-\infty}^{\infty} x f(x)\,dx = \int_l^u \frac{x}{u-l}\,dx = \frac{1}{u-l}\left[\frac{1}{2}x^2\right]_l^u = \frac{u+l}{2}$

Gaussian: for standard normal:
$\langle Z \rangle = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} x e^{-\frac{1}{2}x^2}\,dx + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} x e^{-\frac{1}{2}x^2}\,dx = \frac{1}{\sqrt{2\pi}}\left(\left[e^{-\frac{1}{2}x^2}\right]_{-\infty}^{0} + \left[e^{-\frac{1}{2}x^2}\right]_0^{\infty}\right) = 0$

Exponential: $\langle x \rangle = \int_0^{\infty} x \lambda e^{-\lambda x}\,dx = \left[-x e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx = 0 + \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^{\infty} = \frac{1}{\lambda}$

$$\boxed{\begin{aligned}\langle X \rangle &= \langle \mu + \sigma Z \rangle \\ &= \mu + \sigma \langle Z \rangle \\ &= \mu\end{aligned}}$$

2nd Moment: $m_2 = \langle [X - \langle X \rangle]^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$

Uniform: $\langle x^2 \rangle = \int_l^u x^2 f(x)\,dx = \frac{1}{u-l}\int_l^u x^2\,dx = \frac{1}{u-l}\left[\frac{1}{3}x^3\right]_l^u = \frac{u^2+ul+l^2}{3}$

$\langle x \rangle^2 = \left(\frac{u+l}{2}\right)^2$

$m_2 = Var[X] = \frac{u^2+ul+l^2}{3} + \left(\frac{u+l}{2}\right)^2 = \frac{(u-l)^2}{12}$

Gaussian: for standard normal:

$\langle Z^2 \rangle = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = \frac{1}{\sqrt{2\pi}}\left(\left[-x e^{-\frac{1}{2}x^2}\right]_{-\infty}^{0} + \int_{-\infty}^{0} e^{-\frac{1}{2}x^2}\,dx + \left[-x e^{-\frac{1}{2}x^2}\right]_0^{\infty} + \int_0^{\infty} e^{-\frac{1}{2}x^2}\,dx\right)$

$= \frac{1}{\sqrt{2\pi}}\left(0 + 0 + \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\langle Z \rangle^2 = 0.$

$Var[Z] = 1 - 0 = 1.$

$m_2 = Var[X] = Var[\mu + \sigma Z] = \sigma^2 Var[Z] = \sigma^2.$

Exponential: $\langle x^2 \rangle = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx = \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x}\,dx = 0 + \left[-\frac{2}{\lambda} x e^{-\lambda x}\right]_0^{\infty} + \frac{2}{\lambda}\int_0^{\infty} e^{-\lambda x}\,dx$

$= \frac{2}{\lambda}\left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^{\infty} = \frac{2}{\lambda^2}$

$\langle x \rangle^2 = \frac{1}{\lambda^2}$

$m_2 = Var[x] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$

3rd Moment: $m_3 = \langle [\frac{x - \langle x \rangle}{\sigma}]^3 \rangle$

Uniform: $m_3 = \int_l^u \left(\frac{x - \langle x \rangle}{\sigma}\right)^3 \cdot \frac{1}{u-l}\,dx = \frac{1}{(u-l)\sigma^3}\int_l^u (x - \langle x \rangle)^3\,dx = \frac{1}{(u-l)\sigma^3}\int_l^u \left(x - \frac{u+l}{2}\right)^3 dx$

$\left(y = x - \frac{u+l}{2}\right) = \frac{1}{(u-l)\sigma^3}\int_{-\frac{u-l}{2}}^{\frac{u-l}{2}} y^3\,dy$ $\longrightarrow$ is odd $\Rightarrow$ integral $= 0$

$\therefore m_3 = 0$

Gaussian: $m_3 = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^3 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\infty}^{\infty} \underbrace{\left(\frac{u}{\sigma}\right)^3 e^{-\frac{u^2}{2\sigma^2}}}_{\text{is odd} \Rightarrow \text{integral} = 0} \cdot \frac{1}{\sqrt{2\pi}\sigma}\,du$

$\therefore m_3 = 0$

Exponential: $m_3 = \langle [\frac{(x-\mu)^3}{\sigma^3}] \rangle = \frac{\langle x^3 \rangle - 3\mu\langle x^2 \rangle + \mu^2\langle x \rangle - \mu^3}{\sigma^3} = \frac{\langle x^3 \rangle - 3\mu\sigma^2 - \mu^3}{\sigma^3}$

$\langle x^3 \rangle = \int_0^{\infty} x^3 \lambda e^{-\lambda x}\,dx = \frac{6}{\lambda^3}$

$\therefore m_3 = \frac{\frac{6}{\lambda^3} - 3 \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda^2} - \frac{1}{\lambda^3}}{\frac{1}{\lambda^3}} = 2.$

4th Moment: $m_4 = \langle [\frac{x-\mu}{\sigma}]^4 \rangle = \frac{\langle x^4\rangle - 4\mu\langle x^3\rangle + 6\mu^2\langle x^2\rangle - 3\mu^4}{\sigma 4}$

Uniform: $m_4 = \int_l^u \frac{(x-\mu)^4}{\sigma^4} \cdot \frac{1}{u-l} dx = \frac{1}{\sigma^4 (u-l)} \cdot \int_{-\frac{u-l}{2}}^{\frac{u-l}{2}} y^4 dy = \frac{1}{(\frac{(u-l)^2}{12})^2 (u-l)} \cdot [\frac{1}{5} y^5]_{-\frac{u-l}{2}}^{\frac{u-l}{2}}$

$= \frac{\cancel{\frac{1}{(u-l)^5}}}{12^2} \cdot \frac{2}{5} \cdot \frac{\cancel{(u-l)^5}}{2^5} = \frac{9}{5}.$

Gaussian: $m_3 = \frac{\langle x^3\rangle - 3\mu\sigma^2 - \mu^3}{\sigma^3} = 0 \Rightarrow \langle x^3\rangle = 3\mu\sigma^2 + \mu^3$

$\cancel{\langle x^4\rangle = \int_{-\infty}^{\infty} x^4 f(x) dx = [x^3 f(x)]_{-\infty}^{\infty} + 3\int_{-\infty}^{\infty} x^2 f(x) dx = 0 + 3 Var[x]}$

$\langle X^4\rangle = M_X^{(4)}(0) = 3\sigma^4 + 6\sigma^2(\mu+\sigma^2 t)^2 + (\mu+\sigma^2 t)^4) e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Big|_{t=0}$

$= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

$m_4 = \frac{\langle x^4\rangle - 4\mu(3\mu\sigma^2+\mu^3) + 6\mu^2(\sigma^2+\mu^2) - 3\mu^4}{\sigma^4} = 3.$

Exponential: $M_X^{(n)}(t) = \frac{(-1)^{n+1} n! \frac{1}{\lambda^n}}{(\frac{t}{\lambda}-1)^{n+1}} \Rightarrow M_X^{(4)}(0) = n! \frac{1}{\lambda^n} = 4! \frac{1}{\lambda^4} = \langle X^4\rangle. \quad \langle X^3\rangle = \frac{6}{\lambda^3} \quad \langle X^2\rangle = \frac{2}{\lambda^2}$

$m_4 = \frac{\frac{4!}{\lambda^4} - 4\cdot\frac{1}{\lambda}\cdot\frac{6}{\lambda^3} + 6\cdot\frac{1}{\lambda^2}\cdot\frac{2}{\lambda^2} - 3\cdot\frac{1}{\lambda^4}}{\frac{1}{\lambda^4}} = 4! - 24 + 12 - 3 = 9.$

4(b) $Var(\bar{X}) = Var(\frac{1}{N}\sum_{i=1}^N X_i) = \frac{1}{N^2} Var(\sum_{i=1}^N X_i) = \frac{1}{N^2}\sum_{i=1}^N Var(X_i)$ (if X_i, X_j independent)

$= \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}.$

$Var(X_{med}) = \frac{\pi}{2}\cdot\frac{\sigma^2}{N} > Var(\bar{X})$ consistent with MC simulation.