Dynamics, Simulation and Control of Quadcopters

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Outline

1. Introduction
2. Quadcopter Dynamics
   - Definitions
   - Kinematics
3. References

What is a Copter?

- Copters are flying vehicles which use rapidly spinning rotors make themselves aloft.
- Conventional copters have two rotors.
- There are two ways that rotors can be arranged:
  - As two coplanar rotors, spinning in opposite directions.
  - One main rotor providing thrust and a smaller side rotor oriented laterally.
- The design of copters comes with a control and swashplate mechanism complication.
- The swashplate mechanism was needed to allow the copter to utilize more degrees of freedom.

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Definitions

Degrees of Freedom (DOF)

DOF of a mechanical system is the number of independent parameters that define its configuration.

What is a Quadcopter?

- A quadcopter(quadrotor) is a copter which has four equally spaced rotors.
- With four independent rotors, the need for a swashplate mechanism is alleviated.
- Quadcopters can obtain copters's degrees of freedom by having two more rotors instead of swashplate.
- Quadcopter control is a fundamentally difficult and interesting problem due to six DOF and only four independent inputs.
- In order to achieve six DOF, rotational and translational motion are coupled.

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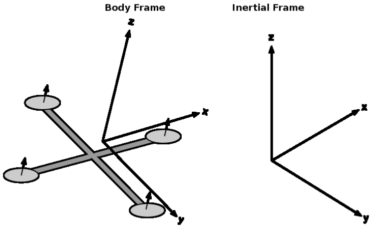
The Body and Inertial Frame

- The inertial frame is defined by the ground, with gravity pointing in the negative z direction.
- The body frame is defined by the orientation of the quadcopter, with the rotor axes pointing in the positive z direction and the arms pointing in the x and y directions.



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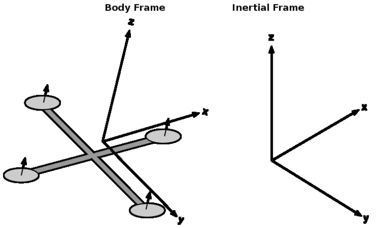
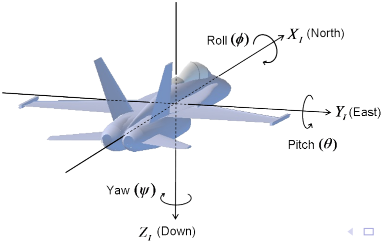


Roll, Pitch and Yaw degrees

- The **roll axis** (or longitudinal axis) passes through the plane from nose to tail – alongside of x axis.
- The **pitch axis** (also called lateral or transverse axis) passes through the plane from wingtip to wingtip – alongside of y axis.
- The vertical **yaw axis** is defined to be perpendicular to the wings with its origin at the center of gravity and directed towards the bottom of the aircraft – alongside of z axis.



Basics of The Kinematic

- Position and velocity of the quadcopter in the inertial frame are defined as $p = (x, y, z)^T$ and $\dot{p} = (\dot{x}, \dot{y}, \dot{z})^T$, respectively.
- The roll, pitch, and yaw angles in the body frame are defined as $\theta = (\phi, \theta, \psi)^T$, with corresponding angular velocities equal to $\dot{\theta} = (\dot{\phi}, \dot{\theta}, \dot{\psi})^T$.

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Basics of The Kinematic

cont.

- In order to convert roll, pitch and yaw velocities into the angular velocity vector, we can use the following relation:

$$\omega = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & s_\theta s_\phi \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} \dot{\theta} \tag{1}$$

where ω is the angular velocity vector in the body frame.

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Basics of The Kinematic

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- Rotation matrix R relates the body and inertial frame, which goes from the body frame to the inertial frame.

$$R = \begin{bmatrix} c_\phi c_\psi - c_\theta s_\phi s_\psi & -(c_\psi s_\phi + c_\phi c_\theta s_\psi) & s_\theta s_\psi \\ c_\theta c_\psi s_\phi + c_\phi s_\psi & c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\psi s_\theta \\ s_\phi s_\theta & c_\phi s_\theta & c_\theta \end{bmatrix} \quad (2)$$

- This matrix is derived by using the *ZYZ Euler angle* conventions.
- For a given vector $\vec{v}$ in the body frame, the corresponding vector is given by $R\vec{v}$ in the inertial frame.

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Physics

- In order to properly model the dynamics of the system, we need an understanding of the physical properties that govern it.
- Topics which are going to discuss in this section are as follow:
  - Motors
  - Forces
  - Torques

Physics
Motors

- For electric motors, the torque produced is given by:

$$\tau = K_t(I - I_0) \tag{3}$$

- Where:
  - τ is the motor torque.
  - I is the input current.
  - I_0 is the current when there is no load on the motor.
  - K_t is the torque proportionality constant.

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Physics
Motors – cont.

- The voltage across the motor is:

$$V = IR_m + K_v\omega \tag{4}$$

- Where:
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  - R_m is the motor resistance.
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Physics

Motors – cont.

- The motor's power consumption is:

$$P = IV = \frac{(\tau + K_t I_0)(K_t I_0 R_m + \tau R_m + K_t K_v \omega)}{K_t^2} \tag{5}$$

- For the purposes of our simple model, we will assume:
  - A negligible motor resistance(i.e $R_m \approx 0$).
  - $K_t I_0 \ll \tau$

$$\begin{aligned} P = IV &= \frac{(\tau + K_t I_0)(\cancelto{0}{K_t I_0 R_m} + \cancelto{0}{\tau R_m} + K_t K_v \omega)}{K_t^2} \\ &= \frac{\cancelto{\tau}{(\tau + K_t I_0)} K_v \omega}{K_t} \\ &= \frac{K_v}{K_t} \tau \omega \end{aligned} \tag{6}$$

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Physics

Forces

- By conservation of energy, we know that:

$$P.dt = F.dx \tag{7}$$

- Equivalently, the power is equal to the thrust times the air velocity:

$$F = \frac{P}{v_h} \xrightarrow{\text{eq.6}} F = \frac{K_p \tau \omega}{v_h} \tag{8}$$

- We assume that:
    - Vehicle speeds are low, so v_h is the air velocity when hovering.
    - The free stream velocity, v_∞ is zero.
    - Since $v_\infty = 0$ this is obvious that $v_h \sim K_m v_m$.

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Forces – cont.

- As torque is defined in The aerodynamic drag, we have:

$$\tau = \frac{1}{2}\rho A v_h^2 \tag{9}$$

- Where:
  - ρ is the density of the surrounding air.
  - A is the area swept out by the rotor.
- From equations 8 and 9, we can derive the following equation:

$$F = \frac{K_p \tau \omega}{v_m} \overset{\text{eq.9}}{=\!=} \frac{K_p}{2}\rho A v_m \omega$$

$$F = T \xrightarrow{\omega = \frac{v_m}{r}} T = k\omega^2 \tag{10}$$

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Forces – cont.

- By eq.10 total thrust on the quadcopter(in the body frame) is given by:

$$T_B = \sum_{i=1}^{4} T_i = k \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} w_i \end{bmatrix} \tag{11}$$

- For modeling the friction, we will assume highly simplified version, which is:

$$F_D = -k_d \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \tag{12}$$

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Physics
Torques

- Each rotor contributes some torque about the body z axis.
- The torque is required to keep the propeller spinning and providing thrust.
- It creates the instantaneous angular acceleration and overcomes the frictional drag forces.
- The drag equation from fluid dynamics gives us the frictional force:

$$F_D = \frac{1}{2}\rho C_D A v^2 \tag{13}$$

- Where:
  - ρ is the surrounding fluid density.
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  - C_D is a dimensionless constant.

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- From eq.13 and eq.14 we have:

$$\tau_D = \frac{1}{2} R\rho C_D A v^2 \xRightarrow{\omega=\frac{v}{R}} \tau_D = \frac{1}{2} R\rho C_D A (R\omega)^2 = b\omega^2 \tag{15}$$

- Where:
  - ω is the angular velocity of the propeller.
  - R is the radius of the propeller.

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Note:

We've assumed that all the force is applied at **the tip of the propeller**, which is certainly <u>inaccurate</u>.

Physics

Torques – cont.

- Torque about the z-axies for the ith motor is eqaul to:

$$\tau_i^z = \tau_{D_i} + I_M \dot{\omega}_i = b\omega_i^2 + I_M \dot{\omega}_i \tag{16}$$

- Where:
  - I_M is the moment of inertia about the motor z-axis.
  - $\dot{\omega}$ is the angular acceleration of the propeller.
  - b drag coefficient constant.
- Note that in eq.16 in steady state flight, $\dot{\omega}_i \approx 0$. So for simplification of our model we will assume $I_M \dot{\omega}_i = 0$.
- Inorder to neutralize the overall torque in quadcopter steady-state, two of the motors should spin clockwise, and the other two should spin counter-clockwise.

Physics

Torques – cont.

- Torque about the z-axies for the ith motor is eqaul to:

$$\tau_i^z = \tau_{D_i} + I_M \dot{\omega}_i = b\omega_i^2 + I_M \dot{\omega}_i \tag{16}$$

- Where:
  - I_M is the moment of inertia about the motor z-axis.
  - $\dot{\omega}$ is the angular acceleration of the propeller.
  - b drag coefficient constant.
- Note that in eq.16 in steady state flight, $\dot{\omega}_i \approx 0$. So for simplification of our model we will assume $I_M \dot{\omega}_i = 0$.
- Inorder to neutralize the overall torque in quadcopter steady-state, two of the motors should spin clockwise, and the other two should spin counter-clockwise.

Physics

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Physics

Torques – cont.

- Eq.16 can be written as following for each propellor:

$$\tau_i^z = (-1)^{i+1} b\omega_i^2 \tag{17}$$

- Which the $(-1)^{i+1}$ term is positive for the ith propeller if the propeller is spinning clockwise and negative if it is spinning counterclockwise.
- The total torque about the axises can be derived from standard mechanics:

$$\tau_\psi = b\sum_{i=1}^{4}(-1)^{i+1}\omega_i^2 \tag{18}$$

$$\tau_\theta = \sum_{i\in\{2,4\}} \vec{r}\times T = Lk(\omega_2^2 - \omega_4^2) \tag{19}$$

$$\tau_\phi = \sum_{i\in\{1,3\}} \vec{r}\times T = Lk(\omega_1^2 - \omega_3^2) \tag{20}$$

- Where the L is the distance from the center of the quadcopter to any of the propellers.

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Physics

Torques – cont.

- From equations 18...20 we can calculate the torque matrix in body frame:

$$\tau_B = \begin{bmatrix} Lk(\omega_1^2 - \omega_3^2) \\ Lk(\omega_2^2 - \omega_4^2) \\ b\sum_{i=1}^{4}(-1)^{i+1}\omega_i^2 \end{bmatrix} \tag{21}$$

Physics

What we have ignored in our modelings?

- Rotational drag forces(our rotational velocities are relatively low).
- Blade flapping(deformation of propeller blades due to high velocities and flexible materials).
- Surrounding fluid velocities(wind).
- The noise distribution model of our motors' speed.
- etc.

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Equations of Motion

Acceleration

- In the inertial frame, the acceleration of the quadcopter is due to thrust, gravity and linear friction.
- We can obtain the thrust vector in the inertial frame by using the rotation matrix R to map the thrust vector from the body frame to the inertial frame:

$$F = m\ddot{x} \tag{22}$$

$$F = m\vec{G} + RT_B + F_D \tag{23}$$

$$\ddot{x} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \frac{RT_B + F_D}{m} \tag{24}$$

- Where:
  - $\vec{x}$ is the position of the quadcopter.
  - g is the acceleration due to gravity.
  - F_D is the drag force.
  - T_B is the thrust vector in the body frame.

Equations of Motion

Rotational Speed – cont.

- Rotational equations can be drived from Euler's equations for rigid body dynamics, Expressed in vector form:

$$I\dot{\omega} + \omega \times (I\omega) = \tau \tag{25}$$

- Where:
  - ω is the angular velocity vector.
  - I is the inertia matrix.
  - τ is a vector of external torques.
  - T_B is the thrust vector in the body frame.
- The eq.25 can rewrite this as:

$$\dot{\omega} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = I^{-1}(\tau - \omega \times (I\omega)) \tag{26}$$

Equations of Motion

Rotational Speed – cont.

- For simplification if we assume that the quadcopter as two thin uniform rods crossed at the origin with a point mass (motor) at the end of each; The inertia matrix would come in form of a diagonal matrix.

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{27}$$

- Therefore, we obtain our final result for the body frame rotational equations of motion:

$$\dot{\omega} = \begin{bmatrix} \tau_\phi I_{xx}^{-1} \\ \tau_\theta I_{yy}^{-1} \\ \tau_\psi I_{zz}^{-1} \end{bmatrix} - \begin{bmatrix} \frac{I_{yy}-I_{zz}}{I_{xx}} \omega_y \omega_z \\ \frac{I_{zz}-I_{xx}}{I_{yy}} \omega_x \omega_z \\ \frac{I_{xx}-I_{yy}}{I_{zz}} \omega_x \omega_y \end{bmatrix} \tag{28}$$

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