The Choquet integral as a tool for aggregating preferences

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Plan

1. Preliminaries
2. MultiAttribute Utility Theory
3. An additive model: the Weighted Arithmetic Mean
4. A non-additive model: the Choquet integral
   - Capacity identification
5. The 2-additive Choquet Integral
6. Elicitation of a 2-additive capacity
   - Binary actions and preferential information
   - A characterization of the 2-additive model
   - How to deal with inconsistencies

Plan

The context: MultiCriteria Decision Aid (MCDA)

Aim: to help a decision-maker (DM) to select one or more alternatives among several alternatives evaluated on $|N|$ criteria often contradictory.

$\Rightarrow$ We need to construct a preference relation over the set of all alternatives X

Example (Evaluation of students in the tv program "Star Academy")

Candidates	1 : Collective Activities	2 : Song	3 : Musical intruments
a : Yvanessa	||||| ||	17	70
b : Michaël	||||| ||||	17	60
c : Jessica	||||| ||	8	70
d : Frank	||||| ||||	8	60
e : Suzanne	||||| ||||| |	10	45
f : Désiré	||||| ||||| ||	10	45

Problem: Give a ranking of all the six students.

Maybe a simple problem if we use the weighted sum as aggregation function.

But how to determine the weight of each criterion in this case?

It is not an easy task!

Notations

- DM: Decision-Maker
- X = the set of all alternatives
- $N = \{1, \ldots, n\}$ the finite set of n criteria
- $X_1, \ldots, X_n$ represent the set of points of view or attributes
- An alternative or option $x = (x_1, \ldots, x_n)$ is identified to an element of $X = X_1 \times \cdots \times X_n$

Example (Evaluation of students in the tv program "Star Academy")

Candidates	1 : Collective Activities	2 : Song	3 : Musical intruments
a : Yvanessa	$\vert\vert\vert\vert\vert\ \vert\vert$	17	70
b : Michaël	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert$	17	60
c : Jessica	$\vert\vert\vert\vert\vert\ \vert\vert$	8	70
d : Frank	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert$	8	60
e : Suzanne	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert\ \vert$	10	45
f : Désiré	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert\ \vert\vert$	10	45

Problem: Give a ranking of all the six students.

$N = \{1, 2, 3\}$; $\quad X$ = the set of all students

$X_1 = [\vert;\ \vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert]$; $\quad X_2 = [0; 20]$; $\quad X_3 = [0; 100]$

$X' = \{a, b, c, d\}$ = a part of X

Plan

MultiAttribute Utility Theory (MAUT)

- Goal: MAUT aims at representing numerically the DM's preferences in the form of a complete preorder $\succsim_X$ with a function $u : X \to \mathbb{R}$ called overall utility function and such that:

$$\forall x, y \in X,\ x \ \succsim_X \ y \Leftrightarrow u(x) \geq u(y)$$

- The function u is constructed so that the larger the overall utility associated to an alternative is, the greater this alternative is "preferred" by the DM.

MultiAttribute Utility Theory (MAUT)

MAUT's hypothesis: $\succsim_X$ is representable by an overall utility function u:

$$x \succsim_X y \Leftrightarrow u(x) \geq u(y)$$

In general, we suppose $u = F \circ U$ where

- $U(x) = (u_1(x_1), \ldots, u_n(x_n))$,
- $u_i : X_i \to \mathbb{R}$ is an utility function on i,
- $F : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function.

Hence we have:

$$\forall (x_1, \ldots, x_n) \in X,\ u(x_1, \ldots, x_n) := F(U(x_1, \ldots, x_n))$$

MultiAttribute Utility Theory (MAUT)

MAUT's hypothesis: $\succsim_X$ is representable by an overall utility function u:

$$\begin{cases} x \succsim_X y \Leftrightarrow u(x) \geq u(y) \\ \forall (x_1, \ldots, x_n) \in X,\ u(x_1, \ldots, x_n) := F(U(x_1, \ldots, x_n)) \end{cases}$$

Remark

Generally, the utility functions u_i and the aggregation function F are not unique.

MultiAttribute Utility Theory (MAUT)

MAUT's hypothesis: $\succsim_X$ is representable by an overall utility function u:

$$\begin{cases} x \succsim_X y \Leftrightarrow u(x) \geq u(y) \\ \forall (x_1, \ldots, x_n) \in X,\; u(x_1, \ldots, x_n) := F(U(x_1, \ldots, x_n)) \end{cases}$$

Remark

Generally, the construction of utility functions u_i and the determination of the aggregation function F are done separately.

- *How to construct u_i?*
  It is not an easy task. For instance some models need commensurability between criteria (will be detailed later!).
- *How to choose the "best" aggregation function?*
  Usually, one use as aggregation function the well-known arithmetic mean (weighted sum).

In practice, using MAUT, how to construct a preference relation $\succsim_X$ over X?

1. People ask to the DM some preferential information $\succsim_{X'}$ on a reference subset $X' \subseteq X$
2. F is generally characterized by a parameter vector (weight vector, probability distribution. . .).
3. the parameter vector is constructed so that $\succsim_X$ is an extension of $\succsim_{X'}$.
4. The model obtained in X' will be then automatically extended to X.

Plan

Definition (Additive model)

The additive model is defined by the existence of utility functions $u_i : X_i \to \mathbb{R}$ such that:

$$\forall (x_1, \ldots, x_n) \in X, \quad u(x_1, \ldots, x_n) := \sum_{i \in N} u_i(x_i). \tag{1}$$

The functions u_i can be determined by some methods like UTA.

Definition (Weighted Arithmetic Mean (WAM))

The Weighted Arithmetic Mean or Weighted Sum is a particular case of an additive model.

It is defined by the existence of utility functions $u_i : X_i \to \mathbb{R}$ and real numbers w_i (weight of criterion i) such that:

$$\forall (x_1, \ldots, x_n) \in X, \quad u(x_1, \ldots, x_n) := \sum_{i \in N} w_i \; u_i(x_i). \tag{2}$$

Example (Evaluation of students in the tv program "Star Academy")

Candidates	1 : Collective Activities	2 : Song	3 : Musical intruments
a : Yvanessa	||||| ||	17	70
b : Michaël	||||| ||||	17	60
c : Jessica	||||| ||	8	70
d : Frank	||||| ||||	8	60
e : Suzanne	||||| |||||		10	45
f : Désiré	||||| ||||| ||	10	45

- Can you give your preference between a and b?
- Can you give your preference between c and d?

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d : Frank	||||| ||||	8	60
e : Suzanne	||||| |||||		10	45
f : Désiré	||||| ||||| ||	10	45

$X' = \{a, b, c, d\}$

- If two students are good in Song and Musical intruments, then the jury prefers strictly the student who have a best evaluation in Collective Activities. $\Rightarrow b \succ_{X'} a$;
- If two students are bad in Song, then the jury prefers strictly the student who have a best evaluation in Musical intruments. $\Rightarrow c \succ_{X'} d$;

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d : Frank	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert$	8	60
e : Suzanne	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert\ \vert$	10	45
f : Désiré	$\vert\vert\vert\vert\vert\ \vert\vert\vert\vert\vert\ \vert\vert$	10	45

$X' = \{a, b, c, d\}$. If $F \equiv$ weighted sum,

$$b \succ_{X'} a \quad \Rightarrow \quad u_1(\vert\vert\vert\vert\vert\vert\vert\vert\vert)\ w_1 + u_3(60)\ w_3 > u_1(\vert\vert\vert\vert\vert\vert\vert)\ w_1 + u_3(70)\ w_3 \tag{3}$$

$$c \succ_{X'} d \quad \Rightarrow \quad u_1(\vert\vert\vert\vert\vert\vert\vert)\ w_1 + u_3(70)\ w_3 > u_1(\vert\vert\vert\vert\vert\vert\vert\vert\vert)\ w_1 + u_3(60)\ w_3 \tag{4}$$

Conclusion: Weighted Sum $\Rightarrow$ criteria are (preferential) independent i.e. no interaction.

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f : Désiré	||||| ||||| ||	10	45

Preferential independence means: the preference of $b = (||||| \ |||; 17; 60)$ over $a = (||||| \ ||; 17; 70)$ is not influenced by values on criterion 2.

i.e.

$$b = (||||| \ |||; 17; 60) \succ_{X'} a = (||||| \ ||; 17; 70)$$

$$\Updownarrow$$

$$c = (||||| \ |||; 8; 60) \succ_{X'} d = (||||| \ ||; 8; 70)$$

Preferential independence

The subset $S \subseteq N$ of criteria is said to be preferentially independent of $N \setminus S$ if

for all $x_S, y_S \in \prod_{i \in S} X_i$, and all $x_{N\setminus S}, z_{N\setminus S} \in \prod_{i \in N\setminus S} X_i$:

$$(x_S, x_{N\setminus S}) \succsim (y_S, x_{N\setminus S}) \Leftrightarrow (x_S, z_{N\setminus S}) \succsim (y_S, z_{N\setminus S}) \tag{5}$$

Roughly speaking, the preference of $(x_S, x_{N\setminus S})$ over $(y_S, x_{N\setminus S})$ is not influenced by the values of $x_{N\setminus S}$.

Remark

This property is necessary but not sufficient to characterize the additive model.

What we have seen until now ...

- The context: MCDA
  - Construct a relation $\succsim_X$ over N.
- We suppose the MAUT's hypothesis:
  - $\succsim_X$ is representable by an overall utility function u:

$$\begin{cases} x \ \succsim_X \ y \Leftrightarrow u(x) \geq u(y) \\ \forall (x_1, \dots, x_n) \in X, \ u(x_1, \dots, x_n) := F(U(x_1, \dots, x_n)) \end{cases}$$

- Limits of additive models

Plan

Definition (Capacity)

A *capacity* (or fuzzy measure) on N is a set function $\mu : 2^N \to [0,1]$ satisfying the three properties:

1. $\mu(\emptyset) = 0$
2. $\mu(N) = 1$ (normality)
3. $\forall A, B \in 2^N,\ [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).

Interpretation

$\mu(S)$ can be interpreted as the "weight" of the coalition of criteria S.

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Additive capacity

$\mu(S)$ is said to be additive if

$$\mu(S \cup T) = \mu(S) + \mu(T) \text{ whenever } S \cap T = \emptyset.$$

In this case it is sufficient to define the n coefficients (weights) $\mu(\{1\}), \ldots, \mu(\{1\})$ to define the capacity entirely.

The Choquet integral

Definition (The Choquet integral)

The Choquet integral of $x := (x_1, ..., x_n) \in \mathbb{R}^n_+$ w.r.t. a capacity μ is defined by:

$$C_\mu(x) := \sum_{i=1}^{n} (x_{\tau(i)} - x_{\tau(i-1)})\ \mu(\{\tau(i), \ldots, \tau(n)\}) \tag{6}$$

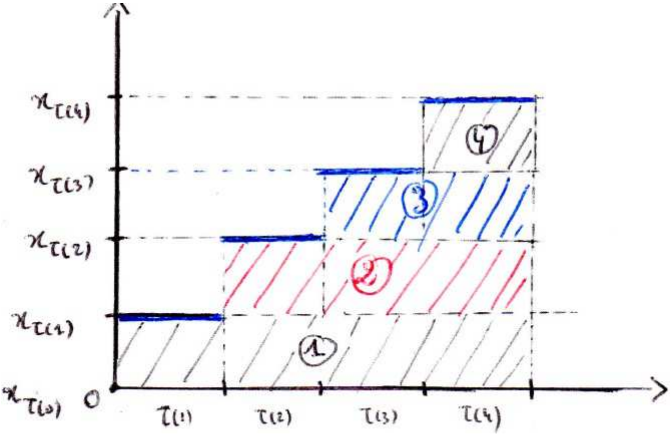
where τ is a permutation on N such that $x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(n-1)} \leq x_{\tau(n)}$, and $x_{\tau(0)} := 0$

Remark

μ additive $\Rightarrow$ $C_\mu \equiv$ Weighted sum.

Grabisch & Labreuche. A decade of Choquet integral (2010). 4OR

The Choquet integral



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where τ is a permutation on N such that $x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(n-1)} \leq x_{\tau(n)}$, and $x_{\tau(0)} := 0$

Remark

Choquet integral $\Rightarrow$ to ensure commensurability between criteria

Definition (The Choquet integral)

The Choquet integral of $x := (x_1, ..., x_n) \in \mathbb{R}^n_+$ w.r.t. a capacity μ is defined by:

$$C_\mu(x) := \sum_{i=1}^{n} (x_{\tau(i)} - x_{\tau(i-1)})\mu(\{\tau(i), \ldots, \tau(n)\}) \tag{8}$$

where τ is a permutation on N such that $x_{\tau(1)} \leq x_{\tau(2)} \leq \cdots \leq x_{\tau(n-1)} \leq x_{\tau(n)}$, and $x_{\tau(0)} := 0$

Commensurability

Commensurability means that one shall be able to compare any element of one point of view with any element of any other point of view:

For $x_i \in X_i$ and $x_j \in X_j$,

$$u_i(x_i) \geq u_j(x_j) \Leftrightarrow \text{DM considers } x_i \text{ at least as good as } x_j$$

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f : Désiré	||||| ||||| ||	10	45

To apply the Choquet integral, we need commensurate scales

Candidates	1 : Collective Activities	2 : Song	3 : Musical intruments
a : Yvanessa	7	17	14
b : Michaël	9	17	12
c : Jessica	7	8	14
d : Frank	9	8	12
e : Suzanne	11	10	9
f : Désiré	12	10	9

Example (Evaluation of students in the tv program "Star Academy")

The same example with commensurate scales

Candidates	1 : Collective Activities	2 : Song	3 : Musical intruments
a : Yvanessa	7	17	14
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If we consider the capacity $\mu : 2^N \rightarrow [0,1]$ defined by:

$\mu(N) = \mu(\{1,2\}) = 1$,
$\mu(\emptyset) = \mu(\{1\}) = 0$,
$\mu(\{2\}) = \mu(\{3\}) = \mu(\{2,3\}) = \mu(\{1,3\}) = \frac{1}{2}$,

then we obtain for the student a:

$$C_\mu(U(a)) = 7 + 7\ \mu(\{2,3\}) + 3\ \mu(\{2\}) = 12$$

Example (Evaluation of students in the tv program "Star Academy")

The same example with commensurate scales

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If we consider the capacity $\mu : 2^N \to [0,1]$ defined by: $\mu(N) = 1$, $\mu(\emptyset) = 0$, $\mu(\{2\}) = \mu(\{3\}) = \mu(\{2,3\}) = \mu(\{1,3\}) = \frac{1}{2}$, $\mu(\{1\}) = 0$, $\mu(\{1,2\}) = 1$,

$$\begin{aligned}
C_\mu(U(a)) &= 7 + 7\ \mu(\{2,3\}) + 3\ \mu(\{2\}) = 12\\
C_\mu(U(b)) &= 9 + 3\ \mu(\{2,3\}) + 5\ \mu(\{2\}) = 13\\
C_\mu(U(c)) &= 7 + 1\ \mu(\{2,3\}) + 6\ \mu(\{3\}) = 10.5\\
C_\mu(U(d)) &= 8 + 1\ \mu(\{1,3\}) + 3\ \mu(\{3\}) = 10
\end{aligned}$$

Hence we have now: $b \succ_X a$ and $c \succ_X d$.

Interaction index

Definition

Given a capacity μ, the interaction index for any subset $A \subseteq N$ is defined by

$$\forall A \subseteq N, \quad I(A) := \sum_{K \subseteq N \setminus A} \frac{(n-k-|A|)!k!}{(n-|A|+1)!} \sum_{L \subseteq A} (-1)^{|A|-|L|} \mu(K \cup L). \qquad (9)$$

Definition

Let μ be a capacity. The interaction index for any pair of criteria i and j is given by the following expression

$$I_{ij} := \sum_{K \subseteq N \setminus \{i,j\}} \frac{(n-k-2)!k!}{(n-1)!} [\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)] \qquad (10)$$

Definition (Importance index)

Let μ be a capacity. The importance index (Shapley value) for a criterion i is given by the following expression:

$$v_i = \sum_{K \subseteq N \setminus i} \frac{(n-k-1)!k!}{n!} (\mu(K \cup i) - \mu(K)). \tag{11}$$

Hypothesis for the next sections

We suppose the utility functions u_i are already constructed using the method given in Grabisch and Labreuche (2003).

Preferential information asked

In the context of MAUT based on the Choquet integral, the preferences, from which the capacity is to be determined, can take the form of:

$$x \; P \; y \Leftrightarrow C_\mu(U(x)) - C_\mu(U(y)) \geq \delta_{X_R}$$

$$x \; I \; y \Leftrightarrow C_\mu(U(x)) = C_\mu(U(y))$$

$$i \succ_{imp} j \Leftrightarrow v_i^\mu - v_j^\mu \geq \delta_{imp}$$

$$i \sim_{imp} j \Leftrightarrow v_i^\mu = v_j^\mu$$

$$ij \succ_{int} kl \Leftrightarrow I_{ij}^\mu - I_{kl}^\mu \geq \delta_{int}$$

$$ij \sim_{int} kl \Leftrightarrow I_{ij}^\mu = I_{kl}^\mu$$

Linear program to solve

Most of the identification methods proposed in the literature can be stated under the form of an optimization problem:

$$
\begin{aligned}
&\text{min or max} \quad \mathcal{F} \\
&\text{s.t.} \quad \begin{cases}
\mu(S \cup i) - \mu(S) \geq 0, \forall i \in N, \forall S \subseteq N - i, \\
\mu(N) = 1, \\
C_\mu(U(x)) - C_\mu(U(y)) \geq \delta_{X_R} \\
\vdots \\
C_\mu(U(x)) = C_\mu(U(y)) \\
\vdots \\
v_i^\mu - v_j^\mu \geq \delta_{imp} \\
\vdots \\
v_i^\mu = v_j^\mu \\
\vdots \\
I_{ij}^\mu - I_{kl}^\mu \geq \delta_{int} \\
\vdots \\
I_{ij}^\mu = I_{kl}^\mu \\
\vdots
\end{cases}
\end{aligned}
\tag{12}
$$

Some existing methods

According to their objective function and the preferential information they require as input, we have:

- A maximum split approach (Marichal & Roubens)
- Minimum variance and minimum distance approaches (Kojadinovic)
- A less constrained approach (Meyer & Roubens)
- Robust approach (Angillela et al.) using necessary and possible binary relations.

Grabisch et al. A review of methods for capacity identification in Choquet integral based multi-attribute utility theory (2008). EJOR

In general

To compute a capacity μ, one needs to define the 2^n coefficients corresponding to the 2^n subsets of N.

$\Rightarrow$ Introduction of k-additive models.

What we have seen until now ...

- The context: MCDA
  - Construct a relation $\succsim_X$ over N.
- We suppose the MAUT's hypothesis:
  - $\succsim_X$ is representable by an overall utility function u:

$$\begin{cases} x \ \succsim_X \ y \Leftrightarrow u(x) \geq u(y) \\ \forall (x_1, \ldots, x_n) \in X, \ u(x_1, \ldots, x_n) := F(U(x_1, \ldots, x_n)) \end{cases}$$

- Limits of additive models
  - Introduction of Choquet integral w.r.t a capacity + identification of a capacity

Plan

k-additive capacity

Definition (Möbius transform)

Let μ be a capacity on N.

The *Möbius transform* of μ is a function $m : 2^N \to \mathbb{R}$ defined by

$$m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K) \quad \forall T \in 2^N.$$

Definition (k-additive capacity (Grabisch))

μ is said to be *k-additive*, $k > 0$, if its *Möbius transform* m satisfied

1. $\forall T \in 2^N,\ m(T) = 0 \text{ if } |T| > k$
2. $\exists B \in 2^N$ such that $|B| = k$ and $m(B) \neq 0$.

2-additive capacity

Definition (2-additive capacity)

μ is said to be 2-*additive* if its *Möbius transform m* satisfied

1. $\forall T \in 2^N,\ m(T) = 0 \text{ if } |T| > 2$
2. $\exists B \in 2^N$ such that $|B| = 2 \text{ and } m(B) \neq 0.$

Lemma

If the coefficients $\mu(\{i\})$ and $\mu(\{i,j\})$ are given for all $i,j \in N$, then the necessary and sufficient conditions that μ is a 2-additive capacity are:

$$\sum_{\{i,j\}\subseteq N} \mu(\{i,j\}) - (n-2)\sum_{i\in N}\mu(\{i\}) = 1 \text{ (normality)} \tag{13}$$

$$\mu(\{i\}) \geq 0,\ \forall i \in N \text{ (nonnegativity)} \tag{14}$$

$\forall A \subseteq N,\ |A| \geq 2,\ \forall k \in A$

$$\sum_{i\in A\setminus\{k\}} (\mu(\{i,k\}) - \mu(\{i\})) \geq (|A|-2)\mu(\{k\}) \text{ (monotonicity).} \tag{15}$$

Notations

$\forall i,j \in N, i \neq j,\ \mu_\emptyset = \mu(\emptyset),\ \mu_i = \mu(\{i\})$ and $\mu_{ij} = \mu(\{i,j\})$

Definition (2-additive Choquet integral)

For any $x := (x_1, ..., x_n) \in X$, the expression of the 2-additive Choquet is:

$$C_\mu((u(x_1), \ldots, u(x_n))) = \sum_{i=1}^{n} v_i u(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij} \, |u(x_i) - u(x_j)| \tag{16}$$

Where

- v_i = the importance of the criterion i ($\equiv$ Shapley index);

$$I_{ij} = \mu_{ij} - \mu_i - \mu_j. \tag{17}$$

- I_{ij} = the interaction index between criteria i and j.

$$v_i = \mu_i + \frac{1}{2} \sum_{k \in N \setminus i} I_{ik}. \tag{18}$$

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$X' = \{a, b, c, d\}$. If we consider the 2-additive capacity $\mu : 2^N \to [0, 1]$ defined such that: $\mu(N) = 1$, $\mu(\emptyset) = 0$, $\mu(\{2\}) = \mu(\{3\}) = \mu(\{2, 3\}) = \mu(\{1, 3\}) = \frac{1}{2}$, $\mu(\{1\}) = 0$, $\mu(\{1, 2\}) = 1$, then we have $I_{12} = \frac{1}{2}$, $I_{13} = 0$, $I_{23} = -\frac{1}{2}$, $v_1 = \frac{1}{4}$, $v_2 = \frac{1}{2}$, $v_3 = \frac{1}{4}$:

$$
\begin{aligned}
C_\mu(U(a)) &= 7\ v_1 + 17\ v_2 + 14\ v_3 - \tfrac{1}{2}(I_{12}\ |7-17| + I_{13}\ |7-14| + I_{23}\ |17-14|) = 12\\
C_\mu(U(b)) &= 9\ v_1 + 17\ v_2 + 12\ v_3 - \tfrac{1}{2}(I_{12}\ |9-17| + I_{13}\ |9-12| + I_{23}\ |17-12|) = 13\\
C_\mu(U(c)) &= 7\ v_1 + 8\ v_2 + 14\ v_3 - \tfrac{1}{2}(I_{12}\ |7-8| + I_{13}\ |7-14| + I_{23}\ |8-14|) = 10.5\\
C_\mu(U(d)) &= 9\ v_1 + 8\ v_2 + 12\ v_3 - \tfrac{1}{2}(I_{12}\ |9-8| + I_{13}\ |9-12| + I_{23}\ |8-12|) = 10
\end{aligned}
$$

Another expression of the 2-additive Choquet integral

For any $x := (x_1, ..., x_n) \in X$, the 2-additive Choquet can be expressed by:

$$C_\mu(U(x)) = \sum_{I_{ij}>0} I_{ij}\ (u(x_i) \wedge u(x_j)) + \sum_{I_{ij}<0} I_{ij}\ (u(x_i) \vee u(x_j)) + \sum_{i=1}^{n} u_i(x_i)(v_i - \frac{1}{2} \sum_{j \neq j} I_{ij})$$

Interaction index

Interpretation of I_{ij}

- $I_{ij} = 0 \Rightarrow$ independence between i and j;
- $I_{ij} > 0 \Rightarrow$ complementary among i and j;

  This means that for the DM, both criteria have to be satisfactory in order to get a satisfactory alternative, the satisfaction of only one criterion being useless.

- $I_{ij} < 0 \Rightarrow$ substitutability or redundance among i and j;

  This means that for the DM, the satisfaction of one of the two criteria is sufficient to have a satisfactory alternative, satisfying both being useless..

Interest of the 2-additive model

The 2-additive Choquet integral

1. is very used in many applications such that
   - the evaluation of discomfort in sitting position (see Grabisch et al. (2002));
   - the construction of performance measurement systems model in a supply chain context (see Berrah and Clivillé (2007), Clivillé et al. (2007));
   - complex system design (Labreuche and Pignon (2007));
2. offers a good compromise between flexibility of the model and complexity;
3. requires to be able to compare any element of one point of view with any element of any other point of view (commensurateness between criteria);
4. The only way to construct the utility functions with the Choquet integral uses the reference levels (Grabisch and Labreuche (2003)).

$$C_\mu((u(x_1),\dots,u(x_n))) = \sum_{i=1}^{n} v_i u(x_i) - \frac{1}{2}\sum_{\{i,j\}\subseteq N} I_{ij}\,|u(x_i)-u(x_j)|$$

Remark

- For all $i,j \in N$,

$$C_\mu((0,\dots,0)) = \mu_\emptyset = 0$$
$$C_\mu((0,\dots,0,\underbrace{1}_{i},\dots,0)) = \mu_i = v_i - \frac{1}{2}\sum_{k\in N,\ k\neq i} I_{ik}$$
$$C_\mu((0,\dots,0,\underbrace{1}_{i},\dots,0,\underbrace{1}_{j},\dots,0)) = \mu_{ij} = v_i + v_j - \frac{1}{2}\sum_{k\in N,\ k\notin\{i,j\}} (I_{ik}+I_{jk})$$

- Therefore we set:

$$\begin{cases} (0,\dots,0) \equiv U(\mathbf{a_0}) \\ (0,\dots,0,\underbrace{1}_{i},\dots,0) \equiv U(\mathbf{a_i}) \\ (0,\dots,0,\underbrace{1}_{i},\dots,0,\underbrace{1}_{j},\dots,0) \equiv U(\mathbf{a_{ij}}) \end{cases}$$

$\mathbf{a_0}$, $\mathbf{a_i}$ and $\mathbf{a_{ij}}$ are called binary actions or binary alternatives.

What we have seen until now ...

- The context: MCDA
  - Construct a relation $\succsim_X$ over N.
- We suppose the MAUT's hypothesis:
  - $\succsim_X$ is representable by an overall utility function u:

$$\begin{cases} x \ \succsim_X \ y \Leftrightarrow u(x) \geq u(y) \\ \forall (x_1, \ldots, x_n) \in X, \ u(x_1, \ldots, x_n) := F(U(x_1, \ldots, x_n)) \end{cases}$$

- Limits of additive models
  - Introduction of Choquet integral w.r.t a capacity + identification of a capacity
  - A particular Choquet integral: a 2-additive Choquet integral
  - In the next section: Elicitation of a 2-additive capacity by using binary actions

Plan

DM can identify two reference levels on i: $\begin{cases} \mathbf{0}_i \in X_i \equiv \text{“neutral” (unsatisfactory)} \\ \mathbf{1}_i \in X_i \equiv \text{satisfactory} \end{cases}$

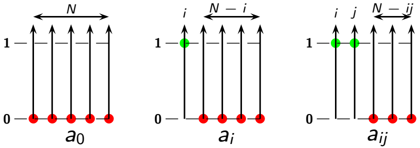
Definition

A *binary action* is an element of the set

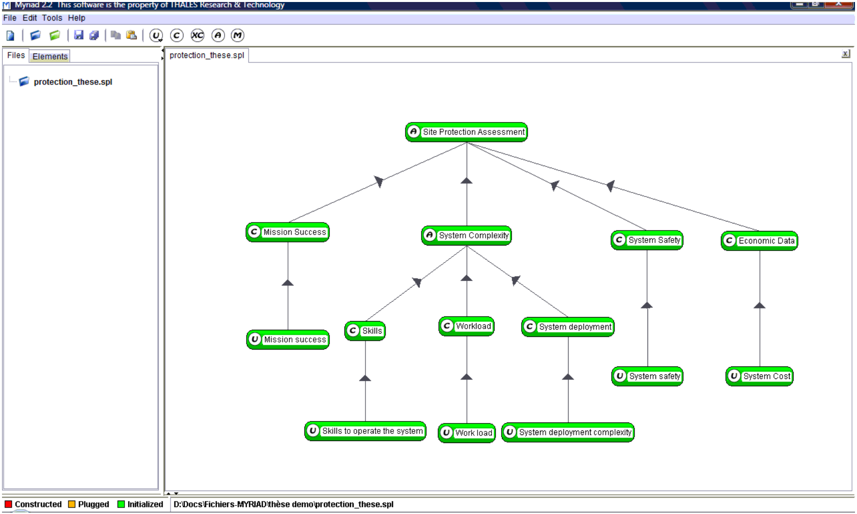
$$\mathcal{B} = \{\mathbf{0}_N,\ (\mathbf{1}_i, \mathbf{0}_{N-i}),\ (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}),\ i, j \in N, i \neq j\}$$

where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) =: a_0$ is the action considered neutral on all criteria.
- $(\mathbf{1}_i, \mathbf{0}_{N-i}) =: a_i$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) =: a_{ij}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.



Binary action in a real application: Site protection



Binary action in a real application: Site protection

Let us consider the aggregation of the subtree System complexity where $1 \equiv$ *Skills*, $2 \equiv$ *Work load*, $3 \equiv$ *System deployment*:

$$\mathcal{B} = \{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\},$$

- $a_{13} \; I \; a_1$: a system requiring a high working load and better on the other criteria is equivalent to a better system solely on the criterion *skills*.
- $a_{12} \; P \; a_3$: DM prefers a system better on *Skills* even its time deployment is important.
- If *Skills* is improved in a_{13}, he prefers a_{13} to a_{12}, i.e $a_{13} \; P \; a_{12}$
- $a_{13} \; I \; a_{23}$: a system requiring a too heavy workload but satisfying on the other criteria are of equal importance to a system requiring very high qualifications for its use, although satisfying on the other criteria.
- $a_1 \; I \; a_2$: A good system on *Skills* is indifferent to a good system on *Work load*.
- $a_3 \; P \; a_0$

Properties of binary actions

For all $i, j \in N$,

$$C_\mu(U(a_0)) = \mu_\emptyset = 0$$

$$C_\mu(U(a_i)) = \mu_i = v_i - \frac{1}{2} \sum_{k \in N,\ k \neq i} I_{ik}$$

$$C_\mu(U(a_{ij})) = \mu_{ij} = v_i + v_j - \frac{1}{2} \sum_{k \in N,\ k \notin \{i,j\}} (I_{ik} + I_{jk})$$

Why binary actions?

They allow us to:

1. have a good specification of the 2-additive model
2. determine:
   - the interaction between two criteria
   - the importance of a criterion

DM's preferential information

Using pairwise comparisons, the DM gives a preferential information on $\mathcal{B}$ allowing the construction of these relations:

$$P = \{(x, y) \in \mathcal{B} \times \mathcal{B} : \text{ DM strictly prefers } x \text{ to } y\}$$
$$I = \{(x, y) \in \mathcal{B} \times \mathcal{B} : \text{ DM is indifferent between } x \text{ and } y\}$$

Definition

The *ordinal information on $\mathcal{B}$* is the structure $\{P,\ I\}$.

Elicitation of a 2-additive capacity

We look for a 2-additive capacity μ such that:

$$\forall x, y \in \mathcal{B},\ x\ P\ y \Rightarrow C_\mu(U(x)) > C_\mu(U(y)), \tag{19}$$

$$\forall x, y \in \mathcal{B},\ x\ I\ y \Rightarrow C_\mu(U(x)) = C_\mu(U(y)), \tag{20}$$

Example

$$N = \{1,2,3\},\ \mathcal{B} = \{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\}$$

$$P = \{(a_{23}, a_2); (a_2, a_0); (a_{23}, a_{12})\}$$
$$I = \{(a_{13}, a_1); (a_3, a_{12})\}$$

$$\begin{cases} C_\mu(U(a_{23})) > C_\mu(U(a_2)) \\ C_\mu(U(a_2)) > C_\mu(U(a_0)) \\ C_\mu(U(a_{23})) > C_\mu(U(a_{12})) \\ C_\mu(U(a_{13})) = C_\mu(U(a_1)) \\ C_\mu(U(a_3)) = C_\mu(U(a_{12})) \\ \left.\begin{array}{l} \mu_\emptyset = 0,\ \mu_1 \geq 0 \\ \mu_2 \geq 0,\ \mu_3 \geq 0 \\ \mu_{12} \geq \mu_1,\ \mu_{12} \geq \mu_2 \\ \mu_{13} \geq \mu_1,\ \mu_{13} \geq \mu_3 \\ \mu_{23} \geq \mu_2,\ \mu_{23} \geq \mu_3 \\ \mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3 \\ \mu_{12} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3 \\ \mu_{13} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3 \end{array}\right\} \text{2-additive Monotonicity constraints} \end{cases}$$

Example

$$N = \{1, 2, 3\},\ \mathcal{B} = \{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\}$$

$$P = \{(a_{23}, a_2); (a_2, a_0); (a_{23}, a_{12})\}$$
$$I = \{(a_{13}, a_1); (a_3, a_{12})\}$$

$$\begin{cases} \mu_{23} > \mu_2 \\ \mu_2 > 0 \\ \mu_{23} > \mu_{12} \\ \mu_{13} = \mu_1 \\ \mu_3 = \mu_{12} \\ \left.\begin{array}{l} \mu_\emptyset = 0,\ \mu_1 \geq 0 \\ \mu_2 \geq 0,\ \mu_3 \geq 0 \\ \mu_{12} \geq \mu_1,\ \mu_{12} \geq \mu_2 \\ \mu_{13} \geq \mu_1,\ \mu_{13} \geq \mu_3 \\ \mu_{23} \geq \mu_2,\ \mu_{23} \geq \mu_3 \\ \mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3 \\ \mu_{12} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3 \\ \mu_{13} + \mu_{23} \geq \mu_1 + \mu_2 + \mu_3 \end{array}\right\} \text{2-additive Monotonicity constraints} \end{cases}$$

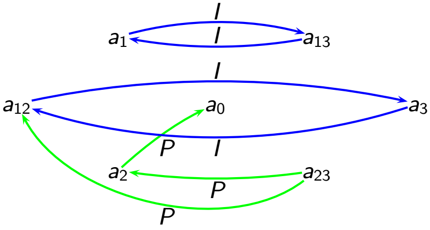
The monotonicity relation M on the pairs of criteria

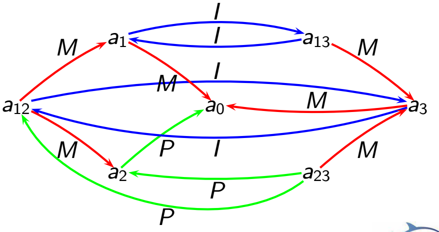
Definition

For $(x, y) \in \{(a_i, a_0), i \in N\} \cup \{(a_{ij}, a_i), i, j \in N, i \neq j\}$,

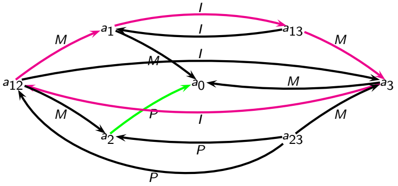
$$x\ M\ y \text{ if } \mathrm{not}(x\ (P \cup I)\ y).$$

M models the natural monotonicity conditions $\mu(\{i\}) \geq 0$ and $\mu(\{i,j\}) \geq \mu(\{i\})$ for a capacity μ





Introduction to the MOPI property



$$\Longrightarrow \left\{ \begin{array}{l} a_{12} \sim a_1 \\ a_{13} \sim a_3 \\ a_2 \; P \; a_0 \end{array} \right. \Longrightarrow (*) \left\{ \begin{array}{l} \mu_{12} = \mu_1 \\ \mu_{13} = \mu_3 \\ \mu_2 > 0 \end{array} \right.$$

$(*)$ leads to a contradiction with the 2-additivity monotonicity constraint for $A = \{1, 2, 3\}$ and state of nature 1 fixed:

$$\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$$

since $\left\{ \begin{array}{l} \mu_{12} = \mu_1 \\ \mu_{13} = \mu_3 \end{array} \right. \Longrightarrow 0 \geq \mu_2.$

MOnotonicity of Preferential Information (MOPI)

Definition

Let $i, j, k \in N$.

1. We call *Monotonicity of Preferential Information in* $\{i, j, k\}$ *w.r.t.* i the following property:

$$\begin{cases} a_{ij} \sim a_i \\ a_{ik} \sim a_k \end{cases} \Rightarrow not(a_j \ TC_P \ a_0)$$
$$\text{and}$$
$$\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_k \end{cases} \Rightarrow not(a_i \ TC_P \ a_0)$$
$$\text{and}$$
$$\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_i \end{cases} \Rightarrow not(a_k \ TC_P \ a_0).$$

   $x \ TC_P \ y \Leftrightarrow \exists$ a path in $(P \cup I \cup M)$ from x to y containing an element of P.

2. $\{i, j, k\}$ satisfies MOPI if $\forall l \in \{i, j, k\}$, $(\{i, j, k\}, l)$-MOPI is satisfied.

We suppose $P \neq \emptyset$.

Theorem (Mayag et al. (Th & Dec 2010))

An ordinal information $\{P, I\}$ is representable by a 2-additive Choquet integral on $\mathcal{B}$ if and only if the following conditions are satisfied:

1. *$(P \cup I \cup M)$ contains no cycle with a P;*
2. *Any subset K of N such that $|K| = 3$ satisfies the MOPI property.*

Corollaire

For any ordinal information $(P \cup I \cup M)$ such that $I = \emptyset$, there exists an ordinal 2-additive scale on X if and only if $(P \cup M)$ has no strict cycle.

Furthermore any ordinal information s.t. $I = \emptyset$ for which $(P \cup M)$ has no strict cycle, can be represented by a 2-additive capacity with nonnegative interactions.

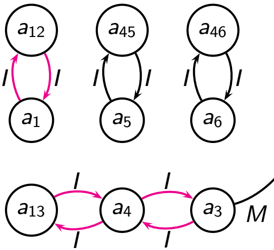
Example

$N = \{1,2,3,4,5,6\}$, $I = \{(a_{12}, a_1), (a_{13}, a_4), (a_3, a_4), (a_{45}, a_5), (a_{46}, a_6)\}$ and $P = \{(a_2, a_0)\}$.

No strict cycle but a violated MOPI property: $\begin{cases} a_{12} \ \mathbf{I} \ a_1 \\ a_{13} \sim a_3 \end{cases}$ and $a_2 \ \mathbf{P} \ a_0$

To solve this problem, DM Modifies $a_{13} \sim a_3$ by changing

- $a_{13} \ \mathbf{I} \ a_4$ to $a_4 \ \mathbf{P} \ a_{13}$
- $a_3 \ \mathbf{I} \ a_4$ to $a_4 \ \mathbf{P} \ a_3$

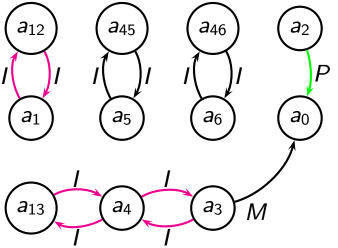


Example

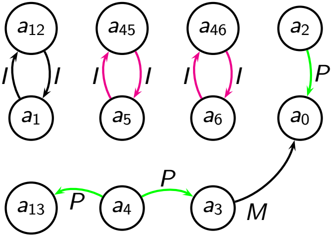
$N = \{1, 2, 3, 4, 5, 6\}$, $I = \{(a_{12}, a_1), (a_{45}, a_5), (a_{46}, a_6)\}$ and
$P = \{(a_2, a_0), (a_4, a_{13}), (a_4, a_3)\}$

After these modifications, we get a new violated MOPI condition:

$$\begin{cases} a_{45} \ \mathbf{I} \ a_5 \\ a_{46} \ \mathbf{I} \ a_6 \end{cases} \text{and } a_4 \ \mathbf{P} \ a_3 \ \mathbf{M} \ a_0$$



$\Rightarrow$



Example

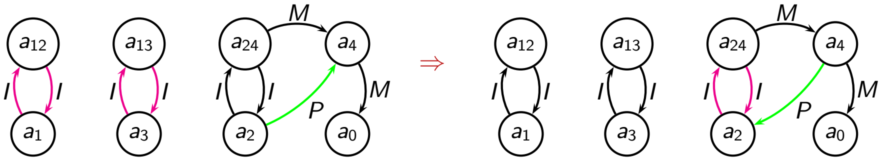
$N = \{1,2,3,4\}$, $I = \{(a_{12}, a_1), (a_{13}, a_3), (a_{24}, a_2)\}$ and $P = \{(a_2, a_4)\}$.

No strict cycle but a violated MOPI property:

$$\begin{cases} a_{12} \;\mathbf{I}\; a_1 \\ a_{13} \;\mathbf{I}\; a_3 \end{cases} \text{and } a_2 \;\mathbf{P}\; a_4 \;M\; a_0$$

If DM changes (a_2 **P** a_4) to (a_4 **P** a_2) then a new strict cycle is created

$$a_4 \;\mathbf{P}\; a_2 \;\mathbf{I}\; a_{24} \;M\; a_4.$$



Example

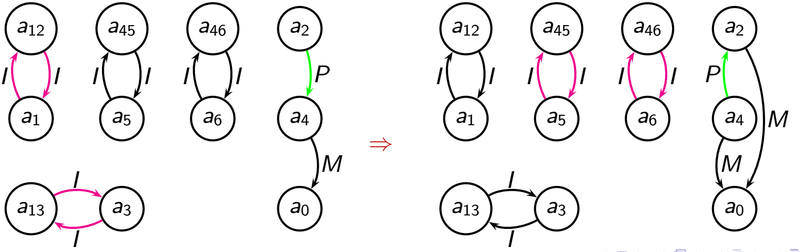
$N = \{1, 2, 3, 4, 5, 6\}$, $I = \{(a_{12}, a_1), (a_{13}, a_3), (a_{45}, a_5), (a_{46}, a_6)\}$ and $P = \{(a_2, a_4)\}$.

No strict cycle but a violated MOPI property:

$$\begin{cases} a_{12} \ \mathbf{I} \ a_1 \\ a_{13} \ \mathbf{I} \ a_3 \end{cases} \text{and } a_2 \ \mathbf{P} \ a_4 \ M \ a_0.$$

If DM changes (a_2 $\mathbf{P}$ a_4) to (a_4 $\mathbf{P}$ a_2) then we get a new violated MOPI condition:

$$\begin{cases} a_{45} \ \mathbf{I} \ a_5 \\ a_{46} \ \mathbf{I} \ a_6 \end{cases} \text{and } a_4 \ \mathbf{P} \ a_2 \ M \ a_0.$$



MOPI property is violated

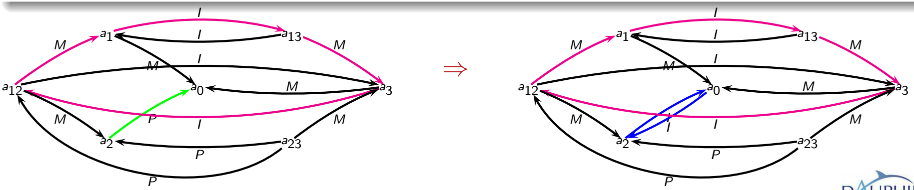
$$\left\{\begin{array}{l} a_{ij} \sim a_j \\ a_{ik} \sim a_k \end{array}\right. \Rightarrow a_i \ TC_P \ a_0$$

Our proposition

1. **Step 1**: Compute the set

$$TC_P(a_i) = \{(x,y) \in P \text{ such that } a_i \ TC \ x \ P \ y\}$$

2. **Step 2**: *Recommendations to DM*
for each $(x,y) \in TC_P(a_i)$, remove P between x and y, replace it by I or don't do anything.



MOPI property is violated

$$\begin{cases} a_{ij} \sim a_j \\ a_{ik} \sim a_k \end{cases} \Rightarrow a_i \ TC_P \ a_0$$

Proposition

If DM follows the recommendations in STEP 2, then no new inconsistencies are created.

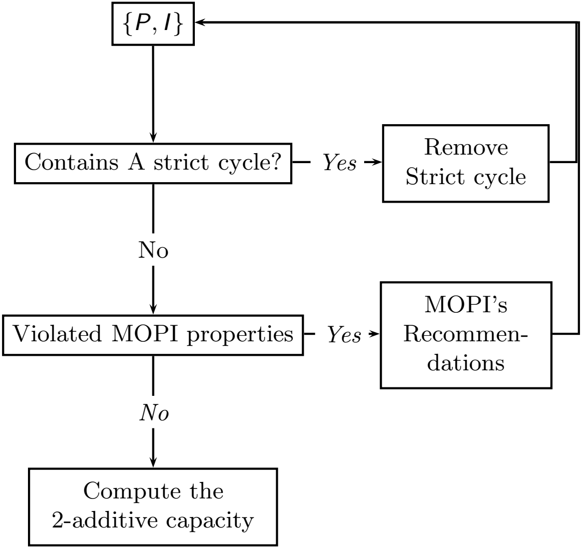


Figure: Algorithm for the treatment of an ordinal information

Thank you !