Fuzzy Integrals

Radko Mesiar[1,2] and Andrea Mesiarová[3]

[1] Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering, Slovak University of Technology
SK-813 68 Bratislava, Slovakia
mesiar@math.sk

[2] Institute of the Theory of Information and Automation
Czech Academy of Sciences
Prague, Czech Republic

[3] Mathematical Institute of SAS
Štefánikova 49, 81473 Bratislava, Slovakia
mesiarova@mat.savba.sk

Abstract. We bring an overview of fuzzy integrals, including historical remarks. Choquet integral can be traced back to 1925. Sugeno integral has a predecessor in Shilkret integral from 1971. Some other fuzzy integrals and the corresponding discrete integrals are also given. An application of Choquet integral to additive impreciseness measuring of fuzzy quantities with interesting consequences for fuzzy measures is presented. Finally, recent development and streaming of fuzzy integrals theory are mentioned.

1 Introduction

The history of integration began with the (σ-additive) measure on the real line/plane assigning to intervals their length / to rectangles their area. This measure is known now as the Lebesgue measure on Borel subsets of $\mathbb{R}$ / $\mathbb{R}^2$, and the first results in this direction were obtained by ancient Greeks. Later generalizations to abstract spaces but still based on a (σ-additive) measure led to the Lebesgue integral and several related integrals with either special domains (Banach spaces, for example) or with special types of additivity (semigroup-valued measures, pseudo-additive measures). For an exhaustive overview we recommend a recent handbook [26].

All above mentioned types of integrals are based on partition-based representation of simple functions (i.e., functions with finite range) and the composition property of measures (i.e., measure of a union $A \cup B$ of disjoint events depends only on measure of A and measure of B).

However, more then 70 years ago, scientists became more and more interested into monotone set functions without composition property of any type. Recall here for example submeasures, supermeasures, sub-(or super-) modular measures, etc. Note also that the first trace of an integral with respect to such set functions goes back to Vitali to 1925 [33], and his integral was independently

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proposed by Choquet in 1953 [9] (and it wears now Choquet's name). In different branches of mathematics, there are different names for the same object of our interest. Monotone set functions vanishing in the empty set (and defined on a σ-algebra, but also on a paving only) are called premeasures [29], capacities (in original [9] with additional requirements, now often abandoned), monotone games, monotone measures. Throughout this contribution, we will use the name fuzzy measure, though again its original definition by Sugeno [30] was more restrictive.

Definition 1. *Let $(X, \mathcal{A})$ be a measurable space, i.e., $X \neq \emptyset$ is a universe and $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. A mapping $m : \mathcal{A} \to [0,1]$ is called a fuzzy measure if*

(i)] $m(\emptyset) = 0$ and $m(X) = 1$ (boundary condition)

and

(ii) $A \subset B \subset X \Rightarrow m(A) \leq m(B)$ (monotonicity)

are fulfilled.

Note also that in the fuzzy measure theory, sometimes the range of m is allowed to be $[0, \infty]$, not forcing the normalization condition $m(X) = 1$. On infinite spaces, also some types of continuity used to be required, especially the left-continuity (lower semi-continuity), i.e.,

(iii) $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$.

If necessary, we will indicate that (iii) is required.

Our aim is to discuss so called fuzzy integrals, i.e., integrals based on fuzzy measures. In general, (measurable) functions to be integrated are $X \to [0,1]$ mappings, and thus they can be treated as fuzzy subsets of X. In such case, the fuzzy integral I is always supposed to be a monotone extension of the underlying fuzzy measure m (acting on crisp subsets of X) which will act on (measurable) fuzzy subsets of X.

2 Basic Fuzzy Integrals

Following Zadeh [36], we will call $\mathcal{A}$-measurable fuzzy subsets of X *fuzzy events.* Note also that Zadeh [36] defined a fuzzy probability measure M for fuzzy events by

$$M(f) = \int_X f \mathrm{d}P,$$

where P is a fixed probability measure on $(X, \mathcal{A})$ and $M(f)$ is the standard Lebesgue integral of f with respect to P.

Obviously, M extends P from crisp subsets of X into fuzzy events on X. However, the requirement that P is a probability restrict Zadeh's proposal efficiently. As already mentioned, the first known approach to fuzzy integrals can be found in [33], but also (and independently) in [9] and [29], and this integral is now known as Choquet integral.

Definition 2. *Let m be a fuzzy measure on $(X, \mathcal{A})$. The Choquet integral $C_m(f)$ of a fuzzy event f with respect to m is given by*

$$C_m(f) = \int_0^1 m(f \geq x) \mathrm{d}x, \tag{1}$$

where the right-hand side of (1) is the Riemann integral.

Note that the Choquet integral is well defined for any fuzzy measure (with no continuity requirement). However, if m is left-continuous then also C_m is a left-continuous functional. Observe that up to the monotonicity and boundary conditions $C_m(0) = 0, C_m(1) = 1$ (i.e., C_m on finite X with n-elements is an n-ary aggregation operator [21], see also [6, 20]), a genuine property characterizing the Choquet integral is the comonotone additivity, i.e.,

$$C_m(f + g) = C_m(f) + C_m(g) \tag{2}$$

whenever $f, g, f + g$ are fuzzy events and f is comonotone to

$$g, (f(x) - f(y))(g(x) - g(y)) \geq 0$$

for all $x, y \in X$. This property was just another source for introducing the Choquet integral, i.e., the only monotone functional I (on events) which is comonotone additive and fulfill the boundary conditions is exactly the Choquet integral C_m with $m(A) = I(\mathbf{1_A})$, see [27].

Observe that the comonotone additivity (with monotonicity) ensures also the homogeneity, $I(cf) = cI(f)$ for any real c such that f and cf are fuzzy events. Finally, note that the comonotone additivity (in representation of functionals) can be replaced by the homogeneity and the horizontal additivity

$$I(f) = I(f \wedge a\ ) + I(f - f \wedge a),$$

$a \in [0, 1]$, f a fuzzy event. Thus, Choquet integral is also (positively) homogenous and horizontal additive. For more details we recommend [10, 2].

Note also that for σ-additive fuzzy measures, i.e., for probability measures, Choquet integral coincide with the usual Lebesgue integral. Moreover, for an ∞-monotone fuzzy measure m, i.e., a belief measure [25, 34], we have $C_m(f) = \inf\{C_P(f) \mid P \geq m\}$. Similarly, for duals to belief measures, $m^d(A) = 1 - m(A^c)$, i.e., for plausibility measures, (∞-alternating fuzzy measures, [34, 25]) we have $C_{m^d}(f) = \sup\{C_P(f) \mid P \leq m^d\}$.

Sugeno integral was introduced in [30].

Definition 3. *Let m be a fuzzy measure on $(X, \mathcal{A})$. The Sugeno integral $S_m(f)$ of a fuzzy event f with respect to m is given by*

$$S_m(f) = \bigvee_{x=0}^{1} \min(x, m(f \geq x)). \tag{3}$$

Note that if m is a possibility measure on X [37, 11] induced by a possibility density $\varphi : X \to [0,1]$, $m(A) = \sup\{\varphi(x) \mid x \in A\}$, then (3) can be rewritten into

$$S_m(f) = \bigvee_{x \in X} \min(\varphi(x), f(x)). \tag{4}$$

Similarly as in the case of Choquet integral, Sugeno integral represents functionals which are monotone, $I(1) = 1, I(0) = 0$, comonotone maxitive (i.e., $I(f \vee g) = I(f) \vee I(g)$ whenever f and g are comonotone) and min-homogenous, i.e., $I(a \wedge f) = a \wedge I(f)$ for all $a \in [0,1]$. For a full proof we recommend overview [2]. Note that the comonotone maxitivity in the above claim can be replaced by a weaker condition of horizontal maxitivity $I(f) = I(f \wedge a) \vee I(f_a), a \in [0,1]$, where

$$f_a(x) = \begin{cases} f(x) & \text{if } f(x) > a, \\ 0 & \text{else} \end{cases}$$

or even by the max-homogeneity $I(a \vee f) = a \vee I(f)$, see [2, 1, 22].

3 Some Other Fuzzy Integrals

In 1971, Shilkret [28] introduced an integral $Shi_m(f)$ with respect to a maxitive fuzzy measure m (i.e., $m(A \cup B) = m(A) \vee m(B)$ for any events A, B), which can be straightforwardly extended to any fuzzy measure m,

$$Shi_m(f) = \bigvee_{x=0}^{1} (x.m(f \geq x)).$$

Similarly, Weber [35] has proposed a generalization of the Sugeno integral. For any t-norm T with no zero divisors (for more details see [18]), Weber integral is given by

$$W_{T,m}(f) = \bigvee_{x=0}^{1} T(x, m(f \geq x)).$$

Note that $W_{\min,m} = S_m$ and that $W_{T_{\mathbf{P}},m} = Shi_m$, where $T_{\mathbf{P}}$ is the product t-norm. Alternative approaches were presented also in [23, 17, 19].

Several more complicated and even peculiar fuzzy integrals are discussed in [2]. Appropriate arithmetical operations for these fuzzy integrals are investigated in [3].

4 Discrete Fuzzy Integrals

In the case when X is finite we will use convention $X = \{1, 2, \ldots, n\}, n \in \mathbb{N}$, and $\mathcal{A} = \mathcal{P}(X)$. Then fuzzy events f are, in fact, n-dimensional vectors $f = (f_1, \ldots, f_n) \in [0,1]^n$.

Fuzzy integrals can be expressed in such case by several equivalent discrete formulas. Some of them are based on the Mőbius transform [13] (namely C_m) or on the possibilistic Mőbius transform [13] (namely S_m). Denote by $(f'_1, \ldots, f'_n)$ a non-decreasing permutation of f. Then (with convention $f'_0 = 0$ and $f'_{n+1} = 1$)

$$C_m(f) = \sum_{i=1}^{n} f'_i \cdot \big(m(\{j \mid f_j \geq f'_i\}) - m(\{j \mid f_j \geq f'_{i+1}\})\big) = \sum_{i=1}^{n} (f'_i - f'_{i-1}) m(\{j \mid f_j \geq f'_i\})$$

and

$$S_m(f) = \bigvee_{i=1}^{n} \min(f'_i, m(\{j \mid f \geq f'_i\})),$$

$$Shi_m(f) = \bigvee_{i=1}^{n} f'_i \cdot m(\{j \mid f \geq f'_i\}),$$

$$W_{T,m}(f) = \bigvee_{i=1}^{n} T(f'_i, m(\{j \mid f \geq f'_i\})).$$

5 Two Applications of Choquet Integral

n-dimensional fuzzy quantities are normal fuzzy subsets of $\mathbb{R}^n$ such that each α-cut, $\alpha \in]0,1]$, is a convex compact subset of $\mathbb{R}^n$. Typical examples are triangular or trapezoidal fuzzy numbers for $n = 1$, conic or pyramidal fuzzy numbers for $n = 2$.

For any fuzzy measure m on Borel subsets of $\mathbb{R}^n$ (not normalized, in general) the Choquet integral $I_m(f) = C_m(f)$ of a fuzzy quantity f is well defined and it can be understood as the impreciseness of f. For the sum $h = f \boxplus g$ of two fuzzy quantities defined by means of the Zadeh extension principle (or, equivalently by means of the sums of corresponding α-cuts), we expect the additivity of impreciseness measure achieved by means of the Choquet integral, i.e.,

$$I_m(f \boxplus g) = I_m(f) + I_m(g).$$

Evidently, not any fuzzy measure m ensures the desired result. A complete description of appropriate fuzzy measures (i.e., fuzzy measures additive in argument, $m(A + B) = m(A) + m(B)$) was recently shown in [5].

In the case $n = 1$, the only convenient m is (up to a multiplicative constant) the standard Lebesgue measure $m = \lambda$, i.e., for compact convex subsets of $\mathbb{R}$ the corresponding length (each such subset is a closed subinterval of $\mathbb{R}$). Moreover, $I_\lambda(f)$ is exactly the area of the surface bounded by f and the real axis. Already for $n = 2$, there are several types of appropriate measure m. Note that any (even infinite) convex combination $m = \sum_{i=1}^{k} w_i m_i$ of fitting fuzzy measures m_i yields a fitting fuzzy measure m (this follows from the additivity in measure of Choquet integral). Two typical examples are:

(i) a fuzzy measure m assigning to each compact convex subset of $\mathbb{R}^2$ its perimeter (note that the additivity of perimeters of compact convex subsets of $\mathbb{R}^2$ was observed already by Cauchy); in this case $I_m(f)$ is the area of the surface of the graph of the function f (lateral surface);

(ii) for arbitrary fixed angle $\varphi \in [0, \pi[$, a fuzzy measure m_φ assigning to each compact convex subset of $\mathbb{R}^2$ the length of its projection into any straight line with slope φ; in this case $I_{m_\varphi}(f)$ is the area of the projection of the body bounded by f and the base plane into any plane perpendicular to the plane with slope φ.

Note that one can expect the volume $V(f)$ of the body bounded by f and the base plane to be an appropriate impreciseness measure. However, V is not an additive function. For example, $V(f \boxplus f) = 4V(f)$.

Another interesting application of Choquet integral is linked to an older result of Tarski [31] showing the existence of an additive fuzzy measure m (but not σ-additive) on an infinite universe such that m is vanishing on singletons (i.e., $m(A) = 0$ for al finite subsets $A \subset X$). Applying the symmetric Choquet integral [29] with respect to Tarski's fuzzy measure m yields a linear operator on (real) measurable functions invariant under finitely many changes over the functions to be evaluated. For more details we recommend [4].

6 Concluding Remarks

The importance of fuzzy integrals towards applications is hidden in their capability to express the possible interaction among single parts of our universe X for a global representation of a function (describing the real acting of an observed system) by means of a single value. This phenomenon can not be captured by the standard Lebesgue integral, though it always play a prominent role also in our discussed framework. Expected properties of our functionals, which extend an evaluation of crisp events (i.e., fuzzy measures) to an evaluation of fuzzy events (i.e., fuzzy integrals) determine our choice of an appropriate fuzzy integral.

Two prominent fuzzy integrals - Choquet and Sugeno integrals - are linked to two different types of arithmetical operations on [0,1] (or, in more general form, on $[0, \infty]$, or even on $[-\infty, \infty]$). Namely to Archimedean operations of (truncated) summation and multiplication in the first case, and to idempotent operations max (sup) and min in the second case. Several attempts to connect these two types of fuzzy integrals (or, equivalently, of arithmetical operations) were published so far, see e.g. [8, 23]. Compare also [17, 19] and [3].

A recent interesting approach of integrating both Choquet and Sugeno integrals into a single functional can be found in [32, 24], compare also [7]. Several other generalizations or extensions can be found , among others, in [29, 14, 10, 25, 15, 16].

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