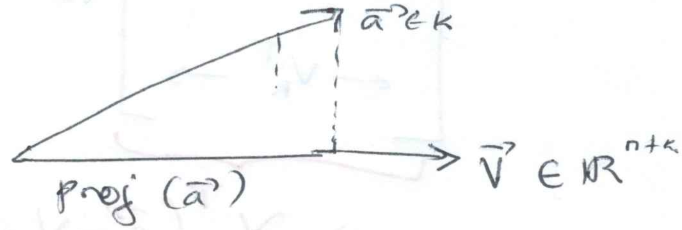
LECT 12 Wed 3/14/18

Recall:

$$\underbrace{\frac{\vec{V}\vec{V}^T}{\|\vec{V}\|^2}}_{H \in \mathbb{R}^{n+k}} \vec{a} = H\vec{a}$$



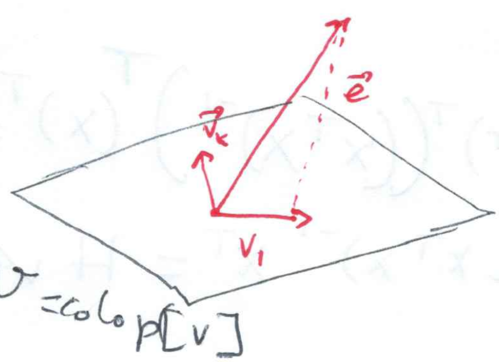
Notes 1

① $\vec{a} = \text{proj}_{\vec{V}}(\vec{a}) + \vec{e} \implies \vec{e} = \vec{a} - \text{proj}_{\vec{V}}(\vec{a})$

② $\text{proj}_{\vec{V}}(\vec{a}) \in \text{colop}(\vec{V}) \implies \text{proj}_{\vec{V}}(\vec{a}) = \cancel{w}\vec{V} = \ell\vec{V}$

③ $\vec{e} \cdot \text{proj}_{\vec{V}}(\vec{a}) = 0 \implies \vec{e} \cdot \vec{V} = 0 \implies \vec{V} \perp \vec{e} = 0$

Matrix $V = [\vec{V}_1 \,\vdots\, \vec{V}_2 \,\vdots\, \dots \,\vdots\, \vec{V}_n] \in \mathbb{R}^{n+k}$



Note 2

#2 $\text{proj}_{\vec{V}}(\vec{a}) \in \text{Colop}(V)$, $\exists \vec{w}$

$\implies \text{proj}_{\vec{V}}(\vec{a}) = w_1\vec{V}_1 + V_2\vec{W}_2 + \dots + W_k\vec{V}_k = V\vec{w}$

#1 $\exists \vec{w}$ $\vec{e} = \vec{a} - V\vec{w}$

#3 $\vec{e} \cdot \vec{V}_1 = 0$

$\vec{e} \cdot V_2 = 0$

$\dots\dots$

$\vec{e} \cdot V_k = 0$

$(a - V\vec{w}) \cdot \vec{V} = 0$

$\vec{V}_1^T(\vec{a} - V\vec{w}) = 0$

$$\underbrace{\begin{bmatrix} \leftarrow & V_1^T & \rightarrow \\ \leftarrow & V_2^T & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & V_k^T & \rightarrow \end{bmatrix}}(\vec{a} - V\vec{w}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}_n$$

$$\Rightarrow V^T(\vec{a} - V\vec{w}) = \vec{0}$$

$$\Rightarrow V^T\vec{a} - V^T V\vec{w} = \vec{0}$$

$$\Rightarrow V^T\vec{a} = \underbrace{V^T V}_{k\times k \text{ matrix}}\vec{w} \qquad \Rightarrow \cancel{\vec{w} = (V^T V)^{-1}\vec{a}}$$

$$\Rightarrow \vec{w} = (V^T V)^{-1} V^T\vec{a}$$

Projection of : $\text{proj}_{\vec{V}}(\vec{a}) = V\vec{w} = \underbrace{V(V^T V)^{-1} V^T}_{H}\vec{a}$

we see this : $\hat{y} = H\bar{y} = \underbrace{X(X^T X)^{-1} X^T}_{H}\vec{Y}$

some Properties :

① Symmetric

$H^T = H \Rightarrow (X(X^T X)^{-1} X^T)^T = (X^T)^T((X^T X)^{-1})^T (X)^T = X$

$= X(X^T X)^{-1} X^T = H$ ✓

$(A^{-1})^T = (A^T)^{-1}$

$A^T(A^{-1})^T = (A^{-1}A)^T = (I)^T = I$

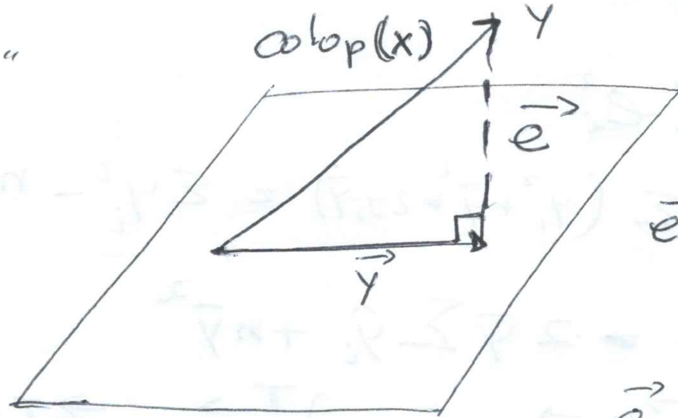
$\Rightarrow A^T B = I \Rightarrow B = (A^T)^{-1}$

② Idempotent

$H \cdot H = X(X^T X)^{-1}(X^T \cancel{X})(X^T X)^{-1} X^T = X(X^T X)^{-1} X^T = H$

$\underbrace{H \cdot H_{\vec{a}}}_{\text{proj}_{\vec{V}}(\vec{a})} = H\,\text{proj}_V(\vec{a}) = \text{proj}_V(\text{proj}_V(\vec{a}))$

Rank of X is "p"



$\vec{e} \cdot \vec{\hat{y}} = 0$

$\vec{e} = \vec{y} - \vec{\hat{y}} = \vec{y} - H\vec{y}$

$= (I - H)\vec{Y}$

$\vec{e} = (I - H)\vec{Y}$

$\vec{\hat{y}} = H\vec{y}$

$(I - H)(I - H) = II - 2HI + HH$

$= I - 2H + H$

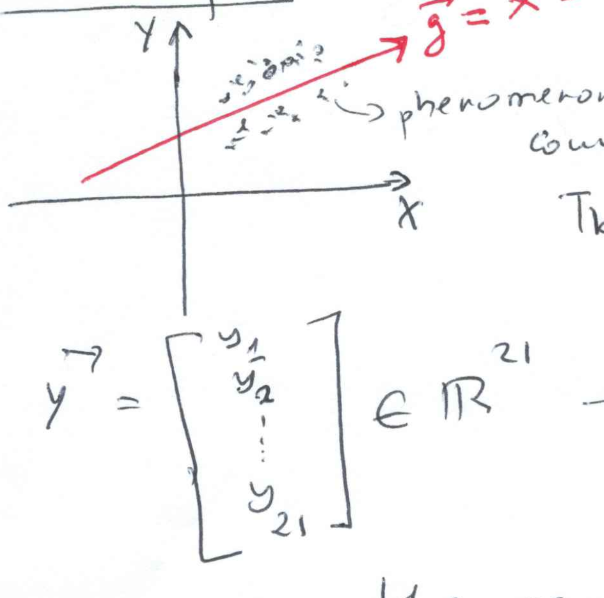
$= I - H$

Model degree of freedom + error degree of freedom = error d° freedom

Rank of (H) is p+1 → by $\hat{y}$

Rank (H) + Rank (I − H) = n b/c y is in "n"

Back to p = 1



→ phenomenom happen in y
count the dots, here is 21 ⇒ 21 dimentional space.
They living in 19 dimentional space.

$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{21} \end{bmatrix} \in \mathbb{R}^{21}$ → squeeze into 2 dimentional space.
$\text{colop}(\vec{1}, \vec{x}) \in \mathbb{R}^{19}$

Pythagorean theorem :

$\|y\|^2 = \|\hat{y}\|^2 + \|e\|^2$

$$\|y\|^2 = \|\vec{\hat{y}}\|^2 + \|\vec{e}\|^2$$

$$\sum y_i^2 = \sum \hat{y}_i^2 + \sum e_i^2$$

<u>Note</u>: $\sum (y_i - \bar{y})^2 = \sum (y_i^2 + \bar{y}^2 - 2y_i\bar{y}) = \sum y_i^2 - n\bar{y}^2$

$$\sum (\hat{y}_i - \bar{y})^2 = \sum \hat{y}_i^2 - 2\bar{y}\sum \hat{y}_i + n\bar{y}^2$$

$$\sum \hat{y}_i = \vec{\hat{y}} \cdot \underset{\text{one}}{\mathbf{1}_n} = \vec{\hat{y}}^T \mathbf{1} = (\underbrace{H\vec{y}}_{\hat{y}})^T \vec{1} = \vec{y}^T H^T \vec{1} = \vec{y}^T H T$$

H: rank: p+1

it projects into space - colop space. Projection of 1 of colop(X

$$\sum y_i^2 = \sum \hat{y}_i^2 + \sum e_i^2 \Rightarrow$$

$$\sum y_i^2 - n\bar{y}^2 = \sum \hat{y}_i^2 - n\bar{y}^2 + \sum e_i^2$$

$$\Rightarrow \sum (y_i - \bar{y})^2 = \underbrace{\sum (\hat{y}_i - \bar{y})^2}_{SSR} + \underbrace{\sum e_i^2}_{SSE}$$