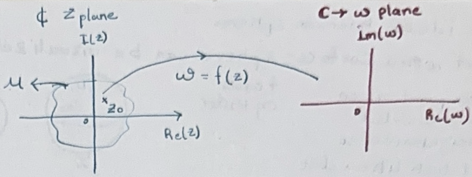
Riemann Surfaces



Goals

- → Idea of Rieman surface
- → Example of Rieman surface

→ Z plane is Complex plain.
→ U is open set in Complex plain (z)
→ f is a function that takes Complex Value and spits Complex values
→ function is define in the open set U

Recall:

$f(z)$ is said to be analytical or holomorphic at z_0 if the following 3 equivalent Conditions holds:

(i) If we write $w = u + iv$ then $u = Re(f)$, $v = Im(f)$; then we want partial derivatives.

$$u_x = \frac{\partial u}{\partial x} ; \quad u_y = \frac{\partial u}{\partial y} ; \quad v_x = \frac{\partial v}{\partial x} ; \quad v_y = \frac{\partial v}{\partial y}$$

exits and are Continious. and furthersatisfies. Cauchy-Riemann Equations

$$u_x = v_y \quad \text{and} \quad v_x = -u_y \quad \forall z \text{ in a neighbord of } z_0$$

If then first Condition is satisfied then f Can be defined as anylatic/holomorphic at z_0

(ii) the limit $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exist

$\forall$ point z in a neighboor hood of z_0

(iii) $\exists$ a power series of the form $\sum_{n\geq 0} a_n (z - z_0)^n$ which is Convergent to $f(z)$ for each point z in the neighboorhood of z_0

→ Connection B/w (i) & (ii) If we want to express the direvatie of a function in the neighboorhood of z_0 then we Can use (i) to write.

$$\cancel{\frac{\partial f}{\partial z}} \quad \frac{df}{dz} = \frac{\partial f}{\partial x} = u_x + i v_x$$

Connection B/w (ii) & (iii) It say that if a function is differential at a point once then it's infintely differentiable.

→ A Convergent power series. allows you to differential it term by term and the resulting series is also differentialable with the same radius.

→ One differentiable $\Rightarrow$ always differentiable in the neighboorhood i.e ∞ differentiable.

$a_n = \frac{f^{(n)}(z_0)}{\lfloor n}$; $f^{(n)}(z_0) = \frac{d^n f}{dz^n}$

So that $\sum a_n (z - z_0)^n$ is th first Tylor expansion of f around z_0

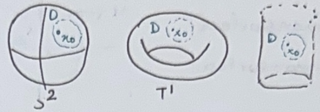
The ide of Riemann Sphereo

start with a Surface; sphere can be visualized in $\mathbb{R}^3$

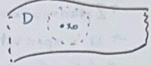
torus

cylinder

→ Given point x_0 on Surface select a neighbour of x_0 that look like a disc.



* Disc is not flat its curved since it sits on surface
* Topologically the disc is flat in Complex plane



Imagine a Complex valued function
↳ (It is always defined on the Disc D) Its takes Complex values.

(neighBoorhood) $D \xrightarrow{f} \mathbb{C}$ is holomorphic? at x_0 (Holomorphic)

Now we want to define when is this function is holomorphic at x_0

So we want to do the same Complex analysis on the plane (which we usually do)

we want to do this Complex analysis on 3d Surfaces.

we want to do Complex analysis on 3D surfaces.

Hence the idea of Riemann Surface is to do Complex analysis on 3D surfaces.

Q_1 How do you define f to be Holo at any point in D (neighboorhood)

One way to do so is:

Topologically identify D with an open subset, say a unit disc.

$$\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$$ this is an open unit discs. Since $|z| < 1$

By chosing a homeomorphism. (Its topological isomorphism)

$\phi : D \to \Delta$ and requiring $f \circ \phi^{-1}$ is holo at $\phi(x_0)$

→ D Contains x_0
→ D is sitting inside the surface X
→ we have function defined on D that take Complex values.
→ Aim is to say f is holomorphic at Point in D

$$\begin{tikzcd}
\mathbb{C} & & X & & \\
U & & \mathcal{U} & & \\
\Delta \arrow[rr, bend left, dashed, "\phi^{-1}"] \arrow[rrrr, bend right, "f \circ \phi^{-1}"'] & & D \arrow[ll, "\sim", "\phi"'] \arrow[rr, "f"] & & \mathbb{C}
\end{tikzcd}$$

Complexplane

Surface

Complex function

Unit disc.

Complex ~~plane~~ Number

To say that f is holo at a point in D we take a topological isomorphism ϕ into the unit disc (Δ) in the complex plain $\mathbb{C}$

we apply Composition ϕ^{-1} ($\because$ the map ϕ is topological isomorphism that means ϕ^{-1} exist)

$f \circ \phi^{-1}$ (to move from one Complex plane to another) Complex number.

→ $f \circ \phi^{-1}$ gives us the map from unit disc of Complex plane to Complex numbers.

→ this means that $f \circ \phi^{-1}$ is holo at a point in unit disc where the point on unit disc is the image of point x_0 that is in D.

→ we have showed that the neighboorhood of a surface is a disc in Complex plane and this plane maps to Complex numbers and most importantly $f \circ \phi^{-1}$ is holo at $\phi(x_0)$ | unit disc ϕ that is mapped from D to Complex plane (disc Δ)

→ In the same way we say that ~~this~~. f is holo on D if $f \circ \phi^{-1}$ is holomorphic on Δ. The pair (D, ϕ) is called a Complex Coordinate chart.

Coordinate chart:

we have seen that the identification of ϕ and this disc like neighboorhood is a pair. and this is called complex Coordinate chart

(D, ϕ) (This provides you with Coordinate chart to do Complex Analysis)

so One intuitive way to see a Riemann Surface is to think them as they come up They help us to do Complex analysis on Surfaces.

Formal defination of Coordinate chart.

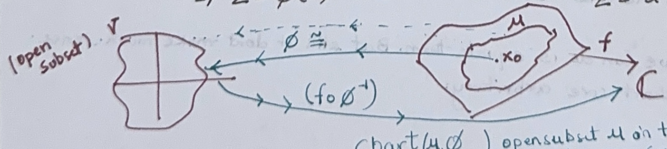
Its a pair $(\mathcal{U}, \phi)$ where $\mathcal{U}$ is an open subset of X,

$\phi : \mathcal{U} \xrightarrow{\sim} V$ is a homeomorphism of $\mathcal{U}$ onto an open subset of V of $\mathbb{C}$

$$\begin{tikzcd}
\mathbb{C} & x \in X & \\
\Delta \arrow[u, phantom, "\cup" description] \arrow[rr, bend right, "f \circ \phi^{-1}"'] & D \arrow[u, phantom, "\cup" description] \arrow[l] \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Call a varying point in $x \in X$.
Then the map of this x in the unit disc Δ is $\phi(x)$

$\Rightarrow z = \phi(x)$ & z is a Coordinate $z \in \mathbb{C}$ plane
$z = a + ib$ $a = Re(z) = Re(\phi(x))$
$b = Im(z) = Im(\phi(x))$



Chart$(\mathcal{U}, \phi)$ opensubset $\mathcal{U}$ on the surface mapped to open subset V (V can be open set not disc)

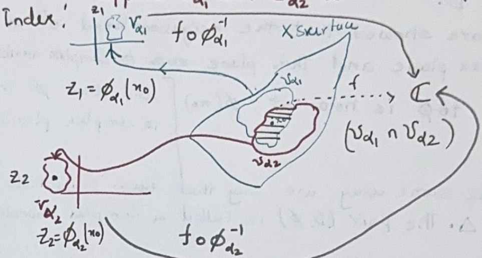
Priliminary Def of Riemann Surface:

A surface X covered by a collection of charts. $\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$

$(X = \bigcup_\alpha U_\alpha)$ I = index. we Run into issue With the above defination

$\alpha \downarrow$ union forall α Belonging to Index

Suppose U_{α_1} and U_{α_2} both contain a point x_0



we see that

$x_0 \in (U_{\alpha_1} \cap U_{\alpha_2})$

then the problem is that

How do we tell f is holomorphic at x_0?

Since the index, chart can be Arbitrary How to define Holo of function f?

$\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ what happens. if x_0 belongs to two (U_1, ϕ_1) & $(U_2 \phi_2)$

F is holomorphic at x_0 ~~then~~

(i) From one subset $(U_{\alpha_1}, \phi_{\alpha_1})$ if $f \circ \phi_{\alpha_1}^{-1}$ is holo at $z_1 = \phi_{\alpha_1}(x_0)$

(ii) from two subset $(U_{\alpha_2}, \phi_{\alpha_2})$ if $f \circ \phi_{\alpha_2}^{-1}$ is holo at $z_2 = \phi_{\alpha_2}(x_0)$

Since the chart are arbritary to define holo at x_0 we have to define the property intrensic to ~~a~~ the function not dependent on the chart or the neighborhood.

As we have seen that the holomorphic is intrensic to the function and does not depend onthe selection of the chart. (Re paramitrization)

How to avoid a situation that where from one chart you get the defination?

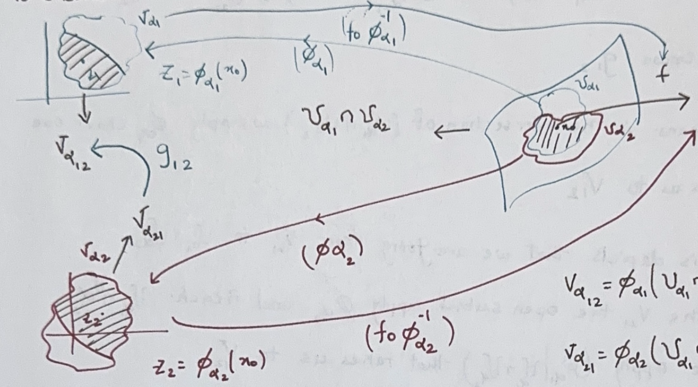
$f \circ \phi_{\alpha_1}^{-1}$ α_1 is the chart selected in this case.

$f \circ \phi_{\alpha_2}^{-1}$ α_2 is the chart seletd

One thing we can do is equate them. But still it dont make much sense. Since the chart are arbritary.

Hence the idea of a function ie holo should be intrensic to the function

To ensure we define Holo independent of chart



$V_{\alpha_{12}} = \phi_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2})$

$V_{\alpha_{21}} = \phi_{\alpha_2}(U_{\alpha_1} \cap U_{\alpha_2})$

→ To ensure the holomorphictity of a (f) function using various charts · happens and to make holomorphicity an intrensic property of the function lets define

$$g_{12} = (\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}}) \circ (\phi_{\alpha_2}^{-1}|_{V_{\alpha_{21}}}) : V_{\alpha_{21}} \longrightarrow V_{\alpha_{12}}$$

we want g_{12} to Be defined as above ~~this~~ ~~ensures~~ · ~~it is~~ and it is an homoeomorphism. Since $(\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}})$ is a open subset

$(\phi_{\alpha_2}^{-1}|_{V_{\alpha_{21}}})$ is again open subset

Requiring g_{12} to be holomorphic should also make it into a holomorphic isomorphism.

Intersection U_{α_1} & $U_{\alpha_2} \Rightarrow (U_{\alpha_1} \cap U_{\alpha_2})$ now lets apply ϕ_{α_1} to intersection.

that takes us to $V_{\alpha_{21}}$

Similarly we apply ϕ_{α_2} to the intersection $(U_{\alpha_1} \cap U_{\alpha_2})$ landing us on $V_{\alpha_{12}}$

Now lest understand what happens in g_{12}

$$g_{12} = \left(\phi_{\alpha_1}\big|_{U_{\alpha_1}\cap U_{\alpha_2}}\right) \circ \left(\phi_{\alpha_2}^{-1}\big|_{V_{\alpha_{21}}}\right) : V_{\alpha_{21}} \to V_{\alpha_{12}}$$

Now lest Break Down g_{12}

$\left(\phi_{\alpha_1}\big|_{U_{\alpha_1}\cap U_{\alpha_2}}\right)$ means in the intersection of $(U_{\alpha_1}\cap U_{\alpha_2})$ we apply ϕ_{α_1} chart one

~~that there~~ takes us to V_{12}

$\left(\phi_{\alpha_2}^{-1}\big|_{V_{\alpha_{21}}}\right)$ this depicts that we are going from V_{21} to $U_{\alpha_1}\cap U_{\alpha_2}$

we take the V_{21} the open subset apply $\phi_{\alpha_2}^{-1}$ and Reach $U_{\alpha_1}\cap U_{\alpha_2}$

and then we Apply $\left(\phi_{\alpha_1}\big|_{U_{\alpha_1}\cap U_{\alpha_2}}\right)$ that takes us to V_{12}

So we Can define $\boxed{g_{12} : V_{21} \to V_{12}}$

we were able to define $g_{12} : V_{21} \to V_{12}$ as homeomorphism since g_{12} is a map from open subset U to V so g_{12} is homeomorphism.

Now since we were able to tell that g_{12} is homeomorphic now we want it holomorphic! (This means its injective)

→ Requiring g_{12} to be holomorphic. should also make it into a holomorphic isomorphism.

$$\boxed{f\circ\phi_{\alpha_1}^{-1}\circ g_{12}} = \boxed{f\circ\phi_{\alpha_2}^{-1}}$$

The above equation tell us that $f\circ\phi_{\alpha}^{-1}$ is holo iff $f\circ\phi_{\alpha_2}^{-1}$ is

Defination:
If we want that the function g_{12} (transition functions) to be holo whichever $(U_{\alpha_1}\cap U_{\alpha_2}\neq\emptyset)$ we get a Compatible. Collections of charts which gives a Riemann Surface structure on "X"

Summarizing on Rieman Surface.

→ started with Complex analysis on Surface in $\mathbb{R}^3$

→ we define Complex Coordinates charts.

$$\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$$ I = indexed charts.
α = one of the arbritary chart.

→ Charts should Cover the whole open set

→ Problem Comes when multiple charst Can represent the same things so to avoid the amiguity. we define a transition function g_{12} to be holomorphic (whenever $U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset$) this gives us Compatible Collections of charts that gives Riemann structure on X

$$f \circ \phi_{\alpha_1}^{-1} \circ g_{12} = f \circ \phi_{\alpha_2}^{-1}$$

↳ g_{12} is holomorphic.