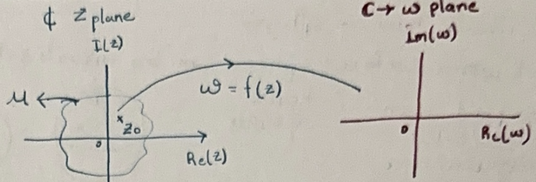
# Riemann Surfaces

## Goals

→ Idea of Rieman surface

→ Example of Rieman surface




→ $Z$ plane is Complex plain.

→ $U$ is open set in Complex plain ($z$)

→ $f$ is a function that takes Complex value and spits Complex values

→ function is define in the open set $U$

Recall:

$f(z)$ is said to be analytical or holomorphic at $z_0$ if the following 3 equivalent Conditions holds:

(i) if we write $\omega = u + iv$ then $u = Re(f)$, $v = Im(f)$ then we want partial derivatives.

$$u_x = \frac{\partial u}{\partial x} ; \; u_y = \frac{\partial u}{\partial y} ; \; v_x = \frac{\partial v}{\partial x} ; \; v_y = \frac{\partial v}{\partial y}$$

Exits and are Continious. and furthersatisfies. Cauchy-Riemann Equations

$$u_x = v_y \quad \text{and} \quad v_x = -u_y \quad \forall z \text{ in a neighbord of } z_0$$

If then first Condition is Satisfied then $f$ Can be defined as anylatic/holomorphic at $z_0$

(ii) the limit $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exist

$\forall$ point $z$ in a neighboor hood of $z_0$

(iii) $\exists$ a power series of the form $\sum_{n \geq 0} a_n (z - z_0)^n$ which is Convergent to $f(z)$ for each point $z$ in the neighboor hood of $z_0$

→ Connection B/w (i) & (ii) If we want to express the direvatie of a function in the neighboorhood of $z_0$ then we Can use (i) to write.

$$\cancel{\frac{\partial f}{\partial z}} \; \frac{df}{dz} = \frac{\partial f}{\partial x} = u_x + i v_x$$

Connection B/w (ii) & (iii) It say that if a function is differential at a point once then it's infintely differentiable.

→ A Convergent power series. allows you to differential it term by term and the resulting Series is also differentialable with the smae radius.

→ One differentiable $\Rightarrow$ always differentiable in the neighboorhood ie $\infty$ differentiable.

$a_n = \frac{f^{(n)}(z_0)}{Ln}$ ; $f^{(n)}(z) = \frac{d^n f}{dz^n}$

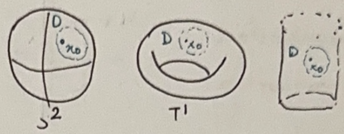
So that $\sum a_n (z - z_0)^n$ is th first Tylor expansion of $f$ around $z_0$

The ide of Riemann sphereo

start with a Surface: sphere can be visualized in $\mathbb{R}^3$

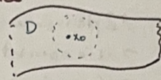
→ Given point $x_0$ on Surface select a neighbour of $x_0$ that look like a disc.

torus
cylinder




*** Disc is not flat its Curved since it sits on Surface**

* Topologically the disc is flat in Complex plane




Imagine a Complex valued function
↳ (It is always defined on the Disc D) Its takes Complex values.

(neighBoor hood) $D \xrightarrow{f} \mathbb{C}$

**Now we want to define when is this function is holomorphic? at $x_0$ (Holomorphic)**

So we want to do the Same Complex analysis on the plane (which we usually do) we want to do this Complex analysis on 3d Surfaces.

Hence the idea of Riemann Surface is to do Complex analysis on 3D Surfaces.

Q, How do you define f to be Holo at any point in D (neighboorhood)

One way to do so is:

Topologically identify D with an open subset, say a unit disc.

$$\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$$ this is an open unit discs. since $|z| < 1$

By chosing a homeomorphism. (Its topological isomorphism)

$\phi : D \to \Delta$ and requiring $f \circ \phi^{-1}$ is holo at $\phi(x_0)$

→ D Contains $X_0$
→ D is ~~stan~~ sitting insde the surface X
→ we have function defined on D that take Complex values.
→ Aim is to say f is holomorphic at Point in D

Complexplane

Surface

Complex function

$$\begin{tikzcd} \mathbb{C} & X & \\ \cup & \cup & \\ \Delta \arrow[r, bend left, dashed, "\phi^{-1}"] \arrow[rr, bend right, "f \circ \phi^{-1}"'] & D \arrow[l, "\phi", "\sim"'] \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

Unit disc

Complex ~~plane~~ Number

To say that f is holo at a point in D we take a topological isomorphism $\phi$ into the unti disc ($\Delta$) in the Complex plain $\mathbb{C}$

we apply Composition $\phi^{-1}$ ($\because$ the map $\phi$ is topological isomorphism that means $\phi^{-1}$ exist)

$f \circ \phi^{-1}$ (to move from one Complex plane to another) Complex number

→ $f \circ \phi^{-1}$ gives us the map from unit disc of complex plane to complex numbers.

→ this means that $f \circ \phi^{-1}$ is holo at a point in unit disc where the point on unit disc is the image of point $x_0$ that is in D.

→ we have showed that the neighboorhood of a surface is a disc in Complex plane and this plane maps to Complex numbers and most importantly

$f \circ \phi^{-1}$ is holo at $\phi(x_0)$ | unit disc $\phi$ that is mapped from D to complex plane (disc $\Delta$)

→ In the same way we say that ~~this~~. f is holo on D if $f \circ \phi^{-1}$ is holomorphic on $\Delta$. The pair $(D, \phi)$ is called a complex coordinate chart.

## Coordinate chart:

we have seen that the identification of $\phi$ and this disc like neighboorhood is a pair. and this is called complex coordinate chart

$(D, \phi)$ (This provides you with coordinate chart to do Complex Analysis)

so one intuitive way to see a Riemann surface is to think them as they come up They help us to do Complex analysis on surfaces.

formal defination of Coordinate chart.

Its a pair $(\mathcal{U}, \phi)$ where $\mathcal{U}$ is an open subset of X,

$\phi : \mathcal{U} \xrightarrow{\sim} V$ is a homeomorphism of $\mathcal{U}$ onto an open subset of V of $\mathbb{C}$

$$\begin{tikzcd}
\mathbb{C} & & X \\
\cup & & \cup \\
\Delta & & D \arrow[ll] \arrow[r] & \mathbb{C} \\
\arrow[from=3-1, to=3-4, bend right, "f \circ \phi^{-1}"']
\end{tikzcd}$$

Call a varying point in $x \in X$

Then the map of this $x$ in the unit disc $\Delta$ is $\phi(x)$

$\Rightarrow Z = \phi(x)$ & Z is a Coordinate $z \in \mathbb{C}$ plane

$z = a + ib$ $\quad a = Re(z) = Re(\phi(x))$

$b = Im(z) = Im(\phi(x))$

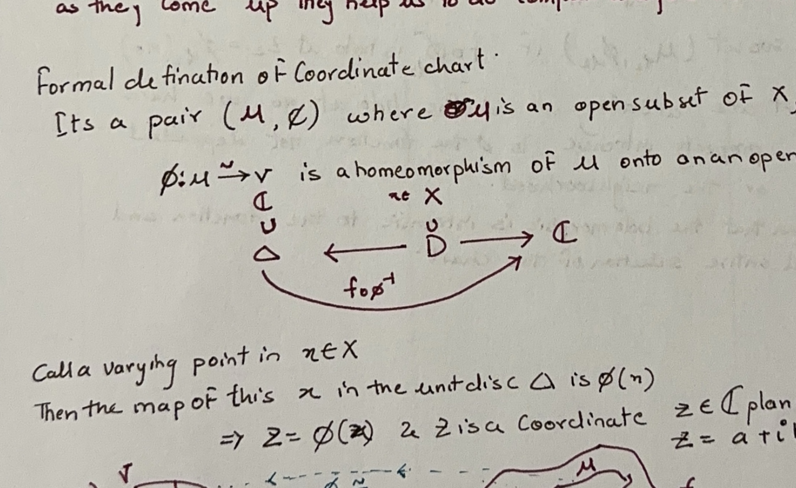



Chart $(\mathcal{U}, \phi)$ open subset $\mathcal{U}$ on the surface mapped to open subset V (V can be open set not disc)

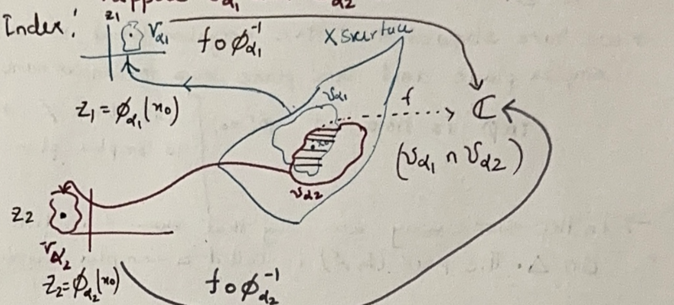
Priliminary Def of Riemann Surface:

A surface X Covered by a Collection of charts. $\{(\mathcal{U}_\alpha, \phi_\alpha) \mid \alpha \in I\}$

$\left(X = \bigcup_\alpha \mathcal{U}_\alpha\right)$ I = index. we Run into issue With the above defination

$\alpha \downarrow$ union forall $\alpha$ Belonging to Index.

Suppose $\mathcal{U}_{\alpha_1}$ and $\mathcal{U}_{\alpha_2}$ both Contain a point $x_0$




we see that
$x_0 \in (\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2})$
then the problem is that
How do we tell f is
holo morphic at $x_0$?

Since the index, chart Can
be Arbritary how to define Holo of function f?

$\{(\mathcal{U}_\alpha, \phi_\alpha) \mid \alpha \in I\}$ what happens. if $x_0$ belongs to two $(\mathcal{U}_1, \phi_1)$ & $(\mathcal{U}_2\, \phi_2)$

F is holo morphic at $x_0$ ~~then~~

(i) From one subset $(\mathcal{U}_{\alpha_1}, \phi_{\alpha_1})$ if $f \circ \phi_{\alpha_1}^{-1}$ is holo at $z_1 = \phi_{\alpha_1}(x_0)$

(ii) from two subset $(\mathcal{U}_{\alpha_2}, \phi_{\alpha_2})$ if $f \circ \phi_{\alpha_2}^{-1}$ is holo at $z_2 = \phi_{\alpha_2}(x_0)$

Since the chart are arbritary to define holo at $x_0$ we have to define the property intrensic to the function not dependent on the chart or the neighboorhood.

As we have seen that the holo morphic is intrensic to the function and does not depend on the selection of the chart. (Re paramitrization)

How to avoid a situation that where from one chart you get the defination?

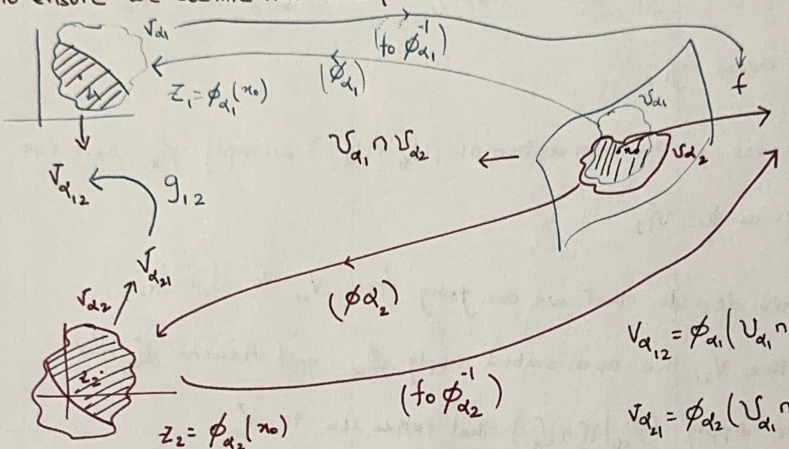
$f \circ \phi_{\alpha_1}^{-1}$ $\alpha_1$ is the chart selected in this case.

$f \circ \phi_{\alpha_2}^{-1}$ $\alpha_2$ is the chart seleted

One thing we Can do is equate them. But still it dont make much sense. since the chart are arbritary.

Hence the idea of a function i.e. holo should be intrensic to the function

To ensure we define Holo indenpendent of chart




$V_{\alpha_{12}} = \phi_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2})$

$V_{\alpha_{21}} = \phi_{\alpha_2}(U_{\alpha_1} \cap U_{\alpha_2})$

→ To ensure the holomorphictity of a (f) function using various charts · happens and to make holomorphicity an intrensic property of the function lets define

$$g_{12} = \left(\phi_{\alpha_1}\big|_{U_{\alpha_1} \cap U_{\alpha_2}}\right) \circ \left(\phi_{\alpha_2}^{-1}\big|_{V_{\alpha_{21}}}\right) : V_{\alpha_{21}} \longrightarrow V_{\alpha_{12}}$$

we want $g_{12}$ to be defined as above ~~this ensures~~ · ~~it is~~ and it is an homoeomorphism. Since $\left(\phi_{\alpha_1}\big|_{U_{\alpha_1} \cap U_{\alpha_2}}\right)$ is a open subset

$\left(\phi_{\alpha_2}^{-1}\big|_{V_{\alpha_{21}}}\right)$ is again open subset

Requiring $g_{12}$ to be holomorphic should also make it into a holomorphic isomorphism.

Intersection $U_{\alpha_1}$ & $U_{\alpha_2} \Rightarrow (U_{\alpha_1} \cap U_{\alpha_2})$ now lets apply $\phi_{\alpha_1}$ to intersection.

that takes us to $V_{\alpha_{21}}$

Similarly we apply $\phi_{\alpha_2}$ to the intersection $(U_{\alpha_1} \cap U_{\alpha_2})$ landing us on $V_{\alpha_{12}}$

Now lest understand what happens in $g_{12}$

$$g_{12} = \left(\phi_{\alpha_1}\big|_{U_{\alpha_1} \cap U_{\alpha_2}}\right) \circ \left(\phi_{\alpha_2}^{-1}\big|_{V_{\alpha_{21}}}\right) : V_{\alpha_{21}} \to V_{\alpha_{12}}$$

Now lest Break Down $g_{12}$

$\left(\phi_{\alpha_1}\big|_{U_{\alpha_1} \cap U_{\alpha_2}}\right)$ means in the intersection of $(U_{\alpha_1} \cap U_{\alpha_2})$ we apply $\phi_{\alpha_1}$ chart one that ~~there~~ takes us to $V_{12}$

$\left(\phi_{\alpha_2}^{-1}\big|_{V_{\alpha_{21}}}\right)$ this depicts that we are going from $V_{21}$ to $U_{\alpha_1} \cap U_{\alpha_2}$
we take the $V_{21}$ the open subset apply $\phi_{\alpha_2}^{-1}$ and Reach $U_{\alpha_1} \cap U_{\alpha_2}$

and then we Apply $\left(\phi_{\alpha_1}\big|_{U_{\alpha_1} \cap U_{\alpha_2}}\right)$ that takes us to $V_{12}$

So we Can define $\boxed{g_{12} : V_{21} \to V_{12}}$

we were able to define $g_{12} : V_{21} \to V_{12}$ as homeomorphism since $g_{12}$ is a map from open subset $U$ to $V$ so $g_{12}$ is homeomorphism.

Now since we were able to tell that $g_{12}$ is homeomorphic now we want it holomorphic! (This means its injective)

→ Requiring $g_{12}$ to be holomorphic should also make it into a holomorphic isomorphism.

$$\boxed{f \circ \phi_{\alpha_1}^{-1}} \circ g_{12} = \boxed{f \circ \phi_{\alpha_2}^{-1}}$$

The above equation tell us that $f \circ \phi_{\alpha}^{-1}$ is holo iff $f \circ \phi_{\alpha_2}^{-1}$ is

Defination:
If we want that the function $g_{12}$ (transition functions) to be holo whichever $(U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset)$ we get a Compatible. Collections of charts which gives a Riemann Surface structure on "$X$"

Summarizing on Rieman Surface.

→ started with Complex analysis on Surface in $\mathbb{R}^3$

→ we define Complex Coordinates Charts.

$$\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$$ $I$ = indexed charts.
$\alpha$ = one of the arbritary chart.

→ Charts should Cover the whole open set

→ Problem Comes when multiple charst Can represent the same things so to avoid the amiguity. we define a transition function $g_{12}$ to be holomorphic (whenever $U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset$) this gives us Compatible collections of charts that gives Riemann structure on X

$$f \circ \phi_{\alpha_1}^{-1} \circ g_{12} = f \circ \phi_{\alpha_2}^{-1}$$

↳ $g_{12}$ is holomorphic.