GEMS

From

THE MATHEMATICS TEACHER

SUB-JUNIOR-III

गणितं मूर्धनि स्थितम्

AMTI

THE ASSOCIATION
OF
MATHEMATICS TEACHERS OF INDIA

GEMS
from
The Mathematics Teacher

SUB JUNIOR III

KAPREKAR CONTEST

Compiled and Edited by

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BETWEEN US

Dear Reader,

The demand for our books based on NMTC questions had been increasing steadily. Hence it is decided to bring all the problems and solutions of NMTC every five or ten years without disturbing the earlier editions. To satisfy the needs of various age groups they are being brought out in four different books – Primary, Sub-Junior, Junior, Inter.

The Present period covered by these books are from 2013 to 2017. We hope and trust the beneficiaries of our service will continue to encourage us to help aspirants of excellence in Mathematics Education.

We are grateful to Dr.S.Muralidharan for having gone through the pages and edited these books in record time.

With best wishes,

Yours sincerely,

(M. MAHADEVAN)
General Secretary, AMTI

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KAPREKAR CONTEST

NMTC at SUB JUNIOR LEVEL
VII & VIII Standards

SCREENING TEST QUESTIONS

1. If $a+b+c=0$ where a,b,c are non zero real numbers, then the value of $(a^2-bc)^2-(b^2-ca)(c^2-ab)$ is

(A) 1 (B) abc (C) $a^2+b^2+c^2$ (D) 0

Solution 1: Using $a+b+c=0$, we have

$$(a^2-bc)^2 = ((b+c)^2-bc)^2 = (b^2+c^2+bc)^2$$

Now,

$$\begin{aligned}(b^2-ca)(c^2-ab) &= (b^2+c(b+c))(c^2+b(b+c)) \\ &= (b^2+c^2+bc)^2\end{aligned}$$

Thus,

$$\begin{aligned}&(a^2-bc)^2-(b^2-ca)(c^2-ab) \\ &\qquad = (b^2+c^2+bc)^2-(b^2+c^2+bc)^2 \\ &\qquad = 0\end{aligned}$$

Answer: (D)

Solution 2:

$$\begin{aligned}&(a^2-bc)^2-(b^2-ca)(c^2-ab) \\ &\qquad = a^4+b^2c^2-2a^2bc \\ &\qquad\qquad -(b^2c^2-c^3a-ab^3+a^2bc) \\ &\qquad = a^4+ab^3+ac^3-3a^2bc \\ &\qquad = a(a^3+b^3+c^3-3abc) \\ &\qquad = a(a+b+c) \\ &\qquad\qquad \times(a^2+b^2+c^2-ab-bc-ca) \\ &\qquad = 0.\end{aligned}$$

2. In the adjoining incomplete magic square the sum of all numbers in any row or column or diagonals is a constant value. The value of x is

17				
23	5			
	6	13	20	x
10			21	
11				9

(A) 18 (B) 24 (C) 22 (D) 16

Solution: Let the missing number in the first column, third row be y. Sum of diagonal elements is $17 + 5 + 13 + 21 + 9 = 65$ and hence sum of numbers in the first column is also 65. Thus $17 + 23 + y + 10 + 11 = 65$ and hence $y = 4$. Now, sum of numbers in third row $= 4 + 6 + 13 + 20 + x = 65$ and hence we have $x = 22$.
Answer: (C)

3. Aruna, Bhanu and Rita have some amount of money. The ratio of the money of Aruna to that of Bhanu is 7:15 and the ratio of the money of Bhanu and Rita is 7:16. If Aruna has Rs. 490, the amount of money Rita has is (in Rupees)

(A) 1500 (B) 1600 (C) 2400 (D) 3600

Solution: Let Aruna have x Rs., Bhanu y Rs. and Rita z Rs. Given that $x : y = 7 : 15$ and $y : z = 7 : 16$ or $\frac{x}{y} = \frac{7}{15}$ and hence $y = \frac{15 \times x}{7} = \frac{15 \times 490}{7} = 1050$. Also $\frac{y}{z} = \frac{7}{16}$ and hence $z = \frac{16 \times y}{7} = \frac{16 \times 1050}{7} = 16 \times 150 = 2400$ Rs.

An easier way is as follows: We have $x : y = 7 : 15$ and $y : z = 7 : 16$ and hence $x : y : z = 49 : 105 : 240$. Since

$x = 490$, it follows that $z = 2400$.
Answer: (C)

4. In Figure 1, all the three semicircles have equal radii of 1 unit. The area of the shaded portion is

(A) $\pi + 2$ (B) 5 (C) $\frac{3\pi}{2} + 1$ (D) 4

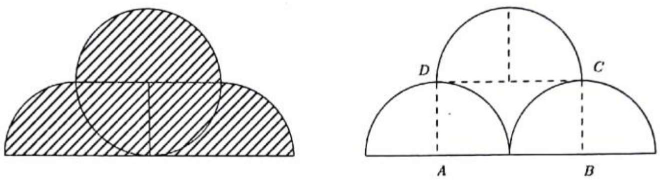

Figure 1

Solution: Total area = Area of rectangle $ABCD + 4\times$ (area of a quadrant) $= 2 + \pi$ square units.

Answer: (A)

5. The sum of three different prime numbers is 40. What is the difference between the two biggest ones among them?

(A) 8 (B) 12 (C) 20 (D) 24

Solution: Since the sum is even, all the three primes can not be odd primes. Hence the sum of two of the primes is $40 - 2 = 38$. Thus the other two primes must be 7 and 31 (of all the positive odd numbers whose sum is 38, the rest of the pairs have at least one composite number). Thus the required difference is 24.

Answer: (D)

6. Peter has written down four natural numbers. If he chooses three of his numbers at a time and adds up each triple, he obtains totals of 186, 206, 215 and 194. The largest number Peter has written is

(A) 93 (B) 103 (C) 81 (D) 73

Solution: Let the numbers be a, b, c, d. Given

$$\begin{aligned} a+b+c &= 186 \\ a+c+d &= 206 \\ b+c+d &= 215 \\ a+b+d &= 194 \end{aligned}$$

Adding the above, we get

$$3(a+b+c+d) = 801$$

and hence $a+b+c+d = 267$. The largest among the numbers is therefore $d = 267 - 186 = 81$.

Answer: (C)

7. There are four non-zero numbers x, y, z and u.
If $x = y - z$, $y = z - u$, $z = u - x$, then the value of
$\frac{x}{y} + \frac{y}{z} + \frac{z}{u} + \frac{u}{x}$
is equal to

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{1}{2}$

Solution: Adding the given equations we get $x+y+z = y - z + z - u + u - x$ and hence $z = -2x$. Now, $x = y - z$ and substituting for z, we get $x = -y$. We have, $y = z - u$ and this yields $u = -x$.

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{u} + \frac{u}{x} = -1 + \frac{1}{2} + 2 - 1 = \frac{1}{2}.$$

Answer: (B)

8. The natural numbers from 1 to 20 are listed below in such a way that the sum of each adjacent pair is a prime number.
$20, A, 16, 15, 4, B, 12, C, 10, 7, 6, D, 2, 17, 14, 9, 8, 5, 18, E.$
The number D is

(A) 1 (B) 3 (C) 11 (D) 13

Solution: The missing numbers in the list are $1, 3, 11, 13, 19$. Since $20+1, 20+13, 20+19$ are not primes, we have either $A = 3$ or $A = 11$. If $A = 11$, then $A + 16 = 27$ is not a prime. Thus $A = 3$.

B must be one of $1, 11, 13, 19$. B can not be 11 ($4+11$ is not a prime) or 13 ($13+12$ is not a prime). Thus $B = 1$ or 19. If $B = 1$, then C must be one of $11, 13, 19$. C can not be 11 ($11+10$ is not a prime) and can not be 13 ($12+13$ is not a prime). Thus $C = 19$. This leaves only $11, 13$ for D. D can not be 13 ($13+2$ is not a prime) and hence $D = 11$ and $E = 13$.

If $B = 19$, then C must be one of $1, 11, 13$. Since C can not be 11 or 13, this gives $C = 1$. As before, we can conclude that $D = 11$. Hence even though B and C are not uniquely determined, D is uniquely determined and $D = 11$.

Answer: (C)

9. How many ordered pairs of natural numbers (a, b) satisfy $a + 2b = 100$?

   (A) 33 (B) 49 (C) 50 (D) 99

   Solution: $a = 100 - 2b$. Clearly $1 \leq b \leq 49$.

   Answer: (B)

10. The value of

$$(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7}) \times (\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{6}+\sqrt{7}-\sqrt{5})$$

   is

   (A) $3\sqrt{210}$ (B) 210 (C) $4\sqrt{210}$ (D) 104

Solution:

$$\begin{aligned}(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})\\ &= (\sqrt{5}+\sqrt{6})^2-(\sqrt{7})^2 = 4+2\sqrt{30}\\ (\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{6}+\sqrt{7}-\sqrt{5})\\ &= (\sqrt{7}+(\sqrt{5}-\sqrt{6}))(\sqrt{7}-(\sqrt{5}-\sqrt{6}))\\ &= (\sqrt{7})^2-(\sqrt{5}-\sqrt{6})^2 = -4+2\sqrt{30}\end{aligned}$$

Given expression $= (4+2\sqrt{30})(-4+2\sqrt{30}) = 104.$

Answer: (D)

11. The least number which, when divided by 52 leaves a remainder 33, when divided by 78 leaves a remainder 59 and when divided by 117 leaves a remainder 98 is

(A) 553 (B) 293 (C) 468 (D) 449

Solution: Let N be the number with the stated properties. First observe that $52-33=19$, $78-59=19$ and $117-98=19$. Since N leaves a remainder 33 when divided by 52, it follows that $N+19$ is divisible by 52. Similarly, $N+19$ is also divisible by 78 and 117. Thus the least number $N+19$ is the least common multiple of 52, 78 and 117 and hence $N+19=468$ and $N=468-19=449$ will satisfy the given conditions.

Answer: (D)

12. A sum of money is divided between two persons in the ratio 3:5. If the share of one person is Rs. 2000 more than that of the other, then the sum of money is (in rupees)

(A) 6000 (B) 8000 (C) 10,000 (D) 12,000

Solution: The shares of the two persons are $3x, 5x$. Given that the difference $2x$ is 2000 Rs. Thus $x = 1000$ and the sum of money is $8x = 8000$ Rs.

Answer: (B)

13. Two numbers are respectively 20% and 50% more than a third number. What percent of the second number is the first number?

(A) 70% (B) 30% (C) 80% (D) 60%

Solution 1: Let a, b be the two numbers and let x be the third number.

Given that $a = 1.2x$ and $b = 1.5x$. Thus

$$\frac{a}{b} = \frac{1.2x}{1.5x} = \frac{4}{5}.$$

Hence a is 80% of b.

Answer: (C)

Solution 2: If the third number is assumed to be 100, then the first number is 120 and the second number is 150. Hence the ratio is $\frac{120}{150} = \frac{4}{5}$ and the first number is 80% of the second number.

14. There are some toys. One third of them are sold at a profit of 15%, one fourth of the total are sold at a profit of 20% and the rest for 24% profit. The total profit is Rs. 3200. The total price of the toys is (in rupees)

(A) 32000 (B) 64000 (C) 16000 (D) 48000

Solution: Let the total price of the toys be x Rs. One third of the toys are sold at a profit of 15%. Hence the profit obtained in this sale is $\frac{x}{3} \times 0.15$. One fourth sold at a profit of 20% fetches a profit of $\frac{x}{4} \times 0.20$. and the rest sold at 24% gives a profit of

$$\left(1 - \frac{x}{3} - \frac{x}{4}\right) \times 0.24$$

The total profit therefore is

$$\frac{x}{3} \times \frac{15}{100} + \frac{x}{4} \times \frac{20}{100} + \left(1 - \frac{x}{3} - \frac{x}{4}\right) \times \frac{24}{100} = 3200.$$

Simplifying, we get $x = 16000$ Rs.

Answer: (C)

15. The radius of a circle is increased by 4 units and the ratio of the areas of the original and the increased circle is 4:9. The radius of the original circle is

(A) 6 (B) 4 (C) 12 (D) 8

Solution: Let R be the radius of the original circle. Given that

$$\frac{\pi R^2}{\pi(4+R)^2} = \frac{4}{9}$$

$$\begin{aligned} 9R^2 - 4(4+R)^2 &= 0 \\ \{(3R-2(4+R)\}\{(3R+2(4+R)\} &= 0 \\ (R-8)(5R+8) &= 0 \end{aligned}$$

Since R can not be negative, we have $R = 8$.

This can also be seen as follows:

$$R^2 : (4+R)^2 = 4:9 \Rightarrow R:(4+R) = 2:3$$

Thus $2(4+R) = 3R$ and $R = 8$.

Answer: (D)

16. If 150% of a certain number is 300, then 30% of the number is

A. 50 B. 60 C. 70 D. 65

Solution: Let the number be x. Since 150% of x is 300, we have $\frac{150}{100} \times x = 300$ and hence $x = 200$. Now 30% of x is $\frac{30}{100} \times 200 = 60$ and the answer is B.

17. The sum of five distinct non negative integers is 90. What can be the second largest number of the five at most?

A. 82 B. 43 C. 34 D. 73

Solution: To get the second largest number, we need to ensure that the three smallest numbers are included in the list of numbers with the sum 90. The first three numbers are $0, 1, 2$ and the remaining is $90 - (0 + 1 + 2) = 87$. Since $87 = 43 + 44$, $0, 1, 2, 43, 44$, are the five distinct non negative integers with sum equal to 90. The second largest is 43.
Answer: B

18. The length of two sides of an isosceles triangle are 5 units and 16 units. The perimeter of the triangle (in the same units) is

A. 26 B. 37 C. 26 or 37 D. none of these

Solution: There are two possibilities 16, 16, 5 or 5, 5, 16. Since $5 + 5 < 16$, lengths $5, 5, 16$ is not possible. Thus the perimeter of the triangle is $16 + 16 + 5 = 37$.

Answer: B

19. $ABCD$ is a rectangle. P is the mid-point of DC and Q is a point on AB such that $AQ = \frac{1}{3}AB$. What fraction of the area of $ABCD$ is $AQPD$?

A. $\frac{1}{2}$ B. $\frac{3}{4}$ C. $\frac{2}{7}$ D. $\frac{5}{12}$

Solution:

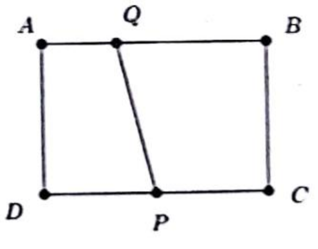

Figure 2

$$\begin{aligned}\text{Area of } AQPD &= \frac{1}{2}AD \times (DP + AQ)\\ &= \frac{1}{2}AD \times \left(\frac{1}{2}DC + \frac{1}{3}AB\right)\\ &= \frac{1}{2}AD \times \left(\frac{1}{2}DC + \frac{1}{3}DC\right)\\ &= \frac{1}{2}AD \times \frac{5}{6}DC\\ &= \frac{5}{12} \times AD \times DC\\ &= \frac{5}{12} \text{ Area of } ABCD\end{aligned}$$

Answer: D

20. The number of digits when $(999999999999)^2$ is expanded is

A. 26 B. 24 C. 32 D. 16

Solution: We have $(10^{12} - 1)^2 = 10^{24} - 2 \times 10^{12} + 1$. Now, 10^{24} has 25 digits and 2×10^{12} has 13 digits. When subtracted, we get a number of 24 digits and 1 added will not affect the number of digits. Hence number of digits is 24.

Answer: B

21. Three equal squares are kept as in the diagram. C, D being the mid points of the respective sides of the lower squares. If $AB = 100\,cm$, area of each square is (in cm^2)

A. 1200 B. 1500 C. 900 D. 1600

Solution: Let the side of each square be $2a$. Clearly $AE = 3a$ and $BE = 4a$. Thus

$$AB^2 = 9a^2 + 16a^2 = 25a^2 = 100^2$$

Hence $a^2 = 400$ and $a = 20$. The side of the square has length 40 and hence the area is 1600 square cms.

Answer: D

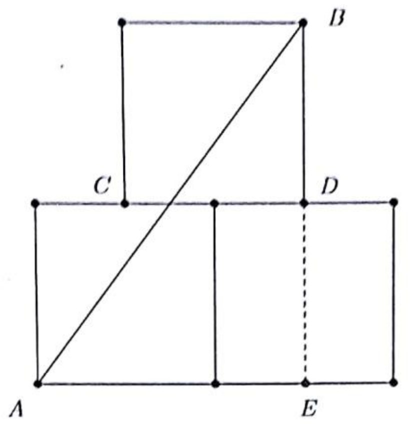

Figure 3

22. Samrud wrote 4 different natural numbers. He chose three numbers at a time and added them each time. He got the sums as 115, 153, 169, 181. The largest of the numbers Samrud first wrote is

A. 37 B. 48 C. 57 D. 91

Solution: Let the numbers be a, b, c, d. Given

$$a+b+c=115 \qquad a+c+d=153$$
$$b+c+d=169 \qquad a+b+d=181$$

Adding all the four equations, we get

$$3(a+b+c+d)=618$$

and hence $a+b+c+d=206$. The largest of the numbers can be obtained by subtracting the smallest sum obtained by Samrud. Thus the largest number is $206-115=91$.

Answer: D

23. Saket wrote a two digit number. He added 5 to the tens digit and subtracted 3 from the units digit of the number. The resulting number is twice the original number. The

original number is

A. 47 B. 74 C. 37 D. 73

Solution: Let $10x+y$ be the two digit number. Given that $2(10x+y)=10(x+5)+(y-3)$. Hence $20x+2y=10x+50+y-3$ and $10x+y=47$.

Answer: A

24. Five consecutive natural numbers cannot add up to

A. 225 B. 222 C. 220 D. 200

Solution: Let the natural umbers be $a, a+1, a+2, a+3, a+4$. The sum of these is $a+a+1+a+2+a+3+a+4=5a+10$ which is divisible by 5. In the options 222 is not divisible by 5. It is clear that the other numbers given can be obtained as sum of five consecutive integers:

$$225 = 43+44+45+46+47$$
$$220 = 42+43+44+45+46$$
$$200 = 38+39+40+41+42$$

Answer: B

25. In Figure 4, the different numbers denote the area of the corresponding rectangle in which the number is there. The value of x is

2014	1007
x	125

A. 3014 B. 1125 C. 2139 D. 250

Solution: Let the lengths of the sides of the rectangles be as shown in the figure.

From the diagram we have $ac=2014$, $bc=1007$, $bd=125$, $ad=x$. Hence

$$2014\times 125 = ac\times bd = bc\times ad = 1007\times x$$

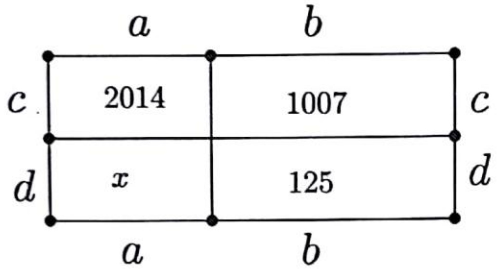

Figure 4

and

$$x = \frac{2014 \times 125}{1007} = 250$$

Answer: D

26. What is the remainder when $2^{87} + 3$ is divided by 7

A. 2 B. 3 C. 4 D. 5

Solution: When $2^3 = 8$ is divided by 7 the remainder is 1. Since $87 = 3 \times 29$ we have $2^{87} = (2^3)^{29}$ leaves a remainder 1 when divided by 7. Thus $2^{87} + 3$ leaves a remainder $1 + 3 = 4$ when divided by 7.

Answer: C

27. If Mahadevan gets 71 in his next examination, his average will be 83. If he gets 99, his average will be 87. How many exams Mahadevan has already taken?

A. 3 B. 4 C. 5 D. 6

Solution: Let the number of exams taken by Mahadevan be $n - 1$. We have $n \times 83 - 71 = n \times 87 - 99$. Thus $n = 7$ and Mahadevan has already taken 6 exams.

Answer: D

28. In Figure 5, AB, CD, EF and GH are straight lines passing through a single point. The value of $\angle x + \angle y + \angle z + \angle u$ is

A. $155°$ B. $164°$ C. $174°$ D. $148°$

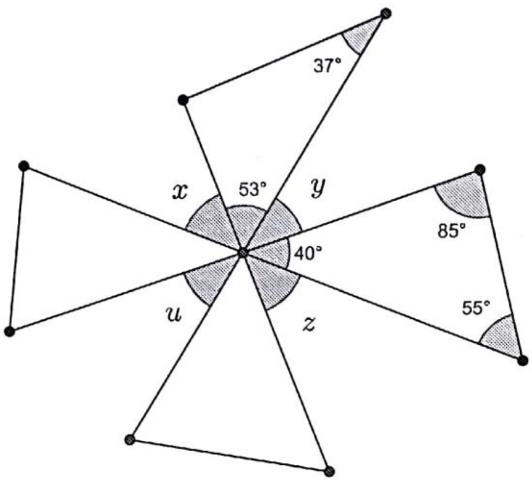

Figure 5

Solution: Clearly $x = z$ and $y = u$. Hence

$$2x + 2y + 2(40° + 53°) = 360°$$
$$2x + 2y = 360° - 186° = 174°$$

Since $x+y+z+u = 2(x+y)$, it follows that $x+y+z+u = 174°$.

Answer: C

29. In Figure 6, $AB = AD$. $\angle DCB = 23°$. The measure of $\angle DBC$ is

A. $55°$ B. $58°$ C. $56°$ D. $45°$

Solution: In $\triangle ABD$, $2x + 44° = 180°$ and hence $x = 68°$. Now, $\angle DBC = x - 23° = 68° - 23° = 45°$

Answer: D

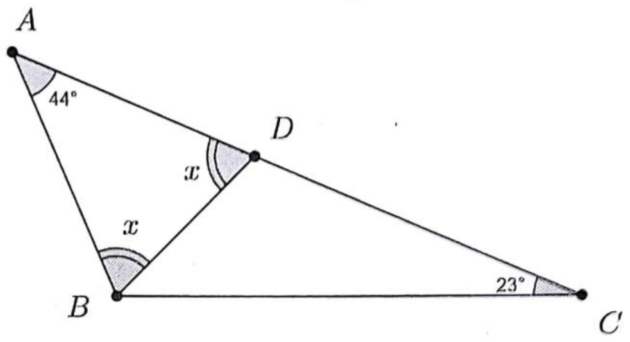

Figure 6

30. In the Figure 7, $ABCD$ is a square. BCE is an equilateral triangle. The measure of $\angle BEA$ is

A. 15° B. 20° C. 18° D. 16°

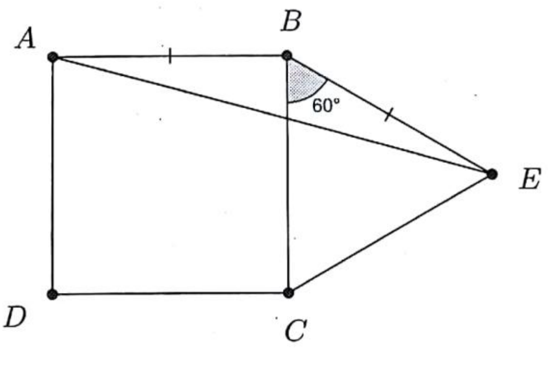

Figure 7

Solution: $\triangle ABE$ is isosceles. Thus

$$2\angle AEB = 180 - (90 + 60) = 30^\circ$$

and $\angle AEB = 15^\circ$.

Answer: A

31. The ratio of the angles of a quadrilateral are in the ratio $7:9:10:10$. Then

    A. One angle of the quadrilateral is greater than $120°$

    B. Only one angle of the quadrilateral is $90°$

    C. The sum of some two angles of the quadrilateral is $100°$

    D. There are exactly two right angles as interior angles

Solution Since the sum of the angles of the quadrilateral is $360°$, it follows that the angles are $70°, 90°, 100°, 100°$. Hence we have exactly one angle that equals $90°$ and the answer is B.

32. The sum of three different integers is 1 and their product is 36. Then

    A. All of them are positive

    B. Only one is negative

    C. Exactly two are negative

    D. All the three are negative

Solution Let a, b, c be the three integers. Since $a \times b \times c = 36$, either all are positive or exactly two of them are negative. If all are positive, then $a, b, c \geq 1$ gives $a + b + c \geq 3$ but we are given that the sum of the integers is 1. Thus exactly two of them must be negative and the answer is C.

33. The value of $\underbrace{2^{2015} + 2^{2015} + \cdots + 2^{2015}}_{256\,terms}$ divided by 2^{2015} is

A. 256 B. 2^{73} C. 2^{2015} D. 2015

Solution

$$\underbrace{2^{2015} + 2^{2015} + \cdots + 2^{2015}}_{256\,terms} = 256 \times 2^{2015}$$

Hence the value of $\underbrace{2^{2015} + 2^{2015} + \cdots + 2^{2015}}_{256\,terms}$ divided by 2^{2015} is 256 and the answer is A.

34. For a, b, define
$$a * b = \frac{ab + ba}{a + b}$$
where by ab we mean writing b after a and interpret the resulting sequence as a decimal number. For example,
$$155 * 60 = \frac{15560 + 60155}{155 + 60}$$
If $a = 2015$ and $b = 5$, $a * b$ lies between
A. 35 and 36 B. 37 and 38 C. 51 and 52 D. 53 and 54

Solution
$$2015 * 5 = \frac{20155 + 52015}{2015 + 5} = \frac{72170}{2020} = 35.72$$
Hence the answer is A.

35. The 2015^{th} letter of the sequence
$$ABCDEDCBAABCDEDCBA\ldots$$
is ——.
Solution Here the sequence of letters $ABCDEDCBA$ is repeated. Since there are 9 letters in this sequence, in the first 2007 letters we will have this sequence of 9 letters repeated 223 times. Now we have 8 more letters to fill and hence the 2015^{th} letter is same as the eighth letter in the sequence $ABCDEDCBA$ and thus is B.

36. n is a natural number. The number of possible reminders when n^2 is divided by 7 is
A. 2 B. 3 C. 4 D. 5

Solution We can write $n = i+7j$ where i, j are integers and $0 \le i < 7$. We have $n^2 = i^2 + 14ij + 49j^2$ and hence n^2 leaves the same reminder as i^2 when divided by 7. Thus it is sufficient to look at the reminders of $0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2$ when divided by 7. These are $0, 1, 4, 2, 2, 4, 1$ and hence the possible reminders are $0, 1, 2, 4$. The answer is C.

37. The ratio of two numbers is $7 : 9$. If each number is decreased by 2, the ratio becomes $3 : 4$. The sum of the two numbers is A. 23 B. 32 C. 48 D. 12

Solution Let the numbers be x, y. We have $\frac{x}{y} = \frac{7}{9}$ and $\frac{x-2}{y-2} = \frac{3}{4}$. Thus $7y = 9x$ and $3(y-2) = 4(x-2)$. Solving, we get $x = 14$, $y = 18$. Thus $x + y = 32$ and answer is B.

38. The speeds of two runners are respectively $15\,km/hr$ and $16\,km/hr$. To cover a distance of $d\,kms$ one takes 16 minutes more than the other. Then d in kms is
A. 32 B. 48 C. 64 D. 128

Solution To cover d kms, the first runner will take $\frac{d}{15}$ hours and the second runner will take $\frac{d}{16}$ hours. We are given that $\frac{d}{15} = \frac{d}{16} + \frac{16}{60}$. Note that since the speed is in km/hr we have converted 16 minutes into hours. Solving for d, we obtain

$$\frac{d}{15} - \frac{d}{16} = \frac{d}{240} = \frac{16}{60}$$

and we obtain $d = 64\,kms$. Answer is C.

39. In the sum $3 + 33 + 333 + 3333 + 33333 + \cdots$ where there are 2015 terms, the number formed by taking the last four digits in order is
A. 6365 B. 6255 C. 6465 D. 6565

Solution Since we are interested only in the last four digits, it is enough to evaluate the last four digits of the sum $3 + 33 + 333 + 3333 + 3333 + \cdots$ where every term after the third term is 3333. Thus the sum is $369 + 2012 \times 3333 = 6706365$. Thus the number formed by the last four digits of the sum is 6365. Answer is A.

40. $a\%$ of the quantity P is added to P. To the increased quantity, $b\%$ of the increased quantity is added. $c\%$ of the result is added to the result and the final quantity is Q. Then P is

A. $\dfrac{Q \times 100 \times 100 \times 100}{(a+b+c)}$

B. $\dfrac{Q}{100(a+b+c)}$

C. $\dfrac{Q \times 100 \times 100 \times 100}{(100+a)+(100+b)+(100+c)}$

D. $\dfrac{Q \times 100 \times 100 \times 100}{(100-a)+(100-b)+(100-c)}$

Solution

If P is increased by $a\%$, we obtain $P\times\left(1+\frac{a}{100}\right)$. Hence we have

$$\begin{aligned} Q &= P \times \left(1+\frac{a}{100}\right) \times \left(1+\frac{b}{100}\right) \times \left(1+\frac{c}{100}\right) \\ &= \frac{P \times (100+a) \times (100+b) \times (100+c)}{100 \times 100 \times 100} \end{aligned}$$

Hence

$$P = \frac{Q \times 100 \times 100 \times 100}{(100+a)+(100+b)+(100+c)}$$

Answer is C.

41. $ABCD$ is a square with area 64 sq cms. The center square has area 16 sq cms. The remaining are four congruent rectangles. The ratio $\frac{length}{breadth}$ of the rectangle is

A. 2 B. 3 C. 4 D. 5

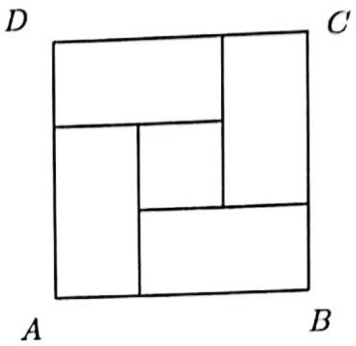

Figure 8

Solution The length of the side of the outer square is 8 cms. If a, b are the length and breadth of the rectangles, then we have $a+b=8$ and the length of the side of the inner square is $a-2b$. Thus $a-2b=4$. Solving for a, b we get $a=\frac{20}{3}$ and $b=\frac{4}{3}$. Thus $\frac{length}{breadth}=\frac{20/3}{4/3}=5$ and the answer is D.

42. If $3^a+3^b=756$, $7^a+2^c=375$ and $5^a+3=128$, the value of $a+b+c$ is

A. 12 B. 14 C. 18 D. 20

Solution Since $5^a+3=128$, it follows that $5^a=125=5^3$ and hence $a=3$. From $7^a+2^c=375$, we obtain $2^c=375-7^3=32$ and consequently, $c=5$. Finally, from $3^a+3^b=756$ we get $3^b=756-3^3=729=3^6$ and $b=6$. Thus $a+b+c=3+6+5=14$ and the answer is B.

43. There are four types of dolls called *Dingle* (D), *Pingle* (P), *Jingle* (J) and *Mingle* (M). All toys of the same category have the same weight. Toys of different category have different weights. They balance as shown in Figure 9. How many *Jingles* will balance a *Mingle*?

A. 2   B. 3   C. 4   D. 5

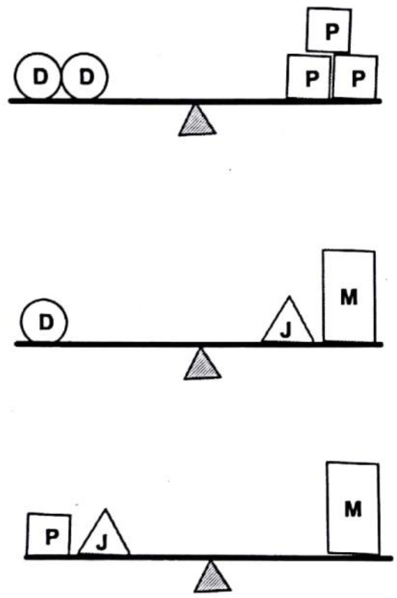

Figure 9

Solution If d, p, j, m respectively denote the weights of the toys, we have $2d = 3p$, $d = j + m$ and $p + j = m$. Hence

$$m = p + j = \frac{2d}{3} + j = \frac{2}{3}(j + m) + j = \frac{5}{3}j + \frac{2}{3}m$$

and $\frac{1}{3}m = \frac{5}{3}j$. Thus $m = 5j$ and the answer is D.

44. A student has to score 30% marks to get through an examination. If he gets 30 marks and fails by 30 marks, the maximum marks set for the examination is

A. 90   B. 200   C. 250   D. 125

Solution Since he fails by 30 marks when he gets 30 marks, the required marks for pass is 60 marks. This is 30% of the total marks. Hence the total marks for the examination is $\frac{60}{30} \times 100 = 200$ marks. Answer is B.

45. a, b, c, d are real numbers such that $1015 \leq a \leq 2015$, $3015 \leq b \leq 4015$, $5015 \leq c \leq 6015$ and $7015 \leq d \leq 8015$. The maximum value of $\dfrac{c+d}{a+b}$ is

A. $\dfrac{1403}{403}$ B. $\dfrac{1402}{403}$ C. $\dfrac{1401}{403}$ D. 2015

Solution Maximum of $c+d$ happens when $c = 6015$ and $d = 8015$. The minimum of $a+b$ happens when $a = 1015$ and $b = 3015$. Thus the maximum value of $\dfrac{c+d}{a+b}$ is

$$\frac{6015+8015}{1015+3015} = \frac{14030}{4030} = \frac{1403}{403}$$

and the answer is A.

46. A black and white photograph is 70% black and 30% white. It is enlarged three times. The percentage of white in the enlargement is

A. 90% B. $66\frac{2}{3}\%$ C. $33\frac{1}{2}\%$ D. 30%

Solution Since in an enlargement, every black dot and white dot is enlarged, the ratio should remain the same. Thus the answer is D.

47. The units digit in the product

$$(5+1)(5^2+1)(5^3+1)\cdots(5^{2015}+1)$$

is A. 9 B. 8 C. 6 D. 4

Solution Since any power of 5 has 5 as the units digit, for any n, the units digit of 5^n+1 is 6. Thus we want the units digit of

$$\underbrace{6 \times 6 \times \cdots \times 6}_{2015\,terms}$$

Again, all powers of 6 end with 6. Hence the units digit of the product given is 6. Answer is C.

48. If the product of the digits of a 4 digit number is 75, the sum of its digits is
A. 12 B. 13 C. 14 D. 15

Solution Since $75 = 1 \times 3 \times 5 \times 5$, it follows that the digits must be $1, 3, 5, 5$ in some order. Thus the sum of the digits is $1 + 3 + 5 + 5 = 14$ and the answer is C.

49. The hypotenuse c and one side a of a right angled triangle are consecutive integers. The square of the third side is
A. $c - a$ B. ca C. $c + a$ D. c/a

Solution Given $c = a + 1$. Thus the square of the other side b is given by $b^2 = c^2 - a^2 = (c - a)(c + a) = c + a$. Answer is C.

50. The fraction $\frac{2121212121210}{1121212121211}$ when reduced to its simplest form is

A. $\frac{73}{70}$ B. $\frac{37}{7}$ C. $\frac{70}{37}$ D. $\frac{70}{13}$

Solution

$$\begin{aligned}\frac{2121212121210}{1121212121211} &= \frac{3 \times 707070707070}{3 \times 373737373737} \\ &= \frac{70 \times 10101010101}{37 \times 10101010101} \\ &= \frac{70}{37}\end{aligned}$$

Answer is C.

51. On the square $ABCD$, point E lies on the side AD and F lies on BC so that $BE = EF = FD = 30\,\text{cm}$. The area of the square (in square cms) is

A. 300 B. 900 C. 810 D. None of these

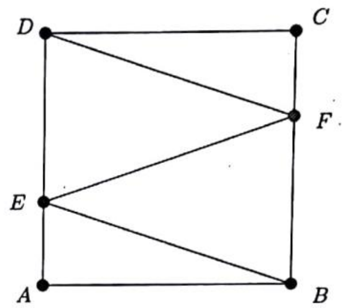

Figure 10

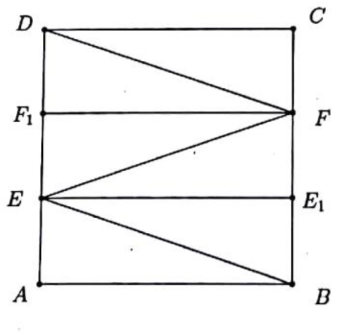

Figure 11

Solution Draw the perpendiculars from E, F to the opposite sides. If these meet the sides at E_1, F_1, clearly, $AE = EF_1 = F_1D$ and $BE_1 = E_1F = FC$. Thus if the side of the square is a, then $AE = \frac{a}{3}$ and from the right angled triangle AEB, we have $a^2 + \frac{a^2}{9} = 900$. Hence the area of the square, $a^2 = 810$.

52. If $2016 = 2^x 3^y 5^z 7^u$ where x, y, z, u are non negative integers, the value of $x + y + 2016z + 3u$ is
A. 10 B. 11 C. 12 D. 13

Solution We have $2016 = 2^5 \times 3^2 \times 7$ and hence $x = 5, y = 2, z = 0, u = 1$. Thus $x+y+2016z+3u = 10$

53. A is the area of triangle of sides 25, 25 and 30. B is the area of a triangle of sides 25, 25 and 40. Then
A. $A = B$ B. $A < B$ C. $A = 3B$ D. $A = 2B$

Solution The triangles are isosceles. The altitude to the base of the first triangle is $\sqrt{25^2 - 15^2} = 20$ and hence the area $A = \frac{1}{2} \times 30 \times 20 = 300$. The altitude to the base of the second triangle is $\sqrt{25^2 - 20^2} = 15$ and its area is $B = \frac{1}{2} \times 40 \times 15 = 300$. Thus $A = B$

54. A number on being divided by 5 leaves a remainder 2 and when divided by 7, leaves a remainder 4. The remainder when the same number is divided by 5×7 is
A. 20 B. 23 C. 32 D. None of these

Solution If the number is N, then $N = 2+5k = 4+7\ell$. Thus $7N = 14 + 35k$ and $5N = 20 + 35\ell$. This gives

$$\begin{aligned} N &= 21N - 20N \\ &= 42 + 35(3k) - 80 - 35(4\ell) \\ &= -38 + 35(3k - 4\ell) \\ &= 32 + 35(3k - 4\ell - 2) \end{aligned}$$

Hence the remainder is 32.

55. The sum of three numbers is 204. If the ratio of the first to second is $2 : 3$ and that of the second to third is $5 : 3$. Then the second number is
A. 60 B. 65 C. 58 D. 90

Solution Let a, b, c be the numbers. We have $a : b = 2 : 3$ and $b : c = 5 : 3$. Hence $a : b : c = 10 : 15 : 9$. The numbers can be written as $10k, 15k, 9k$ and thus $10k+15k+9k = 34k = 204$ and $k = 6$. Thus the second number is $15k = 90$.

56. Anirud goes to the vegetable market to buy onions. He carries money to buy 3 kg onions. But on reaching the market he finds that the price of onions has increased by 20%. If he buys onions for the money he has, the quantity he could buy is

A. 2.5 kg B. 2.2 kg C. 2 kg D. 2.8 kg

Solution If the original price is C per Kg, then Anirud has $3C$ Rs. The increased price is $\frac{120}{100}C = \frac{6}{5}C = 1.2C$. Thus he can buy $\frac{3C}{1.2C} = 2.5$ kg.

57. Two runners cover the same distance at the rate of 15 km per hour and 16 km per hour respectively. The first runner took 16 minutes longer than the second to cover the distance. Then the distance (in km) is

A. 58 B. 64 C. 66 D. 73

Solution If the distance covered is d, then we have

$$\frac{d}{15} = \frac{d}{16} + \frac{16}{60}$$

and $d = 64$

58. The total surface area of a cube is 600 cm^2. The length of its diagonal in cms is

A. 10 B. 30 C. $10\sqrt{2}$ D. $10\sqrt{3}$

Solution If the side of the cube is a, the surface area is $6a^2$. Hence we have $6a^2 = 600$ and thus $a = 10$. The length of the diagonal is $\sqrt{a^2 + a^2 + a^2} = 10\sqrt{3}$

59. In Figure 12, the distance between any two horizontal or vertical dots is one unit. The area of the triangle shown is

A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $\frac{1}{4}$ D. $\frac{1}{6}$

Solution The area of the triangle ABC equals area of triangle AEC − area of triangle ABF − area of

Figure 12

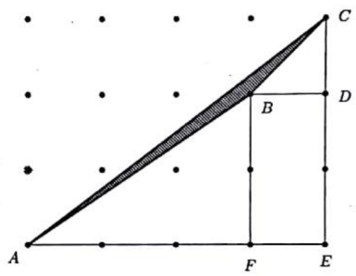

Figure 13

triangle BCD – area of rectangle $BDEF$ and hence equal to

$$\frac{1}{2} \times 4 \times 3 - \frac{1}{2} \times 3 \times 2 - -\frac{1}{2} \times 1 \times 1 - 2 \times 1 = \frac{1}{2}$$

60. The number of natural numbers less than 400 that are not divisible by 17 or 23 is
A. 290 B. 320 C. 360 D. 370

Solution The number of natural numbers less than 400 and divisible by 17 is $\left[\frac{399}{17}\right] = 23$ and the number of natural numbers less than 400 and divisible by 23 is $\left[\frac{399}{23}\right] = 17$ (Here for any real number x, $[x]$ denotes the integral part of x). Since $17 \times 23 = 391$ it follows that the number of natural numbers that are divisible by 17 or 23 is $23 + 17 - 1 = 39$ and thus the number of numbers that are neither divisible by 17 nor by 23 is $399 - 39 = 360$.

61. If a is a % of b and b is b% of c, where a is a positive real number, then the value of c is

A. 120 B. 200 C. 150 D. 100

Solution We have $a = \dfrac{ab}{100}$ and $b = \dfrac{bc}{100}$ and hence $c = 100$.

62. The sum of the reciprocals of all the positive integers that divide 24 is

A. $\dfrac{7}{2}$ B. $\dfrac{1}{2}$ C. $\dfrac{5}{2}$ D. $\dfrac{3}{2}$

Solution The divisors of 24 are $1, 2, 3, 4, 6, 8, 12, 24$. Hence the sum of their reciprocals is

$$\frac{1}{1}+\frac{1}{24}+\frac{1}{2}+\frac{1}{12}+\frac{1}{3}+\frac{1}{8}+\frac{1}{4}+\frac{1}{6} = \frac{25+14+11+10}{24} = \frac{5}{2}$$

63. N is a positive integer and p, q are primes. If $N = pq$ and $\dfrac{1}{N} + \dfrac{1}{p} = \dfrac{1}{q}$, the value of N is

A. 6 B. 7 C. 8 D. 9

Solution We have $\dfrac{1}{pq} = \dfrac{1}{q} - \dfrac{1}{p} = \dfrac{p-q}{pq}$. Thus $p - q = 1$. Since p, q are primes the only possibility is $p = 3, q = 2$. Thus $N = 6$.

64. The number of two digit numbers that leave a remainder 1 when divided by 4 is

A. 20 B. 21 C. 22 D. 23

Solution The smallest two digit number that leaves a remainder 1 when divided by 4 is 13 and the largest such two digit number is 97. Hence the list of such numbers is $13, 17, 21, \ldots, 97$. The number of such numbers is $1 + \frac{97-13}{4} = 22$.

65. If $(a+b)^2 + (b+c)^2 + (c+d)^2 = 4(ab + bc + cd)$, then

A. $a + b$ or $b + c$ or $c + d$ must be zero

B. Two of a, b, c, d are zero and other two non zero

C. $a = b = c = d$

D. None of these

Solution Bringing all the terms to the left hand side, we obtain

$$(a-b)^2 + (b-c)^2 + (c-d)^2 = 0$$

and hence $a = b = c = d$.

66. The fraction $\dfrac{4}{37}$ is written in the decimal form $0.a_1a_2a_3\ldots$. The value of a_{2017} is

A. 8 B. 0 C. 1 D. 5

Solution Computing the first few decimal places, we have $\dfrac{4}{37} = 0.108108108\ldots$. Thus 108 repeats with a period of 3, and $a_{2017} = 1$ (2016 is a multiple of 3).

67. The number of integers x satisfying the equation $(x^2 - 3x + 1)^{x+1} = 1$ is

A. 2 B. 3 C. 4 D. 5

Solution Consider $a^b = 1$. There are three cases: $b = 0, a \neq 0$, $a = 1$, $a = -1$ and b is an even integer.
When $x + 1 = 0$, we have $x^2 - 3x + 1 = 5 \neq 0$ and hence $x = -1$ is a solution.
when $x^2 - 3x + 1 = 1$, we have two solutions $x = 0, 3$.
When $x^2 - 3x + 1 = -1$, we have $(x-2)(x-1) = 0$ and thus $x = 1, 2$. When $x = 2$, $x + 1$ is odd and hence we do not have a solution. When $x = 1$, $x + 1$ is even and this is a solution. Thus we have four solutions $-1, 0, 1, 3$.
Answer C.

68. The number of two digit numbers ab such that the number $ab - ba$ is a prime number is

A. 0 B. 1 C. 2 D. 3

Solution $ab-ba$ has value $(10a+b)-(10b+a)=9(a-b)$ and hence is not a prime for any a,b. Answer A.

69. If $A=\dfrac{5425}{1444}-\dfrac{2987}{3045}-\dfrac{493}{4284}$, then

A. $1<A<2$ B. $2<A<3$ C. $3<A<4$
D. $A<1$

Solution Computing the fractions to three decimal places, we get

$$\frac{5425}{1444}-\frac{2987}{3045}-\frac{493}{4284}\approx 3.757-0.981-0.115=2.661$$

Answer B.

70. What is the 2017^{th} letter in

$$ABRACADABRAABRACADABRA\ldots$$

where the word $ABRACADABRA$ is repeatedly written?
A. A B. B C. C D. R

Solution There are 22 letters in the word. The multiple of 22 nearest to 2017 is 2002. Hence the 2017th letter would be same as the 15th letter in the given word. Thus 2017th letter is A. Answer A.

71. How many of the following statements are true?

(a) A 10% increase followed by another 5% increase is equivalent to a 15% increase

(b) If the radius of a circle is doubled then the ratio of the area of the circle to the circumference is doubled

(c) If a positive fraction is subtracted from 1 and the resulting fraction is again subtracted from 1 we get the original fraction.

A. 0 B. 1 C. 2 D. 3

Solution A 10% increase followed by another 5% increase gives $100 \times 1.1 \times 1.05 = 115.5$ and hence 15.5% increase. Thus (a) is false.

The ratio of the area of a circle of radius R to its circumference is $\dfrac{\pi R^2}{2\pi R} = \dfrac{R}{2}$. Thus when the radius is doubled, the ratio is also doubled. Thus (b) is true.

Subtracting a positive fraction f from 1 we get $1-f$ and subtracting this from 1 again, we get $1-(1-f) = f$, the original fraction. Thus (c) is true. Answer C.

72. In Figure 14, the breadth of the rectangle is 10 units. Two semicircles are drawn on the breadth as diameter. The area of the shaded region is 100 sq units. The shortest distance between the semicircles is

A. $\dfrac{5\pi}{2}$ B. 5π C. $\dfrac{5\pi}{3}$ D. $\dfrac{3\pi}{4}$

Figure 14

Solution Let the length of the rectangle be L. Area of the shaded region is $10L - 25\pi$. Hence $10L - 25\pi = 100$ giving, $L = 10 + \frac{5}{2}\pi$. The shortest distance between thee circles is $L - 10 = \frac{5}{2}\pi$. Answer A.

73. When you arrange the following in descending order

A. 15% of 30

B. 8% of 15

C. 20% of 20

D. 26% of 10

E. 9% of 25

the middle one is

A. 15% of 30

B. 8% of 15

C. 20% of 20

D. 26% of 10

Solution 15% of 30 is $0.15 \times 30 = 4.5$. Similarly, the others are $1.2, 4.0, 2.6, 2.25$. Arranging them in the descending order, we have $4.5, 4.0, 2.6, 2.25, 1.2$ and the middle one is 2.6. Answer D.

74. After simplifying the fraction

$$\left\{\frac{a+\frac{b-a}{1+ab}}{1-\frac{a(b-a)}{1+ab}}\right\}\left\{\frac{\frac{x+y}{1-xy}-y}{1+\frac{y(x+y)}{1-xy}}\right\}$$

we get a term independent of

A. a, y B. b, x C. a, b D. x, y

Solution

$$\begin{aligned}\frac{a+\frac{b-a}{1+ab}}{1-\frac{a(b-a)}{1+ab}} &= \frac{a+a^2b+b-a}{1+ab-ab+a^2}\\ &= \frac{b(1+a^2)}{1+a^2}\\ &= b\end{aligned}$$

Similarly,

$$\frac{\frac{x+y}{1-xy}-y}{1+\frac{y(x+y)}{1-xy}} = x$$

Hence the given expression evaluates to bx and is independent of a, y. Answer A.

75. A boy aims at a target shown in Figure 15. When he hits the center circle he gets 7 points, first annular region 5 points and second annular region 3 points. He shoots six times. Which one of the following is a possible score?
A. 16 B. 26 C. 19 D. 41

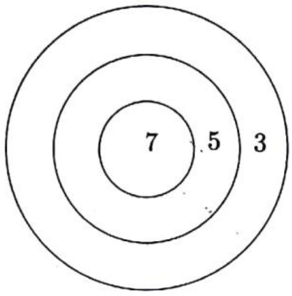

Figure 15

Solution Suppose that he shoots at the second annular region a times, first annular region b times and the center circle a times. Then $a+b+c=6$ and $3a+5b+7c$ will be his total score. If $c=5$, then either $a=0, b=1$ or $a=1, b=0$. In the first case, the total score is $5 \times 1 + 7 \times 5 = 40$ and in the second case, the score is $3 \times 1 + 7 \times 5 = 38$. None of these scores is among the given scores. Let $c=4$. The possible scores are $38, 26$ or 36. Of these, 26 is a given score and hence answer is B.

76. If 7 Rasagullas are distributed to each boy of a group, 10 rasagullas would be left. If 8 are given to each boy then 5 rasagullas would be left. So the person who distributes the rasagullas brought 15 more rasagullas and distributed the same number (x) rasagullas to each. There is no rasagulla left. Then x is

A. 10 B. 11 C. 12 D. 14

Solution Suppose that the number of rasigullas be R and the number of boys B. We are given $7B = R - 10$

and $8B = R - 5$. Hence $B = 5$ and $R = 7B + 10 = 45$. If 15 rasigullas are brought, the total becomes 60 and thus $x = 60/5 = 12$. Answer C.

77. In Figure 16, all squares are of the same size. The total area of the figure is 288 square cms. The perimeter of the figure is (in cm)
A. 86 B. 96 C. 106 D. 92

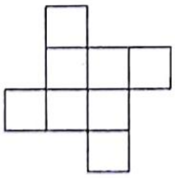

Figure 16

Solution There are eight squares in the diagram. Since the total area is 288 square cms, each individual square has area $288/8 = 36$ cms. Thus the side of the square is $x = 6$ cm. The perimeter is $16x = 96$ cms. Answer B.

78. When Newton was a primary school student he had to multiply a number by 5. But by mistake he divided the number by 5. The percentage error he committed is
A. 95% B. 96% C. 50% D. 75%

Solution If the number to be multiplied is x, then the error is $5x - x/5$ and the percentage error is

$$100 \times \frac{5x - x/5}{5x} = 96$$

Answer B.

79. ABC is an isosceles triangle with sides $AB = AC = 3x - 4 = \frac{3}{4}x + 32$. The area of the equilateral triangle with side length x is
A. $32\sqrt{3}$ B. $36\sqrt{3}$ C. $54\sqrt{3}$ D. $64\sqrt{3}$

Solution Solving for x from $3x - 4 = \frac{3}{4}x + 32$, we get $x = 16$ and hence the area of equilateral triangle with side length x is $\frac{\sqrt{3}}{4}x^2 = 64\sqrt{3}$. Answer D.

80. Two distinct numbers a and b are selected from $1, 2, 3, \ldots, 60$. The maximum value of $\frac{a \times b}{a - b}$ is

A. 6750 B. 5270 C. 4850 D. 3540

Solution We need the numerator to be as large as possible and denominator as small as possible. Clearly, $a = 60, b = 59$ yields the desired maximum as 3540. Answer D.

81. A natural number less than 100 has remainder 2 when divided by 3, remainder 3 when divided by 4 and remainder 4 when divided by 5. The remainder when the number is divided by 7 is ———.

Solution: The number is 1 less than a multiple of 3, 1 less than a multiple of 4 and 1 less than a multiple of 5. Hence the number is 1 less than any multiple of 3, 4 and 5. The least such number is $3 \times 4 \times 5 - 1 = 59$ and the next such number 119 is more than 100. Thus the required number is 59. When this is divided by 7, the remainder is 3.
Answer: 3.

82. The units digit of $(2013)^{2013}$ is ———.

Solution: Observe first that the units digit of 2013^{2013} is the same as the units digit of 3^{2013}.
The various powers of 3 are $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $\ldots$ and hence their units digits are $3, 9, 7, 1, 3, 9, \ldots$.. From this repeating pattern of units digits, we see that if n is of the form $1 + 4k$ for some integer k, then the units digit of 3^n is 3. Since

$2013 = 1 + 503 \times 4$, it follows that the units digit of 3^{2013} is 3.

Answer: 3.

83. In the figure $ABCD$ is a rectangle with sides 24 units, 32 units. SKL, PLN, QNM, RNL are congruent right angled triangles. The area of each triangle is ———.

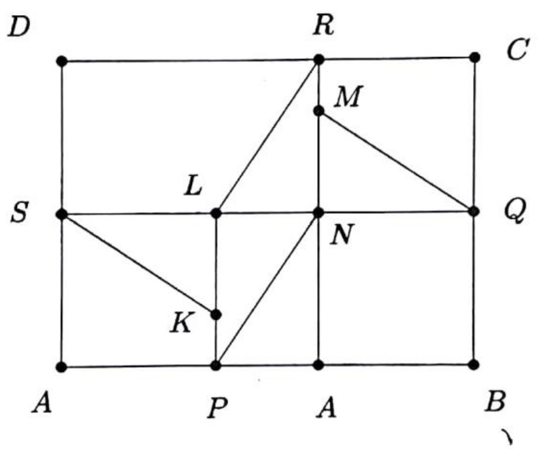

Figure 17

Solution: S, L, N, Q are collinear. Let $LN = LK = NM = x$. $SL = QN = \dfrac{32 - x}{2} = 16 - \dfrac{x}{2} = PL = RN$. But
$PL + RN = AD = 24$ and hence $x = 8$. The base of each triangle is 8 and height is 12. Thus the area is 48 square units.
Answer: 48 sq. units.

84. The number of natural numbers n for which $\dfrac{(n+2)(n+2)}{(n-3)}$ is a natural number is ———.

Solution:

$$\begin{aligned}\frac{(n+2)(n+2)}{n-3} &= \frac{n^2+4n+4}{n-3}\\ &= \frac{n(n-3)+7n+4}{n-3}\\ &= n+\frac{7n+4}{n-3}\\ &= n+\frac{7(n-3)+25}{n-3}\\ &= n+7+\frac{25}{n-3}\end{aligned}$$

$n-3$ must divide 25. The possibilities are $n-3 = \pm 1,\ \pm 5,\ \pm 25$. Thus $n = 2,4,8,28$. When $n = 2$, the given fraction is -4 and hence not a natural number. Thus $n = 4,8,28$.

Answer: 3.

85. The product of two natural numbers a and b divides 48. a,b are not relatively prime to each other. The number of pairs (a,b) where $1 < a+b < 48$, $(a \neq b)$ is ———.

Solution: Since ab divides 48, ab must be one of $2,3,4,6,8,12,16,24,48$.

Since (a,b) are not relatively prime, ab can not be $2,3$ or 6.

$$(a,b) = \begin{cases}(2,2)\\ (2,4),(4,2)\\ (2,6),(6,2)\\ (4,4),(2,8),(8,2)\\ (2,12),(12,2),(6,4),(4,6)\\ (2,24),(4,12),(6,8),(24,2),(12,4),(8,6).\end{cases}$$

There are 18 pairs. Since $a \neq b$ reject $(2,2),(4,4)$. Hence there are 16 pairs.

Answer: 16.

86. The value of the expression

$$\left(\frac{a-a^{-2}}{a^{\frac{1}{2}}-a^{-\frac{1}{2}}}-\frac{1-a^{-2}}{a^{\frac{1}{2}}+a^{-\frac{1}{2}}}-a^{\frac{1}{2}}\right)$$

is $\frac{2}{K^3}$ where $a=(2013)^2$. The value of K is ———.

Solution: Given expression

$$\begin{aligned}
&= \frac{(a^3-1)\sqrt{a}}{a^2(a-1)}-\frac{\sqrt{a}(a^2-1)}{(a+1)a^2}-\sqrt{a}\\
&= \frac{(a-1)(a^2+a+1)\sqrt{a}}{a^2(a-1)}-\frac{\sqrt{a}(a+1)(a-1)}{(a+1)a^2}-\sqrt{a}\\
&= \frac{(a^2+a+1)\sqrt{a}}{a^2}-\frac{\sqrt{a}(a-1)}{a^2}-\sqrt{a}\\
&= \frac{\sqrt{a}}{a^2}\left[a^2+a+1-a+1\right]-\sqrt{a}\\
&= \frac{\sqrt{a}}{a^2}\left[a^2+2\right]-\sqrt{a}\\
&= \frac{\sqrt{a}a^2+2\sqrt{a}-\sqrt{a}a^2}{a^2}\\
&= \frac{2\sqrt{a}}{a^2}=\frac{2}{(\sqrt{a})^3}=\frac{2}{(2013)^3} \text{ since } a=(2013)^2.
\end{aligned}$$

Hence the value of K is 2013.

Answer: 2013.

87. AB and AC are two straight line segments enclosing an angle $70°$. Squares $ABDE$ and $ACFG$ are drawn outside the angle BAC. The diagonal FA is produced to meet the diagonal EB in H. then $\angle EAH =$ ———.

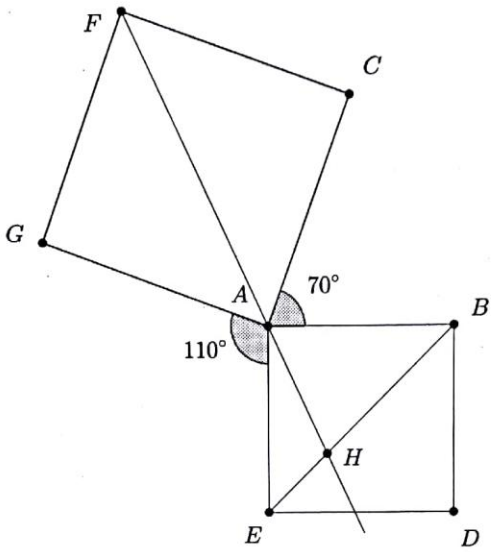

Figure 18

Solution: Angle at $A = 360° = \angle GAE + 90° + 70° + 90°$ and hence $\angle GAE = 110°$.
FAH is a straight line and $\angle FAG = 45°$. Hence

$$180° = \angle FAG + \angle GAE + \angle EAH = 45° + 110 + \angle EAH$$

It follows that $\angle EAH = 25°$.

Answer: $25°$.

88. In the pentagon $ABCDE$, $\angle D = 2\angle B$ and the other angles are each equal to half the sum of the angles $\angle B$ and $\angle D$. The largest interior angle of the pentagon is ________.

Solution: Given that $\angle D = 2\angle B$, and $\angle A + \angle C + \angle E = \frac{3}{2}(\angle B + \angle D)$.

Total of the interior angles equals

$$\frac{5}{2}(\angle B + \angle D) = \frac{3}{2}(3\angle B) = \frac{15}{2}\angle B$$

Since the sum of interior angles in a pentagon is $540°$, it follows that $\frac{15}{2}\angle B = 540°$ and $\angle B = 72°$. So the angles are $108°, 72°, 108°, 144°, 108°$. So, the largest interior angle is $144°$.

Answer: $144°$.

89. The ratio of a two digit number and the sum of its digits is 4:1. If the digit in the units place is 3 more than the digit in the tens place, then the number is ———.

Solution: Let $10a + b$ be the two digit number. Given that $\dfrac{10a+b}{a+b} = \dfrac{4}{1}$ and hence $10a+b = 4a+4b$ or $2a = b$. Also given than $b = a+3$ and hence $a = 3$ and $b = 6$. The number is 36.

Answer: 36.

90. There is a famine in a place. But there is sufficient food for 400 people for 31 days. After 28 days 280 of them left the place. Assuming that each person consumes the same amount of food per day, the number of days for which the rest of the food would last for the remaining people is ———.

Solution: Let F be the quantity of food consumed by one person per day. Then there was $400 \times 31F = 12400F$ units of food to begin with. Out of this, in 28 days, the food consumed by 400 people is $400 \times 28F = 11200F$. The remaining food is hence $12400F - 11200F = 1200F$. The remaining people is $400 - 280 = 120$. Hence the remaining food will be sufficient for $\dfrac{1200F}{120F} = 10$ days.

Answer: 10.

91. A triangle whose sides are integers has a perimeter 8. The area of the triangle is ———.

Solution: Let a, b, c be the sides of the triangle. We have $a + b + c = 8$. Since a, b, c are positive integers and sum of two sides of a triangle must be more than the third side, the only possibility is $3, 3, 2$. Thus the triangle is isosceles with base 2 and height $\sqrt{3^2 - 1} = 2\sqrt{2}$. Hence the area equals $\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2}$ square units.

Answer: $2\sqrt{2}$.

92. If $(10^{2013} + 25)^2 - (10^{2013} - 25)^2 = (\sqrt{10})^n$

then the value of n is ———.

Solution: Since $(a + b)^2 - (a - b)^2 = 4ab$,

$$(10^{2013} + 25)^2 - (10^{2013} - 25)^2 = 4 \times 10^{2013} \times 25 = (10)^{2015}$$

Hence $\frac{n}{2} = 2015$ and $n = 4030$.

Answer: 4030.

93. ABC and ADC are isosceles triangles with $AB = AC = AD$. $\angle BAC = 40°$, $\angle CAD = 70°$. The value of $\angle BCD + \angle BDC =$ ———.

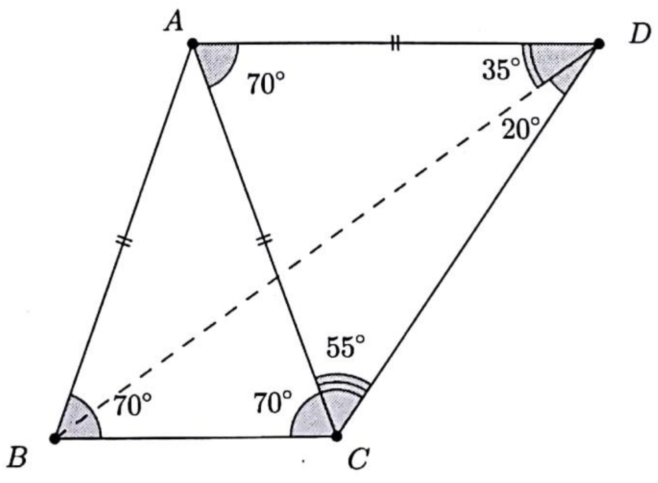

Figure 19

Solution: Since $\angle BAC = 40°$ and $AB = AC$, it follows that $\angle BCA = 70°$. Similarly from the isosceles triangle

ACD, we get $\angle ACD = 55°$. Now from the isosceles triangle ABD, since $\angle BAD = 40° + 70°$, $\angle ABD = 35° = \angle ADB$. Since AD is parallel to BC (alternate angles $\angle DAC = 70° = \angle ACB$), $\angle CBD = \angle ADB = 35°$. Hence

$$\angle BCD + \angle BDC = 180° - \angle DBC = 180° - 35° = 145°.$$

Answer: 145°.

94. There are three persons Samrud, Saket and Vishwa. Samrud is twice the age of Saket and Saket is twice the age of Vishwa. Their total ages will be trebled in 28 years. The present age of Samrud is ————.

Solution: Let the present age of Vishwa be x. Saket's age $= 2x$ and Samrud's age $= 4x$. In 28 years, their ages would be $x + 28, 2x + 28, 4x + 28$. Hence we have

$$(x + 28) + (2x + 28) + (4x + 28) = 3(x + 2x + 4x)$$

Hence $14x = 84$ and $x = 6$. Thus the present age of Samrud is $4x = 24$ years.

Answer: 24 years.

95. The number of three digit numbers of the form $ab5$ which are divisible by 9 is ————.

Solution: Since a number is divisible by 9 only when the sum of its digits is divisible by 9, $a + b + 5$ must be a multiple of 9. Thus $a + b + 5 = 9$ or 18. Consequently, $a + b = 4$ or 13.

When $a + b = 4$ (a, b) is one of

$$(1,3),\ (3,1),\ (2,2),\ (4,0)$$

and when $a + b = 13$ (a, b) is one of

$$(4,9),\ (5,8),\ (6,7),\ (7,6),\ (8,5),\ (9,4).$$

Thus there are 10 such numbers. The numbers are 135, 315, 225, 405, 495, 585, 675, 765, 855, 945.

Answer: 10.

96. The smallest multiple of 9 with no odd digits is ______.

Solution: The multiples of 9 less than 100 are 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99. All of them contain an odd digit. From 100 to 200 the multiples of 9 are 108, 117, 126, 135, 144, 153, 162, 171, 188, 197 and again all of them contain an odd digit. From 200 to 300 the multiples are 216, 225, 234, 243, 252, 261, 270, 279, 288 and the number 288 has no odd digit. Thus 288 is the smallest multiple of 9 with no odd digit.

Answer: 288

97. The sum of the squares of the lengths of the three sides of a right triangle is 800. The length of the hypotenuse is ______.

Solution: Let c be the hypotenuse and a, b other two sides. We have $a^2+b^2 = c^2$ and $a^2+b^2+c^2 = 2c^2 = 800$. Thus $c^2 = 400$ and $c = 20$.

Answer: 20

98. In Figure 20, the biggest square of area $125\,cm^2$ is divided into 5 equal parts of same area. (4 squares and the L shaped figure). The shorter side of the L shape is $k(\sqrt{5} - 2)$, where k is a natural number. Then $k =$______.

Solution:
Total area $= 125\,cm^2$ and area of each part is $25\,cm^2$. The length of the side of the square $= \sqrt{125} = 5\sqrt{5}\,cm$. Length of the shorter side of the L Shape is $= 5\sqrt{5} - 2 \times 5 = 5(\sqrt{5} - 2)$ and thus $k = 5$.

Answer: 5

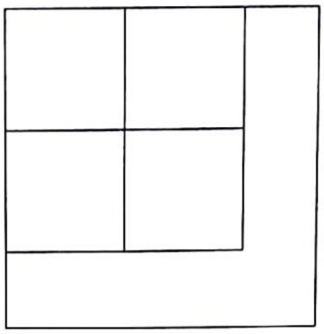

Figure 20

99. Five years ago, the average age of A, B, C and D is 45 years. E joins them now. The average age of all the five now is 49 years. The present age of E is ________.

Solution: Five years ago $A+B+C+D = 180$. Hence the sum of their ages now is $180+20 = 200$ years. Since the average of the ages of A, B, C, D, E now is 49 years, we have $A+B+C+D+E = 245$. Thus age of E now is $245-200 = 45$ years.

100. In the sequence $1, 1, 1, 2, 1, 3, 1, 4, 1, 5 \cdots$ The 2014^{th} term is ________.

Solution: The sequence $1, 1, 1, 2, 1, 3, \cdots$ can be grouped as as $(1, 1)$, $(1, 2)$, $(1, 3), \ldots$. Hence the 2014^{th} term is 1007.

Answer: 1007

101. In Figure 21, $ABCD$ is a rectangle. E is the midpoint of AD. F is the midpoint of EC. The area of the rectangle $ABCD$ is $120 cm^2$.

The area of the triangle BDF is ________cm^2.

Solution: Let the sides of the rectangle have lengths $2a$ and b. Area of the rectangle is $120\, cm^2$ and hence

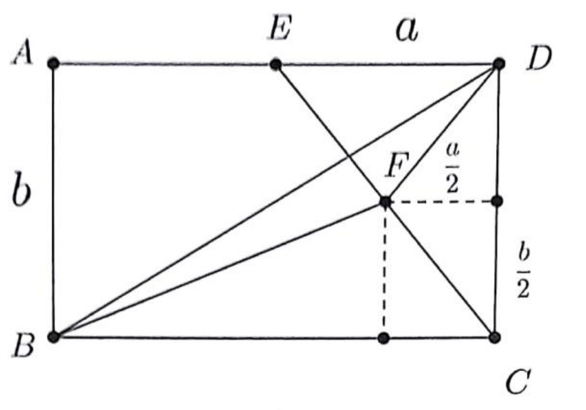

Figure 21

$ab = 60$.

$$\begin{aligned}\triangle DFB &= \triangle BDC - \triangle DFC - \triangle FBC \\ &= 60 - \frac{1}{2} \times b \times \frac{a}{2} - \frac{1}{2}(2a)\left(\frac{b}{2}\right) \\ &= 60 - \frac{ab}{4} - \frac{ab}{2} \\ &= 15\end{aligned}$$

The area of $\triangle DFB$ is 15 cm^2

102. Flag poles are installed along a road side on every 8 meters. Markings are made every $12\,m$ as shown. The length of the road is $240\,m$. The number of markings made beside the flag poles are ________.

Solution: Starting with 0, we have flag posts at $0, 8, 16, \ldots, 240$ meters on the road. Also, starting with 0, we have markings at $12, 24, 36, \ldots, 240$ meters on the road. Thus the markings at the base of the flag posts occur at $0, 24, 48, \ldots, 240$ meters and there are $\frac{240}{24} + 1 = 11$ such markings.

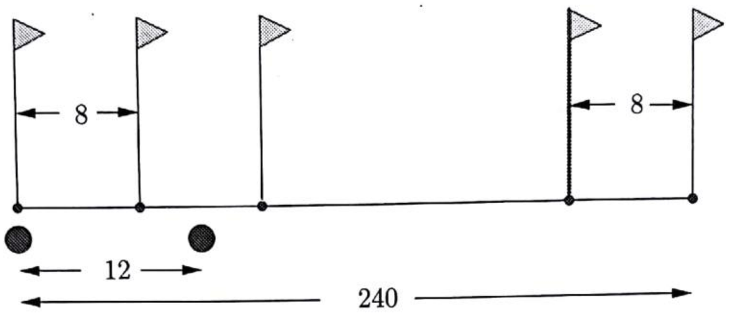

Figure 22

103. The number of different natural numbers n for which $n^2 - 440$ is a perfect square is ______.

Solution: If $n^2 - 440 = m^2$, then $n^2 - m^2 = (n-m)(n+m) = 440$. Since $n - m + n + m = 2n$ is even, either $n - m$ and $n + m$ are both odd or both even. But since 440 is even, $n - m$ and $n + m$ can not be both odd. Thus $n - m$ and $n + m$ are both even. The possibilities are

$n-m$	$n+m$	n	n^2-440
2	220	111	$11881 = 109^2$
4	110	57	$2809 = 53^2$
10	44	27	$289 = 17^2$
20	22	21	$1 = 1^2$

Thus there are four such numbers.

104. Aruna, Bhavani and Christina each wear a saree of different color (blue, yellow or green). It is known that, if Aruna wears blue then Bhavani wears green, if Aruna wears yellow Christina wears green and if Bhavani does not wear yellow, Christina wears blue. The saree Aruna is wearing has the color ______.

Solution: Let A, B, C represent Aruna, Bhavani and Christina. Saree colors blue(b), green (g) and yellow (y).

A	B	C
b	g	y
y	b	g
y	g	b

Answer: Yellow

105. Arish found the value of 3^{19} to be $11a2261467$. He found all the digits correctly except the digit denoted by a. The value of a is ________.

Solution: If a number is divisible by 3^{19}, it has to be divided by 9. Sum of digits is a multiple of 9. Hence $1+1+a+2+2+6+1+4+6+7=30+a$ must be a multiple of 9. Since a can not be more than 9, the only possible multiple of 9 for the sum of the digits is 36. Thus $30+a=36$ and $a=6$.

Answer: 6

106. The daily wages of two persons are in the ratio 3:5. They work in a place and the employer is satisfied with their work and gives Rs 20 more to each. Then the ratio of their wages comes to 13:21. The sum of the original wages of the two persons is ________.

Solution: Let the wages be $3k$ and $5k$. Given that $\frac{3k+20}{5k+20}=\frac{13}{21}$ and hence $k=80$ and total wages in rupees is $3k+5k=8k=640$.

107. The value of x which satisfies the equation

$$\cfrac{5}{6-\cfrac{5}{6-\cfrac{5}{6-x}}}=1$$

Solution: Clearly,

$$6-\cfrac{5}{6-\cfrac{5}{6-x}}=5$$

and hence

$$\cfrac{5}{6-\cfrac{5}{6-x}}=1$$

Now,

$$6-\frac{5}{6-x}=5$$

and

$$\frac{5}{6-x}=1$$

Thus $x=1$.

108. The current age of a father is three times that of his son. Ten years from now, the father's age will be twice that of his son. The fathers age will be 60 after ________years.

Solution: Let F, S be the ages of the father and son now. Given $F=3S$. Ten years from now, we have $F+10=2(S+10)$. Solving, we obtain $F=30$. Thus the father will be 60 years old after 30 more years.

109. The number of (x, y, z) such that

$$xy=6, \quad yz=15, \quad zx=10$$

is ________.

Solution: We have $x^2y^2z^2=6\times 10\times 15=900$. Hence $xyz=\pm 30$. Since x, y, z are all of the same sign, it follows that $(x, y, z)=(2,3,5)$ or $(-2,-3,-5)$.

110. If $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\dfrac{1}{2}$, The value of $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$ in the form $\dfrac{p}{q}$ is ________.

Solution: From $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \dfrac{1}{2}$, it follows that $\dfrac{\sqrt{a}}{\sqrt{b}} = \dfrac{3}{1} = 3$. Hence

$$\frac{a^2+ab+b^2}{a^2-ab+b^2} = \frac{(a/b)^2+(a/b)+1}{(a/b)^2-(a/b)+1} = \frac{81+9+1}{81-9+1} = \frac{91}{72}$$

111. The number of integers in $1, 2, 3, \ldots, 2015$ that are perfect squares and perfect cubes is ———.
Solution If a number is a perfect square and a perfect cube, then it is necessarily a sixth power of an integer. Hence we need to count the integers $1^6, 2^6, 3^6, \ldots n^6$ where $n^6 \leq 2015$. Since $3^6 = 729$ and $4^6 = 4096$, it follows that $1^6, 2^6, 3^6$ are the only sixth powers in the list of numbers given. Thus the answer is 3.

112. $ABCDE$ is a regular pentagon. CDP and EDQ are equilateral triangles. The measure of $\angle QDP$ is ———.
Solution The size of each interior angle in a regular

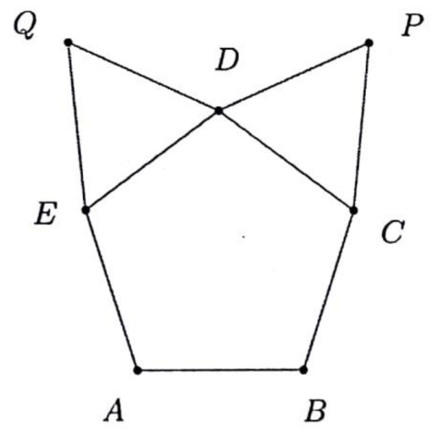

Figure 23

pentagon is $108°$. Since PDC and QDE are equilateral triangles, we have

$$\angle PDQ = 360^\circ - 60^\circ - 60^\circ - 108^\circ = 132^\circ$$

113. The value of $1 - 2 + 3 - 4 + 5 - \cdots + 2015$ is ———.
Solution We have

$$\begin{aligned} S &= 1 \qquad -2+3 \qquad -4-\cdots+2015 \\ &= 2015 \qquad -2014+2013 \qquad -2012+\cdots+1 \end{aligned}$$

Hence

$$\begin{aligned} 2S &= (2016-2016)+\cdots+(2016-2016)+2016 \\ &= 0+0+\cdots+0+2016 = 2016 \end{aligned}$$

Thus $S = 1008$.
We can also see this as follows:

$$\begin{aligned} S &= (1-2)+(3-4)+\cdots+(2013-2014)+2015 \\ &= \underbrace{-1-1-1-\cdots-1}_{1007\,terms}+2015 \\ &= 1008 \end{aligned}$$

114. Using the digits of the number 2015, four digit numbers of different digits are formed. The number of such numbers greater than 2000 and less than 6000 is ———.
Solution We need to find the number of four digit numbers that have only the digits $2, 0, 1, 5$, more than 2000 and less than 6000. Clearly all four digit numbers using only these four digits are less than 6000. Since neither 0 nor 1 can be the first digit, we can fill the first digit in 2 ways (with 2, 5), second digit in 3 ways (digits not used in the first digit), third digit in 2 ways and fourth digit in 1 way. Hence there are $2 \times 3 \times 2 \times 1 = 12$ four digit numbers that satisfy the required properties.

115. *Samrud* got an average mark 85 in his first 8 tests and an average mark 81 in the first 9 tests. His mark in the 9^{th} test is ———.

Solution His total marks in the first 8 tests is $8 \times 85 = 680$ and his total marks in the first 9 tests is $9 \times 81 = 729$. Hence his marks in the 9^{th} test is $729 - 680 = 49$.

116. The remainder when 20150020150002015002015 is divided by 3 is ———.

Solution The remainder when a number is divided by 3 is the same as the remainder when the sum of the digits of the number is divided by 3. The sum of the digits of the given number is 32 and hence the remainder is 2.

117. If

$$\frac{p}{q} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}}$$

where p, q have no common factors, then $p+q =$ ———.

Solution

$$\frac{p}{q} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{5}{21}}}$$

$$= 1 + \frac{1}{2 + \frac{21}{68}} = 1 + \frac{68}{157} = \frac{225}{157}$$

Hence $p + q = 225 + 157 = 382$.

118. If

$$\frac{p}{q} = 1 + \cfrac{5}{1 + \cfrac{4}{1 + \cfrac{3}{1 + \frac{1}{2}}}}$$

where p, q have no common factors, then $p+q =$ ——.

Solution It is easy to see that $\frac{p}{q} = \frac{22}{7}$. Thus $p+q = 29$.

119. In the Figure 24, $ABCD$ is a rectangle. $AD = 2$, $AB = 1$, AE is the arc of the circle with center D. The length BE is equal to ——.

Solution Join DE. In the right angled triangle DEC,

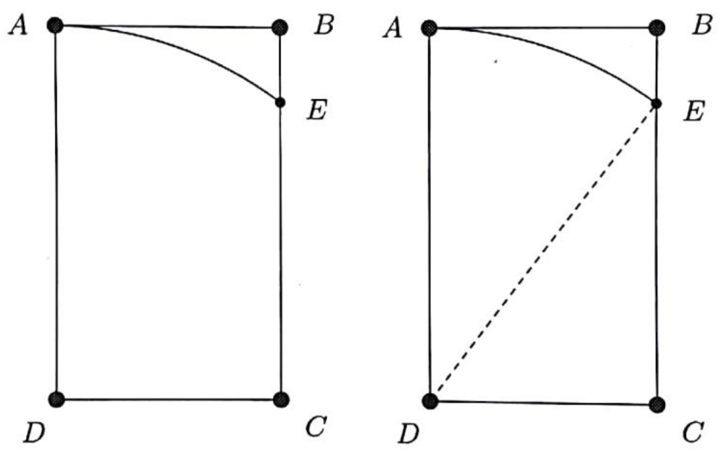

Figure 24

$DE = 2$, $DC = 1$ and hence $EC = \sqrt{DE^2 - DC^2} = \sqrt{3}$. Since $BC = 2$, it follows that $BE = 2 - \sqrt{3}$.

120. In the Figure 25, the squares have are $1\,cm^2$ the rectangles have area $2\,cm^2$. The number of squares with different dimensions in the figure is ——.

Solution From the figure it is clear that the individual rectangles are of size 1×2. There are four squares of size 1×1, three squares with dimension 2×2, two squares of dimension 3×3 and one square with dimension 4×4.

121. A race horse eats $(3a + 2b)$ bags of oats every week. The

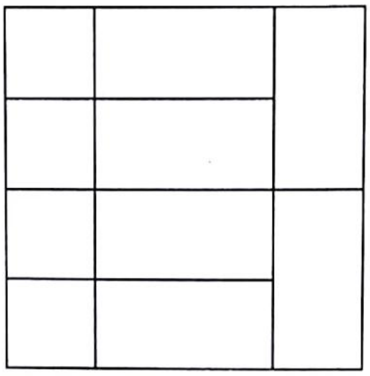

Figure 25

number of weeks in which it can eat $(12a^2 - 7ab - 10b^2)$ bags of oats is ———

Solution The number of weeks is given by

$$\frac{12a^2 - 7ab - 10b^2}{3a + 2b} = 4a - 5b$$

122. Choose 4 digits a, b, c, d from $\{2, 0, 1, 6\}$ and form the number $(10a + b)^{10c+d}$. For example, if $a = 2, b = 0, c = 1, d = 6$, we will get 20^{16}. For all such choices of a, b, c, d the number of distinct numbers that will be formed is —

Solution We can choose a in 4 ways and once a is chosen, b can be chosen in 3 ways. Now there are 2 choices for c and one choice for d. Hence there are 24 possibilities. But not all of these give distinct numbers. When $a = 0, b = 1$, both the choices $c = 2, d = 6$ and $c = 6, d = 2$ give the number 1. For the other choices we obtain distinct numbers and hence the number of distinct integers we obtain is 23.

123. A fraction F becomes $\frac{1}{2}$ when its denominator is

increased by 4 and becomes $\frac{1}{3}$ when its numerator is decreased by 5. Then F equals ———

Solution Let $F = \frac{p}{q}$. We have $\frac{p}{q+4} = \frac{1}{2}$ and $\frac{p-5}{q} = \frac{1}{3}$. Hence $2p = q+4$ and $3p - 15 = q$. Solving for p, q we get $p = 11$ and $q = 18$. The fraction is $\frac{11}{18}$.

124. The average of 5 consecutive positive integers starting with m is n. Then the average of 5 consecutive integers starting with n is (in terms of m) is ———

Solution We have $m + (m+1) + (m+2) + (m+3) + (m+4) = 5m + 10$ and hence $m + 2 = n$. Thus the average of 5 consecutive integers starting with n is $\frac{5n+10}{5} = n + 2 = m + 4$

125. Two boys came to Mahadevan and asked his age. Mahadevan told, "Delete all the vowels and repeated letters from my name. Find the numerical value of the remaining letters (for example, D has value 4, G has 7 etc). Add all of them. Find the number got by interchanging its digits. Add both the numbers. That is my age". One boy ran away. The other boy calculated correctly. The age of Mahadevan is ———

Solution The vowels are A and E and there are no repeated non vowels. Hence after removal, we obtain $MHDVN$ and adding the numerical values of these, we obtain $13 + 8 + 4 + 22 + 14 = 61$. Hence the age is $61 + 16 = 77$.

126. If

$$A = \frac{2^4 + 2^4}{2^{-4} + 2^{-4}}, \quad B = \frac{3^2 + 3^2}{3^{-2} + 3^{-2}}, \quad C = \frac{4^2 + 4^2}{4^{-2} + 4^{-2}}$$

the integral part of $\frac{A+C}{B}$ is ———

Solution $A = \dfrac{2^5}{2^{-3}} = 2^8, B = \dfrac{2 \cdot 3^2}{2 \cdot 3^{-2}} = 3^4$ and $C = \dfrac{2 \cdot 4^2}{2 \cdot 4^{-2}} = 4^4$ Thus $\dfrac{A+C}{B} = \dfrac{2^8 + 4^4}{3^4} = \dfrac{512}{81}$ and the integral part is 6.

127. A six-digit number is formed using the digits 1, 1, 2, 2, 3, 3. The number of 6-digit numbers in which the 1s are separated by one digit, 2s are separated by two digits and 3s are separated by 3 digits is ———

Solution The number 3 can be only in positions 1 and 5 or 2 and 6. Hence the numbers are 312132 and 231213.

128. In the addition shown below, P, I, U are digits. The value of U is ———

$$PI + PI + PI + PI = UP$$

Solution We have $4(10P + I) = 10U + P$ and hence $39P = 10U - 4I$. This implies that P is even. If $P \geq 4$, then $4 \times PI$ will have more than 2 digits. Thus $P = 2$ and $78 = 10U - 4I$. The only possibility is $U = 9$ and $I = 3$.

129. There are four cows, eight hen, a fish, a crow, a girl and a boy in a garden. Outside the garden there is one dog, a peacock and some cats. The number of legs of all of them inside the garden is equal that outside the garden. The number of cats is ———

Solution The number of legs inside the garden is

$$4 \times 4 + 8 \times 2 + 1 \times 0 + 1 \times 2 + 1 \times 2 + 1 \times 2 = 38$$

If the number of cats is c, then the number of legs outside the garden is $1 \times 4 + 1 \times 2 + c \times 4 = 4c + 6$. Hence the number of cats is 8.

130. Two sides of a triangle are 8 cm and 5 cm. The length of the third side in cms is also an integer. The number of

such triangles is ———

Solution If the length of the third side is c, then we must have $c+5>8$ and $8+5>c$. Thus $3<c<13$. Hence $c = 4,5,6,7,8,9,10,11,12$ and there are 9 such triangles.

131. If $a^2-a-10=0$, then $(a+1)(a+2)(a-4)$ is ———

Solution

$$\begin{aligned}(a+1)(a+2)(a-4) &= a^3-a^2-10a-8\\ &= a(a^2-a-10)-8\\ &= -8\end{aligned}$$

132. There are 5 points on the circumference of a circle. The number of chords which can be drawn joining them is ——

Solution If the points are A,B,C,D,E, then the chords are $AB,AC,AD,AE,BC,BD,BE,CD,CE,DE$ and hence there are 10 chords.

133. Each side of an equilateral triangle is 3 cm longer than each side of a square. The total perimeter of the square and the triangle is 51 cm. Then the side of the triangle in cms is ———

Solution Let a be the side of the square. Then the side of the equilateral triangle is $a+3$. The perimeter of the triangle is $3a+9$ and that of the square is $4a$. We have $3a+9+4a=51$ and hence $a=6$. Thus the side of the triangle in cms is 9.

134. The largest three digit number that is a multiple of 3 and 5 is ———

Solution The largest three digit number is 999 and when this number is divided by 15, we get a remainder 9. Hence the largest three digit number that is a multiple of 3 and 5 is 990.

135. Consider the sequence $0, 6, 24, 60, 120, \ldots$. The 6th term of this sequence is ———

Solution Observe the following table:

0		6		24		60		120
	6		18		36		60	
		12		18		24		
			6		6			

The second row is obtained by subtracting consecutive elements in the first row and similarly third row is obtained from the second row. In the fourth row, the numbers are all same and hence we can extend by inserting a 6 in that row and computing backwards. We will obtain 30 as the next term in the third row as 30, the next term in the second row as 90 and the next term in the first row as 210. Thus the sixth term of the sequence is 210.

136. Two cogged wheels of which one has 16 cogs and the other 27 cogs, mesh into each other. If the latter turns 80 times in three quarters of a minute, the number of turns made by the other in 8 seconds is ———

Solution When the first wheel makes 27 turns, the second will make 16 turns. Hence when the second wheel makes 80 turns, the first wheel would have $5 \times 27 = 135$ turns. Since the time taken by the second wheel to make 80 turns is 45 seconds, the first wheel makes 135 turns in 45 seconds. In 8 seconds, it would make $\frac{135}{45} \times 8 = 24$ turns.

137. If n is a positive integer such that $a^{2n} = 2$, then $2a^{6n} - 16$ is ———

Solution $a^{6n} = \left(a^{2n}\right)^3 = 8$. Thus $2a^{6n} - 16 = 0$.

138. The least number of children in a family such that every child has at least one sister and one brother is ———

Solution For any girl, we need another girl and a boy and for any boy, we need a girl and a boy. Thus the minimum number is 4.

139. A water tank is $\frac{4}{5}$ full. When 40 liters of water is removed, it becomes $\frac{3}{4}$ full. The capacity of the tank in liters is ———

Solution From the given data, if the capacity of the tank in liters is C, we have $\frac{4}{5}C - 40 = \frac{3}{4}C$ giving $C = 800$.

140. ABC is an equilateral triangle. Squares are described on the sides AB and AC as shown. The value of x is ———

Solution In the triangle ADE,

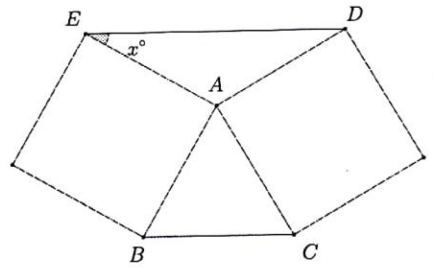

Figure 26

$$\angle EAD = 360^\circ - 90^\circ - 90^\circ - 60^\circ = 120^\circ$$

Hence $2x = 180^\circ - 120^\circ$ and $x = 30^\circ$.

141. $ABCD$ is a trapezium with $AB = 6$ cm, $AD = 8$ cm and $CD = 18$ cms. The sides AB and CD are parallel and AD is perpendicular to AB. P is the point of intersection of AC and BD. The difference between the areas of the triangles PCD and PAB in square cms is ———

Solution The area of the triangle ABD is $\frac{1}{2} \times AB \times AD = 24$ and the area of the triangle ACD is $\frac{1}{2} \times$

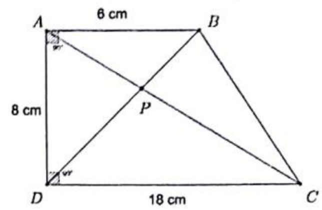

Figure 27

$CD \times AD = 72$. Since $\Delta ABD = \Delta PAB + \Delta APD$ and $\Delta ACD = \Delta APD + \Delta PCD$, it follows that

$$\Delta ACD - \Delta ABD = \Delta PCD - \Delta PAB = 72 - 24 = 48$$

142. The price of cooking oil has increased by 25%. The percentage of reduction that a family should effect in the use of oil so as not to increase the expenditure is ———
Solution Suppose that the current price of the oil is C per Kg and the family uses Y Kgs. The new price of the oil would be $1.25C$ and hence we have $C \times Y = 1.25 \times C \times Y_1$ where Y_1 is the new usage of the family. Hence $Y_1 = \frac{1}{1.25}Y = \frac{4}{5}Y$. Thus the usage must reduce by 20%.

143. The number of natural numbers between 99 and 999 which contains exactly one zero is ——
Solution We need to find the number of three digit numbers that have only one zero. Zero can not occur as the first digit. If the second digit is 0, then we have 9 choices for the first digit and 9 choices for the second digit. Thus we have 81 numbers that have 0 as the second digit. Similarly, we have 81 numbers with 0 as the third digit. Thus the required number of numbers is 162.

144. In Figure 28, we have semicircles and $AB = BC = CD$. The ratio of the unshaded area to the shaded area is ——

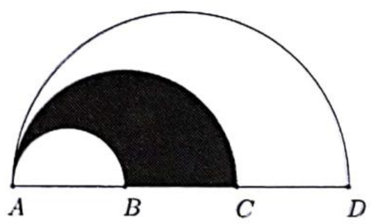

Figure 28

Solution Let $AB = BC = CD = 2r$. Then $AD = 6r$ and the area of the semi circle on AD as diameter is $\frac{1}{2}9\pi r^2$. Also $AC = 4r$ and the area of the semicircle on AC as diameter is $\frac{1}{2}4\pi r^2$. The area of the semi circle on AB as diameter is $\frac{1}{2}\pi r^2$. Thus the shaded area equals $\frac{1}{2}(4-1)\pi r^2 = \frac{3}{2}\pi r^2$ and the unshaded area is $\frac{1}{2}(9-3)\pi r^2 = \frac{6}{2}\pi r^2$. Thus the ratio of the unshaded to the shaded area is $2:1$.

145. Gold is 19 times as heavy as water and copper is 9 times as heavy as water. The ratio in which these two metals be mixed so that the mixture is 15 times as heavy as water is ———

Solution Let the ratio be $x : y$. When x parts of gold and y parts of copper are mixed, then the mixture is $\dfrac{19x+9y}{x+y}$ times as heavy as water. Thus

$$\frac{19x+9y}{x+y} = 15 \Rightarrow 4x = 6y \Rightarrow \frac{x}{y} = \frac{3}{2}$$

and $x : y = 3:2$

146. Five angles of a heptagon (seven sided polygon) are $160°, 135°, 185°, 145°$ and $125°$. If the other two angles are both equal to $x°$, then x is ———

Solution The sum of the angles of the heptagon is $5 \times 180° = 900°$. Thus

$$2x = 900° - 160° - 135° - 185° - 145° - 125° = 150°$$

and $x = 75°$.

147. $ABCD$ is a trapezium with AB parallel to CD and AD perpendicular to AB. If $AB = 23$ cm, $CD = 35$ cm and $AD = 5$ cm. The perimeter of the given trapezium in cms is ———

Solution If E is the foot of the perpendicular from B

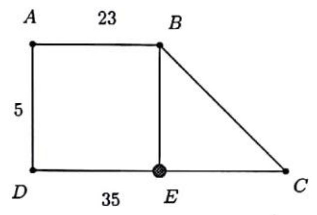

Figure 29

on CD, then $BE = AD = 5$, $EC = CD - DE = 35 - 23 = 12$ and hence $BC = \sqrt{5^2 + 12^2} = 13$. Thus the perimeter is $5 + 23 + 13 + 35 = 76$

148. The number of three digit numbers which are multiples of 11 is ———

Solution The smallest three digit number that is a multiple of 11 is $110 = 10 \times 11$ and the largest three digit number that is a multiple of 11 is $990 = 90 \times 11$. Thus the number of three digit numbers that are multiples of 11 is $90 - 10 + 1 = 81$

149. If a, b are digits, ab denotes the number $10a + b$. Similarly, when a, b, c are digits, abc denotes the number $100a + 10b + c$. If X, Y, Z are digits such that $XX + YY + ZZ = XYZ$, then $XX \times YY \times ZZ$ is ———

Solution From $XX + YY + ZZ = XYZ$, using the remainder when the sides are divided by 10, we deduce that $X + Y + Z \pmod 1)0 = Z$ and hence $X + Y = 0$

(mod 1)0. Note that XX, YY, ZZ are multiples of 11. Hence when we divide the left hand side by 11, we obtain the quotient $X + Y + Z$ and thus $100X + 10Y + Z = 11(X + Y + Z)$. Thus $89X = Y + 10Z$. Since the maximum value of the right hand side in this equation is 99, it follows that $X \leq 1$ and since $X \neq 0$, we have $X = 1$. Since $X + Y$ is a multiple of 10, $Y = 9$ and thus $10Z = 89X - Y = 80$ and $Z = 8$. Thus $XX \times YY \times ZZ = 11 \times 99 \times 88 = 95832$

150. The positive integer n has 2, 5 and 6 as its factors and the positive integer m has 4,8,12 as its factors. The smallest value of $m + n$ is ———

Solution The smallest number with factors $2, 5, 6$ is 30 and the smallest number with factors $4, 8, 12$ is 24. Hence the smallest value of $m + n$ is 54.

FINAL TEST QUESTIONS

1. Prove the algebraic identity

$$a^3 - b^3 = \left\{\frac{a(a^3 - 2b^3)}{a^3 + b^3}\right\}^3 + \left\{\frac{b(2a^3 - b^3)}{a^3 + b^3}\right\}^3.$$

Solution:

$$\text{RHS} = \left\{\frac{a(a^3 + b^3 - 3b^3)}{a^3 + b^3}\right\}^3 - \left\{\frac{b(a^3 + b^3 - 3a^3)}{a^3 + b^3}\right\}^3.$$

Let $a^3 + b^3 = t$. Then RHS

$$\begin{aligned}
&= \frac{(a(t - 3b^3))^3 - (b(t - 3a^3))^3}{t^3} \\
&= \frac{a^3(t^3 - 9t^2b^3 + 27tb^6 - 27b^9)}{t^3} \\
&\quad - \frac{b^3(t^3 - 9t^2a^3 + 27ta^6 - 27a^9)}{t^3} \\
&= \frac{(a^3 - b^3)t^3 + 27ta^3b^3(b^3 - a^3) - 27a^3b^3(b^6 - a^6)}{t^3} \\
&= \frac{(a^3 - b^3)t^3}{t^3} \\
&= a^3 - b^3 = \text{LHS}.
\end{aligned}$$

2. The ratio between a two-digit number and the sum of the digits of that number is $a : b$. If the digits in the units place is n more than the digit in the tens place, prove that the number is given by $\dfrac{9na}{11b - 2a}$.

Solution: Let the number be $10x + y$. Given that

$$\frac{10x + y}{x + y} = \frac{a}{b} \qquad (1)$$

$$y = x + n \qquad (2)$$

Substituting (2) in (1) we get

$$\frac{11x + n}{2x + n} = \frac{a}{b} \qquad (3)$$

Subtracting 1 from both sides of (3), we get

$$\frac{9x}{2x+n} = \frac{a-b}{b} \tag{4}$$

Cross multiplying and simplifying, we obtain

$$x = \frac{n(a-b)}{11b-2a} \tag{5}$$

Thus the number is

$$10x + y = 11x + n = \frac{11n(a-b)}{11b-2a} + n = \frac{9na}{11b-2a}$$

3. If $\dfrac{(a-b)(c-d)}{(b-c)(d-a)} = \dfrac{2012}{2013}$ find the value of $\dfrac{(a-c)(b-d)}{(a-b)(c-d)}$.

Solution: Given

$$\frac{(a-b)(c-d)}{(b-c)(d-a)} = \frac{2012}{2013} \tag{1}$$

Subtracting 1 from both sides, we get

$$\begin{aligned}
\frac{(a-b)(c-d)}{(b-c)(d-a)} - 1 &= \frac{(ac-bc-ad+bd)-(bd-cd-ab+ac)}{(b-c)(d-a)} \\
&= \frac{cd+ab-bc-ad}{(b-c)(d-a)} \\
&= \frac{(b-d)(a-c)}{(b-c)(d-a)} \qquad (2)
\end{aligned}$$

and

$$\frac{2012}{2013} - 1 = -\frac{1}{2013} \tag{3}$$

From (2) and (3) we get

$$\frac{(b-d)(a-c)}{(b-c)(d-a)} = -\frac{1}{2013} \tag{4}$$

Dividing (4) by (1) we get

$$\frac{(a-c)(b-d)}{(a-b)(c-d)} = -\frac{1}{2012}$$

4. Q, R are the midpoints of the sides AC, AB of the isosceles triangle ABC in which $AB = AC$. The median AD is produced to E so that $DE = AD$. EQ and ER are joined to cut BC in N and M respectively. Show that $AMEN$ is a rhombus.

Solution: We will prove that all the four sides of

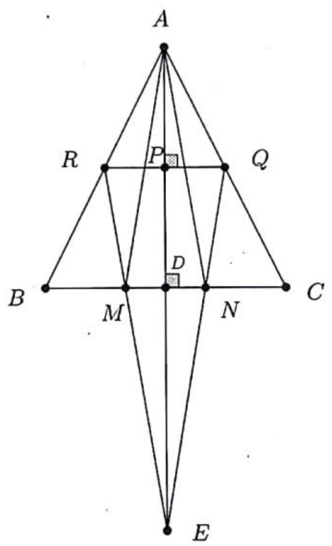

Figure 30

$AMEN$ are equal. Since $AB = AC$, AD is perpendicular to BC. Since QR is parallel to BC, AD is also perpendicular to QR. Clearly, P, the point of intersection of QR and AD is the mid point of QR and D is the mid point of BC.

Hence if we fold the triangle ABC along AE, Q will fall on R and C will fall on B. Hence the line QE will fall on ER and consequently, N will fall on M. Thus AN will fall on AM and hence $AM = AN$ and $EN = EM$. Now if we fold along the line BC, we deduce similarly that $AM = ME$ and $AN = NE$. Thus all the four sides of $AMEN$ are equal and it is a rhombus.

5. $ABCD$ is a square. The diagonals AC, BD, cut at E. From B a perpendicular is drawn to the bisector of $\angle DCA$ and it cuts AC at P and DC at Q. Prove that $DQ = 2PE$.

   Solution: Extend BQ and draw a line through B parallel to EP to meet it at R. Let the bisector of PCD cut the line BQR at S.

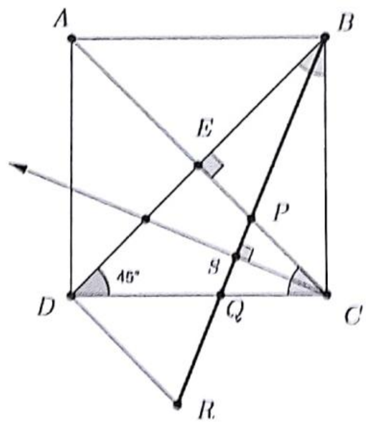

Figure 31

Clearly E is the midpoint of BD and EP is parallel to DR. Hence P is the mid point of BR. By the mid point theorem, $DR = 2EP$.

$\angle DQR = \angle SQC = 90^\circ - 22\frac{1}{2}^\circ = 67\frac{1}{2}^\circ$.

Note that $\angle DBR$ is the angle between DB and DR is equal to the angle between the line EC perpendicular to DB and the line SC perpendicular to DR and thus equal to $22\frac{1}{2}^\circ$. Now, $\angle QRD = 90^\circ - \angle DBR = 67\frac{1}{2}^\circ$

and hence $\angle DQR = \angle QRD$ and $DQ = DR$. Since $DR = 2EP$, it follows that $DQ = 2EP$.

6. (a) A two digit number is equal to six times the sum of its digits. Prove that the two digit number formed by interchanging the digits is equal to five times the sum of its digits.

   (b) Show that $\dfrac{10^{2013}+1}{10^{2014}+1} > \dfrac{10^{3013}+1}{10^{3014}+1}$

   Solution:

   (a) Let $10a + b$ be the two digit number. Given that $10a + b = 6(a + b)$. Thus $4a = 5b$. The number obtained by interchanging the digits of the given number is $10b + a = 5b + 5b + a = 5b + 4a + a = 5(a + b)$.

   (b) **Solution:** Let us put $a = 10^{2013}$ and $b = 10^{1000}$. Then we need to prove that

$$\frac{a+1}{10a+1} > \frac{ab+1}{10ab+1}$$

   This is equivalent to showing

$$(a+1)(10ab+1) > (ab+1)(10a+1)$$

   This holds if and only if

$$10a^2b + 10ab + a + 1 > 10a^2b + ab + 10a + 1$$

   or we must prove $9ab > 9a$ or $b > 1$. Since $b = 10^{1000}$, it is obvious that $b > 1$ and the required inequality holds.

7. (a) For any two natural numbers m, n prove that

   $(m^3 + n^3 + 4)$ cannot be a perfect cube.

   Solution: Any integer N can leave a remainder 0, 1 or 2 when divided by 3 and hence can be written in the form $3k$ or $3k+1$ or $3k+2$ for some integer k. When the number is cubed, then we have $N^3 = 27k^3$ or $N^3 = 27k^3 + 27k^2 + 9k + 1$ or $N^3 = 27k^3 + 54k^2 + 36k + 8$. Thus when N^3 is divided by 9, the remainders can only be $0, 1$ or 8. When m, n take different values, we compute the remainder when $m^3 + n^3 + 4$ is divided by 9. This is given in the following table:

	$m = 3k$	$m = 3k+1$	$m = 3k+2$
$n = 3l$	4	5	3
$n = 3l+1$	5	6	4
$n = 3l+2$	3	4	2

Since none of the entries are $0, 1$ or 8, it follows that $m^3 + n^3 + 4$ can not be a perfect cube.

(b) A circle is divided into six sectors and the six numbers 1, 0, 1, 0, 0, 0 are written clockwise, one in each sector. One can add 1 to the numbers in any two adjacent sectors. Is it possible to make all the numbers equal? If so after how many operations can this be achieved?

Solution: Taking the six numbers as

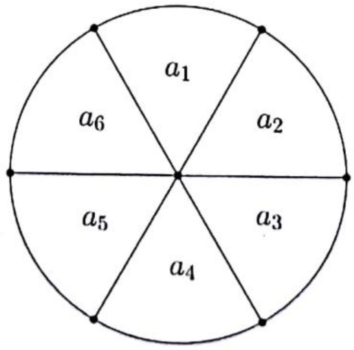

Figure 32

a_1, a_2, a_3, a_4, a_5 and a_6, we find that

$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 1 - 0 + 1 - 0 + 0 - 0 = 2.$

If 1 is added to any two adjacent sectors then the sum $a_1 - a_2 + a_3 - a_4 + a_5 - a_6$ is not altered.

After any number of times if the numbers in the sectors are the same then $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0$ but we started with 2. Hence it is not possible to make the numbers in all sectors the same.

We can not get all the numbers equal even after any number of operations.

8. a) All natural numbers from 1 to 2013 are written in a row in that order. Can you insert + and − signs between them so that the value of the resulting expression is zero? If it is possible, how many + and − signs should be inserted? Justify your answer by giving clear reasoning.

Solution: We have $1 + 2 + 3 + \cdots + 2013 = \frac{2013 \times 2014}{2} = 2013 \times 1007$, odd number.

Changing a plus $(+)$ to minus $(-)$ does not change the parity of the expression.

Thus any number, we get by writing the number 1 to 2013 in a row and placing plus and minus signs between them must be odd. So the result cannot be zero.

b) The natural numbers $1, 2, 3, \cdots$ are partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$ and so on. What are the greatest and least numbers in the set S_{2013} ?

Solution: The set S_i contains i elements. Hence $\bigcup_{i=1}^{k} S_i$ contains $1+2+\cdots+k = \frac{k(k+1)}{2}$ elements. Hence the last number in S_{2013} is $\frac{2013 \times 2014}{2} = 2027091$. Since S_{2013} contains 2013 numbers, the first number in S_{2013} is $2027091 - 2013 + 1 = 2025078$.

9. A hare, pursued by a gray-hound, is 50 of her own leaps ahead of him. In the time hare takes 4 leaps, the gray-hound takes 3 leaps. In one leap the hare goes $1\frac{3}{4}$ meter and the gray-hound $2\frac{3}{4}$ meter. In how many leaps will the gray-hound overtake the hare?

   Solution: Suppose that the hare makes 4 leaps every minute. Then it covers a distance of $4 \times 1.75 = 7$ meters in a minute. In the same one minute, the gray hound makes 3 leaps each of length 2.75 meters. Hence it covers $3 \times 2.75 = 8.25$ meters. In every minute, the gray hound comes by 1.25 meters closer to the hare. The initial distance is 50 hare leaps. This is $50 \times 1.75 = 87.5$ meters. Thus the hound will catch the hare after $\frac{87.5}{1.25} = 70$ minutes. Since the gray hound makes 3 leaps per minute, it will catch the hare after $3 \times 70 = 210$ leaps of its own.

10. If $\sqrt{a-x} + \sqrt{b-x} + \sqrt{c-x} = 0$, show that

$$(a+b+c+3x)(a+b+c-x) = 4(ab+bc+ca)$$

   Solution: Squaring both sides of $\sqrt{a-x} + \sqrt{b-x} = -\sqrt{c-x}$ we get

$$a - x + b - x + 2\sqrt{a-x}\sqrt{b-x} = c - x$$

   Hence

$$2\sqrt{a-x}\sqrt{b-x} = c - a - b + x$$

   Squaring again,

$$4(a-x)(b-x) = (c-a-b)^2 + x^2 + 2x(c-a-b)$$

   Simplifying, we get

$$-3x^2 + 2x(a+b+c) + a^2 + b^2 + c^2 - 2(ab+bc+ca) = 0$$

Adding $4(ab+bc+ca)$ to both sides, we obtain

$$\begin{aligned}-3x^2+2x(a+b+c)+a^2+b^2+c^2+2(ab+bc+ca)\\ =4(ab+bc+ca)\end{aligned}$$

Factoring the left hand side, we get

$$(a+b+c+3x)(a+b+c-x)=4(ab+bc+ca)$$

completing the proof.

11. Some amount of work has to be completed. Anand, Bilal and Charles offered to do the job. Anand would alone take a times as many days as Bilal and Charles working together. Bilal would alone take b times as many days as Anand and Charles together. Charles would alone take c times as many days as Anand and Bilal together. Show that

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=2$$

Solution: Suppose that Anand takes x days to complete the work when he works alone. Similarly let Bhilal and Charles respectively take y days and z days to complete the job when they work alone.

In one day, the quantum of work done by Anand is $\frac{1}{x}$ and similarly the quantum of work done by Bhilal and Charles are respectively $\frac{1}{y}$ and $\frac{1}{z}$. In one day, Bhilal and Charles working together complete $\frac{1}{y}+\frac{1}{z}$ of the work and hence they take $\dfrac{1}{\frac{1}{y}+\frac{1}{z}}$ to complete the work.

Since Anand takes a times as many days as Bhilal and Charles working together, we have

$$x=\frac{a}{\frac{1}{y}+\frac{1}{z}}$$

or

$$a = x\left(\frac{1}{y} + \frac{1}{z}\right) = \frac{xy + xz}{yz}$$

Thus

$$\frac{a}{a+1} = \frac{xy + xz}{xy + yz + zx}$$

Similarly

$$\frac{b}{b+1} = \frac{yz + yx}{xy + yz + zx}$$

and Similarly

$$\frac{c}{c+1} = \frac{zx + zy}{xy + yz + zx}$$

Adding the above three equations, we get

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = \frac{2(xy + yz + zx)}{xy + yz + zx} = 2$$

12. Let P be any point on the diagonal BD of a rectangle $ABCD$. F is the foot of the perpendicular from P to BC. H is a point on the side BC such that $FB = FH$. PC cuts AH in Q. Show that Area of $\triangle APQ =$ Area of $\triangle CHQ$. **Solution:** Since F is the mid point of

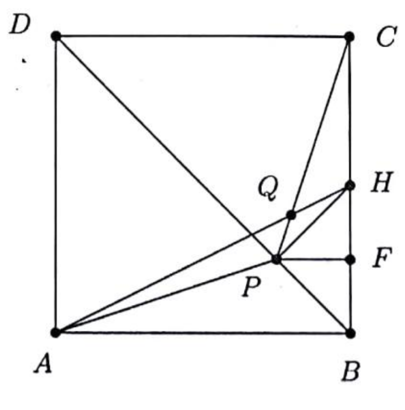

Figure 33

BH and $PF \perp BH$, triangle PBH is isosceles and $\angle FPH = 45°$. It follows that PH is parallel to AC. Since the triangles AHP and PCH are on the same base PH and between the parallels PH and AC, they have equal areas. Hence

$$\begin{aligned}\text{Area of } \triangle APQ &= \text{Area of } \triangle APH - \text{Area of } \triangle PHQ \\ &= \text{Area of } \triangle PHC - \text{Area of } \triangle PHQ \\ &= \text{Area of } \triangle CHQ\end{aligned}$$

13. A three digit number in base 7 when expressed in base 9 has its digits reversed in order. Find the number in base 7 and base 10.

Solution: The three digit number abc in base 7 has the decimal value $7^2a + 7b + c$. Given that

$$7^2a + 7b + c = 9^2c + 9b + a$$

Hence $24a - 40c = b$ and b is a multiple of 4. Since all of a, b, c are at most 6, it follows that either $b = 0$ or $b = 4$. If $b = 4$, then $6a - 10c = 1$, and this is impossible (left hand side is even and right hand side is odd). Thus $b = 0$ and $3a = 5c$. Either $a = c = 0$ or $a = 5$ and $c = 3$. $a = c = 0$ gives the number 0. Since we are looking only for non zero numbers, we deduce that in base 7, the number is 503. In base 10, the number is $49 \times 5 + 0 \times 7 + 3 = 248$.

14. a) Two regular polygons have the number of their sides in the ratio $3 : 2$ and the interior angles in the ratio $10 : 9$. Find the number of sides of the polygons.
b) Find two natural numbers such that their difference, sum and the product is to one another as 1,7 and 24.

Solution: a) Let the number of sides be $3n$ and $2n$. The interior angle of a regular polygon with k sides is

$\frac{(2k-4)90°}{k}$. Hence we have

$$\frac{6n-4}{3n} : \frac{4n-4}{2n} = 10 : 9$$

Simplifying we get $9(3n-2) = 3 \times 10(n-1)$ or $n = 4$. The number of sides of the respective polygons are 12 and 8.

b) Let the numbers be a and b
We have

$$\frac{a-b}{1} = \frac{a+b}{7} = \frac{ab}{24}$$

From the first two equations, we get $a = \frac{4b}{3}$ and from the first and last equations, we get $24(a-b) = ab$. Thus

$$ab = 24(a-b) = 24\left(\frac{4b}{3} - b\right) = 8b$$

Hence $a = 8, b = 6$.

15. Find the value of a, b, c which will make each of the expressions $x^4 + ax^3 + bx^2 + cx + 1$ and $x^4 + 2ax^3 + 2bx^2 + 2cx + 1$ a perfect square.

Solution: If $x^4 + ax^3 + bx^2 + cx + 1$ is a perfect square, then it must be the square of a quadratic in x and examining the terms x^4, ax^3 and the constant term 1, we deduce that

$$x^4 + ax^3 + bx^2 + cx + 1 = \left(x^2 + \frac{a}{2}x + 1\right)^2$$

Similarly, we can write

$$x^4 + 2ax^3 + 2bx^2 + 2cx + 1 = (x^2 + ax + 1)^2$$

Squaring the right hand sides of the above two equations, we get

$$\begin{aligned}
x^4 + ax^3 + bx^2 + cx + 1 = x^4 + ax^3 + &\\
\left(\frac{a^2}{4} + 2\right) x^2 + ax + 1 &\\
x^4 + 2ax^3 + 2bx^2 + 2cx + 1 = x^4 + 2ax^3 + &\\
(a^2 + 2)x^2 + 2ax + 1 &
\end{aligned}$$

Hence $b = \dfrac{a^2}{4} + 2$, $c = a$, and $a^2 + 2 = 2b$. These give $a^2 = 4$ and hence $a = \pm 2$ and $b = 3$. Thus the values of a, b, c are given by $a = 2, b = 3, c = 2$ and $a = -2, b = 3, c = -2$.

16. (a) The diagram below contains 13 boxes. The numbers in the second and twelfth boxes are respectively 175 and 70. Fill up the boxes with natural numbers such that

    i. sum of all numbers in all the 13 boxes is 2015
    ii. sum of the numbers in any three consecutive boxes is always the same

	175										70	

(b) If x, y, z are real and unequal numbers, prove that

$$2015x^2 + 2015y^2 + 6z^2 > 2(2012xy + 3yz + 3zx)$$

Solution

(a) If the number in the box i is x_i, then we have

$$x_i + x_{i+1} + x_{i+2} = x_{i+1} + x_{i+2} + x_{i+3}$$

and hence $x_i = x_{i+3}$ for all i. Thus the entries in the boxes are $x_1, x_2, x_3, x_1, x_2, x_3, \ldots, x_3, x_1$. Since $x_3 = x_{12} = 70$ and $4(x_1 + x_2 + x_3) + x_1 = 2015$, it

follows that $x_1 = 207$. Hence the numbers in the boxes are
$207, 175, 70, 207, 175, 70, 207, 175, 70, 207, 175, 70, 207$.

(b) Since $(x - y)^2 > 0$ when $x \neq y$, and similar inequalities for y, z and z, x, we have

$$x^2 + y^2 > 2xy \tag{1}$$
$$y^2 + z^2 > 2yz \tag{2}$$
$$z^2 + x^2 > 2zx \tag{3}$$

Multiply (1) by 2012, (2) and (3) by 3 and add to get

$$2015x^2 + 2015y^2 + 6z^2 > 2(2012xy + 3yz + 3zx)$$

the desired inequality.

17. Find a, b, c if they are real numbers, $a + b = 4$ and $2c^2 - ab = 4\sqrt{3}c - 10$.

Solution

$$\begin{aligned} 2c^2 - 4\sqrt{3}c - 10 &= ab \\ &= \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \\ &= 4 - \left(\frac{a-b}{2}\right)^2 \end{aligned}$$

Hence

$$\begin{aligned} 2(c^2 - 2\sqrt{3}c - 3) + \left(\frac{a-b}{2}\right)^2 &= 0 \\ 2(c - \sqrt{3})^2 + \left(\frac{a-b}{2}\right)^2 &= 0 \end{aligned}$$

Since a, b, c are real, $c = \sqrt{3}$ and $a = b = 2$.

18. When $a = 2^{2014}$ and $b = 2^{2015}$, prove that

$$\left\{\frac{\frac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\frac{1}{b-a}-\frac{1}{a+b}}\right\} \div \left\{\frac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2}\right\}$$

is divisible by 3.

Solution Put $a+b=x$ and $b-a=y$. The given expression is

$$\left\{\frac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}}\right\} \times \left\{\frac{x^2-y^2}{x^3+y^3}\right\}$$

$$= \frac{x(x^2+y^2-xy)}{x-y} \times \frac{(x-y)(x+y)}{(x+y)(x^2+y^2-xy)}$$

$$= x = a+b = 2^{2014}+2^{2015}$$

$$= 2^{2014}(1+2) = 3 \times 2^{2104}$$

19. Prove that the feet of the perpendiculars drawn from the vertices of a parallelogram onto its diagonals are the vertices of another parallelogram.

Solution Let the diagonals of the given parallelogram

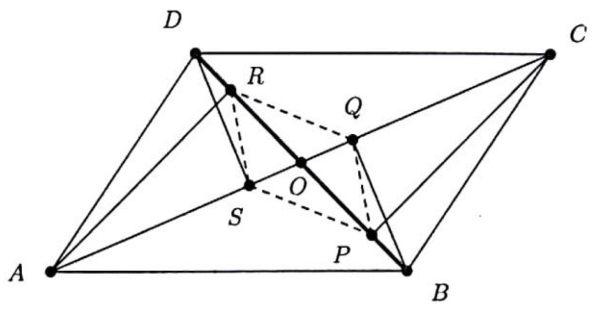

Figure 34

$ABCD$ intersect at O and P, Q, R, S the feet of the perpendiculars from the vertices on the diagonals. In triangles OSD and OQB, we have

$$\angle OSD = \angle OQB = 90^\circ$$

$$\angle SOD = \angle BOQ \quad \text{(vertically opposite angles)}$$

$$OD = OB \quad \text{(in } ABCD \text{ diagonals bisect each other)}$$

Thus triangles OSD and OQB are congruent and hence $OS = OQ$. Similarly, triangles OAR and OCP are congruent and $OR = OP$. Thus in $PQRS$ diagonals bisect each other and consequently, $PQRS$ is a parallelogram.

20. ABC is an acute angled triangle. P, Q are points on AB and AC respectively such that the area of $\triangle APC =$ area of $\triangle AQB$. A line is drawn through B parallel to AC and meets the line through Q parallel to AB at S. QS cuts BC at R. Prove that $RS = AP$.

Solution Given Area $\triangle APC =$ Area $\triangle AQB$. Since

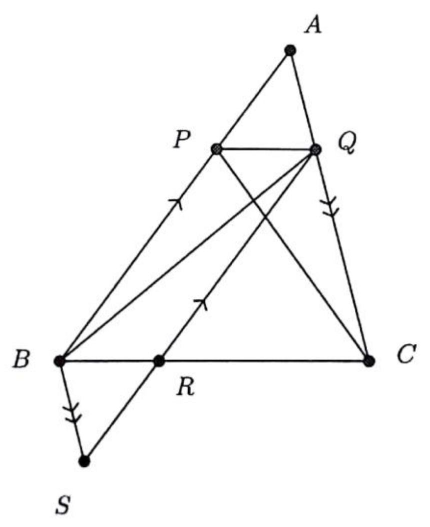

Figure 35

$$\text{Area } \triangle APC = \text{Area } \triangle APQ + \text{Area } \triangle PCQ$$

and

$$\text{Area } \triangle AQB = \text{Area } \triangle APQ + \text{Area } \triangle PQB$$

it follows that

$$\text{Area } \triangle PCQ = \text{Area } \triangle PQB$$

Since these have the same base PQ, it follows that PQ is parallel to BC. Thus $PQRB$ is a parallelogram and $QR = PB$. Also $ABSQ$ is a parallelogram and $AB = QS$. Hence we have

$$AP = AB - PB = QS - PB = QS - QR = RS$$

21. (a) A man is walking from a town A to another town B at a speed of 4 kms per hour. A bus started from town A one hour later and is travelling at a speed of 12 kms per hour. The man on the way got into the bus when it reached him and travelled further two hours to reach the town B. What is the distance between the towns A and B?

(b) A point P is taken within a rhombus $ABCD$ such that $PA = PC$. Show that B, P, D are collinear.

Solution

(a) Suppose that the man has walked x kms before the bus reached him. Then we have $\frac{x}{4} - 1 = \frac{x}{12}$ and hence $x = 6$. After boarding the bus, he travels for 2 hours at the speed of 12 kms per hour and hence travels 24 kms. Thus the distance between A and B is $6 + 24 = 30$ kms.

(b) In a rhombus diagonals bisect each other at right angles. Hence BD is the perpendicular bisector of AC. Since $PA = PC$, it follows that P lies on the perpendicular bisector of AC and hence on BD. Thus B, P, D are collinear.

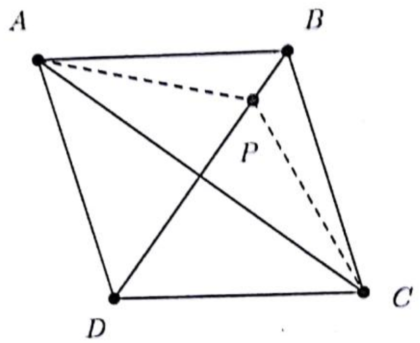

Figure 36

22. If

$$(x+y+z)^3 = (y+z-x)^3+(z+x-y)^3+(x+y-z)^3+kxyz$$

find the numerical value of k.
If $a = 2015, b = 2014, c = \frac{1}{2014}$, prove that

$$\begin{aligned}(a+b+c)^3 &- (a+b-c)^3 \\ &- (b+c-a)^3 - (c+a-b)^3 - 23abc = 2015\end{aligned}$$

Solution Putting $x = y = z = 1$ we get $27 = 1+1+1+k$ and hence $k = 24$. Also

$$\begin{aligned}(a+b+c)^3 &- (a+b-c)^3 \\ &- (b+c-a)^3 - (c+a-b)^3 - 23abc = abc = 2015\end{aligned}$$

23. (a) If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = 2016$, where x, y, z, a, b, c are non zero real numbers, find the value of

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)}$$

(b) Four boys Amar, Benny, Charan, Dany and four girls Azija, Beula, Chitra and Dais have to work on a project. They should form 4 pairs, one boy and one girl in each. They know each other with the following constraints:

i. Amar knows neither Azija nor Buela
ii. Benny does not know Buela
iii. Both Charan and Dany know neither Chitra nor Daisy.

In how many ways can the pairs be formed so that each boy knows the girl in his pair.

Solution (a) We have $\frac{xyz}{abc} = (2016)^3$ and

$$2016 = \frac{x}{a} = \frac{y}{b} = \frac{x+y}{a+b}$$

and hence $\frac{a+b}{x+y} = \frac{1}{2016}$. Similarly, $\frac{b+c}{y+z} = \frac{1}{2016}$ and $\frac{c+a}{z+x} = \frac{1}{2016}$. Thus

$$\frac{xyz(a+b)(b+c)(c+a)}{abc(x+y)(y+z)(z+x)} = (2016)^3\left(\frac{1}{2016}\right)^3 = 1$$

Solution (b) The conditions given can be tabulated as follows:

	Amar	Benny	Charan	Dany
Azija	N	Y	Y	Y
Beula	N	N	Y	Y
Chitra	Y	Y	N	N
Daisy	Y	Y	N	N

Charan and Dany can be paired only with Azija and Beula and Chitra and Daisy can only be paired with Amar and Benny. Thus there are 4 possible ways to form the pairs.

24. In a triangle ABC, $\angle C = 90^\circ$ and $BC = 3AC$. Points D, E lie on CB such that $CD = DE = EB$. Prove that

$$\angle ABC + \angle AEC + \angle ADC = 90^\circ$$

Solution Since $\angle ADC = 45^\circ$, we need to show that

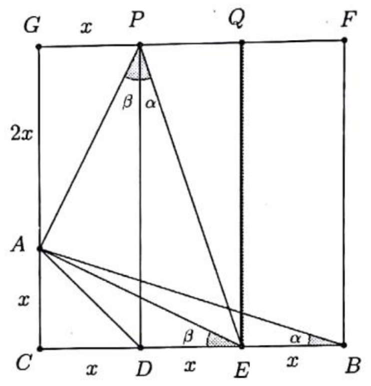

Figure 37

$\angle ABC + \angle AEC = 45^\circ$. Complete the square $CBFG$ and let P, Q be points on GF such that $GP = PQ = QF$. Join AP, PE. Let $CD = x$. We have

$$AP^2 = AG^2 + GP^2 = 4x^2 + x^2 = 5x^2$$
$$AE^2 = AC^2 + CE^2 = 4x^2 + x^2 = 5x^2$$

Also, from triangle PED, we have

$$PE^2 = PD^2 + DE^2 = 9x^2 + x^2 = 10x^2$$

Thus, in triangle APE, $AP = AE$ and $PE^2 = AP^2 + AE^2$ and hence it is an isosceles right angled triangle with $\angle PAE = 90^\circ$ and $\angle APE = 45^\circ$.

Also, noting that the triangles PDE and ACB are congruent, we have $\angle DPE = \angle ABC$ and from the

congruent triangles AGP and ACE, we get $\angle GAP = \angle AEC$. Again, since CG and DP are parallel, we get $\angle GAP = \angle APD$.
Thus

$$\angle ABC + \angle AEC = \angle DPE + \angle APD = \angle APE = 45^\circ$$

25. Let m, n, p be distinct two digit natural numbers. If

$$m = 10a + b, \quad n = 10b + c, \quad p = 10c + a$$

find all possible values of $\gcd(m, n, p)$.
Solution If d is the $\gcd(m, n, p)$ then d also divides $m + n + p = 11(a + b + c)$. If $d = 11$, then $m, n, p \in \{11, 22, \ldots, 99\}$. All these numbers are of the form $10x + x$ and hence we would have $a = b = c$, a contradiction since m, n, p are distinct. Thus $d \neq 11$.
Since d divides $a+b+c$ and the maximum value $a+b+c$ can take is $9 + 9 + 8 = 26$, we have $d \leq 26$.
Since any multiple of 5 ends with 0 or 5 and $a, b, c \neq 0$, if 5 divides d, then we must have $a = b = c = 5$, a contradiction. Thus $5 \nmid d$. Also, if $6 \mid d$, then a, b, c are all even and $a + b, b + c, c + a$ must all be multiples of 3 as well. Thus $a + b, b + c, c + a$ must be multiples of 6. Since $a, b, c \in \{2, 4, 6, 8\}$, this is impossible. Thus $6 \nmid d$. Also, if $9 \mid d$, we need $a + b, b + c, c + a$ must all be multiples of 9 and the only possibility is $(a, b, c) = (0, 0, 0)$ or $(9, 9, 9)$. Hence $9 \nmid d$.
Thus

$$d \in \{1, 2, 3, 4, 7, 8, 13, 14, 16, 17, 19, 21, 23, 26\}$$

Again, if $8 \mid d$, a, b, c are even and hence $m, n, p \in \{24, 48, 64, 88\}$. Clearly this is impossible. Again, one can easily see that $d = 17, 19, 21, 23, 26$ are impossible. Thus $d \in \{1, 2, 3, 4, 7, 13, 14\}$. For each of these cases, we give below m, n, p:

d	(m, n, p)
1	$(11, 12, 21)$
2	$(22, 24, 42)$
3	$(33, 39, 93)$
4	$(44, 48, 84)$
7	$(14, 49, 91)$
13	$(13, 39, 91)$
14	$(28, 84, 42)$

26. If $xy = ab(a+b)$ and

$$x^2 + y^2 - xy = a^3 + b^3$$

find the value of $\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{b} - \frac{y}{a}\right)$

Solution We have

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 + 2xy \\ &= a^3 + b^3 + 3ab(a+b) \\ &= (a+b)^3\end{aligned}$$

and

$$\begin{aligned}\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{b} - \frac{y}{a}\right) &= \frac{x^2+y^2}{ab} - xy\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \\ &= \frac{(x+y)^2}{ab} - \frac{2xy}{ab} - xy\left(\frac{a^2+b^2}{a^2b^2}\right) \\ &= \frac{(a+b)^3}{ab} - xy\left(\frac{(a+b)^2}{a^2b^2}\right) \\ &= \frac{(a+b)^3}{ab} - \left(\frac{xy}{ab}\right)\left(\frac{(a+b)^2}{ab}\right) \\ &= \frac{(a+b)^3}{ab} - \frac{(a+b)^3}{ab} \\ &= 0\end{aligned}$$

27. The square $ABCD$ of side length a cm is rotated about A in the clockwise direction by an angle 45° to become

the square $AB'C'D'$. Show that the shaded area is $(\sqrt{2}-1)a^2$ square cms.

Solution Let $C'D'$ intersect BC at P. Join AC, AC'.

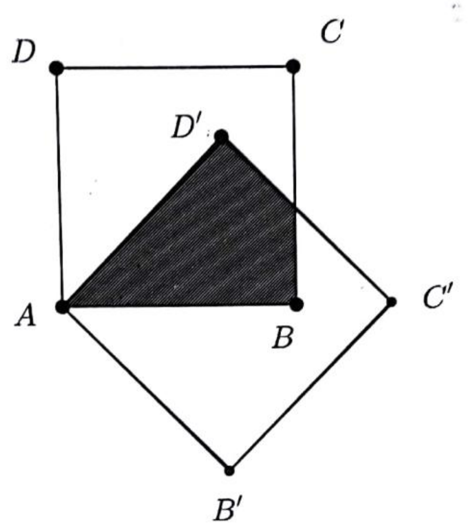

Figure 38

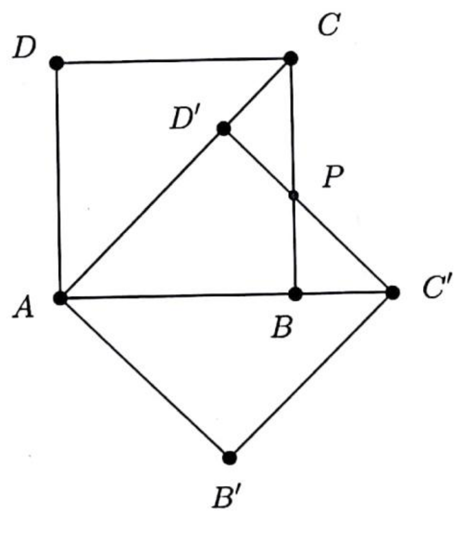

Figure 39

$AD' = a$ and $AC = \sqrt{2}a$. Thus $D'C = D'P = \sqrt{2}a - a$.

Similarly, $BC' = \sqrt{2}a - a$. It follows that the triangles PCD' and $PC'B$ are congruent. Thus $D'P = PB$ and hence the required area is twice the area of triangle $AD'P$. Now, area of triangle $AD'P == \frac{1}{2}a(a\sqrt{2} - a) = (\sqrt{2} - 1)\frac{a^2}{2}$ and the required area is $(\sqrt{2} - 1)a^2$.

28. (a) Find all three digit numbers in which any two adjacent digits differ by 3.

    (b) There are 5 cards. Five positive integers (may be different or equal) are written on these cards, one on each card. Abhiram finds the sum of the numbers on every pair of cards. He obtains only three different totals 57, 70, 83. Find the largest integer written on a card.

Solution

(a) The digits that differ by three are the sets $\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}$. Hence the three digit numbers whose adjacent digits differ by 3 are $630, 147, 741, 258, 852, 369, 963$.

(b) If four of the five numbers are distinct, say $a < b < c < d$, then taking sums of two numbers, we obtain the sums $a + b < a + c < a + d < c + d$. Since this gives 4 different totals, a maximum of three integers can be distinct. Let the distinct ones be a, b, c. The sums of pairs $a+b, b+c, c+a$ are distinct. Thus we need $a + b = 57, b + c = 70, c + a = 83$. This gives $a + b + c = 105$ and hence $c = 48, b = 22, a = 35$. Two other numbers must be equal to one of these. They could not be 48 (since then a sum of 96 must appear) or 22 (we will get 44 as sum). Thus the equal numbers must be 35. Thus the five numbers are $22, 35, 35, 35, 48$

29. (a) ABC is a triangle in which $AB = 24, BC = 10$ and $CA = 26$. P is a point inside the triangle. Perpendiculars are drawn to BC, AB and AC. Length of these perpendiculars respectively are x, y and z. Find the numerical value of $5x + 12y + 13z$.

(b) If

$$\begin{aligned} x^2(y+z) &= a^2, \\ y^2(z+x) &= b^2, \\ z^2(x+y) &= c^2, \\ xyz &= abc \end{aligned}$$

prove that $a^2 + b^2 + c^2 + 2abc = 1$

Solution

(a) The areas of the triangles PBC, PAB, PAC are respectively

$$\frac{1}{2} \times BC \times x, \quad \frac{1}{2} \times AB \times y, \quad \frac{1}{2} \times AC \times z$$

and hence are equal to $5x, 12y, 13z$. Since the sum of these areas is the area of the triangle ABC, we have, using Heron's formula for the area of a triangle,

$$5x + 12y + 13z = \sqrt{30 \times 6 \times 20 \times 4} = 120$$

(b) Multiplying the first three of the given equations, we get

$$x^2y^2z^2(y+z)(z+x)(x+y) = a^2b^2c^2$$

Since $xyz = abc$, we get

$$(y+z)(z+x)(x+y) = 1$$

We have

$$\begin{aligned} a^2 + b^2 + c^2 + 2abc &= x^2(y+z) + y^2(z+x) \\ &\qquad + z^2(x+y) + 2xyz \\ &= (x+y)(y+z)(z+x) \\ &= 1 \end{aligned}$$

30. If

$$\begin{aligned} X &= \frac{a^2-(2b-3c)^2}{(3c+a)^2-4b^2} + \frac{4b^2-(3c-a)^2}{(a+2b)^2-9c^2} \\ &\qquad + \frac{9c^2-(a-2b)^2}{(2b+3c)^2-a^2} \\ Y &= \frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2} + \frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2} \\ &\qquad + \frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2} \end{aligned}$$

find $2017(X+Y)$.

Solution Consider

$$\begin{aligned} &\frac{u^2-(v-w)^2}{(u+w)^2-v^2} + \frac{v^2-(w-u)^2}{(u+v)^2-w^2} + \frac{w^2-(u-v)^2}{(v+w)^2-u^2} \\ &= \frac{(u-v+w)(u+v-w)}{(u+w+v)(u+w-v)} \\ &\qquad + \frac{(v+w-u)(v-w+u)}{(u+v+w)(u+v-w)} \\ &\qquad\quad + \frac{(w+u-v)(w-u+v)}{(v+w+u)(v+w-u)} \\ &= \frac{u+v-w}{u+v+w} + \frac{v+w-u}{u+v+w} + \frac{u+w-v}{u+v+w} \\ &= \frac{u+v+w}{u+v+w} = 1 \end{aligned}$$

Putting $u = a, v = 2b, w = 3c$, we get $X = 1$. Again, putting $u = 2x, v = 3y, w = 4z$, we get $Y = 1$. Thus $2017(X+Y) = 4034$.

31. The sum of the ages of a man and his wife is six times the sum of the ages of their children. Two years ago the sum of their ages was ten times the sum of the ages of their children. Six years hence the sum of their ages will be three times the sum of the ages of their children. How many children do they have?

Solution Let n be the number of children. Let x be the sum of the ages of the man and wife and y, the sum of the ages of their children. Given $x = 6y$, $x-4 = 10(y-2n)$, and $x + 12 = 3(y + 6n)$. Solving, we obtain $n = 3$.

32. (a) a, b, c are three natural numbers such that $a \times b \times c = 27846$. If $\frac{a}{6} = b + 4 = c - 4$, find $a + b + c$.

(b) $ABCDEFGH$ is a regular octagon with side length equal to a. Find the area of the trapezium $ABDG$.

Solution

(a) Let $\frac{a}{6} = b+4 = c-4 = t$. Then $a = 6t, b = t-4, c = t+4$ and $6t \times (t-4) \times (t+4) = 27846 = 6 \times 3 \times 7 \times 13 \times 17$ and $t(t-4)(t+4) = 17 \times 13 \times 21$. This gives $t = 17$ and hence $a + b + c = 102 + 13 + 17 = 132$

(b) Perpendicular distance between the parallel sides AB and DG $a+a\cos 45°$ and $GD = a+2a\cos 45°$. Thus the area is

$$\frac{1}{2}\left(a + \frac{a}{\sqrt{2}}\right)\left(a + a\sqrt{2}\right) = \frac{a^2(3 + 2\sqrt{2})}{2\sqrt{2}}$$

33. (a) If a, b, c are positive real numbers such that no two of them are equal, show that

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)$$

is always positive.

(b) In the figure below, P, Q, R, S are points on the sides of the triangle ABC such that

$$CP = PQ = QB = BA = AR = RS = SC$$

Find the $\angle C$.

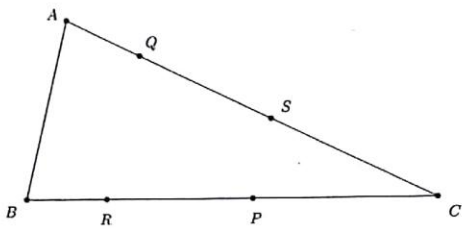

Figure 40

Solution

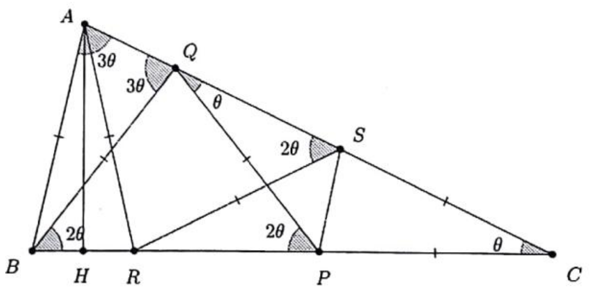

Figure 41

(a) Due to the symmetry of the expression, we can assume that $a > b > c$. We have $a - c > b - c$ and hence $a(a-b)(a-c) > b(a-b)(b-c)$. Also, $c-a < 0$ and $c-b < 0$ and hence $c(c-a)(c-b) > 0$. Thus

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) > 0$$

(b) In Figure 41, let $\angle SCP = \theta$. Let AH be perpendicular to BC from A. Using the equality of the line segments, we obtain $\angle AQB = 3\theta = \angle BAQ$ and $\angle RAQ = 2\theta$. Thus $\angle BAR = \theta$ and $\angle HAR = \dfrac{\theta}{2}$. In the right angled triangle AHC, we have $\angle QAH + \angle QCH = \dfrac{\pi}{2}$. But $\angle QAH = \angle QAB - \angle HAR = 3\theta - \dfrac{\theta}{2} = \dfrac{5\theta}{2}$. Hence $\dfrac{5\theta}{2} + \theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{7}$