IJSO STAGE-I

MATHEMATICS

VOLUME-1

Resonance®
Educating for better tomorrow
Pre-foundation Career Care Programmes (PCCP)
www.pccp.resonance.ac.in

PUBLISHED BY :

Resonance Eduventures Limited

Address : Plot No. A-51 [A], IPIA, Near Resonance CG Tower,
Behind City Mall, Jhalawar Road, Kota (Rajasthan)-324005
Contact. No.:+91-0744-6635569, 2777769
E-mail : pccp@resonance.ac.in
Website : www.pccp.resonance.ac.in

© Resonance Eduventures Limited
ALL RIGHTS RESERVED
All right reserved. Any photocopying, publishing or reproduction of full or any part of this study material is strictly prohibited. This material belongs to only the enrolled student of RESONANCE. Any sale/resale of this material is punishable under law.

Disclaimer

Every care has been taken by the compilers and publishers to give correct, complete and updated information. In case there is any omission, printing mistake or any other error which might have crept in inadvertently, neither the compiler / publisher nor any of the distributors take any legal responsibility.

Edition 2019

PREFACE

Dear Student,

Nothing glitters like success. In today's highly competitive world, a student has to withstand immense pressure in order to succeed. Students aspire to get through various competitive exam to get the exposure at National and International platforms. Also to prepare for your career goals, you will be appearing for various prestigious exams and compete with the country's brilliant minds for limited coveted seats. Students with thorough understanding of the fundamental concepts and logical problem solving skills are able to succeed in that. Now to select an appropriate career path for your future endeavor, a conceptual & systematically designed study material is required to be in the competitive race.

The material we have prepared is not the effort of a single person. It is in fact, written and designed by a well-qualified faculty team of Resonance Pre-foundation Division. I am sure that this material is best in its segment as our faculty team cover all important realms of the related topic followed by a set of extremely good questions of different level. It will not only cater the need of students in school exams but also prepare you to appear in any of the competitive exam at any level. With a quote mentioned below, I wish all the students a great journey ahead.

"You have to dream before your dreams come true"

Pre-foundation Career Care Programmes (PCCP) Division

Every effort has been taken to make our study material error free, however any suggestion to improve is welcome in this regard.

How to use this book?

An Important Note to Parents and Students

Dear Student,

If you are reading this, it means you are serious about performing in your class especially for **National Talent Search Examination (NTSE).** To help you achieve your potential, this book is designed in a way which is highly beneficial for students. Let's see how to use the different components of the book :

- **Theory Part :** The theory part has been redesigned with perfect blend of solved examples, text and important notes. At relevant checkpoints, in theory exercises have been inserted to enhance the reading experience of a student.

- **Board Level Exercise :** This feature has been added in class X module because of the reintroduction of Board examination in CBSE. The exercise covers important questions from NCERT and Exemplar books as well as previous year question papers of class X.

- **Exercise 1 :** This exercise is a **Concept Building Exercise (CBE)**. The questions in this exercise are simple and direct application of the theory which is being covered in the chapter. This helps student in absorbing the concepts included in the chapter (hence the name). The question patterns which are covered in this exercise are according to the patterns being asked in school examinations which give student an edge in the practice of school examinations. The important point is that the homework of this exercise should be neatly maintained in a separate copy.

- **Exercise 2 :** This exercise is **Competitive Level Exercise (CLE).** After attempting exercise 1 and thus absorbing the important concepts, students are ready to implement their learning in slightly higher level questions. These questions are in accordance to the level being asked in the National Level Competitive Examinations. These questions are must for all the students to strengthen their concepts.

- **Exercise 3 :** This exercise consists of questions from previous year papers of competitive examinations especially NTSE. This exercise is included for students to determine their current proficiency level of that chapter with respect to the competitive examination.

After reading the theory and completing the exercises, a student should be able to have a conceptual framework and problem solving aptitude in that particular chapter. For best results, **all exercises should be solved in a fair notebook and all the solutions should be maintained so that when time of revision comes, this notebook proves to be a handy one.**

I would like to request parents to regularly check the homework of student. You don't have to check the full copy. Just check that the solutions of all questions are maintained or not. This simple monitoring serves the dual purpose. First, your child thinks that his/ her education is important for you which is why you are taking time to check his homework. Second, he/ she regularly completes the homework.
In the end, I hope you like this book and sincerely believe that by collective efforts of student, parent and teacher, we can produce maximum results from this book.

CONTENTS

TOPIC		PAGE NO.
1.	Number System	01 - 30
2.	Commercial Mathematics	31 - 71
3.	Polynomials	72 - 93
4.	Linear equation (Two Variable)	94 - 105
5.	Linear Inequation (One Variable)	106 - 114
6.	Quadratic equation	115 - 129
7.	Progressions	130 - 147
8.	Trigonometry	148 - 169
9.	Co-ordinate Geometry	170 - 197
10.	Sets	198 - 215
11.	Plane Geometry - I	216 - 261
12.	Plane Geometry - II	262 - 297
13.	Mensuration	298 - 331
14.	Permutation & Combination	332 – 346
15.	Statistics	347 - 371

NUMBER SYSTEM

A. LOGARITHM

If **'a'** is a positive real number, other than **1** and **x** is a rational number such that $\mathbf{a^x = N}$, then **x** is the logarithm of **N** to the base **a**.

$\therefore$ If $\mathbf{a^x = N}$ then $\mathbf{\log_a N = x}$. [Remember N will be +ve]

(a) Systems of Logarithm :

There are two systems of logarithm which are generally used.

(i) Common logarithm : In this system base is always taken as **10**.

(ii) Natural logarithm : In this system the base of the logarithm is taken as **'e'**. Where **'e'** is an irrational number lying between **2** and **3**. (The approximate value of **e** upto two decimal places is e = 2.73)

Some Useful Results :

(i) If a > 1 then

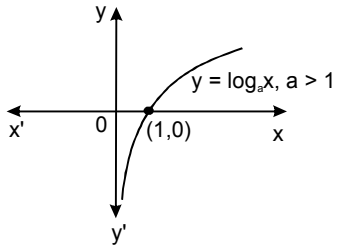
(a) $\log_a x < 0$ [for all x satisfying $0 < x < 1$]

(b) $\log_a x = 0$ for $x = 1$

(c) $\log_a x > 0$ for $x > 1$

(d) $x > y \Rightarrow \log_a x > \log_a y$ i.e. $\log_a x$ is an increasing function.

$\therefore$ Graph of $y = \log_a x$, $a > 1$



(ii) If $0 < a < 1$, then

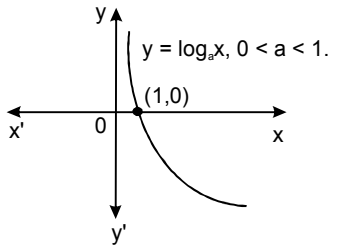
(a) $\log_a x < 0$ for all $x > 1$

(b) $\log_a x = 0$ for $x = 1$

(c) $\log_a x > 0$ for all x satisfying $0 < x < 1$

(d) $x > y \Rightarrow \log_a x < \log_a y$ i.e. $\log_a x$ is a decreasing function.

$\therefore$ Graph of $y = \log_a x$, $0 < a < 1$.



(b) Fundamental Laws of Logarithm :

Logarithm to any base **a** (where $\mathbf{a > 0}$ and $\mathbf{a \neq 1}$).

(i) $\log_a a = 1$

(ii) $\log_a 0$ = not defined [As $a^n = 0$ is not possible, where n is any number]

(iii) $\log_a$ (–ve no.) = not defined. [As in $\log_a N$, N will always be (+ ve)]

(iv) $\log_a (mn) = \log_a m + \log_a n$ [Where m and n are +ve numbers]

(v) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(vi) $\log_a (m)^n = n \log_a m$

(vii) $\log_a m = \frac{\log_b m}{\log_b a}$

(viii) $\log_a m \,.\, \log_m a = 1$

(ix) If **'a'** is a positive real number and **'n'** is a positive rational number, then $a^{\log_a n} = n$

(x) If **'a'** is a positive real number and **'n'** is a positive rational number, then

$$\log_{a^p} n^q = \frac{p}{q}\log_a n$$

(xi) $p^{\log_a q} = q^{\log_a p}$

(xii) $\log_a x = \log_a y \Rightarrow x = y$

Solved Examples

Example.1 If $\log_3 a = 4$, find value of a.

Solution : $\because$ $\log_3 a = 4$

$\therefore$ $a = 3^4$

$\Rightarrow$ $a = 81$.

Example.2 Find the value of $\log\frac{9}{8} - \log\frac{27}{32} + \log\frac{3}{4}$

Solution : Given :

$$\log\frac{9}{8} - \log\frac{27}{32} + \log\frac{3}{4} = \log\left(\frac{9}{8} \div \frac{27}{32}\right) + \log\frac{3}{4}$$

$$= \log\left(\frac{9}{8} \times \frac{32}{27} \times \frac{3}{4}\right)$$

$= \log 1 = 0.$ [$\because \log_a 1 = 0$]

Example.3 If $2\log_4 x = 1 + \log_4(x - 1)$, find the value of x.

Solution : Given $2\log_4 x = 1 + \log_4(x - 1)$

$\Rightarrow$ $\log_4 x^2 - \log_4(x - 1) = 1$

$\Rightarrow$ $\log_4 \frac{x^2}{x-1} = 1$ $\Rightarrow$ $4^1 = \frac{x^2}{x-1}$ $\Rightarrow$ $x^2 = 4x - 4$

$\Rightarrow$ $x^2 - 4x + 4 = 0$ $\Rightarrow$ $(x - 2)^2 = 0$ $\Rightarrow$ $x = 2$.

Example.4 Evaluate : $3^{2-\log_3 5}$.

Solution. Given $3^{2-\log_3 5} = 3^2.3^{-\log_3 5}$ [$\because a^{m+n} = a^m.a^n$]

$= 9.3^{\log_3 5^{-1}} = 9 \times 5^{-1} = \frac{9}{5}$.

Example.5 If $A = \log_{27}625 + 7^{\log_{11}13}$ and $B = \log_9 125 + 13^{\log_{11}7}$, then find the relation between A and B.

Solution. $A = \log_{27}625 + 7^{\log_{11}13} = \log_{3^3} 5^4 + 7^{\log_{11}13}$

or, $A = \frac{4}{3}\log_3 5 + 7^{\log_{11} 13}$ ....(i)

and, $B = \log_9 125 + 13^{\log_{11} 7}$ or, $B = \log_{3^2} 5^3 + 7^{\log_{11} 13}$

or, $B = \frac{3}{2}\log_3 5 + 7^{\log_{11} 13}$ ...(ii)

By (i) and (ii) we have,

$A - \frac{4}{3}\log_3 5 = B - \frac{3}{2}\log_3 5$

$\because \frac{4}{3}\log_3 5 < \frac{3}{2}\log_3 5$

$\therefore A < B.$

Example.6 Find the value of $\log_{25} 125 - \log_8 4$

Solution. Given, $\log_{25} 125 - \log_8 4 = \log_{5^2} 5^3 - \log_{2^3} 2^2 = \frac{3}{2}\log_5 5 - \frac{2}{3}\log_2 2$

$= \frac{3}{2} - \frac{2}{3}$ $[\because \log_a a = 1]$

$= \frac{5}{6}$.

B. FACTORS AND MULTIPLES

Factors : 'a' is a factor of **'b'** if there exists a relation such that **a × n = b**, where **'n'** is any natural number.

- 1 is a factor of all numbers as 1 × b = b.
- Factor of a number cannot be greater than the number (in fact the largest factor will be the number itself). Thus factors of any number will lie between 1 and the number itself (both inclusive) and they are limited.

Multiples : 'a' is a multiple of **'b'** if there exists a relation of the type **b × n = a**. Thus the multiples of 6 are 6 × 1 = 6, 6 × 2 = 12, 6 × 3 = 18, 6 × 4 = 24, and so on.

- The smallest multiple will be the number itself and the number of multiples would be infinite.
- **NOTE :**

To understand what multiples are, let's just take an example of multiples of 3. The multiples are 3, 6, 9, 12, ... so on. We find that every successive multiples appears as the third number after the previous.

So if one wishes to find the number of multiples of 6 less than 255, we could arrive at the number through $\frac{255}{6}$ = 42 (and the remainder 3). The remainder is of no consequence to us. So in all there are 42 multiples. If one wishes to find the multiples of 36, find $\frac{255}{6}$ = 7 (and the remainder is 3).

Hence, there are 7 multiples of 36.

Solved Examples

Example.7 How many numbers from 200 to 600 are divisible by 4, 5, 6 ?

Solution : Every such number must be divisible by L.C.M. of (4, 5, 6) = 60.

$$\left[\frac{600}{60}\right] - \left[\frac{200}{60}\right] = 10 - 3 = 7.$$

Such numbers are 240, 300, 360, 420, 480, 540 and 600.
Clearly, there are 7 such numbers.

Factorisation : It is the process of splitting any number into a form where it is expressed only in terms of the most basic prime factors.
For example, $36 = 2^2 \times 3^2$. It is expressed in the factorised form in terms of its basic prime factors.

Number of factors : For any composite number C, which can be expressed as $C = a^p \times b^q \times c^r \times$....., where a, b, c ..... are all prime factors and p, q, r are positive integers, the number of factors is equal to $(p + 1) \times (q + 1) \times (r + 1)$.... e.g. $36 = 2^2 \times 3^2$. So the factors of 36 = (2 +1) × (2 + 1) = 3 × 3 = 9.

Solved Examples

Example.8 If $N = 12^3 \times 3^4 \times 5^2$, find the total number of even factors of N.

Solution : The factorised form of N is $(2^2 \times 3^1)^3 \times 3^4 \times 5^2 \Rightarrow 2^6 \times 3^7 \times 5^2$.
Hence, the total number of factors of N is (6 + 1) (7 + 1) (2 + 1) = 7 × 8 × 3 = 168.
Some of these are odd multiples and some are even. The odd multiples are formed only with the combination of 3s and 5s.
So, the total number of odd factors is (7 + 1) (2 + 1) = 24.
Therefore, the number of even factors is 168 – 24 = 144.

Example.9 A number N when factorised can be written $N = a^4 \times b^3 \times c^7$. Find the number of perfect squares which are factors of N (The three prime numbers a, b, c > 2).

Solution : In order that the perfect square divides N, the powers of 'a' can be 0, 2 or 4, i.e. 3.
Powers of 'b' can be 0, 2, i.e. 2. Power of 'c' can be 0, 2, 4 or 6, i.e. 4.
Hence, a combination of these powers given 3 × 2 × 4 i.e. 24 numbers.
So, there are 24 perfect squares that divides N.

Example.10 Directions : (i to iv) Answer the questions based on the given information.
There are one thousand lockers and one thousand students in a school. The principal asks thef irst student to go to each locker and open it. Then he asks the second student go to every second locker and close it. The third student goes to every third locker, and if it is closed, he opensit, and it is open, he closes it. The fourth student does it to every fourth locker and so on. The process is completed with all the thousand students.
(i) How many lockers are closed at the end of the process ?
(ii) How many students can go to only one locker ?
(iii) How many lockers are open after 970 students have done their job ?
(iv) How many student go to locker no. 840 ?

Solution : **(i to iv) :** Whether the locker is open or not depends on the number of times it is accessed. If it is accessed odd number of times, then it is open while if it is accessed even number of times then it is closed.
How many times a locker will be accessed depends on the locker no. If it contains odd number of factors, then it will be open and if it contains even number of factors. Then it will be closed. We know that a perfect square contains odd number of factors while a non-perfect square contains even number of factors. Thus the lockers with perfect square number will be open and the number of these perfect squares from 1 to 1000 determines the no. of open lockers.

(i) Number of closed lockers = No. of non-perfect square numbers from 1 to 1000 = 1000 –31 = 969.

(ii) Upto 500 students they can go to two or more than two lockers, while the rest 500 can go to only one locker.

(iii) The 31 perfect squares (the last being 31^2 = 961) will be open while the lockers from 971 to 1000 is yet to be accessed last time so they all are open. The total being = 31 + 30 = 61

(iv) The no. of students that have gone to locker no. 840 is same as the no. of factors of 840.
$840 = 2^3 \times 3 \times 5 \times 7$.
So, the no. of factors = (3 + 1) (1 + 1) (1 + 1) (1 + 1) = 32.

C. HCF AND LCM

(a) LCM (least Common Multiple) :

The LCM of given numbers, as the name suggests is the smallest positive number which is a multiple of each of the given numbers

(b) HCF (Highest Common factor) :

The HCF of given numbers, as the name suggests is the largest factor of the given set of numbers.

Consider the numbers 12, 20 and 30. The factors and the multiples are

Factors	Given numbers	Multiples
1, 2, 3, 4, 6, 12	12	12, 24, 36, 48, 60, 72, 84, 96, 108, 120....
1, 2, 4, 5, 10, 20	20	20, 40, 60, 80, 100, 120.....
1, 2, 3, 5, 6, 10, 15, 30	30	30, 60, 90, 120....

The common factors are 1 and 2 and the common multiples are 60, 120...
Thus the highest common factor is 2 and the least common multiple meaning of HCF it is the largest number that divides all the given numbers.
Also since a number divides its multiple, the meaning of LCM is that it is the smallest number which can be divided by the given numbers.

❖ HCF will be lesser than or equal to the least of the numbers and LCM will be greater than or equal to the greatest of the numbers.

Solved Examples

Example.11 Find a number greater than 3 which when divided by 4, 5, and 6 always leaves the same remainder 3.

Solution : The smallest number which, when divided by 4, 5 and 6, leaves the remainder 3 in each case is LCM (4, 5 and 6) + 3 = 60 + 3 = 63.

Example.12 In a school 437 boys and 342 girls have been divided into classes, so that each class has the same number of students and no class has boys and girls mixed. What is the least number of classes needed?

Solution : We should have the maximum number of students in a class. So we have to find HCF (437, 342) = 19.
HCF is also the factor of difference of the number.

$\therefore$ Number of classes = $\frac{437}{19} + \frac{342}{19}$ = 23 + 18 = 41 classes.

❖ For any two numbers x and y,
x × y = HCF (x, y) × LCM (x, y).

HCF and LCM of fractions :

LCM of fractions = $\dfrac{\text{LCM of numerators}}{\text{HCF of denominators}}$

HCF of fractions = $\dfrac{\text{HCF of numerators}}{\text{LCM of denominators}}$

Make sure the fractions are in the most reducible form.

Solved Examples

Example.13 Find the H.C.F. and L.C.M. of $\frac{8}{9}$, $\frac{16}{81}$, $\frac{2}{3}$ and $\frac{10}{27}$.

Solution : H.C.F. of given fractions = $\dfrac{H.C.F.of(2,8,16,10)}{L.C.M.of(3,9,81,27)} = \dfrac{2}{81}$,

L.C.M. of given fractions = $\dfrac{L.C.M.of(2,8,16,10)}{H.C.F.of(3,9,81,27)} = \dfrac{80}{3}$.

Example.14 Find the least number which when divided by 6, 7, 8, 9 and 10 leaves remainder 1.

Solution : As the remainder is same
Required number = LCM of divisors + Remainder = LCM (6, 7, 8, 9, 10) +1
= 2520 + 1 = 2521.

Example.15 Six bells start tolling together and they toll at intervals of 2, 4, 6, 8, 10, 12 sec. respectively, find
(i) after how much time will all six of them toll together ?
(ii) how many times will they toll together in 30 min ?

Solution : The time after which all six bells will toll together must be multiple of 2, 4, 6, 8, 10, 12.
Therefore, required time = LCM of time intervals = LCM (2, 4, 6, 8, 10, 12) = 120 sec.
Therefore after 120 s all six bells will toll together.
After each 120 s, i.e. 2 min, all bell are tolling together.

Therefore in 30 min they will toll together $\left(\dfrac{30}{2}+1\right)$ = 16 times

1 is added as all the bells are tolling together at the start also, i.e. 0th second.

Example.16 LCM of two distinct natural numbers is 211. What is their HCF ?

Solution : 211 is a prime number. So there is only one pair of distinct numbers possible whose LCM is 211, i.e. 1 and 211. HCF of 1 and 211 is 1.

Example.17 An orchard has 48 apple trees, 60 mango trees and 96 banana trees. These have to be arranged in rows such that each row has the same number of trees and all are of the same type. Find the minimum number of such rows that can be formed.

Solution : Total number of trees are 204 and each of the trees are exactly divisible by 12. HCF of (48, 60, 96).

$\therefore \dfrac{204}{12}$ = 17 such rows are possible.

D. DIVISIBLITY

Division Algorithm : General representation of result is, $\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$

Dividend = (Divisor × Quotient) + Remainder

Solved Examples

Example.18 On dividing 15968 by a certain number, the quotient is 89 and the remainder is 37. Find the divisor.

Solution : Divisor = $\frac{Dividend - Remainder}{Quotient} = \frac{15968 - 37}{89}$ = 179.

❖ **NOTE :**

(i) $(x^n - a^n)$ is divisible by $(x - a)$ for all the values of n.

(ii) $(x^n - a^n)$ is divisible by $(x + a)$ and $(x - a)$ for all the even values of n.

(iii) $(x^n + a^n)$ is divisible by $(x + a)$ for all the odd values of n.

Test of Divisibility :

No.	Divisiblity Test
2	Unit digit should be 0 or even
3	The sum of digits of no. should be divisible by 3
4	The number formed by last 2 digits of given no. should be divisible by 4.
5	Unit digit should be 0 or 5.
6	Number should be divisible by 2 & 3 both
8	The number formed by last 3 digits of given no. should be divisible by 8.
9	Sum of digits of given no. should be divisible by 9
11	The difference between sums of the digits at even & at odd places should be zero or multiple of 11.
25	Last 2 digits of the number should be 00, 25, 50 or 75.

Rule for 7 : Double the last digit of given number and subtract from remaining number the result should be zero or divisible by 7.

Solved Examples

Example.19 Check whether 413 is divisible by 7 or not.

Solution : Last digit = 3, remaining number = 41, 41 – (3 x 2) = 35 (divisible by 7). i.e. 413 is divisible by 7.

This rule can also be used for number having more than 3 digits.

Example.20 Check whether 6545 is divisible by 7 or not.

Solution : Last digit = 5, remaining number 654, 654 – (5 x 2) = 644; 64 – (4 x 2) = 56 divisible by 7. i.e. 6545 is divisible by 7.

Rule for 13 : Four times the last digit and add to remaining number the result should be divisible by 13.

Example.21 Check whether 234 is divisible by 13 or not .

Solution : 234, (4 x 4) + 23 = 39 (divisible by 13), i.e. 234 is divisible by 13.

Rule for 17 : Five times the last digit of the number and subtract from previous number the result obtained should be either 0 or divisible by 17.

Example.22 Check whether 357 is divisible by 17 or not.

Solution : 357, (7 x 5) – 35 = 0, i.e. 357 is divisible by 17.

Rule for 19 : Double the last digit of given number and add to remaining number The result obtained should be divisible by 19.

Example.23 Check whether 589 is divisible by 19 or not.

Solution : 589, (9 x 2) + 58 = 76 (divisible by 19), i.e. the number is divisible by 19.

Example.24 Find the smallest number of six digits which is exactly divisible by 111.

Solution : Smallest number of 6 digits is 100000.
On dividing 100000 by 111, we get 100 as remainder.
$\therefore$ Number to be added = (111 – 100) = 11.
Hence, required number = 100011.

Example.25 Find the largest four digit number which when reduced by 54, is perfectly divisible by all even natural numbers less than 20.

Solution : Even natural numbers less than 20 are 2, 4, 6, 8, 10, 12, 14, 16, 18.
Their LCM = 2 × LCM of first 9 natural numbers = 2 × 2520 = 5040.
This happens to be the largest four-digit number divisible by all even natural numbers less than 20.
54 was subtracted from our required number to get this number.
Hence, (required number – 54) = 5040 $\Rightarrow$ Required number = 5094.

Example.26 Ajay multiplied 484 by a certain number to get the result 3823a. Find the value of 'a'.

Solution : 3823a is divisible by 484, and 484 is a factor of 3823a.
4 is a factor of 484 and 11 is also a factor of 484.
Hence, 3823a is divisible by both 4 and 11.
To be divisible by 4, the last two digits have to be divisible by 4.
'a' can take two values 2 and 6.
38232 is not divisible by 11, but 38236 is divisible by 11.
Hence, 6 is the correct choice.

Example.27 Which digits should come in place of $*$ and \$ if the number 62684$*$\$ is divisible by both 8 and 5 ?

Solution : Since the given number is divisible by 5, so 0 or 5 must come in place of \$. But, a number ending with 5 in never divisible by 8. So, 0 will replace \$.
Now, the number formed by the last three digits is 4$*$0, which becomes divisible by 8, if $*$ is replaced by 4 or 8.
Hence, digits in place of $*$ and \$ are (4 or 8 or 0) and 0 respectively.

(E) REMAINDERS

The method of finding the remainder without actually performing the process of division is termed as **remainder theorem**.

❖ Remainder should always be positive. For example if we divide –22 by 7, generally we get –3 as quotient and –1 as remainder. But this is wrong because remainder is never negative hence the quotient should be – 4 and remainder is +6. We can also get remainder 6 by adding –1 to divisor 7 (7–1 = 6).

Solved Examples

Example.28 Two numbers, x and y, are such that when divided by 6, they leave remainders 4 and 5 respectively. Find the remainder when $(x^2 + y^2)$ is divided by 6.

Solution : Suppose $x = 6k_1 + 4$ and $y = 6k_2 + 5$

$x^2 + y^2 = (6k_1 + 4)^2 + (6k_2 + 5)^2 = 36k_1^2 + 48k_1 + 16 + 36k_2^2 + 60k_2 + 25$

$= 36k_1^2 + 48k_1 + 36k_2^2 + 60k_2 + 41$

Obviously when this is divided by 6, the remainder will be 5.

Example.29 A number when divided by 259 leaves a remainder 139. What will be the remainder when the same number is divided by 37 ?

Solution : Let the number be P.

So, P – 139 is divisible by 259.

Let Q be the quotient then, $\frac{P-139}{259} = Q \Rightarrow P = 259Q + 139 \quad \Rightarrow \frac{P}{37} = \frac{259Q+139}{37}$

$\because$ 259 is divisible by 37,

$\therefore$ When 139 divided by 37, leaves a remainder of 28.

Example.30 A number being successively divided by 3, 5 and 8 leaves remainders 1, 4 and 7 respectively. Find the respective remainders if the order of divisors be reversed.

Solution :

3	x	
5	y	1
8	z	4
	1	7

$\therefore$ z = (8 × 1 + 7) = 15 ; y = (5z + 4) = (5 × 15 + 4) = 79 ; x = (3y + 1) = (3 × 79 + 1) = 238.

Now,

8	238	
5	29	6
3	5	4
	1	2

$\therefore$ Respective remainders are 6, 4, 2.

Example.31 A number was divided successively in order by 4, 5 and 6. The remainders were respectively 2, 3 and 4. Then find out the number.

Solution :

4	x	
5	y	2
6	z	3
	1	4

$\therefore$ z = (6 × 1 + 4) = 10

$\therefore$ y = (5 × z + 3) = (5 × 10 + 3) = 53

$\therefore$ x = (4 × y + 2) = (4 × 53 + 2) = 214

Hence, the required number is 214.

Example.32 In dividing a number by 585, a student employed the method of short division. He divided the number successively by 5, 9 and 13 (factors of 585) and got the remainders 4, 8 and 12. If he had divided number by 585, then find out the remainder.

Solution :

5	x	
9	y	4
13	z	8
	1	12

Now, 1169 when divided by 585 gives remainder = 584.

To find the remainder of big number

❖ **NOTE :**

(i) Binomial Expansion :

$(a + b)^n = a^n + \frac{n}{1!} a^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + ... + b^n$, or

$(a - b)^n = a^n - \frac{n}{1!} a^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 - .....+ (-1)^n b^n$.

Hence, first term is purely of a i.e a^n and last digit is purely of b, i.e. b^n.

(ii) Total number of terms in the expansion of $(a + b)^n$ is $(n + 1)$.

Solved Examples

Example.33 What is the remainder when 7^{38} is divided by 48.

Solution : $\frac{7^{38}}{48} = \frac{(7^2)^{19}}{48} = \frac{(49)^{19}}{48} = \frac{(48+1)^{19}}{48}$ so by using binomial expansion, we can say that 18 terms are completely divisible by 48 but the last term which is $\frac{(+1)^{19}}{48}$ is not divisible.

So, $1^{19} = 1$ is the remainder.

Example.34 What is the remainder if 7^{25} is divided by 4?

Solution : 7^{25} can be written $(8-1)^{25}$. There are 26 terms in all and first 25 terms are divisible by 8, hence also by 4. The last term is $(-1)^{25}$. Hence, $(8 - 1)^{25}$ can be written 8X – 1 or 4Y –1 (where Y = 2X). So, 4Y – 1 divided by 4 leaves the remainder 3.

Example.35 What is the remainder if 3^{45} is divided by 8 ?

Solution : 3^{45} can be written as $9^{22} \times 3$. 9 can be written as (8 + 1). Hence, any power of 9 can be written as 8N + 1. In other words, any power of 9 is 1 more than a multiple of 8. Hence, (8N + 1) × 3 leaves remainder 3 when divided by 8.

Example.36 What is the remainder when $14^{15^{16}}$ is divided by 5 ?

Solution : $14^{15^{16}} = (15 - 1)^{odd} = 15n + (-1)^{odd}$, i.e. a (multiple of 5) –1. Thus when divided by 5 the remainder will be (–1), i.e. 4.

Example.37 What is the remainder when $3^{57} + 27$ is divided by 28?

Solution : $3^{57} = (3^3)^{19}$

$\therefore 3^{57} + 27 = (27)^{19} + 27 = (28 - 1)^{19} + 27$

$= 28M + (-1)^{19} + 27$ [Expand by binomial theorem]

$= 28M - 1 + 27 = 28M + 26$

When 28M + 26 divided by 28, the remainder is 26.

Hence, the required remainder is 26.

Example.38 What is the remainder when $82^{361} + 83^{361} + 84^{361} + 85^{361} + 86^{361}$ is divided by 7?

Solution : $82^{361} + 83^{361} + 84^{361} + 85^{361} + 86^{361} = [(84 - 2)^{361} + (84 - 1)^{361} + 84^{361} + (84 + 1)^{361} + (84 + 2)^{361}]$

Since, 84 is a multiple of 7, then the remainder will be when, $(-2)^{361} + (-1)^{361} + 1^{361} + 2^{361}$ is divided by 7 is $(-2)^{361} + (-1)^{361} + 1^{361} + 2^{361} = 0$. So the remainder is zero.

F. CYCLICITY

We are having 10 digits in our number systems and some of them shows special characteristics like they, repeat their unit digit after a cycle, for example 1 repeat its unit digit after every consecutive power. So, its cyclicity is 1 on the other hand digit 2 repeat its unit digit after every four power, hence the cyclicity of 2 is four. The cyclicity of digits are as follows :

Digit	Cyclicity
0, 1, 5 and 6	1
4 and 9	2
2, 3, 7 and 8	4

So, if we want to find the last digit of 2^{45}, divide 45 by 4. The remainder is 1 so the last digit of 2^{45} would be same as the last digit of 2^1 which is 2.

To Find the Unit Digit in Exponential Expressions :

(i) When there is any digit of cyclicity 4 in unit's place.

Since, when there is 2 in unit's place then in 2^1 unit digit is 2, in 2^2 unit digit is 4, in 2^3 unit digit is 8, in 2^4 unit digit is 6, after that the unit's digit repeats. e.g. unit digit $(12)^{12}$ is equal to the unit digit of, 2^4 i.e.6.

Solved Examples

Example.39 In $(32)^{33}$ unit digit is equal to the unit digit of 2^1 i.e. 2.

Example.40 In $(23)^{15}$ unit digit is equal to the unit digit of 3^3 i.e. 7.

Example.41 In $(57)^9$ unit digit is equal to the unit digit of 7^1 i.e. 7.

Example.42 In $(678)^{22}$ unit digit is equal to the unit digit of 8^2 i.e. 4.

(ii) When there is any digit of cyclicity 2 in unit's place.

Since, when there is 4 in unit's place then in 4^1 unit digit is 4, in 4^2 unit digit is 6 and so on.

Solved Examples

Example.43 In $(34)^{33}$ unit digit is 4.

Example.44 In $(29)^{15}$ unit digit is 9.

Example.45 In $(49)^{18}$ unit digit is 1.

(iii) When there is any digit of cyclicity 1 in unit's place.

Since, when there is 5 in unit's place then in 5^1 unit digit is 5, in 5^2 unit digit is 5 and so on.

Solved Examples

Example.46 In $(25)^{15}$ unit digit is 5.

Example.47 In $(46)^{13}$ unit digit is 6.

Example.48 Find the last digit of

(i) 3^{57} **(ii)** 13^{59}

Solution : **(i)** The cyclicity of 3 is 4. Hence, $\frac{57}{4}$ gives the remainder 1. So, the last digit of 3^{57} is same as the last digit of 3^1, i.e. 3.

(ii) The number of digits in the base will not make a difference to the last digit. It is last digit of the base which decides the last digit of the number itself. For 13^{59}, we find $\frac{59}{4}$ which gives a remainder 3. So the last digit of 13^{59} is same as the last digit of 3^3, i.e. 7.

Example.49 Find unit's digit in $y = 7^{17} + 7^{34}$

Solution : $7^{17} + 7^{34} = 7^1 + 7^2 = 56$, Hence the unit digit is 6

Example.50 What will be the last digit of $(73)^{75^{64^{76}}}$

Solution : Let $(73)^{75^{64^{76}}} = (73)^x$ where $x = 75^{64^{76}} = (75)^{\text{even power}}$

$\because$ Cyclicity of 3 is 4

$\therefore$ To find the last digit we have to find the remainder when x is divided by 4.

$x = (75)^{\text{even power}} = (76 - 1)^{\text{even power}}$, where n is divided by 4 so remainder will be 1.

Therefore, the last digit of $(73)^{75^{64^{76}}}$ will be $3^1 = 3$.

Example.51 What will be the unit digit of $(87)^{75^{63^{55}}}$.

Solution : Let $(87)^{75^{63^{55}}} = (87)^x$ where $x = 75^{63^{55}} = (75)^{\text{odd}}$

$\because$ Cyclicity of 7 is 4.

$\therefore$ To find the last digit we have to find the remainder when x is divided by 4.

$x = (75)^{\text{odd power}} = (76 - 1)^{\text{odd power}}$

where x is divided by 4 so remainder will be –1 or 3, but remainder should be always positive.

Therefore, the last digit of $(87)^{75^{63^{55}}}$ will be $7^3 = 343$.

Hence, the last digit is of $(87)^{75^{63^{55}}}$ is 3.

G. HIGHEST POWER DIVIDING A FACTORIAL

Factorial n : Product of **n** consecutive natural numbers is known as **'factorial n'** it is denoted by **'n!'**.

So, n! = n.(n – 1).(n – 2)...3.2.1. e.g. 5! = 5 × 4 × 3 × 2 × 1 = 120.

❖ The value of factorial zero is equal to the value of factorial one. Hence **0! = 1 = 1!**

The approach to finding the highest power of **x** dividing **y!** is $\left[\frac{y}{x}\right] + \left[\frac{y}{x^2}\right] + \left[\frac{y}{x^3}\right]$ ....., where **[ ]** represents just the integral part of the answer and ignoring the fractional part.

Example.52 What is the highest power of 2 that divides 20! completely?

Solution : 20! = 1 × 2 × 3 × 4 ×....× 18 × 19 × 20 = 1 × (2^1) × 3 × (2^2) × 5 × (2^1 × 3^1) × 7 × (2^3) × ..... so on. In order to find the highest power of 2 that divides the above product, we need to find the sum of the powers of all 2 in this expansion. All numbers that are divisible by 2^1 will contribute 1 to the exponent of 2 in the product $\frac{20}{2^1}$ = 10. Hence, 10 numbers contribute 2^1 to the product. Similarly, all numbers that are divisible by 2^2 will contribute an extra 1 to the exponent of 2 in the product, i.e $\frac{20}{2^2}$ = 5.

Hence, 5 numbers contribute an extra 1 to exponents. Similarly, there are 2 numbers that are divisible by 2^3 and 1 number that is divisible by 2^4. Hence, the total 1s contributed to the exponent of 2 in 20! is the sum of (10 + 5 +2 +1) = 18. Hence, group of all 2s in 20! gives 2^{18} x (N), where N is not divisible by 2.

If 20! is divided by 2^x then maximum value of x is 18.

Example.53 What is the highest power of 5 that divides of x = 100! = 100 × 99 × 98 × ..... × 3 × 2 × 1.

Solution : Calculating contributions of the different powers of 5, we have = $\frac{100}{5^1}$ = 20, $\frac{100}{5^2}$ = 4.

Hence, the total contributions to the power of 5 is 24, or the number 100! is divisible by 5^{24}.

Example.54 How many zeros at the end of first 100 multiples of 10.

Solution : First 100 multiple of 10 are = 10 × 20 × 30 × .....× 1000

= 10^{100} (1 × 2 × 3 × .....× 100)

= 10^{100} × 10^{24} × N

= 10^{124} × N

Where N is not divisible by 10

So, there are 124 zero at the end of first 100 multiple of 10.

Example.55 What is the highest power of 6 that divides 9!

Solution : By the normal method. $\frac{9}{6}$ = 1 and $\frac{9}{6^2}$ = 0. Thus answers we get is 1 which is wrong. True there is just one multiple of 6 from 1 to 9 but the product 2 × 3 = 6 and also 4 × 9 = 36, can further be divided by 6. Thus, when the divisor is a composite number find the highest power of its prime factors and then proceed. In this case, 9! can be divided by 2^7 and 3^4 and thus by 6^4 (In this case we need not have checked power of 2 as it would definitely be greater than that of 3).

Example.56 What is the largest power of 12 that would divide 49! ?

Solution : To check the highest power of 12 in 49!, we need to check the highest powers of 4 and 3 in it.

Highest power of 3 in 49! = 22

Highest power of 2 in 49! = 46

$\therefore$ Highest power of 4 in 49! = $\frac{46}{2}$ = 23

$\therefore$ Highest power of 12 will be 22. (Since the common power between 3 and 4 is 22).

Example.57 How many zeros will be there at the end of $36!^{36!}$?

Solution : Highest power of 5 in 36! is 8.

So, there will be 8 zeros at the end of 36!.

So, at the end of $36!^{36!}$, there will be 8 × 36! zeros.

H. BASE SYSTEM

The number system that we work in is called the **'decimal system'.** This is because there are 10 digits in the system 0-9. There can be alternative system that can be used for arithmetic operations. Some of the most commonly used systems are : binary, octal and hexadecimal.
These systems find applications in computing.

Binary system has 2 digits : 0, 1.

Octal system has 8 digits : 0, 1,..., 7.

Hexadecimal system has 16 digits : 0, 1, 2,..., 9, A , B, C, D, E, F.
After 9, we use the letters to indicate digits. For instance, A has a value 10, B has a value 11, C has a value 12,... so on in all base systems.
The counting sequences in each of the systems would be different though they follow the same principle.

Conversion : Conversion of numbers from (i) decimal system to other base system. (ii) other base system to decimal system.

(i) Conversion from base 10 to any other base :

Solved Examples

Example.58 Convert $(122)_{10}$ to base 8 system.

Solution :

8	122	
8	15	2
8	1	7
	0	1

The number in decimal is consecutively divided by the number of the base to which we are converting the decimal number. Then list down all the remainders in the reverse sequence to get the number in that base.
So, here $(122)_{10} = (172)_8$.

Example.59 Convert $(169)_{10}$ in base 7.

Solution :

7	169	
7	24	1
7	3	3
	0	3

$(169)_{10} = (331)_7$

Example.60 Convert $(0.3125)_{10}$ to binary equivalent.

Solution :

	Integer
$2 \times 0.3125 = [0.625]$	0
$2 \times 0.625 = [1.25]$	1
$2 \times 0.25 = [0.50]$	0
$2 \times 0.50 = [1.00]$	1

Thus
$(0.3125)_{10} = (0.0101)_2$

Example.61 Convert $(1987.725)_{10} \longrightarrow (.....)_8$

Solution : First convert non-decimal part into base 8.

8	1987	
8	248	3
8	31	0
8	3	7
	0	3

$\therefore (1987)_{10} = (3703)_8$

Now we have to convert $(0.725)_{10} \longrightarrow (.....)_8$

Multiply

0.725 × 8 = [5.8] ...5
0.8 × 8 = [6.4] ...6
0.4 × 8 = [3.2] ...3
0.2 × 8 = [1.6] ...1
0.6 × 8 = [4.8] ...4

Keep on accomplishing integral parts after multiplication with decimal part till decimal part is zero.

$\therefore$ $(0.725)_{10} = (0.56314...)_8$

$\therefore$ $(1987.725)_{10} = (3703.56314...)_8$

(ii) Conversion from any other base to decimal system :

Example.62 Convert $(231)_8$ into decimal system.

Solution : $(231)_8$, the value of the position of each of the numbers (as in decimal system) is :

$1 = 8^0 \times 1$
$3 = 8^1 \times 3$
$2 = 8^2 \times 2$
Hence, $(231)_8 = (8^0 \times 1 + 8^1 \times 3 + 8^2 \times 2)_{10}$
$(231)_8 = (1 + 24 + 128)_{10}$
$(231)_8 = (153)_{10}$

Example.63 Convert $(0.03125)_{10}$ to base 16.

Solution : $16 \times 0.03125 = 0.50$
$16 \times 0.5 = 8.08$
So $(0.03125)_{10} = (0.08)_{16}$

Example.64 Convert $(761.56)_8 \longrightarrow (.....)_{16}$

Solution : In such conversion which are standard form conversions, it is easier to
$(761.56)_8 \longrightarrow (.....)_2 \longrightarrow (.....)_{16}$
Converting every digit in base 8 to base 2,
$(111110001.101110)_2 \longrightarrow (1F1.B8)_{16}$

Example.65 Convert $(3C8.08)_{16}$ to decimal

Solution : $(3C8.08)_{16} = 3 \times 16^2 + C \times 16^1 + 8 \times 16^0 + 0 \times 16^{-1} + 8 \times 16^{-2}$
$= 768 + 192 + 8 + 0 + 0.03125 = (968.03125)_{10}$
So, $(3C8.08)_{16} = (968.03125)_{10}$

I. ALPHA NUMERICS PUZZLES

Example.66 If a – b = 2, and $\begin{array}{r} a\ \ a \\ +b\ \ b \\ \hline c\ c\ 0 \end{array}$ then find the value of a, b and c.

Solution : These problems involve basic number
(i) aa + bb = 11(a + b) **(ii)** aa, bb are two-digit numbers.
Hence, their sum cannot exceed 198. So, c must be 1.
(iii) Hence, cc0 = 110. This implies a + b = 10 or a = 6 and b = 4.
Such problems are part of a category of problems called alpha numerics.

Example.67 If $\begin{array}{r} a\ 3\ b \\ -\ a\ c \\ \hline a\ a\ 9 \end{array}$ then find a, b and c if each of them is distinctly different digit.

Solution : (i) since the first digit of (a 3 b) is written as it is after subtracting ac carry over from a to 3.
(ii) there must be a carry over from 3 to b, because if no carry over is there, it means 3 – a = a.

$\Rightarrow \quad 2a = 3 \Rightarrow a = \frac{3}{2}$

which is not possible because a is a digit. For a carry over 1, 2 – a = a

$\Rightarrow \quad a = 1$

(iii) it means b and c are consecutive digit (2, 3), (3, 4),.... (8, 9)

Example.68 The sum of first n natural numbers is a three-digit number, all of whose digits are the same. What is the value of n?

Solution : In 5 seconds, you can solve the equation $\frac{n(n+1)}{2}$ = aaa (111, 222, etc) . How do you proceed next ? If you think it's hit-and-trial from this point, you are wrong. Here goes the simple logic. It might strike you instantly if you have been working with numbers:

$\frac{n(n+1)}{2}$ = aaa = a × 111 = a × 3 × 37 $\Rightarrow$ n(n + 1) = 6a x 37

Look at the L.H.S. of the equation, n(n + 1) is a product of two consecutive natural numbers. Therefore, R.H.S. should also be a product of two consecutive natural numbers. One of the numbers is 37. Therefore, what could the other number 6a, consecutive to 37 be? It can only be 36, giving a = 6 and n = 36. Therefore, 36 numbers have been summed up and their sum is equal to 666.

Example.69 If ABC x CBA = 65125, where A, B and C are single digits, then A + B + C = ?

Solution : As the unit digit of the product is 5, therefore, the unit digit of one of the numbers is 5 and the unit digit of the other number is odd. Therefore, AB5 x 5BA = 65125, where A = 1, 3, 5, 7 or 9.

As the product of two three-digit numbers is a five-digit number, and not a six-digit number, A can only be equal to 1. 1B5 x 5B1 = 65125.

The digits sum of both numbers, 1B5 and 5B1 will be same. Therefore, the product would give digit sum of a perfect square. The digit sum on the R.H.S. is 1. Therefore, the digit sum of each number can be 1or 8. Correspondingly B will be 4 or 2 (as digit sum cannot be equal to 1). Keeping B = 2, we can see that 125 x 521 = 65125.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Logarithm

A-1. Solve for 'x' if $2\log_x 7 + \log_{7x}7+3\log_{49x}7 = 0$.

A-2. If $\log_{10} [\log_{10}(\log_{10}x)] = 0$, then find the value of x.

A-3. Find the value of x, if $\log_3(\log_2 x) + 2 \log_9(\log_7 8) = 2$.

A-4. Find x if $\log_{10} 1250 + \log_{10}80 = x$.

A-5 If log 2 = a and log3 = b then [log(1) + log(1 + 3) + log (1 + 3 + 5) + ...+ .....+ log (1 + 3 + 5 + 7 + ... + 19)] – 2[log 1 + log2 + log3 + ... log7] = p + qa + rb where p,q,r are constants. Find the value of p + 2q + 3r if all logs are in base 10.

Section (B) : Factors and Multiples

B-1. If a 3-digit number 'abc' has 3 factors, then find the total number of factors of the 6-digit number abcabc.

B-2. Find the number of natural numbers less than 100 that cannot be written as multiple of perfect square greater than 1.

B-3. If $N = 3^{15} \times 7^{43}$. Then find the number of factors of N^2 which are less than N but not divide N completely.

B-4. If both 11^2 and 3^3 are factors of number N = a × 65804544. Then find the smallest possible value of a.

B-5. A number N^2 has 15 factors. Find all the possible number of factors of N ?

Section (C) : HCF and LCM

C-1. There are two numbers a and b such that a > b, if HCF of (a,b) = h and LCM of (a,b) = ℓ, then find the LCM of (a – b) and b.

C-2. In a large school auditorium, the students are made to sit to watch the programes. If the teachers make a row of student of 16 each, there will be 12 students left. If they make row of 24 each, there will be 20 students left, if they make rows of 25 each, there will be 21 students left and if they make down of 30 each, there will be 26 students left. Find minimum number of students present in school.?

C-3. Find all pairs of positive integer (x,y,) such that HCF (x,y) + LCM (x,y) = 91.

C-4. How many pairs of positive integers (x,y) exist such that HCF of (x,y) = 35 and x + y = 1085?

C-5. Manas and his friend met at nehru place after a long time. Manas stays at vivek vihar and his friend stays in Gurgaon. Both of them commute by bus. They reached the bus stop, got to know that a bus had left just for each of their destinations. Neither wanted to leave the other alone at bus stop. If frequency of buses to Gurgaon was 7 min and that to vivek vihar was 11 min then.
(A) How long would they wait at the bus stop ?
(B) How many buses going to their destinations would each one decide not to board ?

Section (D) : Divisiblity

D-1 Find the total numbers of natural number 'n' for which 111 divides $16^n - 1$ where 'n' is less than 1000.

D-2 How many Positive integers are there from 1 to 1000 such that leaves remainder of 3 on division by 7 and leaves remainder of 2 on division by 4.

D-3 How many numbers between 1 and 1200 both included are not divisible by any of the numbers 2 , 3 and 5?

D-4 How many five digit number multiples of 11 are there, if the five digit be 7 , 6 , 5 , 4 and 3 are in some order.

D-5 The digits 1 , 2 , 3 , 4 and 5 are each used once to compose a five digit number ABCDE , such that the three digit number ABC is divisible by 4, BCD is divisible by 5 and CDE is divsible by 3.
Find the value of the smallest five digit number ABCDE.

Section (E) : Remainders

E-1. Find the remainder when 1234567..... 4647484950 is divided by 16.

E-2. Find the remainder when the $(1!)^2 + (2!)^2 ..... + (100\,!\,)^2$ is divided by 1152.

E-3. Find the remainder when $3^{21} + 9^{21} + 27^{21} + 81^{21}$ is divided by $(3^{20} + 1)$

E-4. Find the remainder when $\underbrace{222.....2}_{300 \text{ times}}$ is divided by 999.

E-5. Find the remainder when 25^{10} is divided by 576.

Section (F) : Cyclicity

F-1. Find the unit digit of the number equal to sum of 1! + 2! + 3! + 4! + ... 100!.

F-2. Find the unit digit of the number equal to sum of $3! + (3!)^2 + (3!)^3$ ..... $+ (3!)^{100}$.

F-3. Find the unit digit of $(378)^{95^{23^{51}}}$.

F-4. Find the unit digit of $x = (35)^{87} + (93)^{46} + (44)^{91}$.

F-5. Find the unit digit of $(2^2 + 4^2 + 6^2 ..... 100^2)^{64}$.

Section (G) : Base System

G-1 In which base system 65 can be represented as 41 ?

G-2 Represent $0.\overline{n}$ in the form of p/q ($q \neq 0$, p & q are integers). 'n' is the base of the number 2122 which is equal to $(353)_4$.

G-3 A number of three digits in scale 7, when expressed in scale 9 has its digits reversed in order. Find the number in scale 10.

G-4 Find the value of when $(25)_x = (85)_{10}$

G-5 If the decimal number 2^{125} is written in the hexadecimal system then find the 1st digit from left .

Section (H) : Alpha Numerics Puzzles

H-1. In the squence b, a , 4, 6, 22 .... .. each term is sum of the difference of two terms to its left & their product . Find a and b.

H-2. A number AAA when multliplied by 4 and added to a number CCC gives DDD, which is divisible by CCC. Find the numbers AAA, CCC, DDD , where A, C and D are digits from 0 to 9.

H-3. Find the four-digit number ABCD such that ABCD x 4 = DCBA.

H-4. abc is a three digit number. ab, bc, ca are two digit numbers. Determine all three digit numbers abc such that abc = ab + bc + ca

H-5. Find all two digit numbers ab such that $a + b \geq 7$, $a^2 + b^2 \leq 30$ and $ba \leq \frac{1}{2}$ ab.

OBJECTIVE QUESTIONS

Section (A) : Logarithm

A-1. If $\log_x y = 100$ and $\log_2 x = 10$, then the value of y is :
(A) 2^{1000} (B) 2^{100} (C) 2^{2000} (D) 2^{10000}

A-2. $\log_{10}\left(1+\frac{1}{2}\right) + \log_{10}\left(1+\frac{1}{3}\right) + \log_{10}\left(1+\frac{1}{4}\right) + ... + \log_{10}.\left(1+\frac{1}{1999}\right)$ When simplified has the value equal to :
(A) 1 (B) 3 (C) 10 (D) 100

A-3. If $\log_{10}N = 2.5$ then, find out total number of digits in N.
(A) 3 (B) 4 (C) 5 (D) cannot be determine

A-4. If log x = n then 2n is equal to :
(A) $\log (x^2)$ (B) $(\log x)^2$ (C) log (x+2) (D) log 2x

A-5. Given log2 = 0.3010, then log 16 is :
(A) 2.4080 (B) 1.2040 (C) 0.2408 (D) 1.9030

Section (B) : Factors and Multiples

B-1. The number of prime factors of $(3 \times 5)^{12} (2 \times 7)^{10} (10)^{25}$ is :
(A) 47 (B) 60 (C) 72 (D) 94

B-2. Find the total number of prime factors in the expression $(4)^{11} \times (7)^5 \times (11)^2$.
(A) 37 (B) 33 (C) 26 (D) 29

B-3. How many two digits numbers are there having 6 factors:
(A) 16 (B) 17 (C) 20 (D) None of these

B-4. Let N be a natural number having 4 factors, If sum of all factors excluding N is 31. then the number of possible values of N is :
(A) 3 (B) 4 (C) 5 (D) None of these

B-5. Find the Smallest positive integer with six positive odd integer divisors and 12 positive even integer Divisors
(A) 630 (B) 990 (C) 180 (D) None of these

Section (C) : HCF and LCM

C-1. How many three-digit numbers would you find, which when divided by 3, 4, 5, 6, 7 leave the remainders 1, 2, 3, 4, and 5 respectively ?
(A) 4 (B) 3 (C) 2 (D) 1

C-2. Six strings of violin start vibrating simultaneously and they vibrate at 3, 4, 5, 6,10 and 12 times in a minute, find :
i. After how much time will all six of them vibrate together ?
ii. How many times will they vibrate together in 30 min ?
(A) 60 min, 31 times (B) 60 sec, 31 times (C) 120 sec, 15 times (D) None of these

C-3. The HCF of 2 numbers is 11 and their LCM is 693. If their sum is 176, find the numbers.
(A) 99,77 (B) 110, 66 (C) 88,77 (D) 121, 44

C-4. If P is a prime number, then the LCM of P and (P + 1) is
(A) P(P +1) (B) (P + 2)P (C) (P + 1)(P – 1) (D) None of these

C-5. The LCM of two numbers is 567 and their HCF is 9. If the difference between the two numbers is 18, find the two numbers :
(A) 36 and 18 (B) 78 and 60 (C) 63 and 81 (D) 52 and 34

Section (D) : Divisiblity

D-1. How many numbers between 200 and 600 are divisible by 14?
(A) 28 (B) 29 (C) 27 (D) None of these

D-2. How many natural numbers between 200 and 400 are there which are divisible by
i. Both 4 and 5?
ii. 4 or 5 or 8 or 10 ?
(A) 9, 79 (B) 10, 80 (C) 10, 81 (D) None of these

D-3. $4^{61} + 4^{62} + 4^{63} + 4^{64}$ is divisible by :
(A) 3 (B) 10 (C) 11 (D) 13

D-4. If x is a whole number, then $x^2 (x^2 – 1)$ is always divisible by :
(A) 12 (B) 24 (C) 12 – x (D) Multiple of 12

D-5. If 653 xy is exactly divisible by 80, then the find the value of (x + y).
(A) 2 (B) 3 (C) 4 (D) 6

D-6. When a number P is divided by 4 it leaves remainder 3. If the twice of the number P is divided by the same divisor 4 than what will be the remainder ?
(A) 0 (B) 1 (C) 2 (D) 6

D-7. Smallest number which should be added to the number 803642 so that it is divisible by 11.
(A) 40 (B) 4 (C) 23 (D) 7

D-8. If the number 357y25x is divisible by both 3 and 5, then find the missing digit in the unit's place and the thousand place respectively are :
(A) 0, 6 (B) 5, 6 (C) 5, 4 (D) None of these

D-9. A 4-digit number is formed by repeating a 2-digit number such as 2525, 3232 etc. Any number of this from is exactly divisible by :
(A) 7 (B) 11
(C) 13 (D) Smallest 3-digit prime number

D-10. The largest number which exactly divides the product of any four consecutive natural numbers is :
(A) 6 (B) 12 (C) 24 (D) 120

Section (E) : Remainders

E-1. What is the remainder when 9875347 × 7435789 × 5789743 is divided by 4 ?
(A) 1 (B) 2 (C) 3 (D) None of these

E-2. P is a prime number greater than 5. What is the remainder when P is divided by 6?
(A) 5 (B) 1 (C) 1 or 5 (D) None of these

E-3. What is the remainder when 6^{50} is divided by 215?
(A) 1 (B) 36 (C) 5 (D) 214

E-4. What is the remainder when $74^{13} – 41^{13} + 75^{13} – 42^{13}$ is divided by 66?
(A) 2 (B) 64 (C) 1 (D) 0

E-5. Let N = 1421 × 1423 × 1425. What is the remainder when N is divided by 12?
(A) 0 (B) 9 (C) 3 (D) 6

Section (F) : Cyclicity

F-1. Find the unit digit of $(7^{95} – 3^{58})$.
(A) 6 (B) 4 (C) 3 (D) None of these

F-2. If n = 67 then find the unit digit of $[3^n + 2^n]$.
(A) 1 (B) 10 (C) 5 (D) None

F-3. The last digit of $(1^3 + 2^3 + 3^3 + ... 10^3)^{64}$ is :
(A) 2 (B) 5 (C) 9 (D) 0

F-4. Sum of the Unit digit of the number 2^{2006} and 5^{2007} is equal to
(A) 6 (B) 9 (C) 5 (D) 1

F-5. What is the last digit of the number equal to the sum $1 + 6 + 6^2 + ----- + 6^{100}$?
(A) 0 (B) 1 (C) 2 (D) 6

Section (G) : Base System

G-1. When $(55)_{10}$ is represented in base 25 then the expression is :
(A) $(25)_{25}$ (B) $(35)_{25}$ (C) $(55)_{25}$ (D) none of these

G-2. Convert the number 1982 from base 10 to base 12. The results is :
(A) 1182 (B) 1912 (C) 1192 (D) 1292

G-3. What is the decimal equivalent of the 25 digits of hexadecimal number $(100.....001)_{16}$?
(A) $2^{23} + 1$ (B) $2^{24} + 1$ (C) $2^{92} + 1$ (D) $2^{96} + 1$

G-4 The number 35A246772 is in base 9. This number is divisible by 8, then the value of digit A.
(A) 5 (B) 4 (C) 3 (D) 2

G-5 In what base is the equation 53 × 15 = 732 valid?
(A) 15 (B) 14 (C) 13 (D) 12

Section (H) : Alpha Numerics Puzzles

H-1. A 3-digit number 4a3 is added to another 3-digit number 984 to give the four-digit number 13b7, which is divisible by 11. Then ,(a + b) is :
(A) 10 (B) 11 (C) 12 (D) 15

H-2. If aabb is a four digit number and also a perfect square then the value of a + b is :
(A) 12 (B) 11 (C) 10 (D) 9

H-3. The sum of the digits of two digit number is 11, if the digits are reversed the number decreases by 45. The number is :
(A) 38 (B) 65 (C) 74 (D) 83

H-4 The value of A + B + C, if ABC=A!+B!+C!, where A, B , C are digits from 0 to 9 and ABC is a three digit number
(A) 12 (B) 10 (C) 14 (D) None of these

H-5 If the decimal number 2^{111} is written in the octal system, then find its unit place digit ?
(A) 0 (B) 2 (C) 4 (D) None of these

Section (I) MISCELLANEOUS

I-1. Which of the following surds is greatest in magnitude
$\sqrt[6]{17}, \sqrt{2}, \sqrt[12]{25}, \sqrt[3]{4}$.
(A) $\sqrt[3]{17}$ (B) $\sqrt[12]{25}$ (C) $\sqrt[3]{4}$ (D) $\sqrt{2}$

I-2. When **Sholey** screened on the TV there was a commercial break of 5 min after every 15 min of the movie. If from the start of the movie to the end of the movie there was in all 60 min of commercials that was screened what is the duration the movie ?
(A) 180 min (B) 195 min (C) 169 min (D) 165 min

I-3. How many numbers between 400 and 600 begin with or end with a digit of 5 ?
(A) 40 (B) 100 (C) 110 (D) 120

I-4. Anita had to do a multiplication. Instead of taking 35 as one of the multipliers, she took 53. As a result, the product went up by 540. What is the new product?
(A) 1050 (B) 540 (C) 1440 (D) 1590

I-5. What is the value of the following expression $\left(\frac{1}{(2^2-1)}\right)+\left(\frac{1}{(4^2-1)}\right)+\left(\frac{1}{(6^2-1)}\right)+.....+\left(\frac{1}{(20^2-1)}\right)$?
(A) $\frac{9}{19}$ (B) $\frac{10}{19}$ (C) $\frac{10}{21}$ (D) $\frac{11}{21}$

I-6. $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{6}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{8}\right)$ is equal to :
(A) 9 (B) 8 (C) 4.5 (D) None of these

I-7. A rich merchant had collected many gold coins. He did not want any body to know about him. One day, his wife asked, "How many gold coins do we have?" After pausing a moment he replied, "Well ! if divide the coins into two unequal numbers, then 48 times the difference between the two numbers equals the difference between the square of the two numbers. " The wife looked puzzled. Can you help the merchant's wife by finding out how many gold coins the merchant has ?
(A) 96 (B) 53 (C) 43 (D) 48

I-8. Convert 0.225 in to form p/q.
(A) $\frac{3}{10}$ (B) $\frac{9}{40}$ (C) $\frac{9}{50}$ (D) $\frac{9}{400}$

I-9. Arrange the following rational number in ascending order $\frac{3}{7}, \frac{4}{5}, \frac{7}{9}, \frac{1}{2}$.
(A) $\frac{4}{5}, \frac{7}{5}, \frac{3}{9}, \frac{1}{2}$ (B) $\frac{3}{7}, \frac{1}{2}, \frac{7}{9}, \frac{4}{5}$ (C) $\frac{4}{5}, \frac{7}{9}, \frac{1}{2}, \frac{3}{7}$ (D) $\frac{1}{2}, \frac{3}{7}, \frac{7}{9}, \frac{4}{5}$

I-10. Let D be a recurring decimal of the form D = 0. a_1 a_2 a_1 a_2 a_1 a_2 ....., where digits a_1 and a_2 lie between 0 and 9. Further, at most one of them is zero. Which of the following numbers necessarily produces an integer, when multiplied by D?
(A) 18 (B) 108 (C) 198 (D) 288

I-11. If a, a + 2, and a + 4 are prime numbers, then the number of possible solution for a is :
(A) three (B) two (C) one (D) more than three

I-12. If least prime factor of a number m is 3 and least prime factor of another number n is 7, then least prime factor of the number (m + n) is :
(A) 2 (B) 3 (C) 5 (D) 7

I-13. There are four prime numbers written in ascending order. The product of the first three is 385 and that of the last three is 1001. The last number is :
(A) 11 (B) 13 (C) 17 (D) 19

I-14. If $(1^2 + 2^2 + 3^2 + ...+ 10^2) = 385$, then the value of $(2^2 + 4^2 + 6^2 +..... + 20^2)$.
(A) 770 (B) 1155 (C) 1540 (D) (385 × 385)

I-15. If n^2 = 12345678987654321, find the value of n ?
(A) 12344321 (B) 1235789 (C) 11111111 (D) 111111111

Exercise-2

OBJECTIVE QUESTIONS

1. The digit at the 100th place in the decimal representation of $\frac{6}{7}$, is :
(A) 1 (B) 2 (C) 4 (D) 5

2. The number of positive n in the range $12 \le n \le 40$ such that the product (n –1) (n – 2).... 3.2.1 is not divisible by n is :
(A) 5 (B) 7 (C) 13 (D) 14

3. The positive integers A, B, A – B and A + B are all prime numbers. The sum of these four primes is
(A) even (B) divisible by 3 (C) divisible by 5 (D) prime

4. Total number of factors of $(3 \times 5)^{12} (2 \times 7)^{10} (10)^{25}$ is :
(A) 47 (B) 94 (C) 72 (D) 195624

Directions : (5 to 9) Read the following information carefully and answer the questions given below.
In a big hostel, there are 1,000 rooms. In that hostel only even numbers are used for room numbers, i.e. the room numbers are 2, 4, 6, ..., 1998, 2000. All the rooms have one resident each. One fine morning, the warden calls all the residents and tells them to go back to their rooms as well as multiples of their room numbers. When a guy visits a room and finds the door open, he closes it, and if the door is closed, he opens it, All 1,000 guys do this operation. All the doors were open initially.

5. The last room that is closed is room number ?
(A) 1936 (B) 2000 (C) 1922 (D) None of these

6. The 38th room that is open is room number :
(A) 80 (B) 88 (C) 76 (D) None of these

7. If only 500 guys, i.e. residents of room number 2 to 1000 do the task, then the last room that is closed is room number
(A) 2000 (B) 1936 (C) 1849 (D) None of these

8. In the case of the previous question, how many rooms will be closed in all ?
(A) 513 (B) 31 (C) 13 (D) 315

9. If you are a lazy person, you would like to stay in a room whose number is :
(A) more than 500 (B) more than 1000 (C) 500 (D) 2000

10. If $(2^{32} + 1)$ is divisible by a certain number then which of the following is also divisible by that number.
(A) $(2^{16} - 1)$ (B) $2^{16} + 1$ (C) $2^{96} + 1$ (D) None of these

11. $1^{13} + 2^{13} + 3^{13} + ...+ 60^{13}$ is divisible by :
(A) 61 (B) 63 (B) 65 (D) 59

12. A number when divided by 342 gives a remainder 47. When the same number is divided by 19, what would be the remainder ?
(A) 3 (B) 5 (C) 9 (D) None of these

13. In a number system, the product of 44 and 11 is 3414. The number 3111 of this system, when converted to the decimal number system, becomes :
(A) 406 (B) 1086 (C) 213 (D) 691

14. What is remainder when 7^{84} is divided by 2402?
(A) 1 (B) 6 (C) 2401 (D) None of these

15. What is the remainder when 30^{40} is divided by 17?
(A) 1 (B) 16 (C) 13 (D) 4

16. A number when divided successively by 4 and 5 leaves remainders 1 and 4 respectively. When it is successively divided by 5 and 4, then the respective remainders will be :
(A) 1, 2 (B) 2, 3 (C) 3, 2 (D) 4, 1

17. Find out (A + B + C + D) such that AB x CB = DDD, where AB and CB are two-digit numbers and DDD is a three-digit number.
(A) 21 (B) 19 (C) 17 (D) 18

18. If $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$, then :
(A) $M^9 = \frac{9}{N}$ (B) $N^9 = \frac{9}{M}$ (C) $M^3 = \frac{3}{N}$ (D) $N^9 = \frac{3}{M}$

19. Three friends, returning from a movie, stopped to eat at a restaurant. After dinner, they paid their bill and noticed a bowl of mints at the front counter. Sita took 1/3 of the mints, but returned four because she had a monetary pang of guilt. Fatima then took 1/4 of what was left but returned three for similar reasons. Eswari then took half of the remainder but threw two back into the bowl. The bowl had only 17 mints left when the raid was over. How many mints were originally in the bowl?
(A) 38 (B) 31 (C) 41 (D) 48

20. A set of consecutive positive integers beginning with 1 is written on the blackboard. A student came and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What was the number erased?
(A) 7 (B) 8 (C) 9 (D) None of these

21. How many even integers n, where $100 \le n \le 200$, are divisible neither by seven nor by nine ?
(A) 40 (B) 37 (C) 39 (D) 38

22. Three pieces of cakes of weights $4\frac{1}{2}$ lbs, $6\frac{3}{4}$ lbs and $7\frac{1}{5}$ lbs respectively are to be divided into parts of equal weights. Further, each must be as heavy as possible. If one such part is served to each guest, then what is the maximum number of guests that could be entertained ?
(A) 54 (B) 72 (C) 20 (D) 41

23. The largest natural number by which the product of three consecutive even natural numbers is always divisible, is :
(A) 6 (B) 24 (C) 48 (D) 96

24. Let $N = 55^3 + 17^3 - 72^3$, then N is divisible by :
(A) both 7 and 13 (B) both 3 and 13 (C) both 17 and 7 (D) both 3 and 17

25. $7^{6n} - 6^{6n}$, where n is an integer > 0, is divisible by :
(A) 13 (B) 127 (C) 559 (D) All of these

26. Find the square root of $7 - 4\sqrt{3}$.
(A) $2-\sqrt{3}$ (B) $5-\sqrt{3}$ (C) $2-\sqrt{5}$ (D) None of these

27. The value of $\sqrt{10+\sqrt{25+\sqrt{108+\sqrt{154+\sqrt{225}}}}}$ is :
(A) 4 (B) 6 (C) 8 (D) 10

28 On dividing a natural number by 13, the remainder is 3 and on dividing the same number by 21, the remainder is 11. If the numner lies between 500 and 600, then the remainder on dividing the number by 19 is :
(A) 4 (B) 6 (C) 9 (D) 13

29. Expressing $0.\overline{34}+0.3\overline{4}$ as a single decimal , we get
(A) $0.67\overline{88}$ (B) $0.68\overline{9}$ (C) $0.68\overline{78}$ (D) $0.68\overline{7}$

30. The sum of all the possible remainders, which can be obtained when the cube of a natural number is divided by 9, is
(A) 5 (B) 6 (C) 8 (D) 9

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. P, Q and R are three natural numbers such that P and Q are primes and Q divides PR. Then out of the following the correct statement is : **[IJSO-2008]**
(A) Q divides R (B) P divides R (C) P divides QR (D) P divides PQ

2. It is required to decide if 1107 is a prime number or not. The number of trials of division necessary is : **[IJSO-2008]**
(A) 10 (B) 11 (C) 12 (D) 235

3. The number of integers between $-\sqrt{8}$ and $\sqrt{32}$ is : **[NSTSE-2009]**
(A) 5 (B) 6 (C) 7 (D) 8

4. When expanded, the number of zeros in 1000^{10} is : **[NSTSE-2009]**
(A) 13 (B) 30 (C) 4 (D) 10

5. Let N = 28, the sum of All distinct factors of N is : **[IJSO-2009]**
(A) 27 (B) 28 (C) 55 (D) 56

6. The units digit of $(1 + 9 + 9^2 + 9^3 + \text{---------} + 9^{2009})$ is : **[IJSO-2009]**
(A) 0 (B) 1 (C) 9 (D) 3

7. The biggest among the following is : **[IJSO-2009]**
(A) $2^{1/2}$ (B) $3^{1/3}$ (C) $6^{1/6}$ (D) $8^{1/8}$

8. If a, b $\neq$ 1, ab > 0, a $\neq$ b and $\log_b a = \log_a b$, then ab = ? **[IAO- 2009]**
(A) 1/2 (B) 1 (C) 2 (D) 10

9. If $2009 = p^a.q^b$, where "p" and "q" are prime numbers, then find the value of p + q. **[NSTSE 2009]**
(A) 3 (B) 48 (C) 51 (D) 2009

10. If HCF (p, q) = 12 and p × q = 1800 n then LCM (p, q) is : **[NSTSE 2010]**
(A) 3600 (B) 900 (C) 150 (D) 90

11. The value of $\log_{10}(3/2) + \log_{10}(4/3) + \ldots\ldots\ldots$ up to 99 terms. **[IAO 2008]**
(A) 0 (B) 2 (C) 2.5 (D) None of the above

12. In the familiar decimal number system the base is 10. In another number system using base 4, the counting proceeds as 1, 2, 3, 10, 11, 12, 13, 20, 21 .... The twentieth number in this system will be **[IJSO-2010]**
(A) 40 (B) 320 (C) 210 (D) 110

13. Given $\sqrt{600}$ = 24.49, the value of $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ is **[IJSO-2011]**
(A) 9.978 (B) 8.989 (C) 9.989 (D) 9.898

14. If x < 0 and $\log_7(x^2 - 5x - 65) = 0$, then x is **[IJSO-2011]**
(A) –13 (B) –11 (C) – 6 (D) – 5

15. If $2^{2x+1} + 2^{1-2x} = 2$, then the value of x is **[IJSO-2011]**
(A) 0.5 (B) –0.5 (C) 1 (D) 0

16. If the eight digit number 2575d568 is divisible by 54 and 87, the value of the digit 'd' is **[IJSO-2011]**
(A) 4. (B) 7. (C) 0. (D) 8.

17. What will be the remainder if the number 7^{2012} is divided by 25? **[IJSO-2012]**
(A) 24 (B) 18 (C) 7 (D) 1

18. The product of three consecutive natural numbers is 124850054994. What is their average? **[IJSO-2012]**
(A) 4993 (B) 4994 (C) 4997 (D) 4998

19. If $\sqrt[x]{75} = \sqrt[y]{45} = \sqrt[z]{15}$, then which of the statement is true **[IJSO-2012]**
(A) x + y = 2z (B) x + y = 3z (C) x – y = 2z (D) x – y = 3z

20. In a certain number system 363 + 1056 = 1452. Find the value of (654 – 456) in the same number system **[IJSO-2012]**
(A) 156 (B) 165 (C) 178 (D) 198

21. Let T be the number of 4– digit integers, each ending in 3 (in units place) and each divisible by 11.Then **[IJSO-2013]**
(A) $20 \le T \le 79$ (B) $80 \le T \le 89$ (C) $90 \le T \le 99$ (D) $T \ge 100$

22. The number $5\sqrt{41}$ lies between **[IJSO-2013]**
(A) 29 and 30 (B) 30 and 31 (C) 31 and 32 (D) 32 and 33

23. Number plate of a vehicle consists of 4 digits. The first digit is the square of second. The third digit is thrice the second and the fourth digit is twice the second. The sum of all 4 digits is thrice the first. The number is **[IJSO-2014]**
(A) 1132 (B) 4264 (C) 1642 (D) 9396

24. If the highest common factor of a,b and c is 1, where a,b and c belong to the set of natural numbers, then the highest common factor of (a × b) and c is **[IJSO-2014]**
(A) c (B) a × b (C) 1 (D) insufficient data

25. A number x is a rational number if there exists integers p and q such that x = p/q. This is definition of rational numbers in which, **[IJSO-2014]**
(A) both p & q can be zero (B) both p & q should not be zero
(C) q can be zero but not p (D) p can be zero but not q

26. The least positive integer, n, such that 2 divides n, 3 divides n + 1, 4 divides n + 2, 5 divides n + 3 and 6 divides n + 4 is **[IJSO-2014]**
(A) 62 (B) 120 (C) 720 (D) 52

27. The sum of 2 digits x and y is divisible by 7. What can one say about a 3 digit number formed by these two digits. **[IJSO-2014]**
(A) xxy is divisible by 7 (B) xyx is divisible by 7
(C) xyx is divisible by 7^2 (D) yyx is divisible by 7

28. What will be the remainder if the number 7^{2015} is divided by 25? **[IJSO-2015]**
(A) 1 (B) 7 (C) 18 (D) 24

29. What is the sum of all three digit even numbers divisible by seventeen? **[IJSO-2015]**
(A) 18846 (B) 18684 (C) 14688 (D) 16848

30. 8888888 * 8888888 this fifteen digit number is divisible by 22. Find the eighth digit in the number. **[IJSO-2015]**
(A) 1 (B) 3 (C) 5 (D) 8

31. $(41)^{16} - (14)^{16}$ is a multiple of **[IJSO-2015]**
(A) 1485 (B) 1584 (C) 1845 (D) 1854

32. What will be the remainder if the number $(1000000)^3$ is divided by 143? **[IJSO-2015]**
(A) 9 (B) 6 (C) 1 (D) 0

33. If $\sqrt{338} - \sqrt{288} = m$ then m = ? **[IJSO-2015]**
(A) $\sqrt{50}$ (B) $\sqrt{32}$ (C) $\sqrt{18}$ (D) $\sqrt{2}$

34. How many four digit numbers divisible by twenty nine have the sum of their digits 29? **[IJSO-2015]**
(A) 4 (B) 5 (C)) 13 (D) none of these

35. Select any three distinct digits. Form a three digit number. Form the another number by reversing the digits. Find the difference of these two numbers. What is the G.C.D of all such differences? **[IJSO-2015]**
(A) 9 (B) 11 (C) 33 (D) 99

36. 5901AB04 is an eight digit number divisible by 792. Find A + B = ? **[IJSO-2015]**
(A) 6 (B) 7 (C) 8 (D) 9

37. What is the smallest natural number when multiplied by 15 and divided by 63 gives remainder 21? **[IJSO-2015]**
(A) 13 (B) 14 (C) 17 (D) 20

38. How many three digit numbers are divisible by 13 and having middle digit 5 ? **[IJSO-2015]**
(A) 5 (B) 7 (C) 10 (D) 13

39. Number of numbers less than 40 having exactly four divisors is **[IJSO-2016]**
(A) 15 (B) 12 (C) 11 (D) 14

40. Tenth term in the sequence 12, 18, 20, 28, ... is **[IJSO-2016]**
(A) 336 (B) 63 (C) 216 (D) 68

41. Number of integers n such that the number 1 + n is a divisor of the number $1 + n^2$ is **[IJSO-2016]**
(A) 0 (B) 1 (C) 4 (D) 2

42. The number of integers a,b,c for which $a^2 + b^2 - 8c = 3$ is **[IJSO-2016]**
(A) 2 (B) infinite (C) 0 (D) 4

43. How many four digit numbers are there such that when they are divided by 101, they have 99 as remainder ? **IJSO 2017]**
(A) 90 (B) 98 (C) 100 (D) 101

44. By which smallest number we should divide 198396198 to get a perfect square? **[IJSO 2017]**
(A) 14 (B) 18 (C) 22 (D) 28

45. If x = $\left(\sqrt{21}-\sqrt{20}\right)$ and y = $\left(\sqrt{18}-\sqrt{17}\right)$, then **[IJSO 2017]**
(A)x = y (B) x < y (C) x > y (D)x + y = 0

46. What will be the remainder if the number $(7)^{2017}$ is divided by 25? **[IJSO 2017]**
(A) 24 (B) 18 (C) 7 (D) 1

47. The sum of two numbers is 13 and the sum of their cubes is 1066. Find the product of those two numbers. **[IJSO 2017]**
(A) 26 (B) 27 (C) 28 (D) 29

48. What is the largest value of the positive integer k such that k divides $n^2 (n^2 - 1) (n^2 - n - 2)$ for every natural number n ? **[IJSO 2018]**
(A) 6 (B) 12 (C) 24 (D) 48

49. Let n be a positive integer not divisible by 6. Suppose n has 6 positive divisors. The number of positive divisors of 9n is **[IJSO 2018]**
(A) 54 (B) 36 (C) 18 (D) 12

50. Digits a and b are such that the product $\overline{4a1}\times\overline{25b}$ is divisible by 36 (in base 10). The number of ordered pairs (a, b) is **[IJSO 2018]**
(A) 15 (B) 8 (C) 6 (D) 11

51. The integer closest to $\sqrt{111...1-222...2}$, where there are 2018 ones and 1009 twos, is **[IJSO 2018]**
(A) $\frac{10^{1009}-1}{3}$ (B) $\frac{10^{1009}-1}{9}$ (C) $\frac{10^{2018}-1}{3}$ (D) $\frac{10^{2018}-1}{9}$

52. The number of distinct integers in the collection $\left[\frac{10^2}{1}\right],\left[\frac{10^2}{2}\right],\left[\frac{10^2}{3}\right],.....,\left[\frac{10^2}{20}\right]$, where [x] denotes the largest integer not exceeding x, is **[IJSO 2018]**
(A) 20 (B) 18 (C) 17 (D) 15

Answers

Exercise-1

Section (A)

A-1. $x = 7^{-4/3}$, $7^{-1/2}$ **A-2.** $x = 10^{10}$ **A-3.** x = 343. **A-4.** 5. **A-5** 26.

Section (B)

B-1. 16 and 24. **B-2.** 61 numbers. **B-3.** 645 **B-4.** 363 **B-5.** 6 or 8.

Section (C)

C-1. $\frac{(a-b)b}{h}$ **C-2.** 1196 **C-3.** 8

C-4. 15 pairs of x and y **C-5.** **(A)** 77 min. **(B)** 6, 10

Section (D)

D-1 111 **D-2** 36 **D-3** 320 **D-4** 12 **D-5** 12453.

Section (E)

E-1. 6 **E-2.** 41 **E-3.** 60 **E-4.** 222 **E-5.** 241

Section (F)

F-1. 3 **F-2.** 0 **F-3.** 2. **F-4.** 8 **F-5.** 0

Section (G)

G-1 16 **G-2** $\frac{1}{3}$ **G-3** 248 **G-4** 40 **G-5** 2

Section (H)

H-1. a = 2 & b = 2 **H-2.** AAA = {111,222}, CCC = {111,222,444}, DDD = {555,666,888,999}

H-3. 2178 **H-4.** 198 **H-5.** 52

OBJECTIVE QUESTIONS

Section (A)

A-1. (A) **A-2.** (B) **A-3.** (A) **A-4.** (A) **A-5.** (B)

Section (B)

B-1. (D) **B-2.** (D) **B-3.** (A) **B-4.** (B) **B-5.** (C)

Section (C)

C-1. (C) **C-2.** (B) **C-3.** (A) **C-4.** (A) **C-5.** (C)

Section (D)

D-1. (A) **D-2.** (A) **D-3.** (B) **D-4.** (A) **D-5.** (D)

D-6. (C) **D-7.** (D) **D-8.** (B) **D-9.** (D) **D-10.** (C)

Section (E)

E-1.	(A)	**E-2.**	(C)	**E-3.**	(B)	**E-4.**	(D)	**E-5.**	(C)

Section (F)

F-1.	(B)	**F-2.**	(C)	**F-3.**	(B)	**F-4.**	(B)	**F-5.**	(B)

Section (G)

G-1.	(A)	**G-2.**	(C)	**G-3.**	(D)	**G-4**	(B)	**G-5**	(C)

Section (H)

H-1.	(A)	**H-2.**	(B)	**H-3.**	(D)	**H-4**	(B)	**H-5**	(A)

Section (I)

I-1.	(A)	**I-2.**	(B)	**I-3.**	(C)	**I-4.**	(D)	**I-5.**	(C)
I-6.	(C)	**I-7.**	(D)	**I-8.**	(B)	**I-9.**	(B)	**I-10.**	(C)
I-11.	(C)	**I-12.**	(A)	**I-13.**	(B)	**I-14.**	(C)	**I-15.**	(D)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	A	B	D	D	C	B	A	A	B	C	A	C	A	C	A	B	A	B	D	C
Ques.	21	22	23	24	25	26	27	28	29	30										
Ans.	C	D	C	D	D	A	A	A	D	D										

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	AD	B	D	B	D	A	B	B	B	B	D	D	D	C	A	B	D	D	B	B
Ques.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Ans.	B	D	D	D	D	A	B	C	C	C	A	C	D	B	D	C	B	B	D	B
Ques.	41	42	43	44	45	46	47	48	49	50	51	52								
Ans.	C	C	A	C	B	C	D	D	CD	D	A	D								

COMMERICAL MATHEMATICS

A. PERCENTAGE

The word **'percentage'** literally means **'per hundred'** or **'for every hundred.'** Therefore, whenever we calculate something as a part of 100, that part is numerically termed as percentage.
In other words, percentage is a ratio whose second term is equal to 100. i.e. 1 : 4 can be written as 25 : 100 or 25%, 3 : 8 can be written as 37.5 : 100 or 37.5%, 3 : 2 can be written as 150 : 100 or 150%, and so on.

❖ **To express a% as a fraction divide it by 100.**
i.e. a% = a/100

❖ **To express a fraction (x/y) as a percent multiply it by 100.**
i.e. x/y = (x/y x 100)%

Basic Formula of Percentage :

❖ **p% of a number N is = N × $\frac{p}{100}$.**

Solved Examples

Example.1
What is 37.5% of 648 ?
Solution.

37.5% of 648 = $\frac{37.5}{100}$ × 648

= $\frac{375}{1000}$ × 648 = 3 × 81 = 243.

Example.2
What is 20% of 50% of 60% of 200 ?
Solution.

Required percentage : $\left[\left(200\times\frac{60}{100}\right)\times\frac{50}{100}\right]$ × $\frac{20}{100}$ = 12.

❖ **To increase or decrease a number by x %, multiply the number by $\frac{[100 \pm x]}{100}$.**

Where, (+) $\Rightarrow$ Increase, (–) $\Rightarrow$ Decrease.

REMARK :
To solve these type of problems calculate x % of given number & add or subtract the value from given number for increase or decrease respectively.

❖ **To calculate what percentage of a is b, use the formula : Percentage = $\frac{b}{a}$ × 100.**

Example.3
What percentage of 240 is 90 ?
Solution.

Percentage = $\frac{90}{240}$ × 100 = 37.5%.

Example.4

What percentage of 75 is 125 ?

Solution.

Percentage $\frac{125}{75} \times 100 = 166.66\%$.

❖ **Percentage increase/decrease when a quantity 'a' is increased/decreased to become another quantity 'b'.**

Percentage Increase/Decrease

$$= \frac{Increase\ /\ Decrease}{Initial\ Value} \times 100 = \begin{cases} \frac{b-a}{a} \times 100, when\ b > a\ ;(increase) \\ \frac{a-b}{a} \times 100, when\ b < a\ ;(decrease) \end{cases}$$

Therefore new quantity b $= \begin{cases} a \times \left(1 + \frac{percentage\ increase}{100}\right) \\ a \times \left(1 - \frac{percentage\ decrease}{100}\right) \end{cases}$

Example.5

A dealer buys products for Rs.80 and hikes up the price to Rs.125. He sells it to the customer after giving a discount of Rs.5. Find his profit percentage.

Solution.

Profit percentage = Percentage Increase/Decrease in his income.

$$= \frac{Profit}{Cost\ Price} \times 100 = \frac{120-80}{80} \times 100 = 50\%.$$

Example.6

A dealer sells goods priced at Rs.180 after giving a discount of 25%. Find his selling price.

Solution.

Selling price after discount = $180\left(1 - \frac{25}{100}\right) = 135$.

❖ **If one quantity A is x% more or less than another quantity B, then B is less or more than A by : $\left(\frac{\mathbf{x}}{\mathbf{100 \pm x}}\right) \times \mathbf{100}$**

Example.7

The salary of Ramesh is 25% more than that of Anil's salary. By what percentage is Anil's salary less than that of Ramesh's ?

Solution.

Anil's salary is less than that of Ramesh's by

$$= \frac{x}{100 + x} \times 100 = \frac{25}{125} \times 100 = 20\%.$$

Example.8

Vijay's salary was reduced by 50%. Again the reduced salary was increased by 50%. Then, what will be the % loss in salary ?

Solution.

Say, salary was Rs.100

Reduction 50%

Now salary = Rs. 50

Increase = 50% $\Rightarrow \frac{50 \times 150}{100}$ = Rs.75

Loss % = $\frac{100-75}{100} = 25$

Hence, loss is 25%.

Example.9

Entry fee in an exhibition was Rs.1. Later, this was reduced by 25% which increased the sale by 20%. Then, find the percentage of slump in business.

Solution.

Let the total original sale be Rs. 100.

Then, original number of visitors = 100.

New number of visitors = $\frac{120}{0.75}$ = 160.

$\therefore$ Increase % = 60%.

B. IMPORTANT CONCEPTS ASSOCIATED WITH PERCENTAGE

(a) Conversion of Fractions into Percentages

Knowing conversion of common fractions into percentages helps to convert many fractions into percentage immediately. For example, knowing that $\frac{1}{8}$ = 12.5% will help to convert fractions like $\frac{3}{8}$ or $\frac{5}{8}$ into percentages immediately.

❖ **Given below are the fractions converted into percentage.**

Fraction	Percentage	Fraction	Percentage	Fraction	Percentage
$\frac{1}{2}$	50%	$\frac{1}{10}$	10%	$\frac{1}{18}$	5.55%
$\frac{1}{3}$	33.33%	$\frac{1}{11}$	9.09%	$\frac{1}{19}$	5.26%
$\frac{1}{4}$	25%	$\frac{1}{12}$	8.33%	$\frac{1}{20}$	5%
$\frac{1}{5}$	20%	$\frac{1}{13}$	7.69%	$\frac{1}{21}$	4.76%
$\frac{1}{6}$	16.66%	$\frac{1}{14}$	7.14%	$\frac{1}{22}$	4.54%
$\frac{1}{7}$	14.28%	$\frac{1}{15}$	6.66%	$\frac{1}{23}$	4.34%
$\frac{1}{8}$	12.50%	$\frac{1}{16}$	6.25%	$\frac{1}{24}$	4.16%
$\frac{1}{9}$	11.11%	$\frac{1}{17}$	5.88%	$\frac{1}{25}$	4%

Solved Examples

Example.10

The salary of Sachin Tendulkar is 20% more than that of Ricky Pointing. By what percentage is Ricky's salary less than that of Sachin's ?

Solution.

As, Sachin Tendulkar's salary is 20% more than Ricky Ponting's salary then Ricky Ponting's salary is less than Sachin Tendulkar's salary by

$= \frac{20}{20+100} \times 100 = \frac{20}{120} \times 100 = 16.67\%$

Example.11

A ball drops from a height of 4802 m. Thereafter, it bounces every time to a height which is 14.28% less than its previous height. What height will the ball reach on its 4th bounce ?

Solution.

I wonder how many of you will notice that 14.28% = $\frac{1}{7}$. Therefore, the ball is rising up to a height which is $\frac{1}{7}^{th}$ less than the previous height. Or, the ball is rising up to a height which is $\frac{6}{7}$ of the previous height.

Therefore, on its 4th bounce the ball will reach a height

$= 4802 \times \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7} = 2592$ m.

Example.12

A man spends 75% of his income. If his income is increased by 20% and he increased his expenditure by 10%. By what % will saving increased ?

Solution.

Let his income be Rs 100, Expenditure = Rs. 75.

Now, Income is increased by 20%.

New income = 120,

Expenditure is increased by 10% = $\frac{75 \times 110}{100}$ = Rs. 82.50

Saving = 120 – 82.50 = 37.50

Earlier saving = 100 – 75 = 25

Increase in saving = $\frac{37.50 - 25}{25} \times 100 = 50\%$.

Example.13

A students scores 40 marks in an examination and fails by 26 marks. If the passing percentage is 33 then find the maximum marks in the examination.

Solution.

Let, the maximum marks in the examination is 100.

Then he needs 33 marks to pass.

But, passing marks required are 40 + 26 = 66 marks.

33 marks are required to pass if maximum marks are 100.

Here, 66 marks are required to pass, then maximum marks are $\frac{100 \times 66}{33}$ = 200 marks.

C. SUCCESSIVE CHANGES IN PERCENTAGE

If a quantity **x** is increased or decreased successively by A%, B%, C% then the final value of **x** will be = $x\left(1 \pm \frac{A}{100}\right)\left(1 \pm \frac{B}{100}\right)\left(1 \pm \frac{C}{100}\right)$.

- Let the present population of town be **P** and let there be an increase or decrease of **R%** per annum. Then, population after **n** years = $P\left(1 \pm \frac{R}{100}\right)^n$
- If length & breadth of a rectangle is changed by **a %** & **b%** respectively, then **% change** in area will be :

  $= \left\{a \pm b \pm \frac{(a \times b)}{100}\right\}\%$ (use +ve for increase & -ve for decrease)

Solved Examples

Example.14

The population of a variety of tiny bush in an experimental field increased by 10% in the first year, increased by 8% in the second year but decreased by 10% in the third year. If the present number of bushes in the experimental field is 26730, then find the number of bushes in the beginning.

Solution.

Let the number of bushes in the beginning is P so,

$$P \times \left(1+\frac{10}{100}\right)\left(1+\frac{8}{100}\right)\left(1-\frac{10}{100}\right) = 26730 \qquad \Rightarrow P = \frac{26730}{\left(1+\frac{10}{100}\right)\left(1+\frac{8}{100}\right)\left(1-\frac{10}{100}\right)}$$

$$\Rightarrow P = \left(26730 \times \frac{10}{11} \times \frac{25}{27} \times \frac{10}{9}\right) \qquad \Rightarrow P = 25000.$$

D. PROFIT, LOSS AND DISCOUNT

(a) Profit and Loss

Definition :

(i) Cost price (C.P.) : The amount for which an article is bought is called its cost price, abbreviated to CP.

(ii) Selling price (S.P.) : The amount for which an article is sold is called its selling price, abbreviated to SP.

(iii) Gain : When S.P. > C.P. then there is a gain.
Gain = S.P. – C.P.

(iv) Loss : When S.P. < C.P. then there is a loss.
Loss = C.P. – S.P.

❖ **REMARK**

The gain or loss is always calculated on the cost price.

SOME IMPORTANT FORMULAE

(i) Gain = S.P. – C.P.

(ii) Loss = C.P. – S.P.

(iii) $\text{Gain\%} = \left(\frac{\text{Gain}}{\text{C.P.}} \times 100\right)\%$

(iv) $\text{Loss\%} = \left(\frac{\text{Loss}}{\text{C.P.}} \times 100\right)\%$

(v) To find S.P. when C.P. and gain% or loss% are given.

(a) $\text{S.P.} = \frac{(100+\text{Gain\%})}{100} \times \text{C.P.}$

(b) $\text{S.P.} = \frac{(100-\text{Loss\%})}{100} \times \text{C.P.}$

(vi) To find C.P. when S.P. and gain% or loss% are given :

(a) $\text{C.P.} = \frac{100}{100+\text{Gain\%}} \times \text{S.P.}$

(b) $\text{C.P.} = \frac{100}{100-\text{Loss\%}} \times \text{S.P.}$

❖ **Overhead :**

Sometimes, after purchasing an article, we have to pay some more money for things like transportation, labour charges, repairing charges, local taxes, etc. These extra expenses are called **overhead**. For calculating the total cost price, we add overhead to the purchase price.

Solved Examples

Example.15

A grocer buys 20 kg of sugar at a cost of Rs 18 per kg and 30 kg of an inferior sugar at a cost of Rs 15 per kg. He mixes the two kinds of sugar and sells the mixture at a cost of Rs 16.50 per kg. Find his profit or loss percent.

Solution.

C.P. of 20 kg of sugar = 18 × 20 = Rs.360

C.P. of 30 kg of sugar = 15 × 30 = Rs.450

∴ Total C.P. = 360 + 450 = Rs.810

S.P. of (20 + 30) kg = 50 kg of sugar = 16.50 × 50 = Rs.825

∴ Profit = S.P. – C.P. = 825 – 810 = Rs.15

∴ Profit percent = $\frac{15}{810}\times 100 = \frac{50}{27} = 1\frac{23}{27}$.

Hence, the required profit = $1\frac{23}{27}\%$.

Example.16

If the selling price of 20 articles is the same as the cost price of 23 articles, find the profit or loss percent in the transaction.

Solution.

Let the C.P. of an article be Rs x.

Then, C.P. of 23 articles = Rs 23x and C.P. of 20 articles = Rs 20x.

S.P. of 20 articles = C.P. of 23 articles = Rs 23x.

Since, S.P. of 20 articles > C.P. of 20 articles, hence there is a profit in the transaction,

Hence, profit on 20 articles = S.P. – C.P. = Rs (23x – 20x) = Rs 3x.

∴ Profit percent = $\frac{3x}{20x}\times 100$ = 15%

∴ Required profit = 15%.

Example.17

A man bought 2 boxes for Rs.1300. He sold one box at a profit of 20% and other box at a loss of 12%. If the selling price of both the boxes is the same, find the cost price of each box.

Solution.

Let the C.P. of the first box which was sold at a profit of 20% be Rs.x. Then the C.P. of the second box which was sold at a loss of 12% will be Rs.(1300 – x).

Since the first box was sold a profit of 20%, its S.P. = Rs. $\frac{120}{100}x$.

S.P. of second box = Rs. $\frac{88(1300-x)}{100}$

Since, the S.P. of both the boxes are same.

We have,

$\frac{120x}{100} = \frac{88(1300-x)}{100}$ ⇒ 15x = 11 (1300 – x)

⇒ 15x + 11x = 11 × 1300 ⇒ x = $\frac{11\times 1300}{26}$ = 550.

Hence, C.P. of the first box = Rs.550.

And that of the second box = Rs.(1300 – 550) = Rs.750

Example.18

Even after reducing the marked price of a transistor by Rs. 32, a shopkeeper makes a profit of 15 %.If the cost price be Rs. 320, what percentage of profit would he have made if he had sold the transistor at the marked price ?

Solution :

C.P. = Rs. 320, profit = 15%

S.P. = Rs. $\left(\frac{115}{100}\times 320\right)$ = Rs. 368.

Marked price = Rs. (368 + 32) = Rs. 400.

$\therefore$ Required profit% $\left(\frac{80}{320}\times 100\right)$ =% = 25%.

Example.19

A man buys an article and sells it at a profit of 20%. If he would buy it at 20% less and sell it for Rs.75 less, he would have gained 25%. What is the cost price of the article ?

Solution :

Let the C.P. of the article be Rs. x.

He makes a profit of 20%.

Hence, S.P. = Rs. $\frac{120x}{100}$ = Rs. $\frac{6x}{5}$

If he would buy it at 20% less, then

Then, new C.P. = Rs.x $\left(1-\frac{20}{100}\right)$ = Rs. x $\left(1-\frac{1}{5}\right)$ = Rs. $\frac{4x}{5}$

He would sell it for Rs 75 less,

Then, the new S.P. = Rs. $\left(\frac{6x}{5}-75\right)$

He gains 25%, then the new S.P. = Rs. $\frac{125}{100}\times\frac{4x}{5}$ = Rs. x.

Hence, $\frac{6x}{5}-75$ = x $\Rightarrow \frac{6x}{5}-x=75 \Rightarrow \frac{x}{5}=75$

Hence, the required C.P. = Rs 375.

Example.20

A vendor bought oranges at 20 for Rs.56 and sold them at Rs.35 per dozen. Find his gain or loss percent.

Solution :

Let the number of oranges bought = LCM of 20 and 12 = 60

C.P. of 20 oranges = Rs 56.

$\therefore$ C.P. of 1 orange = Rs.. $\frac{56}{20}$

Hence, the C.P. of 60 oranges = Rs. $\left(\frac{56}{20}\times 60\right)$ = Rs.168

S.P. of 12 oranges = Rs.35

$\therefore$ S.P. of 1 orange = Rs.. $\frac{35}{12}$

Hence, the S.P. of 60 oranges = $\left(\frac{35}{12}\times 60\right)$ = Rs.175

Thus, C.P. = Rs 168 and S.P. = Rs.175.

Since, (S.P.) > (C.P.), the vendor has made a gain.

Gain = (175 – 168) = Rs. 7

$\therefore$ Gain% = $\left(\frac{\text{Gain}}{\text{C.P.}}\times 100\right)\%$ = $\left(\frac{7}{168}\times 100\right)\%$ = $4\frac{1}{6}\%$.

(b) Discount

(i) Marked price : In big shops and department stores, every article is tagged with a card and its price is written on it. This is called the **marked price** of that article, abbreviated to MP. For books, the printed price is the marked price.

(ii) List price : Items which are manufactured in a factory are marked with a price according to the list supplied by the factory, at which the retailer is supposed to sell them. This price is known as the **list price** of the article.

(iii) Discount : In order to increase the sale or clear the old stock, sometimes the shopkeepers offer a certain percentage of rebate on the marked price. This rebate is known as **discount**.

❖ **An important fact :** The discount is always calculated on the marked price.
Clearly, **Selling Price = Marked Price – Discount**

Solved Examples

Example.21

The marked price of a woolen coat is Rs. 2000. It is sold at a discount of 15%. The shopkeeper has allowed a further discount of 5% due to off season. Find the selling price of the coat.

Solution :

Marked price = Rs. 2000

I^{st} discount = 15% of Rs. 2000 = Rs. $\frac{15}{100} \times 2000$ = Rs. 300

$\therefore$ The reduced marked price after the 1st discount = Rs.2000 – Rs 300 = Rs. 1700

2nd discount due to off-season = 5% of Rs 1700 = Rs. $\frac{5}{100} \times 1700$ = Rs. 85

Hence, the final reduced price after the 2nd discount = Rs.1700 – Rs.85 = Rs.1615 = S.P.

Hence, the required S.P. of the coat is Rs.1615.

Example.22

Find a single discount equivalent to the discount series 25%, 20% and 10%.

Solution :

Let the marked price of the article be Rs.100

Then a single discount equivalent to the discount series is

$= (100 - \frac{100-25}{100} \times \frac{100-20}{100} \times \frac{100-10}{100} \times 100)\%$

$= (100 - \frac{75}{100} \times \frac{80}{100} \times \frac{90}{100} \times 100)\%$

$= (100 - \frac{3}{4} \times \frac{4}{5} \times \frac{9}{10} \times 100)\%$

$= (100 - 54)\% = 46\%$

Hence, the given discount series is equivalent to a single discount of 46%.

Example.23

A person marks his goods 10 % above his cost price. He then sells it by allowing a discount of 10%. What is his profit or, loss percent?

Solution :

Let his cost price be Rs.x

Then, his marked price = Rs. $\left(x + \frac{10x}{100}\right)$ = Rs.. $\frac{11x}{10}$

He then sells it at a discount of 10% on this marked price.

$\therefore$ Discount = 10% of Rs. $\frac{11x}{10}$ = Rs. $\frac{11x}{10} \times \frac{10}{100}$ = Rs. $\frac{11x}{100}$

$\therefore$ His S.P. = Rs. $\left(\frac{11x}{10}-\frac{11x}{100}\right)$ = Rs. $\frac{110x-11x}{100}$ = Rs. $\frac{99x}{100}$

Since, his C.P. > S.P., hence there will be a loss.

And loss = C.P. – S.P. = Rs. $\left(x-\frac{99x}{100}\right)$ = Rs. $\frac{x}{100}$

$\therefore$ Loss percent = $\frac{x}{100}\times\frac{1}{x}\times 100$ = 1.

Hence, the required loss = 1%.

E. RATIO AND PROPORTION

(a) Ratio

The comparison of two quantities a and b of similar kind is represented as a : b is called a ratio also it can be represented as $\frac{a}{b}$.

In the ratio a : b, we call **a** as the **first term** or **antecedent** and **b**, the **second term** or **consequent.**

For example : The ratio 5 : 9 represents $\frac{5}{9}$, with antecedent = 5 and consequent = 9.

- The multiplication or division of each term of a ratio by the same non-zero number does not affect the ratio.

  For example : 4 : 5 = 8 : 10 = 12 : 15 etc. Also, 4 : 6 = 2 : 3.

(b) Proportion

The equality of two ratios is called **proportion.**
If a : b = c : d, we write, a : b : : c : d and we say that a, b, c, d are in proportion.
where, **a** is called first proportional, **b** is called second proportional, **c** is called third proportional and **d** is called fourth proportional.

- **Law of Proportion :**
  Product of means = Product of extremes
  Thus, if a : b : : c : d $\Rightarrow$ (b × c) = (a × d),
  Here **a** and **d** are called **extremes,** while **b** and **c** are called **mean terms.**
- Mean proportional of two given numbers **a** and **b** is $\sqrt{\mathbf{ab}}$.

Some other ratios :

Compounded Ratio : The compounded ratio of the ratios (a : b), (c : d), (e : f) is (ace : bdf).

Duplicate ratio : The duplicate ratio of (a : b) is (a^2 : b^2).

Sub-duplicate ratio : The sub-duplicate ratio of (a : b) is ($\sqrt{a}$: $\sqrt{b}$).

Triplicate ratio : The triplicate ratio of (a : b) is (a^3 : b^3).

Sub-triplicate ratio : The sub-triplicate ratio of (a : b) is $\left(a^{\frac{1}{3}} : b^{\frac{1}{3}}\right)$.

Componendo : If $\frac{a}{b}=\frac{c}{d}$ then, the componendo is $\frac{a+b}{b}=\frac{c+d}{d}$.

Dividendo : If $\frac{a}{b}=\frac{c}{d}$ then, the dividendo is $\frac{a-b}{b}=\frac{c-d}{d}$.

Componendo and Dividendo : If $\frac{a}{b}=\frac{c}{d}$, then the componendo-dividendo is $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Variation :

(i) We say that **x** is directly proportional to **y**, if **x = ky** for some constant **k** and we write, $\mathbf{x \propto y}$.

(ii) We say that **x** is inversely proportional to **y**, if **xy = k** for some constant **k** and we write, $\mathbf{x \propto \frac{1}{y}}$.

Solved Examples

Example.24

If a : b = 5 : 9 and b : c = 4 : 7, find a : b : c.

Solution :

a : b = 5 : 9 and b : c = 4 : 7 = $\left(4\times\frac{9}{4}\right):\left(7\times\frac{9}{4}\right)$ = 9 : $\frac{63}{4}$

a : b : c = 5 : 9 : $\frac{63}{4}$ = 20 : 36 : 63.

Example.25

Find out :

(i) the fourth proportional to 4, 9, 12.

(ii) the third proportional to 16 and 36.

(iii) the mean proportional between 0.08 and 0.18.

Solution :

(i) Let the fourth proportional to 4, 9, 12 be x.

Then, 4 : 9 : : 12 : x $\Rightarrow$ 4 × x = 9 × 12 $\Rightarrow$ x = $\frac{9\times12}{4}$ = 27.

Fourth proportional to 4, 9, 12 is 27.

(ii) Let the third proportional to 16 and 36 is x.

Then, 16 : 36 : : 36 : x $\Rightarrow$ 16 × x = 36 × 36 $\Rightarrow$ x = $\frac{36\times36}{16}$ = 81.

Third proportional to 16 and 36 is 81.

(iii) Mean proportional between 0.08 and 0.18

$= \sqrt{0.08\times0.18} = \sqrt{\frac{8}{100}\times\frac{18}{100}} = \sqrt{\frac{144}{100\times100}} = \frac{12}{100} = 0.12$

Example.26

If x : y = 3 : 4, find (4x + 5y) : (5x – 2y).

Solution :

$$\frac{x}{y}=\frac{3}{4} \quad\Rightarrow\quad \frac{4x+5y}{5x-2y}=\frac{4\left(\frac{x}{y}\right)+5}{5\left(\frac{x}{y}\right)-2}=\frac{\left(4\times\frac{3}{4}+5\right)}{\left(5\times\frac{3}{4}-2\right)}=\frac{(3+5)}{\left(\frac{7}{4}\right)}=\frac{32}{7}.$$

Example.27

Divide Rs. 1162 among A, B, C in the ratio 35 : 28 : 20.

Solution :

Sum of ratio terms = (35 + 28 + 20) = 83.

A's share = Rs. $\left(1162\times\frac{35}{83}\right)$ = Rs. 490;

B's share = Rs. $\left(1162\times\frac{28}{83}\right)$ = Rs. 392;

C's share = Rs. $\left(1162\times\frac{20}{83}\right)$ = Rs. 280.

Example.28

A bag contains 50 p, 25 p and 10 p coins in the ratio 5 : 9 : 4, amounting to Rs. 206. Find the number of coins of each type.

Solution :

Let the number of 50 p, 25 p and 10 p coins be 5x, 9x and 4x respectively.

Then, $\frac{5x}{2}+\frac{9x}{4}+\frac{4x}{10}$ = 206 $\Rightarrow$ 50x + 45x + 8x = 4120

$\Rightarrow$ 103x = 4120 $\Rightarrow$ x = 40.

Number of 50 p coins = (5 × 40) = 200;
Number of 25 p coins = (9 × 40) = 360;
Number of 10 p coins = (4 × 40) = 160.

Example.29

If a man goes from a place **A** to another place **B** 100 m apart in 4 hours at a certain speed. With the same speed going from **B** to **C** 400 m apart, what time will he take ?

Solution :

d = st, where d is distance in m, s is speed in m/sec., t is time in seconds. Speed is same

$\therefore$ d $\propto$ t.

New distance is 4 times, now the time will be 4 times the time it takes from A to B .So, the time taken from B to C is 4 × 4 = 16 hours.

F. MIXTURE AND ALLIGATION

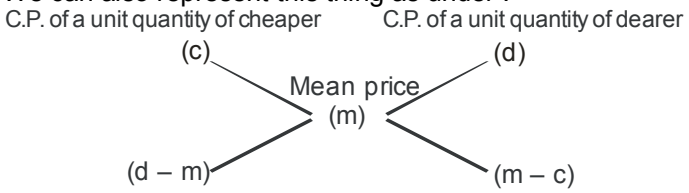
Alligation : It is the rule that enables us to find the ratio in which two or more ingredients at the given price must be mixed to produce a mixture of a desired price.

Mean Price : The cost price of a unit quantity of mixture is called the **mean price**.

Rule of Alligation : If two ingredients are mixed, then,

$$\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}}=\frac{(\text{C.P. of dearer})-(\text{Mean price})}{(\text{Mean price})-(\text{C.P. of cheaper})}$$

We can also represent this thing as under :



$\therefore$ Suppose a container contains x units of liquid from which y units are taken out and replaced by water. After n operations, the quantity of pure liquid :

$$=\left[x\left(1-\frac{y}{x}\right)^n\right] \text{ units.}$$

Solved Examples

Example.30

The cost of Type- 1 rice is Rs.15 per kg and Type-2 rice is Rs.20 per kg. If both Type-1 and Type-2 are mixed in ratio of 2 : 3, then find the price per kg of the mixed variety of rice.

Solution :

Let the price of the mixed variety be Rs. x per kg.

By the rule of alligation, we have :

Cost of 1 kg of Type 1 rice Cost of 1 kg of Type 2 rice

Rs. 15 Rs. 20

Mean price
Rs. x

(20 – x) (x – 15)

$\therefore \frac{(20-x)}{(x-15)} = \frac{2}{3}$ $\Rightarrow$ 60 – 3x = 2x – 30

$\Rightarrow$ 5x = 90 $\Rightarrow$ x = 18.

So, price of the mixture is Rs.18 per kg.

Example.31

A milk vendor has 2 cans of milk. The first contains 25% water and the rest milk. The second contains 50% water. How much milk should he mix from each of the containers so as to get 12 litres of milk such that the ratio of water to milk is 3 : 5 ?

Solution :

Let cost of 1 litre milk be Rs.1.

Milk in 1 litre mixture in 1st can = $\frac{3}{4}$ litre,

C.P. of 1 litre mixture in 1st can = Rs. $\frac{3}{4}$.

Milk in 1 litre mixture in 2nd can = $\frac{1}{2}$ litre,

C.P. of 1 litre mixture in 2nd can = Rs. $\frac{1}{2}$.

Milk in 1 litre of final mixture = $\frac{5}{8}$ litre,

Mean price = Rs. $\frac{5}{8}$.

By the rule of alligation, we have :

$\frac{x}{y} = \frac{3/4 - 5/8}{5/8 - 1/2}$; $\frac{x}{y} = \frac{1/8}{1/8} = \frac{1}{1}$.

C.P. of 1 litre mixture in 1st can C.P. of 1 litre mixture in 2nd can

3/4 1/2

Mean price
5/8

1/8 1/8

$\because$ We will mix 6ℓ from each can.

Example.32

Tea worth Rs.126 per kg and Rs.135 per kg are mixed with a third variety in the ratio 1 : 1 : 2. If the mixture is worth Rs. 153 per kg, then find the price of the third variety per kg.

Solution :

Since first and second varieties are mixed in equal proportions, so their average price

$= Rs.\left(\frac{126+135}{2}\right)$ = Rs.130.50

So, the mixture is formed by mixing two varieties, one at Rs. 130.50 per kg and the other at say, Rs. x per kg in the ratio 2 : 2, i.e., 1 : 1. We have to find x.

By the rule of alligation, we have :

Cost of 1 kg tea of 1st kind Cost of 1 kg tea of 2nd kind

130.50 Rs. x

Mean price
Rs. 153

(x – 153) 22.50

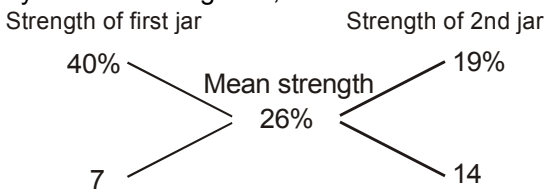
$\Rightarrow$ $1 = \frac{x-153}{22.5}$ $\Rightarrow$ 153 + 22.5 = x $\Rightarrow$ x = Rs.175.50

Example.33

A jar full of whisky contains 40% alcohol . A part of this whisky is replaced by another containing 19% alcohol and now the percentage of alcohol was found to be 26%. Find the quantity of whisky replaced.

Solution :

By the rule of alligation, we have :



So, ratio of 1st and 2nd quantities = 7 : 14 = 1 : 2.

$\therefore$ Required quantity replaced = $\frac{2}{3}$.

Example.34

A vessel is filled with liquid, 3 parts of which are water and 5 parts syrup. How much of the mixture must be drown off and replaced with water so that the mixture may be half water and half syrup ?

Solution :

Suppose the vessel initially contains 8 litres of liquid.

Let x litres of this liquid be replaced with water.

Quantity of water in new mixture = $\left(3-\frac{3x}{8}+x\right)$ litres.

Quantity of syrup in new mixture = $\left(5-\frac{5x}{8}\right)$ litres.

$\therefore \left(3-\frac{3x}{8}+x\right) = \left(5-\frac{5x}{8}\right)$ $\Rightarrow$ 5x + 24 = 40 – 5x

$\Rightarrow$ 10x = 16 $\Rightarrow$ x = $\frac{8}{5}$.

So, part of the mixture replaced = $\left(\frac{8}{5}\times\frac{1}{8}\right) = \frac{1}{5}$.

G. PARTNERSHIP

When two or more persons jointly start a business with an objective to earn money. This is called **partnership**. These persons are called **partners** and the money invested in the business is known as **capital**.

Distribution of Profit/Loss when unequal capital is invested for equal interval of time :

When partners invest different amounts of money, for equal interval of time, then profit/loss is divided in the ratio of their investment.

Solved Examples

Example.35

A and B invested Rs. 3600 and Rs. 4800 respectively to open a shop. At the end of the year B's profit was Rs. 1208. Find A's profit.

Solution :

Profit sharing ratio = 3600 : 4800 = 3 : 4

$$\frac{\text{Profit of } A}{\text{Profit of } B} = \frac{3}{4}$$

$\therefore$ Profit of A = $\frac{3}{4}$ Profit of B $\Rightarrow$ Profit of A = $\frac{3}{4}$ × 1208 = Rs. 906

Distribution of P/L when equal capital is invested for different intervals of time :

Example.36

Govind & Murari started a business with equal capitals. Govind terminated the partnership after 7 months. At the end of the year, they earned a profit of Rs. 7600. Find the profit of each of them.

Solution :

Govind invested for 7 month, Murari invested for 12 month.

Since investment is same for both (Let it be Rs. x)

$\therefore$ Profit sharing ratio = 7x : 12x = 7 : 12

$\therefore$ Govind's profit = $\frac{7}{7+12}$ × 7600 = 2800

Murari's profit = $\frac{12}{7+12}$ × 7600 = 4800.

Example.37

Ramesh started a business by investing Rs. 25000. 3 months later Mahesh joined the business by investing Rs. 25000. At the end of the year Ramesh got Rs. 1000 more than Mahesh out of the profit. Find the total profit.

Solution :

Ramesh invested for 12 month, Mahesh invested for 9 month.

$\therefore$ Profit sharing ratio = 12x : 9x = 12 : 9 = 4 : 3.

Let Capital be Rs P.

Profit of Ramesh = $\frac{4}{7}P$

Profit of Mahesh = $\frac{3}{7}P$ $\Rightarrow$ $\frac{4}{7}P = \frac{3}{7}P + 1000$

$\Rightarrow \frac{4}{7}P - \frac{3}{7}P = 1000$ $\Rightarrow$ $\frac{P}{7} = 1000$ $\Rightarrow$ P = Rs.7000.

Distribution of P/L when capital and time both are unequal :

Example.38

Suresh & Ramesh entered into a partnership by investing Rs.14000 and Rs. 18000 respectively. Suresh with drew his money after 4 months. If the total profit at the end of a year is Rs. 12240, find the profit of each.

Solution :

Profit sharing ratio = 14000 × 4 : 18000 × 12 = 7 : 27

Suresh's profit = $\frac{7}{34}$ × 12240 = Rs. 2520

Ramesh's profit = $\frac{27}{34}$ × 12240 = Rs. 9720

Example.39

David started a business establishment by investing Rs.15000. After 4 months William entered into a partnership by investing a certain amount. At the end of the year; the profit was shared in the ratio 9 : 8. Find how much money was invested by william.

Solution :

Let william invested Rs. x

Profit sharing ratio = 15000 × 12 : 8x = 1,80,000 : 8x

Also profit ratio = 9 : 8

$\therefore$ ATQ, 180000 : 8x = 9 : 8 $\Rightarrow \frac{180000 \times 8}{9 \times 8} = x$

x = Rs. 20,000

Working and Sleeping partner :

Active Partner : A partner who manages the business is known as **active** or **working partner**.

Sleeping Partner : A partner who only invests the money is known as **sleeping partner**.

Example.40

Nitesh & Jitesh invested Rs.15000 and Rs.18000 respectively in a business. If the total profit at the end of the year is Rs. 8800 and Nitesh, being an active partner, gets an additional 12.5% of the profit, find the total profit of Nitesh.

Solution :

Profit sharing ratio = 15000 : 18000 = 5 : 6

Total profit = 8800

Nitesh gets 12.5% of the profit = $\frac{12.5}{100} \times 8800$ = Rs. 1100

Net profit = 8800 – 1100 = Rs. 7700

Nitesh share in profit = $\frac{5}{5+6} \times 7700$ = Rs. 3500

Total profit of Nitesh = 3500 + 1100 = Rs. 4600

Example.41

Tanoj & Manoj started a business by investing Rs. 75000 and Rs. 90000 respectively. It was decided to pay Tanoj a monthly salary of Rs. 1875 as he was the active partner. At the end of the year if the total profit is Rs. 39000, find the profit of each.

Solution :

Profit sharing ratio = 75000 : 90000 = 5 : 6

Total profit = Rs. 39000

Salary of Tanoj = 12 × 1875 = Rs. 22500

Profit left = Rs.39000 – Rs. 22500 = Rs.16500.

Tanoj's profit = $\frac{5}{11} \times 16500$ = 7500.

$\therefore$ Total profit of Tanoj = 22500 + 7,500 = Rs. 30,000

Manoj's profit = $\frac{6}{11} \times 16500$ = Rs. 9,000

Change in invested capital :

Example.42

Rajeev & Sanjeev entered into a partnership and invested Rs. 36000 and Rs. 40000 respectively. After 8 months Rajeev invested an additional capital of Rs. 4000, Sanjeev withdrew Rs. 4000 after 9 months. At the end of the year total profit was Rs. 45800. Find the profit of each.

Solution :

Rajeev's capital = 36000 × 8 + (36000 + 4000) × 4 = Rs. 448000

Sanjeev's capital = 40000 × 9 + (40000 – 4000) × 3 = Rs. 468000

Profit sharing ratio = 448000 : 468000 = 112 : 117

Rajeev's profit = $\frac{112}{229} \times 45800$ = Rs. 22400

Sanjeev's profit = $\frac{117}{229} \times 45800$ = Rs. 23400.

Example.43

A, B and C start a business each investing Rs. 20000. After 5 months A withdrew Rs. 5000, B withdrew Rs. 4000 and C invests Rs. 6000 more. At the end of the year, a total profit of Rs. 69900 was recorded. Find the share of each.

Solution :

Ratio of the capitals of A, B and C

= 20000 × 5 + 15000 × 7 : 20000 x 5 + 16000 × 7 : 20000 × 5 + 26000 × 7

= 205000 : 212000 : 282000 = 205 : 212 : 282.

$\therefore$ A's share = Rs. $(69900 \times \frac{205}{699})$ = Rs. 20500;

B's share = Rs. $(69900 \times \frac{212}{699})$ = Rs. 21200;

C's share = Rs. $(69900 \times \frac{282}{699})$ = Rs. 28200.

H. WORK AND TIME

Work is defined as the amount of job assigned or the amount of job actually done.
Work is always considered as a whole or 1.
Units of work : Work is measured by many units i.e. men-days, men-hours, men-minutes, machine-hours or in general person-time, machine-time.

❖ **If A and B can do a piece of work in x and y days respectively while working alone, then they will take $\left(\frac{xy}{x+y}\right)$ days to complete the work if both are working together.**

Proof : A's one day work = $\frac{1}{x}$ and B's one day work = $\frac{1}{y}$ and (A + B)'s one day work = $\frac{1}{x}+\frac{1}{y}$

(A + B)'s one day work = $\frac{x+y}{xy}$

$\therefore$ Time taken by both A and B (working together) to complete the work = $\frac{xy}{x+y}$.

❖ **If A, B, C can do a piece of work in x, y, z days respectively while working alone, then they will together take $\frac{1}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$ days to complete the work.**

Solved Examples

Example.44

A, B and C together can finish a piece of work in 4 days. A alone can do it in 9 days and B alone in 18 days. How many days will be taken by C to do it alone.

Solution :

Let's time taken by C alone to complete the work in x days

$\therefore \frac{1}{4} = \frac{1}{9}+\frac{1}{18}+\frac{1}{x}$

$\Rightarrow$ x = 12 days.

Example.45

A, B and C can do a piece of work 6, 8 and 12 days respectively. B and C work together for 2 days, then A takes C's place. How long will it take to finish the work.

Solution :

Work done by B & C in 2 days = 2 × $\left\{\frac{1}{8}+\frac{1}{12}\right\} = \frac{5}{12}^{th}$ part of the work.

Remaining work = $\left(1-\frac{5}{12}\right)^{th} = \frac{7}{12}^{th}$ part.

A and B's one day work = $\left(\frac{1}{6}+\frac{1}{8}\right)^{th} = \left(\frac{7}{24}\right)^{th}$ part.

So, $\frac{7}{24}^{th}$ part of work is completed in 1 day.

$\therefore \frac{7}{12}^{th}$ part of work will be completed in = $\frac{1}{7/24}\times\frac{7}{12} = \frac{24}{7}\times\frac{7}{12}$ = 2 days.

❖ **A can finish a work in x days and B is k times as efficient as A (i.e. B will complete the work in $\frac{x}{k}$ days) Then time taken by both A & B working together to finish the job will be $\frac{x}{k+1}$.**

❖ **If A is k times as good as B and takes x days less than B to finish the work. Then the amount of time required by A and B working together is $\frac{kx}{k^2-1}$ days.**

Example.46

A is thrice as good a work man as B and takes 60 days less than B for doing a job. Find the time in which they can do it together.

Solution :

Here k = 3, x = 60

$\therefore$ Time in which they can do it together = $\frac{3\times 60}{3^2-1}=\frac{3\times 60}{8}=22\frac{1}{2}$ days.

Example.47

25 men were employed to do a piece of work in 24 days. After 15 days, 10 more men were engaged and the work was finished a day too soon. In what time could they finish the work if extra men were not employed.

Solution :

Actual work done = (25 × 15) + (25 + 10) × 8 = 655 man days.

Time required by 25 men to complete this work is = $\frac{655}{25}$ = 26.2 days.

Example.48

If 12 men or 18 women can reap a field in 7 days, in what time can 4 men & 8 women reap the same field.

Solution :

12 men = 18 women

$\therefore$ 4 men = 6 women

4 men + 8 women = 6 women + 8 women = 14 women

Total work done = 18 × 7 women-days

$\therefore$ No. of days required to complete this work by 4 men and 8 women = 14 women is = $\frac{18\times 7}{14}$ = 9 days.

I. PIPES AND CISTERNS

Here the work done is in terms of filling or emptying a cistern.

Inlet pipe : It is the pipe connected to cistern which fill the cistern (time taken is in + ve).
Outlet pipe : It is the pipe connected to cistern which empties the cistern. (time taken is – ve).

❖ If an inlet pipe fills a cistern in 'a' hours, then $\left(\frac{1}{a}\right)^{th}$ part is filled in 1 hr.

❖ If two inlet pipes A & B can fill a cistern in 'm' & 'n' hours respectively then together they will take $\left(\frac{mn}{m+n}\right)$ hrs. to fill the cistern.

❖ If an inlet pipe fills a cistern in 'm' hours and an outlet pipe empties it in 'n' hours, then the net part filled in 1 hr. When both the pipes are opened is $\left(\frac{1}{m}-\frac{1}{n}\right)$ hours and the cistern will get filled in $\frac{mn}{n-m}$ hours, for cistern to get filled, m < n.

❖ If m > n, the cistern will never get filled, in this case a completely filled cistern gets emptied in $\left(\frac{mn}{m-n}\right)$ hours.

❖ If an inlet pipe fills a cistern in m hrs. and takes n hrs. longer to fill the cistern due to leak in the cistern, then the time in which the leak will empty the cistern in m × $\left(1+\frac{m}{n}\right)$.

Solved Examples

Example.49

A tank is emptied by 2 pipes and filled by a third. If the 1st two can empty the tank in 2 and 3 hrs. respectively and third can fill it in 4 hours. How much time will it take to empty the tank $\left(\frac{4}{5}\right)^{th}$ full when all three are open.

Solution :

Let the time taken to completely empty the tank is x hrs. $\frac{1}{x}=\frac{1}{2}+\frac{1}{3}+\left(-\frac{1}{4}\right)$

$\Rightarrow \frac{1}{x}=\frac{7}{12}$ $\Rightarrow$ x = $\frac{12}{7}$ hrs.

Complete tank will be emptied in $\frac{12}{7}$ hrs.

$\therefore \left(\frac{4}{5}\right)^{th}$ tank will be emptied in = $\frac{12}{7}$ × $\frac{4}{5}$ = $\frac{48}{35}$ hrs.

Example.50

Two pipes M & N can fill a cistern in 12 & 16 hrs. respectively. If both the pipes are opened together, then after how many minutes N should be closed so that the tank is full in 9 hrs.

Solution :

Let N be closed after x hrs. Then, $x\left(\frac{1}{12}+\frac{1}{16}\right)$ + (9 – x) $\frac{1}{12}$ = 1

$\Rightarrow$ x = 16 × $\frac{3}{12}$ $\Rightarrow$ x = 4 hrs = 240 minutes.

J. TIME, SPEED AND DISTANCE

❖ Speed = $\frac{distance}{time}$

❖ Time = $\frac{distance}{speed}$

❖ Distance = Speed × time

❖ If a certain distance (from A to B) is covered at **u km/hr** and the same distance (from B to A) is covered at **v km/hr**. then the average speed during the whole journey is = $\frac{\mathbf{2uv}}{\mathbf{u+v}}$ km/hr.

Average speed :

If a body travels d_1, d_2, d_3,....., d_n distances with speeds s_1, s_2, ..., s_n,..... respectively, then the average speed of the body through the total distance is given by :

Average speed = $\frac{\text{Total distance covered}}{\text{Total time taken}}$ = $\frac{d_1+d_2+d_3+.....+d_n}{t_1+t_2+t_3+.....+t_n}$

Where, $t_1 = \frac{d_1}{s_1}$, $t_2 = \frac{d_2}{s_2}$ ...

Solved Examples

Example.51

A man travels Ist 50 km at 25 km/hr, next 40 km with 20 km/hr. and then 90 km at 15 km/hr. Then, find his average speed for the whole journey (in km/hr).

Solution :

Avg. Speed = $\dfrac{50+40+90}{\dfrac{50}{25}+\dfrac{40}{20}+\dfrac{90}{15}}$ = 18 km/hr.

Example.52

If a man travels 10 km/hr from A to B and again 15 km/hr. from B to A. Find the average speed of man for complete journey.

Solution :

Avg. speed = $\dfrac{2\times10\times15}{10+15}=\dfrac{2\times10\times15}{25}$ = 12 km/hr.

K. PROBLEMS OF TRAINS

(i) Time taken by a train of length **'a'** metres to pass a pole or a standing man or a signal post is equal to the time taken by the train to cover **'a'** metres.

(ii) Time taken by a train of length **'a'** metres to pass a stationary object of length **'b'** metres is the time taken by the train to cover **(a + b)** metres.

(iii) Suppose two trains or two bodies are moving in the same direction at **u m/s** and **v m/s**, where **u > v**, then their relative speed = **(u – v) m/s**.

(iv) Suppose two trains or two bodies are moving in opposite direction at **u m/s** and **v m/s**, then their relative speed = **(u + v) m/s**.

(v) If two trains of length **'a'** metres and **'b'** metres are moving in opposite directions at **u m/s** and **v m/s**, then time taken by the trains to cross each other = $\dfrac{(a+b)}{(u+v)}$ sec.

(vi) It two trains of length **'a'** metres and **'b'** metres are moving in the same direction at **u m/s** and **v m/s** then the time taken by the faster train to cross the slower train = $\dfrac{(a+b)}{(u-v)}$ sec.

(vii) If two trains (or bodies) start at the same time from points A and B towards each other and after crossing they take **'a'** and **'b'** sec in reaching B and A respectively, then (A's speed) : (B's speed) $=(\sqrt{b}:\sqrt{a})$.

Solved Examples

Example.53

Two trains running in the same direction at 40 km/hr and 22 km/hr completely pass one another in 1 minute. If the length of the Ist train is 125 m., then what will be the length of IInd train.

Solution :

Relative speed of trains = 40 – 22 = 18 km/hr.

$\because$ 18 km/hr. = 5 m/sec.

Let the length of second train = L m.

Time taken to cross each other = $\dfrac{L+125}{5}$

$\therefore$ $\dfrac{L+125}{5}$ = 60 L = 175 m.

Example.54

A train passes a station platform in 36 seconds and a man standing on the platform in 20 seconds. If the speed of the train is 54 km/hr, what is the length of the platform ?

Solution :

Speed = $\left(54 \times \frac{5}{18}\right)$ m/sec = 15 m/sec

Length of the train = (15 × 20) m = 300 m.

Let the length of the platform be x metres.

Then, $\frac{x+300}{36}$ = 15 $\Rightarrow$ x + 300 = 540 $\Rightarrow$ x = 240 m.

Example.55

Two trains are moving in opposite directions 60 km/hr and 90 km/hr. Their lengths are 1.10 km and 0.9 km respectively. Find the time taken by the slower train to cross the faster train in seconds.

Solution :

Relative speed = (60 + 90) km/hr = $\left(150 \times \frac{5}{18}\right)$ m/sec = $\left(\frac{125}{3}\right)$ m/sec.

Distance covered = (1.10 + 0.9) km = 2 km = 2000 m.

Required time = $\left(2000 \times \frac{3}{125}\right)$ sec = 48 sec.

Example.56

A train 125 m long passes a man, running at 5 kmph in the same direction in which the train is going in 10 seconds. Find the speed of the train.

Solution :

Speed of the train relative to man = $\left(\frac{125}{10}\right)$m/sec = $\left(\frac{25}{2}\right)$m/sec = $\left(\frac{25}{2} \times \frac{18}{5}\right)$km/hr = 45 km/hr.

Let the speed of the train be x kmph. Then, relative speed = (x – 5) km/hr

$\therefore$ x – 5 = 45 or x = 50 km/hr.

Example. 57

Two goods train each 500 m long, are running in opposite directions on parallel tracks. Their speeds are 45 km/hr and 30 km/hr respectively. Find the time taken by the slower train to pass the driver of the faster one.

Solution :

Relative speed = (45 + 30) km/hr = $\left(75 \times \frac{5}{18}\right)$ m/sec = $\left(\frac{125}{6}\right)$m/sec

Distance covered = (500 + 500) m = 1000 m.

Required time = $\left(1000 \times \frac{6}{125}\right)$ sec = 48 sec.

L. BOAT AND STREAMS

(i) In water, the direction along the stream is called **downstream**. And, the direction against the stream is called **upstream**.

(ii) If the speed of a boat in still water is u km/hr and the speed of the stream is v km/hr, then :
Speed downstream = (u + v) km/hr.
Speed upstream = (u – v) km/hr.

(iii) If the speed downstream is a km/hr and the speed upstream is b km/hr, then :

Speed of boat in still water = $\frac{1}{2}$ (a + b) km/hr

Rate of stream = $\frac{1}{2}$ (a – b) km/hr.

Solved Examples

Example.58

A man can row three-quarters of a kilometre against the stream in $11\frac{1}{4}$ minutes and covered the same distance with the stream in $7\frac{1}{2}$ min. Find the speed (in km/hr) of the man in still water.

Solution :

Rate upstream = $\left(\frac{750}{675}\right)$ m/sec = $\frac{10}{9}$ m/sec ;

Rate downstream = $\left(\frac{750}{450}\right)$ m/sec = $\frac{5}{3}$ m/sec

$\therefore$ Rate in still water = $\frac{1}{2}\left(\frac{10}{9}+\frac{5}{3}\right)$ m/sec = $\frac{25}{18}$ m/sec = $\left(\frac{25}{18}\times\frac{18}{5}\right)$ km/hr = 5 km/hr.

Example.59

A man's speed with the current is 15 km/hr and the speed of the current is 2.5 km/hr. Find the man's speed against the current.

Solution :

Man's rate in still water = (15 – 2.5) km/hr = 12.5 km/hr.

Man's rate against the current = (12.5 – 2.5) km/hr = 10 km/hr.

Example.60

A man can row $9\frac{1}{3}$ kmph in still water and finds that it takes him thrice as much time to row up than as to row down the same distance in the river. Find the speed of the current.

Solution :

Let speed upstream be x kmph.

Then, speed downstream = 3x kmph.

Speed in still water = $\frac{1}{2}$ (3x + x) kmph = 2x kmph.

$\therefore$ 2x = $\frac{28}{3}$ $\Rightarrow$ x = $\frac{14}{3}$.

So, Speed upstream = $\frac{14}{3}$ km/hr ;

Speed downstream = 14 km/hr.

Hence, speed of the current = $\frac{1}{2}\left(14-\frac{14}{3}\right)$ km/hr = $\frac{14}{3}$ km/hr = $4\frac{2}{3}$ km/hr.

Example.61

A boat covers a certain distance downstream in 1 hour, while it comes back in $1\frac{1}{2}$ hours. If the speed of the stream be 3 kmph, what is the speed of the boat in still water ?

Solution :

Let the speed of the boat in still water be x kmph. Then,

Speed downstream = (x + 3) kmph,

Speed upstream = (x – 3) kmph.

$\therefore$ (x + 3) × 1 = (x – 3) × $\frac{3}{2}$ $\Rightarrow$ 2x + 6 = 3x – 9 $\Rightarrow$ x = 15 kmph.

M. SIMPLE INTEREST AND COMPOUND INTEREST

DEFINITION :

(i) **Principal :** The money borrowed or lent out is called **principal**.

(ii) **Interest :** The additional money paid by the borrower is called the **interest**.

(iii) **Amount :** The total money (interest + principal) paid by the borrower is called the **amount**. [$\therefore A = P + I$]

(iv) **Rate of interest :** If the borrower paid interest of Rs. x on Rs.100 for 1 year, then the rate of interest is x percent per annum.

(v) **Time :** The period for which the sum is borrowed is called the **time**.

(vi) **Conversion Period :** The fixed interval of time at the end of which the interest is calculated and added to the principal at the beginning of the interval is called the conversion period.

(vii) **Simple Interest :** If the principal remains the same throughout the loan period, then the interest paid by the borrower is called **simple interest**.

$$\text{S.I.} = \frac{P \times R \times T}{100}$$

(viii) **Compound Interest :** If the borrower and the lender agree to fix up a certain interval of time (Say,a year or a half year or a quarter of year etc.) so that the Amount (= Principal + Interest) at the end of an interval becomes the principal for the next interval, then the total interest over all the intervals calculated in this way is called the compound interest and is abbreviated as C.I.

NOTE : S.I. and C.I. are equal for Ist year.

(a) Simple interest

To find simple interest and the amount when rate of interest is given as percent per year :

Solved Examples

Example.62

Find the simple interest and the amount on Rs. 2400 for 3 years 5 months and 15 days at the rate of 9%.

Solution :

Given : Principal (P) = Rs. 2400, Rate (R) = 9%.

Time (T) = 3 years 5 months and 15 days = $\frac{83}{24}$ years.

To find : Simple interest and the amount

Simple interest = $\frac{PRT}{100}$ = Rs. 2400 × $\frac{9}{100} \times \frac{83}{24}$ = Rs. 747

And the amount = Rs. 2400 + Rs. 747 = Rs. 3147.

(i) Inverse questions on simple interest

Solved Examples

Example.63

Find the rate of interest when Rs. 640 amounts to Rs. 841 and 60 paise for the period 2 years 7 months and 15 days at a simple rate of interest.

Solution :

Given :

Principal = Rs. 640

Amount = Rs. 841.60

Interest = Rs. (841.60 – 640) = Rs. 201.60

Time = 2 years 7 months 15 days = $\frac{21}{8}$ years

To find : Rate (R)

Simple interest (I) = $\frac{PRT}{100}$ $\Rightarrow$ $201.60 = 640 \times \frac{R}{100} \times \frac{21}{8}$

$\Rightarrow 640 \times \frac{21}{8} \times \frac{R}{100} = \frac{20160}{100}$ $\Rightarrow$ $R = \frac{20160}{100} \times \frac{1}{640} \times \frac{8}{21} \times \frac{100}{1}$

$\Rightarrow$ R = 12 %.

Hence, rate of interest = 12%.

(ii) Some special questions on simple

Solved Examples

Example.64

A sum of money amount to Rs. 1237.50 and Rs. 1443.75 in 4 and 6 years respectively at a simple rate of interest. Find principal and the rate of interest.

Solution :

Principal + interest of 4 years = Rs. 1237.50

Same principal + interest of 6 years = Rs. 1443.75

$\therefore$ 2 years interest on the given principal = Rs (1443.75 – 1237.50) = Rs. 206.25

$\therefore$ 4 years interest on the given principal = Rs. $\frac{206.25}{2} \times 4$ = 412.50

$\therefore$ Principal = Amount of 4 year – Interest of 4 year

= Rs. 1237.50 – 412.50 = Rs. 825

To find : Rate (R)

Given : Principal = 825

Interest = 412.50

Time = 4 years

From the formulae,

Simple interest = $\frac{PRT}{100}$

$\Rightarrow$ $412.50 = 825 \times \frac{R}{100} \times 4$

$\Rightarrow$ $825 \times \frac{R}{100} \times 4 = \frac{41250}{100}$

$\Rightarrow$ $R = \frac{41250}{100} \times \frac{100}{1} \times \frac{1}{825} \times \frac{1}{4}$

$\Rightarrow$ R = 12.5

Hence, the required rate is 12.5%.

Example.65

Madhav lent out Rs. 7953 for 2 years and Rs. 1800 for 3 years at the same rate of simple interest. If he got Rs. 2343.66 as total interest then find the percent rate of interest.

Solution :

Let the percent rate of interest be x % = $\frac{7953 \times 2 \times x}{100} = \frac{15906}{100}$ × x and interest on Rs. 1800 for 3 years at the rate of x% = $\frac{1800 \times 3 \times x}{100}$ = 54 x

According to problem,

$\frac{15906x}{100}$ + 54x = 2343.66 $\Rightarrow \frac{15906x + 5400x}{100} = \frac{234366}{100}$

$\Rightarrow$ 21306x = 234366 $\Rightarrow$ x = $\frac{234366}{21306}$

$\Rightarrow$ x = 11%

Hence, the required rate of interest = 11%.

(b) Compound interest

(i) Computation of Compound Interest when Interest is compounded Annually.

Solved Examples

Example.66

Find the compound interest on Rs. 8000 for 3 year at 5% per annum.

Solution :

Principal for the first year = Rs. 8000, Rate = 5% per annum, T = 1 year.

Interest for the first year = $\frac{P \times R \times T}{100}$ = Rs. $\left[\frac{8000 \times 5 \times 1}{100}\right]$ = Rs. 400

$\therefore$ Amount at the end of the first year = Rs. (8000 + 400) = Rs. 8400

Now, principal for the second year = Rs. 8400

Interest for the second year = $\frac{P \times R \times T}{100}$ = Rs. $\left[\frac{8400 \times 5 \times 1}{100}\right]$ = Rs. 420

$\therefore$ Amount at the end of the second year = Rs. (8400 + 420) = Rs. 8820

Interest for the third year = $\frac{P \times R \times T}{100}$ = Rs. $\frac{8820 \times 5 \times 1}{100}$ = Rs. 441

$\therefore$ Amount at the end of the third year = Rs. (8820 + 441) = Rs. 9261

Now, we know that total C.I. = Amount – Principal = Rs. (9261 – 8000) = Rs. 1261

We can also find the C.I. as follows

Total C.I. = Interest for the first year + Interest for the second year + Interest for third year

= Rs. (400 + 420 + 441) = Rs. 1261

(ii) Computation of Compound Interest When Interest is compounded Half yearly.

Example.67 Find the compound interest on Rs. 8000 for $1\frac{1}{2}$ years at 10% per annum, interest being payable half yearly.

Solution : We have

Rate of interest = 10% per annum = 5% per half year,

Time = $1\frac{1}{2}$ years = 3 half year.

Original principal = Rs. 8000

Interest for the first half year = Rs. $\left[\frac{8000 \times 5 \times 1}{100}\right]$ = Rs. 400

Amount at the end of the first half year = Rs. 8000 + 400 = Rs. 8400
Principal for the second half year = Rs. 8400
Interest for the second half year = Rs. $\left(\frac{8400 \times 5 \times 1}{100}\right)$ = Rs. 420
Amount at the end of the second half year = Rs. 8400 + Rs. 420 = Rs. 8820
Principal for the third half year = Rs. 8820
Interest for the third half year = Rs. $\left(\frac{8820 \times 5 \times 1}{100}\right)$ = Rs. 441
Amount at the end of third half year = Rs. 8820 + Rs. 441 = Rs. 9261
Compound interest = Rs. 9261 – Rs. 8000 = Rs. 1261.

(iii) Computation of compound Interest when Interest is Compounded Quarterly :

Example.68

Find the compound interest on Rs. 10,000 for 1 year at 20% per annum interest being payable quarterly.

Solution :

We have Rate of interest = 20% per annum = $\frac{20}{4}$ = 5% per quarter
Time = 1 year = 4 quarters.
Principal for the first quarter = Rs. 10000
Interest for the first quarter = Rs. $\left(\frac{10000 \times 5 \times 1}{100}\right)$ = Rs. 500
Amount at the end of first quarter = Rs. 10000 + Rs. 500 = Rs. 10500
Principal for the second quarter = Rs. 10500
Interest for the second quarter = Rs. $\left(\frac{10500 \times 5 \times 1}{100}\right)$ = Rs. 525
Amount at the end of second quarter = Rs. 10500 + Rs. 525 = Rs. 11025
Principal for the third quarter = Rs. 11025
Interest for the third quarter = Rs. $\frac{11025 \times 5 \times 1}{100}$ = Rs. 551.25
Amount at the end of the third quarter = Rs. 11025 + Rs. 551.25 = Rs. 11576.25
Principal for the fourth quarter = Rs. 11576.25
Interest for the fourth quarter = Rs. $\left(\frac{11576.25 \times 5 \times 1}{100}\right)$ = Rs. 578.8125
Amount at the end of the fourth quarter = Rs. 11576.25 + Rs. 578.8125 = Rs. 12155.0625
Compound interest = Rs. 12155.0625 – Rs. 10000 = Rs. 2155.0625

(iv) Computation of compound interest by using formulae

(1) Let P be the principal and the rate of interest be R% per annum. If the interest is compounded annually then the amount A and the compound interest C.I. at the end of n years.

Given by A = P $\left(1+\frac{R}{100}\right)^n$ and C.I. = A – P = P $\left\{\left(1+\frac{R}{100}\right)^n - 1\right\}$ respectively.

(2) Let P be the principal and the rate of interest be R% per annum. If the interest is compounded k times in a year annually, then the amount A and the compound interest. C.I. at the end of n years is given by A = P $\left(1+\frac{R}{100k}\right)^{nk}$ and C.I. = A – P = P $\left\{\left(1+\frac{R}{100k}\right)^{nk} - 1\right\}$ respectively.

(3) Let P be the principal and the rate of interest be R_1% for first year, R_2% for second year, R_3% for third year and so on and in the last R_n% for the nth year. Then, the amount A and the compound interest C.I. at the end of n years are given by

$A = P\left(1+\frac{R_1}{100}\right)\left(1+\frac{R_2}{100}\right).....\left(1+\frac{R_n}{100}\right)$ and C.I. = (A – P) respectively

(4) Let P be the principal and the rate of interest be R% per annum. If the interest is compounded annually but time is the fraction of a year, say $5\frac{1}{4}$ year, then amount A is given by

$A = P\left(1+\frac{R}{100}\right)^5\left(1+\frac{R/4}{100}\right)$ and C.I. = A – P.

Example.69

Find the compound interest on Rs. 12000 for 3 years at 10% per annum compounded annually.

Solution :

We know that the amount A at the end of n years at the rate of R % per annum when the interest is compounded annually is given by :

$A = P\left(1+\frac{R}{100}\right)^n$

Here, P = Rs. 12000, R = 10% per annum and n = 3.

$\therefore$ Amount after 3 years

$= P\left(1+\frac{R}{100}\right)^3 = \text{Rs. } 12000 \times \left(1+\frac{10}{100}\right)^3 = \text{Rs. } 12000 \times \left(1+\frac{1}{10}\right)^3$

$= \text{Rs. } 12000 \times \left(\frac{11}{10}\right)^3 = \text{Rs. } 12000 \times \frac{11}{10} \times \frac{11}{10} \times \frac{11}{10} = \text{Rs } (12 \times 11 \times 11 \times 11) = \text{Rs. } 15972.$

Now, Compound interest = A – P

$\Rightarrow$ Compound interest = Rs.15972 – Rs.12000 = Rs. 3972.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Percentage

A-1 The price of an article is cut 20%.To restore it to its former value, by what percentage the new price must be increased ?

A-2 The ratio of the number of boys and girls in a school is 3 : 2. If 20% of the boys and 25% of the girls are scholarship holders, what percentage of the students does not get the scholarship ?

A-3 A students scores 80 marks in an examination and fails by 52 marks. If his passing percentage is 33, then what will be the maximum marks in the examination ?

A-4 The salaries of A, B, C are in the ratio 2 : 3 : 5. If the increments of 15%, 10% and 20% are allowed respectively in their salaries, then what will be the new ratio of their salaries ?

A-5 By what perccentage 10 is more than 5 and by what percentage 5 is less than 10.

Section (B) : Profit, Loss & Discount

B-1 A man bought certain apples at the rate of Rs 15 for 4 and sold them at the rate of Rs 16 for 5. What will be his profit or loss percent?

B-2 A tradesman marks his goods at 25% above their CP and allows for purchasers a discount of 12.5% for cash. What profit % does he make ?

B-3 A shopkeeper cheats both buyers and sellers by 10% by using his false balance. What will be his profit percent?

B-4 A man sold each of two cows at Rs 1955. As a result, he made a profit of 15% in one cow and a loss of 15% in the other. What was his total profit or loss?

B-5 If 140 mangoes are sold at the cost price of 175 mangoes, what will be the profit percent?

Section (C) : Ratio and Proportion

C-1. Two numbers are respectively 20% and 50% more than a third number. Find the ratio of the two numbers

C-2. The ratio of the number of boys and girls in a college is 7 : 8. If the percentage increase in the number of boys and girls be 20 % and 10 % respectively, then find the new ratio.

C-3. The prices of a scooter and a TV are in the ratio 7 : 5. If the scooter costs Rs. 8000 more than a TV set, then find the price of a T.V set.

C-4. Find the ratio whose terms differ by 40 and the measure of which is $\frac{2}{7}$

C-5 The ages of a man and his son is in the ratio of 7 : 2. After 15 years, they would be in the ratio of 2 : 1, Find the father's age when son was born ?

Section (D) : Mixture & Alligation

D-1 In what ratio must rice at Rs. 9.30 per kg mixed with rice at Rs.10.80 per kg so that the mixture be worth Rs.10 per kg ?

D-2 A container contains 40 litres of milk. From this container 4 litres of milk was taken out and replaced by water. This process was repeated further two times. How much milk is now contained by the container ?

D-3 A can contains a mixture of two liquids A and B in the ratio 7 : 5. When 9 litres of mixture are drawn off and the can is filled with B, the ratio A and B becomes 7 : 9. How many litres of liquid A was contained by the can initially ?

D-4 You have a measuring cup with capacity 25 ml and another with capacity 110 ml, the cups have no markings showing intermediate volumes. Using large container an as much tap water as you wish. Find the the smallest amount of water you can measure accurately ?

D-5 A mixture of rice is sold at Rs. 3.00 per kg. This mixture is formed by mixing the rice of Rs. 2.10 and Rs. 2.52 per kg. What is the ratio of price of cheaper to the costlier quality in the mixture if the profit of 25% is being earned.

Section (E) : Partnership

E-1 A, B and C enter into a partnership. They invest Rs.40,000, Rs. 80,000 and Rs.1,20,000 respectively. At the end of the first year, B withdraws Rs.40,000, while at the end of the second year, C withdraws Rs.80,000. In what ratio will the profit be shared at the end of 3 years ?

E-2 A and B started a business jointly. A's investment was thrice the investment of B and the period of his investment was two times the period of investment of B. If B received Rs.4000 as profit, then find their total profit.

E-3 A and B started a business with initial investments in the ratio 14 : 15 and their annual profits were in the ratio 7 : 6. If A invested the money for 10 months, for how many months did B invest his money ?

E-4 A and B started a business in partnership investing Rs. 20,000 and Rs. 15,000 respectively. After six months, C joined them with Rs. 20,000. What will be B's share in the total profit of Rs. 25,000. earned at the end of 2 years from the starting of the business ?

E-5 Arun, Kamal and Vinay invested Rs. 8000, Rs. 4000 and Rs. 8000 respectively in a business. Arun left after six months. If after eight months, there was gain of Rs. 4005, then what will be the share of Kamal ?

Section (F) :Time & Work

F-1 If 270 kg of corn would feed 42 horses for 21 days, for how many days would 360 kg of it feed 21 horses ?

F-2 A, B and C can do a piece of work in 15, 12 and 20 days respectively. They started the work together, but C left after 2 days. In how many days will the remaining work be completed by A and B ?

F-3 Three pipes A, B & C can fill a tank in 8 hours. After working at it together for 2 hours, A is closed and B and C can fill it in 9 hours. The number of hours taken by A alone to fill the cistern ?

F-4 A pipe can fill a cistern in 9 hours. Due to a leak in its bottom, the cistern fills in 10 hours. If the cistern is full, in how much time will it be emptied by the leak ?

F-5 A can do a piece of work in 4 hours; B and C together can do it in 3 hours, while A and C together can do it in 2 hours. How long will B alone take to do it ?

Section (G) :Time,Speed and Distance

G-1 A train 100 m long traveling with speed of 60 km/hr passes of another train 150 m long moving in the opposite direction in 9 sec. What was the speed of second train ?

G-2 Two goods train each 500 m long, are running in opposite directions on parallel tracks. Their speeds are 45 km/hr and 30 km/hr respectively. Find the time taken by the train to cross each other.

G-3 A 400 m long train is running at the speed of 60 km per hour. Find the time taken by train to crosses a bridge of length 800 m.

G-4 Two trains running in the same direction at 40 km/hr and 22 km/hr completely pass one another in 1 minute. If the length of the 1st train is 125 m., then what will be the length of 2nd train.

G-5 Two trains can run at a speed of 54 km/hr and 36 km/hr on a parallel tracks. When they are running in opposite directions, they pass each other in 10 s. When they move in the same direction, a person sitting in the faster train crosses the other train in 30 s. Find the length of the trains (faster and slower respectively)

Section (H) : Simple Interest, Compound Interest

H-1 The C.I. on a sum of money for 2 years is Rs. 832 and the S.I. on the same sum for the same period is Rs. 800. Find the difference between the C.I. and S.I. for 3 years.

H-2 The difference between the S.I. on a certain sum at the rate of 10% per annum for 2 years and C.I. which is compounded every 6 months is Rs. 124.05. Find the principal sum.

H-3 A bacteria reproduces at the rate of 50 % in every 15 min. In how much time will it be $3\frac{3}{8}$ times of itself ?

H-4 The difference between the compound interest and the simple interest on a certain sum for 2 years at 6% per annum is Rs. 18. Find the sum.

H-5 At what rate percent per annum will a sum of Rs. 2000 amount to Rs. 2205 in 2 years, compounded annually ?

OBJECTIVE QUESTIONS

Section (A) : Percentage

A-1. Two-fifth of one-third of three-seventh of a number is 15. What is 40 percent of that number ?
(A) 72 (B) 84 (C) 105 (D) 140

A-2. An organization has 32 members. It hopes to increase the membership by 50% each year. If this is done, how many members will the organization have in 5 years ?
(A) 243 (B) 200 (C) 162 (D) 112

A-3. When any number is divided by 12, then quotient becomes $\frac{1}{4}$th of the other number By how much percent first number is greater than the second number ?
(A) 150% (B) 200% (C) 300% (D) Date inadequate

A-4. Gauri went to the stationers and bought things worth Rs.25, out of which 30 paise went on sales tax on taxable purchases. If the tax rate was 6%, then what was the cost of the tax free items ?
(A) Rs. 15 (B) Rs. 15.70 (C) Rs. 19.70 (D) Rs. 20

A-5. A student has to obtain 33% of the total marks to pass. He got 125 marks and failed by 40 marks. The maximum marks are :
(A) 300 (B) 500 (C) 800 (D) 1000

A-6. A scored 30% marks and failed by 15 marks. B scored 40% marks and obtained 35 marks more than those required to pass. The pass percentage is :
(A) 33% (B) 38% (C) 43% (D) 46%

A-7. In a recent survey, 40% houses contained two or more people. Of those houses containing only one person, 25% were having only a male. What is the percentage of all houses, which contain exactly one female and no males ?
(A) 15 (B) 45 (C) 75 (D) Can't be determined

A-8. $37\frac{1}{2}$% of the candidates in an examination were girls, 75% of the boys and $62\frac{1}{2}$% of the girls passed and 342 girls failed. The number of boys failed was :
(A) 350 (B) 360 (C) 370 (D) 380

A-9. $\frac{5}{9}$ part of the population in a village are males. If 30% of the males are married, the percentage of unmarried females in the total population is :
(A) 20% (B) $27\frac{7}{9}$% (C) 40% (D) 70%

A-10. What percent decrease in salaries would exactly cancel out the 20 percent increase ?
(A) $16\frac{2}{3}$% (B) 18% (C) 20% (D) $33\frac{1}{3}$%

A-11. The price of a T.V. set is decreased by 25% as a result of which the sale increased by 20%. What will be the effect on the total revenue of the shop ?
(A) No effect (B) 5% decrease (C) 10% increase (D) None of these

A-12. In a fraction, if numerator is increased by 40% and denominator is increased by 80%, then what fraction of the original is the new fraction ?
(A) $\frac{1}{2}$ (B) $\frac{7}{9}$ (C) $\frac{7}{18}$ (D) Date inadequate

A-13. The price of wheat falls by 16%. By what percentage a person can increase the consumption of wheat so that his overall budget does not change ?
(A) 16% (B) 18% (C) 18.5% (D) 19%

A-14. A's income is 25% more than B's income B's income in terms of A's income is :
(A) 75% (B) 80% (C) 90% (D) 96%

A-15. If half of x is y and one-third of y is z then :
(A) z = 6% of x (B) z = 16.66% of x (C) z = 60% of x (D) z = 30% of x

A-16. The total income of A and B is Rs 6000. A spends 60 % of his income and B spends 80 % of his income. If their savings are equal, then what is the income of A ?
(A) 2000 (B) 2400 (C) 2600 (D) 2800

A-17. A man earns x % on the first Rs.2,000 and y % on the rest of his income. If he earns Rs. 700 from Rs. 4,000 and Rs. 900 from Rs. 5,000 of income, find x %.
(A) 20 % (B) 15 % (C) 25 % (D) None of these

A-18. A person who has a certain amount with him goes to market. He can buy 50 oranges or 40 mangoes. He retains 10% of the amount for taxi fares and buys 20 mangoes and of the balance left he purchases oranges. Number of oranges he can purchase is :
(A) 36 (B) 40 (C) 15 (D) 20

A-19. If 25% of a number is subtracted from a second number, the second number reduce to its five-sixth. What is the ratio of the first number to the second number ?
(A) 1 : 3 (B) 2 : 3 (C) 3 : 2 (D) Data inadequate

Section (B) : Profit, Loss and Discount

B-1. A person incurs 5 % loss by selling a watch for Rs. 1140. At what price should the watch be sold to earn 5 % profit ?
(A) Rs. 1275 (B) Rs. 1280 (C) Rs. 1210 (D) Rs. 1260

B-2. A book was sold for Rs. 27.50 with a profit of 10%. If it were sold for Rs. 25.75, that what would have been the percentage of profit or loss ?
(A) 4.5% (B) 5.0% (C) 3.0% (D) 3.25%

B-3. The C.P of 21 articles is equal to S.P. of 18 articles. Find the gain or loss percent.
(A) 16% (B) 8% (C) 10% (D) 12%

B-4. By selling 33 metres of cloth, one gains the selling of 11 metres. Find the gain percent.
(A) 17.5% (B) 33.33% (C) 40% (D) 50%

B-5. A vendor bought bananas at 6 for Rs.10 and sold them at 4 for Rs.6. Find his loss percent.
(A) 12% (B) 8% (C) 6% (D) 10%

B-6. After getting two successive discounts, a shirt with a list price of Rs. 150 is available at Rs.105. If the second discount is 12.5%, find the first discount.
(A) 25% (B) 12% (C) 15% (D) 20%

B-7. A shopkeeper expects a gain of $22\frac{1}{2}$ % on his cost price. If in a week, his sale was of Rs. 392, what was his profit ?
(A) Rs. 18.20 (B) Rs. 70 (C) Rs. 72 (D) Rs. 88.25

B-8. Jacob bought a scooter for a certain sum of money. He spent 10% of the cost on repairs and sold the scooter for a profit of Rs. 1100. How much did he spend on repairs if he made a profit of 20% ?
(A) Rs. 400 (B) Rs. 440 (C) Rs.500 (D) Rs.550

B-9. If the selling price of 50 articles is equal to the cost price of 40 articles, then the loss or gain percent is :
(A) 20% gain (B) 20% loss (C) 25% loss (D) 25% gain

B-10. Garima purchased a briefcase with an additional 10 % discount on the reduced price after deducting 20 % on the labelled price. If the labelled price was Rs. 1400, at what price did she purchase the briefcase ?
(A) Rs. 980 (B) Rs. 1008 (C) Rs. 1056 (D) Rs.1120

B-11. By selling an umbrella for Rs. 300, a shopkeeper gains 20%. During a clearance sale, the shopkeeper allows a discount of 10% on the marked price. His gain percent during the sale is :
(A) 7 (B) 7.5 (C) 8 (D) 9

B-12. A trader mixes three varieties of groundnuts costing Rs.50, Rs. 20 and Rs. 30 per kg in the ratio 2 : 4 : 3 in terms of-weight, and sells the mixture at Rs.33 per kg. What percentage of profit does he make ?
(A) 8 % (B) 9 % (C) 10 % (D) None of these

B-13. The cash difference between the selling prices of an article at a profit of 4 % and 6 % is Rs. 3. The ratio of the two selling prices is :
(A) 51 : 52 (B) 52 : 53 (C) 51 : 53 (D) 52 : 55

Section (C) : Ratio and Proportion

C-1. If $\frac{1}{5} : \frac{1}{x} = \frac{1}{x} : \frac{1}{125}$, then the value of x is :
(A) 15 (B) 20 (C) 25 (D) 35

C-2. If x : y = 5 : 2, then (8x + 9y) : (8x + 2y) is :
(A) 22 : 29 (B) 26 : 61 (C) 29 : 22 (D) 61 : 26

C-3. The salaries of A, B, C are in the ratio 2 : 3 : 5. If the increments of 15%, 10% and 20% are allowed respectively in their salaries, then what will be the new ratio of their salaries?
(A) 3 : 3 : 10 (B) 10 : 11 : 20
(C) 23 : 33 : 60 (D) Cannot be determined

C-4. Two numbers are in the ratio 3 : 5. If 9 is subtracted from each, the new numbers are in the ratio 12 : 23. The smaller number is :
(A) 27 (B) 33 (C) 49 (D) 55

C-5. The ratio of three numbers is 3 : 4 : 7 and their product is 18144. The numbers are :
(A) 9, 12, 21 (B) 15, 20, 25 (C) 18, 24, 42 (D) None of these

Section (D) : Mixture & Alligation

D-1. Arun purchased 30 kg of wheat at the rate of Rs.11.50 per kg and 20 kg of wheat at the rate of Rs.14.25 per kg. He mixed the two and sold the mixture. Approximately what price per kg should he sell the mixture to make 30% profit ?
(A) Rs. 16.30 (B) Rs. 8.19 (C) Rs. 163 (D) None of these

D-2. How many kg. of wheat costing Rs.8 per kg must be mixed with 36 kg of rice costing Rs.5.40 per kg so that 20% gain may be obtained by selling the mixture at Rs.7.20 per kg ?
(A) 10.8 kg (C) 10 kg (C) 12.8 kg (D) None of these

D-3. In what ratio must water be mixed with milk costing Rs.12 per litre to obtain a mixture worth of Rs. 8 per litre ?
(A) 1 : 2 (B) 2 : 3 (C) 4:5 (D) 2 : 5

D-4. 5 kg of superior quality of sugar is mixed with 25 kg of inferior quality sugar. The price of superior quality and inferior quality sugar is Rs. 18 and Rs. 12 respectively. Find the average price per kg of the mixture.
(A) Rs. 13 (B) Rs. 15 (C) Rs. 18 (D) Rs. 21

D-5. 16 litres of kerosene is mixed with 5 litres of petrol. The price of kerosene is Rs. 12 per litre and the price of petrol is Rs. 33 per litre. Find the average price of the mixture per litre.
(A) Rs. 15 (B) Rs. 17 (C) Rs. 23 (D) Rs. 27

Section (E) : Partnership

E-1. An amount, of Rs. 2430 is divided among A, B and C such that if their shares be reduced by Rs. 5, Rs. 10 and Rs.15 respectively, the remainders shall be in the ratio of 3 : 4 : 5. Then, B's share was :
(A) Rs. 605 (B) Rs. 790 (C) Rs. 800 (D) Rs. 810

E-2. Sum of Rs. 53 is divided among A, B, C in such a way that A gets Rs. 7 more than what B gets and B gets Rs. 8 more than what C gets. The ratio of their shares is :
(A) 16 : 9 : 18 (B) 25 : 18 : 10 (C) 18 : 25 : 10 (D) 15 : 8 : 30

E-3. A, B and C enter into partnership. A invests 3 times as much as B invests and B invests two-third of what C invests. At the end of the year, the profit earned is Rs. 6600. What is the share of B ?
(A) Rs. 3600 (B) Rs. 1800 (C) Rs. 1200 (D) None of these

E-4. A, B and C enter into partnership A invests some money at the beginning, B invests double the amount after 6 months and C invests thrice the amount after 8 months. If the annual profit be Rs. 27,000, C's share is :
(A) Rs. 8625 (B) Rs. 9000 (C) Rs.10,800 (D) Rs. 11,250

E-5. In a business, A and C invested amounts in the ratio 2 : 1, whereas the ratio between amounts invested by A and B was 3 : 2. If Rs. 1,57,300 was their profit, how much amount did B receive ?
(A) Rs.24,200 (B) Rs.36,300 (C) Rs.48,400 (D) Rs.72,600

E-6. A and B started a partnership business investing some amount in the ratio of 3 : 5. C joined them after six months with an amount equal to that of B. In what proportion should the profit at the end of one year be distributed among A, B and C ?
(A) 3 : 5 : 2 (B) 3 : 5 : 5 (C) 6 : 10 : 5 (D) Data inadequate

E-7. A and B entered into partnership with capitals in the ratio 4 : 5. After 3 months, A withdrew $\frac{1}{4}$ of his capital and B withdrew $\frac{1}{5}$ of his capital. The gain at the end of 10 months was Rs. 760. A's share in this profit is :
(A) Rs.330 (B) Rs.360 (C) Rs.380 (D) Rs.430

E-8. A starts business with Rs. 3500 and after 5 months, B joins with A as his partner. After a year, the profit is divided in the ratio 2 : 3. What is B's contribution in the capital?
(A) Rs.7500 (B) Rs.8000 (C) Rs.8500 (D) Rs.9000

Section (F) : Time & Work

F-1. A tin of oil was $\frac{4}{5}$ full. When six bottles of oil were taken out and four bottles of oil were poured in, it was $\frac{3}{4}$ full. How many bottles of oil did the tin contain initially ?
(A) 16 (B) 40 (C) 32 (D) None of these

F-2. If 5 men can cane 5 chairs in 5 hours, then 1 man shall cane 1 chair in how many hours.
(A) 1 hour (B) 15 hours (C) 5 hours (D) 10 hours

F-3. If 4 workers can dig 20 m long ditch in 6 days, find out the part dug by 2 workers in a day (in metre)?
(A) $1\frac{2}{3}$ (B) $3\frac{2}{4}$ (C) $3\frac{2}{3}$ (D) $1\frac{3}{2}$

F-4. A and B together can complete a work in 12 days. A alone can complete it in 20 days. If B now does the work for half a day daily. Then in how many days A and B together will complete the work ?
(A) 15 days (B) 20 days (C) 45 days (D) 30 days

F-5. A takes 10 days less than the time taken by B to finish a piece of work. If both A and B together can finish the work in 12 days, the time taken by B to finish the work is :
(A) 20 days (B) 30 days (C) 40 days (D) 50 days

F-6. A, B and C can complete a work, working alone in 10, 15 and 20 days respectively. If all of them work together to complete the work, what fraction of the work would have done by B ?
(A) $\frac{4}{13}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{6}{13}$

F-7. A is thrice as efficient as B, and B is twice as efficient as C. If A, B and C work together, how long will they take to complete a job which B completes in 10 days ?
(A) $\frac{20}{9}$ days (B) $\frac{11}{9}$ days (C) 3 days (D) None of these

F-8. A can do a piece of work in 24 days. If B is 60% more efficient then the number of days required by B to do the twice as large as the earlier work is :
(A) 24 (B) 36 (C) 15 (D) 30

F-9. A works twice as fast as B. If B can complete a work in 12 days independently, then the number of days in which A and B can together finish the work is :
(A) 4 days (B) 6 days (C) 8 days (D) 18 days

F-10. Taps A and B can fill a tank in 12 min and 15 min respectively. If both are opened and A is closed after 3 min, how long will it take for B to fill the tank ?
(A) 7 min 45 sec (B) 7 min 15 sec (C) 8 min 5 sec (D) 8 min 15 sec

Section (G) : Time,Speed and Distance

G-1. A train X speeding with 120 kmph crosses another train Y, running in the same direction, in 2 minutes. If the lengths of the trains X and Y be 100 m and 200 m respectively, what is the speed of train Y ?
(A) 111 km/hr (B) 123 km/hr (C) 127 km/hr (D) 129 km/hr

G-2. Two trains running in opposite directions cross a man standing on the platform in 27 seconds and 17 seconds respectively and they cross each other in 23 seconds. The ratio of their speeds is :
(A) 1 : 3 (B) 3 : 2 (C) 3 : 4 (D) None of these

G-3. A man is traveling by car at the rate of 40 km/hr. After every 80 km, he rests for 20 min. How long will he take to cover a distance of 240 km ?
(A) 6 hr 40 min (B) 6 hr (C) 6 hr 20 min (D) 7 hr

Section (H) : Simple Interest, Compound Interest

H-1. At what percent per annum, simple interest will double a sum of money in 12 years ?
(A) $8\frac{1}{3}$ % (B) 24 % (C) $\frac{25}{4}$ % (D) 8.25 %

H-2. Find the amount of Rs 12000 after 2 years compounded annually, the rate of interest being 5 % p.a. during the first year and 6 % p.a. during the second year, also find the compound interest.
(A) Rs. 1356 (B) Rs. 1200 (C) Rs. 1256 (D) None of these

H-3. Find the compound interest on Rs 31250 at 8 % per annum for $2\frac{3}{4}$ years.
(A) Rs. 8637 (B) Rs. 7387 (C) Rs. 1250 (D) None of these

H-4. A certain sum amounts to Rs 72900 in 2 years at 8 % per annum compound interest. Find the sum.
(A) Rs. 72000 (B) Rs. 62500 (C) Rs. 135000 (D) None of these

H-5. In what time will Rs 64000 amount to Rs 68921 at 5% per annum, interest being compounded half-yearly ?
(A) $1\frac{1}{2}$ years (B) $\frac{2}{3}$ years (C) 2 years (D) None of these

H-6. The value of refrigerator which was purchased 2 years ago depreciates at 12 % p.a. If its present value is Rs 9680, for how much was it purchased ?
(A) Rs. 11500 (B) Rs. 10500 (C) Rs. 12000 (D) Rs. 12500

H-7. The compound interest on Rs. 10,000 in 2 years at 4 % per annum, being compounded half yearly is :
(A) Rs. 832.24 (B) Rs. 828.82 (C) Rs. 824.32 (D) Rs. 912.86

H-8. A sum of Rs. 500 was lent for two years at 2 % compound interest. The interest for two years will be :
(A) Rs. 20.00 (B) Rs. 25.00 (C) Rs. 50.20 (D) Rs. 20.20

H-9. Compute the compound interest on Rs. 1000 for 2 years at 10 % per annum when compounded half-yearly.
(A) Rs. 315.15 (B) Rs. 200 (C) Rs. 215.50 (D) None of these

H-10. Find the compound interest on Rs 24000 at 15 % per annum for $2\frac{1}{3}$ years.
(A) Rs. 9327 (B) Rs. 9723 (C) Rs. 9372 (D) Rs. 9237

Exercise-2

OBJECTIVE QUESTIONS

1. A tempo is insured to the extent of $\frac{4}{5}$ of its original value. If the premium on it at the rate of 1.3 percent amounts to Rs.910, the original value of the tempo is :
(A) Rs.78500 (B) Rs.80000 (C) Rs.82500 (D) Rs.87500

2. 10% of the voters did not cast their votes in an election between two candidates. 10% of the votes polled were found invalid. The successful candidate got 54% of the valid votes and won by a margin of 1620 votes. The number of voters enrolled in the voters' list was :
(A) 25000 (B) 33000 (C) 35000 (D) 40000

3. Fresh fruit contains 68% water and dry fruit contains 20% water. How much dry fruit can be obtained from 100 kg of fresh fruits ?
(A) 32 kg (B) 40 kg (C) 52 kg (D) 80 kg

4. Milk contains 5% water. What quantity of pure milk should be added to 10 litres of milk to reduce this to 2%.
(A) 5 liters (B) 7 litres
(C) Cannot be determined (D) None of these

5. One bacteria splits into eight bacteria of the next generation. But due to environment, only 50 % of one generation can produce the next generation. If the seventh generation number is 4096 million, what is the number in first generation ?
(A) 1 million (B) 2 million (C) 4 million (D) 8 million

6. I bought 5 pens, 7 pencils and 4 erasers. Rajan bought 6 pens, 8 erasers and 14 pencils for an amount which was half more what I had paid. What percent of the total amount paid by me was paid for the pens?
(A) 37·5 % (B) 62·5 % (C) 50 % (D) None of these

7. The price of a Maruti car rises by 30 % while the sales of the car comes down by 20 %. What is the percentage change in the total revenue ?
(A) – 4 % (B) – 2 % (C) + 4 % (D) + 2 %

8. A shopkeeper sells 25 articles at Rs. 45 per article after giving 10% discount and earns 50% profit. If the discount is not given, the profit gained is :
(A) 60% (B) $60\frac{2}{3}$ % (C) 66% (D) $66\frac{2}{3}$ %

9. If the manufacturer gains 10%, the wholesale dealer 15% and the retailer 25%, then find the cost of production of a table, the retail price of which is Rs. 1265 ?
(A) 600 (B) 800 (C) 700 (D) 900

10. A man bought a number of clips at 3 for a rupee and an equal number at 2 for a rupee. At what price per dozen should he sell them to make a profit of 20% ?
(A) Rs. 4 (B) Rs. 5 (C) Rs. 6 (D) Rs. 7

11. On selling a T.V. at 5% gain and a fridge at 10% gain, a shopkeeper gains Rs 2000. But if he sells the T.V. at 10% gain and the fridge at 5% loss. He gains Rs 1500 on the transaction. Find the actual prices of T.V. and fridge.
(A) Rs.20,000 & Rs.15,000 (B) Rs.20,000 & Rs.10,000
(C) Rs.10,000 & Rs.12,000 (D) Rs.15,000 & Rs.18,000

12. A shopkeeper purchased 150 identical pieces of calculators at the rate of Rs. 250 each. He spent an amount of Rs. 2500 on transport and packing. He fixed the labelled price of each calculator at Rs. 320. However, he decided to give a discount of 5 % on the labelled price. What is the percentage profit earned by him?
(A) 14 % (B) 15 % (C) 16 % (D) 20 %

13. A shopkeeper sells a badminton racket, whose marked price is Rs.30, at a discount of 15 % and gives a shuttle cock costing Rs.1.50 free with each racket. Even then makes a profit of 20 %. His cost price per racket is :
(A) Rs. 19.75 (B) Rs. 20 (C) Rs. 21 (D) Rs. 21.25

14. An amount of Rs. 735 was divided between A, B and C. If each of them had received Rs. 25 less, their shares would have been in the ratio of 1 : 3 : 2, The money received by C was :
(A) Rs. 195 (B) Rs. 200 (C) Rs. 225 (D) Rs. 245

15. The speeds of three cars are in the ratio 5 : 4 : 6. The ratio between the time taken by them to travel the same distance is :
(A) 5 : 4 : 6 (B) 6 : 4 : 5 (C) 10 : 12 : 15 (D) 12 : 15 : 10

16. Padam purchased 30 kg of rice at the rate of Rs.17.50 per kg and another 30 kg rice at a certain rate. He mixed the two and sold the entire quantity at the rate of Rs.18.60 per kg and made 20% overall profit. At what price per kg did he purchase the lot of another 30 kg rice ?
(A) Rs. 12.50 (B) Rs. 13.50 (C) Rs. 14.50 (D) Rs. 15.50

17. In what ratio must a grocer mix two varieties of tea worth Rs. 60 per kg and Rs. 65 per kg so that by selling the mixture at Rs. 68.20 a kg he may gain 10% ?
(A) 3 : 2 (B) 3 : 4 (C) 3 : 5 (D) 4 : 5

18. A invested Rs. 76000 in a business. After few months, B joined him with Rs. 57000. At the end of the year, the total profit was divided between them in the ratio of 2 : 1. After how many months did B join ?
(A) 4 (B) 8 (C) 9 (D) None of these

19. Simran started a software business by investing Rs. 50,000. After six months, Nanda joined her with a capital of Rs. 80,000. After 3 years, they earned a profit of Rs. 24,500. What was Simran's share in the profit ?
(A) Rs. 9423 (B) Rs. 10,500 (C) Rs. 12,500 (D) Rs. 14,000

20. A, B, C started a business with their investments in the ratio 1 : 3 : 5. After 4 months, A invested the same amount as before and B as well as C withdrew half of their investments. The ratio of their profits at the end of the year is :
(A) 4 : 3 : 5 (B) 5 : 6 : 10 (C) 6 : 5 : 10 (D) 10 : 5 : 6

21. A can do a piece of work in 9 days and B in 18 days. They began the work together but 3 days before the completion of work, A leaves. The time taken to complete the work is :
(A) 7 days (B) 5 days (C) 8 days (D) 11 days

22. Harpal is thrice as good a workman as Kewal and takes 10 days less to do a piece of work than Kewal takes. Then Kewal can do that work in :
(A) 8 days (B) 12 days (C) 13 days (D) 15 days

23. A can do a piece of work in 7 days of 9 hr each, and B can do it in 6 days of 7 hr each. How long will they take to do it working together $\frac{42}{5}$ hr a day ?
(A) 3 days (B) 4 days (C) 4.5 days (D) None of these

24. A leak in the bottom of a tank can empty it in 6 hr. A pipe fills the tank at 4 L/min. When the tank is full, the inlet is opened, but due to the leak the tank is emptied in 8 hr. What is the capacity of the tank ?
(A) 5,260 L (B) 5,760 L (C) 5,846 L (D) 6,970 L

25. Because of a leak, a tank which usually is filled in 3 hrs, now takes 3.5 hrs to be filled. The leak alone will empty the filled tank in :
(A) 18 hrs (B) 21 hrs (C) 15 hrs (D) Cannot be determined

26. In a camp of 100 students, there is ration which lasts for 8 days. After the first 2 days, 50 more students join them. How long will the ration last now ? (Assume all the students have equal eating capacity)
(A) 4 days (B) 6 days (C) 8 days (D) None

27. Two trains of equal length are running on parallel lines in the same direction at 46 km/hr and 36 km/hr. The faster train passes the slower train in 36 seconds. The length of each train is :
(A) 50 m (B) 72 m (C) 80 m (D) 82 m

28. A train overtakes two persons walking along a railway track. The first one walks at 4.5 km/hr. The other one walks at 5.4 km/ hr. The train needs 8.4 and 8.5 seconds respectively to overtake them. What is the speed of the train if both the persons are walking in the same direction as the train ?
(A) 66 km/hr (B) 72 km/hr (C) 78 km/hr (D) 81 km/hr

29. A railway passenger counts the telegraph posts on the line as he passes them while travelling; If the poles are 50 m apart and the train is going at the speed of 48 km/hr. How many posts will be passed by the train per minute ?
(A) 15 (B) 16 (C) 18 (D) 20

30. What sum will become Rs 9826 in 18 months if the rate of interest is $2\frac{1}{2}\%$ per annum and the interest is compounded half-yearly ?
(A) Rs. 9466.54 (B) Rs. 9646.54 (C) Rs. 9566.54 (D) Rs. 9456.54

31. Reena borrowed from Kamila certain sum for two years at simple interest. Reena lent this to Hamid at the same rate for two years compound interest. At the end of two years she received Rs. 110 as compound interest but paid Rs. 100 as simple interest. Find the sum and rate of interest.
(A) Rs 250 and 20 % (B) Rs 250 and 25 % (C) Rs 200 and 20 % (D) Rs 200 and 25 %

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. If a, b, c are in continued proportion, the expression $\frac{a^2+ab+b^2}{b^2+bc+c^2}$ can be simplified to : **[IJSO-2010]**
(A) $\frac{a+c}{a+b}$ (B) $\frac{a}{c}$ (C) $\frac{c}{a}$ (D) $\frac{a-c}{b}$

2. In a school, the average score of all 1400 students at an examination was found to be 69.5 The average score of boys in the school was 68 and that of girls was 72. Hence, the number of boys and the number of girls respectively are **[IJSO-2011]**
(A) 680 and 720 (B) 720 and 680 (C) 875 and 525. (D) 800 and 600.

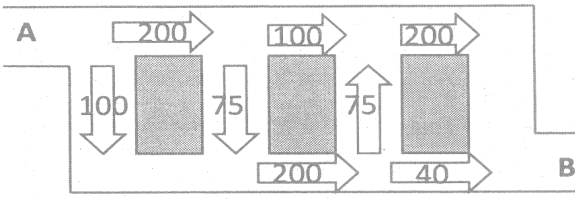
3. The distance between two spots A & B on the same bank of the river is 75km. Speed of the boat in still water is twice as much as that of the speed of the water current of river. The boat travels in the river from A to B and returns back to the spot in 16 hour. What is the speed of the boat in still water? **[IJSO-2012]**
(A) 12.5kmph (B) 15kmph (C) 16kmph (D) 18kmph

4. Y varies inversely as x. If x is increased by 25%, then the value of percentage change to y is **[IJSO-2012]**
(A) 80% (B) 75% (C) 60% (D) 62.5%

5. Scienctists in an R & D company made three design improvements on a car : the first saves 50% of fuel, the second saves 30% of fuel and the third saves 20%. If the company implements all three design changes at once, the new car will consume fuel that is ___% of the fuel consumption of normal car **[IJSO - 2014]**

(A) 50% (B) 100% (C) 28% (D) 20%

6. The diagram shows a road network. All vehicles drive in one direction from A to B. Numbers represent the maximum flow rate (capacity of roads) in vehicles per hour. The maximum number of vehicles that can drive through the network every hour is **[IJSO - 2014]**



(A) 315 (B) 240 (C) 215 (D) 340

7. A water filter advertisement claims to provide 8 litres of water per hour. How much time does it take to fill four bottles of 1.5 litres each ? **[IJSO - 2014]**

(A) 2 hr (B) 1 hr (C) 30 min (D) 45 min

8. If a : b = c : d then how many of the following statements are true? **[IJSO - 2015]**

(i) $c(a + b) = a(c + d)$

(ii) $d(a – b) = b(c – d)$

(iii) $(a^2 + b^2)(ac – bd) = (a^2 – b^2)(ac + bd)$

(iv) $\left(\frac{a^2}{b^2}\right) + \left(\frac{c^2}{d^2}\right) = \left(\frac{2ac}{bd}\right)$

(A) 1 (B) 2 (C) 3 (D) All

9. A train is running at a speed of 54 km/hr. It is not stopping at a certain station. It crosses the person showing green flag in 20 seconds and crosses the platform in 36 seconds. What is the length of the train ? **[IJSO 2017]**

(A) 240 m (B) 300 m (C) 320 m (D) 360 m

10. The value of $\frac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$, when $x = \frac{2ab}{b^2+1}$ is **[IJSO 2018]**

(A) a (B) b (C) x (D) 0

Answers

EXERCISE - 1

SUBJECTIVE QUESTIONS

Section (A)

A-1	25%	**A-2**	78%	**A-3**	400	**A-4**	23 : 33 : 60
A-5	100 % and 50 %						

Section (B)

B-1	$14\frac{2}{3}$% loss	**B-2**	9.4 %	**B-3**	21 %	**B-4**	Rs. 90 loss
B-5	25 %						

Section (C)

C-1.	4 : 5	**C-2.**	21 : 22	**C-3.**	Rs.20000.	**C-4**.	16 : 56
C-5	25.						

Section (D)

D-1	8 : 7.	**D-2**	29.16 litres.	**D-3**	21 litres of A	**D-4**	5 ml.
D-5	2 : 5.						

Section (E)

E-1	3 : 4 : 7.	**E-2**	Rs. 28000.	**E-3**	8 months.	**E-4**	Rs. 7500.
E-5	Rs. 890.						

Section (F)

F-1	56	**F-2**	4 days	**F-3**	24 hrs.	**F-4**	90 Hrs.
F-5	12 Hrs						

Section (G)

G-1	40 km/hr	**G-2**	48 sec.	**G-3**	1 min 12 sec	**G-4**	175 m.
G-5	100 m.						

Section (H)

H-1	Rs. 98.56.	**H-2**	8000.	**H-3**	45 min.	**H-4**	Rs. 5000.
H-5	5% per annum						

OBJECTIVE QUESTIONS

Section (A)

A-1.	(C)	**A-2.**	(A)	**A-3.**	(B)	**A-4.**	(C)	**A-5.**	(B)
A-6.	(A)	**A-7.**	(B)	**A-8.**	(D)	**A-9.**	(B)	**A-10.**	(A)
A-11.	(D)	**A-12.**	(B)	**A-13.**	(D)	**A-14.**	(B)	**A-15.**	(B)
A-16.	(A)	**A-17.**	(B)	**A-18.**	(D)	**A-19.**	(B)		

Section (B)

B-1.	(D)	**B-2.**	(C)	**B-3.**	(A)	**B-4.**	(D)	**B-5.**	(D)
B-6.	(D)	**B-7.**	(C)	**B-8.**	(C)	**B-9.**	(B)	**B-10.**	(B)
B-11.	(C)	**B-12.**	(C)	**B-13.**	(B)				

Section (C)

C-1.	(C)	**C-2.**	(C)	**C-3.**	(C)	**C-4.**	(B)	**C-5.**	(C)

Section (D)

D-1.	(A)	**D-2.**	(A)	**D-3.**	(A)	**D-4**	(A)	**D-5**	(B)

Section (E)

E-1.	(D)	**E-2.**	(B)	**E-3.**	(C)	**E-4.**	(B)	**E-5.**	(C)
E-6.	(C)	**E-7.**	(A)	**E-8.**	(D)				

Section (F)

F-1.	(C)	**F-2.**	(C)	**F-3.**	(A)	**F-4.**	(A)	**F-5.**	(B)
F-6.	(A)	**F-7.**	(A)	**F-8.**	(D)	**F-9.**	(A)	**F-10.**	(D)

Section (G)

G-1.	(A)	**G-2.**	(B)	**G-3.**	(A)

Section (H)

H-1.	(A)	**H-2.**	(A)	**H-3.**	(B)	**H-4.**	(B)	**H-5.**	(A)
H-6.	(D)	**H-7.**	(C)	**H-8.**	(D)	**H-9.**	(C)	**H-10.**	(A)

EXERCISE - 2

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	B	D	A	B	C	D	B	C	B	A	B	D	D	B	A	A	B	B
Ques.	21	22	23	24	25	26	27	28	29	30	31									
Ans.	C	D	A	B	B	A	A	D	B	A	A									

EXERCISE - 3

Ques.	1	2	3	4	5	6	7	8	9	10
Ans.	B	C	A	A	C	C	D	D	B	B

POLYNOMIALS

A. POLYNOMIALS

(a) Definition :

An algebraic expression f(x) of the form $f(x) = a_0 + a_1x + a_2x^2 + ...+ a_nx^n$, Where $a_0, a_1, a_2.....a_n$ are real numbers and all the indices of x are non negative integers is called a **polynomial in x** and the highest index n is called the **degree of the polynomial**, if $a_n \neq 0$. Here a_0 , a_1x, a_2x^2 .....,a_nx^n are called the terms of the polynomial and a_0, a_1, a_2, ... a_n are called various co-efficients of the polynomial f(x). A polynomial in x is said to be in its standard form when the terms are written either in increasing order or decreasing order of the indices of x in various terms.

EXAMPLES :

(i) $2x^3 + 4x^2 + x + 1$ is a polynomial of degree 3.

(ii) $x^7 + x^5 + x^2 + 1$ is a polynomial of degree 7.

(iii) $x^{3/2} + x^2 + 1$ is not a polynomial as the indices of x are not all non negative integer

(iv) $x^2 + \sqrt{2}\,x + 1$ is a polynomial of degree 2.

(v) $x^{-2} + x + 1$ is not a polynomial as –2 is not non negative.

B. TYPES OF POLYNOMIALS

(a) Polynomial Based on Degree :

There are five types of polynomials based on degree.

(i) Constant polynomial :
A polynomial of degree zero is called a zero degree polynomial or constant polynomial.
e.g. $f(x) = 4 = 4x^0$

(ii) Linear polynomial :
A polynomial of degree one is called a linear polynomial. The general form of a linear polynomial is ax + b, where a and b are any real numbers and $a \neq 0$
e.g. 4x + 5, 2x + 3, 5x + 3 etc.

(iii) Quadratic polynomial :
A polynomial of degree two is called a quadratic polynomial. The general form of a quadratic polynomial is $ax^2 + bx + c$ where $a \neq 0$
e.g. $x^2 + x + 1$, $2x^2 + 1$, $3x^2 + 2x + 1$ etc.

(iv) Cubic polynomial :
A polynomial of degree three is called a cubic polynomial. The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where $a \neq 0$

e.g. $x^3 + x^2 + x + 1$, $x^3 + 2x + 1$, $2x^3 + 1$ etc.

(v) Biquadratic polynomial :

A polynomial of degree four is called a biquadratic or quartic polynomial. The general form of biquadratic polynomial is $ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$
e.g. $x^4 + x^3 + x^2 + x + 1$, $x^4 + x^2 + 1$ etc.

NOTE : A polynomial of degree five or more than five does not have any particular name. Such a polynomial is usually called a polynomial of degree five or six or ..... etc.

(b) Polynomial Based on Terms :

There are three types of polynomial based on number of terms.

(i) Monomial : A polynomial is said to be a monomial if it has only one term.
For example, x, $9x^2$, $-5x^2$ are all monomials

(ii) Binomial : A polynomial is said to be a binomial if it contains two terms.
For example $2x^2 + 3x$, $\sqrt{3}\,x + 5x^4$, $-8x^3 + 3$ etc are all binomials.

(iii) Trinomial : A polynomial is said to be a trinomial if it contains three terms.
For example $3x^3 - 8x + \frac{5}{2}$, $\sqrt{7} - x^{10} + 8x^4 - 3x^2$ etc are all trinomials.

REMARKS :
(i) A polynomial having four or more than four terms does not have any particular name. They are simply called polynomials.
(ii) A polynomial whose coefficients are all zero is called a zero polynomial, degree of a zero polynomial is not defined.

C. ZEROS OR ROOTS OF A POLYNOMIAL

(a) Value of a Polynomial :

The value of a polynomial **f(x)** at **x = α** is obtained by substituting x = α in the given polynomial and is denoted by **f(α)**.
Consider the polynomial $f(x) = x^3 - 6x^2 + 11x - 6$,
If we replace x by – 2 everywhere in f(x), we get

$$f(-2) = (-2)^3 - 6(-2)^2 + 11(-2) - 6$$
$$f(-2) = -8 - 24 - 22 - 6$$
$$f(-2) = -60 \neq 0.$$

So, we can say that value of f(x) at x = – 2 is – 60.

(b) Zero or root of a Polynomial :

The real number α is a root or zero of a polynomial **f(x)**, if **f(α) = 0**.
Consider the polynomial $f(x) = 2x^3 + x^2 - 7x - 6$,
If we replace x by 2 everywhere in f(x), we get

$\because$ $f(2) = 2(2)^3 + (2)^2 - 7(2) - 6$

$= 16 + 4 - 14 - 6 = 0$

Hence, x = 2 is a root of f(x).

D. REMAINDER THEOREM

Let **'p(x)'** be any polynomial of degree greater than or equal to one and **a** be any real number and If **p(x)** is divided by **(x – a)**, then the remainder is equal to **p(a)**.

Solved Examples

Example.1 Find the remainder, when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by g(x) = 1 – 2x.

Solution. $f(x) = x^3 - 6x^2 + 2x - 4$

Let, $1 - 2x = 0$ $\quad 2x = 1$ $\quad x = \frac{1}{2}$

Remainder = $f\left(\frac{1}{2}\right)$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4$$

$$= \frac{1}{8} - \frac{3}{2} + 1 - 4 = \frac{1-12+8-32}{8} = -\frac{35}{8}.$$

E. FACTOR THEOREM

Let **p(x)** be a polynomial of degree greater than or equal to 1 and **'a'** be a real number such that **p(a) = 0**, then **(x – a)** is a factor of **p(x)**. Conversely, if **(x – a)** is a factor of **p(x),** then **p(a) = 0**.

Solved Examples

Example.2 Show that x + 1 and 2x – 3 are factors of $2x^3 - 9x^2 + x + 12$.

Solution. To prove that (x + 1) and (2x – 3) are factors of $2x^3 - 9x^2 + x + 12$ it is sufficient to show that p(–1) and $p\left(\frac{3}{2}\right)$ both are equal to zero.

$p(-1) = 2(-1)^3 - 9(-1)^2 + (-1) + 12 = -2 - 9 - 1 + 12 = -12 + 12 = 0.$

and $p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12 = \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4}$

$= \frac{-81 + 81}{4} = 0.$

Hence, (x + 1) and (2x – 3) are the factors $2x^3 - 9x^2 + x + 12$.

Example. 3 Find the values of a and b so that the polynomials $x^3 - ax^2 - 13x + b$ has (x – 1) and (x + 3) as factors.

Solution. Let $f(x) = x^3 - ax^2 - 13x + b$

Because (x – 1) and (x + 3) are the factors of f(x),

$\therefore$ f(1) = 0 and f(– 3) = 0

$f(1) = 0 \Rightarrow (1)^3 - a(1)^2 - 13(1) + b = 0 \Rightarrow 1 - a - 13 + b = 0$

$\Rightarrow -a + b = 12$ .... (i)

f(–3) = 0

$\Rightarrow (-3)^3 - a(-3)^2 - 13(-3) + b = 0$

$\Rightarrow -27 - 9a + 39 + b = 0$

$\Rightarrow -9a + b = -12$ ...(ii)

Subtracting equation (ii) from equation (i)

$(-a + b) - (-9a + b) = 12 + 12 \Rightarrow -a + 9a = 24 \Rightarrow 8a = 24 \Rightarrow a = 3.$

Put a = 3 in equation (i)

$-3 + b = 12 \Rightarrow b = 15.$

Hence, a = 3 and b = 15.

F. DIVISION ALGORITHM FOR POLYNOMIALS

If p(x) and g(x) are any two polynomials with $g(x) \neq 0$, then we can find polynomials r(x) and q(x) such that $p(x) = g(x) \times q(x) + r(x)$

i.e. Dividend = (Divisor x Quotient) + Remainder

where r(x)=0 or degree of r(x) < degree of g(x).

(i) If r(x) = 0, g(x) is a factor of p(x)

(ii) If deg(p(x)) > deg(g(x)),
then deg(q(x)) = deg(p(x)) –deg(g(x))

(iii) If deg(p(x)) = deg(g(x)),
then deg(q(x)) = 0 and deg(r(x)) < deg(g(x))

Solved Examples

Example.4 What must be subtracted from $x^3 - 6x^2 - 15x + 80$ so that the result is exactly divisible by $x^2 + x - 12$.

Solution. Let ax + b be subtracted from $p(x) = x^3 - 6x^2 - 15x + 80$ so that it is exactly divisible by $x^2 + x - 12$.

$\therefore s(x) = x^3 - 6x^2 - 15x + 80 - (ax + b) = x^3 - 6x^2 - (15 + a)x + (80 - b)$

Dividend = Divisor × quotient + remainder

But remainder will be zero.

$\therefore$ Dividend = Divisor × quotient

$\Rightarrow s(x) = (x^2 + x - 12) \times$ quotient $\Rightarrow$ $s(x) = x^3 - 6x^2 - (15 + a)x + (80 - b)$

$$
\begin{array}{r|l}
 & x-7 \\
\hline
x^2+x-12 & x^3-6x^2-x(15+a)+80-b \\
 & \underline{{}_{-}x^3 \;{}_{-}^{+}x^2 \;{}_{-}^{+}12x} \\
 & -7x^2+12x-(15+a)x+80-b \\
 & \qquad or \\
 & -7x^2+x(-3-a)+80-b \\
 & \underline{{}_{+}^{-}7x^2 \;{}_{+}^{-}7x \qquad {}_{-}^{+}84} \\
 & x(4-a)+(-4-b)=0
\end{array}
$$

Hence, $x(4 - a) + (-4 - b) = 0.x + 0$

$\Rightarrow$ $4 - a = 0$ & $(-4 - b) = 0$ $\Rightarrow$ $a = 4$ and $b = -4$

Hence, if in p(x) we subtract 4x – 4 then it is exactly divisible by $x^2 + x - 12$.

Example.5 Find all the zeros of the polynomial $f(x) = 2x^4 - 2x^3 - 7x^2 + 3x + 6$, if two of its zeros are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$.

Solution. Since, $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$ are zeros of f(x).

Therefore, $\left(x+\sqrt{\frac{3}{2}}\right)\left(x-\sqrt{\frac{3}{2}}\right) = \left(x^2-\frac{3}{2}\right) = \frac{2x^2-3}{2}$ or $2x^2 - 3$ is a factor of f(x).

$$
\begin{array}{r|l}
 & x^2-x-2 \\
\hline
2x^2-3 & 2x^4-2x^3-7x^2+3x+6 \\
 & \underline{{}_{-}2x^4 \qquad {}_{-}^{+}3x^2} \\
 & -2x^3-4x^2+3x+6 \\
 & \underline{{}_{+}^{-}2x^3 \qquad {}_{-}^{+}3x} \\
 & -4x^2+6 \\
 & \underline{-4x^2+6} \\
 & {}_{+}\quad {}_{-} \\
 & 0
\end{array}
$$

$\therefore 2x^4 - 2x^3 - 7x^2 + 3x + 6 = (2x^2 - 3)(x^2 - x - 2) = (2x^2 - 3)(x - 2)(x + 1)$

$= 2\left(x+\sqrt{\frac{3}{2}}\right)\left(x-\sqrt{\frac{3}{2}}\right)(x - 2)(x + 1)$

So, the zeros are $-\sqrt{\frac{3}{2}}$, $\sqrt{\frac{3}{2}}$, 2, – 1.

G. ALGEBRAIC IDENTITIES

Some important identities are

(i) $(a + b)^2 = a^2 + 2ab + b^2$

(ii) $(a – b)^2 = a^2 – 2ab + b^2$

(iii) $a^2 – b^2 = (a + b) (a – b)$

(iv) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2 bc + 2ca$

(v) $a^3 + b^3 = (a + b) (a^2 – ab + b^2)$

(vi) $a^3 – b^3 = (a – b) (a^2 + ab + b^2)$

(vii) $(a + b)^3 = a^3 + b^3 + 3ab (a + b)$

(viii) $(a – b)^3 = a^3 – b^3 – 3ab (a – b)$

(ix) $a^3 + b^3 + c^3 – 3abc = (a + b + c) (a^2 + b^2 + c^2 – ab – bc – ac)$

Special case : if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$.

Value Form :

(i) $a^2 + b^2 = (a + b)^2 – 2ab$, if **a + b** and **ab** are given.

(ii) $a^2 + b^2 = (a – b)^2 + 2ab$, if **a – b** and **ab** are given.

(iii) $a + b = \sqrt{(a-b)^2 + 4ab}$, if **a – b** and **ab** are given.

(iv) $a – b = \sqrt{(a+b)^2 - 4ab}$, if **a + b** and **ab** are given.

(v) $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 – 2$, if **a +** $\frac{1}{a}$ is given.

(vi) $a^2 + \frac{1}{a^2} = \left(a - \frac{1}{a}\right)^2 + 2$, if **a –** $\frac{1}{a}$ is given.

(vii) $a^3 + b^3 = (a + b)^3 – 3ab (a + b)$, if **(a + b)** and **ab** are given.

(viii) $a^3 – b^3 = (a – b)^3 + 3ab (a – b)$, if **(a – b)** and **ab** are given.

(ix) $a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 – 3\left(a + \frac{1}{a}\right)$, if **a +** $\frac{1}{a}$ is given.

(x) $a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 + 3\left(a - \frac{1}{a}\right)$, if **a –** $\frac{1}{a}$ is given.

(xi) $a^4 – b^4 = (a^2 + b^2) (a^2 – b^2) = [(a + b)^2 – 2ab](a + b) (a – b)$.

Solved Examples

Example.6 Expand :

(i) $\left(2x - \frac{1}{3x}\right)^2$ **(ii)** $(3x^2 + 5y)^2$

(iii) $\left(\sqrt{2}x - 3y\right)\left(\sqrt{2}x + 3y\right)$ **(iv)** $\left(\frac{1}{4}a - \frac{1}{2}b + 1\right)^2$

Solution. **(i)** $\left(2x - \frac{1}{3x}\right)^2 = (2x)^2 – 2(2x)\left(\frac{1}{3x}\right) + \frac{1}{(3x)^2} = 4x^2 – \frac{4}{3} + \frac{1}{9x^2}$.

(ii) $(3x^2 + 5y)^2 = (3x^2)^2 + 2(3x^2)(5y) + (5y)^2 = 9x^4 + 30x^2y + 25y^2$

(iii) $(\sqrt{2}\,x – 3y)(\sqrt{2}\,x + 3y) = (\sqrt{2}\,x)^2 – (3y)^2 = 2x^2 – 9y^2$

(iv) $\left(\frac{1}{4}a-\frac{1}{2}b+1\right)^2$

$= \left(\frac{1}{4}a\right)^2 + \left(-\frac{1}{2}b\right)^2 + (1)^2 + 2\left(\frac{1}{4}a\right)\left(-\frac{1}{2}b\right) + 2\left(-\frac{1}{2}b\right)(1) + 2(1)\left(\frac{1}{4}a\right)$

$= \frac{1}{16}a^2 + \frac{1}{4}b^2 + 1 - \frac{ab}{4} - b + \frac{a}{2}$.

Example. 7 Simplify :

(i) $\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right)$ **(ii)** $(2x + y)(2x - y)(4x^2 + y^2)$

Solution. **(i)** $\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right) = \left(x^2-\frac{1}{x^2}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right)$

$= \left[(x^2)^2-\left(\frac{1}{x^2}\right)^2\right]\left(x^4+\frac{1}{x^4}\right) = \left(x^4-\frac{1}{x^4}\right)\left(x^4+\frac{1}{x^4}\right) = (x^4)^2-\left(\frac{1}{x^4}\right)^2 = x^8 - \frac{1}{x^8}$.

(ii) $(2x + y)(2x - y)(4x^2 + y^2) = [(2x)^2 - (y)^2](4x^2 + y^2)$
$= (4x^2 - y^2)(4x^2 + y^2) = (4x^2)^2 - (y^2)^2 = 16x^4 - y^4$.

Example.8 Find the value of $x - y$ when $x + y = 9$ & $xy = 14$:

Solution. $x + y = 9$

On squaring both sides $x^2 + y^2 + 2xy = 81$

Putting value of $xy = 14$

$x^2 + y^2 + 28 = 81$

$x^2 + y^2 = 81 - 28 = 53$ ...(i)

$(x - y)^2 = x^2 + y^2 - 2xy$

Putting $xy = 14$ and (i)

$(x - y)^2 = 53 - 2(14) = 53 - 28 \Rightarrow (x - y)^2 = 25 \Rightarrow x - y = \pm\sqrt{25} = \pm 5$

Example.9 If $x^2 + \frac{1}{x^2} = 23$, find the values of $\left(x+\frac{1}{x}\right)$, $\left(x-\frac{1}{x}\right)$ and $\left(x^4+\frac{1}{x^4}\right)$.

Solution. $x^2 + \frac{1}{x^2} = 23$ …(i)

$\Rightarrow x^2 + \frac{1}{x^2} + 2 = 25$ [Adding 2 on both sides of (i)]

$\Rightarrow (x^2) + \left(\frac{1}{x}\right)^2 + 2 \cdot x \cdot \frac{1}{x} = 25 \Rightarrow \left(x+\frac{1}{x}\right)^2 = (5)^2 \Rightarrow x + \frac{1}{x} = \pm 5$

$\left(x-\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 \Rightarrow \left(x-\frac{1}{x}\right)^2 = 23 - 2 = 21 \Rightarrow \left(x-\frac{1}{x}\right) = \pm\sqrt{21}$.

$\left(x^2+\frac{1}{x^2}\right)^2 = \left(x^4+\frac{1}{x^4}\right) + 2 \Rightarrow \left(x^4+\frac{1}{x^4}\right) = \left(x^2+\frac{1}{x^2}\right)^2 - 2$

$\Rightarrow \left(x^4+\frac{1}{x^4}\right) = (23)^2 - 2 = 529 - 2 \Rightarrow \left(x^4+\frac{1}{x^4}\right) = 527.$

Example.10 Find the value of $\left[\frac{a^2-5ab}{a^2-6ab+5b^2} \times \frac{a^2-b^2}{a^2+ab}\right]$.

Solution. $\frac{a^2-5ab}{a^2-6ab+5b^2} \times \frac{a^2-b^2}{a^2+ab} = \frac{a(a-5b)}{(a-b)(a-5b)} \times \frac{(a-b)(a+b)}{a(a+b)} = 1$

Example.11 Find the value of $\dfrac{7.98\times7.98-2.02\times2.02}{5.96}$

Solution. $\dfrac{7.98\times7.98-2.02\times2.02}{5.96} = \dfrac{(7.98+2.02)(7.98-2.02)}{5.96} = \dfrac{10\times5.96}{5.96} = 10.$

Example.12 Simplify :

(i) $(3x + 4)^3 - (3x - 4)^3$ **(ii)** $\left(x+\dfrac{2}{x}\right)^3+\left(x-\dfrac{2}{x}\right)^3$

Solution. **(i)** $(3x + 4)^3 - (3x - 4)^3 = [(3x)^3 + (4)^3 + 3(3x)(4)(3x + 4)] - [(3x)^3 - (4)^3 - 3(3x)(4)(3x - 4)]$
$= [27x^3 + 64 + 36x(3x + 4)] - [27x^3 - 64 - 36x(3x - 4)]$
$= [27x^3 + 64 + 108x^2 + 144x] - [27x^3 - 64 - 108x^2 + 144x]$
$= 27x^3 + 64 + 108x^2 + 144x - 27x^3 + 64 + 108x^2 - 144x = 128 + 216x^2.$

(ii) $\left(x+\dfrac{2}{x}\right)^3+\left(x-\dfrac{2}{x}\right)^3 = x^3+\left(\dfrac{2}{x}\right)^3+3(x)\left(\dfrac{2}{x}\right)\left(x+\dfrac{2}{x}\right)+x^3-\left(\dfrac{2}{x}\right)^3-3(x)\left(\dfrac{2}{x}\right)\left(x-\dfrac{2}{x}\right)$

$= x^3+\dfrac{8}{x^3}+6x+\dfrac{12}{x}+x^3-\dfrac{8}{x^3}-6x+\dfrac{12}{x} = 2x^3+\dfrac{24}{x}$.

Example.13 Evaluate :

(i) $(1005)^3$ **(ii)** $(997)^3$

Solution. **(i)** $(1005)^3 = (1000 + 5)^3 = (1000)^3 + (5)^3 + 3(1000)(5)(1000 + 5)$
$= 1000000000 + 125 + 15000(1000 + 5) = 1000000000 + 125 + 15000000 + 75000$
$= 1015075125.$

(ii) $(997)^3 = (1000 - 3)^3 = (1000)^3 - (3)^3 - 3 \times 1000 \times 3 \times (1000 - 3)$
$= 1000000000 - 27 - 9000 \times (1000 - 3) = 1000000000 - 27 - 9000000 + 27000$
$= 991026973.$

Example.14 If $x - \dfrac{1}{x} = 5$, find the value of $x^3 - \dfrac{1}{x^3}$.

Solution. We have, $x - \dfrac{1}{x} = 5$ ...(i)

$\Rightarrow \left(x-\dfrac{1}{x}\right)^3 = (5)^3$ [Cubing both sides of (i)]

$\Rightarrow x^3 - \dfrac{1}{x^3} - 3x\cdot\dfrac{1}{x}\cdot\left(x-\dfrac{1}{x}\right) = 125$

$\Rightarrow x^3 - \dfrac{1}{x^3} - 3\left(x-\dfrac{1}{x}\right) = 125$

$\Rightarrow x^3 - \dfrac{1}{x^3} - 3 \times 5 = 125$ [Substituting $\left(x-\dfrac{1}{x}\right) = 5$]

$\Rightarrow x^3 - \dfrac{1}{x^3} - 15 = 125$

$\Rightarrow x^3 - \dfrac{1}{x^3} = (125 + 15) = 140.$

Example.15 Find the products of the following expression :

(i) $(4x + 3y)(16x^2 - 12xy + 9y^2)$

(ii) $(5x - 2y)(25x^2 + 10xy + 4y^2)$

Solution. **(i)** $(4x + 3y)(16x^2 - 12xy + 9y^2) = (4x + 3y)[(4x)^2 - (4x) \times (3y) + (3y)^2]$
$= (a + b)(a^2 - ab + b^2)$ [Where $a = 4x$, $b = 3y$]
$= a^3 + b^3 = (4x)^3 + (3y)^3 = 64x^3 + 27y^3.$

(ii) $(5x - 2y)(25x^2 + 10xy + 4y^2) = (5x - 2y)[(5x)^2 + (5x) \times (2y) + (2y)^2]$
$= (a - b)(a^2 + ab + b^2)$ [Where $a = 5x$, $b = 2y$] $= a^3 - b^3$
$= (5x)^3 - (2y)^3 = 125x^3 - 8y^3.$

Example. 16 If a + b + c = 9 and ab + bc + ac = 26, find the value of $a^3 + b^3 + c^3 - 3abc$.

Solution. We have a + b + c = 9 ...(i)

$\Rightarrow (a + b + c)^2 = 81$ [On squaring both sides of (i)]

$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ac) = 81$

$\Rightarrow a^2 + b^2 + c^2 + 2 \times 26 = 81$ [$\because$ ab + bc + ac = 26]

$\Rightarrow a^2 + b^2 + c^2 = (81 - 52)$ $\Rightarrow a^2 + b^2 + c^2 = 29$.

Now, we have

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

$= (a + b + c)[(a^2 + b^2 + c^2) - (ab + bc + ac)] = 9 \times [(29 - 26)] = (9 \times 3) = 27$.

Example.17 Simplify : $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

Solution. Here, $(a^2-b^2)+(b^2-c^2)+(c^2-a^2)=0$

$\therefore (a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3 = 3(a^2-b^2)(b^2-c^2)(c^2-a^2)$

Also, $(a-b)+(b-c)+(c-a)=0$

$\therefore (a-b)^3+(b-c)^3+(c-a)^3 = 3(a-b)(b-c)(c-a)$

$\therefore$ Given expression

$= \dfrac{3(a^2-b^2)(b^2-c^2)(c^2-a^2)}{3(a-b)(b-c)(c-a)} = \dfrac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$

$= (a+b)(b+c)(c+a)$.

Example.18 Find the value of $(28)^3 - (78)^3 + (50)^3$.

Solution. Let a = 28, b = – 78, c = 50

Then, a + b + c = 28 – 78 + 50 = 0

$a^3 + b^3 + c^3 = 3abc$.

So, $(28)^3 + (-78)^3 + (50)^3 = 3 \times 28 \times (-78) \times 50 = -327600$.

G. FACTORS OF A POLYNOMIAL

To express a given polynomial as the product of polynomials, each of degree less than that of the given polynomial such that no such a factor has a factor of lower degree, is called **factorization**.

(a) Factorization by taking out the common factor :

When each term of an expression has a common factor, divide each term by this factor and take out as a multiple.

Solved Examples

Example.19 Factorize : $6x^3 + 8x^2 - 10x$

Solution. $6x^3 + 8x^2 - 10x = 2x(3x^2 + 4x - 5)$

(b) Factorization by grouping :

Example.20 Factorize : ax + by + ay + bx

Solution. ax + by + ay + bx = ax + ay + bx + by = a (x + y) + b(x+y) = (x + y) (a + b)

(c) Factorization by making a perfect square :

Example.21 Factorize : $4x^2 + 12x + 9$

Solution. $4x^2 + 12x + 9 = (2x)^2 + 2(2x)(3) + 3^2 = (2x + 3)^2$

(d) Factorization the difference of two squares :

Example.22 Factorize : $4x^2 - 25$.

Solution. $4x^2 - 25 = (2x)^2 - (5)^2 = (2x - 5)(2x + 5)$

(e) Factorization of a quadratic polynomial by splitting the middle term :

Example.23 Factorize : $x^2 + 4\sqrt{2}\,x + 6$

Solution. $x^2 + 4\sqrt{2}\,x + 6 = x^2 + 3\sqrt{2}\,x + \sqrt{2}\,x + 6$

$= x(x + 3\sqrt{2}) + \sqrt{2}(x + 3\sqrt{2}) = (x + 3\sqrt{2})(x + \sqrt{2})$

(f) Factorization of a algebraic expression as the sum or difference of two cubes :

Example.24 Factorize : $16a^3b - 250b^4$.

Solution. $16a^3b - 250b^4 = 2b(8a^3 - 125b^3) = 2b\{(2a)^3 - (5b)^3\} = 2b\{(2a-5b)(4a^2+25b^2+10ab)\}$

(g) Factorization of a algebraic expression of the form $a^3 + b^3 + c^3 - 3abc$:

Example.25 Factorize :

(i) $2\sqrt{2}a^3 + 8b^3 - 27c^3 + 18\sqrt{2}abc$ **(ii)** $(x - y)^3 + (y - z)^3 + (z - x)^3$

Solution. **(i)** $2\sqrt{2}a^3 + 8b^3 - 27c^3 + 18\sqrt{2}abc = \left(\sqrt{2}a\right)^3 + (2b)^3 + (-3c)^3 - 3(\sqrt{2}a)(2b)(-3c)$

$= \left(\sqrt{2}a + 2b - 3c\right)\left[\left(\sqrt{2}a\right)^2 + (2b)^2 + (-3c)^2 - \left(\sqrt{2}a\right)(2b) - (2b)(-3c) - (-3c)\left(\sqrt{2}a\right)\right]$

$= \left(\sqrt{2}a + 2b - 3c\right)\left[2a^2 + 4b^2 + 9c^2 - 2\sqrt{2}ab + 6bc + 3\sqrt{2}ac\right]$

(ii) $(x - y)^3 + (y - z)^3 + (z - x)^3$

Let $a = x - y$, $b = y - z$ and $c = z - x$

So, $(x - y)^3 + (y - z)^3 + (z - x)^3 = a^3 + b^3 + c^3$

Now, $a + b + c = x - y + y - z + z - x = 0$

So, $a^3 + b^3 + c^3 = 3abc$

$\therefore (x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$

H. H.C.F. AND L.C.M. OF POLYNOMIALS

- A polynomial D(x) is a divisor of the polynomial P(x) if it is a factor of P(x). Where Q(x) is another polynomial such that P(x) = D(x) × Q(x)
- **HCF/GCD (Greatest Common Divisor) :** A polynomial h(x) is called the HCF or GCD of two or more given polynomials, if h(x) is a polynomial of highest degree dividing each of one of the given polynomials.
- **L.C.M. (Least Common Multiple) :** A polynomial P(x) is called the LCM of two or more given polynomials, if it is a polynomial of smallest degree which is divided by each one of the given polynomials. For any two polynomials P(x) and Q(x). We have :
  P(x) × Q(x) = [HCF of P(x) and Q(x)] × [LCM of P(x) and Q (x)]

Solved Examples

Example.26 If $p(x) = (x + 2)(x^2 - 4x - 21)$, $Q(x) = (x - 7)(2x^2 + x - 6)$ find the HCF and LCM of P(x) and Q(x).

Solution. $P(x) = (x + 2)(x^2 - 4x - 21) = (x + 2)(x^2 - 7x + 3x - 21) = (x + 2)(x - 7)(x + 3)$

$Q(x) = (x - 7)(2x^2 + x - 6) = (x - 7)(2x^2 + 4x - 3x - 6) = (x - 7)[2x(x + 2) - 3(x + 2)]$

$= (x - 7)(2x - 3)(x + 2)$

HCF = $(x + 2)(x - 7)$

LCM = $(x + 3)(x - 7)(2x - 3)(x + 2)$.

Example.27 If HCF & LCM of P(x) and Q(x) are $(x + 2)$ and $(x + 3)(x^2 + 9x + 14)$ respectively if $P(x) = x^2 + 5x + 6$, find Q(x).

Solution. $P(x) = (x^2 + 5x + 6) = (x + 2)(x + 3)$

LCM = $(x + 3)(x^2 + 9x + 14) = (x + 3)(x + 7)(x + 2)$

We know that HCF $\times$ LCM = P(x) $\times$ Q(x)

$$Q(x) = \frac{(x+2)(x+3)(x+7)(x+2)}{(x+2)(x+3)} = (x + 7)(x+2) = x^2 + 9x + 14.$$

I. RELATIONSHIP BETWEEN ZEROS AND COEFFICIENT OF A QUADRATIC POLYNOMIALS

Let α and β be the zeros of a quadratic polynomial $f(x) = ax^2 + bx + c$. By factor theorem $(x - \alpha)$ and $(x - \beta)$ are the factors of f(x).

$\therefore$ $f(x) = k(x - \alpha)(x - \beta)$ are the factors of f(x)

$\Rightarrow ax^2 + bx + c = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$ $\Rightarrow$ $ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta$

Comparing the coefficients of x^2, x and constant terms on both sides, we get

$a = k$, $b = -k(\alpha + \beta)$ and $c = k\alpha\beta$

$\Rightarrow \alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

$$\Rightarrow \quad \alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} \quad \text{and} \quad \alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence, Sum of the zeros $= -\dfrac{b}{a} = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of the zeros $= \dfrac{c}{a} = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$.

❖ **REMARKS :**

If α and β are the zeros of a quadratic polynomial f(x). Then, the polynomial f(x) is given by

$f(x) = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$ or $f(x) = k\{x^2 - (\text{Sum of the zeros})x + \text{Product of the zeros}\}$

Solved Examples

Example.28 Find a quadratic polynomial whose zeros are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

Solution. Given : $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$

Sum of zeros $= \alpha + \beta = 3 + \sqrt{5} + 3 - \sqrt{5} = 6$

Products of zeros $= \alpha\beta$

$= (3 + \sqrt{5})(3 - \sqrt{5}) = 9 - 5 = 4$

So, quadratic polynomial is $k\{x^2 - 6x + 4\}$, where k is any constant.

J. RELATIONSHIP BETWEEN ZEROS AND COEFFICIENT OF A CUBIC POLYNOMIAL

Let α, β, γ be the zeros of a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then, by factor theorem, $x - \alpha$, $x - \beta$ and $x - \gamma$ are factors of f(x) . Also, f(x) being a cubic polynomial, cannot have more than three linear factors.

$\therefore$ $f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$

$\Rightarrow$ $ax^3 + bx^2 + cx + d = k(x - \alpha)(x - \beta)(x - \gamma)$

$\Rightarrow$ $ax^3 + bx^2 + cx + d = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\}$

$\Rightarrow$ $ax^3 + bx^2 + cx + d = kx^3 - k(\alpha + \beta + \gamma)x^2 + k(\alpha\beta + \beta\gamma + \gamma\alpha)x - k\alpha\beta\gamma$

Comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$a = k$, $b = -k(\alpha + \beta + \gamma)$, $c = k(\alpha\beta + \beta\gamma + \gamma\alpha)$ and $d = -k(\alpha\beta\gamma)$

$\Rightarrow$ $\alpha + \beta + \gamma = -\dfrac{b}{a}$ $\Rightarrow$ $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$ And, $\alpha\beta\gamma = -\dfrac{d}{a}$

$\Rightarrow$ Sum of the zeros $= -\dfrac{b}{a} = -\dfrac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

$\Rightarrow$ Sum of the products of the zeros taken two at a time $= \dfrac{c}{a} = \dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

$\Rightarrow$ Product of the zeros $= -\dfrac{d}{a} = -\dfrac{\text{Constant term}}{\text{Coefficient of } x^3}$

❖ **REMARKS :**

Cubic polynomial having α, β and γ as its zeros is given by

$f(x) = k(x - \alpha)(x - \beta)(x - \gamma)$

$f(x) = k\{x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma\}$, where k is any non-zero real number.

Solved Examples

Example.29 α, β, γ are zeros of cubic polynomial $x^3 - 12x^2 + 44x + c$. If α, β, γ are in A.P., find the value of c.

Solution. $f(x) = x^3 - 12x^2 + 44x + c$.

α, β, γ are in A.P.

Let $\alpha = a - d$, $\beta = a$, $\gamma = a + d$

Sum of zeros $= \alpha + \beta + \gamma = a - d + a + a + d = 3a$

So, $3a = -(-12)$

$a = 4$.

Sum of product of zeros taken two at a time

$\Rightarrow$ $\alpha\beta + \beta\gamma + \gamma\alpha = 44$ $\Rightarrow$ $(a - d)a + (a)(a + d) + (a + d)(a - d) = 44$

$\Rightarrow$ $a^2 - ad + a^2 + ad + a^2 - d^2 = 44$ $\Rightarrow$ $3a^2 - d^2 = 44$

$\Rightarrow$ $3(4)^2 - d^2 = 44 \Rightarrow d^2 = 4$ $\Rightarrow$ $d = \pm 2$

product of zeros $= \alpha\beta\gamma = -c$

$\Rightarrow$ $(a - d)a(a + d) = -c$ $\Rightarrow$ $(4 - 2)(4)(4 + 2) = -c$

$\Rightarrow$ $(2)(4)(6) = -c$ $\Rightarrow$ $c = -48$.

K. GRAPH OF POLYNOMIALS

In algebraic or in set theoretic language the graph of a polynomial f(x) is the collection (or set) of all points **(x, y)**, where **y = f(x).** In geometrical or in graphical language the graph of a polynomial f(x) is a smooth free hand curve passing through points (x_1, y_1), (x_2, y_2), (x_3, y_3),... **etc,** where y_1, y_2, y_3, ... are the values of the polynomial **f(x)** at x_1, x_2, x_3, ... respectively.

In order to draw the graph of a polynomial **f(x)**, follow the following algorithm.

ALGORITHM :

Step (i) Find the values $y_1, y_2, ..., y_n, ...$ of polynomial f(x) on different points $x_1, x_2, ..., x_n, ...$ and prepare a table that gives values of **y** or **f(x)** for various values of **x**.

x :	x_1	x_2	...	x_n	x_{n+1}	...
y = f(x) :	$y_1 = f(x_1)$	$y_2 = f(x_2)$	···	$y_n = f(x_n)$	$y_{n+1} = f(x_{n+1})$	···

Step (ii) Plot that points $(x_1, y_1), (x_2, y_2), (x_3, y_3), ... (x_n, y_n), ...$ on rectangular co-ordinate system. In plotting these points use different scales on the X and Y axes.

Step (iii) Draw a free hand smooth curve passing through points plotted in **step 2** to get the graph of the polynomial **f(x)**.

(a) Graph of a Linear Polynomial :

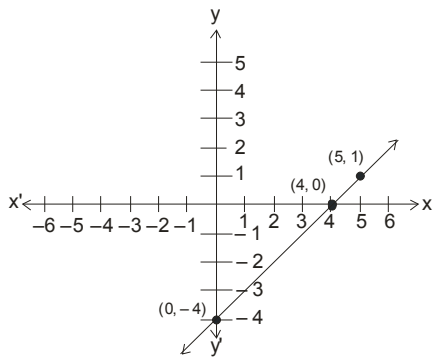
Consider a linear polynomial **f(x)= ax + b**, $a \neq 0$. Graph of **y = ax + b** is a straight line. That is why f(x) = ax + b is called a **linear polynomial**. Since two points determine a straight line, so only two points need to plotted to draw the line **y = ax + b**. The line represented by **y = ax + b** crosses the X-axis at exactly one point, namely $\left(-\frac{b}{a}, 0\right)$.

Solved Examples

Example.30 Draw the graph of y = x – 4.

Solution. y = x – 4

x	0	4	5
y	– 4	0	1



(b) Graph of a Quadratic Polynomial :

Let a, b, c be real numbers and $a \neq 0$. Then $f(x) = ax^2 + bx + c$ is known as a **quadratic polynomial** in x. Graph of the quadratic polynomial i.e. the curve whose equation is $y = ax^2 + bx + c$, $a \neq 0$. Graph of a quadratic polynomial is always a parabola.

Let $y = ax^2 + bx + c$, where $a \neq 0$

$\Rightarrow$ $4ay = 4a^2x^2 + 4abx + 4ac$

$\Rightarrow$ $4ay = 4a^2x^2 + 4abx + b^2 - b^2 + 4ac$

$\Rightarrow$ $4ay = (2ax + b)^2 - (b^2 - 4ac)$

$\Rightarrow$ $4ay + (b^2 - 4ac) = (2ax + b)^2$

$\Rightarrow$ $4ay + (b^2 - 4ac) = 4a^2(x + b/2a)^2$

$\Rightarrow$ $4a\left\{y + \frac{b^2 - 4ac}{4a}\right\} = 4a^2\left(x + \frac{b}{2a}\right)^2$

$\Rightarrow$ $\left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$ **....(i)**

where, **D ≡ b² – 4ac** is the **discriminant** of the quadratic equation.

❖ **REMARKS :**

Shifting the origin at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$,

we have $X = x - \left(-\frac{b}{2a}\right)$ and $Y = y - \frac{(-D)}{4a}$.

Substituting these values in (i), we obtain **$Y = aX^2$ ...(ii)**
which is the standard equation of parabola.

Clearly, this is the equation of a parabola having its vertex at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.

The parabola opens upwards or downwards according as a > 0 or a < 0.

L. SIGN OF QUADRATIC EXPRESSIONS

Let α be a real root of **$ax^2 + bx + c = 0$**. Then, $a\alpha^2 + b\alpha + c = 0$. Point $(\alpha, 0)$ lies on $y = ax^2 + bx + c$. Thus, every real root of $ax^2 + bx + c = 0$ represents a point of intersection of the parabola with the X-axis.

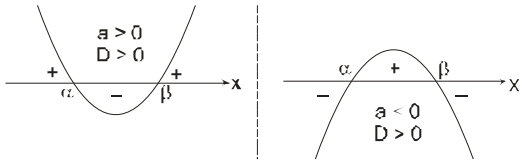
Conversely, if the parabola $y = ax^2 + bx + c$ intersects the X-axis at a point $(\alpha, 0)$, then $(\alpha, 0)$ satisfies the equation $y = ax^2 + bx + c$

$\Rightarrow a\alpha^2 + b\alpha + c = 0$ [α is a real root of $ax^2 + bx + c = 0$]

Thus, the intersection of the parabola $y = ax^2 + bx + c$ with X-axis gives all the real roots of $ax^2 + bx + c = 0$. Following conclusions may be drawn :

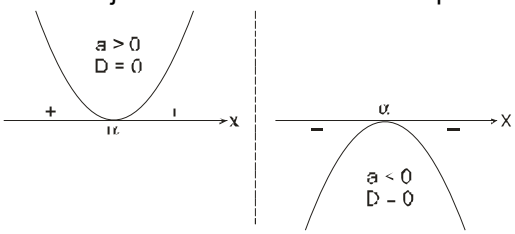
(i) If D > 0, the parabola will intersect the x-axis in two distinct points and vice-versa.

The parabola meets x-axis at $\alpha = \frac{-b-\sqrt{D}}{2a}$ and $\beta = \frac{-b+\sqrt{D}}{2a}$.



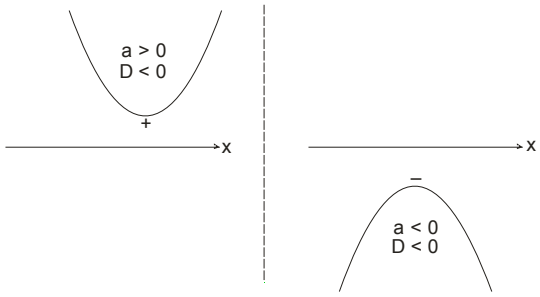
Roots are real & distinct

(ii) If D = 0, the parabola will just touch the x-axis at one point and vice-versa.



Roots are equa

(iii) If D < 0, the parabola will not intersect x-axis at all and vice-versa.



Roots are imaginary

❖ **REMARKS :**

★ $\forall$ x $\in$ R, y > 0 only if a > 0 & D ≡ b² – 4ac < 0

★ $\forall$ x $\in$ R, y < 0 only if a < 0 & D ≡ b² – 4ac < 0

Solved Examples

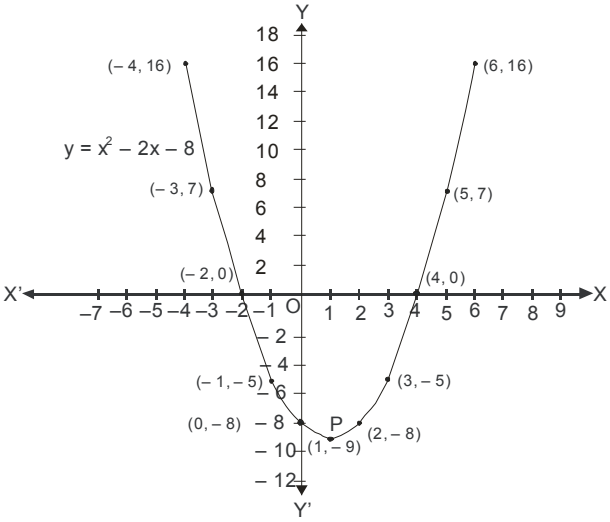
Example.31 Draw the graph of the polynomial $f(x) = x^2 - 2x - 8$.

Solution. Let $y = x^2 - 2x - 8$.

The following table gives the values of y or f(x) for various values of x.

x	– 4	– 3	– 2	– 1	0	1	2	3	4	5	6
$y = x^2 - 2x - 8$	16	7	0	– 5	– 8	– 9	– 8	– 5	0	7	16

Let us plot the points (–4,16), (–3, 7), (–2, 0), (–1, –5), (0, –8), (1, –9), (2, –8), (3, –5), (4, 0),(5, 7) and (6, 16) on a graphs paper and draw a smooth free hand curve passing through these points. The curve thus obtained represents the graphs of the polynomial $f(x) = x^2 - 2x - 8$. This is called a parabola. The lowest point P, called a minimum point, is the vertex of the parabola.Vertical line passing through P is called the axis of the parabola. Parabola is symmetric about the axis. So, it is also called the line of symmetry.



❖ **Observations :**

From the graphs of the polynomial $f(x) = x^2 - 2x - 8$, following observations can be drawn :

(i) The coefficient of x^2 in $f(x) = x^2 - 2x - 8$ is 1 (a positive real number) and so the parabola opens upwards.

(ii) $D = b^2 - 4ac = 4 + 32 = 36 > 0$. So, the parabola cuts X-axis at two distinct points.

(iii) On comparing the polynomial $x^2 - 2x - 8$ with $ax^2 + bx + c$, we get a = 1, b = – 2 and c = – 8. The vertex of the parabola has coordinates (1, – 9) i.e. $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, where $D \equiv b^2 - 4ac$.

(iv) The polynomial $f(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$ is factorizable into two distinct linear factors (x – 4) and (x + 2). So, the parabola cuts X-axis at two distinct points (4, 0) and (– 2, 0). The x-coordinates of these points are zeros of f(x).

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Factor theorem and remainder theorem

A-1 Determine whether x – 3 is a factor of polynomial $p(x) = x^3 - 3x^2 + 4x - 12$.

A-2 Using factor theorem, prove that p(x) is divisible by g(x) if $P(x) = 4x^4 + 5x^3 - 12x^2 - 11x + 5$, $g(x) = 4x + 5$

A-3 Find the remainder, when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $g(x) = 1 - 2x$.

A-4 The polynomials $ax^3 + 3x^2 - 3$ and $2x^3 - 5x + a$ when divided by (x – 4) leaves remainder R_1 & R_2 respectively then find the value of 'a' if $2R_1 - R_2 = 0$.

A-5 If $ax^3 + bx^2 + x - 6$ has x + 2 as a factor and leaves a remainder 4 when divided by (x – 2), find the values of a and b ?

Section (B) : Algebraic identities and factorization

B-1. Evaluate :

(i) $(5x + 4y)^2$ **(ii)** $(4x - 5y)^2$ **(iii)** $\left(2x - \frac{1}{x}\right)^2$

B-2. If $a^4 + \frac{1}{a^4} = 119$, then find the value of $a^3 - \frac{1}{a^3}$.

B-3. If x + y = 3 and xy = – 18, find the value of $x^3 + y^3$.

B-4. Prove that : $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

B-5. If a + b + c = 15, $a^2 + b^2 + c^2 = 83$, then find the value of $a^3 + b^3 + c^3 - 3abc$.

Section (C) : Relationship between zeros AND coefficients and graph of polynomial

C-1 If α, β are the zeroes of the polynomial $2y^2 + 7y + 5$, write the value of $\alpha + \beta + \alpha\beta$.

C-2 If α, β are zeros of $x^2 + 5x + 5$, find the value of $\alpha^{-1} + \beta^{-1}$.

C-3 If α, β are zeros of quadratic polynomial $kx^2 + 4x + 4$, find the value of k such that $(\alpha + \beta)^2 - 2\alpha\beta = 24$.

C-4 Find a quadratic polynomial whose zeros are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

C-5 Draw the graph of following polynomials.

(i) f(x) = – 3 **(ii)** f(x) = x – 4

(iii) $f(x) = x^2 - 9$ **(iv)** $f(x) = 2x^2 - 4x + 5$

OBJECTIVE QUESTIONS

Section (A) : Factor theorem and remainder theorem

A-1. The remainder obtained when $t^6 + 3t^2 + 10$ is divided by $t^3 + 1$ is :
(A) $t^2 - 11$ (B) $3t^2 + 11$ (C) $t^3 - 1$ (D) $1 - t^3$

A-2. If $(x + a)$ is a factor of $x^2 + px + q$ and $x^2 + mx + n$ then the value of a is :
(A) $\frac{m-p}{n-q}$ (B) $\frac{n-q}{m-p}$ (C) $\frac{n+q}{m+p}$ (D) $\frac{m+p}{n+q}$

A-3. If $x^2 - 4$ is a factor of $2x^3 + ax^2 + bx + 12$, where a and b are constant. Then the values of a and b are :
(A) – 3, 8 (B) 3, 8 (C) –3, – 8 (D) 3, – 8

A-4. If $x^{51} + 51$ is divided by $(x + 1)$ the remainder is :
(A) 0 (B) 1 (C) 49 (D) 50

A-5. The polynomial $p(x) = 2x^4 - x^3 - 7x^2 + ax + b$ is divisible by $x^2 - 2x - 3$ for certain values of a and b. The value of $(a + b)$, is :
(A) – 34 (B) – 30 (C) – 26 (D) – 18

Section (B) Algebraic identities and factorization

B-1. If $a^2 - b^2 = 21$ and $a^2 + b^2 = 29$, which of the following could be the value of ab ?
I. –10 **II.** $5\sqrt{2}$ **III.** 10
(A) I only (B) II only (C) III only (D) I and III only

B-2. On simplifying $(a + b)^3 + (a - b)^3 + 6a(a^2 - b^2)$ we get :
(A) $8a^2$ (B) $8a^2b$ (C) $8a^3b$ (D) $8a^3$

B-3. Find the value of $\frac{a^3 + b^3 + c^3 - 3abc}{ab + bc + ca - a^2 - b^2 - c^2}$, when a = – 5, b = – 6, c = 10.
(A) 1 (B) –1 (C) 2 (D) –2

B-4. If $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$ then which one of the following expression is correct :
(A) $x^3 + y^3 + z^3 = 0$ (B) $x + y + z = 3\, x^{\frac{1}{3}} y^{\frac{1}{3}} z^{\frac{1}{3}}$
(C) $x + y + z = 3xyz$ (D) $x^3 + y^3 + z^3 = 3xyz$

B-5. Evaluate : $\frac{(a-b)^2}{(b-c)(c-a)} + \frac{(b-c)^2}{(a-b)(c-a)} + \frac{(c-a)^2}{(a-b)(b-c)}$.
(A) 0 (B) 1 (C) 2 (D) 3

B-6. The value of $\frac{0.76 \times 0.76 \times 0.76 + 0.24 \times 0.24 \times 0.24}{0.76 \times 0.76 - 0.76 \times 0.24 + 0.24 \times 0.24}$ is :
(A) 0.52 (B) 1 (C) 0.01 (D) 0.1

B-7. If $x = \frac{1}{2}$, then the value of $x + \frac{1}{1 + \frac{1}{1 + \frac{1}{x}}}$ is :
(A) $\frac{5}{4}$ (B) $\frac{4}{5}$ (C) $\frac{3}{4}$ (D) None of these

B-8. $m^5 + m^4 + m^3 + m^2 + m + 1 = (m^3 + 1) \times$ _____
(A) $m^5 + m^4 + m^2 + m$ (B) $m^2 + m^3$ (C) $m^3 + m^3 + m + 1$ (D) $m^2 + m + 1$

B-9. If x = 0.50, then the value of the expression $\left\{(1+x+x^2)+\frac{x^3}{1-x}\right\}$ is :
(A) 4 (B) 2 (C) 1.50 (D) 1

B-10. If $p = 2^{2/3} + 2^{1/3}$, then :
(A) $p^3 - 6p + 6 = 0$ (B) $p^3 - 3p - 6 = 0$ (C) $p^3 - 6p - 6 = 0$ (D) $p^3 - 3p + 6 = 0$

Section (C) Relationship between zeros and coefficients and graph of polynomial

C-1. If one zero of $2x^2 - 3x + k$ is reciprocal to the other, then the value of k is :
(A) 2 (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) – 3

C-2. If α and β are the zeroes of $x^2 - 4x + 1$, then $\frac{1}{\alpha}+\frac{1}{\beta}-\alpha\beta$ is :
(A) 3 (B) 5 (C) – 5 (D) – 3

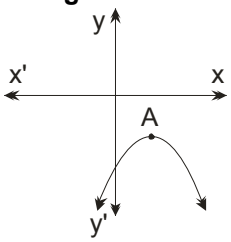
C-3. The equation $x^2 + Bx + C = 0$ has 5 as the sum of its roots, and 15 as the sum of the square of its roots. The value of 'C' is :
(A) 5 (B) 7.5 (C) 10 (D) 12.5

C-4. If α, β are the zero's of polynomial $f(x) = x^2 - p(x + 1) - c$ then $(\alpha + 1)(\beta + 1)$ is equal :
(A) c – 1 (B) 1 – c (C) c (D) 1 + c

C-5. If the sum of the zeros of the quadratic polynomial $f(t) = kt^2 + 2t + 3k$ is equal to their product, then the value of k is :
(A) $-\frac{3}{2}$ (B) $\frac{3}{2}$ (C) $-\frac{2}{3}$ (D) none of these

C-6. If α, β are the zeroes of $x^2 - 6x + k = 0$. What is the value of k if $3\alpha + 2\beta = 20$.
(A) –16 (B) 8 (C) – 2 (D) – 8

C-7. The graph of $y = ax^2 + bx + c$ is given in **figure** then identify the signs of a, b and c :



(A) a < 0, b > 0, c < 0 (B) a < 0, b < 0, c > 0 (C) a < 0, b > 0, c > 0 (D) a < 0, b < 0, c < 0

C-8. Minimum value for the polynomial $4x^2 - 6x + 1$ is :
(A) $-\frac{3}{4}$ (B) $-\frac{5}{4}$ (C) $-\frac{5}{16}$ (D) $-\infty$

Exercise-2

OBJECTIVE QUESTIONS

1. A quadratic polynomial is exactly divisible by (x + 1) & (x + 2) and leaves the remainder 4 after division by(x + 3) then that polynomial is :
(A) $x^2 + 6x + 4$ (B) $2x^2 + 6x + 4$ (C) $2x^2 + 6x - 4$ (D) $x^2 + 6x - 4$

2. What is the remainder when the polynomial $p(x) = x^{200} - 2x^{199} + x^{50} - 2x^{49} + x^2 + x + 1$ is divided by $(x - 1)(x - 2)$?
(A) 1 (B) 7 (C) $2x + 1$ (D) $6x - 5$

3. A cubic polynomial p(x) is such that p(1)=1, p(2)=2, p(3)= 3 and p(4) = 5, then the value of p(6) is :
(A) 16 (B) 13 (C) 10 (D) 7

4. If $(x + y + z) = 1$, $xy + yz + zx = -1$, $xyz = -1$, then value of $x^3 + y^3 + z^3$ is :
(A) –1 (B) 1 (C) 2 (D) –2

5. If $a^4 + \frac{1}{a^4} = 119$, then find the value of $a^3 - \frac{1}{a^3}$.
(A) 11 (B) 36 (C) 33 (D) 12

6. If $x + y = 3$ and $xy = 2$, then the value of $x^3 - y^3$ is equal to
(A) 6 (B) 7 (C) 8 (D) 0

7. If $4x - 5z = 16$ and $xz = 12$, $64x^3 - 125z^3 =$
(A) 14512 (B) 15676 (C) 25833 (D) 15616

8. $\dfrac{x^{-3} - y^{-3}}{x^{-3}y^{-1} + (xy)^{-2} + y^{-3}x^{-1}}$
(A) $x + y$ (B) $y - x$ (C) $\frac{1}{x} - \frac{1}{y}$ (D) $\frac{1}{x} + \frac{1}{y}$

9. If $\dfrac{\left(\sqrt{a} - \sqrt{b}\right)^2 + 4\sqrt{ab}}{a - b} = \dfrac{5}{3}$, then the value of a : b is :
(A) 1 : 16 (B) 1 : 4 (C) 4 : 1 (D) 16 : 1

10. If α, β are zeros of quadratic polynomial $kx^2 + 4x + 4$, find the value of k such that $(\alpha+\beta)^2 - 2\alpha\beta = 24$.
(A) – 1 (B) $\frac{2}{3}$ (C) both (A) & (B) (D) None of these

11. If α, β and γ are the three zeroes of the polynomial $p(x) = x^3 - 64x - 14$, what is the value of $\alpha^3 + \beta^3 + \gamma^3$?
(A) 36 (B) 40 (C) 42 (D) 64

12. If $(a^2 + b^2)^3 = (a^3 + b^3)^2$ then $\frac{a}{b} + \frac{b}{a} =$
(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) $\frac{5}{6}$ (D) $\frac{6}{5}$

13. If $x = \sqrt{2 + \sqrt{2}}$, then $x^4 + \frac{4}{x^4}$ is :
(A) $2(3 - \sqrt{2})$ (B) $6\sqrt{2} - 2$ (C) $6 - \sqrt{2}$ (D) 12

14. If $f(x) = x^4 - 2x^3 + 3x^2 - ax + b$ is a polynomial such that when it is divided by $(x - 1)$ and $(x + 1)$ the remainders are 5 and 19 respectively. If f(x) is divided by $(x - 2)$, then remainder is :
(A) 0 (B) 5 (C) 10 (D) 2

15. The cubic polynomial P(x) satisfies the condition that $(x - 1)^2$ is a factor of $P(x) + 2$, and $(x + 1)^2$ is a factor of $P(x) - 2$.Then P(3) equals.
(A) 27 (B) 18 (C) 12 (D) 6

16. If $(x-1)^3 + (y-2)^3 + (z-3)^3 = 3(x-1)(y-2)(z-3)$ and $x-1 \neq y-2 \neq z-3$ then $x+y+z$ is equal to
(A) 2 (B) – 6 (C) 6 (D) 4

17. If $2x^3 - 5x^2 + x + 2 = (x-2)(ax^2 - bx - 1)$, then a & b are respectively
(A) 2, 1 (B) 2, – 1 (C) 1, 2 (D) – 1, 1/2

18. The number of real roots of the equation $(x-1)^2 + (x-2)^2 + (x-3)^2 = 0$ is
(A) 0 (B) 1 (C) 2 (D) 3

19. If a, b, c are real, then $a(a-b) + b(b-c) + c(c-a) = 0$, only if
(A) $a+b+c=0$ (B) $a=b=c$
(C) $a=b$ or $b=c$ or $c=a$ (D) $a-b-c=0$

20. If $x^3 - 6x^2 + 11x - 6 = (x-a)(x-b)(x-c)$, then $2a+b+c$ is equal to, where $a<b<c$
(A) 6 (B) 8 (C) 7 (D) –7

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. Let $f(x) = x^2 + x - 6$. For what values of "t" does $f(t-5) = 0$? **[NSTSE-2009]**
(A) – 3 and 2 (B) – 2 and 3 (C) 5 (D) 2 and 7

2. If α, β, γ are the roots of the equation $(x-2)(x^2+6x-11) = 0$, therefore, $(\alpha+\beta+\gamma)$ equals : **[IJSO-2010]**
(A) – 4 (B) 23/6 (C) 13 (D) – 8

3. If $x^2 - 5x + 1 = 0$, then $\dfrac{x^{10}+1}{x^5}$ has the value **[IJSO-2010]**
(A) 2524 (B) 2525 (C) 2424 (D) 2010

4. The graph of the equation $y = 2x^2 + 4x + 3$ has its lowest point at : **[IJSO-2010]**
(A) (– 1, 9) (B) (1, 9) (C) (– 1, 1) (D) (0, 3)

5. Given that $a(a+b) = 36$ and $b(a+b) = 64$, where a and b are positive, $(a-b)$ equals **[IJSO-2011]**
(A) 2.8 (B) 3.2 (C) –2.8 (D) –2.5

6. When the polynomial $(6x^4 + 8x^3 + 17x^2 + 21x + 7)$ is divided by $(3x^2 + 4x + 1)$, the remainder is $(ax - b)$. Therefore, **[IJSO-2011]**
(A) $a=1, b=2$ (B) $a=1, b=-2$ (C) $a=2, b=1$ (D) $a=-1, b=-2$

7. If $a+b+c=1$, $a^2+b^2+c^2 = 21$ and $abc = 8$ then find the value of $(1-a)(1-b)(1-c)$ **[IJSO-2012]**
(A) –10 (B) –18 (C)–24 (D)–30

8. Find $x^2+y^2+z^2$ if $x^2+xy+xz = 135$, $y^2+yz+yx = 351$ and $z^2+zx+zy = 243$ **[IJSO-2012]**
(A) 225 (B) 250 (C) 275 (D) 300

9. If $x^3 = a+1$ and $x + (b/x) = a$; then x equals **[IJSO-2013]**
(A) $\dfrac{a(b+1)}{a^2-b}$ (B) $\dfrac{ab+1}{a^2-b}$ (C) $\dfrac{ab+a+1}{a^2-b}$ (D) $\dfrac{ab-a-1}{a^2-b}$

10. Let a, b, c be positive real numbers such that $abc \neq 1$, $(ab)^2 = (bc)^4 = (ca)^x = abc$. Then x equals **[IJSO-2013]**
(A) 1 (B) 2 (C) 3 (D) 4/5

11. Sucharitha purchases x pencils at Rs x each, y pens at Rs y each and z notebooks at Rs z each. She purchases altogether 50 items and pays Rs. 1000. The cost of y pencils, z pens and x notebooks is **[IJSO-2013]**

(A) Rs 600 (B) Rs.750 (C) Rs.500 (D) Rs.350

12. The number of real values of a for which the cubic equation $x^2 - 3ax^2 + 3ax - a = 0$ has all real roots one of which is a itself, is **[IJSO-2013]**

(A) 0 (B) 1 (C) 2 (D) 3

13. If $xy^2 = a^3$, $yz^2 = b^3$ and $zx^2 = c^3$ then z^3 equals **[IJSO-2013]**

(A) $\frac{bc^4}{a^2}$ (B) $\frac{b^4c}{a^2}$ (C) $\frac{b^2c^4}{b^2}$ (D) $\frac{ab^4}{c^2}$

14. If $3x + 3y - 1$, $4x^2 + y - 5$, $4x + 2y$ are the sides of an equilateral triangle, its area is closest to the integer **[IJSO-2013]**

(A) 84 (B) 85 (C) 86 (D) 87

15. The number $3^8(3^{10} + 6^5) + 2^3(2^{12} + 6^7)$ is **[IJSO-2016]**

(A) A perfect square and a perfect cube (B) Neither a perfect square nor a perfect cube
(C) A perfect cube but not a perfect square (D) A perfect square but not a perfect cube

16. If $x^2 + xy + xz = 135$, $y^2 + yz + xy = 351$ and $z^2 + xz + yz = 243$, then $x^2 + y^2 + z^2 =$ _____ **[IJSO 2017]**

(A) 300 (B) 275 (C) 250 (D) 225

17. If $p + q + r = 2$, $p^2 + q^2 + r^2 = 30$ and $pqr = 10$, the value of $(1 - p)(1 - q)(1 - r)$ will be **[IJSO 2017]**

(A) -18 (B) -24 (C) -27 (D) -35

18. If $\left(x + \frac{1}{x}\right) = 5$, then $\left(x^3 + \frac{1}{x^3}\right) - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) =$ _____. **[IJSO 2017]**

(A) 10 (B) 5 (C) 0 (D) -5

19. If $x^2 - 3x + 2$ is a factor of $x^4 - px^2 + q$, then p, q are: **[IJSO 2017]**

(A) 2, 3 (B) 4, 5 (C) 5, 4 (D) 0, 0

20. If $(a + b + c + d) = 4$, then

$$\frac{1}{(1-a)(1-b)(1-c)} + \frac{1}{(1-b)(1-c)(1-d)} + \frac{1}{(1-c)(1-d)(1-a)} + \frac{1}{(1-d)(1-a)(1-b)} = _____$$

[IJSO 2017]

(A) 4 (B) 1 (C) 1 / 4 (D) 0

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 yes,(x – 3) is a factor of p(x). **A-3** $\frac{35}{8}$ **A-4** $\frac{18}{127}$. **A-5** a = 0 and b = 2.

Section (B)

B-1. **(i)** $25x^2 + 16y^2 + 40xy$ **(ii)** $16x^2 + 25y^2 - 40xy$ **(iii)** $4x^2 + \frac{1}{x^2} - 4$.

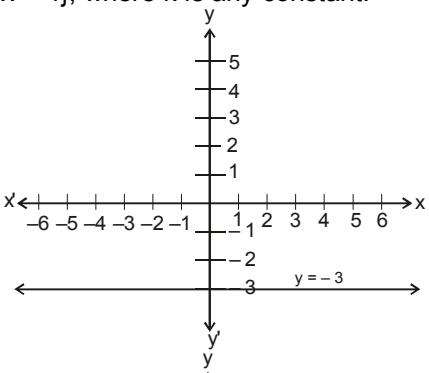
B-2. ± 36. **B-3.** 189. **B-5.** 180.

Section (C)

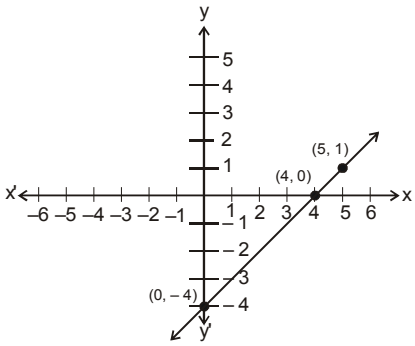
C-1 – 1. **C-2** – 1. **C-3** k = – 1 or k = $\frac{2}{3}$.

C-4 k $\{x^2 - 6x + 4\}$, where k is any constant.

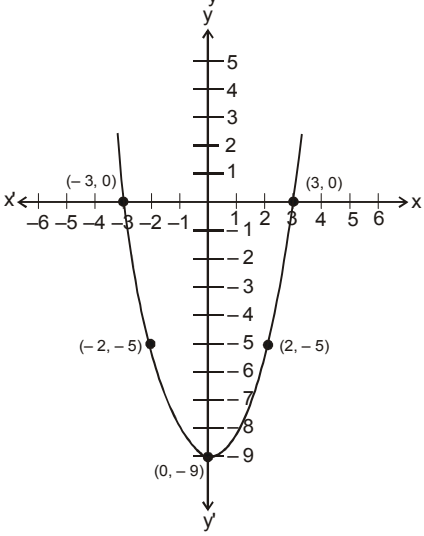
C-5 **(i)**



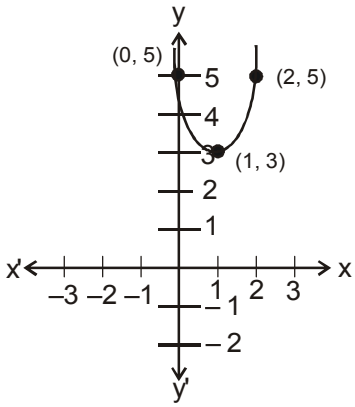
(ii)



(iii)



(iv)



OBJECTIVE QUESTIONS

Section (A)

A-1. (B) **A-2.** (B) **A-3.** (C) **A-4.** (D) **A-5.** (A)

Section (B)

B-1. (D) **B-2.** (D) **B-3.** (A) **B-4.** (B) **B-5.** (D)

B-6. (B) **B-7.** (A) **B-8.** (D) **B-9.** (B) **B-10.** (C)

Section (C)

C-1. (A) **C-2.** (A) **C-3.** (A) **C-4.** (B) **C-5.** (C)

C-6. (A) **C-7.** (A) **C-8.** (B)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	B	D	A	B	B	B	D	B	D	C	C	A	D	C	B	C	A	A	B	C

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	B	C	C	A	B	C	C	D	B	C	B	B	C	B	B	C	C	D

LINEAR EQUATION (TWO VARIABLE)

A. LINEAR EQUATION IN TWO VARIABLE

An equation of the form Ax + By + C = 0 is called a **linear equation**.
Where **A** is called coefficient of **x**, **B** is called coefficient of **y** and **C** is the constant term (free from x & y)
A, B, C $\in$ R [$\in \rightarrow$ belongs to, R $\rightarrow$ Real No.]
But A and B can not be simultaneously zero.
If A $\neq$ 0, B = 0, equation will be of the form Ax + C = 0.
If A = 0, B $\neq$ 0, equation will be of the form By + C = 0.
If A $\neq$ 0 , B $\neq$ 0, C = 0 equation will be of the for A x + By = 0. [line passing through origin]
If A $\neq$ 0 , B $\neq$ 0 , C $\neq$ 0 equation will be of the form A x + By + C = 0.
It is called a linear equation, because the two unknowns (x & y) occur only in the first power, and the product of two unknown quantities does not occur.
Since it involves two variables, therefore a single equation will have infinite set of solution i.e. indeterminate solution. So we require a pair of equation i.e. simultaneous equations.

Standard form of linear equation :

Standard form refers to all positive coefficients.
$a_1x + b_1y + c_1 = 0$ ...(i)
$a_2x + b_2y + c_2 = 0$ ...(ii)
For solving such equations, we have three methods :
(i) Elimination by Substitution.
(ii) Elimination by equating the coefficients.
(iii) Elimination by Cross multiplication.

(a) Elimination By Substitution :

Solved Examples

Example.1 Solve : x + 4y = 14 & 7x – 3y = 5.

Solution. x + 4y = 14
x = 14 – 4y ....(i)
7x – 3y = 5 ....(ii)
Substitute the value of x in equation (ii)
$\Rightarrow$ 7 (14 – 4y) – 3y = 5 $\Rightarrow$ 98 – 28y – 3y = 5
$\Rightarrow$ 98 – 31y = 5 $\Rightarrow$ 93 = 31y $\Rightarrow$ $y = \frac{93}{31} \Rightarrow y = 3.$
Now substitute value of y in equation (ii)
$\Rightarrow$ 7x – 3 (3) = 5 $\Rightarrow$ 7x – 3 (3) = 5
$\Rightarrow$ 7x = 14 $\Rightarrow$ $x = \frac{14}{7} = 2$
So, solution is x = 2 and y = 3.

(b) Elimination by Equating the Coefficients :

Example.2 Solve : 9x – 4y = 8 & 13x + 7y = 101.
Solution. 9x – 4y = 8 ..... (i)
13x + 7y = 101 .... (ii)

Multiply equation (i) by 7 and equation (ii) by 4, we get

$$\begin{array}{lrl} & 63x - 28y & = 56 \\ \text{Add} & 52x + 28y & = 404 \\ \hline & 115x & = 460 \end{array} \Rightarrow \quad x = \frac{460}{115} \Rightarrow x = 4.$$

Substitute x = 4 in equation (i)

$9(4) - 4y = 8 \quad \Rightarrow \quad 36 - 8 = 4y$

$\Rightarrow \quad 28 = 4y \quad \Rightarrow \quad y = \frac{28}{4} = 7.$

So, solution is x = 4 and y = 7.

(c) Elimination by Cross Multiplication :

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$$\begin{matrix} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{matrix}$$

(Write the coefficient in this manner)

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \Rightarrow \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

Also $\frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$

$$\therefore \quad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

Example.3 Solve : 3x + 2y + 25 = 0 & x + y + 15 = 0.

Solution. Here, $a_1 = 3$, $b_1 = 2$, $c_1 = 25$

$a_2 = 1$, $b_2 = 1$, $c_2 = 15$

$$\therefore \begin{matrix} 2 & 25 & 3 & 2 \\ 1 & 15 & 1 & 1 \end{matrix}$$

$$\frac{x}{2\times15 - 25\times1} = \frac{y}{25\times1 - 15\times3} = \frac{1}{3\times1 - 2\times1} \Rightarrow \frac{x}{30-25} = \frac{y}{25-45} = \frac{1}{3-2}$$

$$\Rightarrow \frac{x}{5} = \frac{y}{-20} = \frac{1}{1} \qquad \text{.....(i)}$$

$$\Rightarrow \frac{x}{5} = 1, \ \frac{y}{-20} = \frac{1}{1} \qquad x = 5, y = -20$$

So, solution is x = 5 and y = – 20.

Example.4 Solve the following system of equations : $\frac{6}{x+y} = \frac{7}{x-y} + 3$, $\frac{1}{2(x+y)} = \frac{1}{3(x-y)}$

where $x + y \neq 0$ and $x - y \neq 0$.

Solution. $\frac{6}{x+y} = \frac{7}{x-y} + 3$ ...(i)

Put x + y = a, x – y = b

Now, the given equation reduces to

$\frac{6}{a} = \frac{7}{b} + 3$ ...(ii)

Another equation is $\frac{1}{2(x+y)} = \frac{1}{3(x-y)}$ ...(iii)

$\frac{1}{2a} = \frac{1}{3b}$

or, $a = \frac{3b}{2}$ ...(iv)

Put (iv) in (ii)

$\frac{6}{3b} \times 2 = \frac{7+3b}{b}$

$4 = 7 + 3b$

$b = -1$ ...(v)

Put (v) in (iv)

$a = -\frac{3}{2}$

From our assumptions,

$a = x + y = -\frac{3}{2}$

$b = x - y = -1$

Adding the two equations given above,

$2x = -\frac{5}{2} \Rightarrow x = -\frac{5}{4}$

$\therefore \quad y = -\frac{5}{4} + 1 \Rightarrow y = -\frac{1}{4}$.

Example.5 If $\frac{1}{2(x+2y)} + \frac{5}{3(3x-2y)} = -\frac{3}{2}$, $\frac{5}{4(x+2y)} - \frac{3}{5(3x-2y)} = \frac{61}{60}$,

where $x + 2y \neq 0$ and $3x - 2y \neq 0$. Then what will be the values of x and y ?

Solution. Let $\frac{1}{x+2y} = U$, $\frac{1}{3x-2y} = V$ $\quad \frac{U}{2} + \frac{5V}{3} = -\frac{3}{2}$

$\Rightarrow 3U + 10V = -9$ ... (i)

$\Rightarrow \frac{5U}{4} - \frac{3V}{5} = \frac{61}{60}$

$\Rightarrow 75U - 36V = 61$ ... (ii)

Equation (i) is multiplied by 25

$75U + 250V = -225$

$75U - 36V = 61$

$- \quad + \quad -$

Subtracting, $286V = -286$

$V = -1$, $U = \frac{1}{3}$

$\therefore \frac{1}{x+2y} = \frac{1}{3}$, $\frac{1}{3x-2y} = -1$

$x + 2y = 3$, $-3x + 2y = 1$

$x + 2y = 3$

$-3x + 2y = 1$

$+ \quad - \quad -$

Subtracting, $4x = 2$

So, $x = \frac{1}{2}$, $y = \frac{5}{4}$.

Example.6 Solve : $3\sqrt{2}\,x - 5\sqrt{3}\,y + \sqrt{5} = 0$

$2\sqrt{3}\,x + 7\sqrt{2}\,y - 2\sqrt{5} = 0.$

Solution. $3\sqrt{2}\,x - 5\sqrt{3}\,y + \sqrt{5} = 0$ ....(i)

$2\sqrt{3}\,x + 7\sqrt{2}\,y - 2\sqrt{5} = 0$ ....(ii)

Multiplying (i) by $2\sqrt{3}$ and (iii) by $3\sqrt{2}$

$$6\sqrt{6}\,x - 10 \times 3y = -\,2\sqrt{15}$$
$$6\sqrt{6}\,x + 21 \times 2y = 6\sqrt{10}$$
$$- \qquad - \qquad -$$

Subtracting, $-72y = -(6\sqrt{10} + 2\sqrt{15})$

or, $72y = 6\sqrt{10} + 2\sqrt{15}$

$$y = \frac{6\sqrt{10} + 2\sqrt{15}}{72}$$

$$\therefore \quad x = \frac{10\sqrt{15} - 7\sqrt{10}}{72}.$$

B. CONDITIONS FOR SOLVABILITY OF SYSTEM OF EQUATIONS

Let the two equations be :

$a_1x + b_1y + c_1 = 0$ ...(i)

$a_2x + b_2y + c_2 = 0$ ...(ii)

(a) Unique Solution :

If the Denominator $a_1b_2 - a_2\,b_1 \neq 0$, then the given system of equations have unique solution (i.e. only one solution)

$$\therefore \quad a_1b_2 - a_2\,b_1 \neq 0 \;\Rightarrow\; \frac{a_1}{a_2} \neq \frac{b_1}{b_2}.$$

For two lines : lines are said to be consistent (i.e. they meet at one point) when the given system of equation has unique solution.

(b) No Solution :

If the Denominator $a_1b_2 - a_2\,b_1 = 0$, then the given system of equations have no solution.

i.e. $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

For two lines : Lines are said to be inconsistent (i.e. they does not meet) when the given the system of equation has no solution.

(c) Many Solutions (Infinite Solutions) :

If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ then system of equations has many solutions and lines are said to be consistent.

For two lines : Two lines are coincident when they have many solutions.

Solved Examples

Example.7 Find the value of 'M' for which the given system of equation has only one solution (i.e. unique solution). $Mx - 2y = 9$ & $4x - y = 7$.

Solution. $a_1 = M,\ b_1 = -2,\ c_1 = 9$

$a_2 = 4,\ b_2 = -1,\ c_2 = 7$

Condition for unique solution is $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\Rightarrow \frac{M}{4} \neq \frac{+2}{+1} \Rightarrow M \neq \frac{8}{1} \Rightarrow M \neq 8.$

$\therefore$ M can have all real values except 8.

Example.8 What is the value of a, for which the system of linear equations ax + 3y = a – 3 ; 12x + ay = a has no solution.

Solution. Condition for no solution is $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$\frac{a}{12} = \frac{3}{a} \neq \frac{a-3}{a}$

$a^2 = 36$ or, $a = \pm 6$

a = + 6 is not possible because it gives

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

or, a = – 6 is the answer.

Example.9 Find the values of α and β for which the following system of linear equations has infinite number of solution :2x + 3y = 7 & $2\alpha x + (\alpha + \beta) y = 28$.

Solution. For infinite solution :

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\Rightarrow \frac{2}{2\alpha} = \frac{3}{\alpha+\beta} = \frac{7}{28}$

$\Rightarrow \frac{1}{\alpha} = \frac{3}{\alpha+\beta} = \frac{1}{4}$

$\Rightarrow \alpha = 4$ [From the first & third term]

$\Rightarrow \frac{3}{4+\beta} = \frac{1}{4} \Rightarrow 4 + \beta = 12$

$\Rightarrow \beta = 8.$

C. WORD PROBLEMS

Solved Examples

Example.10 A two digit number is such that product of its digits is 18. When 63 is subtracted from the number, the digits interchange their place. Find the number.

Solution. xy = 18 ...(i)

Let the given number be 10x + y

As per the question,

(10x + y) – 63 = 10y + x $\Rightarrow$ 10x – 10y – x + y = 63

$\Rightarrow$ 9x – 9y = 63

$\Rightarrow$ x – y = 7 ...(ii)

Put $x = \frac{18}{y}$ in (ii)

$\Rightarrow \frac{18}{y} - y = 7 \Rightarrow 18 - y^2 = 7y \Rightarrow y^2 + 7y - 18 = 0$

$\Rightarrow$ $y^2 + 9y - 2y - 18 = 0$ $\Rightarrow$ $y(y + 9) - 2(y + 9) = 0$

$\Rightarrow$ $(y + 9)(y - 2) = 0$ $\Rightarrow$ $y = 2, -9$

$y = -9$ is not valid

$\therefore$ $y = 2$, $x = 9$.

So, the number = $10x + y = 10(9) + 2 = 92$.

Example.11 The sum of two numbers is 2490. If 6.5 % of one number is equal to 8.5 % of the other, find the numbers.

Solution. Let, the numbers be x & y.

Then, $x + y = 2490$ ...(i)

$\Rightarrow \frac{x \times 6.5}{100} = \frac{8.5}{100} \times y$ $\Rightarrow$ $x = \frac{8.5}{6.5} y$ $\Rightarrow$ $x = \frac{17}{13} y$ ...(ii)

Put (ii) in (i)

$\frac{17}{13} y + y = 2490$ $\Rightarrow$ $30y = 13 \times 2490$ $\Rightarrow$ $y = \frac{13 \times 2490}{30} = 1079$

$\therefore$ $x = 1411$

So, the numbers are $x = 1411$ & $y = 1079$.

Example.12 A and B each has a certain number of mangoes. A says to B, 'if you give 30 of your mangoes I will have twice as many as left with you.' B replies 'if you give me 10, I will have thrice as left with you.' Find how many mangoes does each have.

Solution. Say, A has x mangoes & B has y mangoes initially.

As per the statement of A to B,

$x + 30 = 2(y - 30)$ or, $x - 2y = -90$ ...(i)

and as per statement of B to A,

$3(x - 10) = y + 10$ or, $3x - y = 40$ ...(ii)

Now, we have

$x - 2y = -90$

$3x - y = 40$

$3x - 6y = -270$

$3y - y = 40$

$- \quad + \quad -$

$-5y = -310$

$\Rightarrow$ $y = 62$

$\therefore$ $x = 34$. So, A have 34 mangoes and B have 62 mangoes.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Method of solving

A-1 Solve : $px + qy = r$ and $qx - py = 1 + r$.

A-2 Solve for x and y : $(a - b)x + (a + b)y = a^2 - 2ab - b^2$
$(a + b)(x + y) = a^2 + b^2$

A-3 Solve for x and y :
$37x + 43y = 123$
$43x + 37y = 117$

A-4 Solve for x and y : $\frac{x}{3} + y = 1.7$ and $\frac{11}{x+\frac{y}{3}} = 10 \ \forall \left[x+\frac{y}{3} \neq 0\right]$.

A-5 Solve : $119x - 381y = 643$
$381x - 119y = -143$

Section (B) : Condition of solvability

B-1 Find the number of solutions of the following pair of linear equations :
$x + 2y - 8 = 0$
$2x + 4y = 16$.

B-2 Find the value of k for which the given system of equations has a unique solution.
$3x + 5y = 12$; $4x - 7y = k$.

B-3 Find the value of k for which the following system of linear equation becomes infinitely many solution or represent the coincident lines.
$6x + 3y = k - 3$; $2kx + 6y = k$.

B-4 Find the value of k or which the following systems of equations be Inconsistent.
$2x + ky + k + 2 = 0$; $kx + 8y + 3k = 0$.

B-5 For what value of a and b, the following system of equations have an infinite number of solutions.
$2x + 3y = 7$; $(a - b)x + (a + b)y = 3a + b - 2$.

Section (C) : Word problem

C-1 If a certain number is divided by the sum of its two digits , the quotient is 6 and remainder is 3. If the digits are interchanged and the resulting number is divided by the sum of the digits, then the quotient is 4 and the remainder is 9. Find the number

C-2 2 men and 3 boys together can do a piece of work in 8 days. The same work is done in 6 days by 3 men and 2 boys together. How long would 1 boy alone or 1 man alone take to complete the work.

C-3 A man sold a chair and a table together for Rs. 1520 thereby making a profit of 25% on chair and 10% on table. By selling them together for Rs. 1535 he would have made a profit of 10% on the chair and 25% on the table. Find cost price of each.

C-4 Abdul travelled 300 km by train and 200 km by taxi taking 5 hours 30 minutes. But, if he travels 260 km by train and 240 km by taxi, he takes 6 minutes longer. Find the speed of the train and that of the taxi.

C-5 A train covered a certain distance at a uniform speed. If the train would have been 6 km/h faster, it would have taken 4 hours less than the scheduled time. And, if the train were slower by 6 km/h, it would have taken 6 hours more than the scheduled time. Find the length of the journey.

OBJECTIVE QUESTIONS

Section (A) : Method of solving

A-1. Solve the following system of equation for x and y : $\frac{2}{x}+\frac{3}{y}=2;\quad \frac{8}{x}-\frac{6}{y}=2$.

(A) $\frac{1}{2},\frac{1}{3}$ (B) $\frac{1}{3},\frac{1}{2}$ (C) 2, 3 (D) 3, 2

A-2. The solution of the following pairs of simultaneous equation is
: $\frac{2x-1}{3}+\frac{y+2}{4}=4$ and $\frac{x+3}{2}-\frac{x-y}{3}=3$.

(A) x = – 5, y = – 4 (B) x = 5, y = 2 (C) x = – 5, y =2 (D) x = 5, y = 4

A-3. Solve for x & y : $\frac{7}{3^x}-\frac{6}{2^y}=15$ and $\frac{8}{3^x}=\frac{9}{2^y}$.

(A) – 3, – 2 (B) – 2, – 3 (C) $\frac{-1}{2}$, $\frac{-1}{3}$ (D) $\frac{-1}{3}$, $\frac{-1}{2}$

A-4. If 173x + 197y = 149 and 197x + 173y = 221, then find (x, y)

(A) (3,–2) (B) (2, 1) (C) (1, –2) (D) (2, –1)

A-5. A pair of equations x = a and y = b graphically represent lines which are :

(A) Intersecting at (a, b) (B) intersecting at (b, a)
(C) parallel (D) coincident

(B) Condition of solvability

B-1. Match the coloumn

(i) $2x+5y=10$, $3x+4y=7$	(A) Unique solution
(ii) $2x+5y=10$, $6x+15y=20$	(B) Infinetly many solution
(iii) $5x+2y=10$, $10x+4y=20$	(C) No solution

(A) (i) – A, (ii) – B, (iii) – C (B) (i) – B, (ii) – C, (iii)– A
(C) (i) – A, (ii) – C, (iii) – B (D) (i) – C, (ii) – A, (iii) – B

B-2. For what value of K, the system of linear equations 2x + ky = 1 ; 3x – 5y = 7 has a unique solutions :

(A) $K \neq -\frac{10}{3}$ (B) $K \neq \frac{10}{3}$ (C) $K \neq -35$ (D) $K = -\frac{10}{3}$

B-3. If the system of the given equation has no solution then find the value of k.
$2x + ky = 7$ & $2kx + 3ky = 20$.
(A) 1 (B) 6 (C) 3 (D) None of these

B-4. The number of solutions of the following pair of linear equations is $x + 2y - 8 = 0$ and $2x + 4y = 16$.
(A) 1 (B) 0
(C) 2 (D) infinitely many solutions.

B-5. If the system of equations $3x + 4y = 12$ and $(a + b)x + 2(a - b)y = 5a - 1$ has infinitely many solutions then a & b satisfy the equation
(A) $a - 5b = 0$ (B) $5a - b = 0$ (C) $a + 5b = 0$ (D) $5a + b = 0$

(C) Word problem

C-1. Sum of two numbers is 35 and their difference is 13. Find the numbers.
(A) 23, 12 (B) 24, 11 (C) 23, 10 (D) 22, 9

C-2. The digit in the ten's place of a two-digit number is three times that in the one's place. If the digits are reversed the new number will be 36 less than the original number. Find the number.
(A) 64 (B) 52 (C) 62 (D) 42

C-3. Denominator of a rational number is 4 less than its numerator. If 11 is added to numerator and 1 is subtracted from denominator, the new number becomes $\frac{7}{3}$. Find the rational number.
(A) $\frac{23}{27}$ (B) $\frac{27}{23}$ (C) $\frac{13}{17}$ (D) $\frac{17}{13}$

C-4. The sum of the present ages of father and his son is 60 years. 6 years ago, father's age was five times the age of the son. After six years son's age will be :
(A) 20 years (B) 14 years (C) 12 years (D) 18 years

C-5. Four years ago father's age was 6 times that of his son. Twelve years from now, father's age will be twice that of the son. What is the ratio of father and son's present ages ?
(A) 6 : 1 (B) 7 : 1 (C) 8 : 2 (D) 7 : 2

Exercise-2

OBJECTIVE QUESTIONS

1. How many positive integer solutions are there of equation $3x + 4y = 63$
(A) 3 (B) 4 (C) 5 (D) None of these

2. Find the value of K for which the following system of linear equations has no solution
$kx + 12y = k$; $3x + ky = k - 3$
(A) 6 (B) –6 (C) 6 and –6 both (D) None of these

3. The graphs of $2x + 3y - 6 = 0$, $4x - 3y - 6 = 0$, $x = 2$ and $y = \frac{2}{3}$ intersects in
(A) Four points (B) one point (C) two point (D) infinite no. of points

4. The pair of equations $3^{x+y} = 81$, $81^{x-y} = 3$ has :

(A) No solution
(B) The solution $x = 2\frac{1}{2}$, $y = 2\frac{1}{2}$
(C) The solution x = 2, y = 2
(D) The solution $x = 2\frac{1}{8}$, $y = 1\frac{7}{8}$

5. The solution of the equations $\frac{m}{x} + \frac{n}{y} = a$ and $\frac{n}{x} + \frac{m}{y} = b$ is given by :

(A) $x = \frac{n^2 + m^2}{am - bn}$, $y = \frac{m^2 - n^2}{bm - an}$
(B) $x = \frac{m^2 - n^2}{am - bn}$, $y = \frac{n^2 - m^2}{bm - an}$
(C) $x = \frac{m^2 - n^2}{am - bn}$, $y = \frac{m^2 - n^2}{bm - an}$
(D) $x = \frac{n^2 - m^2}{am - bn}$, $y = \frac{n^2 - m^2}{bm - an}$

6. Find the values of a and b for which the following system of linear equations has infinite number of solutions : $2x - 3y = 7$; $(a + b)x - (a + b - 3)y = 4a + b$
(A) a = – 5, b = – 1 (B) a = 1, b = 5 (C) a = – 1, b = – 5 (D) a = 5, b = 1

7. The sum of a two digit number and the number formed by interchanging its digits is 110.If 10 is subtracted from the first number, the new number is 4 more than 5 times the sum of the digits in the first number. Find the first number.
(A) 46 (B) 64 (C) 55 (D) 65

8. After covering a distance of 30 km with a uniform speed there is some defect in a train engine and therefore, its speed is reduced to 4/5 of its original speed. Consequently, the train reaches its destination late by 45 minutes. Had it happened after covering 18 kilometres more, the train would have reached 9 minutes earlier. Find the speed of the train and the distance of journey.
(A) Speed = 40 km/hr, Distance = 160 km
(B) Speed = 20 km/hr, Distance = 100 km
(C) Speed = 30 km/hr, Distance = 120 km
(D) Speed = 35 km/hr, Distance = 140 km

9. A railway half ticket costs half the full fare and the reservation charge is the same on half ticket as on full ticket. One reserved first class ticket from Mumbai to Ahmedabad costs Rs. 216 and one full and one half reserved first class tickets cost Rs. 327. What is the basic first class full fare and what is the reservation charge ?
(A) Full Fare = Rs. 210, Reservation charge = Rs. 6
(B) Full Fare = Rs. 144, Reservation charge = Rs. 72
(C) Full Fare = Rs. 180, Reservation charge = Rs. 36
(D) Full Fare = Rs. 200, Reservation charge = Rs. 16

10. A wizared having powers of mystic in candations and magical medicines seeing a cock, fight going on, spoke privately to both the owners of cocks. To one he said; if your bird wins, than you give me your stake-money, but if you do not win, I shall give you two third of that'. Going to the other, he promised in the same way to give three fourths. From both of them his gain would be only 12 gold coins. Find the stake of money each of the cock-owners have.
(A) 27 gold coins & 30 gold coins respectively.
(B) 12 gold coins & 20 gold coins respectively.
(C) 33 gold coins & 30 gold coins respectively.
(D) 42 gold coins & 40 gold coins respectively.

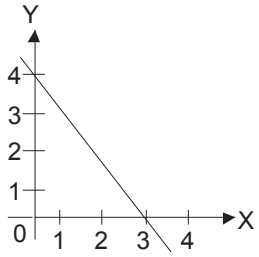
Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. The equations 2x – 3y + 5 = 0 and 6y – 4x = 10, when solved simultaneously, have : **[IJSO-2008]**
(A) only one solution (B) no solution
(C) only two solutions (D) infinite number of solutions

2. Given 3x – 4y = 7 and x + cy = 13, for what value of "c" will the two equation not have a solution ? **[NSTSE-2009]**
(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) – 4 (D) $\frac{-4}{3}$

3. The below diagram shows the graph of : **[NSTSE-2009]**



(A) y = x – 3 (B) y = 2x + 3 (C) y = –x – 3 (D) y = $\frac{-4x}{3}$ + 4

4. Let "b" be a positive number such that the system $\begin{cases} ax + 3y = 1 \\ 5x + ay = b \end{cases}$ has an infinite number of solutions. By rounding to the nearest hundredth, the value of "b" equals : **[NSTSE 2009]**
(A) 0.60 (B) 1.29 (C) 1.67 (D) 3.87

5. The solution set of the system of equation $\frac{4}{x}$ + 5y = 7, $\frac{3}{x}$ + 4 y = 5 is : **[NSTSE-2010]**
(A) $\left(\frac{1}{3},-1\right)$ (B) $\left(\frac{-1}{3},1\right)$ (C) $\left(\frac{-1}{3},-1\right)$ (D) $\left(\frac{1}{3},1\right)$

6. If 2a = b, the pair of equations ax + by = $2a^2 - 3b^2$, x + 2y = 2a – 6b possess : **[NSTSE 2010]**
(A) no solution (B) only one solution (C) only two solutions
(D) an infinite number of solutions

7. In a rectangle ABCD the lengths of sides AB, BC, CD and DA are (5x + 2y + 2) cm, (x + y + 4) cm, (2x + 5y – 7) cm and (3x + 2y – 11) cm respectively. Which of the following statements is /are true ? **[IJSO-2009]**
(A) One of the sides of the rectangle is 15 cm long. (B) Each diagonal of the rectangle is 39 cm long.
(C) Perimeter of the rectangle is 102 cm. (D) Area of the rectangle is 560 cm^2

8. If 4a – 18b + 13c = 0 and 3a + 3b – 4c = 0, then the ratio a : b : c will be : **[IJSO-2010]**
(A) 3 : 6 : 5 (B) 5 : 3 : 6 (C) 3 : 5 : 6 (D) 5 : 6 : 3

9. A pen costs Rs 13 and a note book costs Rs. 35 Let m be the maximum number of items that can be bought for Rs 1000 and n be the minimum number of items that can be bought for the same amount. Then m + n is **[IJSO-2013]**
(A) 76 (B) 88 (C) 96 (D) 98

10. A box contains some red and some yellow balls. If one red ball is removed, one seventh of the remaining balls would be red ; if one yellow ball is removed, one-sixth of the remaining balls would be red. If n denotes the total number of balls in the box, then the sum of the digits of n is **[IJSO-2018]**
(A) 6 (B) 7 (C) 8 (D) 9

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 $x = \frac{q + r(p+q)}{p^2 + q^2}$ and $y = \frac{r(q-p) - p}{p^2 + q^2}$

A-2 $x = a + b$ and $y = -\frac{2ab}{a+b}$.

A-3 x = 1 and y = 2.

A-4 x = 0.6 and y = 1.5

A-5 x = –1 and y = – 2.

Section (B)

B-1 $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$

B-2 $k \in R$

B-3 k = 6

B-4 k = – 4

B-5 a = 5 , b = 1.

Section (C)

C-1 75.

C-2 one boy 120 days and one man 20 days.

C-3 Cost price of chair = Rs. 600 and cost price of table = Rs. 700.

C-4 Speed of train = 100 km/hr, Speed of taxi = 80 km/hr.

C-5 720 km.

OBJECTIVE QUESTIONS

Section (A)

A-1. (C) **A-2.** (B) **A-3.** (B) **A-4.** (D) **A-5.** (A)

Section (B)

B-1. (C) **B-2.** (A) **B-3.** (C) **B-4.** (D) **B-5.** (A)

Section (C)

C-1. (B) **C-2.** (C) **C-3.** (D) **C-4.** (A) **C-5.** (D)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10
Ans.	C	B	B	D	C	A	B	C	A	D

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10
Ans.	D	D	D	B	A	D	ABC	C	D	B

LINEAR INEQUATION

A. INEQUATIONS

A statement involving variable (s) and the sign of inequality viz, >,<, ≥ or ≤ is called an **inequation**. An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

(i) $3x - 2 < 0$ **(ii)** $2x^2 + 3x + 4 > 0$ **(iii)** $2x + 5y \geq 4$

(a) Properties of inequalities

(i) If '**a**' is a positive no. i.e. **a > 0** then for x < y $\Rightarrow$ $\frac{x}{a} < \frac{y}{a}$ & ax < ay.

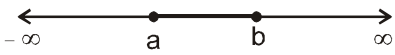
(ii) If '**a**' is –ve i.e. **a < 0** then for x < y $\Rightarrow$ $\frac{x}{a} > \frac{y}{a}$ & ax > ay.

(iii) If '**a**' is a +ve no. i.e. **a > 0** then for x > y $\Rightarrow$ $\frac{x}{a} > \frac{y}{a}$ & ax > ay.

(iv) If '**a**' is a –ve no. i.e. **a < 0** then for x > y $\Rightarrow$ $\frac{x}{a} < \frac{y}{a}$ & ax < ay.

B. TYPES OF INTERVALS

(i) Closed interval : Let **a** and **b** be two given real numbers such that a < b. Then the set of all real numbers x such that $a \leq x \leq b$ is called **closed interval** and is denoted by [a, b] may be graphed as :



(ii) Open interval : If **a** and **b** are two real numbers such that a < b, then the set of all real numbers x satisfying a < x < b is called an **open interval** and is denoted by (a, b) or]a, b[and may be graphed as :

(iii) Semi-closed or semi-open interval : If **a** and **b** are two real numbers such that a < b, then the sets (a, b] = {x ∈ R : a < x ≤ b} and [a, b) = {x ∈ R : a ≤ x < b} are known as **semi-open** or **semi-closed intervals**. (a, b] and [a, b) are also denoted by]a, b] and [a, b[respectively.

C. LINEAR INEQUATIONS IN ONE VARIABLE

Let **a** be a non-zero real number and **x** be a variable. Then inequations of the form ax + b < 0, ax + b ≤ 0, ax + b > 0 and ax + b ≥ 0 are known as linear inequations in one variable **x**.

For example : 9x – 15 > 0, 5x – 4 ≥ 0, 3x + 2 < 0 and 2x – 3 ≤ 0 are linear inequations in one variable.

(a) Solving linear inequations in one variable

Rule 1: Same number may be added to (or subtracted from) both side of an inequation without changing the sign of inequality.

Rule 2 : Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.

Rule 3 : Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.

Solved Examples

Example.1

Solve the inequality ax > a.

Sol. This inequality has the parameter **a** that needs to be investigated further.

If a > 0, then x > 1

If a < 0, then x < 0

Example.2

Solve : 24 x < 100 when

(i) x is a natural number **(ii)** x is an integer.

Sol. We are given 24 x < 100

$\Rightarrow \quad \frac{24x}{24} < \frac{100}{24} \quad \Rightarrow \quad x < \frac{25}{6}$

(i) When x is a natural number, the following values of x make the statement true. 1, 2, 3, 4.

(ii) When x is an integer, the solutions of the given equations are ....,– 3, – 2, – 1, 0, 1, 2, 3, 4.

The solution set of the equation is : { ...,– 3,– 2,– 1, 0,1, 2, 3, 4}.

Example.3

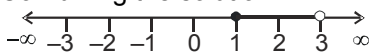
Solve & graph the solution set of $3x + 6 \geq 9$ and $– 5x > –15$, $x \in R$.

Sol. $3x + 6 \geq 9$ and $–5x > –15$

$\Rightarrow \quad 3x \geq 3$ and $– x > – 3$

$\Rightarrow \quad x \geq 1$ and $x < 3$

Combining the solution

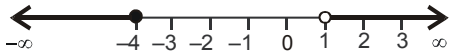


So, the solution is x[1,3).

Example.4

Solve & graph the solution set of $– 2 < 2x – 4$ and $– 2x + 5 \geq 13$, $x \in R$.

Sol.



$– 2 < 2x – 4$

$\Rightarrow \quad 2x – 4 > –2$ and $–2x + 5 \geq 13$

$\Rightarrow \quad 2x > 2$ and $–2x \geq 13 – 5$

$\Rightarrow \quad x > 1$ and $–2x \geq 8$

$\Rightarrow \quad x > 1$ and $–x \geq 4$

$\Rightarrow \quad x > 1$ and $x \leq –4$

$\therefore$ x > 1 and $x \leq – 4$ or $x \in (– \infty, – 4] \cap (1, \infty)$.

$\therefore x \in \Phi$.

Example.5

Solve the following equation : $2(2x + 3) – 10 \leq 6 (x – 2)$

Sol. We have,

$\Rightarrow \quad 2(2x + 3) – 10 \leq 6 (x – 2)$

$\Rightarrow \quad 4x + 6 – 10 \leq 6x – 12$

$\Rightarrow \quad 4x – 4 \leq 6x – 12$

$\Rightarrow \quad 4x – 6x \leq – 12 + 4$ [Transposing – 4 to RHS and 6x to LHS]

$\Rightarrow \quad – 2x \leq – 8$

$\Rightarrow \quad \frac{–2x}{–2} \geq \frac{–8}{–2}$

$\Rightarrow \quad x \geq 4$

$\Rightarrow \quad x \in [4, \infty)$

Hence, the solution set of the given inequation is $[4, \infty)$ which can be graphed on real line as shown in Figure.

Example.6

Solve the following inequations : $\frac{5x-2}{3}-\frac{7x-3}{5}>\frac{x}{4}$.

Sol. $\frac{5x-2}{3}-\frac{7x-3}{5}>\frac{x}{4}$

$\frac{5(5x-2)-3(7x-3)}{15}>\frac{x}{4}$

$\frac{25x-10-21x+9}{15}>\frac{x}{4}$

$\frac{4x-1}{15}>\frac{x}{4}$

4 (4x – 1) > 15 x [Multiplying both sides by 60 i.e. LCM of 15 and 4]

16x – 4 > 15x

16x – 15x > 4 [Transposing 15 x to LHS and – 4 to RHS]

x > 4

$x \in (4, \infty)$

Hence, the solution set of the given inequation is (4, ∞). This can be graphed on the real number line as shown in figure.

Example.7

Solve the following inequations : $\frac{2x+4}{x-1}\geq 5$.

Sol. We have, $\frac{2x+4}{x-1}\geq 5$

$\Rightarrow \frac{2x+4}{x-1}-5\geq 0$

$\Rightarrow \frac{2x+4-5(x-1)}{x-1}\geq 0$

$\Rightarrow \frac{2x+4-5x+5}{x-1}\geq 0$

$\Rightarrow \frac{-3x+9}{x-1}\geq 0$ [Multiplying both sides by – 1]

$\Rightarrow \frac{3x-9}{x-1}\leq 0$

$\Rightarrow \frac{3(x-3)}{x-1}\leq 0$ [Dividing both sides by 3]

$\Rightarrow \frac{x-3}{x-1}\leq 0$ $\Rightarrow$ $1 < x \leq 3$ $\Rightarrow$ $x \in (1, 3]$

Hence, the solution set of the given inequations is (1, 3].

Example.8

Solve 5x - 3 < 3x + 1 when

(i) x is an integer, **(ii)** x is a real number.

Sol. We have, 5x – 3 < 3x + 1

$\Rightarrow$ 5x – 3 + 3 < 3x + 1 + 3 $\Rightarrow$ 5x < 3x + 4

$\Rightarrow$ 5x – 3 x < 3x + 4 – 3x $\Rightarrow$ 2x < 4

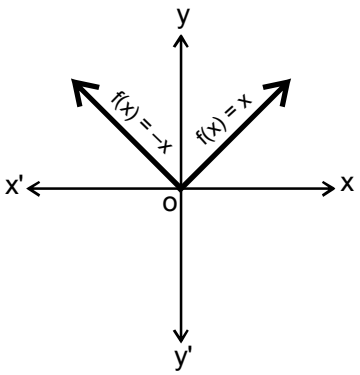
$\Rightarrow$ x < 2

(i) When x is an integer, the solutions of the given inequality are {....., – 4, –3, –2, –1, 0,1}
(ii) When x is a real number, the solutions of the inequality are given by x < 2, i.e., all real number x which are less than 2. Therefore, the solution set of the inequality is x (– ∞, 2).

D. ABSOLUTE VALUE OF A REAL NUMBER

The function f(x) defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \ge 0 \\ -x, & \text{when } x < 0 \end{cases}$

is called the modulus function. It is also called the absolute value function.



Solved Examples

Example.9

Solve : |x – 4 | = 7

Sol. $|x-4| = \begin{cases} x-4, & \text{where } x-4 \ge 0 \Rightarrow x \ge 4 \\ -(x-4), & \text{where } x-4 < 0 \Rightarrow x < 4 \end{cases}$

∴ x – 4 = 7 and – (x – 4) = 7
x = 11 and – x + 4 = 7
⇒ – x = 3 ⇒ x = – 3 ∴ **Ans.** $\begin{cases} x = -3 \\ x = 11 \end{cases}$

Example.10

Evaluate : $|3| + |-2-3| - 3 - |-7|$.

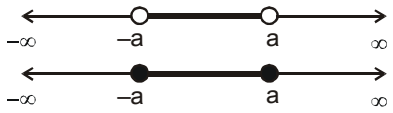
Sol. $|3| + |-2-3| - 3 - |-7|$ = 3 + – 3 – {–(–7)}
= 3 + {–(–5)} – 3 – 7 = 3 + 5 – 10 = 8 – 10 = – 2.

(a) Inequations involving absolute value

Result 1. If a is a positive real number, then

(i) | x | < a ⇔ – a < x < a i.e. x ∈ (– a, a)

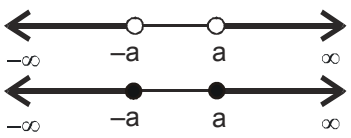
(ii) | x | ≤ a ⇔ – a ≤ x ≤ a i.e. x ∈ [–a, a]



Result 2. If a is a positive real number, then

(i) | x | > a ⇔ x < – a or x > a

(ii) | x | ≥ a ⇔ x ≤ – a or x ≥ a



Solved Examples

Example.11 Find x from $1 < |x| < 2$ and represent it on number line.

Sol. $1 < |x| \Rightarrow |x| > 1 \Rightarrow$ x > 1 or x < –1

$\therefore$ $x \in (-\infty, -1) \cup (1, \infty)$ ...(i)

also $|x| < 2 \Rightarrow$ x < 2 or x > – 2

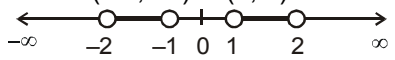
$\therefore$ x lies between – 2 & 2

$\Rightarrow$ $x \in (-2, 2)$ ...(ii)

Combining the two results, we get

$1 < |x| < 2 \Rightarrow$ {– 2 < x < –1} $\cup$ {1 < x < 2}

i.e. $x \in (-2, -1) \cup (1, 2)$



Example.12

Find x satisfying $|x - 5| \le 3$.

Sol. as $|x - a| \le r \Leftrightarrow a - r \le x \le a + r$ i.e. $x \in [a - r, a + r]$

$\therefore$ $|x - 5| \le 3 \Leftrightarrow 5 - 3 \le x \le 5 + 3$ i.e. $2 \le x \le 8$ i.e. $x \in [2, 8]$

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Problems based on inequality

A-1 $\frac{5x-1}{x^2+3} < 1$

A-2 $\frac{x^4+x^2+1}{x^2-4x-5} < 0$

A-3 $\frac{x^2-1}{x^2+x+1} < 1$

A-4 $\frac{x^2-1}{2x+5} < 3$

A-5 $2 + \frac{3}{x+1} > \frac{2}{x}$

Section (B) : Problems based on modulus function

B-1 $|x| - |x-2| \geq 1$

B-2 $|x^3 - 1| \geq 1 - x$

B-3 $\left|x^2 - 4x + 4\right| \geq 1$

B-4 $\frac{|x+2|-x}{x} < 2$

B-5 $\frac{|x-3|}{x^2-5x+6} \geq 2$

OBJECTIVE QUESTIONS

Section (A) : Problems based on inequality

A-1. If $-5x + 1 \geq 0$, then :

(A) $x \geq -\frac{1}{5}$ (B) $x \leq -\frac{1}{5}$ (C) $x \geq \frac{1}{5}$ (D) $x \leq \frac{1}{5}$

A-2. If x, y are any two positive real numbers and x > y, then which of the following statement is not true ?

(A) $-x < y$ (B) $-x < -y$ (C) $\frac{1}{x} > \frac{1}{y}$ (D) $\frac{1}{x} < \frac{1}{y}$

A-3. Solution of inequality $\frac{2x-1}{x+1} < 1$ is :

(A) $-1 < x < 3$ (B) $-1 < x < 2$ (C) $-2 < x < 2$ (D) $-3 < x < 2$

A-4. Solve : $\frac{1}{x} < 1$.

(A) $(-\infty, 0) \cup (1, +\infty)$ (B) $(-\infty, 0) \cup [1, +\infty)$ (C) $(1, +\infty)$ (D) $(-\infty, 0)$

A-5. Solve : $\frac{x}{x+2} \leq \frac{1}{x}$.

(A) $-3 < x \leq 2$ (B) $-2 < x \leq 2$ (C) $-2 < x \leq 3$ (D) $(-2, -1] \cup (0, 2]$

Section (B) : Problems based on modulus function

B-1. Solve : $|2x - 3| \leq 5$.

(A) $-2 \leq x \leq 5$ (B) $-2 \leq x \leq 4$ (C) $-1 \leq x \leq 4$ (D) $-1 \leq x \leq 5$

B-2. Solve : $|1 - x| > 3$.

(A) $x > 4$ or $x > -2$ (B) $x > 4$ (C) $x < -2$ (D) $x > 4$ or $x < -2$

B-3. The solution the equation $|3x - 10| \geq 5$ is :

(A) $x \geq 5$ or $x \leq \frac{5}{3}$ (B) $x \leq 5$ or $x \geq \frac{5}{3}$ (C) $x \geq -5$ or $x \leq -\frac{5}{3}$ (D) $x \leq -5$ or $x \geq -\frac{5}{3}$

B-4. Which of the following is a solution of $3|x+1| - 5 = -2$?

(A) 2 (B) 1 (C) $\frac{4}{3}$ (D) None of these

B-5. Find x satisfying $|x - 5| \leq 3$

(A) $2 \leq x \leq 8$ (B) $3 \leq x \leq 10$ (C) $2 \geq x \geq 8$ (D) None of these

Exercise-2

OBJECTIVE QUESTIONS

1. Solution of the system of inequalities is : $3x - 4 < 8x + 6$; $2x - 1 > 5x - 4$ & $11x - 9 \leq 15x + 3$

(A) $-2 < x < 1$ (B) $-4 < x < 1$ (C) $-1 < x < 1$ (D) $-2 < x < 2$

2. Solution of inequality $(x - 1)^2 (x + 1)^3 (x - 4) \leq 0$ is :

(A) $-1 < x < 3$ (B) $-2 \leq x \leq 4$ (C) $-1 \leq x \leq 4$ (D) $-1 < x < 2$

3. Solve : $\frac{(x-1)^2(x+1)^3}{x^4(x-2)} \leq 0.$

(A) $-1 \leq x < 1$ (B) $-1 \leq x < 3$ (C) $-2 \leq x < 2$ (D) $-1 \leq x < 2 - \{0\}$

4. The marks obtained by a student of Class X in first and second unit test are 35 and 21, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 30 marks.

(A) $x \leq 34$ (B) $x \geq 34$ (C) $x > 34$ (D) $x < 34$

5. The solution set of the inequality $0 < \frac{x}{x+1} < 1, x \in R$ is

(A) Set of all positive real numbers (B) set of all real numbers except – 1

(C) set of all non-negative real numbers (D) set of all numbers satisfying $0 \leq x \leq 1, x \in R$

6. Solve : $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.

(A) $[-8, \infty)$ (B) $(7, \infty)$ (C) $(-8, \infty)$ (D) $[8, \infty)$

7. Solve : $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1.$

(A) $x \leq 1$ (B) $x > 1$ (C) $x \geq 1$ (D) $x < 2$

8. Find pair of consecutive odd natural numbers, both of which are larger than 13, such that their sum is less than 40.

(A) (15,17) (B) (13,17) (C) (19,21) (D) (21,17)

9. The value of the function $y = |2x + 1| + 2|x - 2|$ in the interval $-\frac{1}{2} \leq x < 2$, is

(A) $4x - 3$ (B) $3x - 1$ (C) 5 (D) 1

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. If a,b,c are positive, $\frac{a+c}{b+c}$ is **[IJSO - 2011]**

(A) always smaller than $\frac{a}{b}$ (B) always greater than $\frac{a}{b}$

(C) greater than $\frac{a}{b}$ only if a > b. (D) greater than $\frac{a}{b}$ only if a < b.

2. Let a, b be two positive real numbers such that $a < b < \frac{1}{a}$ and let $x = \left(a+\frac{1}{a}\right) - \left(b+\frac{1}{b}\right)$.Then **[IJSO - 2013]**

(A) x is always greater than zero
(B) x is always less than zero
(C) x = 0
(D) No such definite conclusion can be drawn about x

3. The solution set of the inequality $0 < \frac{x}{x+1} < 1$, $x \in R$ is **[IJSO - 2014]**

(A) Set of all positive real numbers (B) set of all real numbers except – 1
(C) set of all non-negative real numbers (D) set of all numbers satisfying $0 \le x \le 1$, $x \in R$

4. The statement "a is not less than 4" is correctly represented by **[IJSO - 2016]**

(A) a < 4 (B) a > 4 (C) $a \ge 4$ (D) $a \le 4$

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 $x \in (-\infty, 1) \cup (4, \infty)$ **A-2** $x \in (-1, 5)$ **A-3** $x \in (-2, \infty)$

A-4 $x \in \left(-\infty, -\frac{5}{2}\right) \cup (-2, 8)$ **A-5** $(-\infty, -2) \cup (-1, 0) \cup (1/2, +\infty)$

Section (B)

B-1 $x \in [3/2, \infty)$ **B-2** $(-\infty, -1] \cup [0, \infty)$ **B-3** $x \in (-\infty, -1] \cup [3, \infty)$

B-4 $x \in (-\infty, -1/2) \cup (0, \infty)$ **B-5** so $x \in [3/2, 2)$

OBJECTIVE QUESTIONS

Section (A)

A-1. (D) **A-2.** (C) **A-3.** (B) **A-4.** (A) **A-5.** (D)

Section (B)

B-1. (C) **B-2.** (D) **B-3.** (A) **B-4.** (D) **B-5.** (A)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9
Ans.	A	C	D	B	A	D	C	A	C

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4
Ans.	D	A	A	C

QUADRATIC EQUATION

A. QUADRATIC EQUATIONS

If P(x) is quadratic expression in variable x, then P(x) = 0 is known as a quadratic equation.

(a) General form of a Quadratic Equation :

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

Since quadratic equations, in general, are of the following types :

(i) $b = 0,\ c \neq 0$, i.e., of the type $ax^2 + c = 0$.

(ii) $b \neq 0,\ , c = 0$, i.e., of the type $ax^2 + bx = 0$.

(iii) $b = 0,\ c = 0$, i.e., of the type $ax^2 = 0$.

(iv) $b \neq 0,\ c \neq 0$, i.e., of the type $ax^2 + bx + c = 0$.

B. ROOTS OF A QUADRATIC EQUATION

The value of x which satisfies the given quadratic equation is known as its **root**. The roots of the given equation are known as its solution.

General form of a quadratic equation is :

$ax^2 + bx + c = 0$

$\Rightarrow$ $4a^2x^2 + 4abx + 4ac = 0$ [Multiplying by 4a]

$\Rightarrow$ $4a^2x^2 + 4abx = -4ac$ [By adding b^2 both sides]

$\Rightarrow$ $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$

$\Rightarrow$ $(2ax + b)^2 = b^2 - 4ac$

Taking square root of both the sides

$\Rightarrow$ $2ax + b = \pm\sqrt{b^2 - 4ac}$ $\Rightarrow$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hence, roots of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

NOTE: A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation, $ax^2 + bx + c = 0$ is :

★ A quadratic equation if $a \neq 0$: Two roots.

★ A linear equation if $a = 0,\ b \neq 0$: One root.

★ A contradiction if $a = b = 0,\ c \neq 0$: No root.

★ An identity if $a = b = c = 0$: Infinite roots.

★ A quadratic equation cannot have more than two roots.

★ It follows from the above statement that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.

C. NATURE OF ROOTS

Consider the quadratic equation, $a x^2 + b x + c = 0$ having α, β as its roots and $b^2 - 4ac$ is called discriminant of roots of quadratic equation. It is denoted by D or Δ.

Roots of the given quadratic equation may be

(i) Real and unequal **(ii)** Real and equal **(iii)** Imaginary and unequal.

Let the roots of the quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$, b, c $\in$ **R**) be α and β then

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{... (i)} \qquad \text{And} \qquad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a} \quad \text{... (ii)}$$

The nature of roots depends upon the value of expression **'$b^2 - 4ac$'** with in the square root sign. This is known as **discriminant** of the given quadratic equation.

Consider the Following Cases :

Case-1 When $b^2 - 4ac > 0$, (D > 0)

In this case roots of the given equation are real and distinct and are as follows

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

(i) When a($\neq$ 0), b, c $\in$ Q and $b^2 - 4ac$ is a perfect square
In this case both the roots are rational and distinct.

(ii) When a($\neq$ 0), b, c $\in$ Q and $b^2 - 4ac$ is not a perfect square
In this case both the roots are irrational and distinct. **[See remarks also]**

Case-2 When $b^2 - 4ac = 0$, (D = 0)

In this case both the roots are real and equal to $-\frac{b}{2a}$.

Case-3 When $b^2 - 4ac < 0$, (D < 0)

In this case $b^2 - 4ac < 0$, then $4ac - b^2 > 0$

$$\therefore \quad \alpha = \frac{-b+\sqrt{-(4ac-b^2)}}{2a} \text{ and } \beta = \frac{-b-\sqrt{-(4ac-b^2)}}{2a}$$

$$\text{or} \quad \alpha = \frac{-b+i\sqrt{4ac-b^2}}{2a} \text{ and } \beta = \frac{-b-i\sqrt{4ac-b^2}}{2a} \quad [\because \sqrt{-1} = i]$$

i.e. in this case both the roots are imaginary and distinct.

❖ **REMARKS :**

★ If **a, b, c** $\in$ **Q** and **$b^2 - 4ac$** is positive **(D > 0)** but not a perfect square, then the roots are irrational and they always occur in conjugate pairs like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if **a, b, c** are irrational numbers and **$b^2 - 4ac$** is positive but not a perfect square, then the roots may not occur in conjugate pairs.

★ If **$b^2 - 4ac$** is negative **(D < 0)**, then the roots are complex conjugate of each other. In fact, complex roots of an equation with real coefficients always occur in conjugate pairs like 2 + 3i and 2 – 3i. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.

★ If **a** and **c** are of the same sign and b has a sign opposite to that of a as well as c, then both the roots are positive, the sum as well as the product of roots is positive **(D $\geq$ 0)**.

★ If **a, b, c** are of the same sign then both the roots are negative, the sum of the roots is negative but the product of roots is positive **(D $\geq$ 0).**

Solved Examples

Example.1

Find the roots of the equation $x^2 - x - 3 = 0$.

Solution.

$x^2 - x - 3 = 0$

From the quadratic formula we can find the value of x,

$$x = \frac{+1 \pm \sqrt{1 - 4 \times 1 \ (-3)}}{2 \times 1} = \frac{1 \pm \sqrt{13}}{2}$$

So, $x = \frac{1+\sqrt{13}}{2}, \frac{1-\sqrt{13}}{2}$

Hence, the roots are Irrational.

Example.2

Determine the value of K for which the x = – a is a solution of the equation :

$x^2 - 2(a + b)x + 3K = 0$.

Solution.

Putting x = – a in the given equation, we have $(-a)^2 - 2(a + b)(-a) + 3K = 0$

$\Rightarrow \quad a^2 + 2a^2 + 2ab + 3K = 0 \Rightarrow \quad 3K = -3a^2 - 2ab \quad \Rightarrow \quad K = -\frac{a}{3}(3a + 2b)$.

D. METHODS OF SOLVING QUADRATIC EQUATION

(a) By Factorization :

ALGORITHM :

Step (i) Factorize the constant term of the given quadratic equation.

Step (ii) Express the coefficient of middle term as the sum or difference of the factors obtained in step (i). Clearly, the product of these two factors will be equal to the product of the coefficient of x^2 and constant term.

Step (iii) Split the middle term in two parts obtained in step (ii).

Step (iv) Factorize the quadratic equation obtained in step (iii).

Solved Examples

Example.3

Solve the following quadratic equation by factorization method : $x^2 - 2ax + a^2 - b^2 = 0$.

Solution.

Here, Factors of constant term $(a^2 - b^2)$ are (a – b) and (a + b).

Also, Coefficient of the middle term = – 2a = – [(a – b) + (a + b)]

$\therefore \quad x^2 - 2ax + a^2 - b^2 = 0$

$\Rightarrow \quad x^2 - \{(a - b) + (a + b)\}x + (a - b)(a + b) = 0$

$\Rightarrow \quad x^2 - (a - b)x - (a + b)x + (a - b)(a + b) = 0$

$\Rightarrow \quad x\{x - (a - b)\} - (a + b)\{x - (a - b)\} = 0$

$\Rightarrow \quad \{x - (a - b)\}\{x - (a + b)\} = 0$

$\Rightarrow \quad x - (a - b) = 0$ or, $x - (a + b) = 0$

$\Rightarrow \quad x = a - b$ or $x = a + b$

Example.4

Solve the quadratic equation by the method of factorisation : $x^2 + 8x + 7 = 0$.

Solution.

We have

$x^2 + 8x + 7 = 0$

$\Rightarrow \quad x^2 + 7x + x + 7 = 0$

$\Rightarrow \quad x(x + 7) + 1(x + 7) = 0$

$\Rightarrow \quad (x + 7)(x + 1) = 0$

$\therefore \quad$ Either $x + 7 = 0$

$\Rightarrow \quad x = -7$

or $\quad x + 1 = 0$

$\Rightarrow \quad x = -1$

Hence, the required solutions are $x = -7$ and $x = -1$.

Example.5

Solve the quadratic equation $16x^2 - 24x = 0$.

Solution.

The given equation may be written as $8x(2x - 3) = 0$.

This gives $x = 0$ or $x = \frac{3}{2}$.

$x = 0, \frac{3}{2}$ are the required solutions.

Example.6

Find the solutions of the quadratic equation $x^2 + 6x + 5 = 0$.

Solution.

The quadratic polynomial $x^2 + 6x + 5$ can be factorised as follows :

$x^2 + 6x + 5$

$= x^2 + 5x + x + 5$

$= x(x + 5) + 1(x + 5)$

$= (x + 5)(x + 1)$

Therefore, the given quadratic equation becomes $(x + 5)(x + 1) = 0$.

This gives $x = -5$ or $x = -1$.

Therefore, $x = -1, -5$ are the required solutions of the given equation.

(b) By Completion of Square Method :

ALGORITHM :

Step-(i) Obtain the quadratic equation. Let the quadratic equation be $ax^2 + bx + c = 0$, $a \neq 0$.

Step-(ii) Make the coefficient of x^2 unity, if it is not unity. i.e., obtain $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Step-(iii) Shift the constant term $\frac{c}{a}$ on R.H.S. to get $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Step-(iv) Add square of half of the coefficient of x, i.e. $\left(\frac{b}{2a}\right)^2$ on both sides to obtain :

$$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Step-(v) Write L.H.S. as the perfect square of a binomial expression and simplify R.H.S. to get

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Step-(vi) Take square root of both sides to get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Step (vii) Obtain the values of x by shifting the constant term $\frac{b}{2a}$ on RHS.

Solved Examples

Example.7

Solve: $25x^2 - 30x + 9 = 0$.

Solution.

$25x^2 - 30x + 9 = 0 \quad \Rightarrow \quad (5x)^2 - 2(5x)\times 3 + (3)^2 = 0 \quad \Rightarrow \quad (5x - 3)^2 = 0$

This gives $x = \frac{3}{5}, \frac{3}{5}$ or simply $x = \frac{3}{5}$ as the required solution.

Example.8

Solve: $x^2 + 3x + 1 = 0$.

Solution.

We have $x^2 + 3x + 1 = 0$

Add and subtract ½ (coefficient of x)^2 in L.H.S. and get

$$x^2 + 3x + 1 + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = 0 \quad \Rightarrow \quad x^2 + 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(x + \frac{3}{2}\right)^2 - \frac{5}{4} = 0 \quad \Rightarrow \quad \left(x + \frac{3}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 \quad \Rightarrow \quad x + \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$$

This gives $x = \frac{-\left(3 + \sqrt{5}\right)}{2}$ or $x = \frac{-3 + \sqrt{5}}{2}$

Therefore, $x = -\frac{3 + \sqrt{5}}{2}, \ \frac{-3 + \sqrt{5}}{2}$ are the solutions of the given equation.

Example.9

Solve the quadratic equation by completing the squares : $3x^2 - 2\sqrt{15}\,x - 2 = 0$.

Solution.

We have

$$3x^2 - 2x\sqrt{15} - 2 = 0 \quad \Rightarrow \quad x^2 - 2\frac{\sqrt{15}}{3}x - \frac{2}{3} = 0 \quad \text{[Dividing both sides by the coefficient of } x^2\text{]}$$

$$\Rightarrow \quad x^2 - 2\frac{\sqrt{15}}{3}x + \left(\frac{\sqrt{15}}{3}\right)^2 - \left(\frac{\sqrt{15}}{3}\right)^2 - \frac{2}{3} = 0$$

[Adding and subtracting the square of half the coefficient of x]

$$\Rightarrow \quad \left(x - \frac{\sqrt{15}}{3}\right)^2 - \frac{15}{9} + \frac{2}{3} = 0 \Rightarrow \quad \left(x - \frac{\sqrt{15}}{3}\right)^2 - \left(\frac{15 + 6}{9}\right) = 0$$

$$\Rightarrow \quad \left(x - \frac{\sqrt{15}}{3}\right)^2 - \frac{21}{9} = 0 \quad \Rightarrow \quad x - \frac{\sqrt{15}}{3} = \pm\frac{\sqrt{21}}{3} \text{ [Taking the square roots on both sides]}$$

$\Rightarrow \quad x = \frac{\sqrt{15}}{3} \pm \frac{\sqrt{21}}{3} = \frac{\sqrt{15} \pm \sqrt{21}}{3}$. Hence, the required solutions are

$x = \frac{\sqrt{15} + \sqrt{21}}{3}$ and $x = \frac{\sqrt{15} - \sqrt{21}}{3}$.

(c) By Using Quadratic Formula :

Solve the quadratic equation in general form viz. $ax^2 + bx + c = 0$.

We have, $ax^2 + bx + c = 0$

Step (i) By comparison with general quadratic equation, find the value of a, b and c.

Step (ii) Find the discriminant of the quadratic equation.

$D = b^2 - 4ac$

Step (iii) Now find the roots of the equation by given equation

$$x = \frac{-b+\sqrt{D}}{2a}, \quad \frac{-b-\sqrt{D}}{2a}$$

❖ **REMARK :**

★ If $\mathbf{b^2 - 4ac < 0}$, i.e., negative, then $\sqrt{b^2 - 4ac}$ is not real and therefore, the equation does not have any real roots.

Solved Examples

Example.10

Solve the quadratic equation $x^2 - 7x - 5 = 0$.

Solution.

Comparing the given equation with $ax^2 + bx + c = 0$, we find that a = 1, b = – 7 and c = – 5.

Therefore, $D = (-7)^2 - 4 \times 1 \times (-5) = 49 + 20 = 69 > 0$

Since, D is positive, the equation has two roots given by $\frac{7+\sqrt{69}}{2}, \quad \frac{7-\sqrt{69}}{2}$

$\Rightarrow \quad x = \frac{7+\sqrt{69}}{2}, \quad \frac{7-\sqrt{69}}{2}$ are the required solutions.

Example.11

If the roots of the equation : $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, show that $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$.

Solution.

Since the roots of the given equations are equal, so discriminant will be equal to zero.

$\Rightarrow \quad b^2(c - a)^2 - 4a(b - c)c(a - b) = 0$

$\Rightarrow \quad b^2(c^2 + a^2 - 2ac) - 4ac(ba - ca - b^2 + bc) = 0,$

$\Rightarrow \quad a^2b^2 + b^2c^2 + 4a^2c^2 + 2b^2ac - 4a^2bc - 4abc^2 = 0,$

$\Rightarrow \quad (ab + bc - 2ac)^2 = 0 \quad \Rightarrow \quad ab + bc - 2ac = 0$

$\Rightarrow \quad ab + bc = 2ac \quad \Rightarrow \quad \frac{1}{c} + \frac{1}{a} = \frac{2}{b}$

$\Rightarrow \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}.$ **Hence Proved.**

Example.12

If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal , then prove that 2b = a + c.

Solution.

If the roots of the given equation are equal, then discriminant is zero i.e

$(c - a)^2 - 4(b - c)(a - b) = 0$

$\Rightarrow \quad c^2 + a^2 - 2ac + 4b^2 - 4ab + 4ac - 4bc = 0$

$\Rightarrow \quad c^2 + a^2 + 4b^2 + 2ac - 4ab - 4bc = 0$

$\Rightarrow \quad (c + a - 2b)^2 = 0$

$\Rightarrow \quad c + a = 2b.$ **Hence Proved.**

Example.13

If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real and distinct, then find all possible values of a .

Solution.

Since the roots of the given equation are real and distinct, we must have D > 0

$\Rightarrow$ $64 - 4(a^2 - 6a) > 0 \Rightarrow 4[16 - a^2 + 6a] > 0$

$\Rightarrow$ $-4(a^2 - 6a - 16) > 0 \Rightarrow a^2 - 6a - 16 < 0$

$\Rightarrow$ $(a - 8)(a + 2) < 0$

$\Rightarrow$ $-2 < a < 8$.

Hence, the roots of the given equation are real if **'a'** lies between – 2 and 8.

E. APPLICATIONS OF QUADRATIC EQUATIONS

ALGORITHM :

The method of problem solving consists of the following three steps :

Step (i) Translating the word problem into symbolic language (mathematical statement) which means identifying relationships existing in the problem and then forming the quadratic equation.

Step (ii) Solving the quadratic equation thus formed.

Step (iii) Interpreting the solution of the equation, which means translating the result of mathematical statement into verbal language.

REMARKS :

★ Two consecutive odd natural numbers be $2x - 1$, $2x + 1$ where $x \in N$.

★ Two consecutive even natural numbers be $2x$, $2x + 2$ where $x \in N$.

★ Two consecutive even positive integers be $2x$, $2x + 2$ where $x \in Z^+$.

★ Consecutive multiples of 5 be $5x$, $5x + 5$, $5x + 10$, ...

Solved Examples

Example.14

The sum of the squares of two consecutive positive integers is 545. Find the integers.

Solution.

Let x be one of the positive integers. Then the other integer is $x + 1$, where $x \in z^+$.

Since the sum of the squares of the integers is 545, we get $x^2 + (x + 1)^2 = 545$

$\Rightarrow$ $2x^2 + 2x - 544 = 0$

$\Rightarrow$ $x^2 + x - 272 = 0$

$\Rightarrow$ $x^2 + 17x - 16x - 272 = 0$

$\Rightarrow$ $x(x + 17) - 16(x + 17) = 0$

$\Rightarrow$ $(x - 16)(x + 17) = 0$.

Here, x = 16 or x = – 17. But, x is a positive integer. Therefore, reject x = – 17 and take x = 16. Hence, two consecutive positive integers are 16 and (16 + 1), i.e., 16 and 17.

Example.15

The sum of two numbers is 48 and its product is 432. Find the numbers.

Solution.

Let the two numbers be x and 48 – x so that their sum is 48. It is given that the product of the two numbers is 432. Hence, we have

$x(48 - x) = 432$

$\Rightarrow$ $48x - x^2 - 432 = 0$

$\Rightarrow$ $x^2 - 48x + 432 = 0$

$\Rightarrow$ $x^2 - 36x - 12x + 432 = 0$

$\Rightarrow$ $x(x - 36) - 12(x - 36) = 0$

$\Rightarrow$ $(x - 36)(x - 12) = 0$

$\therefore$ Either $x - 36 = 0$

$\Rightarrow$ $x = 36$ or $x - 12 = 0 \Rightarrow x = 12$.

$\therefore$ When one number is 12 another number is 48 – 12 = 36 and when one number is 36, another number is 48 – 36 = 12.

Example.16

The length of a hall is 5 m more than its breadth. If the area of the floor of the hall is 84 m^2, what are the length and the breadth of the hall ?

Solution.

Let the breadth of the hall be x metres.

Then the length of the hall is (x + 5) metres.

The area of the floor = $x(x + 5)$ m^2

Therefore, $x(x + 5) = 84$ $\Rightarrow$ $x^2 + 5x - 84 = 0$ $\Rightarrow$ $(x + 12)(x - 7) = 0$

This gives $x = 7$ or $x = -12$.

Since, the breadth of the hall cannot be negative, we reject $x = -12$ and take $x = 7$ only.

Thus, breadth of the hall = 7 metres, and length of the hall = (7 + 5), i.e., 12 metres.

Example.17

Out of a group of swans, $\frac{7}{2}$ times the square root of the total number are playing on the shore of a tank. The two remaining ones are playing, in deep water. What is the total number of swans ?

Solution.

Let us denote the number of swans by x.

Then, the number of swans playing on the shore of the tank = $\frac{7}{2}\sqrt{x}$.

There are two remaining swans.

Therefore, $x = \frac{7}{2}\sqrt{x} + 2$ $\Rightarrow$ $x - 2 = \frac{7}{2}\sqrt{x}$

$\Rightarrow$ $(x - 2)^2 = \left(\frac{7}{2}\right)^2 x$ $\Rightarrow$ $4(x^2 - 4x + 4) = 49x$

$\Rightarrow$ $4x^2 - 65x + 16 = 0$ $\Rightarrow$ $4x^2 - 64x - x + 16 = 0$

$\Rightarrow$ $4x(x - 16) - 1(x - 16) = 0$ $\Rightarrow$ $(x - 16)(4x - 1) = 0$

This gives $x = 16$ or $x = \frac{1}{4}$. We reject $x = \frac{1}{4}$ and take $x = 16$.

Hence, the total number of swans is 16.

Example.18

The sum 'S' of first n natural numbers is given by the relation $S = \frac{n(n+1)}{2}$. Find n, if the sum is 276.

Solution.

We have $S = \frac{n(n+1)}{2} = 276$ $\Rightarrow$ $n^2 + n - 552 = 0$

This gives $n = \frac{-1+\sqrt{1+2208}}{2}, \frac{-1-\sqrt{1+2208}}{2}$ $\Rightarrow$ $n = \frac{-1+\sqrt{2209}}{2}, \frac{-1-\sqrt{2209}}{2}$

$\Rightarrow$ $n = \frac{-1+47}{2}, \frac{-1-47}{2}$ $\Rightarrow$ $n = 23, -24$

We reject $n = -24$, since –24 is not a natural number. Therefore, n = 23.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Method of finding roots

A-1 Solve the following quadratic equation by factorisation method : $x^2 + 2ax + a^2 - b^2 = 0$.

A-2 Solve : $9x^2 - 30x + 25 = 0$.

A-3 Solve the following quadratic equation by completing the square method :- $x^2 + 9x + 1 = 0$.

A-4 Solve : $\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(x-3)(2x+3)} = 0$.

A-5 Solve the following equation for x : $9x^2 - 9(a + b)x + (2a^2 + 5ab + 2b^2) = 0$.

Section (B) : Nature of roots

B-1 Find the value of k for which quadratic equation $(k - 2)x^2 + 2(2k - 3)x + 5k - 6 = 0$ has equal roots.

B-2. If the roots of the equations $px^2 + 2qx + r = 0$ and $qx^2 - 2\sqrt{pr}\,x + q = 0$ be real then prove that $q^2 = pr$.

B-3. If the roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in G.P. then find the roots.

B-4 If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real and distinct, then find all possible values of a .

B-5 For what value of k, $(4 - k)x^2 + (2k + 4)x + (8k + 1)$ is a perfect square.

Section (C) : Word problem

C-1 A motor boat whose speed is 18 km/h in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

C-2 Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

C-3 A two digit number is four times the sum and three times the product of its digits, find the number.

C-4 The numerator of a fraction is 1 less than its denominator. If 3 is added to each of the numerator and denominator, the fraction is increased by $\frac{3}{28}$. Find the fraction.

C-5 An aeroplane left 30 minutes later than its scheduled time and in order to reach its destination 1500 km away in time, it has to increase its speed by 250 km/h from its usual speed. Determine its usual speed.

OBJECTIVE QUESTIONS

Section (A) : Method of finding roots

A-1. The roots of the equation $x^{2/3} + x^{1/3} - 2 = 0$ are :
(A) 1,4 (B) 1,– 4 (C) 1, – 8 (D) 1, 8

A-2. Evaluate : $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+}}}}.....\infty$.
(A) 2.45 (B) 2.59 (C) 2.79 (D) 3

A-3. Solve : $\sqrt{2x+9} + x = 13$
(A) 8 (B) 20 (C) 16 (D) 3

A-4. Solve for x : $2\left(x^2+\frac{1}{x^2}\right) - 9\left(x+\frac{1}{x}\right) + 14 = 0$.
(A) $\frac{1}{2}$, 1, 2 (B) $\frac{1}{2}$, 1,– 2 (C) $\frac{1}{2}$, 1, 4 (D) $\frac{1}{2}$, 1,– 4

A-5. Solve the quadratic equation : $6\sqrt{2}x^2 + 5x - 3\sqrt{2} = 0$.
(A) x= $\frac{\sqrt{2}}{3}$ or x = $\frac{-3}{2\sqrt{2}}$ (B) x= $\frac{1}{2}$ or x = $\frac{-3}{2\sqrt{2}}$ (C) x= $\frac{1}{2}$ or x = 1 (D) None of these

Section (B) : Nature of roots

B-1. If the roots of the quadratic equation $12x^2 - mx + 5 = 0$ are in the ratio 3 : 4, then m =
(A) $7\sqrt{5}$ (B) $8\sqrt{5}$ (C) $6\sqrt{10}$ (D) None of these

B-2. If the equation $x^2 - 2kx - 2x + k^2 = 0$ has equal roots, the value of k must be :
(A) zero (B) either zero or $-\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) either $\frac{1}{2}$ or $-\frac{1}{2}$

B-3. For what value of k will $x^2 - (3k - 1)x + 2k^2 + 2k = 11$ have equal roots ?
(A) 9, – 5 (B) – 9, 5 (C) 9, 5 (D) – 9, – 5

B-4. A student notices that the roots of the equation $x^2 + bx + a = 0$ are each 1 less than the roots of the equation $x^2 + ax + b = 0$. Then a + b is :
(A) possibly any real number (B) – 2 (C) – 4 (D) – 5

B-5. Both roots of equation $(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$ are always:
(A) Positive (B) Negative (C) Real (D) None of these

Section (C) : Word problem

C-1. The sum of the squares of two consecutive odd positive integers is 290. Find them.
(A) 11, 13 (B) 9, 11 (C) 13, 15 (D) 7, 9

C-2. The sum of the squares of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers.
(A) 14, 10 (B) 12, 6 (C) 12, 8 (D) 16, 12

C-3. The length of a hypotenuse of a right triangle exceeds the length of its base by 2 cm and exceeds twice the length of the altitude by 1 cm. Find the length of each side of the triangle (in cm) :
(A) 6, 8, 10 (B) 7, 24, 25 (C) 8, 15, 17 (D) 7, 40, 41

C-4. Two square have sides x cm and (x + 4) cm. The sum of their areas is 656 cm^2 . Find the sides of the square
(A) 8 cm, 12 cm (B) 12 cm, 15 cm (C) 6 cm, 10 cm (D) 16 cm, 20 cm

C-5. If the list price of a book is reduced by Rs. 5, a person can buy 5 more books for Rs. 300. Find the original list price of the book.
(A) 15 (B) 10 (C) 20 (D) 25

Section (D) : Miscellaneous

D-1. If the roots of the equation $x^2 + px - 6 = 0$ are 6 and – 1, then the value of p is :
(A) 2 (B) 3 (C) – 5 (D) 5

D-2. The quadratic equation having roots $1+\sqrt{5}$ and $1-\sqrt{5}$ is :
(A) $x^2 - 2x - 4 = 0$ (B) $x^2 + 2x + 4 = 0$ (C) $x^2 + 2x - 4 = 0$ (D) $x^2 - 2x + 4 = 0$

D-3. If a and c are such that the quadratic equation $ax^2 - 5x + c = 0$ has 10 as the sum of the root and also as the product of the roots, find a and c respectively.
(A) $\frac{1}{2}$, 6 (B) $\frac{1}{2}$, 5 (C) $\frac{1}{2}$, 8 (D) None of these

D-4. The equation $\sqrt{x-2\sqrt{x-1}}+\sqrt{x+3-4\sqrt{x-1}} = 1$ has
(A) No solution (B) Only one solution (C) Only two solution (D) More than two solution

D-5. For all real value of x, $\frac{x^2+2x+p}{x^2+4x+3}$ can take all real value if
(A) $0 < p < 2$ (B) $0 \le p \le 1$ (C) $-1 < p < 1$ (D) $-3 \le p \le 1$

Exercise-2

OBJECTIVE QUESTIONS

1. If p and q are the roots of the equation $x^2 + px + q = 0$, then :
(A) p = 1 (B) p = 1 or 0 (C) p = – 2 (D) p = – 2 or 0

2. If the equation $\frac{a}{x-a} + \frac{b}{x-b} = 1$ has roots equal in magnitude but opposite in sign, then the value of a + b is
(A) – 1 (B) 0 (C) 1 (D) none of these

3. The roots of the quadratic equation $(a + b - 2c)x^2 - (2a - b - c)x + (a - 2b + c) = 0$ are
(A) a + b + c & a – b + c (B) 1/2 & a – 2b + c
(C) a – 2b + c & 1/(a + b – 2c) (D) none of these

4. The least integral positive value of 'm' for which the expression $mx^2 - 4x + 3m + 1$ is positive for every $x \in R$ is :
(A) 1 (B) 3 (C) 4 (D) 2

5. If the quadratic equations, $ax^2 + 2cx + b = 0$ & $ax^2 + 2bx + c = 0$ $(b \neq c)$ have a common root, then $a + 4b + 4c$ is equal to :
(A) -2 (B) -2 (C) 0 (D) 1

6. If $x = 1 + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + ----}}}$, then the value of x will be
(A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{3}{2}$

7. The solution of the equation $4^x + 2^x = 6$ is/are :
(A) 0, 1 (B) 2, 1 (C) 1, 0 (D) 1

8. The equation $\sqrt{x+1} + \sqrt{x-1} = 0$ has :
(A) one real solution (B) two real solutions (C) No solution (D) None of these

9. The roots of the equation $\sqrt{3x+1} - 1 = \sqrt{x}$ are :
(A) 0 (B) 1 (C) 0,1 (D) None

10. The roots of the equation $x^2 + ax - 4 = 0$ are, where $a \in R$:
(A) real and distinct (B) equal (C) imaginary (D) real

11. A shopkeeper buys a number of books for Rs. 80. If he had bought 4 more for the same amount, each book would have cost Rs. 1 less. How many books did he buy ?
(A) 8 (B) 16 (C) 24 (D) 28

12. Solve the system of equations : $\begin{cases} 1/x + 1/y = 3/2, \\ 1/x^2 + 1/y^2 = 5/4 \end{cases}$
(A) (1,2) (B) (1,3) (C) (2, 3) (D) (2, 2)

13. If one root of the quadratic equation $ax^2 + bx + c = 0$ is double the other then which one of the following is correct :
(A) $b^2 = 3ac$ (B) $2b^2 = 5ac$ (C) $2b^2 = 9ac$ (D) $2b^2 > 9ac$

COMPREHENSION (For Q.No. 14. to 16)

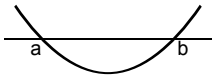
The coordinates of the vertex of the parabola $f(x) = 2x^2 + px + q$ are (–3, 1), then

14. The value of p is
(A) 12 (B) – 12 (C) 19 (D) – 19

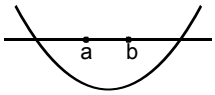
15. The value of q is
(A) – 19 (B) 19 (C) – 12 (D) none of these

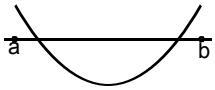
16. The parabola
(A) touches the x-axis
(B) intersect the x-axis in two real and distinct points
(C) lies completely above the x-axis
(D) lies completely below the x-axis

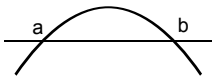
17. Solution set of the equation $3^{2x^2} - 2.3^{x^2+x+6} + 3^{2(x+6)} = 0$ is
(A) {–3, 2} (B) {6, –1} (C) {–2, 3} (D) {1, – 6}

18. If graph of y = (x – a)(x – b) is 

, then graph of y = (x – a) (x – b) – 2 is

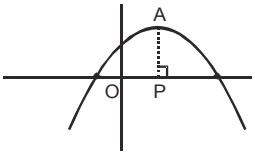
(A) 

(B) 

(C) 

(D) None of these

19. If $y = ax^2 + bx + c$ represents the curve given in the figure and $b^2 = 2(b + 2ac)$, where $a \neq 0$ and AP = 3 units, then OP =



(A) 3/2 (B) 3/4 (C) 3 (D) 6

20. If b > 0, b ≠ 1, then the equation $2\log_x b + \log_{bx} b + 3\log_{b^2x} b = 0$ has
(A) Exactly one real root (B) Two real roots
(C) No real roots (D) Infinite number of real roots

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. The quadratic equation $a x^2 + bx + c = 0$ has real roots α and β. If a, b, c real and of the same sign, then : **[IJSO-2008]**
(A) α and β are both positive
(B) α and β are both negative
(C) α and β are of opposite sign
(D) Nothing can be said about the signs of α and β as the information is insufficient.

2. $(2x^2 + 3x + 5)^{1/2} + (2x^2 + 3x + 20)^{1/2} = 15$, therefore x is : **[IJSO-2009]**
(A) (–8 / 3) (B) (14/5) (C) (–11/2) (D) 4

3. The sum of all roots of the equations $|x - 1|^2 - 5|x - 1| + 6 = 0$ is : **[IJSO-2009]**
(A) 5 (B) 4 (C) 2 (D) 6

4. Solve for x, given $y = x^2 - 1$ and $x = 2y + 1$. **[NSTSE 2009]**
(A) $x \in \{0, 1\}$ (B) $x \in \left\{\frac{-1}{2}, \frac{-3}{4}\right\}$ (C) $x \in \left\{\frac{-1}{2}, 1\right\}$ (D) $x \in \left\{0, \frac{-3}{4}\right\}$

5. Find the sum of all values of "x", so that $16^{(x^2+3x-1)} = 8^{(x^2+3x+2)}$. **[NSTSE 2009]**
(A) 0 (B) 3 (C) – 3 (D) – 5

6. The product of the roots of the equation $\sqrt[3]{8+x} + \sqrt[3]{8-x} = 1$ is **[IJSO - 2010]**
(A) – 21 (B) – 189 (C) 9 (D) – 5

7. If one of the roots of the equation $x^2 - px + q = 0$ is m times the other root then $m/(1 + m^2)$ is equal to **[IJSO - 2012]**

(A) $\frac{q}{p^2-2q}$ (B) $\frac{p}{q^2-2p}$ (C) $\frac{q}{q^2-2p}$ (D) $\frac{p}{p^2-2q}$

8. If a and b are two positive real numbers such that $\frac{a^2+b^2}{ab} = 6$, then a positive value of $\frac{a}{b}$ lies between **[IJSO - 2013]**

(A) 2 and 3 (B) 3 and 4 (C) 4 and 5 (D) 5 and 6

9. The number of natural numbers $n \le 30$ for which $\sqrt{n+\sqrt{n+\sqrt{n+.....}}}$ is natural number is

(A) 30 (B) zero (C) 6 (D) 5

10. The number of natural numbers $n \le 30$ for which $\sqrt{n+\sqrt{n+\sqrt{n+..}}}$ is a prime number is **[IJSO - 2016]**

(A) Three (B) Zero (C) Nine (D) Two

11. Two regular polygons of different number of sides are taken. In one of them, its sides are coloured red and diagonals are coloured green; in the other, sides are coloured green and diagonals are coloured red. Suppose there are 103 red lines and 80 green lines. The total number of sides the two polygons together have is **[IJSO - 2018]**

(A) 23 (B) 28 (C) 33 (D) 38

12. If $\sin\theta$ and $\cos\theta$ are roots of the equation $px^2 + qx + r = 0$, then : **[IJSO - 2018]**

(A) $p^2 - q^2 + 2pr = 0$ (B) $(p + r)^2 = q^2 - r^2$ (C) $p^2 + q^2 - 2pr = 0$ (D) $(p - r)^2 = q^2 + r^2$

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 x = –(a – b) or x = –(a + b)

A-2 $x = \frac{5}{3}, \frac{5}{3}$

A-3 $x = \frac{-9 \pm \sqrt{77}}{2}$

A-4 x = – 1

A-5 $x = \frac{2a+b}{3}, \frac{a+2b}{3}$.

Section (B)

B-1 1,3

B-3. $1, \frac{1}{2}, \frac{1}{4}$

B-4 – 2 < a < 8

B-5 k = 0 or k = 3.

Section (C)

C-1 6 km/hr.

C-2 Smaller tap = 25 hr , larger tap = 15 hrs.

C-3 10 x + y = 10 × 2 + 4 = 24.

C-4 $\frac{3}{4}$.

C-5 750 km/hr.

OBJECTIVE QUESTIONS

Section (A)

A-1. (C) **A-2.** (D) **A-3.** (A) **A-4.** (A) **A-5.** (A)

Section (B)

B-1. (A) **B-2.** (C) **B-3.** (C) **B-4.** (C) **B-5.** (C)

Section (C)

C-1. (A) **C-2.** (C) **C-3.** (C) **C-4.** (D) **C-5.** (C)

Section (D)

D-1. (C) **D-2.** (A) **D-3.** (B) **D-4.** (D) **D-5.** (D)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	B	B	D	D	C	D	D	C	C	A	B	A	C	A	B	C	C	A	C	B

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10	11	12
Ans.	B	B,D	B	C	C	B	A	D	D	A	B	A

PROGRESSION

A. SEQUENCE

A sequence is an arrangement of numbers in a definite order according to some rule.
e.g. **(i)** 2, 5, 8, 11, ... **(ii)** 4, 1, – 2, – 5, ... **(iii)** 3, –9, 27, – 81, ...

Types of Sequence

On the basis of the number of terms there are two types of sequence :

(i) Finite sequences : A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences : A sequence is said to be infinite if it has infinite number of terms.

Solved Examples

Example.1

Write down the sequence whose n^{th} term is :

(i) $\frac{2^n}{n}$ **(ii)** $\frac{3+(-1)^n}{3^n}$

Solution.

(i) Let $t_n = \frac{2^n}{n}$

Put n = 1, 2, 3, 4, ... we get

$t_1 = 2, t_2 = 2, t_3 = \frac{8}{3}, t_4 = 4$

So the sequence is $2, 2, \frac{8}{3}, 4, ...$

(ii) Let $t_n = \frac{3+(-1)^n}{3^n}$

Put n = 1, 2, 3, 4, ...

So the sequence is

$\frac{2}{3}, \frac{4}{9}, \frac{2}{27}, \frac{4}{81}, ...$

B. PROGRESSION

Those sequence whose terms follow certain patterns are called progressions. Generally there are three types of progressions.

(i) Arithmetic Progression **(A.P.)**
(ii) Geometric Progression **(G.P.)**
(iii) Harmonic Progression **(H.P.)**

C. ARITHMETIC PROGRESSION

A sequence is called an **A.P.**, if the difference of a term and the previous term is always same. **i.e. $d = t_{n+1} - t_n$ = Constant** for all **$n \in N$**. The constant difference, generally denoted by **'d'** is called the common difference.

Solved Examples

Example.2

Find the common difference of the following A.P. : 1, 4, 7, 10, 13, 16,.....

Solution.

4 – 1 = 7 – 4 = 10 – 7 = 13 – 10 = 16 – 13 = 3 (constant).

$\therefore$ Common difference (d) = 3.

D. GENERAL FORM OF AN A.P.

If we denote the starting number i.e. the 1st number by **'a'** and a fixed number to be added is **'d'** then **a, a + d, a + 2d, a + 3d, a + 4d,.....** forms an **A.P.**

Solved Examples

Example.3

Find the A.P. whose 1st term is 10 & common difference is 5.

Solution.

Given :First term (a) = 10 & Common difference (d) = 5.

$\therefore$ A.P. is 10, 15, 20, 25, 30,.....

E. n^{th} FORM OF AN A.P.

Let A.P. be a, a + d, a + 2d, a + 3d,.....

Then, First term (a_1) = a + 0.d

Second term (a_2) = a + 1.d

Third term (a_3) = a + 2.d

. .

. .

. .

n^{th} term (a_n) = a + (n – 1) d

$\therefore$ **a_n = a + (n – 1) d is called the nth term.**

Solved Examples

Example.4

Determine the A.P. whose third term is 16 and the difference of 5th term from 7th term is 12.

Solution.

Given : $a_3 = a + (3 - 1)\,d = a + 2d = 16$ .... (i)

$a_7 - a_5 = 12$ ..... (ii)

$\Rightarrow$ (a + 6d) – (a + 4d) = 12

$\Rightarrow$ a + 6d – a – 4d = 12

$\Rightarrow$ 2d = 12

$\Rightarrow$ d = 6

Put d = 6 in equation (i)

a = 16 – 12 $\Rightarrow$ a = 4.

$\therefore$ A.P. is 4, 10, 16, 22, 28,.....

Example.5

Which term of the sequence 72, 70, 68, 66,..... is 40 ?

Solution.

Here 1st term a = 72 and common difference d = 70 – 72 = – 2.

For finding the value of n

$a_n = a + (n - 1)d$

$\Rightarrow$ $40 = 72 + (n - 1)(-2)$ $\Rightarrow$ $40 - 72 = -2n + 2$

$\Rightarrow$ $-32 = -2n + 2$ $\Rightarrow$ $-34 = -2n$

$\Rightarrow$ $n = 17$

17th term is 40.

Example.6

Is 184, a term of the sequence 3, 7, 11,..... ?

Solution.

Here 1st term (a) = 3 and common difference (d) = 7 – 3 = 4.

nth term $(a_n) = a + (n - 1)d$

$\Rightarrow$ $184 = 3 + (n - 1)4$ $\Rightarrow$ $181 = 4n - 4$

$\Rightarrow$ $185 = 4n$ $\Rightarrow$ $n = \frac{185}{4}$

Since, n is not a natural number.

$\therefore$ 184 is not a term of the given sequence.

F. m^{th} TERM OF AN A.P. FROM THE END

Let **'a'** be the 1st term and **'d'** be the common difference of an **A.P.** having **n** terms. Then **mth** term from the end is **(n – m + 1)th** term from beginning or **{n – (m – 1)}th** term from beginning.

Solved Examples

Example.7

Find 20th term from the end of an A.P. 3, 7, 11.....407.

Solution.

Let $a_n = 407$, 1st term = 3 , common difference = 4

$407 = 3 + (n - 1)4 \Rightarrow n = 102$

20th term from end $\Rightarrow m = 20$

$a_{102-(20-1)} = a_{102-19} = a_{83}$ from the beginning.

$a_{83} = 3 + (83 - 1)4 = 331$.

G. SELECTION OF TERMS IN AN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

No. of Terms	Terms	Common Difference
For 3 terms	a – d, a, a + d	d
For 4 terms	a – 3d, a – d, a + d, a + 3d	2d
For 5 terms	a – 2d, a – d, a, a + d, a + 2d	d
For 6 terms	a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d	2d

Solved Examples

Example.8

The sum of three numbers in A.P. is – 3 and their product is 8. Find the numbers.

Solution.

Let three no.'s in A.P. be a – d, a, a + d

$a - d + a + a + d = -3 \Rightarrow 3a = -3 \Rightarrow a = -1$

and $(a - d)\, a\, (a + d) = 8 \Rightarrow a(a^2 - d^2) = 8$

$\Rightarrow (-1)(1 - d^2) = 8 \Rightarrow 1 - d^2 = -8$

$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

If a = – 1 & d = 3 numbers are – 4, –1, 2.

If a = – 1 & d = –3 numbers are 2, –1, – 4.

H. SUM OF n TERMS OF AN A.P.

Let A.P. be **a, a + d, a + 2d, a + 3d,....., a + (n – 1)d**

Then,

$S_n = a + (a + d) + ... + \{a + (n - 2)d\} + \{a + (n - 1)d\}$ ..(i)

also

$S_n = \{a + (n - 1)d\} + \{a + (n - 2)d\} + ... + (a + d) + a$ ..(ii)

Add (i) & (ii)

$\Rightarrow 2S_n = 2a + (n - 1)d + 2a + (n - 1)d + ... + 2a + (n - 1)d$

$\Rightarrow 2S_n = n[2a + (n - 1)d]$

$\Rightarrow S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_n = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[a + \ell]$

$S_n = \frac{n}{2}[a + \ell]$ where, ℓ is the last term.

★ **r^{th} term of an A.P. when sum of first r terms is S_r is given by, $t_r = S_r - S_{r-1}$.**

Solved Examples

Example.9

Find the sum of 20 terms of the A.P. 1,4,7,10.....

Solution.

a = 1, d = 3

$S_n = [2a + (n - 1)d]$

$S_{20} = \frac{20}{2}[2(1) + (20 - 1)3] = 590.$

Example.10

Find the sum of all three digit natural numbers, which are divisible by 7.

Solution.

1st no. is 105 and last no. is 994.

$994 = 105 + (n - 1)7$

$\therefore n = 128$

$\because$ Sum, $S_{128} = \frac{128}{2}[105 + 994] = 70336.$

I. PROPERTIES OF A.P.

(i) For any real number **a** and **b**, the sequence whose **n**th term is a_n **= an + b** is always an **A.P.** with common difference **'a'** (i.e. coefficient of term containing **n**).

(ii) If a constant term is added to or subtracted from each term of an **A.P.** then the resulting sequence is also an **A.P.** with the same common difference.

(iii) If each term of a given **A.P.** is multiplied or divided by a non-zero constant **K,** then the resulting sequence is also an **A.P.** with common difference **Kd** or $\frac{d}{K}$ respectively. Where **d** is the common difference of the given **A.P.**

(iv) In a finite **A.P.** the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of **1st** and **last term**.

(v) If three numbers **a, b, c** are in **A.P.** , then **2b = a + c**.

Solved Examples

Example.11

Check whether $a_n = 2n^2 + 1$ is an **A.P.** or not.

Solution.

$a_n = 2n^2 + 1$

Then, $a_{n+1} = 2(n + 1)^2 + 1$

$\therefore \quad a_{n+1} - a_n = 2(n^2 + 2n + 1) + 1 - 2n^2 - 1$

$= 2n^2 + 4n + 2 + 1 - 2n^2 - 1$

$= 4n + 2$, which is not constant

$\therefore$ The above sequence is not an A.P.

Arithmetic Mean (Mean or Average) (A.M.)

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

A.M. for any n number $a_1, a_2, ..., a_n$ is;

$$A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}.$$

n - Arithmetic Means Between Two Numbers :

If a, b are any two given numbers & a, $A_1, A_2, ..., A_n$, b are in A.P. then $A_1, A_2, ... A_n$ are the n-A.M.'s between a & b.

Total terms are n + 2.

Last term b = a + (n+2–1)d.Now, $d = \frac{b-a}{n+1}$.

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1} ....., A_n = a + \frac{n(b-a)}{n+1}.$$

NOTE :

Sum of all n-A.M.'s inserted between a & b is equal to n times the single A.M. between a & b.

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

Solved Examples

Example.12

Between two numbers whose sum is $\frac{13}{6}$, an even number of A.M.s are inserted, the sum of these means exceeds their number by unity. Find the number of means.

Solution.

Let a and b be two numbers and 2n A.M.s are inserted between a and b then

$\frac{2n}{2}$ (a + b) = 2n + 1 $\Rightarrow$ n $\left(\frac{13}{6}\right)$ = 2n + 1 $\left[\text{Given } a+b=\frac{13}{6}\right]$

$\Rightarrow$ n = 6.

$\therefore$ Number of means = 12.

Example.13

Insert 20 A.M. between 2 and 86.

Solution.

Here 2 is the first term and 86 is the 22nd term of A.P. So, 86 = 2 + (21)d

$\Rightarrow$ d = 4

So, the series is 2, 6, 10, 14,....., 82, 86

$\therefore$ required means are 6, 10, 14,...82.

J. GEOMETRIC PROGRESSION (G.P.)

G.P. is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the preceding term multiplied by a constant. Thus in a G.P. the ratio of successive terms is constant. This constant factor is called the **common ratio** of the series & is obtained by dividing any term by that which immediately precedes it.

Example : 2, 4, 8, 16 ... & $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$...are in G.P.

(i) Therefore a, ar, ar^2, ar^3, ar^4,..... is a G.P. with 'a' as the first term and 'r' as common ratio.
n^{th} term = $a\,r^{n-1}$

(ii) **Sum of the first n terms.**

$$S_n = \begin{cases} \dfrac{a\left(r^n-1\right)}{r-1}, & r>1 \\[2ex] \dfrac{a\left(1-r^n\right)}{1-r}, & r<1 \end{cases}$$

(iii) **Sum of an infinite G.P.** when $|r| < 1$. When $n \to \infty$, $r^n \to 0$ if $|r| < 1$ therefore,

$S_\infty = \frac{a}{1-r}(|r|<1)$.

Solved Examples

Example.14

If the first term of G.P. is 7, its nth term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Solution.

Given : a = 7

$t_n = ar^{n-1} = 7(r)^{n-1} = 448$ $\Rightarrow$ $7r^n = 448\,r$

Also $S_n = \frac{a(r^n-1)}{r-1} = \frac{7(r^n-1)}{r-1}$ $\Rightarrow$ $889 = \frac{448r-7}{r-1} \Rightarrow r = 2$

Hence $T_5 = ar^4 = 7(2)^4 = 112$.

Example.15

Let $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ find the sum of

(i) first 20 terms of the series

(ii) infinite terms of the series.

Solution.

(i) $S_{20} = \dfrac{\left(1-\left(\frac{1}{2}\right)^{20}\right)}{1-\frac{1}{2}} = \dfrac{2^{20}-1}{2^{19}}$. **(ii)** $S_{\infty} = \dfrac{1}{1-\frac{1}{2}} = 2$.

K. PROPERTIES OF G.P.

(i) If a, b, c are in G.P. , then $b^2 = ac$, in general if $a_1, a_2, a_3, a_4, ... a_{n-1}, a_n$ are in G.P., then $a_1a_n = a_2a_{n-1} = a_3 a_{n-2} = ...$

(ii) Any three consecutive terms of a G.P. can be taken as $\frac{a}{r}$, a , ar.

(iii) Any four consecutive terms of a G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3.

(iv) If each term of a G.P. be multiplied or divided or raised to the power by some non–zero quantity, the resulting sequence is also a G.P.

L. GEOMETRIC MEANS (MEAN PROPORTIONAL(G.M.))

If a, b, c are in G.P., b is the G.M. between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0.

n-Geometric Means Between a, b :

If a, b are two given numbers & a, $G_1, G_2, ..., G_n$, b are in G.P.. Then,

$G_1, G_2, G_3, ..., G_n$ are n-G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,....., $G_n = a(b/a)^{n/n+1}$

❖ **NOTE :**

The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\prod_{r=1}^{n} G_r = \left(\sqrt{ab}\right)^n = G^n$ where G is the single G.M. between a & b.

Solved Examples

Example.16

Insert 4 G.M.s between 2 and 486.

Solution.

Common ratio of the series is given by

$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = (243)^{1/5} = 3$

Hence four G.M.s are 6, 18, 54, 162.

M. HARMONIC PROGRESSION (H.P.)

A sequence is said to be H.P. if the reciprocals of its terms are in A.P. If the sequence $a_1, a_2, a_3, ..., a_n$ is an H.P. then $1/a_1, 1/a_2, ..., 1/a_n$ is an A.P. Here we do not have the formula for the sum of the n terms of a H.P. For H.P. whose first term is a and second term is b, the nth term is t_n

$$t_n = \frac{ab}{b+(n-1)(a-b)}.$$

If a, b, c are in H.P. $\Rightarrow$ $b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

NOTE :

(i) If a, b, c are in A.P. $\Rightarrow$ $\frac{a-b}{b-c} = \frac{a}{a}$

(ii) If a, b, c are in G.P. $\Rightarrow$ $\frac{a-b}{b-c} = \frac{a}{b}$.

N. HARMONIC MEAN (H.M)

If a, b, c are in H.P., b is the H.M. between a & c, then $b = \frac{2ac}{a+c}$.

If $a_1, a_2, ... a_n$ are 'n' non-zero numbers then H.M. denoted by H of these numbers is given by :

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right].$$

Solved Examples

Example.17

If m^{th} term of H.P. is n, while n^{th} term is m, find its $(m + n)^{th}$ term.

Solution.

Given $T_m = n$ or $\frac{1}{a+(m-1)\ d} = n$, where a is the first term and d is the common difference of the corresponding A.P.

So, $a + (m - 1) d = \frac{1}{n}$ and $a + (n - 1) d = \frac{1}{m}$

$\Rightarrow$ $(m - n) d = \frac{m-n}{mn}$ $\Rightarrow$ $d = \frac{1}{mn}$

So, $a = \frac{1}{n} - \frac{(m-1)}{mn} = \frac{1}{mn}$

Hence, $T_{(m+n)} = \frac{1}{a+(m+n-1)\ d} = \frac{mn}{1+m+n-1} = \frac{mn}{m+n}$.

Example.18

Insert 4 H.M between $\frac{2}{3}$ and $\frac{2}{13}$.

Solution.

Let d be the common difference of corresponding A.P.

So, $d = \frac{\frac{13}{2} - \frac{3}{2}}{5} = 1.$

$\therefore$ $\frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2}$ $\Rightarrow$ $H_1 = \frac{2}{5}$

$$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2} \quad \Rightarrow \quad H_2 = \frac{2}{7}$$

$$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2} \quad \Rightarrow \quad H_3 = \frac{2}{9}$$

$$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2} \quad \Rightarrow \quad H_4 = \frac{2}{11}.$$

Relation between means :

If A, G, H are respectively A.M., G.M., H.M. between a & b both being unequal & positive then, **$G^2 = AH$** i.e. A, G, H are in G.P.

Example.19

The A.M. of two numbers exceeds the G.M. by $\frac{3}{2}$ and the G.M. exceeds the H.M. by $\frac{6}{5}$; find the numbers.

Solution.

Let the numbers be a and b, now using the relation G^2 = A.H.

$$\Rightarrow \quad G^2 = \left(G + \frac{3}{2}\right)\left(G - \frac{6}{5}\right) \quad \Rightarrow \quad G^2 = G^2 + \frac{3}{10}G - \frac{9}{5} \quad \Rightarrow \quad G = 6 \quad \& \ A = \frac{15}{2}$$

i.e. ab = 36, also a + b = 15

Hence the two numbers are 3 and 12.

- **A.M. ≥ G.M. ≥ H.M.**

Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. = $\frac{a_1 + a_2 + a_3 + ... + a_n}{n}$,

G.M. = $(a_1 a_2 a_3 .....a_n)^{1/n}$ and

H.M. = $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$.

It can be shown that A.M. ≥ G.M. ≥ H.M. and equality holds at either places if $a_1 = a_2 = a_3 = ...= a_n$.

Example.20

If a, b, c, > 0 prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

Solution.

Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}\right)^{\frac{1}{3}} \quad \Rightarrow \quad \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Arithmetic progression (n^{th} term, selection of terms, sum of n terms, arithmetic mean)

A-1 Find the middle term of the progression 4,9,14,.....104.

A-2 Find the sum of all the even positive integers less than 200 which are not divisible by 6.

A-3 Find the value of x satisfying the equation $\frac{x-1}{x}+\frac{x-2}{x}+\frac{x-3}{x}+ .....+ \frac{1}{x} = 3$.

A-4. Find the number of terms which are identical In the following two A.P.'s.
2, 5, 8, 11 .... to 60 terms and 3, 5, 7, ... 50 terms

A-5 If the sum of four numbers in A.P. be 48 and that the product of the extremes is to the product of the means is 27 to 35 then find the numbers.

A-6. There are m A.M. between 1 and 31. If the ratio of the 7th and $(m-1)^{th}$ means is 5 : 9, then find the value of $\frac{m}{7}$.

A-7. If $a_1, a_2, a_3, ..., a_n$ are in A.P. and $a_i > 0$ for all i, then prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}} + ... + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

A-8. Prove that the number of terms in the A.P. a + b + ...+ c is $\frac{b+c-2a}{b-a}$

A-9. Jasleen saved Rs 5 in the first week of the year and then increased her weekly savings by Rs1.75 each week. In what week will her weekly savings be Rs 20.75?

A-10. If the mth term of an A.P. be 1/n and n^{th} term be 1/m, then show that its $(mn)^{th}$ term is 1.

Section (B) : Geometric progression (n^{th} term, selection of terms, sum of n terms, geometric mean)

B-1 The fifth term of a GP is 81 and its 8th term is 2187, then find its third term .

B-2 Which term of the progression 18, –12, 8, ... is $\frac{512}{729}$?

B-3. 4 GM are insested between 3 and 729 then find the 3rd GM.

B-4. The A.M. of two numbers is 34 and GM is 16, find the numbers.

B-5. Find the product of 6 geometric means between 8 and $\frac{1}{16}$.

B-6. If $3 + 3a + 3a^2 + ...\infty = \frac{45}{8}$ (a > 0), then find the value of a .

B-7. If A.M. between positive numbers p and q ($p \geq q$) is two times the GM, then find the value of p : q.

B-8. If the sum of an infinite GP is 4/3 and the sum of the series obtained on squaring each term is 16/27, then find its common ratio.

B-9. In a G.P. the first term is a, second term is b and the last term is c then prove that sum of the series will be $\frac{a^2 - bc}{a - b}$.

B-10. A golf ball is dropped from a height of 80 m. Each time the ball hits the ground, it rebounds to 1/3 of the height through which it has fallen. Then find the total distance travelled by the ball.

Section (C) : Harmonic progression (n^{th} term, selection of terms, sum of n terms, harmonic mean)

C-1. If the mth term of a H.P. be n and n^{th} term be m, then find the rth term .

C-2. If first and second terms of a HP are a and b, then find its nth.

C-3. If a, b, c are in H.P. then find the value of $\frac{b+a}{b-a} + \frac{b+c}{b-c}$.

C-4. The sum of three consecutive number in HP is 37 and the sum of the reciprocals is 1/4 then find the number.

C-5. If a, b, c are in G.P. where a, b, c > 0 then prove that $\frac{1}{1+\log_{10} a}, \frac{1}{1+\log_{10} b}, \frac{1}{1+\log_{10} c}$ are in H.P.

Section (D) : Miscellaneous

D-1 Prove that the sum of the series $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + ... + \frac{1}{1+2+3+.....+n}$ is $\frac{2n}{n+1}$

D-2 Prove that the sum of the series $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$ to n terms is $\frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$.

D-3 Find the sum to infinity of the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ...$

D-4. Find the number of terms in the sequence 1,3,6,10,15,21,.....5050 ?

D-5. If A,G, H be respectively the AM, GM, and H.M between two positive numbers if x A = yG = zH where x, y, z are non-zero positive real number then prove that x,y,z are in G.P.

OBJECTIVE QUESTIONS

(A) Arithmetic progression (n^{th} term, selection of terms, sum of n terms, arithmetic mean)

A-1. If t_{54} of an A.P. is – 61 and t_4 = 64, find t_{10}.
(A) 59 (B) 49 (C) 36 (D) 64

A-2. Find the number of terms in the sequence 4, 12, 20, ...108.
(A) 12 (B) 19 (C) 13 (D) 14

A-3. Find the sum of all natural numbers divisible by 5, but less than 100.
(A) 950 (B) 925 (C) 880 (D) 1050

A-4. Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.
(A) 70441 (B) 70821 (C) 50821 (D) 40821

A-5. If sum of n terms of a sequence is given by $S_n = 2n^2 + 3n$, find its 50th term.
(A) 250 (B) 225 (C) 201 (D) 205

A-6. If 9th term of an A.P. be zero then the ratio of its 29th and 19th term is :
(A) 2 : 1 (B) 1 : 2 (C) 1 : 3 (D) 3 : 1

A-7. If four numbers in A. P. are such that their sum is 50 and the greatest number is 4 times the least, then the numbers are :
(A) 5, 10 ,15, 20 (B) 4, 10, 16, 22 (C) 3, 7, 11, 15 (D) None of these

A-8. If S_1 is the sum of an arithmetic progression of 'n' odd number of terms and S_2 the sum of the terms of the series in odd places, then $\frac{S_1}{S_2} =$
(A) $\frac{2n}{n+1}$ (B) $\frac{n}{n+1}$ (C) $\frac{n+1}{2n}$ (D) $\frac{n+1}{n}$

A-9. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers.
(A) 3, 4, 5 (B) 2, 4, 6 (C) 2, 5, 8 (D) 3, 6, 9

A-10. The angles of a quadrilateral are in A.P. whose common difference is 10º. Find the angles.
(A) 70º, 80º, 90º, 100º (B) 60º, 80º, 100º, 120º (C) 72º, 85º, 98º, 111º (D) 75º, 85º, 95º, 105º

Section (B) : Geometric progression (n^{th} term, selection of terms, sum of n terms, geometric mean)

B-1. The value of $\sum_{h=1}^{\infty}(-1)^{n+1}\left(\frac{n}{5^n}\right)$ equals
(A) $\frac{5}{12}$ (B) $\frac{5}{24}$ (C) $\frac{5}{36}$ (D) $\frac{5}{16}$

B-2. 3 geometric means between 1/9 and 9 is :
(A) $\frac{2}{3}$, 1, 3 (B) $\frac{1}{3}$, 2, 3 (C) $\frac{1}{3}$, 1, 6 (D) $\frac{1}{3}$, 1, 3

B-3. The first term of a G.P. is 7, the last term is 567 and sum of terms is 847. Find the common ratio of the G.P.
(A) 3 (B) 2 (C) 4 (D) 6

B-4. Find the sum the infinite G.P.: $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$
(A) $\frac{3}{5}$ (B) $\frac{3}{2}$ (C) $\frac{49}{27}$ (D) $\frac{8}{5}$

B-5. The eighth term of G.P. is 128 and common ratio is 2. The product of its first five terms is
(A) 4^6 (B) 4^3 (C) 4^5 (D) 4^4

Section (C) : Harmonic progression (n^{th} term, selection of terms, sum of n terms, harmonic mean)

C-1. If H is the harmonic mean between a and b then $\frac{H+a}{H-a} + \frac{H+b}{H-b}$ is equal to
(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 2 (D) None of these

C-2. If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in H.P. then $\frac{a_1}{a_2+a_3+\ldots+a_n}, \frac{a_2}{a_1+a_3+\ldots+a_n}, \ldots\ldots,$ $\frac{a_n}{a_1+a_2+\ldots+a_{n-1}}$ are in
(A) A.P. (B) G.P. (C) H.P. (D) None of these

C-3. If a, b, c, d are in H.P. then ab + bc + cd is
(A) 3 a d (B) (a + b)(c + d) (C) 3 a c (D) 3 b d

C-4. Greatest positive term of a H.P. whose first two terms are 2/5 and 12/23 is–
(A) 6 (B) 5 (C) 1/6 (D) 37/7

C-5. If a, b, c are three distinct numbers in G.P. and a + x, b + x, c + x are in H.P. then x is equal to
(A) a (B) b (C) c (D) None of these

Section (D) : Miscellaneous

D-1. Which one of the following is true about A.M., G.M. and H.M. :
(A) H.M. $\geq$ G.M. $\geq$ A.M. (B) A.M. $\geq$ H.M. $\geq$ G.M.
(C) A.M. $\geq$ G.M. $\geq$ H.M. (D) G.M. $\geq$ H.M. $\geq$ A.M.

D-2. If the numbers a, b, c are in A.P. , b, c, d are in G.P. and c, d, e are in H.P., then a, c, e are in
(A) A.P. (B) G.P. (C) H.P. (D) None of these

D-3. The sum of first p-terms of a sequence is p(p + 1) (p + 2). The 10th term of the sequence is
(A) 396 (B) 600 (C) 330 (D) 114

D-4. If a > 0, b > 0, c > 0 and a, b, c are distinct then (a + b) (b + c) (c + a) is greater than
(A) 2(a + b + c) (B) 3(a + b + c) (C) 6 a b c (D) 8 a b c

D-5. If 2a + 3b + c = 3 ; a > 0, b > 0, c > 0, then the greatest value of $a^2 b^5 c^2$
(A) $\frac{5^5 \quad 2^2}{3^{23}}$ (B) $\frac{5^5 \ . \ 2^2}{3^{14}}$ (C) $\frac{4 \ . \ 5^5}{9^9}$ (D) $\frac{5^6 \ . \ 2^2}{3^4 . \ 9^{10}}$

Exercise-2

OBJECTIVE QUESTIONS

1. The sum of three numbers in A.P. is 27 and the sum of their squares is 293, find them.
(A) 4, 9, 14 (B) 6, 9, 12 (C) 7, 9, 11 (D) 8, 9, 10

2. The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11th terms.
(A) $\frac{4}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) $\frac{4}{7}$

3. The sum of 3rd and 15th elements of an arithmetic progression is equal to the sum of 6th, 11th and 13th elements of the same progression. Then which element of the series should necessarily be equal to zero ?
(A) 1st (B) 9th (C) 12th (D) None of the above

4. If p^{th}, q^{th} and r^{th} term of an A.P. are a, b, c respectively, then a (q – r) + b (r – p) + c (p – q) = ?
(A) 2 (B) 1 (C) 0 (D) None of these

5. If m times the mth term of an A.P. is equal to n times its nth term. Find the value of $(m + n)^{th}$ term of the A.P.
(A) 2 (B) 1 (C) 0 (D) None of these

6. Divide 600 biscuits among 5 boys so that their shares are in Arithmetic progression and the two smallest shares together make one-seventh of what the other three boys get. What is the sum of the shares of the two boys who are getting lesser number of biscuits, than the remaining three ?
(A) 75 (B) 85 (C) 185 (D) 90

7. If S_r denotes the sum of the first r terms of an A.P. Then, $S_{3n} : (S_{2n} - S_n)$ is
(A) n (B) 3n (C) 3 (D) None of these

8. If $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the A.M. between a and b. Then, find the value of n.
(A) 1 (B) 2 (C) 0 (D) 3

9. If $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ and ab = c, what is the average (arithmetic mean) of a and b ?
(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{a+b}{2c}$

10. The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.
(A) 1, 2, 4, 8, ... (B) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$ (C) $1, \frac{1}{4}, \frac{1}{8}, ...$ (D) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$

11. Find the 10th term of the H.P. :$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, ...$
(A) 19 (B) $\frac{1}{19}$ (C) 21 (D) $\frac{1}{21}$

12. If A.M. and G.M. of two numbers is 10 and 8 respectively then find their H.M. :
(A) 9 (B) 12 (C) 6 (D) 6.4

13. If a, b, c are in A.P. a, mb, c are in G.P., then a, m^2b, c are in
(A) A.P. (B) G.P. (C) H.P. (D) None of these

14. Find the sum of the sequence : $\frac{1}{9}+\frac{1}{18}+\frac{1}{30}+\frac{1}{45}+\frac{1}{63}+.....\infty$:
(A) $\frac{1}{3}$ (B) 1 (C) $\frac{2}{3}$ (D) 2

15. The value of the sum $\frac{1}{3^2+1}+\frac{1}{4^2+2}+\frac{1}{5^2+3}+\frac{1}{6^2+4}$ .... ∞ is equal to
(A) 13/36 (B) 12/36 (C) 15/36 (D) 18/36

16. Let S_1, S_2, S_3 be the sums of n terms of three series in A.P., the first term of each being 1 and the common differences 1, 2, 3 respectively. If $S_1 + S_3 = \lambda S_2$. then the value of λ is
(A) 1 (B) 2 (C) 3 (D) none of these

17. The nth term of series $\frac{1}{1} + \frac{1+2}{2} + \frac{1+2+3}{3}$ +.....will be

(A) $\frac{n+1}{2}$ (B) $\frac{n-1}{2}$ (C) $\frac{n^2+1}{2}$ (D) $\frac{n^2-1}{2}$

18. The sum of infinity of the series $1 + 2\left(1-\frac{1}{n}\right) + 3\left(1-\frac{1}{n}\right)^2 + \ldots\ldots\ldots$ is

(A) n^2 (B) $n(n + 1)$ (C) $n\left(1-\frac{1}{n}\right)$ (D) None of these

19. Find the sum to 'n' terms and the sum to infinite terms of the series

$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+$.....upto n terms

(A) $\frac{n}{n+1}$ (B) $\frac{6n}{n+1}$ (C) $\frac{6n}{n-1}$ (D) None of these

20. If n^{th} term of the series $3\frac{1}{3}, 2, 1\frac{3}{7}, 1\frac{1}{9}$,..... is $\frac{an+10}{bn+c}$, $\forall$ n $\in$ N, then find the value of (a + b + c)

(A) 1 (B) 2 (C) 3 (D) 4

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. Consider the sequence 1, – 2, 3, – 4, 5, – 6, ..., $n(-1)^{n+1}$. What is the average of the first 300 terms of the sequence ? **[NSTSE 2009]**

(A) – 1 (B) 0.5 (C) 0 (D) – 0.5

2. If the first four terms of an arithmetic sequence are : a, 2a, b and a – 6 – b for some numbers "a" and "b", then the value of the 100th term is : **[NSTSE 2009]**

(A) – 100 (B) – 300 (C) 150 (D) – 150

3. If $l^2 + m^2 + n^2 = 5$, then (lm + mn + ln) is : **[IJSO-2009]**

(A) $\geq (-5/2)$ (B) $\geq (-1)$ (C) ≤ 5 (D) ≤ 3

4. Let S_n denote the sum of the first 'n' terms of an A.P. and $S_{2n} = 3S_n$. Then, the ratio $S_{3n} : S_n$ is equal to : **[NSTSE 2010]**

(A) 4 : 1 (B) 6 : 1 (C) 8 : 1 (D) 10 : 1

5. If $a^2 + b^2 + c^2 = 1$ and p = ab + bc + ca, then **[IJSO-2010]**

(A) $\frac{1}{2} \leq p \leq 2$ (B) $-\frac{1}{2} \leq p \leq \frac{1}{2}$ (C) $-\frac{1}{2} \leq p \leq 1$ (D) $-1 \leq p \leq \frac{1}{2}$

6. If $a^2 + b^2 + c^2 + d^2 = 25$, out of the following statements the correct one is **[IJSO-2011]**

(A) ab + bc + cd + da $\leq$ 25 (B) ab + bc +cd + da $\geq$ 25.

(C) ab + bc + cd + da $\leq \frac{25}{2}$ (D) ab + bc + cd + da $\geq \frac{25}{2}$

7. If $\frac{b+c-a}{a}$, $\frac{c+a-b}{b}$ and $\frac{a+b-c}{c}$ are in A.P. and a + b + c $\neq$ 0, then **[IJSO-2011]**

(A) b = $\frac{2ac}{a-c}$ (B) b = $\frac{2ac}{a+c}$ (C) b = $\frac{a+c}{2}$ (D) b = $\sqrt{ac}$

8. Find the value $\frac{2}{15}+\frac{2}{35}+\frac{2}{63}+\frac{2}{99}+.....+\frac{2}{9999}$ **[IJSO-2012]**

(A) 8/33 (B) 2/11 (C) 98/303 (D) 222/909

9. x + y + z = 1 ; $x \neq z$; x, y, z are real numbers and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ = m. How many of the following values of m are always possible. **[IJSO-2012]**

(i) m = 6, **(ii)** m = 8, **(iii)** m = 10 **(iv)** m = 12

(A) 1 (B) 2 (C) 3 (D) all

10. The sum $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+.....-\frac{1}{2012}+\frac{1}{2013}$ equals **[IJSO-2013]**

(A) $\frac{1}{1006}+\frac{1}{1007}+\frac{1}{1008}+.....+\frac{1}{2013}$ (B) $\frac{1}{1007}+\frac{1}{1008}+\frac{1}{1009}+.....+\frac{1}{2013}$

(C) $\frac{1}{1006}+\frac{1}{1007}+\frac{1}{1008}+.....+\frac{1}{2012}$ (D) $\frac{1}{1007}+\frac{1}{1008}+\frac{1}{1009}+.....+\frac{1}{2012}$

11. A number of bacteria are placed in a glass. 1 second later each bacterium divides in three, the next second each of the resulting bacteria divides in three again, and so on. After one minute the glass, is full. When was 1/9th of the glass full ? **[IJSO-2014]**

(A) 15 sec (B) 45 sec (C) 58 sec (D) 38 sec

12. A number is said to be triangular number if it is the sum of consecutive numbers beginning with 1. Which one of the following is not a triangular number : **[IJSO-2014]**

(A) 1431 (B) 190 (C) 28 (D) 506

13. The houses of a row are numbered consecutively from 1 to 49. Find the value of x such that the sum of the numbers of houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. **[IJSO-2014]**

(A) 25 (B) 35 (C) 37 (D) No such value exists

14. Along a road lie an odd number of stones placed at intervals of 10 metre. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. If a man starts from one of the end stones, and by carrying them in succession he covers 3 km to pile all stones at the centre. The number of stones is then : **[IJSO-2014]**

(A) 12 (B) 15 (C) 30 (D) 25

15. $\frac{3}{4}+\frac{3}{28}+\frac{3}{70}+\frac{3}{130}+....+\frac{3}{9700}=?$ **[IJSO-2015]**

(A) 0.97 (B) 0.99 (C) 1 (D) 1.03

16. The sum of first four terms of an A.P is 56. The sum of last four terms of same A.P is 112. The first term of the A.P is 11. Find the number of terms in that A.P. **[IJSO-2015]**

(A) 7 (B) 8 (C) 11 (D) 13

17. There are ten numbers in a certain A.P. The sum of first three terms is 321. The sum of last three numbers is 405. Find the sum of all the ten numbers. **[IJSO-2015]**

(A) 1165 (B) 1210 (C) 1221 (D) 1252

18.. If a, b > 0 then **[IJSO-2016]**

(A) $a + b \leq \sqrt{ab}$ (B) $a + b > \sqrt{ab}$

(C) $a + b \leq \sqrt{2ab}$ (D) None of the above inequalities will hold

19. A triangle has perimeter 316 and its sides are of integer length. The maximum possible area for such a triangle is achieved for **[IJSO-2016]**

(A) Single triangle (B) Two triangles
(C) Three triangles (D) More than three triangle

20. If for x, y > 0 we have $\frac{1}{x}+\frac{1}{y}$ = 2 then the minimum value of xy is **[IJSO-2016]**

(A) 2 (B) 1 (C) 4 (D)

21. $1\frac{1}{2}+1\frac{1}{6}+1\frac{1}{12}+1\frac{1}{20}+1\frac{1}{30}$ + ...+$1\frac{1}{380}$ = _____ **[IJSO 2017]**

(A) 20.25 (B) 20.05 (C) 19.95 (D) 19.85

22. What is the sum of all odd numbers between 500 and 600? **[IJSO 2017]**

(A) 26000 (B) 27000 (C) 27500 (D) 29500

23. Let T_k denote the k-th term of an arithmetic progression. Suppose there are positive integers $m \neq n$ such that $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$. Then T_{mn} equals **[IJSO 2018]**

(A) $\frac{1}{mn}$ (B) $\frac{1}{m}+\frac{1}{n}$ (C) 1 (D) 0

24. For a regular k-sided polygon, let a(k) denotes its interior angle. Suppose n > 4 is such that a(n – 2), a(n),a(n + 3) forms an arithmetic progression. The sum of digits of n is **[IJSO 2018]**

(A) 2 (B) 3 (C) 4 (D) 5

25. The sum of 5 numbers in geometric progression is 24. The sum of their reciprocals is 6. The product of the terms of the geometric progression is **[IJSO 2018]**

(A) 36 (B) 32 (C) 24 (D) 18

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 54 **A-2** 6534 **A-3** x = 7 **A-4.** 17 identical terms

A-5 6, 10, 14, 18 **A-6.** 2 **A-9.** 10th week

Section (B)

B-1 9 **B-2** 9th **B-3.** 81 **B-4.** 64 and 4.

B-5. $\frac{1}{8}$ **B-6.** $\frac{7}{15}$ **B-7.** $\frac{2+\sqrt{3}}{2-\sqrt{3}}$ **B-8.** ½ **B-10.** 160 m.

Section (C)

C-1. $\frac{mn}{r}$ **C-2.** $\frac{ab}{b+(n-1)(a-b)}$. **C-3.** 2 **C-4.** 15, 12, 10

Section (D)

D-3 $1-\frac{1}{n+1}$ **D-4.** 100.

OBJECTIVE QUESTIONS

Section (A)

A-1. (B) **A-2.** (D) **A-3.** (A) **A-4.** (B) **A-5.** (C)

A-6. (A) **A-7.** (A) **A-8.** (A) **A-9.** (B) **A-10.** (D)

Section (B)

B-1. (C) **B-2.** (D) **B-3.** (A) **B-4.** (B) **B-5.** (C)

Section (C)

C-1. (C) **C-2.** (A) **C-3.** (A) **C-4.** (A) **C-5.** **(B)**

Section (D)

D-1. (C) **D-2.** (B) **D-3.** (C) **D-4.** (D) **D-5.** (B)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	A	A	C	C	C	A	C	C	B	D	B	D	C	A	A	B	A	A	B	C

Exercise-3

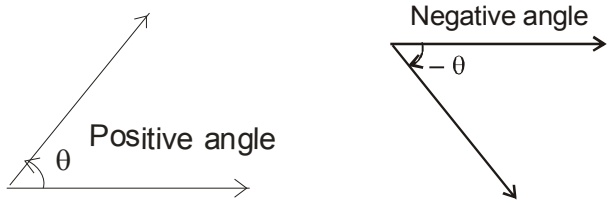
IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	AC	B	C	A	B	C	B	B	C	D	B	D	B	C	B	B	A	B
Ques.	21	22	23	24	25															
Ans.	C	C	C	B	B															

TRIGONOMETRY

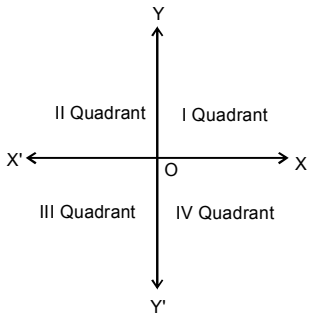
A. ANGLE

An angle is the amount of rotation of a revolving line with respect to a fixed line. If the rotation is in **anticlock-wise sense**, then the angle measured is **positive** and if the rotation is in **clock-wise sense**, then the angle measured is **negative**.



B. QUADRANTS

Let **X'OX** and **YOY'** be two lines at right angles in a plane. These lines divide the plane into four equal parts are known as **quadrants**. The lines **X'OX** and **YOY'** are known as **X-axis** and **Y-axis** respectively. These two lines taken together are known as the **co-ordinate axe**s. The regions **XOY, YOX', X'OY'** and **Y'OX** are known as **first**, **second**, **third** and **fourth quadrants** respectively.



(a) Systems of measurement of angles :

(i) Sexagesimal system **(ii)** Centesimal system **(iii)** Circular system

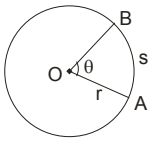
(i) Sexagesimal system : In this system a **right angle** is divided into **90** equal parts called **degrees**. Each **degree** is divided into **60** equal parts called **minutes** and each **minute** is divided into **60** equal parts called **seconds**.

Thus, **1 right angle = 90 degrees** (**90º**)
1º = 60 minutes (**60'**)
1' = 60 seconds (**60"**)

(ii) Centesimal system : In this system a **right angle** is divided into **100** equal parts, called **grades**. Each **grade** is sub divided into **100 minutes**, and each **minute** into **100 seconds**.

Thus, **1 right angle = 100 grades (100^g)**
1 grade = 100 minutes (100')
1 minute = 100 seconds (100")

(iii) Circular system : In this system the unit of measurement is radian. **One radian**, written as **1^c**, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



The number of radians is an angle subtended by an arc of a circle at the centre is equal to $\frac{\text{length of arc}}{\text{radius}}$.

$\therefore \quad \theta = \frac{s}{r}$

Where, θ = angle in radian, s = arc length and r = radius.

(b) Relation Between Three System of Measurement of Angles :

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Where, D = number of degrees,
G = number of grades,
and R = number of radians.

NOTE : (i) The angle between two consecutive digits in a clock is **30º = (π/6 radians)**.

(ii) The **hour hand** rotates through an angle of **30º** in **one hour**, i.e. **(1/2)º** in **one minute**.

(iii) The **minute hand** rotates through an angle of **6º** in **one minute**.

Solved Examples

Example.1

Express in radians 47º 25' 36".

Solution.

$$47^\circ\, 25' \left(\frac{36}{60}\right)' = 47^\circ \left(25\frac{3}{5}\right)' = 47^\circ \left(\frac{128}{5}\right)' = 47^\circ \left(\frac{128}{5}\times\frac{1}{60}\right)^\circ = \left(47\frac{32}{75}\right)^\circ$$

$$= \left(\frac{3557}{75}\right)^\circ = \frac{3557}{75}\times\frac{\pi^c}{180} = \frac{3557}{13500}\pi^c.$$

Example.2

Express in degrees :

(a) $\left(\frac{2\pi}{15}\right)^c$ **(b)** $(-2)^c$.

Solution.

(a) $\left(\frac{2\pi}{15}\right)^c = \left(\frac{2\pi}{15}\times\frac{180}{\pi}\right)^\circ = 24^\circ$

(b) $(-2)^c = \left(-2\times\frac{180}{\pi}\right)^\circ = \left(\frac{180}{22}\times 7\times(-2)\right)^\circ = \left(-114\frac{6}{11}\right)^\circ$

$$= -114^\circ \left(\frac{6}{11}\times 60\right)' = -114^\circ\left(32\frac{8}{11}\right)'$$

$$= -114^\circ\, 32' \left(\frac{8}{11}\times 60\right)'' = -114^\circ\, 32'\, 44''.$$

Example.3

Express in radians $345^g\, 25'\, 36''$.

Solution.

$$345^g\, 25'\, 36'' = 345.2536^g = \frac{3452536}{2000000}\pi^c$$

$$= 1.726268\, \pi^c$$

Example.4

One angle of a triangle is $\frac{2x}{3}$ grades another is $\frac{3x}{2}$ degrees, whilst the third is $\frac{\pi x}{75}$ radians ; express all angles in degrees.

Solution.

$\frac{2}{3}x^g = \frac{2}{3} \times \left(\frac{9^o}{10}\right) = \frac{3}{5}x^o$

And $\frac{\pi x^c}{75} = \frac{x}{75} 180^o = \frac{12x^o}{5}$

But $\frac{3}{5}x^o + \frac{3}{2}x^o + \frac{12}{5}x^o = 180^o$

$\because$ $6x^o + 15x^o + 24x^o = 1800$

$45x^o = 1800$

$x = 40^o$

Hence, three angles of the triangle are 24º, 60º and 96º.

Example.5

The angles of a triangle are in A.P. and the number of degrees in the least is to the number of radians in the greatest is 60 to π^c. Find the angles in degrees.

Solution.

The three angles in A.P. ; if y is common difference, let these angles be $(x - y)^o$, x^o and $(x + y)^o$.

$\therefore$ $x + y + x + x - y = 180^o$

$\therefore$ $x = 60^o$

According to the question.

$\frac{(x-y)}{(x+y)\frac{\pi^c}{180}} = \frac{60}{\pi}$ or $\pi(x - y) = (x + y)\frac{\pi}{180^o} \times 60^o$

$\Rightarrow$ $3(x - y) = x + y$ $\Rightarrow$ $4y = 2x$

$\Rightarrow$ $y = x/2$

$\therefore$ $y = \frac{60^o}{2} = 30^o$.

Hence three angles are 30º, 60º and 90º.

Example.6

The angles in one regular polygon is to that in another as 3 : 2, also the number of sides in the first is twice that in the second ; how many sides do the polygons have ?

Solution.

Suppose the second regular polygon has number of side = x.

$\therefore$ The first regular polygon will have number of side = 2x.

$\therefore$ Each angle of the first polygon = $\frac{(4x-4)}{2x}$ right angle

And each angle of the second polygon = $\frac{(2x-4)}{x}$ right angle

$\therefore$ $\frac{4x-4}{2x} : \frac{2x-4}{x} = 3 : 2$

$\Rightarrow$ $\frac{4x-4}{x} = \frac{6x-12}{x}$

$\Rightarrow$ $4x - 4 = 6x - 12$

$\Rightarrow$ $2x = 8 \Rightarrow x = 4$.

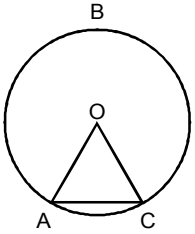
$\therefore$ The number of sides in the first and second polygons respectively are 8 and 4.

Example.7

The radius of a certain circle is 30 cm, find the approximately length of an arc of this circle ; if the length of the chord of the arc be 30 cm.

Solution.

Let ABC be the circle whose centre is O and AC is chord.



In ΔAOC, AO = OC = AC = 30 cm.

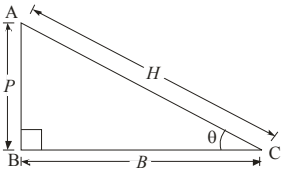
$\because \quad \angle AOC = 60^o = \frac{\pi}{3}$

Hence, arc AC = radius × $\frac{\pi}{3}$ = 30 × $\frac{\pi}{3}$ = 10π = 31.4159 cm.

C. TRIGONOMETRY

Trigonometry means, the science which deals with the measurement of triangles.

Trigonometric Ratios :



A right angled triangle is shown in **Figure**. $\angle$ **B** is of **90º**. Side opposite to $\angle$ **B** is called **hypotenuse**. There are two other angles i.e. $\angle$**A** and $\angle$**C**. If we consider $\angle$**C** as **θ**, then opposite side to this angle is called **perpendicular** and side adjacent to θ is called **base**.

(i) Six Trigonometric Ratios are :

$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{P}{H} = \frac{AB}{AC}$ $\qquad$ $\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{H}{P} = \frac{AC}{AB}$

$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{B}{H} = \frac{BC}{AC}$ $\qquad$ $\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{H}{B} = \frac{AC}{BC}$

$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{P}{B} = \frac{AB}{BC}$ $\qquad$ $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{B}{P} = \frac{BC}{AB}$

(ii) Interrelationship in Basic Trigonometric Ratios :

$\tan\theta = \frac{1}{\cot\theta} \quad \Rightarrow \quad \cot\theta = \frac{1}{\tan\theta}$

$\cos\theta = \frac{1}{\sec\theta} \quad \Rightarrow \quad \sec\theta = \frac{1}{\cos\theta}$

$\sin\theta = \frac{1}{\text{cosec}\,\theta} \quad \Rightarrow \quad \text{cosec}\,\theta = \frac{1}{\sin\theta}$

We also observe that

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Solved Examples

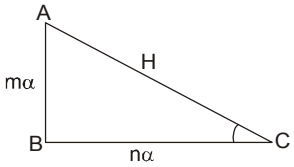
Example.8

If $\tan\theta = \frac{m}{n}$, then find $\sin\theta$.

Solution.

Let $P = m\alpha$ and $B = n\alpha$

$\therefore \quad \tan\theta = \frac{P}{B} = \frac{m}{n}$



$H^2 = P^2 + B^2$

$H^2 = m^2\alpha^2 + n^2\alpha^2$

$H = \alpha\sqrt{m^2 + n^2}$

$\therefore \quad \sin\theta = \frac{P}{H} = \frac{m\alpha}{\alpha\sqrt{m^2+n^2}}$

$\sin\theta = \frac{m}{\sqrt{m^2+n^2}}$.

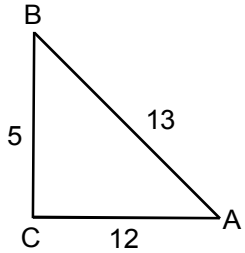
Example.9

If $\text{cosec } A = \frac{13}{5}$, then prove that : $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.

Solution.

We have $\text{cosec } A = \frac{13}{5} = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$.

So, we draw a right triangle ABC, right angled at C such that Hypotenuse AB = 13 units and perpendicular BC = 5 units



By pythagoras theorem,

$AB^2 = BC^2 + AC^2 \quad \Rightarrow (13)^2 = (5)^2 + AC^2$

$AC^2 = 169 - 25 = 144$

$AC = \sqrt{144} = 12$ units

$\tan A = \frac{BC}{AC} = \frac{5}{12}$ and $\sin A = \frac{BC}{AB} = \frac{5}{13}$

and $\quad \sec A = \frac{AB}{AC} = \frac{13}{12}$

L.H.S $\tan^2 A - \sin^2 A$

$$= \left(\frac{5}{12}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{25}{144} - \frac{25}{169} = \frac{25(169-144)}{144\times169} = \frac{25\times25}{144\times169}$$

R.H.S. $\sin^4 A \times \sec^2 A$

$$= \left(\frac{5}{13}\right)^4 \times \left(\frac{13}{12}\right)^2 = \frac{5^4 \times 13^2}{13^4 \times 12^2}$$

$$= \frac{5^4}{13^2 \times 12^2} = \frac{25 \times 25}{144 \times 169}$$

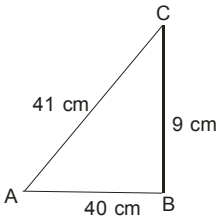
So, L.H.S = R.H.S **Hence Proved.**

Example.10

In Δ ABC, right angled at B, AC + AB = 81 cm and BC = 9 cm. Determine the value of cot C, cosec C, sec C.

Solution.

In Δ ABC, we have



$(AC)^2 = (AB)^2 + BC^2$

$\Rightarrow$ $(81 - AB)^2 = AB^2 + (9)^2$ [$\because$ AC + AB = 9cm $\Rightarrow$ AC = 9 – AB]

$\Rightarrow$ $6561 + AB^2 - 162\ AB = AB^2 + 81$

$\Rightarrow$ $6480 - 162\ AB = 0$

$\Rightarrow$ $AB = \frac{6480}{162} = 40$ cm.

Now, AC + AB = 81 cm

AC = 81 – 40 = 41 cm

So, cot C = $\frac{BC}{AB} = \frac{9}{40}$, cosec C = $\frac{AC}{AB} = \frac{41}{40}$, sec C = $\frac{AC}{BC} = \frac{41}{9}$.

D. TRIGONOMETRIC TABLE

$\theta \rightarrow$	0º	30º	45º	60º	90º
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cot	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cosec	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Solved Examples

Example.11

Given that cos (A – B) = cos A cos B + sin A sinB, find the value of cos15º.

Solution.

Putting A = 45º and B = 30º

We get

cos (45º – 30º) = cos 45º cos 30º + sin 45º sin 30º

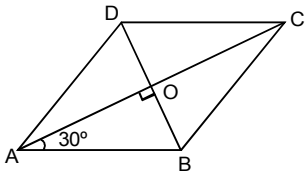
$\Rightarrow$ cos 15º = $\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$ $\Rightarrow$ cos 15º = $\frac{\sqrt{3}+1}{2\sqrt{2}}$.

Example.12

A Rhombus of side of 10 cm has two angles of 60º each. Find the length of diagonals and also find its area.

Solution.

Let ABCD be a rhombus of side 10 cm and $\angle$BAD = $\angle$BCD = 60º. Diagonals of parallelogram bisect each other. So, AO = OC and BO = OD



In right triangle AOB

sin 30º = $\frac{OB}{AB}$ $\Rightarrow$ $\frac{1}{2} = \frac{OB}{10}$ $\Rightarrow$ OB = 5 cm

$\therefore$ BD = 2(OB) $\Rightarrow$ BD = 2 (5) $\Rightarrow$ BD = 10 cm

cos 30º = $\frac{OA}{AB}$ $\Rightarrow$ $\frac{\sqrt{3}}{2} = \frac{OA}{10}$ $\Rightarrow$ OA = $5\sqrt{3}$

$\therefore$ AC = 2(OA) $\Rightarrow$ AC = 2 ($5\sqrt{3}$) = $10\sqrt{3}$ cm

So, the length of diagonals AC = $10\sqrt{3}$ cm & BD = 10 cm.

Area of Rhombus = $\frac{1}{2}$ × AC × BD = $\frac{1}{2}$ × $10\sqrt{3}$ × 10 = $50\sqrt{3}$ cm^2.

E. TRIGONOMETRIC RATIO OF COMPLEMENTARY ANGLES

sin (90 – θ) = cos θ cos (90 – θ) = sin θ

tan (90 – θ) = cot θ cot (90 – θ) = tan θ

sec (90 – θ) = cosec θ cosec (90 – θ) = sec θ

Solved Examples

Example.13

$\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2.$

Solution.

$\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2 = \frac{\cot(90^\circ - 36^\circ)}{\tan 36^\circ} + \frac{\tan(90^\circ - 70^\circ)}{\cot 70^\circ} - 2$

$= \frac{\tan 36^\circ}{\tan 36^\circ} + \frac{\cot 70^\circ}{\cot 70^\circ} - 2$ [$\therefore$cot (90 – θ) = tanθ and tan (90 – θ) = cot θ]

= 1 + 1 – 2 = 0.

Example.14.

$$\frac{2\sin 68^\circ}{\cos 22^\circ} - \frac{2\cot 15^\circ}{5\tan 75^\circ} - \frac{3\tan 45^\circ \tan 20^\circ \tan 40^\circ \tan 50^\circ \tan 70^\circ}{5}.$$

Solution.

$$\frac{2\sin 68^\circ}{\cos 22^\circ} - \frac{2\cot 15^\circ}{5\tan 75^\circ} - \frac{3\tan 45^\circ \tan 20^\circ \tan 40^\circ \tan 50^\circ \tan 70^\circ}{5}$$

$$= \frac{2\sin(90^\circ - 22^\circ)}{\cos 22^\circ} - \frac{2\cot(90^\circ - 75^\circ)}{5\tan 75^\circ} - \frac{3(1)(\tan 20^\circ \tan 70^\circ)(\tan 40^\circ \tan 50^\circ)}{5}$$

$$= \frac{2\cos 22^\circ}{\cos 22^\circ} - \frac{2\tan 75^\circ}{5\tan 75^\circ} - \frac{3[\tan(90^\circ - 70^\circ)\tan 70^\circ][\tan(90^\circ - 50^\circ)\tan 50^\circ]}{5}$$

$$= 2 - \frac{2}{5} - \frac{3}{5} \text{ (cot 70º tan 70º) (cot 50º tan 50º)}$$

[$\therefore$ tan (90º – θ) = cotθ, cot (90º – θ) = tan θ & sin (90º – θ) = cosθ]

$$= 2 - \frac{2}{5} - \frac{3}{5} = 2 - 1 = 1.$$

Example.15

If sin 3A = cos (A – 26º) where 3A is an acute angle, find the value of A.

Solution.

sin 3A = cos (A – 26º)

$\Rightarrow$ cos (90º – 3A) = cos (A – 26º) [$\because$ sinθ = cos(90º – θ]

$\Rightarrow$ 90º – 3A = A – 26º

$\Rightarrow$ 4A = 116º

$\Rightarrow$ A = 29º

Example.16

If tan 2A = cot (A – 18º), where 2A is an acute angle, find the value of A.

Solution.

tan 2A = cot (A – 18º)

$\Rightarrow$ cot (90º – 2A) = cot (A – 18º)

$\Rightarrow$ 90º – 2A = A – 18º

$\Rightarrow$ 90º + 18º = A + 2A

$\Rightarrow$ 3A = 108º

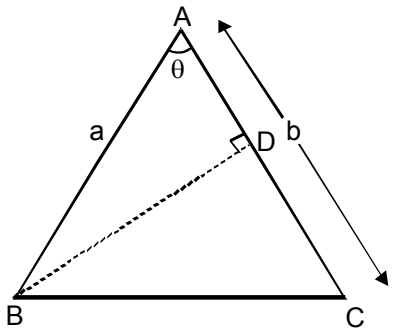
$\Rightarrow$ A = 36º.

F. AREA OF TRIANGLE

In a **ABC**, a & b are the length of 2 sides of triangle and **θ** is the included angle between them.

Then, **Area of triangle =** $\frac{1}{2}$ **ab sinθ.**

Proof :



Const. : Draw a line from B perpendicular to AC, **i.e.** BDAC

$\sin\theta = \frac{BD}{a} \Rightarrow BD = a\sin\theta$

$\therefore$ Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times b \times BD = \frac{1}{2} \times b \times a\sin\theta = \frac{1}{2}ab\sin\theta.$

G. TRIGONOMETRIC IDENTITIES

(i) $\mathbf{\sin^2\theta + \cos^2\theta = 1}$

(A) $\sin^2\theta = 1 - \cos^2\theta$

(B) $\cos^2\theta = 1 - \sin^2\theta$

(ii) $\mathbf{1 + \tan^2\theta = \sec^2\theta}$ [where $\theta \neq 90^o$]

(A) $\sec^2\theta - 1 = \tan^2\theta$

(B) $\sec^2\theta - \tan^2\theta = 1$

(C) $\tan^2\theta - \sec^2\theta = -1$

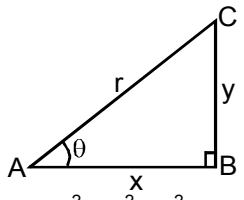
(iii) $\mathbf{1 + \cot^2\theta = \text{cosec}^2\theta}$ [where $\theta \neq 0^o$]

(A) $\text{cosec}^2\theta - 1 = \cot^2\theta$

(B) $\text{cosec}^2\theta - \cot^2\theta = 1$

(C) $\cot^2\theta - \text{cosec}^2\theta = -1$

Proof : Consider a right-angled ΔABC in which $\angle B = 90^o$ and $\angle A = \theta^o$. Let AB = x units, BC = y units and AC = r units.



Then, by Pythagoras theorem, we have $x^2 + y^2 = r^2$

Now,

(i) $\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \left(\frac{y^2}{r^2} + \frac{x^2}{r^2}\right) = \frac{(x^2+y^2)}{r^2} = \frac{r^2}{r^2} = 1$ $[\because x^2 + y^2 = r^2]$

$\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = 1 + \left(\frac{y}{x}\right)^2 = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2} = \left(\frac{r}{x}\right)^2 = \sec^2\theta$

$\therefore$ $1 + \tan^2\theta = \sec^2\theta.$

(iii) $1 + \cot^2\theta = 1 + \left(\frac{x}{y}\right)^2 = \frac{y^2+x^2}{y^2} = \frac{r^2}{y^2} = \left(\frac{r}{y}\right)^2 = \text{cosec}^2\theta$

$\therefore$ $1 + \cot^2\theta = \text{cosec}^2\theta.$

Solved Examples

Example.17

Prove : $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\theta + \cot\theta$

Solution.

LHS $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sin\theta} = \text{cosec}\,\theta + \cot\theta.$

Example.18

Prove that : $\dfrac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)}$ = cot θ

Solution.

LHS $\dfrac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)} = \dfrac{1-\sin^2\theta+\cos\theta}{\sin\theta(1+\cos\theta)}$

$= \dfrac{\cos^2\theta+\cos\theta}{\sin\theta(1+\cos\theta)} = \dfrac{\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \cot\theta$. RHS

Example.19

Prove that : $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}$ = tanθ.

Solution.

LHS $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta} = \dfrac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta-1)} = \dfrac{\tan\theta(1-2\sin^2\theta)}{(1-2\sin^2\theta)}$

= tanθ **R.H.S. Hence proved**

Example.20

If 7 $\sin^2\theta$ + 3 $\cos^2\theta$ = 4, show that tan θ = $\dfrac{1}{\sqrt{3}}$.

Solution.

7 $\sin^2\theta$ + 3 $\cos^2\theta$ = 4

Divide by $\cos^2\theta$

$\Rightarrow \dfrac{7\sin^2\theta}{\cos^2\theta} + \dfrac{3\cos^2\theta}{\cos^2\theta} = \dfrac{4}{\cos^2\theta} \Rightarrow 7\tan^2\theta + 3 = 4\sec^2\theta$

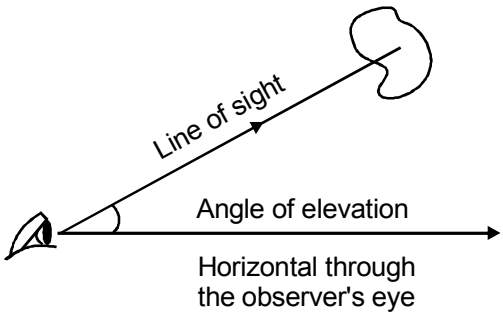
$\Rightarrow 7\tan^2\theta + 3 = 4(1+\tan^2\theta) \Rightarrow 7\tan^2\theta + 3 = 4 + 4\tan^2\theta$

$\Rightarrow 3\tan^2\theta = 1 \Rightarrow \tan^2\theta = \dfrac{1}{3}$

$\Rightarrow \tan\theta = \dfrac{1}{\sqrt{3}}$.

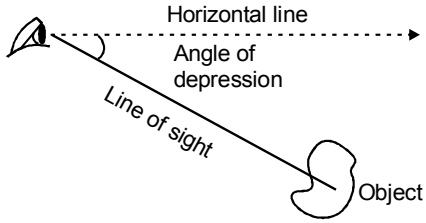
H. ANGLE OF ELEVATION

In order to see an object which is at a higher level compared to the ground level we are to look up. The line joining the object and the eye of the observer is known as the line of sight and the angle which this line of sight makes with the horizontal drawn through the eye of the observer is known as the **angle of elevation**. Therefore, the angle of elevation of an object helps in finding out its height **(Figure)**.



I. ANGLE OF DEPRESSION

When the object is at a lower level than the observer's eyes, he has to look downwards to have a view of the object. In that case, the angle which the line of sight makes with the horizontal through the observer's eye is known as the **angle of depression (Figure).**



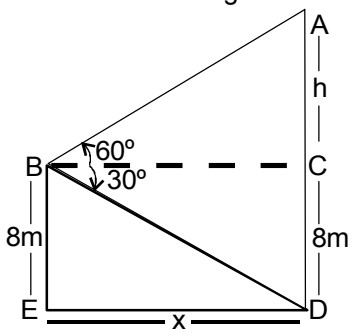
Solved Examples

Example.21

A man is standing on the deck of a ship, which is 8 m above water level. He observes the angle of elevation of the top of a hill as 60º and the angle of depression of the base of the hill as 30º. Calculate the distance of the hill from the ship and the height of the hill.

Solution.

Let x be distance of hill from man and h + 8 be height of hill which is required.



In right triangle ACB,

$\Rightarrow \quad \tan 60^\circ = \frac{AC}{BC} = \frac{h}{x} \quad \Rightarrow \quad \sqrt{3} = \frac{h}{x}$

In right triangle BCD,

$\tan 30^\circ = \frac{CD}{BC} = \frac{8}{x} \quad \Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{8}{x} \quad \Rightarrow \quad x = 8\sqrt{3}$

$\therefore$ Height of hill = h + 8 = $\sqrt{3}.x + 8 = \sqrt{3}\left(8\sqrt{3}\right) + 8 = 32$ m.

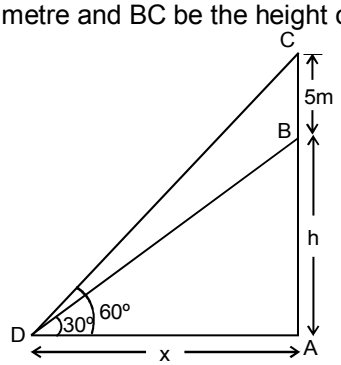
Distance of ship from hill = x = $8\sqrt{3}$ m.

Example.22

A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height 5 meters. At a point on the plane, the angle of elevation of the bottom and the top of the flag staff are respectively 30º and 60º. Find the height of tower.

Solution.

Let AB be the tower of height h metre and BC be the height of flag staff surmounted on the tower.



Let the point on the plane be D at a distance x meter from the foot of the tower

In Δ ABD

$\tan 30^\circ = \frac{AB}{AD}$ $\Rightarrow$ $\frac{1}{\sqrt{3}} = \frac{h}{x}$ $\Rightarrow$ $x = \sqrt{3}\ h$ .....(i)

In Δ ADC

$\tan 60^\circ = \frac{AC}{AD}$ $\Rightarrow$ $\sqrt{3} = \frac{5+h}{x}$

$\Rightarrow$ $x = \frac{5+h}{\sqrt{3}}$ .....(ii)

From (i) and (ii)

$\Rightarrow$ $\sqrt{3}\ h = \frac{5+h}{\sqrt{3}}$ $\Rightarrow$ $3h = 5 + h$

$\Rightarrow$ $2h = 5$ $\Rightarrow$ $h = \frac{5}{2} = 2.5$ m

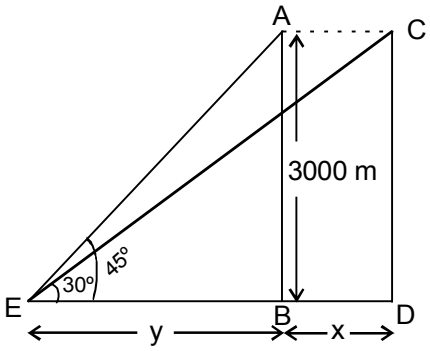
So, the height of tower = 2.5 m

Example.23

The angle of elevation of an aeroplane from a point on the ground is 45º. After a flight of 15 sec, the elevation changes to 30º. If the aeroplane is flying at a height of 3000 metres, find the speed of the aeroplane.

Solution.

Let the point on the ground is E which is y metres from point B and let after 15 sec flight it covers x metres distance.



In Δ AEB

$\tan 45^\circ = \frac{AB}{EB}$ $\Rightarrow$ $1 = \frac{3000}{y}$

$\Rightarrow$ $y = 3000$ m .....(i)

In Δ CED

$\Rightarrow$ $\tan 30^\circ = \frac{CD}{ED}$

$\Rightarrow$ $\frac{1}{\sqrt{3}} = \frac{3000}{x+y}$ ($\because$ AB = CD)

$\Rightarrow$ $x + y = 3000\sqrt{3}$ .....(ii)

From equation (i) and (ii)

$\Rightarrow$ $x + 3000 = 3000\sqrt{3}$ $\Rightarrow$ $x = 3000\sqrt{3} - 3000$ $\Rightarrow$ $x = 3000(\sqrt{3} - 1)$

$\Rightarrow$ $x = 3000 \times (1.732 - 1)$ $\Rightarrow$ $x = 3000 \times 0.732$ $\Rightarrow$ $x = 2196$ m.

Speed of Aeroplane $= \frac{\text{Distance covered}}{\text{Time taken}} = \frac{2196}{15}$ m/sec $= \frac{2196}{15} \times \frac{18}{5}$ Km/hr = 527.04 Km/hr

Hence, the speed of aeroplane is 527.04 Km/hr.

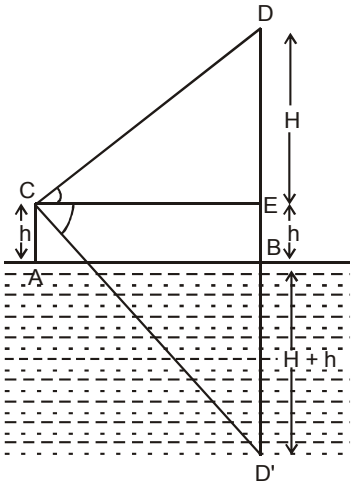
Example.24

If the angle of elevation of a cloud from a point h metres above a lake is α and the angle of depression of its reflection in the lake is β, prove that the distance of the cloud from the point of observation is $\frac{2h \sec\alpha}{\tan\beta - \tan\alpha}$.

Solution.

Let AB be the surface of the lake and let C be a point of observation such that AC = h metres. Let D be the position of the cloud and D' be its reflection in the lake. Then BD = BD'.

In Δ DCE



$\tan\alpha = \frac{DE}{CE}$

$\Rightarrow \quad CE = \frac{H}{\tan\alpha}$ .....(i)

In Δ CED'

$\tan\beta = \frac{ED'}{EC}$

$\Rightarrow \quad CE = \frac{h+H+h}{\tan\beta}$

$\Rightarrow \quad CE = \frac{2h+H}{\tan\beta}$ .....(ii)

From (i) & (ii)

$\Rightarrow \quad \frac{H}{\tan\alpha} = \frac{2h+H}{\tan\beta}$

$\Rightarrow \quad H\tan\beta = 2h\tan\alpha + H\tan\alpha \quad \Rightarrow \quad H\tan\beta - H\tan\alpha = 2h\tan\alpha$

$\Rightarrow \quad H(\tan\beta - \tan\alpha) = 2h\tan\alpha$

$\Rightarrow \quad H = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha}$ .....(iii)

In Δ DCE

$\sin\alpha = \frac{DE}{CD} \quad \Rightarrow \quad CD = \frac{DE}{\sin\alpha} \quad \Rightarrow \quad CD = \frac{H}{\sin\alpha}$

Substituting the value of H from (iii)

$CD = \frac{2h\tan\alpha}{(\tan\beta - \tan\alpha)\sin\alpha} \Rightarrow \quad CD = \frac{2h\frac{\sin\alpha}{\cos\alpha}}{(\tan\beta - \tan\alpha)\sin\alpha} \quad \Rightarrow \quad CD = \frac{2h\sec\alpha}{\tan\beta - \tan\alpha}$

Hence, the distance of the cloud from the point of observation is $\frac{2h\sec\alpha}{\tan\beta - \tan\alpha}$. **Hence Proved.**

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : System of measurement of angle/conversion

A-1 Reduce 63° 14' 51" to centesimal Measure.

A-2 The diameter of a sheet is 28 cm, through what distance does its centre move during one revolution of the wheel along the ground ?

A-3 What must be the radius of a circular running path, round which an athlete must run 5 times in order to describe 1760 meters ?

A-4. The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest as 40 : π. find the angle in degrees

A-5. Express in the three systems of angular measurement the magnitude of the angle of a regular decagon.

Section (B) : Trigonometric ratios

B-1 In the given triangle AB = 3 cm and AC = 5 cm. Find all trigonometric ratios.

B-2 If $\tan\theta = \frac{m}{n}$, then find $\sin\theta$.

B-3 If $\operatorname{cosec} A = \frac{13}{5}$, then prove that $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.

B-4 In Δ ABC, right angled at B, AC + AB = 9 cm and BC = 3 cm. Determine the value of cot C, cosec C, sec C.

B-5 If $\cot\theta = \frac{3}{4}$, prove that $\sqrt{\frac{\sec\theta - \operatorname{cosec}\theta}{\sec\theta + \operatorname{cosec}\theta}} = \frac{1}{\sqrt{7}}$.

Section (C) : Trigonometric angle

C-1 Evaluate : $\frac{2}{3}(\cos^4 30^\circ - \sin^4 45^\circ) - 3(\sin^2 60^\circ - \sec^2 45^\circ) + \frac{1}{4}\cot^2 30^\circ$

C-2 Find the value of $2\tan^2 45^\circ + 3\cos^2 30^\circ - \sin^2 60^\circ$.

C-3 Find the value of $\frac{\cos 45^\circ}{\sec 30^\circ + \operatorname{cosec} 30^\circ}$

C-4 A Rhombus of side of 10 cm has two angles of 60º each. Find the length of diagonals and also find its area.

C-5 Find the value of tan60º, geometrically.

Section (D) : Complementary angle

D-1 If A + B = 90º, prove that : $\sqrt{\dfrac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \dfrac{\sin^2 B}{\cos^2 A}}$ = tan A.

D-2 If A, B, C are the interior angles of a Δ ABC, show that :

(i) $\sin \dfrac{B+C}{2} = \cos \dfrac{A}{2}$ **(ii)** $\cos \dfrac{B+C}{2} = \sin \dfrac{A}{2}$

D-3 Evaluate : $\dfrac{\sin\theta\cos\theta\sin(90^\circ-\theta)}{\cos(90^\circ-\theta)} + \dfrac{\cos\theta\sin\theta\cos(90^\circ-\theta)}{\sin(90^\circ-\theta)} + \dfrac{\sin^2 27^\circ + \sin^2 63^\circ}{\cos^2 40^\circ + \cos^2 50^\circ}$

D-4 Without using trigonometric tables, evaluate the following :

$\dfrac{\cos 70^\circ}{3\sin 20^\circ} + \dfrac{4(\sec^2 59^\circ - \cot^2 31^\circ)}{3} - \dfrac{2}{3}\sin 90^\circ$

D-5 If sin 3θ = cos (θ – 6º) and 3θ and θ – 6º are acute, find the value of θ.

Section (E) : Trigonometric Identities

E-1 Prove that : $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$

E-2 Prove that : $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = \sec\theta \operatorname{cosec}\theta + 1$

E-3 If $\sec^2\theta\,(1 + \sin\theta)\,(1 - \sin\theta) = k$, then find the value of k.

E-4 Prove that : $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1+\sin\theta}{\cos\theta}$

E-5 If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, then Prove that : $x^2 + y^2 + z^2 = r^2$.

Section (F) : Height and distance

F-1. Two pillars of equal height stand on either side of a roadway which is 60 m wide. At a point in the roadway between the pillars, the angle of elevation of the top of pillars are 60º and 30º. Find the height of the pillars.

F-2. If the angles of elevation of the top of a tower from two points distant a and b from the base and in the same straight line with it are complementary, then find the height of the tower.

F-3. From the top of a cliff 25 m high the angle of elevation of a top of tower is found to be equal to the angle of depression of the foot of the tower. Find the height of the tower.

F-4. A tree 12 m high, is broken by the wind in such a way that its top touches the ground and makes an angle 60º with the ground. Find the height from the bottom of the tree from where it is broken by the wind is approximately.

F-5 AB is a vertical pole and C is the middle point. The end A is on the level ground and P is any point on the level ground other than A. The portion CB subtends an angle β at P. If AP : AB = 2 : 1, then find β .

OBJECTIVE QUESTIONS

Section (A) : System of measurement of angle/conversion

A-1. Radian measure of 175º 45' is :

(A) $\frac{700}{720}\pi$ (B) $\frac{703}{720}\pi$ (C) $\frac{705}{720}\pi$ (D) $\frac{710}{720}\pi$

A-2. Degree measure of $\left(\frac{1}{4}\right)^c$ is :

(A) 15º 19'5" (B) 14º 19' 5" (C) 15º 18' 6" (D) 14º 18' 6"

A-3. Degree measure of $\frac{7\pi}{6}$ is :

(A) 210° (B) 240° (C) 270° (D) None

Section (B) : Trigonometric ratios

B-1. If $\tan\theta = \frac{a}{b}$ then value of is $\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta}$ =

(A) $\frac{a^2+b^2}{a^2-b^2}$ (B) $\frac{a^2-b^2}{a^2+b^2}$ (C) $\frac{a}{\sqrt{a^2+b^2}}$ (D) $\frac{1}{\sqrt{a^2+b^2}}$

B-2. If $\alpha + \beta = \frac{\pi}{2}$ and $\sin\alpha = \frac{1}{3}$, then $\sin\beta$ is :

(A) $\frac{\sqrt{2}}{3}$ (B) $\frac{2\sqrt{2}}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

B-3. If $7\sin\alpha = 24\cos\alpha$; $0 < \alpha < \frac{\pi}{2}$, then value of $14\tan\alpha - 75\cos\alpha - 7\sec\alpha$ is equal to :

(A) 1 (B) 2 (C) 3 (D) 4

B-4. The area of a triangle is 12 sq. cm. Two sides are 6 cm and 12 cm. The included angle is :

(A) $\cos^{-1}\left(\frac{1}{3}\right)$ (B) $\cos^{-1}\left(\frac{1}{6}\right)$ (C) $\sin^{-1}\left(\frac{1}{6}\right)$ (D) $\sin^{-1}\left(\frac{1}{3}\right)$

Section (C) : Trigonometric angle

C-1. If $a = \frac{4}{3}\cot^2 30^\circ + 3\sin^2 60^\circ - 3\,\text{cosec}^2 60^\circ - \frac{3}{4}\tan^2 30^\circ$ and $b = 3\tan^2 45^\circ + \cos 0^\circ - \cot 90^\circ$ then $\log_b(a)$ is :

(A) 2 (B) –1 (C) $\frac{1}{2}$ (D) $\frac{-1}{2}$

C-2. If α and β are angles in the first quadrant, $\tan\alpha = \frac{1}{7}$, $\sin\beta = \frac{1}{\sqrt{10}}$, then using the formula sin (A + B) = sin A cos B + cos A sin B, one can find the value of $(\alpha + 2\beta)$ to be :
(A) 0º (B) 45º (C) 60º (D) 90º

C-3. The value of the expression $\frac{4}{3}\cot^2 30^\circ + 3\sin^2 60^\circ - 2\text{cosec}^2 60^\circ - \frac{3}{4}\tan^2 30^\circ$ is :
(A) 1 (B) $-\frac{20}{3}$ (C) $\frac{10}{3}$ (D) 5

C-4. The value of the expression $\frac{5\sin^2 30^\circ + \cos^2 45^\circ + 4\tan^2 60^\circ}{2\sin 30^\circ \cos 60^\circ + \tan 45^\circ}$ is :
(A) 4 (B) 9 (C) $\frac{53}{12}$ (D) $\frac{55}{6}$

C-5. Find the value of x from the equation : $x\sin\frac{\pi}{6}\cos^2\frac{\pi}{4} = \frac{\cot^2\frac{\pi}{6}\sec\frac{\pi}{3}\tan\frac{\pi}{4}}{\text{cosec}^2\frac{\pi}{4}\,\text{cosec}\frac{\pi}{6}}$
(A) 4 (B) 6 (C) – 2 (D) 0

Section (D) : Complementary angle

D-1. The value of tan 5º tan 10º tan 15º tan 20º..... tan 85º is :
(A) 1 (B) 2 (C) 3 (D) None

D-2. If $\alpha + \beta = 90^\circ$ and $\alpha = 2\beta$, then $\cos^2\alpha + \sin^2\beta$ equals to :
(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) 2

D-3. If tan (A – B) = $\frac{1}{\sqrt{3}}$ and tan (A + B) = $\sqrt{3}$, $0^\circ < A + B \le 90^\circ$, A > B. Then the value of A and B is :
(A) 45º, 30º (B) 45º, 15º (C) 60º, 30º (D) none

D-4. If A, B, C are the interior angles of a triangle ABC, then $\cos\left(\frac{A+B}{2}\right)$ equals to :
(A) $\cos\frac{C}{2}$ (B) $\sec\frac{C}{2}$ (C) $\text{cosec}\frac{C}{2}$ (D) $\sin\frac{C}{2}$

D-5. If tan 2A = cot (A – 18º), where 2A is an acute angle, find the value of A.
(A) 36º (B)63º (C)26º (D) 62º

Section (E) : Trigonometric Identities

E-1. $\frac{\tan^2\alpha - \sin^2\alpha}{\cot^2\alpha - \cos^2\alpha} = (\tan\alpha)^k$ where $k \in N$, the value of 'k' is equal to :
(A) 8 (B) 6 (C) 4 (D) 2

E-2. If sec A + tanA = p, then the value of sin A is
(A) $\frac{1-2p}{1+p^2}$ (B) $\frac{p^2-1}{p^2+1}$ (C) $\frac{1+p^2}{2(1-p^2)}$ (D) $\frac{p^2+1}{p^2-1}$

E-3. If $a\cos\theta + b\sin\theta = 3$ & $a\sin\theta - b\cos\theta = 4$ then $a^2 + b^2$ has the value =
(A) 25 (B) 14 (C) 7 (D) none

E-4. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then $\cos\theta - \sin\theta =$?
(A) $\sqrt{2}\tan\theta$ (B) $\sqrt{2}\sin\theta$ (C) $\dfrac{\sqrt{2}}{\cos\theta + \sin\theta}$ (D) none of these.

E-5. If $\alpha\cos^2 3\theta + \beta\cos^4\theta = 16\cos^6\theta + 9\cos^2\theta$ is an identity then
(A) $\alpha = 1, \beta = 18$ (B) $\alpha = 1, \beta = 24$ (C) $\alpha = 3, \beta = 24$ (D) $\alpha = 4, \beta = 2$

Section (F) : Height and distance

F-1. The shadow of a tower is 30 metres when the sun's altitude is 30º. When the sun's altitude is 60º, then the length of shadow will be :
(A) 60 m (B) 15 m (C) 10 m (D) 5m

F-2. The angles of elevation of the top of a vertical tower from two points 30 metres apart, and on the same straight line passing through the base of tower, are 30^0 and 60^0 respectively. The height of the tower is :
(A) 10 m (B) 15 m (C) $15\sqrt{3}$ m (D) 30 m

F-3. When a eucalyptus tree is broken by strong wind, its top strikes the ground at an angle of 30º to the ground and at a distance of 15 m from the foot. What is the height of the tree?
(A) $15\sqrt{3}$ m (B) $10\sqrt{3}$ m (C) 20 m (D) 10 m

Exercise-2

OBJECTIVE QUESTIONS

1. The difference between two angles is 19º and their sum is $\dfrac{890}{9}$ grades. Find the greater angle.
(A) 63º (B) 35º (C) 27º (D) 54º

2. If $\tan\theta = 4$, then $\left(\dfrac{\tan\theta}{\dfrac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta}\right)$ is equal to :
(A) 0 (B) $2\sqrt{2}$ (C) $\sqrt{2}$ (D) 1

3. If $\tan 15^\circ = 2 - \sqrt{3}$, then the value of $\cot^2 75^\circ$ is :
(A) $7 + \sqrt{3}$ (B) $7 - 2\sqrt{3}$ (C) $7 - 4\sqrt{3}$ (D) $7 + 4\sqrt{3}$

4. The angles of the triangles ABC and DEF are given as follows : A = 90^0, B = 30^0, D = 90^0 and E = 30^0. If the side BC is twice the side EF, which of the following statement is true?
(A) Sin B = 2 Sin E (B) Sin E = 2 Sin B (C) Sin B = Sin E (D) Sin A = Sin B

5. Evaluate : $\dfrac{\sec^2 54^\circ - \cot^2 36^\circ}{\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ} + 2\sin^2 38^\circ \sec^2 52^\circ - \sin^2 45^\circ + \dfrac{2}{\sqrt{3}}\tan 17^\circ \tan 60^\circ \tan 73^\circ$.
(A) $\dfrac{9}{2}$ (B) $\dfrac{7}{2}$ (C) $\dfrac{3}{2}$ (D) $\dfrac{1}{2}$

6. If $\cos^4\theta + \sin^2\theta = m$ then
(A) $1 \le m \le 2$ (B) $\dfrac{3}{4} \le m \le 1$ (C) $\dfrac{1}{2} \le m \le 1$ (D) $\dfrac{3}{4} \le m \le \dfrac{13}{16}$

7. If $\sin\phi$ and $\cos\phi$ are the roots of the equation $ax^2 + bx + c = 0$, then
(A) $(a-c)^2 + c^2 = b^2$ (B) $(a-c)^2 - c^2 = b^2$ (C) $(a+c)^2 + c^2 = b^2$ (D) $(a + c)^2 - c^2 = b^2$

8. If sinx + cosecx = 2, then $\sin^n x + \operatorname{cosec}^n x$ =

(A) n(sinx + cosecx) (B) 2 (C) $\dfrac{n}{\sin x + \operatorname{cosec} x}$ (D) n

9. An aeroplane when flying at a height 2500 m from the ground, passes vertically above another aeroplane. At an instant when the angles of elevation of the two aeroplanes from the same point on the ground are 45º and 30º respectively, then the vertical distance between the two aeroplanes at that instant is :
(A) 1158 m (B) 1058 m (C) 1008 m (D) none

10. If the angle of elevation of a cloud from a point 200 metres above a lake is 30º and the angle of depression of its reflection in the lake is 60º, then the height of the cloud (in metres) above the lake is
(A) 200 (B) 300 (C) 500 (D) none

11. The angle of elevation of the top of tower from the top and bottom of a building h metre high are α and β, then the height of tower is :
(A) h sin α cos β / sin (α + β) (B) h cos α cos β / sin(β–α)
(C) h cos α sin β / sin(β–α) (D) None of these

12. A man at the top of a vertical lighthouse, observes a boat coming directly towards it.If it takes 20 minutes for the angle of depression to change from 30º to 60º, the time taken by the boat to reach the lighthouse from the point when the angle of depression was 30º, is :
(A) 30 minutes (B) 20 minutes (C) 10 minutes (D) 5 minutes

13. If each of α, β and γ is a positive acute angle such that sin ($\alpha + \beta - \gamma$) = $\frac{1}{2}$, cos($\beta + \gamma - \alpha$) = $\frac{1}{2}$ and tan ($\gamma + \alpha - \beta$) = 1, then the values of α, β and γ is :
(A) 45º, 45º & 90º (B) 60º, 45º & 75º
(C) $37\frac{1}{2}°$, 45° and $52\frac{1}{2}°$ (D) none

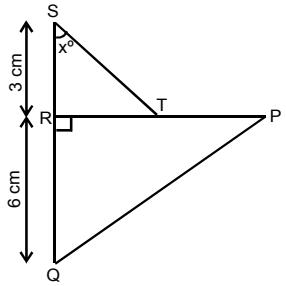
Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. The expression (1 – tan A + sec A) (1 – cot A + cosec A) has value : **[IJSO-2008]**
(A) – 1 (B) 0 (C) + 1 (D) + 2

2. A person on the top of a tower observes a scooter moving with uniform velocity towards the base of the tower. He finds that the angle of depression changes from 30º to 60º in 18 minutes. The scooter will reach the base of the tower in next : **[IJSO-2008]**
(A) 9 minutes (B) 18 / ($\sqrt{3}$ – 1) minutes
(C) 6 $\sqrt{3}$ minutes (D) the time depends upon the height of the tower

3. In the diagram, PTR and QRS are straight lines. Given that, tan xº = $\frac{4}{3}$ and "T" is the midpoint of PR, calculate the length of PQ, in cm. **[NSTSE 2009]**



(A) $\sqrt{8}$ (B) 9 (C) $\sqrt{59}$ (D) 10

4. The trigonometric expression $\cot^2\theta\left[\frac{\sec\theta-1}{1+\sin\theta}\right]+\sec^2\theta\left[\frac{\sin\theta-1}{1+\sec\theta}\right]$ has the value **[IJSO-2009]**

(A) –1 (B) 0 (C) 1 (D) 2

5. $(1 + \tan^2\theta) / (1 + \cot^2\theta) =$ **[IJSO-2009]**

(A) $\tan^2\theta$ (B) $\cot^2\theta$ (C) $\sec^2\theta$ (D) $\text{cosec}^2\theta$

6. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then $\cos\theta - \sin\theta = ?$ **[NSTSE 2009]**

(A) $\sqrt{2}\tan\theta$ (B) $\sqrt{2}\sin\theta$ (C) $\frac{\sqrt{2}}{\cos\theta+\sin\theta}$ (D) none of these.

7. The tops of two poles of heights 20m and 14m are connected by a wire. If the wire makes an angle of 30º with the horizontal, then the length of the wire is : **[NSTSE 2009]**

(A) 40 m (B) 12 m (C) 28 m (D) 68 m

8. If $\sin 2\left(\frac{2}{1}.\frac{3}{2}.\frac{4}{3}.....\frac{x-1}{x-2}\right) = 1$, 0º < x < 100º, then the value of x is equal to **[NSTSE 2010]**

(A) 91º (B) 80º (C) 49º (D) 46º

9. If $p = \sqrt{\frac{1-\sin x}{1+\sin x}}$, $q = \frac{1-\sin x}{\cos x}$, $r = \frac{\cos x}{1+\sin x}$, then Which one of the following statement is correct ? **[NSTSE-2010]**

(A) $p = q \neq r$ (B) $q = r \neq p$ (C) $r = p \neq q$ (D) $p = q = r$

10. If $\sin\theta + \text{cosec}\,\theta = 2$, then $[\sin^8\theta + \text{cosec}^8\theta]$ will have the value **[IJSO-2010]**

(A) 2 (B) 2^4 (C) 2^6 (D) 2^8

11. An aeroplane is flying horizontally at a height of 3150 m above a horizontal plane ground. At a particular instant it passes another aeroplane vertically below it. At this instant, the angles of elevation of the planes from a point on the ground are 30º and 60º. Hence, the distance between the two planes at that instant is **[IJSO-2011]**

(A) 1050 m. (B) 2100 m. (C) 4200 m. (D) 5250 m.

12. If $\cot^2\theta(1-3\sec\theta + 2\sec^2\theta) = 1$ **[IJSO-2012]**

(A) 120º (B) 210º (C) 300º (D) 330º

13. If $\sin x + \sin y = a$ and $\cos x - \cos y = b$.Then find the value of $\frac{1}{2}(2 - a^2 - b^2)$ **[IJSO-2012]**

(A) $\cos(x + y)$ (B) $\cos(x - y)$ (C) $\sin(x + y)$ (D) $\sin(x - y)$

14. Two friends A and B watched a car from the top of their buildings. Angle of depression for A was 10° more than angle of depression for B, then **[IJSO-2016]**

(A) A's apartment is taller than B's apartment
(B) B's apartment is taller than A's apartment
(C) A's apartment and B's apartment have same height
(D) We cannot compare the heights of the two apartments

15. In a square ABCD, a point P is inside the square such that ABP is an equilateral triangle. The segment AP cuts the diagonal BD in E. Suppose AE = 2. The area of ABCD is **[IJSO-2018]**

(A) $4+2\sqrt{3}$ (B) $5+2\sqrt{3}$ (C) $4+4\sqrt{3}$ (D) $5+4\sqrt{3}$

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 $70^g\ 27'50''$ **A-2** 88 cm. **A-3** 56 metres. **A-4.** 20°, 60° and 100°.

A-5. $144^o, 160^g, \frac{4\pi}{5}$ radians.

Section (B)

B-1 $\sec\theta = \frac{5}{3}$, $\text{cosec}\,\theta = \frac{5}{4}$, $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{4}$ **B-2** $\sin\theta = \frac{m}{\sqrt{m^2+n^2}}$.

B-4 $\cot C = \frac{3}{4}$, $\text{cosec}\,C = \frac{5}{4}$, $\sec C = \frac{5}{3}$.

Section (C)

C-1 $\frac{113}{24}$. **C-2** $\frac{7}{2}$ **C-3** $\frac{3-\sqrt{3}}{4\sqrt{2}}$ **C-4** $50\sqrt{3}$ cm^2.

C-5 tan 60º is $\sqrt{3}$.

Section (D)

D-3 2 **D-4** 1. **D-5** $\theta = 24^o$.

Section (E)

E-3 1

Section (F)

F-1. $15\sqrt{3}$ m **F-2.** $h = \sqrt{ab}$ **F-3.** 50 meter **F-4.** 5.568 m

F-5 $\tan^{-1}\left(\frac{2}{9}\right)$

OBJECTIVE QUESTIONS

Section (A)

A-1. (B) **A-2.** (B) **A-3.** (A)

Section (B)

B-1. (A) **B-2.** (B) **B-3.** (B) **B-4.** (D)

Section (C)

C-1. (C) **C-2.** (B) **C-3.** (C) **C-4.** (D) **C-5.** (B)

Section (D) :

D-1. (A) **D-2.** (A) **D-3.** (B) **D-4.** (D) **D-5.** (A)

Section (E) :

E-1. (B) **E-2.** (B) **E-3.** (A) **E-4.** (B) **E-5.** (B)

Section (F) :

F-1. (C) **F-2.** (C) **F-3.** (A)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	D	D	C	C	A	B	B	B	C	D	C	A	C

Exercise-3

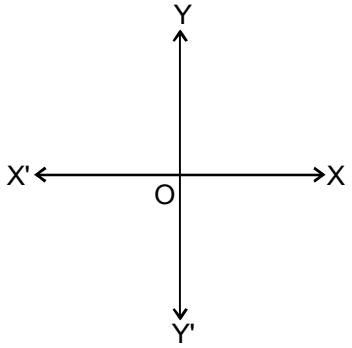
IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	D	A	D	B	A	B	B	D	D	A	B	C	A	D	A

CO-ORDINATE GEOMETRY

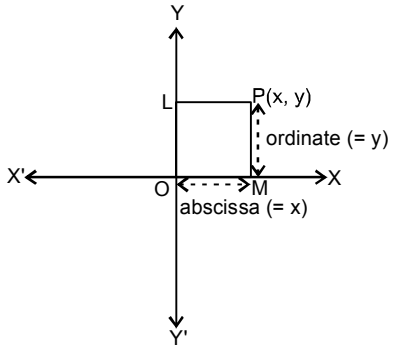
A. RECTANGULAR CO-ORDINATES

Take two perpendicular lines **X'OX** and **Y'OY** intersecting at the point **O**. **X'OX** and **Y'OY** are called the co-ordinate axes. **X'OX** is called the **X-axis**, **Y'OY** is called the **Y-axis** and **O** is called the origin. Lines **X'OX** and **Y'OY** are sometimes also called rectangular axes.



(a) Co-ordinates of a Point :

Let **P** be any point as shown in figure. Draw **PL** and **PM** perpendiculars on **Y- axis** and **X - axis**, respectively. The length **LP (or OM)** is called the x **- coordinate** or the **abscissa of point P** and **MP** is called the **y-coordinate** or the **ordinate of point P**. A point whose abscissa is **x** and ordinate is **y** named as the point **(x, y) or P (x, y).**



The two lines **X'OX** and **Y'OY** divide the plane into four parts called **quadrants**. **XOY, YOX', X'OY'** and **Y'OX** are, respectively, called the first, second, third and fourth quadrants. The following table shows the signs of the coordinates of points situated in different quadrants:

Quadrant	X - coordinate	Y - coordinate	Point
First quadrant	+	+	(+, +)
Second quadrant	–	+	(–, +)
Third quadrant	–	–	(–, –)
Fourth quadrant	+	–	(+, –)

REMARKS :

(i) Abscissa is the perpendicular distance of a point from **y-axis**. (i.e., positive to the right of **y-axis** and negative to the left of **y-axis**).

(ii) Ordinate is the perpendicular distance of a point from **x-axis**. (i.e., positive above **x-axis** and negative below **x-axis**).

(iii) Abscissa of any point on **y-axis** is zero.

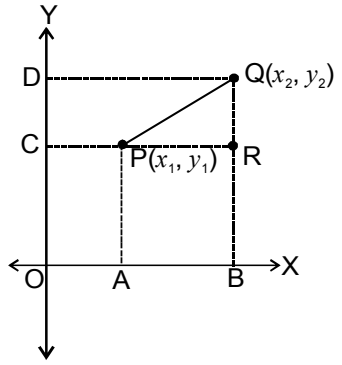
(iv) Ordinate of any point on **x-axis** is zero.

(v) Co-ordinates of the **origin** are **(0,0)**.

B. DISTANCE BETWEEN TWO POINTS

Let two points be **P (x_1, y_1)** and **Q (x_2, y_2)**.

Take two mutually perpendicular lines as the coordinate axis with **O** as origin. Mark the points **P (x_1, y_1)** and **Q (x_2, y_2)**. Draw lines **PA**, **QB** perpendicular to **X-axis**, from the points **P** and **Q**, which meet the **X-axis** in points A and B, respectively.



Draw lines PC and QD perpendicular to **Y-axis**, which meet the**Y-axis** in C and D, respectively. Produce CP to meet BQ in R.

Now, OA = abscissa of P = x_1

Similarly, OB = x_2, OC = y_1 and OD = y_2

Therefore, we have

PR = AB = OB – OA = $x_2 - x_1$

Similarly, QR = QB – RB = QB – PA = $y_2 - y_1$

Now, using Pythagoreus Theorem, in right angled triangle PRQ, we have

$PQ^2 = PR^2 + RQ^2$

or $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Since the distance or length of the line-segment PQ is always non-negative, on taking the positive square root, we get the distance as

PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

This result is known as **distance formula**.

Corollary : The distance of a point **P (x_1, y_1)** from the origin **(0, 0)** is given by **OP =** $\sqrt{x_1^2 + y_1^2}$.

SOME USEFUL POINTS :

1. In questions relating to geometrical figures, take the given vertices in the given order and proceed as indicated.

(i) For an **isosceles triangle** : We have to prove that at least two sides are equal.

(ii) For an **equilateral triangle** : We have to prove that three sides are equal.

(iii) For a r**ight-angled triangle** : We have to prove that the sum of the squares of two sides is equal to the square of the third side.

(iv) For a **square** : We have to prove that the four sides are equal, two diagonals are equal.

(v) For a **rhombus** : We have to prove that four sides are equal (and there is no need to establish that two diagonals are unequal as the square is also a rhombus).

(vi) For a **rectangle** : We have to prove that the opposite sides are equal and two diagonals are equal.

(vii) For a **parallelogra**m : We have to prove that the opposite sides are equal (and there is no need to establish that two diagonals are unequal as the rectangle is also a parallelogram).

2. For three points to be **collinear** : We have to prove that the sum of the distances between two pairs of points is equal to the third pair of points.

Solved Examples

Example.1 Find the distance between the points (8, –2) and (3, –6).

Solution. Let the points (8, –2) and (3, –6) be denoted by P and Q, respectively.
Then, by distance formula, we obtain the distance PQ as

PQ = $\sqrt{(3-8)^2+(-6+2)^2} = \sqrt{(-5)^2+(-4)^2} = \sqrt{41}$ unit.

Example.2 Prove that the points (1, –1), $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and (1, 2) are the vertices of an isosceles triangle.

Solution. Let the point (1, –1), $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and (1, 2) be denoted by P, Q and R, respectively. Now,

PQ = $\sqrt{\left(-\frac{1}{2}-1\right)^2+\left(\frac{1}{2}+1\right)^2} = \sqrt{\frac{18}{4}} = \frac{3}{2}\sqrt{2}$

QR = $\sqrt{\left(1+\frac{1}{2}\right)^2+\left(2-\frac{1}{2}\right)^2} = \sqrt{\frac{18}{4}} = \frac{3}{2}\sqrt{2}$

PR = $\sqrt{(1-1)^2+(2+1)^2} = \sqrt{9} = 3$

From the above, we see that PQ = QR.
$\therefore$ The triangle is isosceles.

Example.3 Using distance formula, show that the points (–3, 2), (1, –2) and (9, –10) are collinear.

Solution. Let the given points (–3, 2), (1, –2) and (9, –10) be denoted by A, B and C, respectively. Points A, B and C will be collinear, if the sum of the lengths of two line-segments is equal to the third.

Now, AB = $\sqrt{(1+3)^2+(-2-2)^2} = \sqrt{16+16} = 4\sqrt{2}$

BC = $\sqrt{(9-1)^2+(-10+2)^2} = \sqrt{64+64} = 8\sqrt{2}$

AC = $\sqrt{(9+3)^2+(-10-2)^2} = \sqrt{144+144} = 12\sqrt{2}$

Since, AB + BC = $4\sqrt{2} + 8\sqrt{2} = 12\sqrt{2}$ = AC, the points A, B and C are collinear.

Example.4 Find a point on the X-axis which is equidistant from the points (5, 4) and (–2, 3).

Solution. Since the required point (say P) is on the X-axis, its ordinate will be zero. Let the abscissa of the point be x.
Therefore, coordinates of the point P are (x, 0).
Let A and B denote the points (5, 4) and (– 2, 3), respectively.
Since we are given that AP = BP, we have
$AP^2 = BP^2$
i.e., $(x-5)^2 + (0-4)^2 = (x+2)^2 + (0-3)^2$
$\Rightarrow$ $x^2 + 25 - 10x + 16 = x^2 + 4 + 4x + 9$
$\Rightarrow$ $-14x = -28$ $\Rightarrow$ $x = 2$.
Thus, the required point is (2, 0).

Example.5 The vertices of a triangle are (– 2, 0), (2, 3) and (1, – 3). Is the triangle equilateral, isosceles or scalene ?

Solution. Let the points (–2, 0), (2, 3) and (1, –3) be denoted by A, B and C respectively. Then,

AB = $\sqrt{(2+2)^2+(3-0)^2} = 5$

BC = $\sqrt{(1-2)^2+(-3-3)^2} = \sqrt{37}$ and AC = $\sqrt{(1+2)^2+(-3-0)^2} = 3\sqrt{2}$

Clearly, AB $\neq$ BC $\neq$ AC

Therefore, ABC is a scalene triangle.

Example.6

The length of a line-segment is 10. If one end is at (2, – 3) and the abscissa of the second end is 10, show that its ordinate is either 3 or – 9.

Solution.

Let (2, – 3) be the point A. Let the ordinate of the second end B be *y*. Then its coordinates will be (10, y).

$\therefore$ AB = $\sqrt{(10-2)^2+(y+3)^2} = 10$ (Given)

$\Rightarrow$ 64 + 9 + y^2 + 6y = 100 $\Rightarrow$ y^2 + 6y + 73 – 100 = 0

$\Rightarrow$ y^2 + 6y – 27 = 0 $\Rightarrow$ (y + 9)(y – 3) = 0

Therefore, y = – 9 or y = 3.

i.e., The ordinate is 3 or – 9.

Example.7 Show that the points (– 2, 5), (3, – 4) and (7, 10) are the vertices of a right triangle.

Solution. Let the three points be A (– 2, 5), B (3, – 4) and C (7, 10).

Then, $AB^2 = (3 + 2)^2 + (-4 - 5)^2 = 106$

$\Rightarrow$ $BC^2 = (7 - 3)^2 + (10 + 4)^2 = 212$

$\Rightarrow$ $AC^2 = (7 + 2)^2 + (10 - 5)^2 = 106$

We see that

$\Rightarrow$ $BC^2 = AB^2 + AC^2$ $\Rightarrow$ 212 = 106 + 106

$\Rightarrow$ 212 = 212

$\therefore$ $\angle A = 90^0$.

Thus, ABC is a right triangle, right angled at A.

Example.8 If the distance of P (x, y) from A (5, 1) and B (–1, 5) are equal, prove that 3x = 2y.

Solution. P (x, y), A (5, 1) and B (–1, 5) are the given points.

AP = BP [Given]

$\therefore$ $AP^2 = BP^2 \Rightarrow AP^2 - BP^2 = 0$

$\{(x-5)^2 + (y-1)\}^2 - \{(x+1)^2 + (y-5)^2\} = 0$

$x^2 + 25 - 10x + y^2 + 1 - 2y - x^2 - 1 - 2x - y^2 - 25 + 10y = 0$

$-12x + 8y = 0$ 3x = 2y.

C. SECTION FORMULA

(a) Formula for Internal Division :

The coordinates of the point **(x, y)** which divides the line segment joining the points **(x_1, y_1)** and **(x_2, y_2)** internally in the ratio **m : n** are given by

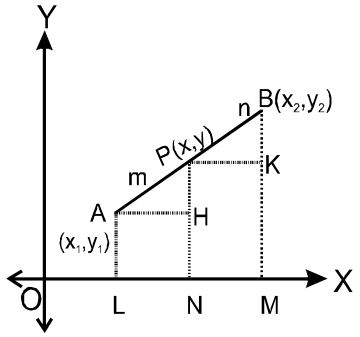
$$x = \frac{mx_2 + nx_1}{m+n}, \; y = \frac{my_2 + ny_1}{m+n}.$$

Proof :

Let O be the origin and let OX and OY be the X-axis and Y-axis respectively. Let **A** (x_1, y_1) and **B** (x_2, y_2) be the given points. Let **(x, y)** be the coordinates of the point **P** which divides **AB** internally in the ratio **m : n**. Draw AL ⊥ OX, BM ⊥ OX, PN ⊥ OX. Also, draw **AH** and **PK** perpendiculars from **A** and **P** on **PN** and **BM** respectively. Then,

OL = x_1, ON = x, OM = x_2, AL = y_1, PN = y and BM = y_2.

∴ AH = LN = ON – OL = $x - x_1$, PH = PN – HN



PN – AL = $y - y_1$, PK = NM = OM – ON = $x_2 - x$

And BK = BM – MK = BM – PN = $y_2 - y$.

Clearly, Δ AHP and Δ PKB are similar.

$$\frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK} \quad \Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Now, $\frac{m}{n} = \frac{x - x_1}{x_2 - x} \quad \Rightarrow \quad mx_2 - mx = nx - nx_1$

$\Rightarrow \quad mx + nx = mx_2 + nx_1 \quad \Rightarrow \quad x = \frac{mx_2 + nx_1}{m + n}$

and $\frac{m}{n} = \frac{y - y_1}{y_2 - y} \quad \Rightarrow \quad my_2 - my = ny - ny_1$

$\Rightarrow \quad my + ny = my_2 + ny_1 \quad \Rightarrow \quad y = \frac{my_2 + ny_1}{m + n}$

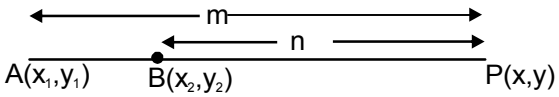
Thus, the coordinates of P are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$.

REMARK :

If **P** is the mid-point of **AB**, then it divides **AB** in the ratio **1 : 1**, so its coordinates are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

(b) Formula for External Division :

The coordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio **m : n** are given by :



$$x = \frac{mx_2 - nx_1}{m - n}, \quad y = \frac{my_2 - ny_1}{m - n}.$$

Solved Examples

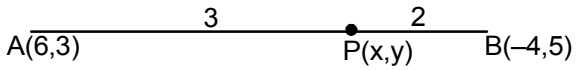
Example.9 Find the coordinates of the point which divides the line segment joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 (i) internally (ii) externally.

Solution. Let P(x, y) be the required point.

(i) For internal division, we have

$x = \dfrac{3\times(-4)+2\times 6}{3+2}$ And $y = \dfrac{3\times 5+2\times 3}{3+2}$

$\Rightarrow$ $x = 0$ and $y = \dfrac{21}{5}$.

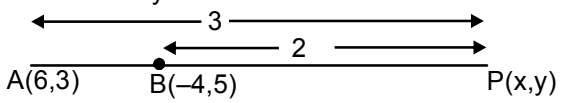


So the coordinates of P are $\left(0, \dfrac{21}{5}\right)$.

(ii) For external division, we have

$x = \dfrac{3\times -4-2\times 6}{3-2}$ and $y = \dfrac{3\times 5-2\times 3}{3-2}$

$x = -24$ and $y = 9$



So the coordinates of P are (–24, 9).

Example.10 In what ratio does the point (–1, –1) divides the line segment joining the points (4, 4) and (7, 7)?

Solution. Suppose the point C (–1, –1) divides the line joining the points A(4, 4) and B(7, 7) in the ratio k : 1.

Then, the coordinates of C are $\left(\dfrac{7k+4}{k+1}, \dfrac{7k+4}{k+1}\right)$

But, we are given that the coordinates of the point C are (–1, –1).

$\therefore$ $\dfrac{7k+4}{k+1} = -1 \Rightarrow k = -\dfrac{5}{8}$

Thus, C divides AB externally in the ratio 5 : 8.

Example.11 In what ratio does the X-axis divide the line segment joining the points (2, –3) and (5, 6)?

Solution. Let the required ratio be λ : 1. Then the coordinates of the point of division are $\left(\dfrac{5\lambda+2}{\lambda+1}, \dfrac{6\lambda-3}{\lambda+1}\right)$.

But, it is a point on X-axis on which y-coordinate of every point is zero.

$\therefore$ $\dfrac{6\lambda-3}{\lambda+1} = 0$ $\Rightarrow$ $\lambda = \dfrac{1}{2}$

Thus, the required ratio is $\dfrac{1}{2}$: 1 or 1 : 2.

Example.12 Determine the ratio in which the line 3x + y – 9 = 0 divides the segment joining the points (1, 3) and (2, 7).

Solution. Suppose the line 3x + y – 9 = 0 divides the line segment joining A (1, 3) and B (2, 7) in the ratio k : 1 at point C. Then, the coordinates of C are $\left(\dfrac{2k+1}{k+1}, \dfrac{7k+3}{k+1}\right)$. But, C lies on 3x + y – 9 = 0, therefore $3\left(\dfrac{2k+1}{k+1}\right)+\dfrac{7k+3}{k+1}-9=0$

$\Rightarrow \quad 6k + 3 + 7k + 3 - 9k - 9 = 0 \quad \Rightarrow \quad k = \frac{3}{4}$

So, the required ratio is 3 : 4 internally.

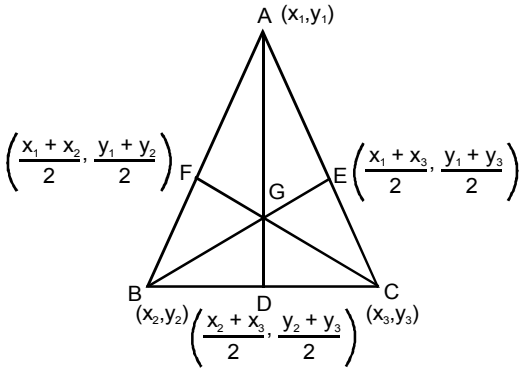
D. CENTROID OF A TRIANGLE

Prove that the coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$. Also, deduce that the medians of a triangle are concurrent.

Proof :

Let **A** (x_1, y_1), **B** (x_2, y_2) and **C** (x_3, y_3) be the vertices of Δ **ABC** whose medians are AD, BE and CF respectively. So D, E and F are respectively the mid-points of BC, CA and AB.

Coordinates of **D** are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$. Coordinates of a point dividing AD in the ratio **2 : 1** are $\left(\frac{1.x_1+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1.y_1+2\left(\frac{y_2+y_3}{2}\right)}{1+2}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$



The coordinates of E are $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$. The coordinates of a point dividing BE in the ratio 2 : 1 are $\left(\frac{1.x_2+\frac{2(x_1+x_3)}{2}}{1+2}, \frac{1.y_2+\frac{2(y_1+y_3)}{2}}{1+2}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

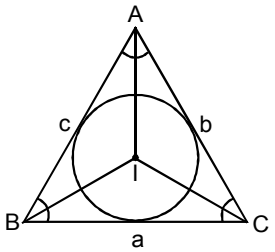
Similarly the coordinates of a point dividing CF in the ratio 2 : 1 are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Thus, the point having coordinates $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ is common to AD, BE and CF and divides them in the ratio 1 : 2.

Hence, medians of a triangle are concurrent and the coordinates of the centroid are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

E. IN-CENTRE OF A TRIANGLE

The coordinates of the **in-centre (intersection point of angle bisector segment)** of a triangle whose vertices **(x_1, y_1), (x_2, y_2)** and **(x_3, y_3)** are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$.



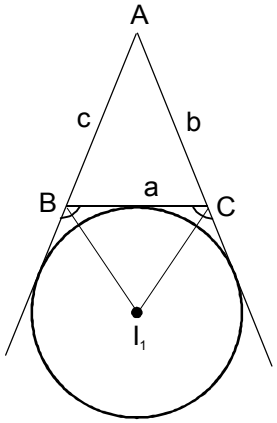
Where **a, b, c** be the lengths of the sides **BC, CA, AB** respectively.

F. EX-CENTRE OF A TRIANGLE

Let **A (x_1, y_1), B (x_2, y_2), C (x_3, y_3)** be the vertices of the **triangle ABC** and let **a, b, c** be the lengths of the sides **BC, CA, AB** respectively.

The coordinates of **ex-centre I_1** (centre of exscribed circle opposite to the angles A) are given by

$$\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$



The coordinates of **I_2** and **I_3** (centres of exscribed circles opposite to the angles **B** and **C** respectively) are given by

$I_2\left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$ and $I_3\left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$ respectively.

❖ **NOTE :**

(i) Incentre divides the angle bisectors in the ratio, (b + c) : a; (c + a) : b & (a + b) : c.

(ii) Orthocenter, Centroid & Circumcenter are always collinear & centroid divides the line joining orthocenter & circumcenter in the ratio 2 : 1 respectively.

(iii) In an isosceles triangle Centrod (G), Orthocenter (O), Incenter (I) & Circumcenter (C) lie on the same line and in an equilateral triangle, all these four points coincide.

Solved Examples

Example.13

Find the coordinates of (i) centroid (ii) in-centre of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

Solution.

(i) We know that the coordinates of the centroid of a triangle whose angular points are

$(x_1, y_1), (x_2, y_2)$ (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

So the coordinates of the centroid of a triangle whose vertices are (0, 6), (8, 12) and (8, 0) are $\left(\frac{0+8+8}{3}, \frac{6+12+0}{3}\right)$ or $\left(\frac{16}{3}, 6\right)$.

(ii) Let A (0, 6), B (8, 12) and C (8, 0) be the vertices of triangle ABC.

Then c = AB = $\sqrt{(0-8)^2+(6-12)^2}$ = 10, b = CA = $\sqrt{(0-8)^2+(6-0)^2}$ = 10

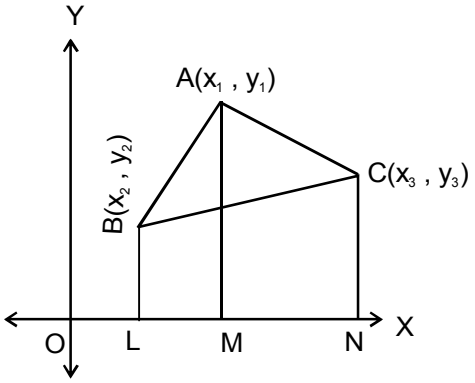
And a = BC = $\sqrt{(8-8)^2+(12-0)^2}$ = 12.

The coordinates of the in-centre are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

or $\left(\frac{12\times0+10\times8+10\times8}{12+10+10}, \frac{12\times6+10\times12+10\times0}{12+10+10}\right)$ or $\left(\frac{160}{32}, \frac{192}{32}\right)$ or (5, 6).

G. AREA OF A TRIANGLE

Let **ABC** be any triangle whose vertices are **A (x_1, y_1), B (x_2, y_2)** and **C(x_3, y_3)**. Draw BL, AM and CN perpendiculars from B, A and C respectively, to the X – axis. ABLM, AMNC and BLNC are all trapeziums.



Area of ΔABC = Area of trapezium ABLM + Area of trapezium AMNC – Area of trapezium BLNC

We know that, Area of trapezium = $\frac{1}{2}$ (Sum of parallel sides) (distance b/w them)

Therefore, Area of ΔABC = $\frac{1}{2}$ (BL + AM) (LM) + $\frac{1}{2}$ (AM + CN) MN – $\frac{1}{2}$ (BL + CN) (LN)

Area of ΔABC = $\frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$

Area of ΔABC = $\frac{1}{2}\left|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]\right|$.

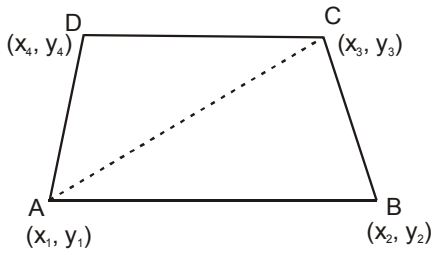
(a) Condition for collinearity :

Three points **A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3)** are collinear if **Area of ΔABC = 0**.

H. AREA OF QUADRILATERAL

Let the vertices of Quadrilateral ABCD are A (x_1, y_1), B (x_2, y_2) , C (x_3, y_3) and D (x_4, y_4)

So, Area of quadrilateral ABCD = Area of Δ ABC + Area of Δ ACD.



Solved Examples

Example.14 The vertices of Δ ABC are (–2, 1), (5, 4) and (2, – 3) respectively. Find the area of triangle.

Solution. A (–2, 1), B (5, 4) and C (2, – 3) be the vertices of triangle.

So, $x_1 = -2$, $y_1 = 1$; $x_2 = 5$, $y_2 = 4$; $x_3 = 2$, $y_3 = -3$

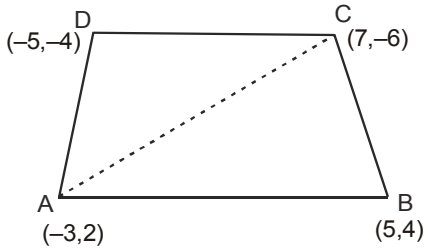
Area of Δ ABC = $\frac{1}{2}\left|\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]\right|$

$= \frac{1}{2}\left|\left[(-2)(4+3) + (5)(-3-1) + 2(1-4)\right]\right| = \frac{1}{2}\left|\left[-14 + (-20) + (-6)\right]\right|$

$= \frac{1}{2}\left|-40\right| = 20$ Sq. unit.

Example.15 Find the area of quadrilateral whose vertices, taken in order, are A(–3, 2), B(5, 4), C (7, – 6) and D (–5, –4).

Solution. Area of quadrilateral = Area of ΔABC + Area of Δ ACD



So, Area of ΔABC

$= \frac{1}{2}\left|(-3)(4+6) + 5(-6-2) + 7(2-4)\right| = \frac{1}{2}\left|-30 - 40 - 14\right| = \frac{1}{2}\left|-84\right| = 42$ Sq. units

Area of Δ ACD

$= \frac{1}{2}\left|-3(-\ \ 6+4) + 7(-4-2) + (-5)(2+6)\right| = \frac{1}{2}\left|+6 - 42 - 40\right| = \frac{1}{2}\left|-76\right| = 38$ Sq. units

So, Area of quadrilateral ABCD = 42 + 38 = 80 Sq. units.

H. LOCUS AND EQUATION OF THE LOCUS

LOCUS : The curve described by a point which moves under given condition or conditions is called the **locus**. For example, suppose C is a point in the plane of the paper and **P** is a variable point in a plane of the paper such that its distance from **C** is always equal to **a** (say). It is clear that all the positions of the moving point **P** lie on the circumference of a circle whose centre is **C** and whose radius is **a**.

EQUATION OF THE LOCUS OF A POINT : The equation to the locus of a point is the relation which is satisfied by the co-ordinates of every point on the locus of the point.

ALGORITHM TO FIND THE LOCUS OF A POINT :

STEP 1 : Assume the co-ordinates of point say **(h, k)** whose locus is to be found.

STEP 2 : Write the given condition in mathematical form involving **h, k**.

STEP 3 : Eliminate the variables if any.

STEP 4 : Replace **h** by **x** and **k** by **y i**n the result obtained in **step 3**.
The equation obtained is the locus of the point which moves under some stated condition(s).

Solved Examples

Example.16 The sum of the squares of the distances of a moving point from two fixed points (a, 0) and (–a, 0) is equal to a constant quantity $2c^2$. Find the equation of its locus.

Solution. Let P (h, k) be any position of the moving point and let A (a, 0), B (–a, 0) be the given points. Then $PA^2 + PB^2 = 2c^2$ (Given)

$\Rightarrow$ $(h - a)^2 + (k - 0)^2 + (h + a)^2 + (k - 0)^2 = 2c^2$

$\Rightarrow$ $h^2 - 2ah + a^2 + k^2 + h^2 + 2ah + a^2 + k^2 = 2c^2$

$\Rightarrow$ $2h^2 + 2k^2 + 2a^2 = 2c^2$

$\Rightarrow$ $h^2 + k^2 = c^2 - a^2$

Hence, locus of (h, k) is $x^2 + y^2 = c^2 - a^2$.

Example.17 Find the locus of a point, so that the join of (– 5, 1) and (3, 2) subtends a right angle at the moving point.

Solution. Let P (h, k) be a moving point and let A (– 5, 1) and B (3, 2) be given points. By the given condition.

$\angle APB = 90^o$

$\therefore$ ΔPAB is a right angled triangle

$\Rightarrow$ $AB^2 = AP^2 + PB^2$

$\Rightarrow$ $(3 + 5)^2 + (2 - 1)^2 = (h + 5)^2 + (k - 1)^2 + (h - 3)^2 + (k - 2)^2$

$\Rightarrow$ $65 = 2 (h^2 + k^2 + 2h - 3k) + 39$

$\Rightarrow$ $h^2 + k^2 + 2h - 3k - 13 = 0$

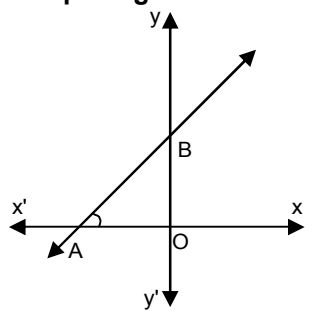
Hence, locus of (h, k) is $x^2 + y^2 + 2x - 3y - 13 = 0$.

I. STRAIGHT LINE

A **straight line** is a curve such that every point on the line segment joining any two points on it lies on it.

(a) Slope (Gradient) of a Line :

The trigonometrical tangent of the angle that a line makes with the positive direction of the **x-axis** in **anticlockwise sense** is called the **slope** or **gradient** of the line.



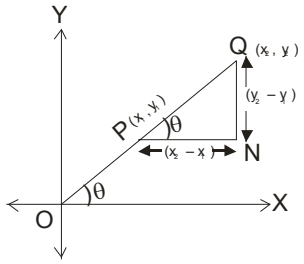
The slope of a line is generally denoted by **m**. Thus, **m = tan θ**.

Since a line parallel to **x-axis** makes an angle of **0º** with **x-axis**, therefore its slope is **tan 0º = 0**. A line parallel to **y-axis** i.e., perpendicular to x-axis makes an angle of **90º** with **x-axis**, so its slope is **tan** $\frac{\pi}{2} = \infty$. Also the slope of a line **equally inclined** with axis is **1** or **–1** as it makes **45º** or **135º** angle with **x-axis**.

The angle of inclination of a line with the positive direction of **x-axis** in **anticlockwise sense** always lies between **0º** and **180º.**

(b) Slope of a Line in Term of Coordinates of any two Points on it :

Let **P** (x_1, y_1) and **Q** (x_2, y_2) be two points on a line making an angle θ with the positive direction of **x-axis**.



In Δ PQN, $\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$

Thus, if (x_1, y_1) and (x_2, y_2) are coordinates of any two points on a line, then its slope is

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissa}}$.

Solved Examples

Example.18 What is the slope of a line whose inclination with the positive direction of X-axis is :

(i) 0º **(ii)** 90º **(iii)** 120º **(iv)** 150º

Solution. **(i)** Here, θ = 0º

Slope = tan θ = tan 0º = 0. [line is parallel to x –axis] (ii) Here θ = 90º

(ii) Slope = tan θ = tan 90º = ∞.

∴ The slope of line is not defined. [line is parallel to y – axis]

(iii) Here θ = 120º

∴ Slope = tan θ = tan 120º = tan (180º – 60º) = – tan 60º = **–** $\sqrt{3}$

(iv) Here θ = 150º

∴ Slope = tan θ = tan 150º = tan (180º – 30º) = – tan 30º = – $\frac{1}{\sqrt{3}}$.

Example.19 Find the slope of the line passing through the points

(i) (1, 6) and (– 4, 2) **(ii)** (5, 9) and (2, 9)

Solution. **(i)** Let A (1, 6) and B (– 4, 2)

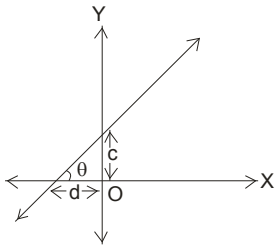
∴ Slope of AB = $\frac{2-6}{-4-1} = \frac{-4}{-5} = \frac{4}{5}$ $\left(\text{Using slope} = \frac{y_2 - y_1}{x_2 - x_1}\right)$

(ii) Let A (5, 9), B (2, 9)

∴ Slope of AB = $\frac{9-9}{2-5} = \frac{0}{-3} = 0$.

J. EQUATION OF STRAIGHT LINES IN VARIOUS FORMS

(a) The Slope Intercept Form of a Line :



The equation of a line with slope **m** and making an intercept **c** on **Y-axis** is **y = mx + c**.
The equation of a line with slope **m** and making an intercept **d** on **x-axis** is **y = m(x – d).**

Solved Examples

Example.20 Find the equation of a line with slope – 1 and cutting off an intercept of 4 units on negative direction of y-axis

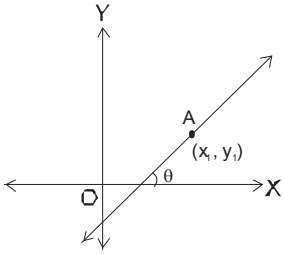
Solution. Here m = –1 and c = – 4. So, the equation of the line is y = mx + c, y = – x – 4 or x + y + 4 = 0.

Example.21 Find the equation of a line which cuts off intercept 4 at x - axis and makes an angle 60º with positive direction of the x-axis.

Solution. Slope m of the line = tan 60º = $\sqrt{3}$ and the x - intercept = 4.

Therefore, the equation of the line is y = $\sqrt{3}$ (x – 4).

(b) The Point-Slope Form of a Line :



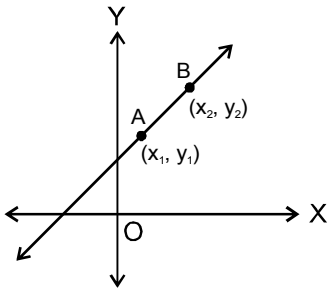
The equation of a line which passes through the point **(x_1, y_1)** and has the slope **'m'** is **$y - y_1 = m(x - x_1)$.**

Solved Examples

Example.22 Find the equation of a line passing through (2, –3) and inclined at an angle of 135º with the positive direction of x-axis.

Solution. Here, m = slope of the line = tan 135º = tan (90º + 45º) = – cot 45º = –1,
and x_1 = 2, y_1 = –3.
So, the equation of the line is $y - y_1 = m(x - x_1)$
i.e. y – (–3) = –1(x – 2) or y + 3 = – x + 2 or x + y + 1 = 0.

(c) The Two-Point Form of a Line :



The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$.

Solved Examples

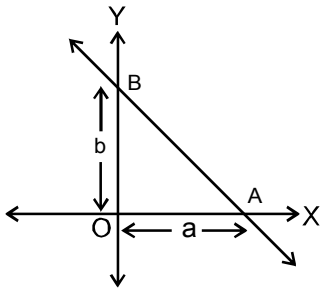
Example.23 Find the equation of the line joining the points (– 1, 3) and (4, – 2).

Solution. Here the two points are $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

So, the equation of the line in two-point form is

$$y - 3 = \frac{-3-2}{4-(-1)}(x + 1) \Rightarrow \quad y - 3 = -x - 1 \Rightarrow \quad x + y - 2 = 0.$$

(d) The Intercept Form of a Line :



The equation of a line which cuts off intercepts **a** and **b** respectively from the **x** and **y-axis** is $\frac{x}{a} + \frac{y}{b} = 1$.

Solved Examples

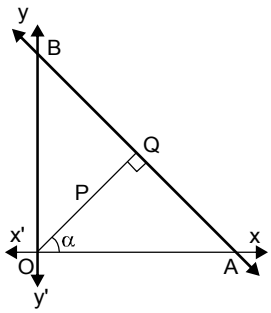
Example.24 Find x-intercept & y-intercept of the line $2x - 3y + 5 = 0$.

Solution. Here, $a = 2$, $b = -3$, $c = 5$

x-intercept $= -\frac{c}{a} = -\frac{5}{2}$ and y-intercept $= \frac{5}{3}$.

(e) Perpendicular / Normal form :

x cos α + y sin α = p (where $p > 0$, $0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the **origin O** on the line is **p** and this perpendicular makes an angle α with positive **x–axis**.



Solved Examples

Example.25 Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30º with the positive direction of the x-axis.

Solution. Here p = 3, α = 30º

$\therefore$ Equation of the line in the normal form is x cos 30º + y sin 30º = 3

or $\frac{\sqrt{3}}{2}x + \frac{y}{2} = 3$ or $\sqrt{3}\,x + y = 6$.

(f) Parametric Form :

P (r) = (x, y) = (x_1 + r cos θ, y_1 + r sin θ) or $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$ **= r** is the equation of the line in parametric form, where **'r'** is the parameter whose absolute value is the distance of any point **(x, y)** on the line from the fixed point **(x_1, y_1)** on the line.

(g) Equations of straight lines passing through A given point and making a given angle with a given line

Equations of the straight lines which pass through a given point (x_1, y_1) and make a given α with the given straight line y = mx + c are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Solved Examples

Example.26 Find the equation of the line through the point A (2, 3) and making an angle of 45º with the x axis. Also determine the length of intercept on it between A and the line x + y + 1 = 0.

Solution. The equation of a line through A and making an angle of 45º with the x-axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} \text{ or } \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} \quad \text{or} \quad x - y + 1 = 0$$

Suppose this line meets the line x + y + 1 = 0 at P such that AP = r. Then the coordinates of P are given by $\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$

$\Rightarrow$ x = 2 + r cos 45º, y = 3 + r sin 45º $\Rightarrow$ $x = 2 + \frac{r}{\sqrt{2}}$, $y = 3 + \frac{r}{\sqrt{2}}$

Thus, the coordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

Since P lies on x + y + 1 = 0,

so $\quad 2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$

$\Rightarrow \quad \sqrt{2}\, r = -6 \Rightarrow r = -3\sqrt{2}$

$\Rightarrow \quad$ length AP = | r | = $3\sqrt{2}$

Thus, the length of the intercept = $3\sqrt{2}$.

K. ANGLE BETWEEN TWO LINES

The angle θ between the lines having slopes $\mathbf{m_1}$ and $\mathbf{m_2}$ is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Solved Examples

Example.27 If A (– 2,1), B (2, 3) and C (– 2, – 4) are three points, find the angle between BA and BC.

Solution. Let m_1 and m_2 be the slopes of BA and BC respectively. Then, $m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$ and

$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$

Let θ be the angle between BA and BC. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right| = \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \pm\frac{2}{3} \qquad \theta = \tan^{-1}\left(\frac{2}{3}\right).$$

(a) Condition of Parallelism of Lines :

It two lines $\mathbf{y = m_1x + c_1}$ and $\mathbf{y = m_2x + c_2}$ of slopes $\mathbf{m_1}$ and $\mathbf{m_2}$ are parallel then the angle θ between them is of **0º**.

$\therefore \quad \tan\theta = \tan 0^\circ = 0 \Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = 0 \quad \Rightarrow \quad m_2 = m_1$

(b) Condition of Perpendicularity of Two Lines :

If two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ of slopes m_1 and m_2 are perpendicular, then the angle θ between them is of 90º.

$$\cot\theta = 0 \Rightarrow \frac{1 + m_1 m_2}{m_2 - m_1} = 0 \Rightarrow m_1 m_2 = -1$$

Thus when two lines are perpendicular, the product of their slopes is –1. If m is the slope of a line, then the slope of a line perpendicular to it is – (1/m).

Solved Examples

Example.28 A line passing through the points (a, 2a) and (– 2, 3) is perpendicular to the line 4x + 3y + 5 = 0, find the value of 'a'.

Solution. Let m_1 be slope of the line joining A (a, 2a) and B (–2, 3). Then $m_1 = \frac{2a-3}{a+2}$.

Let m_2 be slope of the line 4x + 3y + 5 = 0. Then, $m_2 = -\frac{4}{3}$.

Since the two lines are perpendicular, then $m_1m_2 = -1$.

$\Rightarrow \frac{2a-3}{a+2} \times -\frac{4}{3} = -1 \Rightarrow$ 8a – 12 = 3a + 6 $\Rightarrow$ a = $\frac{18}{5}$.

L. TRANSFORMATION OF GENERAL EQUATION IN DIFFERENT STANDARD FORMS

The general equation of a straight line is **Ax + By + C = 0** which can be transformed to various standard forms as discussed below :

(a) Transformation of Ax + By + C = 0 in the Slope Intercept Form (y = mx + c) :

$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$,

This is of the form y = mx + c, where $m = -\frac{A}{B}$, $c = -\frac{C}{B}$.

Thus, for the straight line Ax + By + C = 0, m = slope = $-\frac{A}{B} = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}$,

and Intercept on y-axis = $-\frac{C}{B} = -\frac{\text{Const. term}}{\text{Coeff. of } y}$.

(b) Transformation of Ax + By + C = 0 in Intercept Form $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$:

We have, Ax + By + C = 0 $\Rightarrow$ Ax + By = –C

$\Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1 \Rightarrow \frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1.$

(c) Transformation of Ax + By + C = 0 in the normal form (x cosa + ysina = p) :

We have Ax + By + C = 0 ....(i)

Let x cosα + y sinα – p = 0 ....(ii)

Equation (i) & (ii) represent the same straight line

$\therefore \frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p} \Rightarrow \sin\alpha = \frac{-Bp}{C}, \cos\alpha = \frac{-Ap}{C}$

$\Rightarrow p = \pm\frac{C}{\sqrt{A^2+B^2}}, \cos\alpha = \frac{-A}{\sqrt{A^2+B^2}}, \sin\alpha = \frac{-B}{\sqrt{A^2+B^2}}$

So, equation (ii) take the form

$\frac{-A}{\sqrt{A^2+B^2}}x + \frac{-B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$

This is the equation required normal form of the Ax + By + C = 0.

Solved Examples

Example.29 Transform the equation of the line $\sqrt{3}\,x + y + 8 = 0$ to (i) slope intercept form and find its slope and y - intercept (ii) intercept form and find intercepts on the coordinate axis (iii) normal form and find the inclination of the perpendicular segment from the origin on the line with the axis and its length.

Solution. **(i)** $\sqrt{3}\,x + y + 8 = 0 \quad \Rightarrow \quad y = -\sqrt{3}\,x - 8$

This is the slope intercept form of the given line.

$\therefore$ Slope $= -\sqrt{3}$, y - intercept = – 8.

(ii) $\sqrt{3}\,x + y + 8 = 0 \quad \Rightarrow \quad \sqrt{3}\,x + y = -8 \quad \Rightarrow \quad \dfrac{x}{-8/\sqrt{3}} + \dfrac{y}{-8} = 1$

So, x - intercept $= -\dfrac{8}{\sqrt{3}}$ and y - intercept = – 8.

(iii) $\sqrt{3}\,x + y + 8 = 0 \quad \Rightarrow \quad -\sqrt{3}\,x - y = 8$

$$\Rightarrow \quad -\frac{\sqrt{3}}{\sqrt{\left(\sqrt{3}\right)^2 + 1^2}}\,x - \frac{1}{\sqrt{\left(\sqrt{3}\right)^2 + 1^2}} = \frac{8}{\sqrt{\left(\sqrt{3}\right)^2 + 1^2}}$$

$$\Rightarrow \quad -\frac{\sqrt{3}}{2}x - \frac{1}{2}y = \frac{8}{2} \quad \Rightarrow \quad -\frac{\sqrt{3}}{2}x - \frac{1}{2}y = 4$$

This is the normal form of the line. So,

$\cos\alpha = -\dfrac{\sqrt{3}}{2}$, $\sin\alpha = -\dfrac{1}{2}$ and p = 4.

Since, $\sin\alpha$ and $\cos\alpha$ both are negative, therefore α is in the third quadrant and is equal to $\pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$. Hence, for a given line $\alpha = \dfrac{7\pi}{6}$ and p = 4.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Distance formula

A-1 Find the value of x, if the distance between the points (x, – 1) and (3, 2) is 5.

A-2 The abscissa of a point is twice its ordinate and the sum of the abscissa and the ordinate is – 6. What are the coordinates of the point?

A-3 If the point P(x, y) is equidistant from the points A (3, 6) and B (– 3, 4), prove that 3x + y – 5 = 0.

A-4 Find a point on X-axis which is equidistant from the points (7, 6) and (– 3, 4).

A-5 Prove that (2, – 2), (– 2, 1) and (5, 2) are the vertices of a right angled triangle. Find the area of the triangle and the length of the hypotenuse.

Section (B) : Section formula

B-1 If the mid point of the line-segment joining the points (– 7, 14) and (K, 4) is (a, b), where 2a + 3b = 5, find the value of K.

B-2 Determine the ratio in which the point (– 6, a) divides the join of A (– 3, – 1) and B (– 8, 9). Also find the value of a.

B-3 If A (– 2, – 1), B (a, 0), C (4, b) and D (1, 2) are the vertices of a parallelogram, find the values of a and b.

B-4 The line segment joining the points (3, – 4) and (1, 2) is trisected at the points P and Q. If the coordinates of P and Q are (p, – 2) and (5/3, q) respectively. Finds the value of p and q.

B-5 If A (4, – 8), B (3, 6) and C (5, – 4) are the vertices of a ΔABC, D is the mid-point of BC and P is point on AD joined such that $\frac{AP}{PD} = 2$, find the coordinates of P.

Section (C) : Centroid, circumcentre, orthocentre, incentre and excentre

C-1 If two vertices of a triangle are (3, 7) and (– 1, 5) and its centroid is (1, 3), find the coordinates of the third vertex.

C-2 Find the co-ordinates of (i) centroid (ii) in-centre of the triangle whose vertices are (0, 6), (8, 12) and (8, 0).

C-2. If the circumcentre of a triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and (2a, – 2a) ; then prove that the orthocentre lies on the line $(a - 1)^2 x - (a + 1)^2 y = 0$

C-3. If $\frac{2}{1!\ 9!} + \frac{2}{3!\ 7!} + \frac{1}{5!\ 5!} = \frac{2^m}{n!}$, then prove that orthocentre of the triangle having sides x – y + 1 = 0, x + y + 3 = 0 and 2x + 5y – 2 = 0 is (2m – 2n, m – n).

C-4. The mid points of the sides of a triangle are A (5, 0), B(5, 12) and C(0, 12).Then find the coordinates of the orthocentre of this triangle.

C-5. Let P = (–1, 0) Q = (0, 0) and R = $(3, 3\sqrt{3})$ be three points. Find the equation of the bisector of the $\angle PQR$

Section (D) : Area of triangle and quadrilateral

D-1 Prove that the points (a, 0), (0, b) and (1, 1) are collinear if, $\frac{1}{a}+\frac{1}{b}$ = 1.

D-2 The co-ordinates of two points A & B are (3, 4) and (5, – 2) respectively. Find the co-ordinate of point P if PA = PB, the area of ΔAPB = 10.

D-3 Four points A(6, 3), B(–3, 5), C (4, – 2) and D (x, 3x) are given in such a way that $\frac{\text{Area}(\Delta DBC)}{\text{Area}(\Delta ABC)}=\frac{1}{2}$, find x.

D-4 Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, – 1), (2, 1) and (0, 3).

D-5 (5, 10), (–15, 15) and (5,5) are the coordinates of vertices A, B and C respectively of ΔABC and P is a point on median AD such that AP : PD = 2 : 3. Find the Ratio of the areas of the triangle PBC and ABC

Section (E) : Straight line

E-1 Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.

E-2 Find slope, x-intercept & y-intercept of the line 2x – 3y + 5 = 0.

E-3 Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line 2x – 3y = 7.

E-4 One side of a rectangle lies along the line 4x + 7y + 5 = 0. Two of its vertices are (–3, 1) and (1, 1). Then find the equations of other sides.

E-5 Find the equation of the perpendicular bisector of the line joining the points (2, 1) and (4, 3).

OBJECTIVE QUESTIONS

Section (A) : Distance formula

A-1. Find the point P(x, y), if its distance from (–3, 0) & (3, 0) is 4 units individually :
(A) $\left(0,\sqrt{5}\right)$ (B) $\left(0,-\sqrt{5}\right)$ (C) $\left(0,-\sqrt{7}\right)$ (D) (1, 0)

A-2. The point on the y - axis which is equidistant from A (–5, –2) and B (3, 2) is :
(A) (– 4, 0) (B) (0, – 2) (C) (–2, 0) (D) (0, – 4)

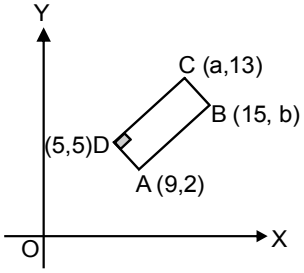
A-3. If two vertices of an isosceles triangle are (2, 0) and (2, 5) and length of the equal sides is 3, then the third vertex is :
(A) (2, 6) & (– 5, 3) (B) (8, 3) & (5, 1) (C) $(2 \pm \frac{\sqrt{11}}{2}, \frac{5}{2})$ (D) $(3 \pm \frac{\sqrt{14}}{2}, \frac{7}{2})$

A-4. If the point (0, 2) is equidistant from the points (3, k) and (k, 5), then the value of k is :
(A) 0 (B) 2 (C) – 2 (D) None of these

A-5. If the distance between the points (a, 2) and (3, 4) be 8 then a =
(A) $2+3\sqrt{15}$ (B) $2-3\sqrt{15}$ (C) $2\pm3\sqrt{15}$ (D) $3\pm2\sqrt{15}$

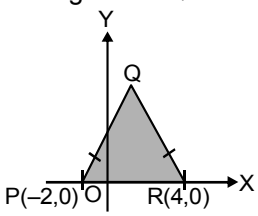
A-6. Value of 'a', when the distance between the points (3, a) and (4, 1) is $\sqrt{10}$ is :
(A) 4 or – 2 (B) 2 or 4 (C) 6 or 2 (D) None

A-7. In the rectangle shown, the value of a – b is :



(A) – 3 (B) – 1 (C) 3 (D) 1

A-8. In the diagram, PQR is an isosceles triangle and QR = 5 units.



The coordinates of Q are :
(A) (1,5) (B) (3,4) (C) (2,4) (D) (1,4)

A-9. The points A(–4, –1), B (–2, –4), C (4, 0) and D(2, 3) are the vertices of :
(A) Parallelogram (B) Rectangle (C) Rhombus (D) None of these

A-10. The three points (– 2, 2), (8, – 2) and (– 4, –3) are the vertices of :
(A) An isosceles triangle (B) An equilateral triangle
(C) A right angled triangle (D) None of these

Section (B) : Section formula

B-1. If the y – axis divides the line joined by (2, 4) & (–3, 5), then the ratio is :
(A) 2 : 3 (B) 2 : 5 (C) 3 : 2 (D) 5 : 2

B-2. The ratio in which the line segment joining (3, 4) and (– 2, – 1) is divided by the x-axis is :
(A) 3 : 2 (B) 4 : 1 (C) 4 : 3 (D) None of these

B-3. The co–ordinates of one end of a diameter of a circle are (5 , – 7) . If the co–ordinates of the centre be (7 , 3), the co–ordinates of the other end of the diameter are :
(A) (6 , – 2) (B) (9 , 13) (C) (– 2 , 6) (D) (13 , 9)

B-4. The point (11 , 10) divides the line segment joining the points (5 , – 2) and (9 , 6) in the ratio :
(A) 1 : 3 internally (B) 1 : 3 externally (C) 3 : 1 internally (D) 3 : 1 externally

B-5. If A & B are the points $(-3, 4)$ and $(2, 1)$, then the co–ordinates of the point C on produced AB such that $AC = 2BC$ are :

(A) (2, 4) (B) (3, 7) (C) (7, –2) (D) $\left(-\frac{1}{2}, \frac{5}{2}\right)$

B-6. If (3, – 4) and (– 6, 5) are the extremities of the diagonal of a parallelogram and (– 2, 1) is its third vertex, then its fourth vertex is :
(A) (– 1, 0) (B) (0, – 1) (C) (– 1, 1) (D) None of these.

B-7. The points which trisect the line segment joining the points (0, 0) and (9, 12) are :
(A) (3, 4) (B) (8, 6) (C) (6, 8) (D) (4, 0)

B-8. A (1,1) and B (2, –3) are two points and D is a point on AB produced such that AD = 3 AB. Then the co–ordinates of D is :
(A) (4, 11) (B) (4, – 11) (C) (– 2, 5) (D) (– 4, – 11)

B-9. Four points are on a line segment. If AB : BC = 1 : 2 and BC : CD = 8 : 5, then AB : BD equals :
(A) 4 : 13 (B) 1 : 13 (C) 1 : 7 (D) 3 : 13

B-10. If the line joining A (2, 3) and B(– 5, 7) is cut by x-axis at P, then AP : PB is :
(A) 3 : 7 (B) – 3 : 7 (C) 7 : 3 (D) 7 : – 3

Section (C) : Centroid, circumcentre, orthocentre incentre and excentre

C-1. If the coordinates of the centroid of a triangle are (1, 3) and two of its vertices are (– 7, 6) and (8, 5), then the third vertex of the triangle is :

(A) $\left(\frac{2}{3}, \frac{14}{3}\right)$ (B) $\left(-\frac{2}{3}, -\frac{14}{3}\right)$ (C) (2, – 2) (D) (– 2, 2)

C-2. If the centroid and circumcenter of a triangle are (3, 3) and (6, 2) respectively, then the orthocentre is :
(A) (–3, 5) (B) (–3, 1) (C) (3, –1) (D) (9, 5)

Section (D) : Area of triangle and quadrilateral

D-1. The points which are not collinear are :
(A) (0, 1), (8, 3) and (6, 7) (B) (4, 3), (5, 1) and (1, 9)
(C) (2, 5), (–1, 2) and (4, 7) (D) (–3, 2), (1, – 2) and (9, – 10)

D-2. Value of p, for which the points (– 5, 1), (1, p) and (4, – 2) are collinear is :
(A) 0 (B) 2 (C) – 1 (D) None of these

D-3. If the points A (6, 1), B (8, 2), C (9, 4) and D (P, 3) are the vertices of a parallelogram taken in order. Then the value of P is :
(A) 7 (B) 8 (C) 4 (D) 9

D-4. The area of the triangle whose vertices are (a,a), (a + 1, a + 1) and (a + 2, a) is :
(A) a^3 (B) 1 (C) 2a (D) $2^{1/2}$

D-5. Area of the triangle with vertices P(5, 2), Q(– 9, –3) and R(–3, –5) (in sq. units) is :
(A) 58 (B) 14 (C) 29 (D) None

D-6. Area of the quadrilateral (in sq. units) with vertices A(–1, 6), B(–3, –9), C(5, –8) and D(3, 9) is :
(A) 48 (B) 96 (C) 24 (D) None of these

D-7. If the points (k, 2 – 2k), (1 – k, 2k) and (–k –4, 6 – 2k) be collinear, the possible values of k are :
(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) 1 (D) –1

D-8. The points (– a, – b), (0, 0), (a, b) and (a^2, ab) are :
(A) Collinear (B) Vertices of a parallelogram
(C) Vertices of a rectangle (D) None of these.

D-9. The area of a triangle whose vertices are (a, c + a), (a, c) and (– a, c – a) are :
(A) a^2 (B) b^2 (C) c^2 (D) $a^2 + c^2$

D-10. Find the value for "x", so that the three points, (2, 7), (6, 1), (x, 0) are collinear.
(A) 7 (B) $4\frac{1}{2}$ (C) 10 (D) $6\frac{2}{3}$

Section (E) : Straight line

E-1. If the point (2, –3) lies on $kx^2 - 3y^2 + 2x + y - 2 = 0$, then k is equal to :
(A) $\frac{1}{7}$ (B) 16 (C) 7 (D) 12

E-2. Equation to the locus of a point equidistant from the points A (1, 3) and B (– 2, 1) is :
(A) 4x + 6y = 5 (B) 6x + 4y = 5 (C) – 4x + 6y = 8 (D) None

E-3. Equation of a line with slope – 6 and cutting off an intercept of 7 units on negative direction of y-axis is :
(A) 6x – 6y – 4 = 0 (B) 6x – y – 7 = 0 (C) 6x – y – 4 = 0 (D) 6x + y + 7 = 0

E-4. Equation of a line passing through (3, 2) and inclined at an angle of 120º with the positive direction of x-axis is :
(A) $\sqrt{3}x + y = 2 + 3\sqrt{3}$ (B) 2x – 3y – 4 = 0 (C) $\sqrt{3}x - y = 2 - 3\sqrt{3}$ (D) 3x – 2y + 4 = 0

E-5. Find the equation of the line which cuts off an intercept 4 on the positive direction of x-axis and an intercept 3 on the negative direction of y-axis .
(A) x + 3y + 3 = 0 (B) 3x – 4y = 12 (C) 4x +3 y + 1 = 0 (D) None

E-6. If the straight line, 2x + 3ay – 1 = 0 & 3x + 4y + 1= 0 are mutually perpendicular, then the value of 'a' will be
(A) 8/9 (B) – 1/2 (C) – 8/9 (D) 1/2

E-7. If the length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis, then the equation of the line is :
(A) $x + \sqrt{3}y - 14 = 0$ (B) $x - y - \sqrt{3} = 0$
(C) $y - \sqrt{3}x - 14 = 0$ (D) $\sqrt{3}x + \sqrt{3}y + 4 = 0$

E-8. Equation of the straight line that has y-intercept 4 and is parallel to the straight line 2x – 3y = 7 is :
(A) 2x + 3y + 12 = 0 (B) 2x – 3y + 16 = 0 (C) 2x – 3y + 12 = 0 (D) 2x – 3y – 12 = 0

E-9. A line passes through (2, 2) and is perpendicular to the line 3x + y = 3 . Its y – intercept is :
(A) 1/3 (B) 2/3 (C) 1 (D) 4/3

E-10. The point (7,2) and (–1,0) lie on a line.
(A) 7y = 3x – 7 (B) 4y = x +1 (C) y = 7x + 7 (D) x = 4y + 1

Exercise-2

OBJECTIVE QUESTIONS

1. If two opposite vertices of a square are (5, 4) and (1, –6), then the coordinates of its remaining two vertices is :
(A) (–2, 2) & (5, 3) (B) (8, –3) & (–2, 1) (C) (8, 6) & (3, 5) (D) (1, –3) & (2, 5)

2. If the middle points of the sides of a triangle be (–2, 3), (4, – 3) and (4, 5), then centroid of triangle is
(A) $\left(\frac{5}{3}, 2\right)$ (B) $\left(\frac{5}{6}, 1\right)$ (C) $\left(1, \frac{5}{6}\right)$ (D) $\left(2, \frac{5}{3}\right)$

3. If the vertices of a triangle be (2, 1), (5, 2) and (3, 4) then its circumcenter is :
(A) $\left(\frac{13}{2}, \frac{9}{2}\right)$ (B) $\left(\frac{13}{4}, \frac{9}{4}\right)$ (C) $\left(\frac{9}{4}, \frac{13}{4}\right)$ (D) None

4. The orthocentre of the triangle ABC is 'B' and the circumcentre is 'S' (a, b). If A is the origin then the co–ordinates of C are :
(A) (2a, 2b) (B) $\left(\frac{a}{2}, \frac{b}{2}\right)$ (C) $\left(\sqrt{a^2+b^2}, 0\right)$ (D) None

5. If P (1, 2), Q (4, 6), R (5, 7) and S (a, b) are the vertices of a parallelogram PQRS, then :
(A) a = 2, b = 4 (B) a = 3, b = 4 (C) a = 2, b = 3 (D) a = 3, b = 5

6. If A (2, 2), B (–4, –4), C(5, –8) are the vertices of any triangle, the length of median passes through C will be :
(A) $\sqrt{65}$ (B) $\sqrt{117}$ (C) $\sqrt{85}$ (D) $\sqrt{113}$

7. If the mid point of the line-segment joining the points (– 7, 14) and (K, 4) is (a, b), where 2a + 3b = 5, then the value of K is :
(A) K = 15 (B) K = – 7 (C) K = – 15 (D) K = 10

8. Value of m for which the point P (m, 6) divides the join of A (– 4, 3) and B (2, 8) is :
(A) 5 (B) $\frac{3}{2}$ (C) $-\frac{2}{5}$ (D) None

9. The incentre of the triangle formed by (0, 0), (5, 12), (16, 12) is :
(A) (7, 9) (B) (9, 7) (C) (–9, 7) (D) (–7, 9)

10. The incentre of the triangle with vertices (1, $\sqrt{3}$), (0, 0) and (2, 0) is :
(A) $\left(1, \frac{\sqrt{3}}{2}\right)$ (B) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ (C) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (D) $\left(1, \frac{1}{\sqrt{3}}\right)$

11. Orthocentre of triangle with vertices (0, 0), (3, 4) and (4, 0) is :

(A) $\left(-3, \frac{3}{4}\right)$ (B) (3, 12) (C) $\left(3, \frac{3}{4}\right)$ (D) (3, 9)

12. If four points A (6, 3), B (–3, 5), C (4, – 2) and D (x, 3x) are given in such a way that $\frac{\text{Area}(\Delta DBC)}{\text{Area}(\Delta ABC)} = \frac{1}{2}$, then the value of x is :

(A) $\frac{3}{8}$ or $\frac{-14}{8}$ (B) 2 or – 3 (C) $\frac{11}{8}$ or $\frac{-3}{8}$. (D) None of these.

13. If the co-ordinates of two points A and B are (3, 4) and (5, –2) respectively, then the co-ordinates of any point P if PA = PB and Area of Δ PAB = 10 is :

(A) (7, 2) or (1, 0) (B) (– 7, 2) or (3, 0) (C) (7, – 2) or (5, 0) (D) (7, –2) or (–1, 0)

14. A straight line through P (1, 2) is such that its intercept between the axis is bisected at P. Its equation is :

(A) x + 2y = 5 (B) x – y + 1 = 0 (C) x + y – 3 = 0 (D) 2x + y – 4 = 0

15. A line L is perpendicular to the line 5x – y = 1 and the area of the triangle formed by the line L and coordinate axes is 5. The equation of the line L is :

(A) x + 5y = 5 (B) x + 5y = $\pm 5\sqrt{2}$ (C) x – 5y = 5 (D) x – 5y = $5\sqrt{2}$

16. If PS is the median of the triangle with vertices P (2, 2), Q (6, –1) and R (7, 3). The equation of the line passing through (1, –1) and parallel to PS is :

(A) 2x – 9y – 7 = 0 (B) 2x – 9y – 11 = 0 (C) 2x + 9y – 11 = 0 (D) 2x + 9y + 7 = 0

17. Locus of a point such that the sum of its distance from the points (0, 2) and (0, – 2) is 6 :

(A) $4x^2 + 6y^2 = 5$ (B) $5x^2 + 9y^2 = 50$ (C) $9x^2 + 5y^2 = 45$ (D) None

18. Two opposite vertices of a rectangle are (1, 3) and (5, 1). If the other two vertices of the rectangle lie on the line y – x + λ = 0, then λ =

(A) – 1 (B) 2 (C) 1 (D) None of these

19. The points on the line x + y = 4 which lie at a unit distance from the line 4x + 3y = 10, are :

(A) (3, 1) (B) (7, 11) (C) (–7, 11) (D) (1, 3)

20. Which of the following pair(s) of straight lines are mutually perpendicular ?

(A) 3x – 4y + 12 = 0 and 4x + 3y – 6 = 0 (B) 4x + 5y – 8 = 0 and 3x – 2y + 4 = 0

(C) $\frac{x - x_1}{2} = \frac{y - y_1}{3}$ & $\frac{x - x_2}{3} = \frac{y - y_2}{-2}$ (D) $\frac{x - x_1}{5} = \frac{y - y_1}{-4}$ & $\frac{x - x_2}{4} = \frac{y - y_2}{5}$

21. Given the points A (0, 4) and B (0, –4), the equation of the locus of the point P (x, y) such that $|AP - BP| = 6$ is :

(A) $9x^2 - 7y^2 + 63 = 0$ (B) $9x^2 - 7y^2 - 63 = 0$

(C) $7x^2 - 9y^2 + 63 = 0$ (D) $7x^2 - 9y^2 - 63 = 0$

22. The line segment joining the points (3, – 4) and (1, 2) is trisected at the points P and Q. If the coordinates of P and Q are (p, – 2) and (5/3, q) respectively, then find the value of p and q.

(A) 0 (B) 1 (C) 2 (D) 3

23. If the three vertices of a parallelogram are (a + b, a – b), (2a + b, 2a – b) and (a – b, a + b), then find the coordinates of fourth vertex.
(A) (–a, a) (B) (–b, b) (C) (–a, b) (D) (–b, a)

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. Consider the point A(a, b+c), B(b, c+a) and C(c, a + b). The area of ΔABC is **[IJSO-2011]**
(A) $2(a^2 + b^2 + c^2)$ (B) $\frac{a^2+b^2+c^2}{6}$ (C) 2(ab + bc + ca) (D) none of these

2. The centre of a clock is taken as origin. At 4.30 pm, the equation of line along minute hand is x = 0. Therefore, at this instant the equation of line along the hour hand will be **[IJSO-2011]**
(A) x – y = 0 (B) x + y = 0 (C) $y = \sqrt{2x}$ (D) $y = \frac{x}{\sqrt{2}}$

3. Find the equation of the line parallel to 4x + 3y = 5 and having x-intercept (-3) **[IJSO-2012]**
(A) 3x + 4y + 12 = 0 (B) 3x + 4y = 12 (C) 4x + 3y –12 = 0 (D) 4x + 3y + 12 = 0

4. In the xy-plane let A be the point (5,0) and L be the line $y = \frac{x}{3}$. The number of points P on the line L such that triangle OAP is isosceles is (O being the origin) **[IJSO-2013]**
(A) 2 (B) 3 (C) 4 (D) 5

5. If A(p, q + r), B(q, r + p) and C(r, p + q) are points then area of triangle ABC **IJSO-2015]**
(A) $p^2+q^2+r^2$ (B)$(p + q + r)^2$ (C) $\frac{1}{2}$ (pq + qr + rp) (D) zero

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 x = 7 or x = – 1 **A-2** (– 4, – 2). **A-3** 3x + y – 5 = 0.

A-4 (3, 0). **A-5** $5\sqrt{2}$ unit, $\frac{25}{2}$ sq.unit

Section (B)

B-1 K = – 15. **B-2** 3 : 2 and a = 5. **B-3** a = 1 and b = 3

B-4 p = 7/3 and q = 0. **B-5** (4, – 2).

Section (C)

C-1 (1,– 3). **C-2** (i) $\left(\frac{16}{3}, 6\right)$. (ii) (5, 6)

C-5. (0,0) **C-6.** $\sqrt{3}$ x + y = 0

Section (D)

D-2 (7, 2) or (1, 0). **D-3** x = $\frac{11}{8}$ or x = $\frac{-3}{8}$. **D-4** $\frac{35}{2}$ sq. unit **D-5** $\frac{3}{5}$.

Section (E)

E-1 x + y = 7 and 4x + 3y = 24

E-2 slope = $\frac{2}{3}$,x-intercept = $-\frac{5}{2}$,y-intercept = $\frac{5}{3}$

E-3 2x – 3y + 12 = 0 **E-4** 7x – 4y + 25 = 0, 7x – 4y – 3 = 0, 4x + 7y = 11

E-5 x + y – 5 = 0

OBJECTIVE QUESTIONS

Section (A)

A-1. (C) **A-2.** (B) **A-3.** (C) **A-4.** (D) **A-5.** (D)

A-6. (A) **A-7.** (D) **A-8.** (D) **A-9.** (B) **A-10.** (C)

Section (B)

B-1. (A) **B-2.** (B) **B-3.** (B) **B-4.** (D) **B-5.** (C)

B-6. (A) **B-7.** (A, C) **B-8.** (B) **B-9.** (A) **B-10.** (B)

Section (C)

C-1. (C) **C-2.** (A)

Section (D)

D-1. (A) **D-2.** (C) **D-3.** (A) **D-4.** (B) **D-5.** (C)

D-6. (B) **D-7.** (B, D) **D-8.** (A) **D-9.** (A) **D-10.** (D)

Section (E) : Straight line

E-1. (C) **E-2.** (B) **E-3.** (D) **E-4.** (A) **E-5.** (B)

E-6. (B) **E-7.** (C) **E-8.** (C) **E-9.** (D) **E-10.** (B)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	B	D	B	A	C	C	C	C	A	D	C	C	A	D	B	D	C	C	AC	ACD
Ques.	21	22	23																	
Ans.	A	A	B																	

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5
Ans.	D	B	D	C	D

SETS

A. SETS

A well defined collection of objects is known as **sets**. If **a** is an element of a set **A**, then we write **a** $\in$ **A** and say **a** belongs to **A**. If **a** does not belong to **A**, then **a** $\notin$ **A** is written.

For Example : The collection of all states in the Indian union is a set but collection of good cricket players of India is not a set, since the term "good player is vague and it is not well defined.

Some letters are reserved for the sets as listed below :

N : For the set of Natural numbers.

Z : For the set of Integers.

Z^+ : For the set of all positive Integers.

Q : For the set of all Rational numbers.

Q^+ : For the set of all positive Rational numbers.

R : For the set of all Real numbers.

R^+ : For the set of all Positive real numbers.

C : For the set of all Complex numbers.

B. DESCRIPTION OF SETS

A set is often described in the following two ways :

(a) Roster Method :

In this method a set is described by listing elements, separated by commas, within braces **{ }**.

Solved Examples

Example.1
Write the set of vowels of English alphabet in roster form.
Sol. A = {a, e, i, o, u}.

Example.2
Write the set of even natural numbers in roster form.
Sol. B = {2,4,6,.....}.

Example.3
Write the set of all prime numbers less than 11 in roster form.
Sol. C = {2,3,5,7}.

Example.4
Write the set A = {x $\in$ z, $x^2 < 20$} in the roster from.
Sol. We observe that the squares of integers 0, ± 1, ± 2, ± 3, ± 4 are less than 20. Therefore, the set A in roster form is A = { – 4, – 3, – 2, –1, 0, 1, 2, 3, 4}.

NOTE : The order in which the element are written in a set makes no difference.

(b) Set Builder Method :

In this method, a set is described by a characterizing property **P(x)** of its elements **x**. In such a case the set is described by **{x : P(x) holds }** or, **{x | P(x) holds,}** which is read as 'the set of all **x** such that **P(x)** holds'. The symbol **'|'** or **':'** is read as **'such that'**.

Example.5

Write the set A = {0,1,4,9,16,.....} in set builder form.

Sol. A = $\{x^2 : x \in Z\}$.

Example.6

Write the set X = $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...\}$ in the set builder form.

Sol. We observe that the elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set builder form is X = $\left\{\frac{1}{n^2};\ n \in N\right\}$.

C. TYPES OF SETS

(a) Empty Set :

A set is said to be **empty** or **null** or **void** set if it has no element and it is denoted by **ϕ or { }**.

Solved Examples

Example.7

Write $\{x \in N : 5 < x < 6\}$ in roster form.

Sol. A = { }.

(b) Singleton Set :

A set consisting of a single element is called a **singleton set**.

Example.8

Write the set $\{x : x \in N \text{ and } x^2 = 9\}$ in roster form.

Sol. Let B is the set. So B is a singleton set equal to {3}.

(c) Finite Set :

A set is called a **finite set** if it is either void set or its element can be listed (counted labelled) by natural numbers **1, 2, 3 .....** and the process of listing terminates at a certain natural number **n** (say).

For Example : Set of all persons on the earth is a finite set.

(d) Infinite Set :

A set whose elements cannot be listed by natural numbers **1, 2, 3,.....** for any natural number **n** is called an **infinite set**.

For Example : Set of all points in a plane is an infinite set.

Solved Examples

Example.9

Which of the following sets are finite and which are infinite ?

(a) Set of concentric circle in a plane.
(b) Set of letters of English alphabets.
(c) $\{ x \in N, x > 5 \}$
(d) $\{ x \in R, 0 < x < 1 \}$
(e) $\{ x \in N, x < 200\}$

Sol. **(a)** Infinite set **(b)** Finite set **(c)** Infinite set **(d)** Infinite set
(e) Finite set

(e) Cardinal Number of a Finite Set :

The number **n** in the above definition is called the **cardinal number** or **order** of a finite set **A** and is denoted by **n(A)**.

(f) Equivalents Set :

Two finite sets **A** and **B** are **equivalent** if their cardinal numbers are same. i.e. **n(A) = n(B)**.
For Example : A = {1,2,3} and B = {a,b,c} are equivalent sets.

(g) Equal Set :

Two sets **A** and **B** are said to be equal if every element of **A** is a member of **B**, and every element of **B** is member of **A**.

NOTE : Equal sets are equivalents but equivalent sets need not be equal.

Solved Examples

Example. 10

Are the following sets equal ?
A = { x : x is a letter in the word reap }
B = { x : x is a letter in the word paper }.

Sol. A = { r, e, a, p}
B = { p, a, e, r }
So, A and B are equal sets.

(h) Subset :

Let **A** and **B** be two sets. If every element of **A** is an element of **B**, then **A** is called a subset of **B**. If **A** is a subset of **B**, we write $\mathbf{A} \subseteq \mathbf{B}$, which is read as **"A is a subset of B"** or **"A is contained in B"**. Thus, $\mathbf{A} \subseteq \mathbf{B}$ if $\mathbf{a} \in \mathbf{A} \Rightarrow \mathbf{a} \in \mathbf{B}$. The symbol **" $\Rightarrow$ "** stands for **"implies".** If **A** is not a subset of **B**, we write
$A \not\subset B$.

NOTE : The **subset** consisting of all elements of a given set is called **improper subset**. Every set is a subset of itself and it is consider to be an improper subset . A subsets **A** of a set **B** is called a **proper subse**t of B if $\mathbf{A} \neq \mathbf{B}$ and we write $\mathbf{A} \subset \mathbf{B}$. So empty set ($\phi$) is consider as a proper subset.

SOME RESULTS ON SUBSET :

(i) Every set is a subset of itself
(ii) The empty set is a subset of every set.
(iii) The total number of subsets of a finite set containing n element is $\mathbf{2^n}$.
(iv) The total number of proper subsets of a finite set containing n element is $\mathbf{2^n - 1}$.

(i) Universal Set :

A set that contains all sets in a given context is called the **Universal Set**.

Solved Examples

Example.11

If A = {1,2,3}, B = {2,4,5,6} and C = {1,3,5,7}, then find the universal set.

Sol. U = {1, 2, 3, 4, 5, 6, 7} can be taken as the universal set.

(j) Power Set

Let **A** be a set. Then the collection or family of all subsets of **A** is called the **power set** of **A** and is denoted by **P(A)**.

Example.12

Let A = {1,2,3}. Then find the power set of A.

Sol. Subset of A are : ϕ, {1}, {2}, {3}, {1,2}, {1,3}, {2,3} and {1,2,3}.
Hence, total number of subset are = $2^3 = 8$.
Hence, P(A) ={ ϕ, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }.

Example.13

Let A = {1,2,3,4}, B = {1,2,3} and C = {2,4}. Find sets X satisfying each pair of conditions.

(i) $X \subseteq B$ and $X \not\subset C$ **(ii)** $X \subseteq B$, $X \neq B$ and $X \not\subset C$

(iii) $X \subseteq A$, $X \subseteq B$ and $X \subseteq C$

Sol. We have

(i) $X \subseteq B$ and $X \not\subset C$

$\Rightarrow$ X is subset of B but X is not a subset of C

$\Rightarrow$ $X \in P(B)$ but $X \notin P(C)$

$\Rightarrow$ X = {1}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}.

(ii) We have,

$X \subseteq B$, $X \neq B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B other than B itself and X is not a subset of C

$\Rightarrow$ $X \in P(B)$, $X \notin P(C)$ and $X \neq B$

$\Rightarrow$ X = {1}, {3}, {1,2}, {1,3}, {2,3}.

(iii) We have,

$X \subseteq A$, $X \subseteq B$ and $X \subseteq C$

$\Rightarrow$ $X \in P(A)$, $X \in P(B)$ and $X \in X(C)$

$\Rightarrow$ X is a subset of A, B and C

$\Rightarrow$ X = ϕ, {2}

D. VENN DIAGRAM

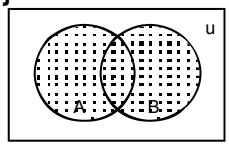
Diagram drawn to represent sets are called **Venn-Euler diagram** or simply **Venn diagram**. In Venn-diagram the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle.

E. OPERATION ON SETS

(a) Union of Sets :

Let **A** and **B** be two sets. The **union** of **A** and **B** is the set of all those elements which belong **either to A or to B or to both A and B**.

Thus, $\mathbf{A \cup B = \{ x : x \in A \text{ or } x \in B\}}$.



Solved Examples

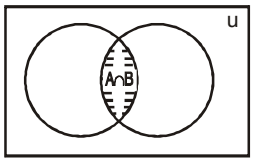
Example.14

If A {1, 2, 3} and B = {1, 3, 5, 7}, then find $A \cup B$.

Sol. $A \cup B$ = {1, 2, 3, 5, 7}.

(b) Intersection of Sets

Let **A** and **B** be two sets. The **intersection** of **A** and **B** is the set of all those elements that belong to **both A and B**.



Thus, $\mathbf{A \cap B = \{x : x \in A \text{ and } x \in B\}}$.

Solved Examples

Example.15

If A = { 1, 2, 3, 4, 5 } and B = { 1, 3, 9, 12 }, then find $A \cap B$.

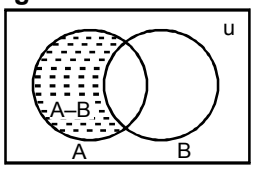
Sol. $A \cap B$ = { 1, 3 }.

(c) Disjoint Sets

Two sets **A** and **B** are said to be **disjoint**, if $\mathbf{A \cap B = \phi}$. If $\mathbf{A \cap B \neq \phi}$, then A and B are said to be intersecting or overlapping sets.

(d) Difference of Sets

Let **A** and **B** be two sets. The **difference** of **A and B**, written as **A – B**, is the set of all those elements of **A which do not belong to B**.

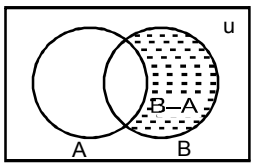


Thus, $\mathbf{A - B = \{x : x \in A \text{ and } x \notin B\}}$

or, $\mathbf{A - B = \{x \in A : x \notin B\}}$.

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$.

Similarly, the difference **B – A** is the set of all those elements of **B** that do not belong to **A** i.e. $\mathbf{B - A = \{x \in B : x \notin A\}}$.



Solved Examples

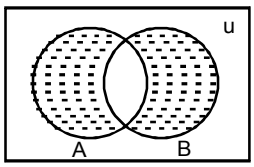
Example.16

If A = { 2, 3, 4, 5, 6, 7 } and B = { 3, 5, 7, 9, 11, 13 }, then find A – B and B – A.

Sol. A – B = { 2, 4, 6 } and B – A = { 9, 11, 13 }.

(e) Symmetric Difference of Two Sets :

Let **A** and **B** be two sets. The **symmetric difference** of sets **A** and **B** is the set **(A – B)** $\cup$ **(B – A)** and is denoted by **A** Δ **B**.



Solved Examples

Example.17

If $A = \{ x \in R : 0 < x < 3 \}$, $B = \{ x \in R : 1 \le x \le 5 \}$, then find $A \Delta B$.

Sol. $A - B = \{ x \in R : 0 < x < 1 \}$, $B - A = \{ x \in R : 3 \le x \le 5 \}$

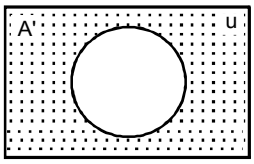
$A \Delta B = \{ x \in R : 0 < x < 1 \} \cup \{ x \in R : 3 \le x \le 5 \}$

$A \Delta B = \{ x \in R : 0 < x < 1 \text{ or } 3 \le x \le 5 \}$.

(f) Complement of a Set :

Let **U** be the universal set and let **A** be a set such that **A** $\subset$ **U**. Then, the **complement of A** with respect to **U** is denoted by **A'** or A^c or **U– A** and is defined the set of all those elements of **U** which are not in **A**.

Thus, $\mathbf{A' = \{x \in U : x \notin A\}}$.



Clearly, $\mathbf{x \in A' \Leftrightarrow x \notin A}$.

Solved Examples

Example.18

Let U = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = { 1, 2, 3, 4 }, B = { 2, 4, 6, 8 } and C = { 3, 4, 5, 6 }. Find

(i) A^c **(ii)** $(A \cap C)^c$ **(iii)** $(A \cup B)^c$ **(iv)** $(B - C)^c$

Sol. **(i)** $A^c = U - A = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} - \{ 1, 2, 3, 4 \} = \{ 5, 6, 7, 8, 9 \}$.

(ii) $(A \cap C)^c = U - (A \cap C)$

$= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} - \{ 3, 4 \}$

$= \{ 1, 2, 5, 6, 7, 8, 9 \}$.

(iii) $(A \cup B)^c = U - (A \cup B)$

$= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} - \{ 1, 2, 3, 4, 6, 8 \} = \{ 5, 7, 9 \}$.

(iv) $(B - C)^c = U - (B - C)$

$= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} - \{ 2, 8 \} = \{ 1, 3, 4, 5, 6, 7, 9 \}$.

E. LAWS OF ALGEBRA OF SETS

Some fundamental laws of algebra of sets are :

(a) Idempotent Laws :

For any set **A**, we have **(i) $A \cup A = A$ (ii) $A \cap A = A$**.

Proof : (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

(b) Identity Laws :

For any set **A**, we have **(i) $A \cup \phi = A$ (ii) $A \cap U = A$**.

i.e. ϕ and **U** are identity elements for union and intersection respectively.

Proof : (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

(c) Commutative Laws :

For any two set **A** and **B,** we have

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

i.e. **union and intersection are commutative**.

Proof : Two set X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$. Also, $X \subseteq Y$ if every element of X belongs to Y.

(i) Let x be an arbitrary element of $A \cup B$. Then

$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B \Rightarrow x \in B \text{ or } x \in A \Rightarrow x \in B \cup A$

$\therefore A \cup B \subseteq B \cup A$.

Similarly, $B \cup A \subseteq A \cup B$.

Hence, $A \cup B = B \cup A$.

(d) Associative Laws :

If **A**, **B** and **C** are any three sets, then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$)

i.e. **union and intersection are associative**.

Proof :

(i) Let x be an arbitrary element of $(A \cup B) \cup C$. Then, $x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B) \text{ or } x \in C$

$\Rightarrow$ $(x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Rightarrow$ $x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow$ $x \in A \text{ or } (x \in B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$.

(ii) Let x be an arbitrary element of $A \cap (B \cap C)$. Then,

$x \in (A \cap B) \cap C$

$\Rightarrow$ $x \in A \text{ and } x \in (B \cap C)$

$\Rightarrow$ $x \in A \text{ and } (x \in B \text{ and } x \in C)$

$\Rightarrow$ $(x \in A \text{ and } x \in B) \text{ and } x \in C$

$\Rightarrow$ $x \in (A \cap B) \text{ and } x \in C$

$\Rightarrow$ $x \in (A \cap B) \cap C$

$\therefore$ $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) Distributive Laws :

If **A**, **B** & **C** are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

i.e. union and intersection are distributive over intersection and union respectively.

Proof :

(i) Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Rightarrow$ $x \in A$ or ($x \in B$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$) [$\because$ 'or' is distributive over 'and']

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow$ $x \in ((A \cup B) \cap (A \cup C))$

$\therefore$ $A \cup (B \cap C) \subseteq (A \cap B) \cap (A \cup C)$.

Similarly, $(A \cup B) \cap (A \cap C) \subseteq A \cup (B \cap C)$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) Let x be an arbitrary element of $A \cap (B \cup C)$. Then,

$x \in A \cap (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cup C)$

$\Rightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

$\therefore$ $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Similarly, $(A \cap B) \cap (A \cap C) \subseteq A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(f) De - Morgans Laws :

If **A** and **B** are any two sets, then

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

Proof :

(i) Let x be an arbitrary element of $(A \cup B)'$. Then,

$x \in (A \cup B)'$

$\Rightarrow$ $x \notin (A \cup B)$

$\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A'$ and $x \in B'$

$\Rightarrow$ $x \in A' \cap B'$.

$\therefore$ $(A \cup B)' \subseteq A' \cap B'$.

Again, let y be an arbitrary element of $A' \cap B'$. Then,

$y \in A' \cap B'$

$\Rightarrow$ $y \in A'$ and $y \in B'$

$\Rightarrow$ $y \notin A$ and $y \notin B$

$\Rightarrow$ $y \notin A \cup B$.

$\Rightarrow$ $y \in (A \cup B)'$.

$\therefore$ $A' \cap B' \subseteq (A \cup B)'$.

Hence, $(A \cup B)' = A' \cap B'$

(ii) Let x be an arbitrary element of $(A \cap B)'$. Then,

$x \in (A \cap B)'$

$\Rightarrow$ $x \notin (A \cap B)$

$\Rightarrow$ $x \notin A$ or $x \notin B$

$\Rightarrow$ $x \in A'$ or $x \in B'$

$\Rightarrow$ $x \in A' \cup B'$.

$\Rightarrow$ $(A \cap B)' \subseteq A' \cup B'$.

Again, let y be an arbitrary element of $A' \cup B'$. Then,

$y \in (A' \cup B')$

$\Rightarrow$ $y \in A'$ or $y \in B'$
$\Rightarrow$ $y \notin A$ or $y \notin B$
$\Rightarrow$ $y \notin (A \cap B)$.
$\Rightarrow$ $y \in (A \cap B)'$.
$\therefore$ $A' \cup B' \subseteq (A \cap B)'$. Hence, $(A \cap B)' = A' \cup B'$.

Solved Examples

Example.19

Let A and B be sets, If $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, prove that A = B.

Sol. $A \cup X = B \cup X$ for some set X
$\Rightarrow$ $A \cap (A \cup X) = A \cap (B \cup X)$
$\Rightarrow$ $(A \cap A) \cup (A \cap X) = A \cap (B \cup X)$
$\Rightarrow$ $A = (A \cap B) \cup (A \cap X)$
$\Rightarrow$ $A = (A \cap B) \cup \phi$ [$\because A \cap X = \phi$ (given)]
$\Rightarrow$ $A = A \cap B$
$\Rightarrow$ $A \subset B$ .....(i)
Again, $A \cup X = B \cup X$
$\Rightarrow$ $B \cap (A \cup X) = B \cap (B \cup X)$
$\Rightarrow$ $(B \cap A) \cup (B \cap X) = (B \cap B) \cup (B \cap X)$
$\Rightarrow$ $(B \cap A) \cup (B \cap X) = B$
$\Rightarrow$ $(B \cap A) \cup \phi = B$ [$\because B \cap X = \phi$ (given)]
$\Rightarrow$ $B \cap A = B$
$\Rightarrow$ $B \subset A$ .....(ii)
From (i) and (ii) A = B.

Example.20

Using properties of sets, show that for any two sets A and B, $(A \cup B) \cap (A \cap B') = A$.

Sol. $(A \cup B) \cap (A \cap B') = ((A \cup B) \cap A) \cup ((A \cup B) \cap B')$
$= A \cup ((A \cup B) \cap B')$
$= A \cup ((A \cap B') \cup (B \cap B'))$
$= A \cup (A \cap B') = A$.

F. MORE RESULTS ON OPERATIONS ON SETS

If **A**, **B** and **C** are **finite sets** and **U** be the **finite universal set**, then

(i) $A - B = A \cap B' = A - (A \cap B)$
(ii) $A - B = A \Leftrightarrow A \cap B = \phi$
(iii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
(vi) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(vii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint non-void sets.
(viii) $n(A - B) = n(A) - n(A \cap B)$.

Solved Examples

Example. 21

In a group of 800 people, 550 can speak Hindi and 450 can speak English. How many can speak both Hindi and English ?

Sol. Let H denote the set of people speaking Hindi and E denote the set of people speaking English.
$n(H) = 550$, $n(E) = 450$ and $n(H \cup E) = 800$.
$n(H \cup E) = n(H) + n(E) - n(H \cap E)$
$\Rightarrow$ $n(H \cap E) = n(H) + n(E) - n(H \cup E)$
$\Rightarrow$ $n(H \cap E) = 550 + 450 - 800 = 200$.
Hence, 200 people can speak both Hindi and English.

Example.22

There are 40 students in a chemistry class and 60 students in a physics class. Find the number of students which are either in physics class or chemistry class in the following cases:

(i) the two classes meet at the same hour.

(ii) the two classes meet at different hours and 20 students are enrolled in both the subjects.

Sol. Let A be the set of students in chemistry class and B be the set of students in physics class.

It is given that n (A) = 40 and n(B) = 60.

(i) If two classes meet at the same hour, then there will not be a common student sitting in both the classes. Therefore, $n(A \cap B) = 0$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad n(A \cup B) = 40 + 60 - 0 = 100$

(ii) If two classes meet at different timings then there can be some student sitting in both the classes. Therefore, $n(A \cap B) = 20$

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 40 + 60 - 20 = 80.$

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Representation of sets, types of sets, subsets

A-1 Examine whether the following statements are true or false :

(i) $\{a, b\} \not\subseteq \{b, c, a\}$ **(ii)** $\{a, e\} \not\subseteq \{x : x \text{ is a vowel in the English alphabet}\}$

(iii) $\{1, 2, 3\} \subseteq \{1, 3, 5\}$ **(iv)** $\{a\} \in \{a, b, c\}$

A-2 Write the set of all the possible subsets (power set) of the set :

(i) $A = \{a, b\}$ **(ii)** $B = \{a, b, c\}$

A-3 Classify the following as a finite or infinite set :

(i) $A = \{x \in N : (x - 1)(x - 2) = 0\}$ **(ii)** $B = \{x \in N : x \text{ is odd}\}$

A-4. Describe the following set by set property method {0, 3, 8, 15, 24, 35}

A-5 Write the following set by roster method : The set of all natural numbers 'x' such that $4x + 9 < 50$.

Section (B) : Algebra of sets

B-1 Let A = {2, 4, 6, 8} and B = {6, 8, 10, 12} then find $A \cup B$

B-2 Let A = {1, 2, 3, 4, 5, 6}, B = {2, 4, 6, 8}. Find $A - B$ and $B - A$.

B-3 State true or false :

(i) $A \cup A' = \phi$ **(ii)** $\phi' \cap A = A$

B-4 If A = {1, 2, 3, 4, 5}, B = {1, 3, 5, 8}, C = {2, 5, 7, 8} verify that $A - (B \cup C) = (A - B) \cap (A - C)$.

B-5 Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {2, 4, 6, 8} and B = {2, 3, 5, 7}. Verify that $(A \cap B)' = A' \cup B'$.

Section (C) : Theorem on cardinal number

C-1 In a group of 40 students, 26 take tea, 18 take coffee and 8 take neither of the two. How many take both tea and coffee ?

C-2 In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

(i) How many drink tea and coffee both ?

(ii) How many drink coffee but not tea ?

Comprehension # 1 (C-3 to C-5)

In a group of 1000 people, there are 750 people, who can speak Hindi and 400 people, who can speak Bengali.

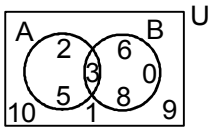
C-3. Find the number of people who can speak Hindi only.

C-4. Find the number of people who can speak Bengali only.

C-5. Find the number of people who can speak both Hindi and Bengali.

Section (D) : Venn diagram

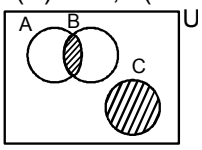
D-1 From the adjoining Venn diagram, determine the following sets :



(i) $A \cup B$ **(ii)** $A \cap C$ **(iii)** $A - B$ **(iv)** $(A \cap B)'$

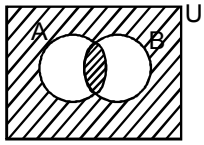
D-2 A and B are two sets such that $n(A - B) = 14 + x$, $n(B - A) = 3x$ and $n(A \cap B) = x$. Draw a Venn diagram to illustrate information and if $n(A) = n(B)$ then find the value of x.

D-3 Express the shaded region of the adjoining Venn diagram in terms of union and intersection of the sets A, B and C :
Also if $n(U) = 50$, $n(A) = 26$
$n(C) = 12$, $n(A \cap B) = 20$ and



$n(A' \cap B') = 27$, find $n(B)$ and $n(A' \cap B' \cap C')$

D-4 Represent the shaded part of the Venn diagram in terms of union and intersection of the sets A and B and their complements.



D-5 Draw the Venn diagram of $((A'\cap B')'\cap(A'\cup B')')'$ is where A' is compliment of A

OBJECTIVE QUESTIONS

Section (A) : Representation of sets, types of sets, subsets

A-1. If $A = \{x \in N \mid x^2 \le 36\}$, then A is equal to :
(A) {1, 2, 3, 4, 5, 6} (B) {6}
(C) {0, 1, 3, 4, 5, 6} (D) {– 6, – 5.... – 1, 0, 1, 2,...6}

A-2. The number of subsets of an empty set is :
(A) 0 (B) 1 (C) 2 (D) 3

A-3. Which of the following sets are disjoint ?
(A) The set of natural numbers and set of integers.
(B) The set of natural numbers divisible by 3 and set of natural numbers divisible by 7.
(C) The set of all multiples of 3 and set of all multiples of 5.
(D) The set of odd natural numbers and set of even natural numbers.

A-4. How many subsets can be formed from the set {p, q, r} is ?
(A) 6 (B) 7 (C) 8 (D) 9

A-5. Which one of the following statements is correct ?
(A) Finite sets are equal
(B) Two sets are disjoint, if there are no common elements in both the sets
(C) The set of integers is also called the set of natural number.
(D) A set having no element is called infinite set.

Section (B) : Algebra of sets

B-1. $A \cap B = A$ is true if :
(A) $A \subset B$ (B) $A \not\subset B$ (C) $B \subset A$ (D) $B = \phi$

B-2. A' is the complement of a set A in the universal set U. Then which of the following statements is not true ?
(A) $A \cup A' = U$ (B) $A \cap A' = \phi$
(C) If $A' = \phi$, then $A' \cap U = A$ (D) If $A' = \phi$, then $A' \cup U = A$

B-3. If A = {1, 2, 3,.....9} and B = {2, 3, 4, 5, 7, 8}, then A – B is given by :
(A) {1, 6, 7, 8} (B) {1, 6, 9} (C) {1, 9} (D) {6, 9}

B-4. Which one of the following statements is true ?
(A) $A \cup \phi = A$ (B) $A \cup \phi = \phi$ (C) $A \cup U = \phi$ (D) $A \cup \phi = U$

B-5. The set of all those elements of A and B which are common to both is called :
(A) Union of two sets (B) Intersection of two sets
(C) Disjoint sets (D) None of these.

Section (C) : Theorem on cardinal number

C-1. In a group of 500 people, 200 can speak Hindi alone while only 125 can speak English alone. The number of people who can speak both Hindi and English is :
(A) 175 (B) 325 (C) 300 (D) 375

C-2. A small town has a population 8000 out of which 3500 people read 'Times of India' and 3000 people read 'Indian Express' and 800 people read both. How many neither read both newspapers.
(A) 800 (B) 1500 (C) 2300 (D) 2850

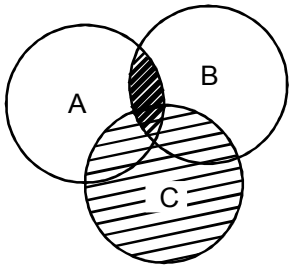
C-3. If n(A) = 115, n(B) = 326, n(A – B) = 47 then $n(A \cup B)$ is equal to :
(A) 373 (B) 165 (C) 370 (D) None

C-4. Let n(U) = 700, n(A) = 200, n(B) = 300 and $n(A \cap B) = 100$, then $n(A^c \cap B^c) =$
(A) 400 (B) 600 (C) 300 (D) 200

C-5. In a hotel, 60% had vegetarian lunch while 30% had non-vegetarian lunch and 15% had both types of lunch. If 96 people were present, how many did not eat either type of lunch ?
(A) 20 (B) 24 (C) 26 (D) 28

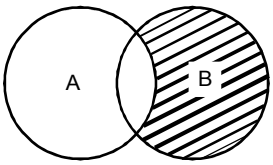
Section (D) : Venn diagram

D-1. The shaded region in the venn's diagram given below represents :



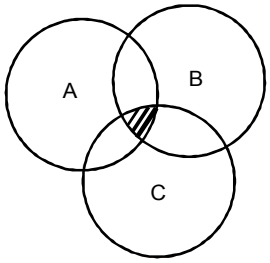
(A) $(A \cup B) \cap C$ (B) $(A \cap B) \cup C$ (C) $(A \cup B) \cup C$ (D) $(A \cap B) \cap C$

D-2. The shaded region in venn's diagram given below represents :



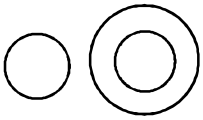
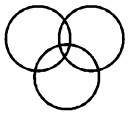
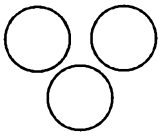
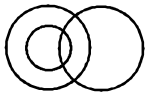
(A) A – B (B) B – A (C) A' – B (D) A – B'

D-3. The shaded region in venn's diagram given below represents :

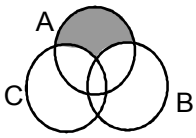


(A) $A \cap B \cap C$ (B) $A \cup B \cap C$ (C) $A \cup B \cup C$ (D) None of these

D-4. Which of the following venn-diagrams best represents the sets of females, mothers and doctors ?

(A)  (B)  (C)  (D) 

D-5. The shaded region in the given figure is



(A) $A \cap (B \cup C)$ (B) $A \cup (B \cap C)$ (C) $A \cap (B - C)$ (D) $A - (B \cup C)$

Exercise-2

OBJECTIVE QUESTIONS

1. If A is a void set then n[P(P(P(A)))] is :
(A) 0 (B) 1 (C) 3 (D) 4

2. If A be a finite set having n elements and P(A) is its power set, then total number of elements of P (P(A)) is:
(A) 2^n (B) 4^n (C) 2^{2^n} (D) 2^{2n}

3. If A = { ϕ, { ϕ }, 1, {1, ϕ }, 2 } then which of the following is true ?
(A) { 1 } A (B) $2 \subset A$
(C) { {2}, {1} } $\subset$ A (D) { ϕ, { ϕ }, {1, ϕ } } $\subset$ A

Directions : Q. 4 - 9

In a survey of 25 students, it was found that 15 had taken mathematics, 12 had taken physics and 11 had taken chemistry, 5 had taken mathematics and chemistry, 9 had taken mathematics and physics, 4 had taken physics and chemistry and 3 had taken all the three subjects.

4. Find the number of students that had taken only mathematics.
(A) 2 (B) 3 (C) 4 (D) 5

5. Find the number of students that had taken physics and chemistry but not mathematics.
(A) 1 (B) 2 (C) 3 (D) 10

6. Find the number of students that had taken at least one of the three subjects.
(A) 20 (B) 22 (C) 23 (D) 25

7. Find the number of students that had taken none of the subjects.
(A) 0 (B) 2 (C) 3 (D) 5

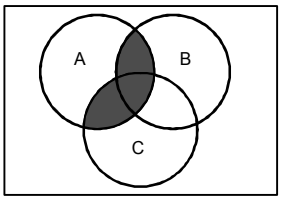
8. Find the number of students that had taken only one of the subjects.
(A) 10 (B) 11 (C) 12 (D) 15

9. Find the number of students that had taken exactly two of the three subjects.
(A) 9 (B) 10 (C) 11 (D) 12

10. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics and 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is :
(A) 20 (B) 21 (C) 23 (D) 22

11. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Of the total, 64 played both basketball and hockey ; 80 played cricket and basketball and 40 played cricket and hockey 24 player all the three games. The number of boys who did not play any game is
(A) 128 (B) 216 (C) 240 (D) 160

12. 3 sets A, B, C are represented by 3 circles in the given venn diagram. What does the shaded region represents :



(A) $(A \cup B) \cap C$ (B) $A \cup (B \cap C)$ (C) $(A \cap B) \cup C$ (D) $A \cap (B \cup C)$

13. Given K = { B, A, N, T, I }. Then the number of subsets of K, that contain both A, N is :
(A) 8 (B) 16 (C) 24 (D) 32

14. A = {1, 2, {3, 4}, {5}}. Which of the following is true ?
(A) $\{3, 4\} \subset A$ (B) $3 \in A$ (C) n(A) = 5 (D) $\{5\} \in A$

15. A = { 1, 2, {3, 4}}, number of proper subsets of A is :
(A) 16 (B) 15 (C) 3 (D) 7

16. The relation $A \cup B = A \cap B$ can be true ; when :
(A) A = B (B) $A = \phi$
(C) $A \subset B$ (D) A and B are disjoint sets

17. If U = {1, 2, 3, 4, 5, 6, 7}, A = {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} then A' – B is equal to :
(A) {1, 3, 5} (B) {4, 6} (C) {3, 4, 6} (D) {6}

18. A – B is equal to :
(A) $A \cap B$ (B) $A \cap B'$ (C) $A' \cap B'$ (D) $A' \cup B'$

19. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then
(A) A = C (B) B = C (C) $A \cap B = \phi$ (D) A = B

20. Let X = {1, 2, 3, 4, 5}. The number of different ordered pairs (Y, Z) that can formed such that $Y \subseteq X$, $Z \subseteq X$ and $Y \cap Z$ is empty, is :
(A) 5^2 (B) 3^5 (C) 2^5 (D) 5^3

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. If set X consists of three elements then the number of elements in the power set of power set of X is **[IJSO-2016]**
(A) 3^3 (B) 2^3 (C) 3^8 (D) 2^8

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 **(i)** False as {a, b} is subset of {b, c, a}
(ii) False as a, e are vowels
(iii) False as element 2 is not in the set {1, 3, 5}
(iv) False as a $\in$ {a, b, c} and {a} $\subseteq$ {a, b, c}

A-2 **(i)** P (A) = {ϕ, {a}, {b}, {a, b}} **(ii)** B = {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c}}

A-3 **(i)** Finite **(ii)** Infinite

A-4. $\{x : x = n^2 - 1, n \in N, n \le 6\}$ **A-5** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Section (B)

B-1 $A \cup B = \{2, 4, 6, 8, 10, 12\}$ **B-2** $A - B = \{1, 3, 5\}$, $B - A = \{8\}$

B-3 **(i)** false because $A \cup A' = U$ **(ii)** true as $\phi' \cap A = U \cap A = A$

B-4 {4} **B-5** $A' \cup B' = \{1, 3, 4, 5, 6, 7, 8, 9\}$

Section (C)

C-1 12 **C-2** **(i)** 16 **(ii)** 20 **C-3.** 600 **C-4.** 250

C-5. 150

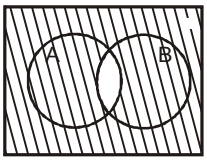
Section (D)

D-1 **(i)** $A \cup B = \{0, 2, 3, 5, 6, 8\}$ **(ii)** $A \cap C = \{3\}$
(iii) $A - B = \{2, 5\}$ **(iv)** $(A \cap B)' = \{0, 1, 2, 5, 6, 8, 9, 10, 12\}$

D-2 x = 7 **D-3** $n(B) = 7$, $n(A' \cap B' \cap C') = 15$

D-4 $(A \cup B)' \cup (A \cap B)$ or $(A - B)' \cap (B - A)'$

D-5



OBJECTIVE QUESTIONS

Section (A)

A-1. (A) **A-2.** (B) **A-3.** (D) **A-4.** (C) **A-5.** (B)

Section (B)

B-1. (A) **B-2.** (C) **B-3.** (B) **B-4.** (A) **B-5.** (B)

Section (C)

C-1. (A) **C-2.** (C) **C-3.** (A) **C-4.** (C) **C-5.** (B)

Section (D)

D-1. (B) **D-2.** (B) **D-3.** (A) **D-4.** (D) **D-5.** (D)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	C	D	C	A	C	B	B	A	D	D	D	A	D	D	A	D	B	B	B

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1
Ans.	D

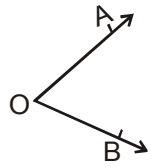
PLANE GEOMETRY - I

A. LINE

A line has length but no width and no thickness.

B. ANGLE

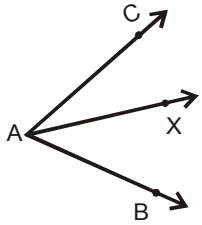
An angle is the union of two non-collinear rays with a common initial point. The common initial point is called the **'vertex'** of the angle and two rays are called the **'arms'** of the angles**.**



❖ **REMARK :**

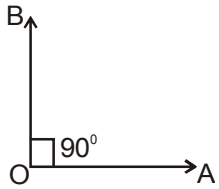
Every angle has a measure and unit of measurement is degree.
One right angle = 90°
1° = 60' (minutes)
1' = 60" (Seconds)

Angle addition axiom : If **X** is a point in the interior of ∠BAC, then m ∠BAC = m ∠BAX + m ∠XAC.

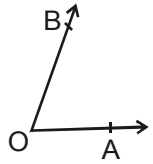


(a) Types of Angles :

(i) Right angle : An angle whose measure is 90° is called a **right angle**.

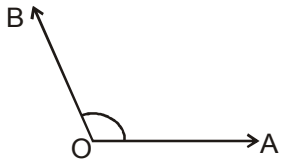


(ii) Acute angle : An angle whose measure is less than 90° is called an **acute angle**.



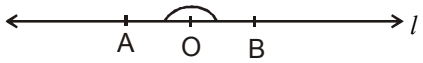
$0^0 < \angle BOA < 90^0$

(iii) Obtuse angle : An angle whose measure is more than 90° but less than 180° is called an **obtuse angle**.

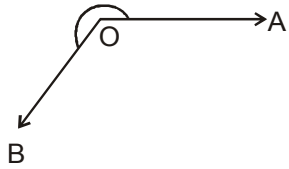


$90^0 < \angle AOB < 180^0$.

(iv) Straight angle : An angle whose measure is 180º is called a **straight angle**.

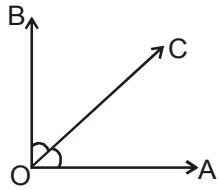


(v) Reflex angle : An angle whose measure is more than 180º is called a **reflex angle**.



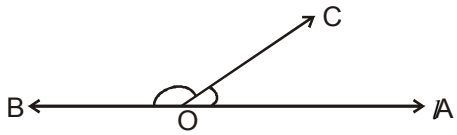
180º < ∠AOB < 360º.

(vi) Complementary angles : Two angles, the sum of whose measures is 90º are called **complementary angles**.



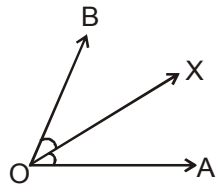
∠AOC & ∠BOC are complementary as their sum is 90°.

(vii) Supplementary angles : Two angles, the sum of whose measures is 180°, are called the **supplementary angles**.

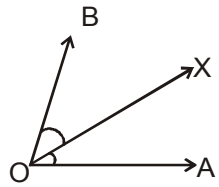


∠AOC & ∠BOC are supplementary as their sum is 180°.

(viii) Angle Bisectors : A ray OX is said to be the bisector of ∠AOB , if X is a point in the interior of ∠AOB, and ∠AOX = ∠BOX.

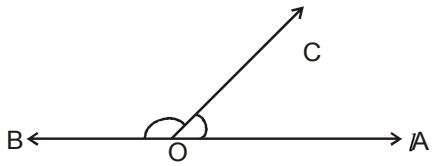


(ix) Adjacent angles : Two angles are called **adjacent angles**, if
(A) they have the same vertex,
(B) they have a common arm,
(C) non common arms are on either side of the common arm.



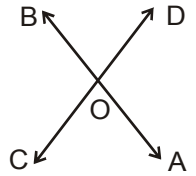
∠AOX and ∠BOX are adjacent angles, OX is common arm, OA and OB are non common arms and lies on either side of OX.

(x) Linear pair of angles : Two adjacent angles are said to form a linear pair of angles, if their non common arms are two opposite rays.



∠AOC + ∠BOC = 180º.

(xi) Vertically opposite angles : Two angles are called a pair of **vertically opposite angles**, if their arms form two pairs of opposite rays.

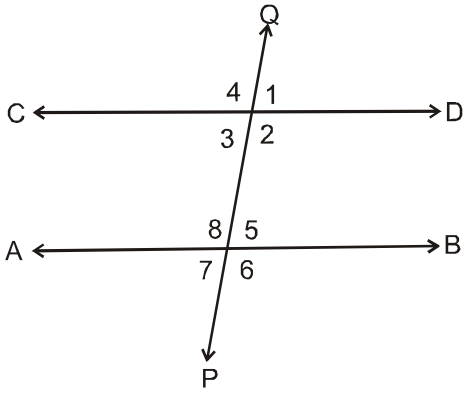


$\angle$ AOC & $\angle$ BOD form a pair of vertically opposite angles. Also $\angle$ AOD & $\angle$ BOC form a pair of vertically opposite angles.

If two lines intersect, then the vertically opposite angles are equal i.e. $\angle AOC = \angle BOD$ and $\angle BOC = \angle AOD$.

(b) Angles Made by a Transversal with two Parallel Lines :

Transversal : A line which intersects two or more given parallel lines at distinct points is called a **transversal** of the given lines.



(i) Corresponding angles : Two angles on the same side of a transversal are known as the **corresponding angles** if both lie either above the two lines or below the two lines, in figure $\angle 1$ & $\angle 5$, $\angle 4$ & $\angle 8$, $\angle 2$ & $\angle 6$, $\angle 3$ & $\angle 7$ are the pairs of corresponding angles.

If a transversal intersects two parallel lines then the corresponding angles are equal i.e. $\angle 1 = \angle 5$, $\angle 4 = \angle 8$, $\angle 2 = \angle 6$ and $\angle 3 = \angle 7$.

(ii) Alternate interior angles : $\angle 3$ & $\angle 5$, $\angle 2$ & $\angle 8$, are the pairs of **alternate interior angles**.

If a transversal intersects two parallel lines then the each pair of alternate interior angles are equal i.e. $\angle 3 = \angle 5$ and $\angle 2 = \angle 8$.

(iii) Co- interior angles : The pair of interior angles on the same side of the transversal are called pairs of **consecutive or co - interior angles**. In figure $\angle 2$ & $\angle 5$, $\angle 3$ & $\angle 8$, are the pairs of co-interior angles.

If a transversal intersects two parallel lines then each pair of consecutive interior angles are supplementary i.e. $\angle 2 + \angle 5 = 180^\circ$ and $\angle 3 + \angle 8 = 180^\circ$.

Solved Examples

Example.1.

Find the measure of an angle, if six times its complement is 12º less than twice its supplement.

Solution.

Let the measure of the required angle be xº.

Then, measure of its complement = (90 – x)º.

Measure of its supplement = (180 – x)º.

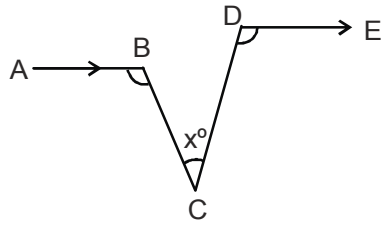
$\therefore$ $6(90 - x) = 2(180 - x) - 12$ $\Rightarrow$ $540 - 6x = 360 - 2x - 12$

$\Rightarrow$ $4x = 192$ $\Rightarrow$ $x = 48^\circ$.

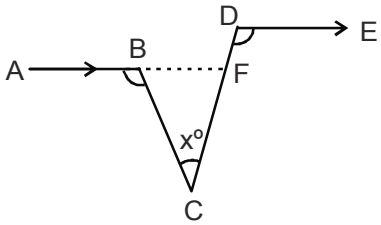
Hence, the required angle is 48º.

Example.2.

In the adjoining figure, ∠ABC = 100º, ∠EDC = 120º and AB || DE. Then, find ∠BCD .



Solution.



Produce AB to meet CD at F.

∠BFD = ∠EDF = 120° [alternate interior ∠s]

⇒ ∠BFC = (180° – 120°) = 60°

⇒ ∠CBF = (180° – 100°) = 80°

∴ ∠BCF = 180°– (60° + 80°) = 40°.

C. POLYGON

A closed plane figure bounded by line segments is called a **polygon**.

A polygon is named according to the number of sides it has :

No. of sides	3	4	5	6	7	8	10
Figure	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon	Octagon	Decagon

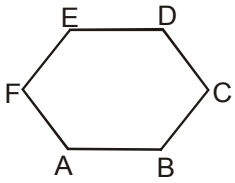
In general, a polygon having **n** sides is called 'n' sided polygon.

❖ **Diagonal of Polygon** :

Line segment joining any two non-consecutive vertices of a polygon is called its **diagona**l.

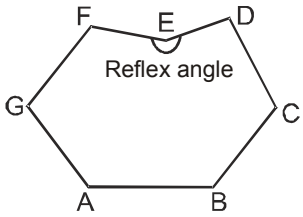
❖ **Convex Polygon** :

If all the interior angles of a polygon are less than 180^0, it is called a **convex polygon**.



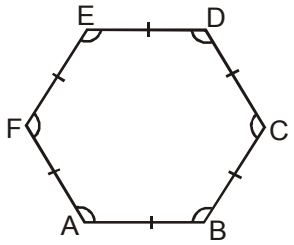
❖ **Concave Polygon :**

If one or more of the interior angles of a polygon is greater than 180^0 i.e. reflex, it is called a **concave polygon**.



❖ **Regular Polygon :**

A polygon is called a regular polygon if all its sides have equal length and all the angles have equal measure.



❖ **REMARKS :**

(1) The sum of the interior angles of a convex polygon of n sides is (2n – 4) right angles or (2n – 4) 90º.

(2) The sum of the **exterior angles** of a convex polygon is 4 right angles or 360º.

(3) Each interior angle of a n-sided regular polygon is $\frac{(2n-4)\times 90^\circ}{n}$

(4) Each exterior angle of a regular polygon of n sides = $\left(\frac{360^0}{n}\right)$

(5) If a polygon has n sides, then the number of diagonals of the polygon = $\frac{n(n-3)}{2}$.

Solved Examples

Example.3

If the sum of interior angles of a polygon is 1620º, find its number of sides.

Solution.

We know that sum of all interior angles of a n sided polygon = (2n – 4) right angles

= [(2n – 4) × 90º]

but given sum of interior angles = 1620º

$(2n - 4) \times 90^0 = 1620^\circ$

$2n - 4 = \frac{1620^\circ}{90^\circ}$

2n – 4 = 18

2n = 22

n = 11.

D. CLOCK TEST

❖ Minute hand and hour hand coincides once in every hour. They **coincide** 11 times in 12 hours and 22 times in 24 hours.

❖ They coincide only one time between 11 to 1 O' clock. at 12 O' clock.

❖ Minute hand and hour hand are **opposite** once in every hour. They do it 11 times in 12 hours and 22 times in 24 hours.

❖ They opposite only one time between 5 to 7 O' clock. at 6 O' clock.

❖ Both hands (minute and hour) are **perpendicular** twice in every hour. 22 times in 12 hours and 44 times in 24 hours.

❖ In one minute, hour hand moves 1/2º and minute hand moves 6º. In one hour, hour hand moves 30º and minute hand moves 360º.

❖ In an hour, minute hand moves 55 minutes ahead of hour hand.

Solved Examples

Example.4

At what time between 3 O'Clock and 4 O'Clock will the two hands coincide ?

Solution.

At 3 O'clock the distance between the two hands is 15 minutes when they coincide with each other the distance between the two hands will be 0 min.

So, the time taken (15 + 0) = 15 minutes.

Minute hand is 55 min. ahead of hour hand in 60 min.

$\therefore$ Minute hand is 1 min. ahead of hour hand in $\frac{60}{55}$ min.

$\therefore$ Minute hand is 15 min. ahead of hour hand in $\frac{60 \times 15}{55} = \frac{180}{11} = 16\frac{4}{11}$ min.

Hence the right time is $16\frac{4}{11}$ minute past 3.

Example.5

At what time between 2 O'clock and 3 O'clock will the two hands be opposite ?

Solution.

At 2 O'clock the distance between the two hands is 10 minutes. When they are at 30 minutes distance, they are opposite to each other. The time taken (30 + 10) = 40 min.

Minute hand is 55 min. ahead of hour hand in 60 min.

$\therefore$ Minute hand is 1 min. ahead of hour hand in $\frac{60}{55}$ min.

$\therefore$ Minute hand is 40 minutes ahead of hour hand in $\frac{60 \times 40}{55} = \frac{480}{11} = 43\frac{7}{11}$ min.

Hence, the right time is $43\frac{7}{11}$ min. past 2.

Example.6

At what time between 4 O'clock and 5 O'clock will the hands are perpendicular?

Solution.

At 4 O'clock the distance between the two hands is 20 min. When they are at 15 minutes distance, they are perpendicular to each other.

Case-I When the time taken (20 – 15) = 5 min.

Minute hand is 55 min. ahead of hour hand in 60 min.

$\therefore$ Minute hand is 5 min. ahead of hour hand in $\frac{60\times5}{55} = \frac{60}{11} = 5\frac{5}{11}$ min.

Hence, the right time is $5\frac{5}{11}$ min. past 4.

Case-II When the time taken (20 + 15) = 35 min.

Minute hand is 55 min. ahead of hour hand in 60 min.

$\therefore$ Minute hand is 35 min. ahead of hour hand in $\frac{60\times35}{55} = \frac{420}{11} = 38\frac{2}{11}$ min.

Hence, the right time is $38\frac{2}{11}$ min. past 4.

E. MIRROR IMAGE OF CLOCK

- If the time is between 1 O'clock to 11 O'clock, then to find the mirror image, time is subtracted from 11 : 60.
- If the time is between 11 O'clock to 1 O'clock, then to find the mirror image, time is subtracted from 23 : 60.

Example.7

The time in the clock is 4 : 46, what is the mirror image ?

Solution.

(11 : 60) – (4 : 46) = 7 : 14.

Example.8

The time in the clock is 12 : 35, then find its mirror image.

Solution.

(23 : 60) – (12 : 35) = 11 : 25.

F. ANGLE BETWEEN TWO HANDS

- **Angle are of two types :**

  Positive angle : It is obtained by moving from hour hand to minute hand moving in clockwise direction.

  Negative angle : It is obtained by moving from minute hand to hour hand.
- Both types of angles are 360º in total. If one angle is known, other can be obtained by subtracting from 360º.

Solved Examples

Example.9 At 4 : 30, what is the angle formed between hour hand and minute hand ?

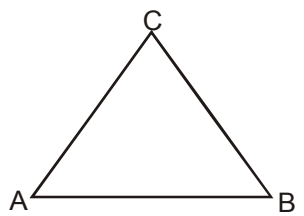
Solution. At 4 O' clock angle between hour and min. hand is of 120º.

∵ In 30 min. minute hand make an angle of 180º.

So, the resultant angle is 180º – 120º = 60º.
But in 30 min. hour hand will also cover an angle of 15º.
Hence, the final angle between both hands is 60º – 15º = 45º.

G. TRIANGLE

A plane figure bounded by three lines in a plane is called a **triangle**. Every triangle have three sides and three angles. If ABC is any triangle then AB, BC & CA are three sides and ∠A, ∠B and ∠C are three angles.



- **Types of triangles :**

(a) On the basis of sides we have three types of triangle.

1. **Scalene triangle** – A triangle in which no two sides are equal is called a **scalene triangle**.
2. **Isosceles triangle** – A triangle having two sides equal is called an **isosceles triangle**.
3. **Equilateral triangle** – A triangle in which all sides are equal is called an **equilateral triangle**.

(b) On the basis of angles we have three types :

1. **Right triangle** – A triangle in which any one angle is right angle is called **right triangle**.
2. **Acute triangle** – A triangle in which all angles are acute is called an **acute triangle**.
3. **Obtuse triangle** – A triangle in which any one angle is obtuse is called an **obtuse triangle**.

❖ **SOME IMPORTANT THEOREMS :**

Theorem : The sum of interior angles of a triangle is 180º.

Theorem : If the bisectors of angles $\angle ABC$ and $\angle ACB$ of a triangle ABC meet at a point O, then $\angle BOC = 90^\circ + \frac{1}{2}\angle A$.

Exterior Angle of a Triangle :

If the side of the triangle is produced, the exterior angle so formed is equal to the sum of two interior opposite angles.

Corollary : An exterior angle of a triangle is greater than either of the interior opposite angles.

Theorem : The sides AB and AC of a ΔABC are produced to P and Q respectively. If the bisectors of $\angle PBC$ and $\angle QCB$ intersect at O, then $\angle BOC = 90^\circ - \frac{1}{2}\angle A$.

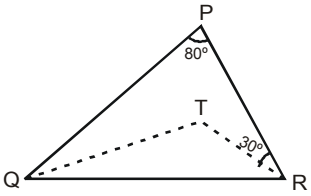
Solved Examples

Example.10

In figure, TQ and TR are the bisectors of $\angle Q$ and $\angle R$ respectively. If $\angle QPR = 80^\circ$ and $\angle PRT = 30^\circ$, determine $\angle TQR$ and $\angle QTR$.

Solution.

Since the bisectors of $\angle Q$ and $\angle R$ meet at T.



$\therefore \quad \angle QTR = 90^\circ + \frac{1}{2}\angle QPR \quad \Rightarrow \quad \angle QTR = 90^\circ + \frac{1}{2}(80^\circ)$

$\Rightarrow \quad \angle QTR = 90^\circ + 40^\circ = 130^\circ$

In ΔQTR, we have

$\angle TQR + \angle QTR + \angle TRQ = 180^\circ$

$\Rightarrow \quad TQR + 130^\circ + 30^\circ = 180^\circ \quad [\because \angle TRQ = \angle PRT = 30^\circ]$

$\Rightarrow \quad \angle TQR = 20^\circ$. Thus, $\angle TQR = 20^\circ$ and $\angle QTR = 130^\circ$.

H. CONGRUENT FIGURES

The figures are called congruent if they have same shape and same size. In other words, two figures are called congruent if they are having equal length, width and height.

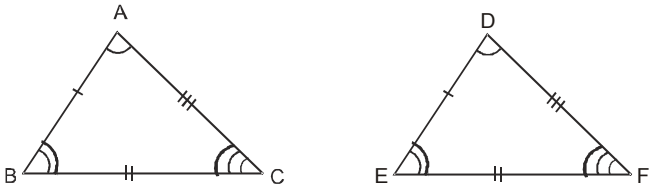
Fig.(i)

Fig.(ii)

In the above figures {fig.(i) and fig.(ii)} both are equal in length, width and height, so these are congruent figures.

(a) Congruent Triangles :

Two triangles are congruent if and only if one of them can be made to superimposed on the other, so as to cover it exactly.



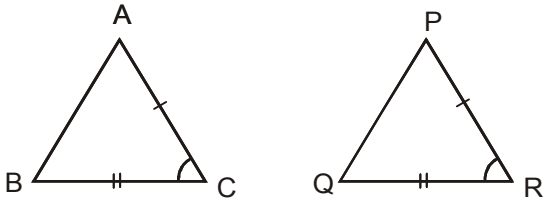
If two triangles ΔABC and ΔDEF are congruent then $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and AB = DE, BC = EF, AC = DF.

If two ΔABC & ΔDEF are congruent then we write $\Delta ABC \cong \Delta DEF$, we can not write as $\Delta ABC \cong \Delta DFE$ or $\Delta ABC \cong \Delta EDF$.

Hence, we can say that "two triangles are congruent if and only if there exists a one-one correspondence between their vertices such that the corresponding sides and the corresponding angles of the two triangles are equal.

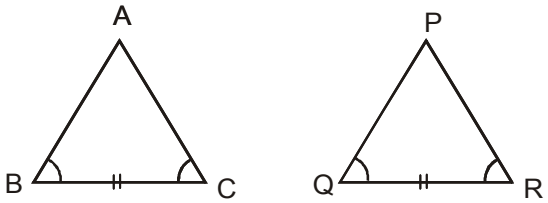
(b) Sufficient Conditions for Congruence of two Triangles :

(i) SAS Congruence Criterion :



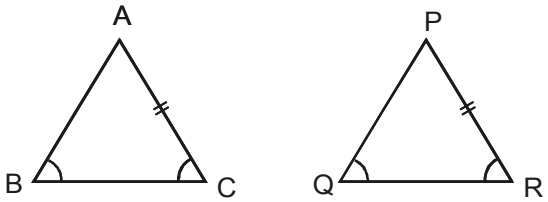
Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

(ii) ASA Congruence Criterion :



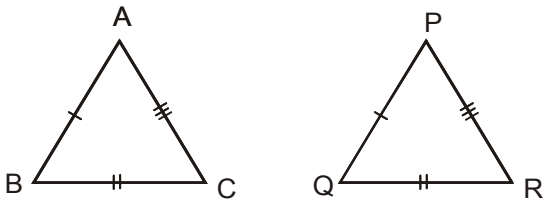
Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

(iii) AAS Congruence Criterion :



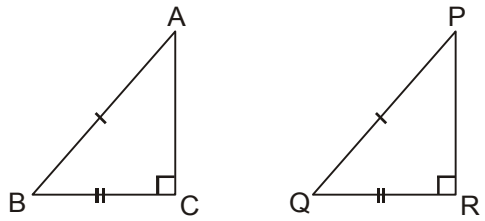
If any two angles and a non included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

(iv) SSS Congruence Criterion :



Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

(v) RHS Congruence Criterion :



Two right triangles are congruent if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other triangle.

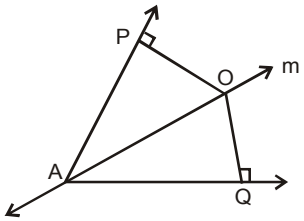
❖ **NOTE :**

If two triangles are congruent then their corresponding sides and angles are also congruent by **CPCT (corresponding parts of congruent triangles are also congruent).**

Theorem : If the bisector of the vertical angle bisects the base of the triangle, then the triangle is isosceles.

Solved Examples

Example.11 In figure, line m is the bisector of an angle $\angle A$ and O is any point on m. OQ and OP are perpendiculars from O to the arms of $\angle A$. Prove that :



(i) $\Delta AOQ \cong \Delta AOP$ **(ii)** OQ = OP.

Solution. **(i)** In ΔAOQ and ΔAOP,

$\angle OAQ = \angle OAP$ [$\because$ line m is the bisector of $\angle A$]

$\angle AQO = \angle APO = 90^o$ [Given]

and AO = AO [Common]

By AAS congruency

$\therefore \Delta AOQ \cong \Delta AOP$

(ii) OQ = OP [C.P.C.T.]

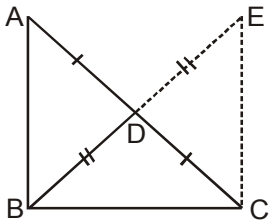
Example.12 If D is the mid-point of the hypotenuse AC of a right triangle ABC, prove that BD = $\frac{1}{2}$ AC.

Solution. **Given :** ΔABC is a right triangle such that $\angle B = 90^0$ and D is mid point of AC.

To prove : BD = $\frac{1}{2}$ AC.

Construction : Produce BD to E such that BD = DE and join EC.

Proof :



In ΔADB and ΔCDE

AD = DC [Given]

BD = DE [By construction]

And, $\angle ADB = \angle CDE$ [Vertically opposite angles]

By SAS criterion of congruence we have

$\Delta ADB \cong \Delta CDE$

EC = AB and $\angle CED = \angle ABD$ ... (i) [By CPCT]]

But $\angle CED$ & $\angle ABD$ are alternate interior angles

CE || AB

$\angle ABC + \angle ECB = 180^0$ [Consecutive interior angles]

$90 + \angle ECB = 180^0$ $\angle ECB = 90^0$.

Now, In ΔABC & ΔECB we have

AB = EC [By (i)]

BC = BC [Common]

And, $\angle ABC = \angle ECB = 90^0$

By SAS criterion of congruence

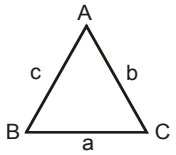
$\Delta ABC \cong \Delta ECB$

$\Rightarrow$ AC = EB [By CPCT]

$\Rightarrow$ $\frac{1}{2}AC = EB$ $\Rightarrow$ $BD = \frac{1}{2}AC$. **Hence Proved.**

I. SOME INEQUALITY RELATIONS IN A TRIANGLE

In triangle ABC, AB = c, BC = a & CA = b then,

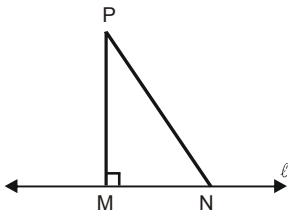


(i) The sum of any two sides of a triangle is greater than its third side. i.e. in ΔABC,

(A) $a + b > c$ (B) $b + c > a$ (C) $a + c > b$

(ii) If two sides of a triangle are unequal, then the longer side has greater angle opposite to it i.e. in ΔABC, if AB > AC then $\angle$**C** > $\angle$**B**.

(iii) Of all the line segments that can be drawn to a given line, from a point, not lying on it, the perpendicular line segment is the shortest. i.e. in ΔPMN,



(A) PM < PN.

(iv) The difference between any two sides of a triangle is less than its third side. i.e. in ΔABC,

(A) $a - b < c$ (B) $b - c < a$ (C) $a - c < b$

(v) In a right angle triangle the sum of squares of two smaller sides is equal to the square of its third side.i.e. in ΔABC, $a^2 + b^2 = c^2$.

(vi) If sum of squares of two smaller sides is greater than the square of its third side then that triangle is acute angled triangle. i.e in ΔABC, $a^2 + b^2 > c^2$.

(vii) If sum of squares of two smaller sides is lesser than the square of its third side then that triangle is obtuse angled triangle. i.e in ΔABC, $a^2 + b^2 < c^2$.

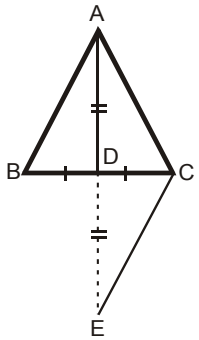
Solved Examples

Example.13 From which given triplet we can make the sides a triangle.
(i) {15, 7, 8} **(ii)** {3.5, 4.5, 5.5}

Solution. As we know sum of two sides is always greater then third side.
(i) we can not make the triangle because
here the sum of two side is equal to third side
i.e 7 + 8 = 15
(ii) We can make the triangle because sum of two sides is always greater then third side
3.5 + 4.5 >5.5
3.5 + 5.5 >4.5
4.5 + 5.5 >3.5

Example.14 Prove that any two sides of the triangle are together greater than twice the median drawn to the third side.

Solution. **Given :** ΔABC and AD is the median.
To prove : AB + AC > 2AD
Construction : Produce AD to E such that AD = DE. Join EC.
Proof : In ΔADB and ΔCDE



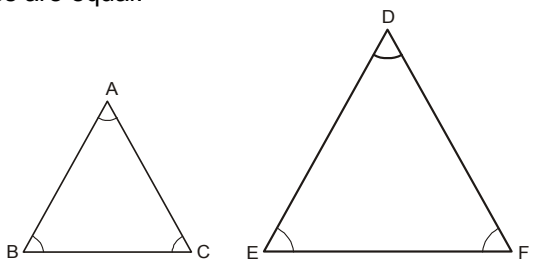
AD = DE	[By construction]
BD = DC	[AD is the median]
$\angle$ ADB = $\angle$ CDE	[Vertically opposite angles]
$\Delta ADB \cong \Delta CDE$	[By SAS congruency]
So, by CPCT	
AB = EC	
In ΔAEC	
AC + EC > 2AD.	[Sum of two sides of a triangle is always greater than the third side]
So, AC + AB > 2AD.	[As EC = AB].

J. SIMILAR FIGURES

Two geometric figures having the same shape and size are known as **congruent figures**. Geometric figures having the same shape but different sizes are known as **similar figures**.

K. SIMILAR TRIANGLES

Two triangles **ABC** and **DEF** are said to be similar if their
(i) Corresponding angles are equal.



i.e. $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

(ii) Corresponding sides are proportional.

i.e. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

(a) Characteristic Properties of Similar Triangles :

(i) (AAA Similarity) If two triangles are equiangular, then they are similar.

(ii) (SSS Similarity) If the corresponding sides of two triangles are proportional, then they are similar.

(iii) (SAS Similarity) If in two triangles, two pairs of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(b) Results Based Upon Characteristic Properties of Similar Triangles :

(i) If two triangles are equiangular, then the ratio of the corresponding sides is the same as the ratio of the corresponding medians.

(ii) If two triangles are equiangular, then the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.

(iii) If two triangles are equiangular then the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.

(iv) If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite side in the same ratio, then the triangles are similar.

(v) If two sides and a median bisecting one of these sides of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then the triangles are similar.

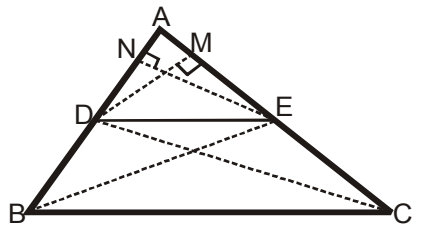
(vi) If two sides and a median bisecting the third side of a triangle are respectively proportional to the corresponding sides and the median of another triangle, then two triangles are similar.

NOTE : If the ratio of sides of triangle is a : b : c, then ratio of their altitudes is $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

L. THALES THEOREM (BASIC PROPORTIONALITY THEOREM)

Statement : If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

Given : A triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively.



To Prove : $\frac{AD}{DB} = \frac{AE}{EC}$.

Construction : Join BE and CD and draw DM $\perp$ AC and EN $\perp$ AB.

Proof : Area of Δ ADE $= \frac{1}{2}$ base × height $= \frac{1}{2}$ AD × EN.

Area of Δ ADE is denoted as ar(ADE)

So, ar(ADE) $= \frac{1}{2}$ AD × EN

And $\quad ar(BDE) = \frac{1}{2}\ DB \times EN,$

Therefore, $\frac{ar(ADE)}{ar(BDE)} = \frac{\frac{1}{2}AD\times EN}{\frac{1}{2}DB\times EN} = \frac{AD}{DB}$ ... (i)

Similarly, $ar(ADE) = \frac{1}{2}\ AE \times DM$ and $ar(DEC) = \frac{1}{2}\ EC \times DM.$

And $\quad \frac{ar(ADE)}{ar(DEC)} = \frac{\frac{1}{2}AE\times DM}{\frac{1}{2}EC\times DM} = \frac{AE}{EC}$ ... (ii)

Note that Δ BDE and Δ DEC are on the same base DE and between the two parallel lines BC and DE.

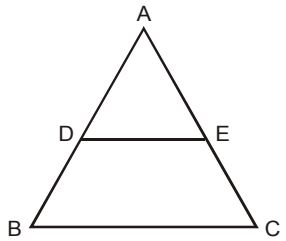
So, $\quad ar(BDE) = ar(DEC)$ ... (iii)

Therefore, from (i), (ii) and (iii), we have :

$\frac{AD}{DB} = \frac{AE}{EC}$ **Hence Proved.**

COROLLARY

If in a ΔABC, a line DE || BC, intersects AB in D and AC in E, then



(i) $\frac{DB}{AD} = \frac{EC}{AE}$ **(ii)** $\frac{AB}{AD} = \frac{AC}{AE}$ **(iii)** $\frac{AD}{AB} = \frac{AE}{AC}$

(iv) $\frac{AB}{DB} = \frac{AC}{EC}$ **(v)** $\frac{DB}{AB} = \frac{EC}{AC}$

(a) Converse of Basic Proportionality Theorem :

If a line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

(b) Some Important Results and Theorems :

(i) The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(ii) In a triangle **ABC**, if **D** is a point on **BC** such that **D** divides **BC** in the ratio **AB : AC**, then **AD** is the bisector of $\angle$**A**.

(iii) The external bisector of an angle of a triangle divides the opposite sides externally in the ratio of the sides containing the angle.

(iv) The line drawn from the mid-point of one side of a triangle parallel to another side bisects the third side.

(v) The line joining the mid-points of two sides of a triangle is parallel to the third side.

(vi) The diagonals of a trapezium divide each other proportionally.

(vii) If the diagonals of a quadrilateral divide each other proportionally, then it is a trapezium.

(viii) Any line parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.

(ix) If three or more parallel lines are intersected by two transversals, then the intercepts made by them on the transversals are proportional.

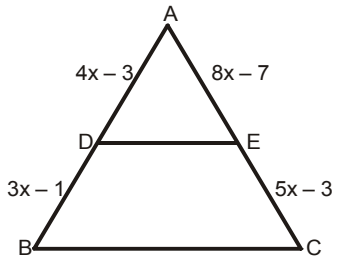
Solved Examples

Example.15 In a ΔABC, D and E are points on the sides AB and AC respectively such that DE || BC. If AD = 4x – 3, AE = 8x – 7, BD = 3x – 1 and CE = 5x – 3, find the value of x.

Solution. In ΔABC, we have

DE || BC

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$ [By Basic Proportionality Theorem]



$\Rightarrow \frac{4x-3}{3x-1} = \frac{8x-7}{5x-3}$

$\Rightarrow 20x^2 - 15x - 12x + 9 = 24x^2 - 21x - 8x + 7$

$\Rightarrow 20x^2 - 27x + 9 = 24x^2 - 29x + 7$

$\Rightarrow 4x^2 - 2x - 2 = 0$

$\Rightarrow 2x^2 - x - 1 = 0$

$\Rightarrow (2x + 1)(x - 1) = 0$

$\Rightarrow x = 1$ or $x = -\frac{1}{2}$

So, the required value of x is 1. [$x = -\frac{1}{2}$ is neglected as length can not be negative].

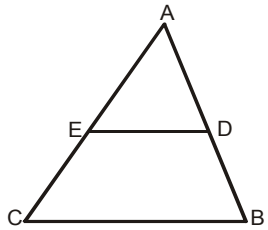
Example.16 D and E are respectively the points on the sides AB and AC of a ΔABC such that AB = 12 cm, AD = 8 cm, AE = 12 cm and AC = 18 cm, show that DE || BC.

Solution. We have,

AB = 12 cm, AC = 18 cm, AD = 8 cm and AE = 12 cm.

$\therefore$ BD = AB – AD = (12 - 8) cm = 4 cm

CE = AC - AE = (18 – 12) cm = 6 cm



Now, $\frac{AD}{BD} = \frac{8}{4} = \frac{2}{1}$

And, $\frac{AE}{CE} = \frac{12}{6} = \frac{2}{1} \Rightarrow \frac{AD}{BD} = \frac{AE}{CE}$

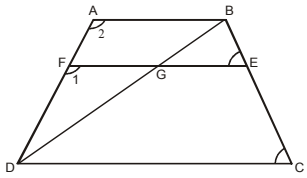
Thus, DE divides sides AB and AC of ΔABC in the same ratio. Therefore, by the converse of basic proportionality theorem, we have DE || BC.

Example.17 In a trapezium ABCD, AB || DC and DC = 2AB. EF drawn parallel to AB cuts AD in F and BC in E such that $\frac{BE}{EC} = \frac{3}{4}$. Diagonal DB intersects EF at G. Prove that 7FE = 10AB.

Solution. In ΔDFG and ΔDAB,

$\angle 1 = \angle 2$ [Corresponding $\angle$s AB || FG]

$\angle$ FDG = $\angle$ ADB [Common]



$\therefore$ ΔDFG ~ ΔDAB [By AA rule of similarity]

$\therefore$ $\frac{DF}{DA} = \frac{FG}{AB}$ … (i)

$\angle$FDG = $\angle$ADB [Common]

$\therefore$ ΔDFG ~ ΔDAB [By AA rule of similarity]

$\therefore$ $\frac{DF}{DA} = \frac{FG}{AB}$ … (i)

Again in trapezium ABCD

EF || AB || DC

$\Rightarrow$ $\frac{AF}{DF} = \frac{3}{4}$ $\left[\frac{BE}{EC} = \frac{3}{4}\text{(given)}\right]$

$\Rightarrow$ $\frac{AF}{DF} + 1 = \frac{3}{4} + 1$

$\Rightarrow$ $\frac{AF + DF}{DF} = \frac{7}{4}$

$\Rightarrow$ $\frac{AD}{DF} = \frac{7}{4}$ $\Rightarrow$ $\frac{DF}{AD} = \frac{4}{7}$ … (ii)

From (i) and (ii), we get

$\frac{FG}{AB} = \frac{4}{7}$ i.e., FG = $\frac{4}{7}$AB … (iii)

In ΔBEG and ΔBCD, we have

$\angle$BEG = $\angle$BCD [Corresponding angle EG || CD]

$\angle$GBE = $\angle$DBC [Common]

$\therefore$ ΔBEG ~ ΔBCD [By AA rule of similarity]

$\therefore$ $\frac{BE}{BC} = \frac{EG}{CD}$

$\therefore$ $\frac{3}{7} = \frac{EG}{CD}$ $\left[\frac{BE}{EG} = \frac{3}{7} \text{ i.e., } \frac{EC}{BE} = \frac{4}{3} \Rightarrow \frac{EC + BE}{BE} = \frac{4+3}{3} \Rightarrow \frac{BC}{BE} = \frac{7}{3}\right]$

$\therefore$ EG = $\frac{3}{7}$CD = $\frac{3}{7}$(2 AB) $[\because \text{CD} = 2\text{AB (given)}]$

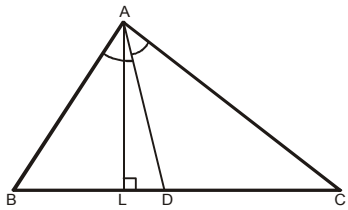
$\therefore$ EG = $\frac{6}{7}$AB ... (iv)

Adding (iii) and (iv), we get

FG + EG = $\frac{4}{7}$AB + $\frac{6}{7}$AB = $\frac{10}{7}$AB $\Rightarrow$ EF = $\frac{10}{7}$ AB i.e., 7EF = 10AB. **Hence proved.**

Example.18 In ΔABC, if AD is the bisector of $\angle$A, prove that $\frac{\text{Area } (\Delta ABD)}{\text{Area } (\Delta ACD)} = \frac{AB}{AC}$.

Solution. In ΔABC, AD is the bisector of $\angle$A.



$\therefore \dfrac{AB}{AC} = \dfrac{BD}{DC}$ .....(i) [By internal bisector theorem]

From A draw $AL \perp BC$

$\therefore \dfrac{\text{Area } (\Delta ABD)}{\text{Area } (\Delta ACD)} = \dfrac{\frac{1}{2}BD.AL}{\frac{1}{2}DC.AL} = \dfrac{BD}{DC} = \dfrac{AB}{AC}$ [From (i)]

Hence Proved.

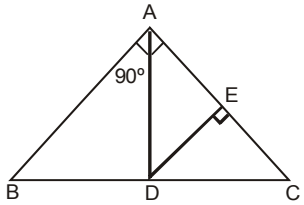
Example.19 In triangle ABC, $\angle BAC = 90^o$, AD is its bisector. If $DE \perp AC$, prove that $DE \times (AB + AC) = AB \times AC$.

Solution. It is given that AD is the bisector of $\angle A$ of Δ ABC.

$\therefore \dfrac{AB}{AC} = \dfrac{BD}{DC}$

$\Rightarrow \dfrac{AB}{AC} + 1 = \dfrac{BD}{DC} + 1$ [Adding 1 on both sides]

$\Rightarrow \dfrac{AB+AC}{AC} = \dfrac{BD+DC}{DC}$



$\Rightarrow \dfrac{AB+AC}{AC} = \dfrac{BC}{DC}$ .....(i)

In Δ's CDE and CBA, we have

$\angle DCE = \angle BCA$ [Common]

$\angle DEC = \angle BAC$ [Each equal to 90º]

So, by AA-criterion of similarity

Δ CDE ~ Δ CBA

$\Rightarrow \dfrac{CD}{CB} = \dfrac{DE}{BA}$

$\Rightarrow$ = ...(ii)

From (i) and (ii), we have

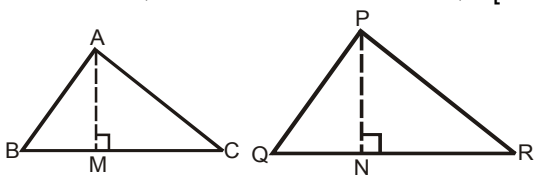
$\Rightarrow$ =

$\Rightarrow$ DE × (AB + AC) = AB × AC.

M. AREAS OF SIMILAR TRIANGLES

Theorem : The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Given : Two triangles ABC and PQR such that Δ ABC ~ Δ PQR [Shown in the figure]



To prove : $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{CA}{RP}\right)^2$.

Construction : Draw altitudes AM and PN of the triangle ABC and PQR.

Proof : ar(ABC) = $\frac{1}{2}$ BC × AM

And ar(PQR) = $\frac{1}{2}$ QR × PN

So, $\frac{ar(ABC)}{ar(PQR)} = \frac{\frac{1}{2}BC\times AM}{\frac{1}{2}QR\times PN} = \frac{BC\times AM}{QR\times PN}$ ... (i)

Now, in Δ ABM and Δ PQN,

And ∠B = ∠Q [As Δ ABC ~ Δ PQR]

∠M = ∠N [90º each]

So, Δ ABM ~ Δ PQN [AA similarity criterion]

Therefore, $\frac{AM}{PN} = \frac{AB}{PQ}$ ... (ii)

Also, Δ ABC ~ Δ PQR [Given]

So, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$ ... (iii)

Therefore, $\frac{ar(ABC)}{ar(PQR)} = \frac{BC}{QR}\times\frac{AB}{PQ}$ [From (i) and (ii)]

$= \frac{AB}{PQ} \times \frac{AB}{PQ}$ [From (iii)]

$= \left(\frac{AB}{PQ}\right)^2$

Now using (iii), we get

$$\frac{\mathbf{ar(\Delta ABC)}}{\mathbf{ar(\Delta PQR)}} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$$

(a) Properties of Areas of Similar Triangles :

(i) The areas of two similar triangles are in the ratio of the squares of corresponding altitudes.

(ii) The areas of two similar triangles are in the ratio of the squares of the corresponding medians.

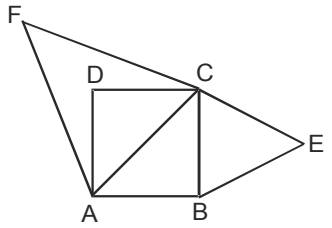
(iii) The area of two similar triangles are in the ratio of the squares of the corresponding angle bisector segments.

Example.20 Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.

Solution. **Given :** A square ABCD. Equilateral triangles ΔBCE and ΔACF have been described on side BC and diagonal AC respectively.

To Prove : Area (ΔBCE) = $\frac{1}{2}$. Area (ΔACF)

Proof : Since ΔBCE and ΔACF are equilateral. Therefore, they are equiangular (each angle being equal to 60º) and hence ΔBCE ~ ΔACF.



[ABCD is a square
Diagonal = $\sqrt{2}$ (side)
AC = $\sqrt{2}$ BC]

$\Rightarrow \dfrac{Area(\Delta BCE)}{Area(\Delta ACF)} = \dfrac{BC^2}{AC^2}$ $\qquad \Rightarrow \dfrac{Area(\Delta BCE)}{Area(\Delta ACF)} = \dfrac{BC^2}{\left(\sqrt{2}BC\right)^2} = \dfrac{1}{2}$

$\Rightarrow \dfrac{Area(\Delta BCE)}{Area(\Delta ACF)} = \dfrac{1}{2}$. **Hence Proved.**

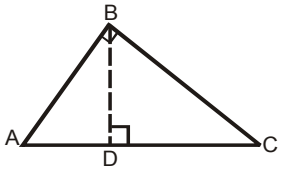
O. PYTHAGORAS THEOREM

Statement : In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given : A right triangle ABC, right angled at B.

To prove : $AC^2 = AB^2 + BC^2$

Construction : $BD \perp AC$



Proof : Δ ADB & Δ ABC

$\angle DAB = \angle CAB$ [Common]

$\angle BDA = \angle CBA$ [90º each]

So, $\Delta ADB \sim \Delta ABC$ [By AA similarity]

$\dfrac{AD}{AB} = \dfrac{AB}{AC}$ [Sides are proportional]

or, $AD \cdot AC = AB^2$ ... (i)

Similarily $\Delta BDC \sim \Delta ABC$

So, $\dfrac{CD}{BC} = \dfrac{BC}{AC}$

or $CD \cdot AC = BC^2$ ... (ii)

Adding (i) and (ii),

$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$

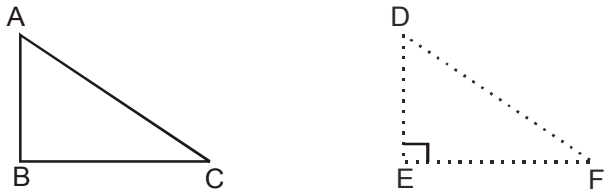
or, $AC (AD + CD) = AB^2 + BC^2$

or, $AC. AC = AB^2 + BC^2$

or, $\mathbf{AC^2 = AB^2 + BC^2}$ **Hence Proved.**

(a) Converse of Pythagoras Theorem :

Theorem : In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.



Given : A triangle ABC such that $AC^2 = AB^2 + BC^2$

Construction : Construct a triangle DEF such that DE = AB, EF = BC and E = 90º.

Proof : In order to prove that B = 90º, it is sufficient to show Δ ABC ~ Δ DEF. For this we proceed as follows.

Since Δ DEF is a right-angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$DF^2 = DE^2 + EF^2 \quad \Rightarrow \quad DF^2 = AB^2 + BC^2$ [$\because$ DE = AB and EF = BC (By construction)]

$\Rightarrow \quad DF^2 = AC^2$ [$\because AB^2 + BC^2 = AC^2$ (Given)]

$\Rightarrow \quad DF = AC$ .....(i)

Thus, in Δ ABC and Δ DEF, we have

AB = DE, BC = EF [By construction]

And AC = DF [From equation (i)]

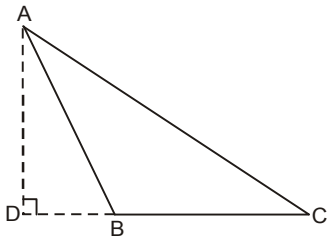
$\therefore$ $\Delta ABC \cong \Delta DEF$ [By SSS criteria of congruency]

$\Rightarrow$ $\angle B = \angle E = 90^o$

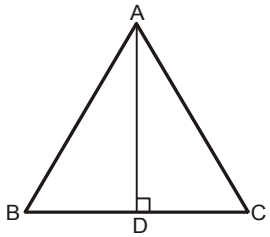
Hence, Δ ABC is a right triangle, right angled at B.

(b) Some Results Deduced From Pythagoras Theorem :

(i) In the given figure **ΔABC** is an obtuse triangle, obtuse angled at **B**.
If **AD $\perp$ CB**, then **$AC^2 = AB^2 + BC^2 + 2BC.\ BD$**



(ii) In the given figure, if $\angle$ **B** of **ΔABC** is an acute angle and **AD $\perp$ BC**,
then **$AC^2 = AB^2 + BC^2 - 2BC\ .\ BD$**



(iii) In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

(iv) Three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians of the triangle.

Solved Examples

Example.21 BL and CM are medians of ΔABC right angled at A. Prove that $4\ (BL^2 + CM^2) = 5\ BC^2$.

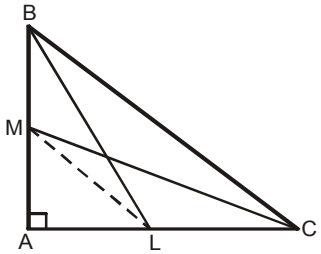
Solution. In ΔBAL

$\Rightarrow$ $BL^2 = AL^2 + AB^2$ … (i) [Using Pythagoras theorem]

and, In ΔCAM

$CM^2 = AM^2 + AC^2$ … (ii) [Using Pythagoras theorem]

Adding (1) and (2) and then multiplying by 4, we get



$4(BL^2 + CM^2) = 4(AL^2 + AB^2 + AM^2 + AC^2)$

$= 4\{AL^2 + AM^2 + (AB^2 + AC^2)\}$ [$\therefore$ Δ ABC is a right triangle]

$= 4(AL^2 + AM^2 + BC^2) = 4(ML^2 + BC^2)$ [$\therefore$ ΔLAM is a right triangle]

$= 4ML^2 + 4\ BC^2$

[A line joining mid-points of two sides is parallel to third side and is equal to half of it, ML =BC/2]

$= BC^2 + 4BC^2 = 5BC^2$. **Hence proved.**

Example.22 In the given figure, BC ⊥ AB, AE ⊥ AB and DE ⊥ AC. Prove that DE. BC = AD. AB.

Solution. In ΔABC and ΔEDA,

We have

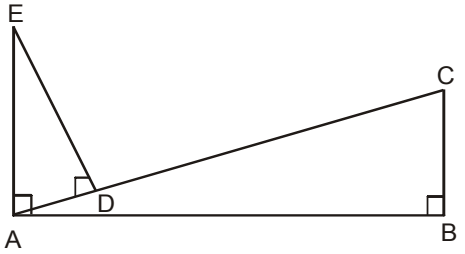
∠ ABC = ∠ ADE [Each equal to 90°]

∠ ACB = ∠ EAD [Alternate angles]

∴ By AA Similarity

ΔABC ~ ΔEDA

$\Rightarrow \frac{BC}{AB} = \frac{AD}{DE}$



⇒ DE . BC = AD. AB. **Hence Proved.**

Example.23 O is any point inside a rectangle ABCD (shown in the figure).

Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

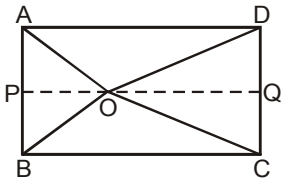
Solution. Through O, draw PQ || BC so that P lies on AB and Q lies on DC.

Now, PQ || BC

Therefore,

PQ ⊥ AB and PQ ⊥ DC [∠ B = 90º and ∠ C = 90º]

So, ∠ BPQ = 90º and ∠ CQP = 90º



Therefore, BPQC and APQD are both rectangles.

Now, from Δ OPB,

$OB^2 = BP^2 + OP^2$ ... (i)

Similarly, from Δ ODQ,

$OD^2 = OQ^2 + DQ^2$ ... (ii)

From Δ OQC, we have

$OC^2 = OQ^2 + CQ^2$ ... (iii)

And from Δ OAP, we have

$OA^2 = AP^2 + OP^2$ ... (iv)

Adding (i) and (ii)

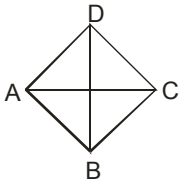
$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$

$= CQ^2 + OP^2 + OQ^2 + AP^2$ [As BP = CQ and DQ = AP]

$= CQ^2 + OQ^2 + OP^2 + AP^2 = OC^2 + OA^2$ [From (iii) and (iv)] **Hence Proved.**

M. QUADRILATERAL

A **quadrilateral** is a four sided closed figure.



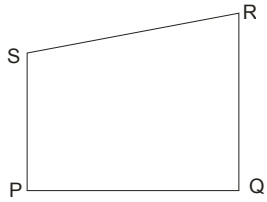
Let **A**, **B**, **C** and **D** be four points in a plane such that :

(i) No three of them are collinear.

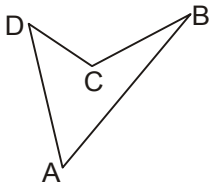
(ii) The line segments AB, BC, CD and DA do not intersect except at their end points, then **figure** obtained by joining A, B, C & D is called a **quadrilateral**.

Convex and Concave Quadrilaterals :

(i) A quadrilateral in which the measure of each interior angle is less than 180° is called a **convex quadrilateral**. In figure, PQRS is convex quadrilateral.

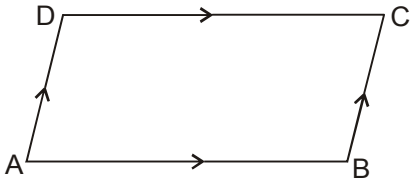


(ii) A quadrilateral in which the measure of one of the interior angles is more than 180° is called a **concave quadrilateral**. In figure, ABCD is concave quadrilateral**.**



Special Quadrilaterals :

(i) Parallelogram : A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel. In figure, AB || DC, AD || BC therefore, ABCD is a parallelogram.



Properties :

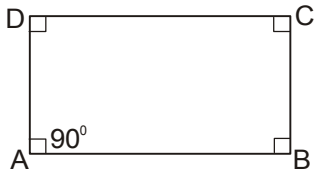
(a) A diagonal of a parallelogram divides it into two congruent triangles.

(b) In a parallelogram, opposite sides are equal.

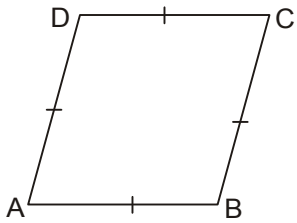
(c) The opposite angles of a parallelogram are equal.

(d) The diagonals of a parallelogram bisect each other.

(ii) Rectangle : A r**ectangle** is a parallelogram, in which each of its angle is a right angle. If ABCD is a rectangle then $\angle A = \angle B = \angle C = \angle D = 90°$, AB = CD, BC = AD and diagonals AC = BD.

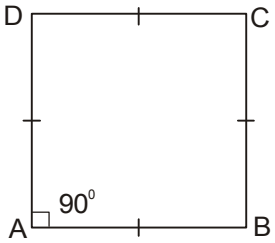


(iii) Rhombus : A **rhombus** is a parallelogram in which all its sides are equal in length. If ABCD is a rhombus then, AB = BC = CD = DA.



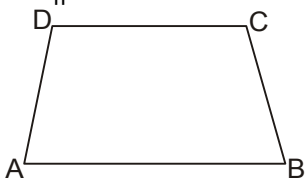
The diagonals of a rhombus are perpendicular to each other.

(iv) Square : A **square** is a parallelogram having all sides equal and each angle equal to right angle. If ABCD is a square then AB = BC = CD = DA, diagonal AC = BD and $\angle A = \angle B = \angle C = \angle D = 90°$.

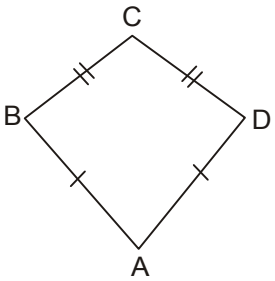


The diagonals of a square are perpendicular to each other.

(v) Trapezium : A **trapezium** is a quadrilateral with only one pair of opposite sides parallel. In figure, ABCD is a trapezium with AB || DC.



(vi) Kite : A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. If ABCD is a kite then AB = AD and BC = CD.



(vii) Isosceles trapezium : A trapezium is said to be an **isosceles trapezium**, if its non-parallel sides are equal. Thus a quadrilateral ABCD is an isosceles trapezium, if AB || DC and AD = BC.

In isosceles trapezium **$\angle A = \angle B$ and $\angle C = \angle D$.**

P. PROPERTIES

Theorem 1 : The sum of the four angles of a quadrilateral is 360°.

Theorem 2 : A diagonal of a parallelogram divides it into two congruent triangles.

Theorem 3 : In a parallelogram, opposite sides are equal.

Theorem 4 : The opposite angles of a parallelogram are equal.

Theorem 5 : The diagonals of a parallelogram bisect each other.

Theorem 6 : Each of the four angles of a rectangle is a right angle.

Theorem 7 : Each of the four sides of a rhombus is of the same length.

Theorem 8 : Each of the angles of a square is a right angle and each of the four sides is of the same length.

Theorem 9 : The diagonals of a rectangle are of equal length.

Theorem 10 : The diagonals of a rhombus are perpendicular to each other.

Theorem 11 : The diagonals of a square are equal and perpendicular to each other.

Theorem 12 : Parallelogram and Triangles on the same base (or equal bases) and between the same parallels, then area of parallelogram is twice the area of triangle.

Theorem 13 : Two triangles on the same base (or equal bases) and between the same parallels are equal in area.

Theorem 14 : Median of a triangle divides it into two triangles of equal area.

Theorem 15 : Parallelograms on the same base and between the same parallels are equal in area.

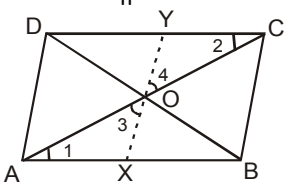
A quadrilateral become a parallelogram when :
(i) Opposite angles are equal.
(ii) Both the pair of opposite sides are equal
(iii) A pair of opposite side is equal as well as parallel
(iv) Diagonals of quadrilateral bisect each other.

❖ **REMARK :**
(i) Square, rectangle and rhombus are all parallelograms.
(ii) Kite and trapezium are not parallelograms.
(iii) A square is a rectangle.
(iv) A square is a rhombus.
(v) A parallelogram is a trapezium.

Solved Examples

Example.26 The diagonals of a parallelogram ABCD intersect at O. A line through O intersects AB at X and DC at Y. Prove that OX = OY.

Solution. Since AB || CD.



In ΔOAX and ΔOCY

$\angle 1 = \angle 2$ [Alternate angles]

OA = OC and, $\angle 3 = \angle 4$ [Vertically opposite angles]

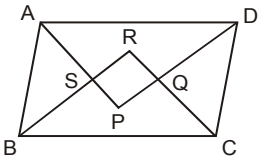
So, by ASA criterion of congruence,

$\Delta OAX \cong \Delta OCY$

OX = OY.

Example.27 The angle bisectors of a parallelogram form a rectangle.

Solution. A parallelogram ABCD in which bisectors of angles A, B, C, D intersect at P, Q, R, S to form a quadrilateral PQRS.



Since, ABCD is a parallelogram, so AD || BC and transversal AB intersects them at A and B respectively.

$\angle A + \angle B = 180^o \Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^o$

$\angle BAS + \angle ABS = 90^o$ ....(i) [AS and BS are bisectors of $\angle A$ and $\angle B$ respectively]

$\angle BAS + \angle ABS + \angle ASB = 180^o$ $90^o + \angle ASB = 180^o$

$\angle ASB = 90^o$ $\angle RSP = 90^o$ [$\angle ASB$ and $\angle RSP$ are vertically opposite angles]

Similarly, $\angle SRQ = 90^o$, $\angle RQP = 90^o$ and $\angle SPQ = 90^o$.

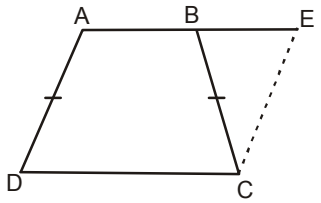
Hence, PQRS is rectangle.

Example.28 ABCD is a trapezium in which AB || CD and AD = BC. Show that :

(i) $\angle A = \angle B$ **(ii)** $\angle C = \angle D$

(iii) $\Delta ABC \cong \Delta BAD$ **(iv)** diagonal AC = diagonal BD.

Solution.



(i) Extend AB and draw a line through C parallel to DA intersecting AB produced at E.
Since, AD || CE and transversal AE cuts them at A and E respectively.

$\therefore$ $\angle A + \angle E = 180^o$ $\Rightarrow 180^o - \angle E = \angle A$

Since, AB || CD and AD || CE

$\therefore$ AECD is a parallelogram. $\Rightarrow$ AD = CE $\Rightarrow$ BC = CE

Thus, in ΔBCE

BC = CE $\Rightarrow$ $\angle CBE = \angle CEB$ $\Rightarrow 180^o - \angle B = \angle E$

$\Rightarrow$ $180^o - \angle E = \angle B$ $\Rightarrow \angle A = \angle B$.

(ii) Consecutive interior angles on the same side of a transversal are supplementary.

$\therefore$ $\angle A + \angle D = 180^o$ and $\angle E + \angle C = 180^o$ $\Rightarrow$ $\angle A + \angle D = \angle E + \angle C$

$\Rightarrow$ $\angle B + \angle D = \angle E + \angle C$ $\Rightarrow$ $\angle D = \angle C$ [$\angle B = \angle E$]

(iii) In ΔABC and ΔBAD

AB = BA

$\angle B = \angle A$ and BC = AD

So, by SAS congruence criterion

$\Delta ABC \cong \Delta BAD$

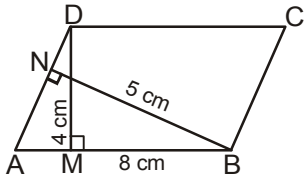
(iv) Since, $\Delta ABC \cong \Delta BAD$

AC = BD

Hence, diagonal AC = diagonal BD.

Example.29 In a parallelogram ABCD, AB = 8 cm. The altitudes corresponding to sides AB and AD are respectively 4 cm and 5 cm. Find AD.

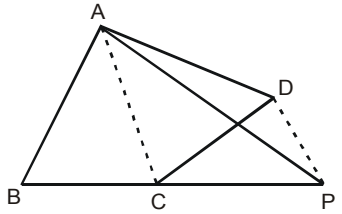
Solution. Area of a $||^{gm}$ = Base × corresponding altitude



$\therefore$ Area of parallelogram ABCD = AD × BN = AB × DM

$\Rightarrow$ AD × 5 = 8 × 4 $\Rightarrow$ AD = $\frac{8\times4}{5}$ = 6.4 cm.

Example.30 ABCD is a quadrilateral. A line through D, parallel to AC, meets BC produced in P as shown in figure. Prove that ar (ΔABP) = ar(quad. ABCD).



Solution. Since Δs ACP and ACD are on the base AC and between the same parallels AC and DP.

$\therefore$ ar(ΔACP) = ar(ΔACD) $\Rightarrow$ ar(ΔACP) + ar(ΔABC) = ar(ΔACD) + ar(ΔABC)

$\Rightarrow$ ar(ΔABP) = ar(quad. ABCD).

Example.31 If the medians of a ΔABC intersect at G,

show that $ar(\Delta AGB) = ar(\Delta AGC) = ar(\Delta BGC) = \frac{1}{3} ar(\Delta ABC)$.

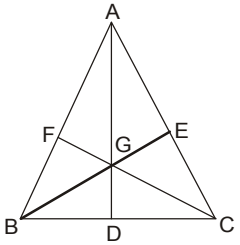
Solution. **Given :** A Δ ABC and its medians AD, BE and CF intersect at G.

To prove : $ar(\Delta AGB) = ar(\Delta AGC) = ar(\Delta BGC) = \frac{1}{3} ar(\Delta ABC)$.

Proof : A median of a triangle divides it into two triangles of equal area.

In ΔABC, AD is the median.

$\therefore ar(\Delta ABD) = ar(\Delta ACD)$ ...(i)



In ΔGBC, GD is the median.

$\therefore$ $ar(\Delta GBD) = ar(\Delta GCD)$ ...(ii)

Subtract equation (ii) from (i), we get

$ar(\Delta ABD) - ar(\Delta GBD) = ar(\Delta ACD) - ar(\Delta GCD)$

$\therefore ar(\Delta AGB) = ar(\Delta AGC)$ ..(iii)

Similarly, $ar(\Delta AGB) = ar(\Delta BGC)$ ..(iv)

From (iii) & (iv)

$ar(\Delta AGB) = ar(\Delta AGC) = ar(\Delta BGC)$

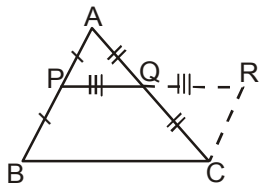
But, $ar(\Delta ABC) = ar(\Delta AGB) + ar(\Delta AGC) + ar(\Delta BGC) = 3\ ar(\Delta AGB)$

$ar(\Delta AGB) = \frac{1}{3} ar(\Delta ABC)$.

Hence, $ar(\Delta AGB) = ar(\Delta AGC) = ar(\Delta BGC) = \frac{1}{3} ar(\Delta ABC)$.

O. MID POINT THEOREM

In a triangle, the line segment joining the mid-points of any two sides is parallel to the third side and is half of it.



Given : A triangle ABC in which P is the mid-point of side AB and Q is the mid-point of side AC.

To Prove : PQ is parallel to BC and is half of it

i.e., PQ || BC and $PQ = \frac{1}{2}$ BC.

Construction : Produce PQ upto point R such that PQ = QR. Join R and C.

Proof : In ΔAPQ and ΔCRQ

PQ = QR [By construction]

AQ = QC [Given]

And, $\angle AQP = \angle CQR$ [Vertically opposite angles]

So, $\Delta APQ \cong \Delta CRQ$ [By SAS]

$\therefore$ AP = CR [By cpct]

And, $\angle APQ = \angle CRQ$ [By cpct]

But, $\angle APQ$ and $\angle CRQ$ are alternate angles and whenever the alternate angles are equal, the lines are parallel.

$\Rightarrow$ AP || CR $\Rightarrow$ AB || CR $\Rightarrow$ BP || CR

AP = BP [Given, P is mid-point of AB]

$\Rightarrow$ CR = BP [As, AP = CR]

Now, BP = CR and BP || CR $\Rightarrow$ BCRP is a parallelogram.

[When any pair of opposite sides are equal and parallel, the quadrilateral is a parallelogram]

BCRP is a parallelogram and opposite sides of a parallelogram are equal and parallel.

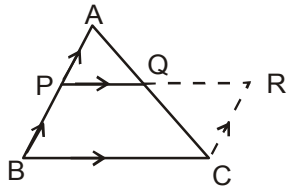
PR = BC and PR || BC

Since, PQ = QR $\Rightarrow$ PQ = $\frac{1}{2}$ PR = $\frac{1}{2}$ BC [As, PR = BC]

Also, PQ || BC [As, PR || BC]

$\therefore$ PQ || BC and PQ = $\frac{1}{2}$ BC **Hence Proved.**

Converse of the Mid-Point Theorem : The line drawn through the mid-point of one side of a triangle parallel to the another side; bisects the third side.



Given : A triangle ABC in which P is the mid-point of side AB and PQ is parallel to BC.

To prove : PQ bisects the third side AC i.e., AQ = QC.

Construction : Through C, draw CR parallel to BA, which meets PQ produced at point R.

Proof : Since, PQ || BC i.e., PR || BC [Given]

and CR || BA i.e., CR || BP. [By construction]

$\therefore$ Opposite sides of quadrilateral PBCR are parallel.

$\Rightarrow$ PBCR is a parallelogram

$\Rightarrow$ BP = CR

Also, BP = AP [As, P is mid-point of AB]

$\therefore$ CR = AP

AB || CR and AC is transversal, $\angle PAQ = \angle RCQ$ [Alternate angles]

$\therefore$ AB || CR and PR is transversal, $\angle APQ = \angle CRQ$ [Alternate angles]

In APQ and CRQ

CR = AP, $\angle PAQ = \angle RCQ$ and $\angle APQ = \angle CRQ$

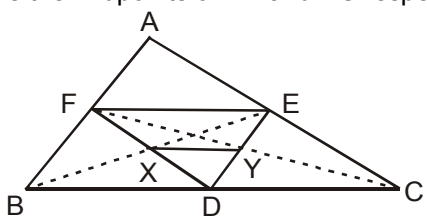
$\Rightarrow$ $\Delta APQ \cong CRQ$ [By ASA]

$\Rightarrow$ AQ = QC **Hence Proved.**

Solved Examples

Example.32 In the adjoining figure, D, E, F are the midpoints of the sides BC, CA and AB of ΔABC. If BE and DF intersect at X while CF and DE intersect at Y, prove that XY = $\frac{1}{4}$ BC.

Solution. In ΔABC, F and E are the midpoints of AB and AC respectively.



$\therefore$ FE || BC and FE = $\frac{1}{2}$ BC = BD.

$\therefore$ FE || BD and FE = BD.

So, BDEF is a parallelogram whose diagonals BE and DF intersect each other at X.

$\therefore$ X is the midpoint of DF.

Similarly, Y is the midpoint of DE.

Thus, in ΔDEF, X and Y are the midpoints of DF and DE respectively.

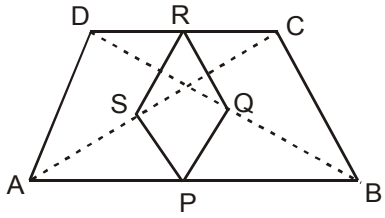
So, XY || FE and XY = $\frac{1}{2}$FE = $\frac{1}{2} \times \frac{1}{2}$ BC = $\frac{1}{4}$ BC.

Example.33 In the adjoining figure, ABCD is a trapezium in which AB || DC and AD = BC. If P, Q, R, S be respectively the midpoints of BA, BD and CD, CA then show that PQRS is a rhombus.

Solution. In ΔBDC, Q and R are the midpoints of BD and CD respectively.

$\therefore$ QR || BC and QR = $\frac{1}{2}$BC.

Similarly, PS || BC and PS = $\frac{1}{2}$BC.



$\therefore$ PS || QR and PS = QR [Each equal to $\frac{1}{2}$BC.]

$\therefore$ PQRS is a parallelogram.

In ΔACD, S and R are the midpoints of AC and CD respectively.

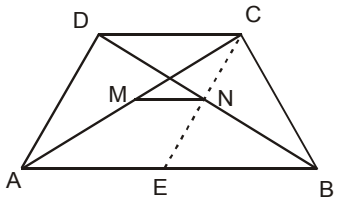
$\therefore$ SR || AD and SR = $\frac{1}{2}$ AD = $\frac{1}{2}$BC [$\because$ AD = BC]

$\therefore$ PS = QR = SR = PQ.

Hence, PQRS is a rhombus.

Example.34 Prove that the line segment joining the midpoints of the diagonals of a trapezium is parallel to theparallel sides and equal to half of their difference.

Solution. Let ABCD be a trapezium in which AB || DC, and let M and N be the midpoints of the diagonals AC and BD respectively.



Join CN and produce it to meet AB at E.

In ΔCDN and ΔEBN, we have

DN = BN [$\because$ N is midpoint of BD]

$\angle$DCN = $\angle$BEN [Alt. Int. $\angle$]

$\angle$CDN = $\angle$EBN [Alt. Int. $\angle$]

$\therefore$ ΔCDN $\cong$ ΔEBN [AAS-criteria]

$\therefore$ DC = EB and CN = NE [c.p.c.t]

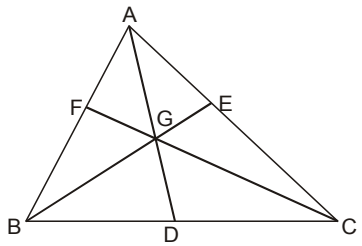
Thus; in ΔCAE, the points M and N are the midpoints of AC and CE respectively.

$\therefore$ MN || AE and MN = $\frac{1}{2}$ AE $\Rightarrow$ MN || AB || DC

and MN = $\frac{1}{2}$ AE = $\frac{1}{2}$ (AB – EB) = $\frac{1}{2}$ (AB – DC) [$\because$ EB = DC].

Example.35 AD, BE and CF be medians of a ΔABC.
Prove that 2(AD + BE + CF) < 3(AB + BC + CA) < 4(AD + BE + CF).

Solution. In ΔABC, AD, BE and CF are medians. We know that these are concurrent.Let these meet at G.
Also AG : GD = BG : GE = CG : GF = 2 : 1
So, AG = $\frac{2}{3}$AD, BG = $\frac{2}{3}$BE, CG = $\frac{2}{3}$CF.
Because in triangle, the sum of two sides is greater than the third,
So, in ΔAGB, AG + BG > AB
in ΔBGC, BG + CG > BC
in ΔCGA, CG + AG > CA
Adding these, we get
2(AG + BG + CG) > AB + BC + CA
or $2\left(\frac{2}{3}AD+\frac{2}{3}BE+\frac{2}{3}CF\right)$ > AB + BC + CA
or $\frac{4}{3}$ (AD + BE + CF) > AB + BC + CA

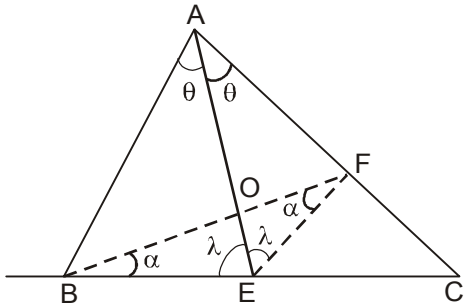


or 4(AD + BE + CF) > 3(AB + BC + CA) ... (i)
Now, in ΔABD
AB + BD > AD
or, AB + $\frac{1}{2}$BC > AD (Since AD is a median)
Similarly, BC + $\frac{1}{2}$CA > BE and CA + $\frac{1}{2}$AB > CF
Adding these, we get $\frac{3}{2}$ (AB + BC + CA) > AD + BE + CF
or 3(AB + BC + CA) > 2(AD + BE + CF) ... (ii)
Combining (i) and (ii), we get
2(AD + BE + CF) < 3(AB + BC + CA) < 4(AD + BE + CF).

Example.36 In a ΔABC, AC > AB. The bisector of $\angle A$ meets BC at E. Prove that CE > BE.

Solution. Given : A ΔABC, in which AC > AB and AE bisects $\angle A$
To Prove : CE > BE.
Construction : Take a point F on AC, such that AF = AB. Join BF and EF. Let BF intersect AE at O.
Proof : ΔABF is an isosceles (Since AB = AF, by construction)
AE bisects $\angle A$ (given)
$\therefore$ AE is the perpendicular bisector of BF, so any point on AE is equidistant from B and F. In particular,
BE = FE ... (i)
So, in ΔBEF,
$\angle EBF = \angle EFB = \alpha$, say (angles opposite equal sides)
So, $\angle BEO = 90° - \alpha = \angle FEO$ (= λ, say)

Thus AE also bisects $\angle BEF$.



Now, because in a triangle, exterior angle is the sum of interior opposite angles, so

$\angle EFC = \theta + \lambda$ and ext $\angle B = \theta + \lambda$

So, $\angle EFC$ = ext $\angle B$. But $\angle B = 2\theta + \angle C$, so $\angle EFC = 2\theta + \angle C$.

Therefore $\angle EFC > \angle C$

So, CE > FE

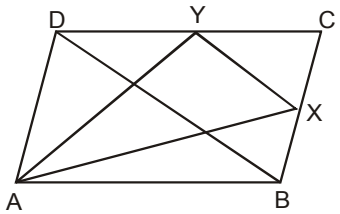
Hence CE > BE (using (i)).

Example.37 ABCD is a parallelogram. X and Y are mid–points of BC and CD respectively.

Prove that area (ΔAXY) = $\frac{3}{8}$ area(parallelogram ABCD).

Solution. Since X and Y are the mid–points of sides BC and CD respectively in ΔBCD.

$\therefore$ XY || BD and XY = $\frac{1}{2}$BD.



area (ΔCYX) = $\frac{1}{4}$ area(ΔDBC)

= area (parallelogram ABCD). .. (i) [Since ar(ΔDBC) = $\frac{1}{2}$ area (parallelogram ABCD)]

Parallelogram ABCD and ΔABX are between the same parallels AD and BC and BX = $\frac{1}{2}$BC.

$\therefore$ area (ΔABX) = $\frac{1}{4}$ area (parallelogram ABCD) ... (ii)

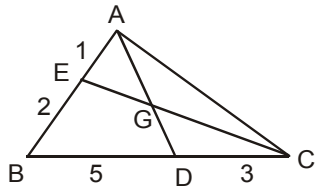
Similarly, area (ΔAYD) = $\frac{1}{4}$ area (parallelogram ABCD)

Now, ar(ΔAXY) = ar(parallelogram ABCD) – [ar(ΔABX) + ar(ΔAYD) + ar(ΔCYX)]

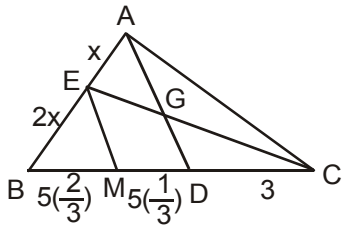
ar(ΔAXY) = ar (parallelogram ABCD) $-\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{8}\right)$ ar(ABCD)

ar(ΔAXY) = $\left(1-\frac{5}{8}\right)$ ar(parallelogram ABCD) = $\frac{3}{8}$ ar(parallelogram ABCD).

Example.38 In the figure AE : EB = 1: 2 and BD : DC = 5 : 3. Find EG : GC.



Solution. Through E, we draw a line parallel to AD, intersecting BC at M.



BM = $5\left(\frac{2}{3}\right)$ and MD = $5\left(\frac{1}{3}\right)$

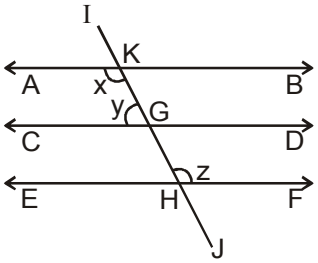
EG : GC = MD : DC = $\frac{5}{3}$: 3 = 5 : 9.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Lines and angles

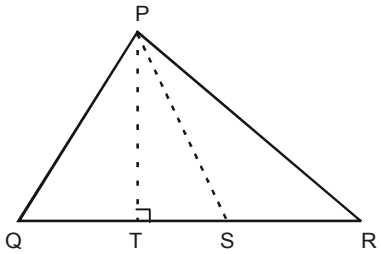
A-1 In figure, if and y : z = 3 : 7, find x.



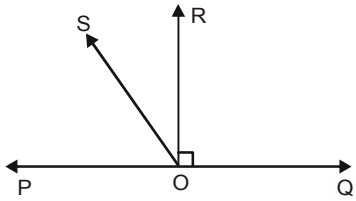
A-2 Two complementary angles are such that two times the measure of one is equal to three times measure of the other. Find the measure of the large angle.

A-3 Write the supplementary angles of the following angles.
(a) 54^0 28' **(b)** 98^0 35' 20"

A-4 In figure, PS is the bisector of $\angle QPR$ and $PT \perp QR$. Show that $\angle TPS = \frac{1}{2}(\angle Q - \angle R)$.

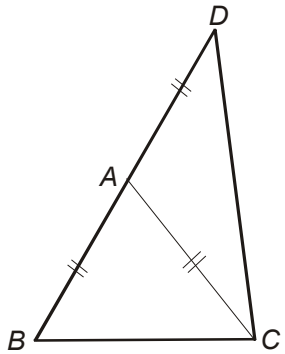


A-5 In figure, POQ is a line. Ray OR is perpendicular to line PQ. OS is another ray lying between rays OP and OR. Prove that $\angle ROS = \frac{1}{2}(\angle QOS - \angle POS)$.



Section (B) : Congruent Triangle and Similar Triangle

B-1 In the figure, AB = AC = AD, prove that $\angle BCD$ is a right angle.



B-2 ABC is a triangle and D is the mid point of BC. The perpendiculars from D to AB and AC are equal. Prove that the triangle is isosceles.

B-3 Prove that the perimeter of a triangle is greater than the sum of its three medians.

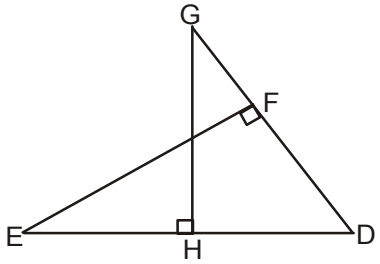
B-4 O is any point in the interior of a triangle ABC. Prove that :

(i) AB + AC > OB + OC **(ii)** AB + BC + CA > OA + OB + OC

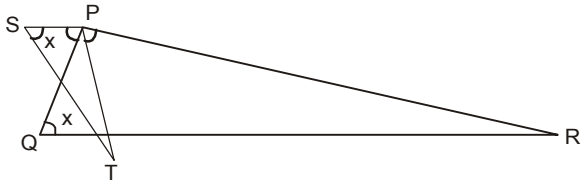
(iii) OA + OB + OC > $\frac{1}{2}$ (AB + BC + AC).

B-5 Prove that the sum of three altitudes of a triangle is less than the sum of three sides of the triangle.

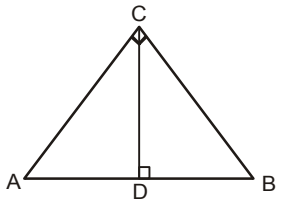
B-6 Given : $\angle GHE = \angle DFE = 90^o$, DH = 8, DF = 12, DG = 3x – 1 and DE = 4x + 2.



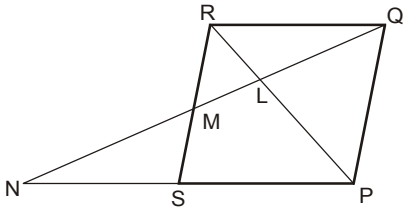
B-7 In figure, $\angle QPS = \angle RPT$ and $\angle PST = \angle PQR$. Prove that $\Delta PST \sim \Delta PQR$ and hence find the ratio ST : PT, if PR : QR = 4 : 5.



B-8 In figure, $\angle ACB = 90^o$, CD $\perp$ AB, prove that $CD^2 = BD.AD$.

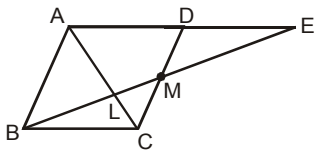


B-9 In the figure, PQRS is a parallelogram with PQ = 16 cm and QR = 10 cm. L is a point on PR such that RL : LP = 2 : 3. QL produced meets RS at M and PS produced at N.



Find the lengths of PN and RM.

B-10 In figure, M is mid-point of side CD of a parallelogram ABCD. The line BM is drawn intersecting AC at L and AD produced at E. Prove that EL = 2BL.



Section (C) : Pythagoras theorem

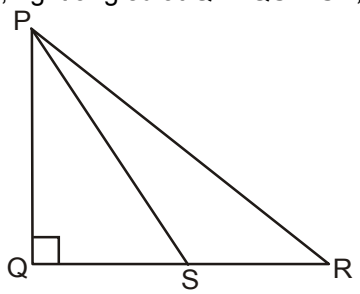
C-1 O is any point inside a rectangle ABCD. If OB = 3, OD = 5, OA = 4, then find OC.

C-2 ABC is a right triangle, right-angled at C. Let BC = a, CA = b, AB = c and let p be the length of perpendicular form C on AB, prove that

(i) cp = ab **(ii)** $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

C-3 In an equilateral triangle ABC, the side BC is trisected at D. Prove that $9\,AD^2 = 7AB^2$.

C-4 In a right triangle, prove that the square of the hypotenuse is equal to sum of the squares of the other two sides.
Using the above result, prove the following :
In figure PQR is a right triangle, right angled at Q. If QS = SR, show that $PR^2 = 4PS^2 - 3PQ^2$.

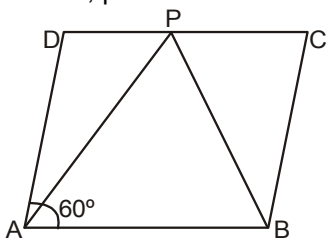


C-5 In Δ ABC, $\angle ABC = 135^o$. Prove that $AC^2 = AB^2 + BC^2 + 4ar(\Delta ABC)$.

Section (D) : Quadrilaterals

D-1 ABCD is a parallelogram. P is a point on AD such that AP = $\frac{1}{3}$AD. Q is a point on BC such that CQ = $\frac{1}{3}$BC. Prove that AQCP is a parallelogram.

D-2 In the following figure, ABCD is a parallelogram in which $\angle A = 60^o$. If the bisectors of $\angle A$ and $\angle B$ meet at P, prove that $\angle APB = 90^o$. Also, prove that AD = DP, PC = BC and DC = 2AD.



D-3 In ΔABC, AD is the median through A and E is the mid-point of AD. BE produced meets AC in F. Prove that AF = $\frac{1}{3}$AC.

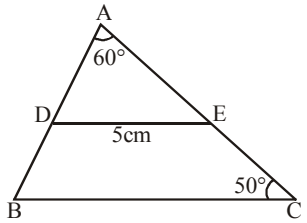
D-4 Let ABCD be a trapezium in which AB || DC, and let M and N be the midpoints of the diagonals AC and BD respectively. If AB = 15 cm and CD = 5 cm, then find MN.

D-5. Prove that quadrilateral form by line segments joining the midpoints of the sides of an isosceles trapezium is a rhombus.

OBJECTIVE QUESTIONS

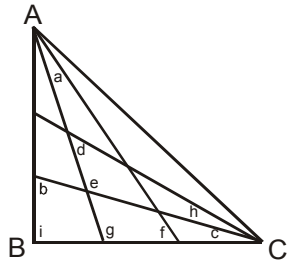
Section (A) : Lines and angles

A-1. In Figure, D and E are the mid-points of sides AB and AC respectively of ΔABC.Find $\angle EDB$.



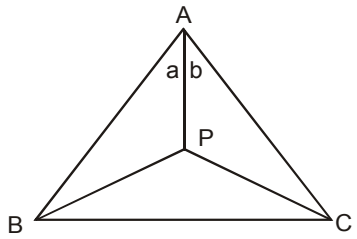
(A) 110º (B) 120º (C) 70º (D) 80º

A-2. In ΔABC, what is sum of the angles a + b + c + d + e + f + g + h + i ?



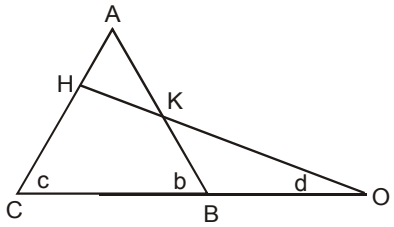
(A) 360º (B) 540º (C) 600º (D) Cannot be determined

A-3. P is a point inside ΔABC. If $\angle PBA = 20º$, $\angle BAC = 50º$ and, $\angle PCA = 35º$, then the measure of $\angle BPC$ is :



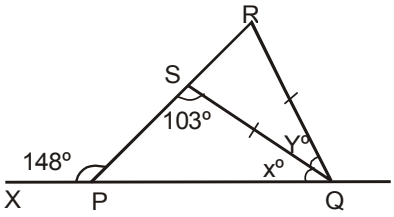
(A) 65º (B) 75º (C) 90º (D) 105º

A-4. In the given figure OBC and OKH are straight lines. If AH = AK, b = 80º and c = 30º then the value of d is :



(A) 20º (B) 25º (C) 30º (D) 45º

A-5. In given figure find the values of x and y, if QS = RQ.



(A) x = 36º, y = 32º (B) x = 45º, y = 32º (C) x = 32º, y = 45º (D) x = 45º, y = 26º

Section (B) : Congruent triangles & similar triangles

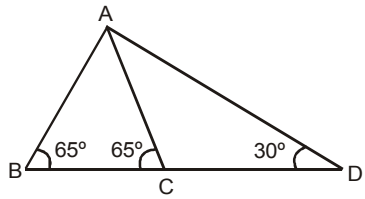
B-1. In a right angled triangle ABC, P is mid point of AC. Which one is true ?

(A) PA = $\frac{AC}{2}$ (B) PB = $\frac{AC}{2}$ (C) PA = PB (D) All of these

B-2. The sides of a triangle are in the ratio 4 : 6 : 11. Which of the following words best described the triangle?

(A) obtuse (B) isosceles (C) acute (D) impossible

B-3. In the given diagram $\angle B = \angle C = 65^o$ and $\angle D = 30^o$, then the true statement is :

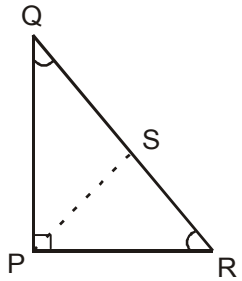


(A) BC = CA (B) CA > CD (C) BD > AD (D) AC = AD

B-4. In a ΔPQR, PS is bisector of $\angle P$, $\angle Q = 70^o$ and $\angle R = 30^o$, then :

(A) QR < PR > PQ (B) QR > PR > PQ (C) QR = PR = PQ (D) QR < PR = PQ

B-5. In the right-angled triangle QPR given below, PS is the altitude to the hypotenuse. The figure is followed by three possible inferences.



I. Triangle PQS and Triangle RPS are similar.

II. Triangle PSQ and Triangle RSP are congruent.

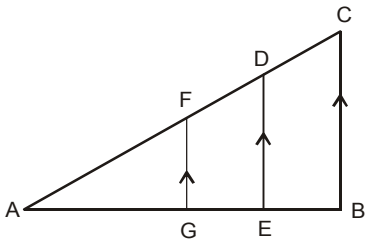
III. Triangle PSQ and triangle RPQ are similar.

Mark the correct option

(A) I and II are correct (B) I and III are correct

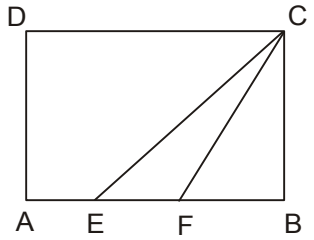
(C) only II is correct (D) All three are correct

B-6. In the given figure, ABC is a right angled triangle. Also FG || DE || BC and AG = GE = EB. If DE = 12 cm, then the measure of BC is :



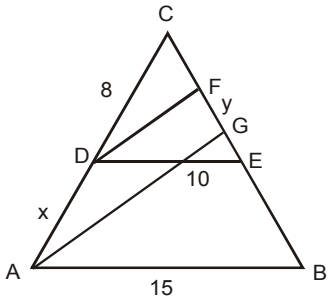
(A) 12 cm (B) 18 cm (C) 24 cm (D) 30 cm

B-7. In the below diagram, ABCD is a rectangle with AE = EF = FB. What is the ratio of the areas of ΔCEF and that of the rectangle ?



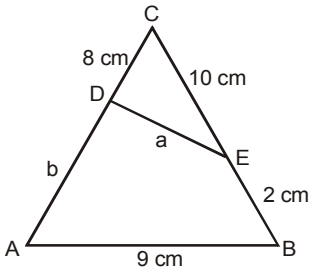
(A) $\frac{1}{6}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) None of these

B-8. In the figure DF || AG, DE || AB, AB = 15, CD = 8, AD = x, DE = 10, FG = y and CG = 6. The ratio x : y equal to :



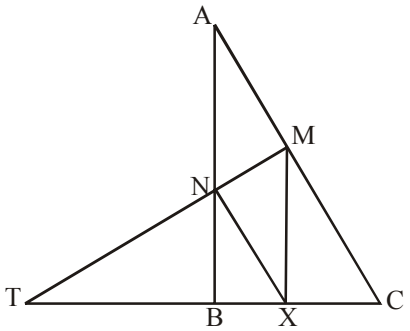
(A) 1 : 2 (B) 1: 3 (C) 2 : 1 (D) 3 : 2

B-9. In the figure $\angle$A = $\angle$CED, CD = 8 cm CE = 10 cm, BE = 2cm, AB = 9cm, AD = b and DE = a. The value of a + b is :



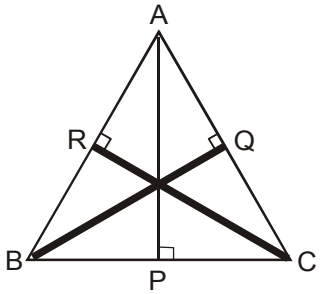
(A) 13 cm (B) 15 cm (C) 12 cm (D) 9 cm

B-10. Let X be any point on the side BC of a triangle ABC. If XM, XN are drawn parallel to BA and CA meeting CA, BA in M,N respectively ; MN meets CB produced in T. Then



(A) TB^2 = TX TC (B) TC^2 = TB TX
(C) TX^2 = TB TC (D) TX^2 = 2(TB TC)

B-11. The altitude of the given triangle are AP = 3, BQ = 4 and CR = 5. What is the ratio of BC : AC : AB ?



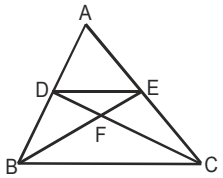
(A) 20 : 15 : 12 (B) 20 : 12 : 15 (C) 15 : 12 : 20 (D) 3 :4 : 5

B-12. The perimeters of two similar triangles are 25 cm and 15 cm respectively. If one side of first triangle is 9 cm, then the corresponding side of the other triangle is :
(A) 6.2 cm (B) 3.4 cm (C) 5.4 cm (D) 8.4 cm

B-13. In a ΔABC, AB = AC = 2.5 cm, BC = 4 cm. Find its height from A to the opposite base :
(A) 1.5 cm (B) 1 cm (C) 2 cm (D) 3 cm

B-14. Two triangles ABC and PQR are similar, if BC : CA : AB = 1 : 2 : 3, then $\frac{QR}{PR}$ is :
(A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$

B-15. In the given figure, DE || BC and AD : DB = 5 : 4, find $\frac{area(\Delta DFE)}{area(\Delta CFB)}$.



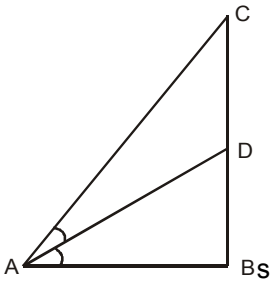
(A) 5 : 9 (B) 25 : 16 (C) 25 : 81 (D) None of these

Section (C) : Pythagoras theorem

C-1. In a triangle ABC, if $\angle$B = 90º and D is the point in BC such that BD = 2 DC, then :
(A) $AC^2 = AD^2 + 3\ CD^2$ (B) $AC^2 = AD^2 + 5\ CD^2$
(C) $AC^2 = AD^2 + 7\ CD^2$ (D) $AC^2 = AB^2 + 5\ BD^2$

C-2. Let ΔXOY be a right angled triangle with $\angle$XOY = 90º. Let M and N be the midpoints of legs OX and OY, respectively. Given that XN = 19 and YM = 22, the length XY is equal to
(A) 24 (B) 26 (C) 28 (D) 34

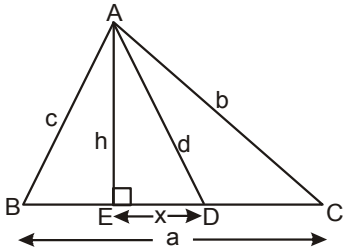
C-3. If CD = 15, DB = 9, AD bisects $\angle$A, $\angle$ABC = 90º, then AB has length :



(A) 32 (B) 18 (C) 7 (D) 24

C-4. If the sides of a right triangle are 9, 12 and 15 cm long, then the sum of squares of medians is :
(A) 227.5 (B) 337.5 (C) 537.5 (D) None of these

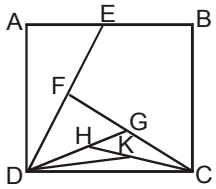
C-5. In the following figure, AE ⊥ BC, D is the mid point of BC, then x is equal to



(A) $\frac{1}{a}\left[b^2-d^2-\frac{a^2}{4}\right]$ (B) $\frac{h+d}{3}$ (C) $\frac{c+d-h}{2}$ (D) $\frac{a^2+b^2+d^2-c^2}{4}$

Section (E) : Quadrilaterals

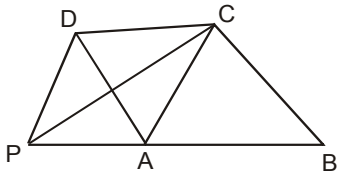
E-1. In the figure, the area of square ABCD is 4 cm² and E any point on AB. F, G, H and K are the mid point of DE, CF, DG, and CH respectively. The area of ΔKDC is -



(A) $\frac{1}{4}$ cm² (B) $\frac{1}{8}$ cm² (C) $\frac{1}{16}$ cm² (D) $\frac{1}{32}$ cm²

E-2. The line joining the mid points of the diagonals of a trapezium has length 3. If the longer base is 97, then the shorter base is :
(A) 94 (B) 92 (C) 91 (D) 90

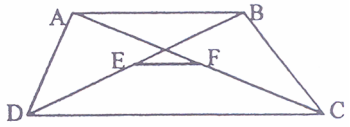
E-3. In the adjoining figure DP is parallel to AC, then the ratio of area of triangle PCB and quadrilateral ABCD is



(A) 1 : 1 (B) 1 : 2 (C) 1 : 4 (D) 2 : 3

E-4. One side of a parallelogram has length 3, and another side has length 4. Let a and b denote the lengths of the diagonals of the parallelogram. Which of the following quantities can be determined from the given information?
I. a + b **II** $a^2 + b^2$ **III** $a^3 + b^3$
(A) Only I (B) Only II (C) Only III (D) Only I and II

E-5. In the trapezium shown, AB II DC, and E and F are the midpoints of the two diagonals. If DC = 60 and EF = 5 then the length of AB is equal to :

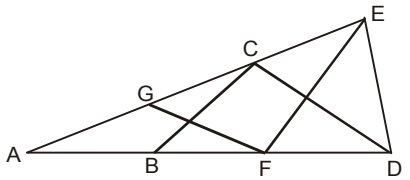


(A) 40 (B) 45 (C) 50 (D) 55

Exercise-2

OBJECTIVE QUESTIONS

1. In the figure, AB = BC = CD = DE = EF = FG = GA, then find ∠DAE (approximately)

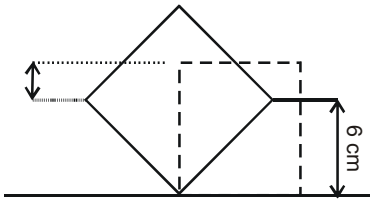


(A) 24º (B) 25º (C) 26º (D) None of these

2. If the sum of all angles except one of a convex polygon is 2180º, then the number of sides of the polygon is

(A) 19 (B) 17 (C) 15 (D) 13

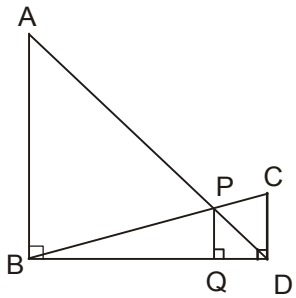
3. A square board side 10 centimeters, standing vertically, is tilted to the left so that the bottom-right corner is raised 6 centimeters from the ground.



By what distance is the top-left corner lowered from its original position ?

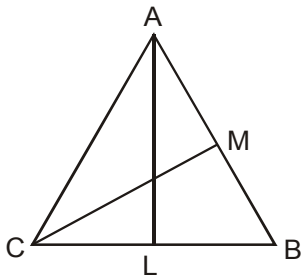
(A) 1 cm (B) 2 cm (C) 3 cm (D) 0.5 cm

4. In the diagram given below, ∠ABD = ∠CDB = ∠PQD = 90º. If AB : CD = 3 : 1, the ratio of CD : PQ is :



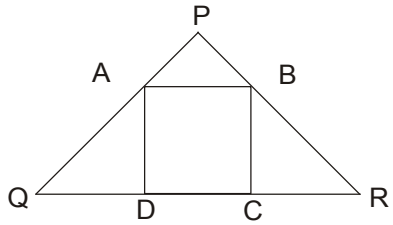
(A) 1 : 0.69 (B) 1 : 0.75 (C) 1 : 0.72 (D) None of the above.

5. In the figure below, AL is perpendicular to BC and CM is perpendicular to AB. If CL = AL = 2BL, find MC/AM.



(A) 2 (B) 3 (C) 4 (D) Cannot be determined

6. A square ABCD is constructed inside a triangle PQR having sides 10, 17 and 21 as shown in figure. Find the perimeter of the square ABCD.



(A) 28 (B) 23.2 (C) 25.4 (D) 28.8

7. ABCD is a trapezium in which AB || CD. If $\angle ADC = 2\angle ABC$, AD = a cm and CD = b cm, then the length (in cm) of AB is :

(A) $\frac{a}{2} + 2b$ (B) a + b (C) $\frac{2}{3}a + b$ (D) $a + \frac{2}{3}b$

8. E is the midpoint of diagonal BD of a parallelogram ABCD. If the point E is joined to a point F on DA such that DF = $\frac{1}{3}$ DA, then the ratio of the area of ΔDEF to the area of quadrilateral ABEF is :

(A) 1 : 3 (B) 1: 4 (C) 1 : 5 (D) 2 : 5

9. The sides of rectangle are all produced in order, in such a way that the length of each side is increased by 'k' times itself. The area of the new quadrilateral formed becomes $2\frac{1}{2}$ times the area of the original rectangle. Find the value of 'k'.

(A) (B) (C) (D) None of these

10. In the figure, the quadrilateral ABCD is a rectangle, P lies on AD and Q on AB. The triangles PAQ, QBC and PCD all have the same area, and BQ = 2. The length of AQ, is :

(A) $3+\sqrt{5}$ (B) $2\sqrt{3}$

(C) $\sqrt{5}+1$ (D) not uniquely determined

11. Find the angle between the two hands of a clock of 15 minutes past 4 O'clock

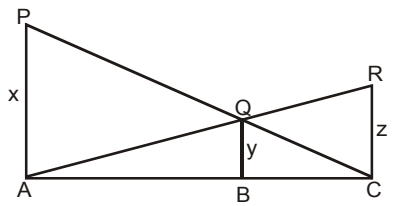
(A) 38.5^0 (B) 36.5^0 (C) 37.5^0 (D) None of these

12. At what time between 5 O'clock & 5 : 30 will the hands of a clock be at right angle ?

(A) $10\frac{10}{11}$ minutes past 5. (B) $11\frac{5}{11}$ minutes past 5.

(C) $9\frac{10}{11}$ minutes past 5. (D) $10\frac{9}{11}$ minutes past 5.

13. In figure, PA, QB and RC are each perpendicular to AC. Then : $\frac{1}{x}+\frac{1}{z}=$



(A) 1 (B) y (C) (D) None

14. In a triangle ABC, the internal bisector of the angle A meets BC at D. If AB = 4, AC = 3 and $\angle A = 60^o$, then the length of AD is :

(A) $2\sqrt{3}$ (B) $\frac{12\sqrt{3}}{7}$ (C) $\frac{15\sqrt{3}}{8}$ (D) None of these

15. In a triangle ABC, a straight line parallel to BC intersects AB and AC at point D and E respectively. If the area of ADE is one-fifth of the area of ABC and BC = 10 cm, then DE equals :
(A) 2 cm (B) 2 cm (C) 4 cm (D) 4 cm

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. L, M and N are mid points of sides AB, BC and CA of triangle ABC. If area of triangle ABC is 48 units, the area of triangle LMN will be : **[IJSO-2008]**
(A) 6 units (B) 8 units (C) 12 units (D) 24 units

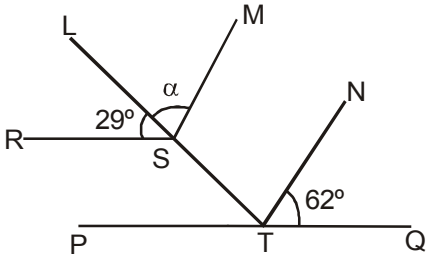
2. The adjacent sides of a parallelogram are 15 cm and 10cm. If length of one diagonal of this parallelogram is 20 cm, the length of other diagonal will be : **[IJSO-2008]**
(A) 30 cm (B) $5\sqrt{10}$ (C) $10\sqrt{10}$ cm (D) 4 cm

3. In the parallelogram, the value of "x" is : **[NSTSE-2009]**

(A) 30º (B) 50º (C) 70º (D) 80º

4. Given two similar triangles one of which has twice the perimeter of the other, by what factor is the area of the larger triangle bigger than the smaller ? **[NSTSE-2009]**
(A) 2 (B) 4 (C) (D) 2

5. In the figure shown, PQ || RS and SM || TN. Then measure of angle α is : **[IJSO-2009]**



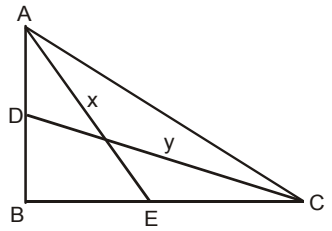
(A) 58º (B) 118º (C) 89º (D) 91º

6. The diagonals of a quadrilateral are equal and bisect each other. The quadrilateral has to be : **[IJSO-2009]**
(A) any parallelogram (B) any rectangle (C) any trapezium (D) any rhombus

7. ABCD is a parallelogram. 'P' is a point on AD such that AP = $\frac{1}{3}$ AD and 'Q' is a point on BC such that CQ = $\frac{1}{3}$ BC. Then AQCP is a : **[NSTSE-2010]**
(A) Parallelogram (B) Rhombus (C) Rectangle (D) Square

8. The degree measure of each of the three angles of a triangle is an integer. Which of the following could not be the ratio of their measures ? **[NSTSE-2010]**
(A) 2: 3 : 4 (B) 3 : 4 : 5 (C) 5 : 6 : 7 (D) 6: 7 : 8

9. ΔABC is a right angled triangle, where $\angle B = 90^o$. CD and AE are medians. If AE = x and CD = y then, correct statement is : **[NSTSE-2010]**

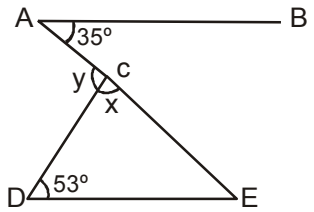


(A) $x^2 + y^2 = AC^2$ (B) $x^2 + y^2 = 2AC^2$ (C) $x^2 + y^2 = \frac{3}{2}AC^2$ (D) $x^2 + y^2 = \frac{5}{4}AC^2$

10. If the hypotenuse of a right angled triangle is 41 cm and the area of the triangle is 180 sq cm, then the difference between the lengths of the legs of the triangle must be : **[NSTSE 2010]**
(A) 22 cm (B) 25 cm (C) 27 cm (D) 31 cm

11. The sides of a triangle are in the ratio 6 : 8 : 9 . Therefore: **[IJSO-2010]**
(A) Angles of the triangle are in the ratio 6 : 8 : 9 (B) It is an acute angled triangle
(C) It is a right angled triangle (D) It is an obtuse angled triangle

12. In the figure shown AB is parallel to DE. The difference between angles x and y is : **[IJSO-2010]**



(A) 0° (B) 4° (C) 10° (D) 12°

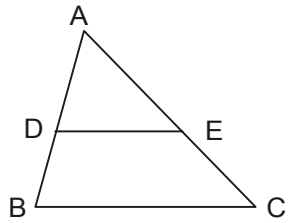
13. AD, BE and CF are the medians of ΔABC. The sum of lengths of segments BE and CF is : **[IJSO-2010]**

(A) $< \frac{3}{2}BC$ (B) $> \frac{5}{3}BC$ (C) $> \frac{3}{2}BC$ (D) $< \frac{2}{3}BC$

14. All the three sides of a Δ ABC have lengths in integral units, with AB = 2001 units and BC = 1002 = units. The possible number of triangles with this condition is : **[IJSO-2010]**
(A) 2001 (B) 2002 (C) 2003 (D) 2004

15. If the altitudes of a triangle are in the ratio 2 : 3 : 4, then the lengths of the corresponding sides are in the ratio: **[IJSO-2010]**
(A) 2 : 3 : 4 (B) 6 : 4 : 3 (C) 3 : 2 : 4 (D) 3 : 2 : 1

16. In th figure shown below, DE || BC and AD = 3x – 2, AE = 5x – 4, BD = 7x – 5 and CE = 5x – 3. Therefore , the value of x is **[IJSO-2011]**



(A) only 1 (B) only $\frac{7}{10}$ (C) 1 or $\frac{7}{10}$ (D) $\frac{7}{10}$

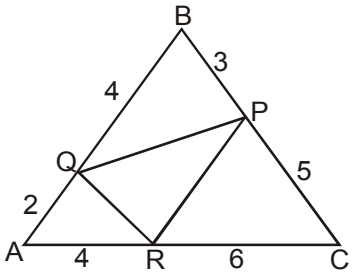
17. In Δ ABC, D is a point on side AC such that $\angle ABD = \frac{1}{2} \angle ABC$. If AB = 36, BC = 48, CD = 28, then the length DA will be **[IJSO-2011]**

(A) 20 (B) 21 (C) 22 (D) 24

18. Consider triangles having integer sides such that no side is greater than 4 units. How many such triangles are possible? **[IJSO-2012]**

(A) 13 (B) 17 (C) 24 (D) 64

19. In the adjoining figure AQ = 2, QB = 4, BP = 3, PC = 5, CR = 6 and RA = 4. Find the area of triangle PQR.



[IJSO-2012]

(A) 4.8 (B) 5.2 (C) 5.8 (D) 6.2

20. Let ABC be a triangle in which AB = AC. Let D be a point on BC such that AD bisects angle A. Value of the ratio $\frac{BD}{DC}$ is **[IJSO-2013]**

(A) 1:1 (B) 1:2 (C) 2:1 (D) none of these

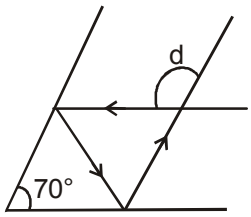
21. Let m be the number of distinct (non congruent) integer-sided triangles each with perimeter 15 and n be the number of distinct (non congruent) integer sided triangles each with perimeter 16. Then m-n equals **[IJSO-2013]**

(A) –2 (B) 0 (C) 2 (D) – 4

22. At what time (to the nearest second) immediately after 4 O' clock will angle between the hands of the clock be the same as that at 4 O' clock ? **[IJSO-2013]**

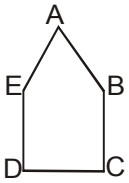
(A) $4^h\ 42^m\ 50^s$ (B) $4^h 43^m 38^s$ (C) $4^h 43^m 40^S$ (D) $5^h\ 5^m 27^s$

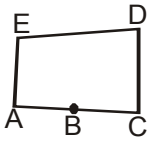
23. A ray of light is incident on system of mirror as shown in the adjacent figure. What is the total deflection (d) of the ray when it emerges out after two reflections ? **[IJSO-2014]**

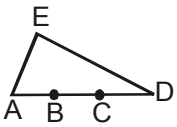


(A) 220° (B) 180° (C) 120° (D) 140°

24. If the distance between A and B is 230 km, B and C is 120 km, C and A is 350 km. Also, if the distance between C and D is 200 km, distance between D and B is 330 km and distance from A to E is 100 km and distance between D and E is 570 km. The diagram (not drawn to scale) that represents this graphically is: **[IJSO-2014]**

(A) 

(B) 

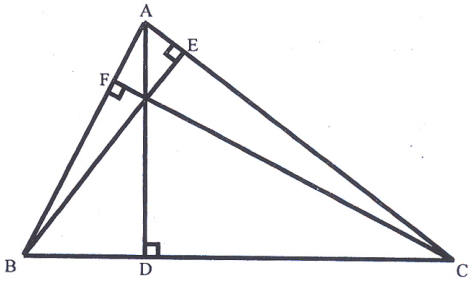
(C) 

(D) none of these

25. The adjacent sides of a parallelogram are 30 cm and 20 cm. The length of one of the diagonal is 40 cm. What is the length of the other diagonal? **[IJSO-2015]**

(A) 60 cm (B) $10\sqrt{10}$ cm (C) $20\sqrt{5}$ cm (D) $8\sqrt{30}$ cm

26. In the adjoining figure segment AD, BE and CF are the altitudes of triangle ABC. Find AD × BC if AB × AC = 409.6, BE × CF = 202.5. **[IJSO-2015]**



(A) 225 (B) 256 (C) 288 (D) 312

27. The angle between the hour arm and the minute arm of a clock at 2:10 a.m. is **[IJSO-2016]**

(A) zero (B) 4º (C) 5º (D) 6º

28. If ABCD is a rhombus and ∠ ABC = 60 º then **[IJSO-2016]**

(A) The point A, B, C, D are concylic

(B) The quadrilateral has exactly half the area of the square with same sides as ABCD

(C) The quadrilateral has area $\frac{\sqrt{3}}{2}$ AB^2

(D) The diagonals of the quadrilateral ABCD are equal and bisect each other at right angle

29. Diagonals of a quadrilateral bisect each other. Therefore the quadrilateral must be a **[IJSO 2017]**

(A) rectangle (B) rhombus (C) parallelogram (D) square

30. What is the radius of the circumcircle of a triangle whose sides are 30 cm, 36 cm and 30 cm. **[IJSO 2017]**

(A) 15 cm (B) 16 cm (C) 17 cm (D) 18 cm

31. In triangle ABC, segment AD, segment BE and segment CF are altitudes. If AB × AC = 172.8 cm^2 and BE × CF = 108.3 cm^2 then AD × BC = **[IJSO 2017]**

(A) 136.8 cm^2 (B) 132.4 cm^2 (C) 129.2 cm^2 (D) 128.6 cm^2

32. Let ABCD be a rectangle. Let X and Y be points respectively on AB and CD such that AX : XB = 1 : 2 = CY : YD. Join AY and CX ; let BY intersect CX in K; let DX intersect AY in L. If $\frac{m}{n}$ denotes the ratio of the area XKYL to that of ABCD, then m + n equals **[IJSO 2018]**

(A) 9 (B) 11 (C) 13 (D) 15

33. In a triangle ABC, let AD be the median from A; Let E be a point on AD such that AE : ED = 1 : 2 ; and let BE extended meets AC in F. The ratio of AF/FC is **[IJSO 2018]**

(A) 1/6 (B) 1/5 (C) ¼ (D) 1/3

34. In a triangle ABC, a point D on AB is such that AD : AB = 1 : 4 and DE is parallel to BC with E on AC. Let M and N be the mid points of DE and BC respectively. What is the ratio of the area of the quadrilateral BNMD to that of triangle ABC? **[IJSO 2018]**

(A) $\frac{1}{4}$ (B) $\frac{9}{32}$ (C) $\frac{7}{32}$ (D) $\frac{15}{32}$

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 126º **A-2** 54º **A-3** **(a)** 125º 32´ **(b)** 81º 24' 40´´

Section (B)

B-6 30 unit **B-7** 5 : 4 **B-9** PN = 15 cm and RM = 10.67 cm

Section (C)

C-1 $3\sqrt{2}$

Section (D)

D-4 5 cm.

OBJECTIVE QUESTIONS

Section (A)

A-1. (A) **A-2.** (B) **A-3.** (D) **A-4.** (B) **A-5.** (D)

Section (B)

B-1. (D) **B-2.** (D) **B-3.** (C) **B-4.** (B) **B-5.** (B)

B-6. (B) **B-7.** (A) **B-8.** (C) **B-9.** (A) **B-10.** (C)

B-11. (A) **B-12.** (C) **B-13.** (A) **B-14.** (B) **B-15.** (C)

Section (C)

C-1. (B) **C-2.** (B) **C-3.** (B) **C-4.** (B) **C-5.** (A)

Section (D)

D-1. (B) **D-2.** (C) **D-3.** (A) **D-4.** (B) **D-5.** (C)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	C	C	B	B	B	B	B	C	A	C	C	A	C	B	B

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

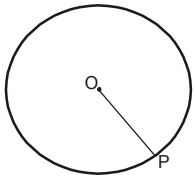
Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	C	B	C	B	A	D	D	D	B	B	C	C	B	A	B	A	C	A
Ques.	21	22	23	24	25	26	27	28	29	30	31	32	33	34						
Ans.	C	B	D	D	B	C	C	C	C	D	A	A	C	D						

PLANE GEOMETRY - II

A. DEFINITIONS

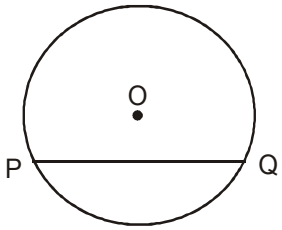
❖ **Circle :** The collection of all the points in a plane, which are at a fixed distance from a fixed point in the plane, is called a **circle**.

The fixed point is called the **centre** of the circle and the fixed distance is called the **radius** of the circle.

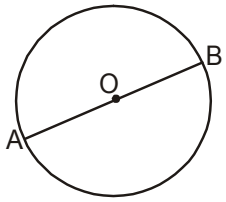


In figure, **O** is the **centre** and the length **OP** is the **radius** of the circle. So the line segment joining the centre and any point on the circle is called a radius of the circle.

❖ **Chord :** If we take two points P and Q on a circle, then the line segment PQ is called a **chord** of the circle.

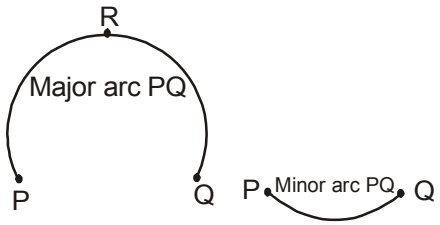


❖ **Diameter :** The chord which passes through the centre of the circle, is called the **diameter** of the circle.



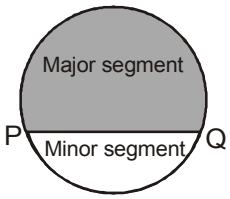
A diameter is the longest chord and all diameters of same circle have the same length, which is equal to two times the radius. In figure, AOB is a **diameter** of circle.

❖ **Arc :** A piece of a circle between two points is called an **arc**. The longer one is called the **major arc PQ** and the shorter one is called the **minor arc PQ**. The minor arc PQ is also denoted by $\overset{\frown}{PQ}$ and the major arc PQ by $\overset{\frown}{QP}$. When P and Q are ends of a diameter, then both arcs are equal and each is called a **semi circle**.

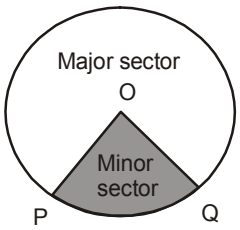


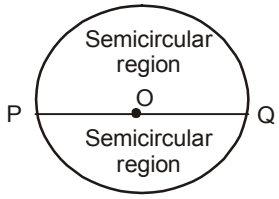
❖ **Circumference :** The length of the complete circle is called its **circumference**.

Segment : The region between a chord and either of its arcs is called a **segment** of the circular region or simply a segment of the circle. There are two types of segments which are the **major segment** and the **minor segment** (as in **figure**).

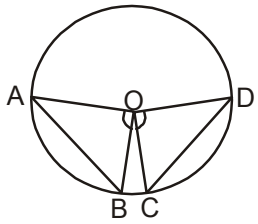


Sector : The region between an arc and the two radii, joining the centre to the end points of an arc is called a **sector**. Minor arc corresponds to the **minor sector** and the major arc corresponds to the **major sector**. When two arcs are equal, then both segments and both sectors become the same and each is known as a **semicircular region**.





Theorem : Equal chords of a circle subtend equal angles at the centre.
Given : AB and CD are the two equal chords of a circle with centre O.

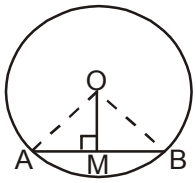


To Prove : $\angle AOB = \angle COD$.
Proof : In ΔAOB and ΔCOD,

	OA = OC	[Radii of a circle]
	OB = OD	[Radii of a circle]
	AB = CD	[Given]
$\therefore$	$\Delta AOB \cong \Delta COD$	[By SSS congruency]
$\therefore$	$\angle AOB = \angle COD$.	[By CPCT] **Hence Proved.**

Converse : If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.

Theorem : The perpendicular from the centre of a circle to a chord bisects the chord.



Given : A circle with centre O. AB is a chord of this circle. OM $\perp$ AB.
To Prove : MA = MB.
Construction : Join OA and OB.
Proof : In right triangles OMA and OMB,

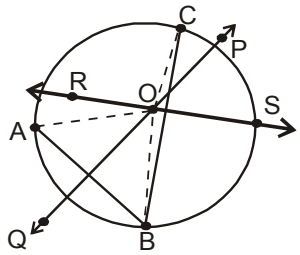
	OA = OB	[Radii of a circle]
	OM = OM	[Common]
	$\angle OMA = \angle OMB$	[90º each]
$\therefore$	$\Delta OMA \cong \Delta OMB$	[By RHS]
$\therefore$	MA = MB	[By cpct]

Hence Proved.

Converse : The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

Theorem : There is one and only one circle passing through three given non-collinear points.

Proof : Take three points A, B and C, which are not in the same line, or in other words, they are not collinear [as in figure]. Draw perpendicular bisectors of AB and BC say, PQ and RS respectively. Let these perpendicular bisectors intersect at one point O.(Note that PQ and RS will intersect because they are not parallel) [as in figure].



$\because$ O lies on the perpendicular bisector PQ of AB.

$\therefore$ OA = OB

[$\because$ Every point on the perpendicular bisector of a line segment is equidistant from its end points]

Similarly,

$\because$ O lies on the perpendicular bisector RS of BC.

$\therefore$ OB = OC

[$\because$ Every point on the perpendicular bisector of a line segment is equidistant from its end points]

So, OA = OB = OC

i.e., the points A, B and C are at equal distances from the point O.

So, if we draw a circle with centre O and radius OA it will also pass through B and C. This shows that there is a circle passing through the three points A, B and C. We know that two lines (perpendicular bisectors) can intersect at only one point, so we can draw only one circle with radius OA. In other words, there is a unique circle passing through A, B and C.

Hence Proved.

Solved Examples

Example. 1

The radius of a circle is 25 cm and the length of one of its chords is 14 cm. Find the distance of the chord from the centre.

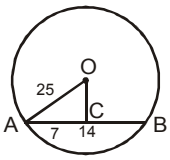
Solution

Let O be the center of the circle of radius 25 cm and AB is the chord of length 14 cm

$OC \perp AB$

$AC = \frac{AB}{2} = \frac{14}{2} = 7$ [Line perpendicular from centre to chord bisect the chord]

In ΔAOC



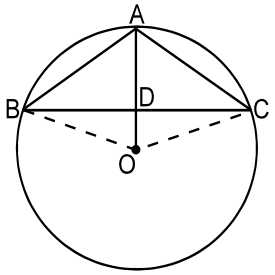
$(OC)^2 + (AC)^2 = (AO)^2$

$(OC)^2 = (25)^2 - (7)^2$

OC = 24 cm.

Example.2

In figure, $\overset{\frown}{AB} \cong \overset{\frown}{AC}$ and O is the centre of the circle. Prove that OA is the perpendicular bisector of BC.



Solution.

Given : In figure, $\overset{\frown}{AB} \cong \overset{\frown}{AC}$ O is the centre of the circle.
To Prove : OA is the perpendicular bisector of BC.
Construction : Join OB and OC.
Proof :

$\because$ $\overset{\frown}{AB} \cong \overset{\frown}{AC}$ [Given]

chord AB = chord AC.

[$\because$ If two arcs of a circle are congruent, then their corresponding chords are equal]

$\therefore$ $\angle AOB = \angle AOC$ ...(i)

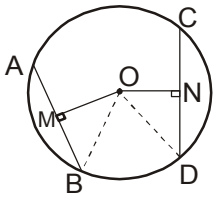
[$\because$ Equal chords of a circle subtend equal angles at the centre]

In ΔOBD and ΔOCD,

$\angle DOB = \angle DOC$ [From (i)]
OB = OC [Radii of the same circle]
OD = OD [Common]
$\therefore$ $\Delta OBD \cong \Delta OCD$ [By SAS congruency]
$\therefore$ $\angle ODB = \angle ODC$ ...(ii) [By CPCT]
And, BD = CD ...(iii) [By CPCT]
But $\angle BDC = 180^o$
$\therefore$ $\angle ODB + \angle ODC = 180^o$
$\Rightarrow$ $\angle ODB + \angle ODB = 180^o$ [From equation(ii)]
$\Rightarrow$ $2\angle ODB = 180^o$
$\Rightarrow$ $\angle ODB = 90^o$
$\therefore$ $\angle ODB = \angle ODC = 90^o$ ...(iv) [From (ii)]

So, by (iii) and (iv), OA is the perpendicular bisector of BC. **Hence Proved.**

Theorem : Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).



Given : A circle have two equal chords AB & CD. i.e. AB = CD and OM $\perp$ AB, ON $\perp$ CD.
To Prove : OM = ON
Construction : Join OB & OD
Proof : AB = CD (Given)

[$\because$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

$\therefore$ AB = CD $\Rightarrow$ BM = DN

In ΔOMB & ΔOND

$\angle OMB = \angle OND = 90^o$	[Given]
OB = OD	[Radii of same circle]
BM = DN	[Proved above]
$\therefore$ Δ OMB $\cong$ Δ OND	[By R.H.S. congruency]
$\therefore$ OM = ON	[By CPCT] **Hence Proved.**

❖ **REMARK :**

Chords equidistant from the centre of a circle are equal in length.

Solved Examples

Example.3

AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E. Prove that EB = ED.

Solution

Given : AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E.

To Prove : EB = ED.

Construction : From O draw OP $\perp$ AB and OQ $\perp$ CD. Join OE.

Proof : $\because$ AB = CD [Given]

$\therefore$ OP = OQ [Equal chords of a circle are equidistant from the centre]

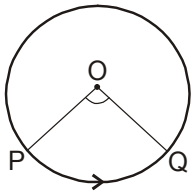
Now in ΔOPE and ΔOQE,

$\angle OPE = \angle OQE$	[Each 90º]
OE = OE	[Common]
OP = OQ	[Proved above]
$\therefore$ $\Delta OPE \cong \Delta OQE$	[By RHS congruency]
$\therefore$ PE = QE	[By CPCT]
$\Rightarrow$ $PE - \frac{1}{2}AB = QE - \frac{1}{2}CD$	[$\because$ AB = CD (Given)]
$\Rightarrow$ PE – PB = QE – QD	$\Rightarrow$ EB = ED. **Hence Proved.**

❖ **REMARK :**

Angle Subtended by an Arc of a Circle :

In figure, the angle subtended by the minor arc PQ at O is $\angle POQ$ and the angle subtended by the major arc PQ at O is reflex angle $\angle POQ$.

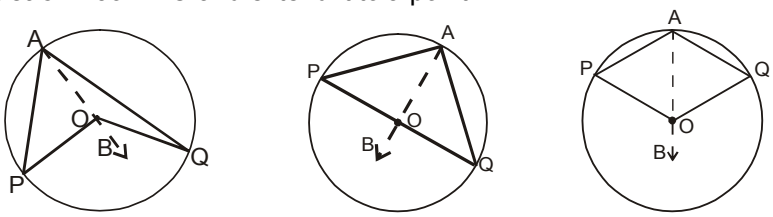


Theorem : The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Given : An arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle.

To Prove : $\angle POQ = 2\angle PAQ$.

Construction : Join AO and extend it to a point B.



(A) **(B)** **(C)**

Proof : There arises three cases:-
(A) arc PQ is minor
(B) arc PQ is a semi-circle
(C) arc PQ is major.
In all the cases,

$\angle BOQ = \angle OAQ + \angle AQO$ ...(i)

[$\because$ An exterior angle of a triangle is equal to the sum of the two interior opposite angles]

In ΔOAQ,

$OA = OQ$ [Radii of a circle]

$\therefore$ $\angle OQA = \angle OAQ$ ...(ii)

[Angles opposite equal sides of a triangle are equal]

From (i) and (ii)

$\angle BOQ = 2\angle OAQ$ ...(iii)

Similarly,

$\angle BOP = 2\angle OAP$ ...(iv)

Adding (iii) and (iv), we get

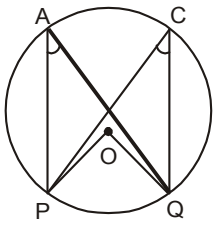
$\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$

$\Rightarrow$ $\angle POQ = 2\angle PAQ.$ ...(v)

❖ **Note :**

For the case (C), where PQ is the major arc, (v) is replaced by reflex angles.
Thus, reflex $\angle POQ = 2\angle PAQ$.

Theorem : Angles in the same segment of a circle are equal.
Proof : Let P and Q be any two points on a circle to form a chord PQ, A and C any other points on the remaining part of the circle and O be the centre of the circle. Then,



$\angle POQ = 2\angle PAQ$ ...(i)

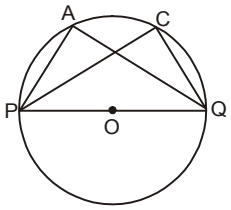
And $\angle POQ = 2\angle PCQ$ ...(ii)

[Angle subtended at the centre is double than the angle subtended by it on the remaining part of the circle]

From equation (i) & (ii)

$2\angle PAQ = 2\angle PCQ$

$\Rightarrow$ $\angle PAQ = \angle PCQ.$ **Hence Proved.**



Theorem : Angle in the semicircle is a right angle.
Proof : $\angle PAQ$ is an angle in the segment, which is a semicircle.

$\therefore$ $\angle PAQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180^\circ = 90^\circ$ [$\because$ POQ is straight line angle or $\angle POQ = 180^\circ$]

If we take any other point C on the semicircle, then again we get

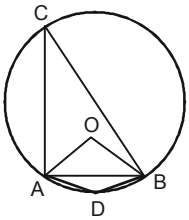
$\angle PCQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180^\circ = 90^\circ.$ **Hence Proved.**

Solved Examples

Example.4.

A chord of a circle is equal to the radius of the circle, find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

Solution



Let AB be a chord of the circle with centre at O such that OA = OB = AB.

ΔOAB is equilateral.

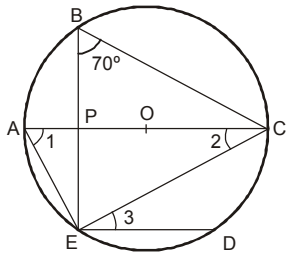
$\therefore \quad \angle AOB = 60^o \quad \Rightarrow \quad \angle ACB = \frac{1}{2}\angle AOB = 30^o$

Consider arc ACB.

Clearly, it makes 360º – 60º = 300º at the centre O.

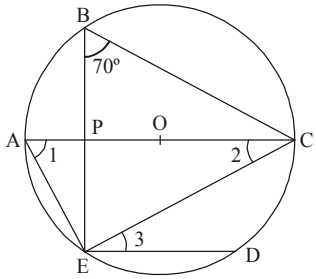
Example.5.

In the given figure, the chord ED is parallel to the diameter AC. Find $\angle$CED.



Solution

$\angle CBE = \angle 1$		[Angles in the same segment]
$\angle 1 = 70^o$	…(i)	[$\because \angle CBE = 70^o$]
$\angle AEC = 90^o$	..(ii)	[Angle in a semicircle is a right angle]



Now, in ΔAEC,

$\angle 1 + \angle AEC + \angle 2 = 180^o$

$\therefore \quad 70^o + 90^o + \angle 2 = 180^o \quad \Rightarrow \quad \angle 2 = 180^o - 160^o = 20^o$

Thus $\angle 2 = 20^o$ ...(iii)

Also, ED || AC [Given]

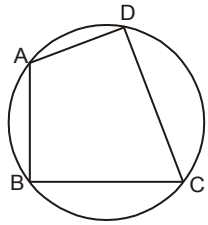
$\therefore \quad \angle 2 = \angle 3$ [Alternate angles]

$\therefore \quad \angle 3 = 20^o$

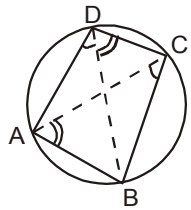
Hence, $\angle$ CED = 20º.

B. CYCLIC QUADRILATERAL

A quadrilateral ABCD is called **cyclic** if all the four vertices of it lie on a circle.



Theorem : The sum of either pair of opposite angles of a cyclic quadrilateral is 180º.
Given : A cyclic quadrilateral ABCD.
To Prove : ∠A + ∠C = ∠B + ∠D = 180º.
Construction : Join AC and BD.
Proof : ∠ACB = ∠ADB ...(i)



And ∠BAC = ∠BDC ...(ii)
[Angles of same segment of a circle are equal]
Adding equation (i) & (ii)
∴ ∠ACB + ∠BAC = ∠ADB + ∠BDC
∠ACB + ∠BAC = ∠ADC.
Adding ∠ABC to both sides, we get
∠ACB + ∠BAC + ∠ABC = ∠ADC + ∠ABC.
∴ ∠ADC + ∠ABC = 180º
i.e., ∠D + ∠B = 180º
∴ ∠A + ∠C = 360º – (∠B + ∠D) = 180º [∵ $\angle A + \angle B + \angle C + \angle D = 360^\circ$] **Hence Proved.**

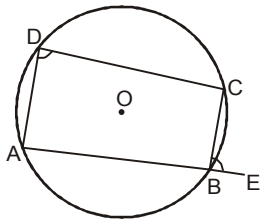
Solved Examples

Example.6

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Solution

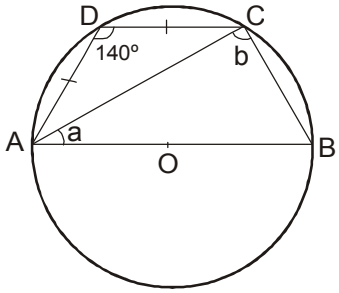
Let ABCD be a cyclic quadrilateral inscribed in a circle with centre O. The side AB of quadrilateral ABCD is produced to E. Then, we have to prove that
∠CBE = ∠ADC.



Since the sum of opposite pairs of angles of a cyclic quadrilateral is 180°
∠ABC + ∠ADC = 180°
But ∠ABC + ∠CBE = 180° [∵∠ABC and ∠CBE form a linear pair]
∴ ∠ABC + ∠ADC = ∠ABC + ∠CBE
⇒ ∠ADC = ∠CBE
or ∠CBE = ∠ADC. **Hence proved**

Example.7

Find the value of a and b.



Solution

In ΔADC

AD = DC

$\angle DAC = \angle DCA = x$ (say)

$140^\circ + x + x = 180^\circ$

$2x = 40^\circ \quad \Rightarrow \quad x = 20^\circ$

$\angle ADC + \angle ABC = 180^\circ$ [Opposite angles in a cyclic quadrilateral are supplementary]

$140^\circ + \angle ABC = 180^\circ$

$\angle ABC = 40^\circ$

$\angle b = 90^\circ$ [Angle in a semicircle]

$\angle a = 180^\circ - \angle ABC - \angle b = 180^\circ - 40^\circ - 90^\circ$

$\angle a = 180^\circ - 130^\circ = 50^\circ$.

So, $a = 50^\circ$ and $b = 90^\circ$.

Example.8

If the non-parallel side of a trapezium are equal, prove that it is cyclic.

Solution

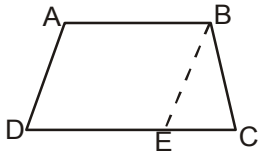
Given : ABCD is a trapezium whose two non-parallel sides AD and BC are equal.

To Prove : Trapezium ABCD is a cyclic.

Construction : Draw BE || AD.

Proof : $\because$ AB || DE [Given]

and AD || BE [By construction]



$\therefore$ Quadrilateral ABED is a parallelogram.

$\therefore$ $\angle BAD = \angle BED$ ...(i) [Opp. angles of a $||^{gm}$]

And, AD = BE ...(ii) [Opp. sides of a $||^{gm}$]

But AD = BC ...(iii) [Given]

From (ii) and (iii),

BE = BC

$\therefore$ $\angle BCE = \angle BEC$ ...(iv) [Angles opposite to equal sides]

$\angle BEC + \angle BED = 180^\circ$ [Linear Pair Axiom]

$\angle BCE + \angle BAD = 180^\circ$ [From (iv) and (i)]

Trapezium ABCD is cyclic.

[$\because$ If a pair of opposite angles of a quadrilateral is 180°, then the quadrilateral is cyclic]

Hence Proved.

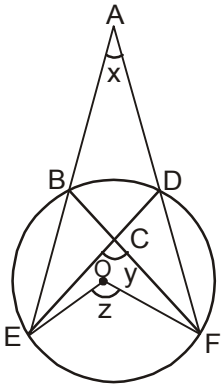
Example.9

In figure, O is the centre of the circle. Prove that $\angle x + \angle y = \angle z$.

Solution

$\angle EBF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$

[$\because$ Angle subtended by an arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]



$\angle ABF = 180^\circ - \frac{1}{2}\angle z$ ...(i) [Linear Pair Axiom]

$\angle EDF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$

[$\because$ Angle subtended by any arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$\therefore$ $\angle ADE = 180^\circ - \frac{1}{2}\angle z$ ...(ii) [Linear Pair Axiom]

$\angle BCD = \angle ECF = \angle y$ [Vertically Opp. Angles]

$\angle BAD = \angle x$

In quadrilateral ABCD

$\angle ABC + \angle BCD + \angle CDA + \angle BAD = 2 \times 180^\circ$

$\Rightarrow$ $180^\circ - \frac{1}{2}\angle z + \angle y + 180^\circ - \frac{1}{2}\angle z + \angle x = 2 \times 180^\circ$

$\Rightarrow$ $\angle x + \angle y = \angle z$. **Hence Proved.**

Example.10

AB is a diameter of the circle with centre O and chord CD is equal to radius OC. AC and BD produced meet at P. Prove that $\angle CPD = 60^\circ$.

Solution

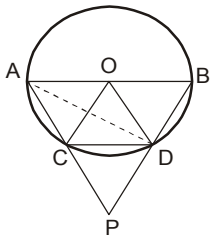
Given : AB is a diameter of the circle with centre O and chord CD is equal to radius OC. AC and BD produced meet at P.

To Prove : $\angle CPD = 60^\circ$.

Construction : Join AD.

Proof : In OCD,

OC = OD ...(i)



[Radii of the same circle]

OC = CD ...(ii) [Given]

From (i) and (ii),

OC = OD = CD

ΔOCD is an equilateral triangle.

$\angle COD = 60^o$

$\angle CAD = \frac{1}{2}\angle COD = \frac{1}{2}\angle(60^o) = 30^o$

[$\because$ Angle subtended by any arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$\Rightarrow$ $\angle PAD = 30^o$ ...(iii)

And, $\angle ADB = 90^o$ ...(iv) [Angle in a semi-circle]

$\Rightarrow$ $\angle ADB + \angle ADP = 180^o$ [Linear Pair Axiom]

$\Rightarrow$ $90^o + \angle ADP = 180^o$ [From (iv)]

$\Rightarrow \angle ADP = 90^o$ ...(v)

In ΔADP,

$\angle APD + \angle PAD + \angle ADP = 180^o$

$\Rightarrow \angle APD + 30^o + 90^o = 180^o$ [From (iii) and (v)]

$\Rightarrow \angle APD + 120^o = 180^o$

$\Rightarrow \angle APD = 180^o - 120^o = 60^o$

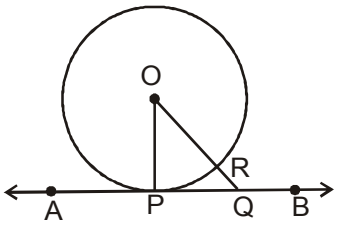
$\Rightarrow \angle CPD = 60^o$. **Hence Proved.**

C. SECANT AND TANGENT

Secant to a circle is a line which intersects the circle in two distinct points.

A **tangent** to a circle is a line that intersects the circle in exactly one point.

Theorem : A tangent to a circle is perpendicular to the radius through the point of contact.



Given : A circle C (O, r) and a tangent AB at a point P.

To prove : $OP \perp AB$

Construction : Take any point Q, other than P on the tangent AB. Join OQ. Suppose OQ meets the circle at R.

Proof : Among all line segments joining the point O to a point on AB, the shortest one is perpendicular to AB. So, to prove that $OP \perp AB$, it is sufficient to prove that OP is shorter than any other segment joining O to any point of AB.

Clearly OP = OR (Radius)

Now, OQ = OR + RQ

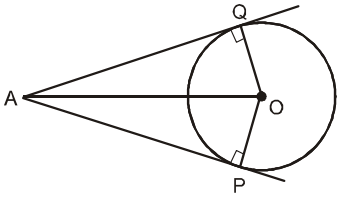
$\Rightarrow$ OQ > OR

$\Rightarrow$ OQ > OP ($\because$ OP = OR)

Thus, OP is shorter than any other segment joining O to any point of AB.

Hence, $OP \perp AB$.

Theorem : Lengths of two tangents drawn from an external point to a circle are equal.



Given : AP and AQ are two tangents drawn from a point A to a circle C (O, r).

To prove : AP = AQ

Construction : Join OP, OQ and OA.

Proof : In ΔAOQ and ΔAPO

$\angle OQA = \angle OPA$ [Tangent at any point of a circle is perp. to radius through the point of contact]

AO = AO [Common]

OQ = OP [Radius]

So, by R.H.S. criterion of congruency $\Delta AOQ \cong \Delta AOP$

$\therefore$ AQ = AP [By CPCT] **Hence Proved.**

RESULTS :

(i) If two tangents are drawn to a circle from an external point, then they subtend equal angles at the centre. $\angle$**QOA =** $\angle$**POA** [By CPCT]

(ii) If two tangents are drawn to a circle from an external point, they are equally inclined to the segment, joining the centre to that point $\angle$**OAQ =** $\angle$**OAP** [By CPCT]

Solved Examples

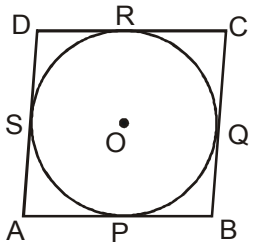
Example.11

If all the sides of a parallelogram touches a circle, show that the parallelogram is a rhombus.

Solution

Given : Sides AB, BC, CD and DA of a $||^{gm}$ ABCD touch a circle at P, Q, R and S respectively.

To prove : $||^{gm}$ ABCD is a rhombus.



Proof : AP = AS ....(i)

BP = BQ ....(ii)

CR = CQ ....(iii)

DR = DS ....(iv) [Tangents drawn from an external point to a circle are equal]

Adding (i), (ii), (iii) and (iv), we get

$\Rightarrow$ AP + BP + CR + DR = AS + BQ + CQ + DS

$\Rightarrow$ (AP + BP) + (CR + DR) = (AS + DS) + (BQ + CQ)

$\Rightarrow$ AB + CD = AD + BC

$\Rightarrow$ AB + AB = AD + AD [In a $||^{gm}$ ABCD, opposite sides are equal]

$\Rightarrow$ 2AB = 2AD or AB = AD

But AB = CD and AD = BC [Opposite sides of a || gm]

$\therefore$ AB = BC = CD = DA

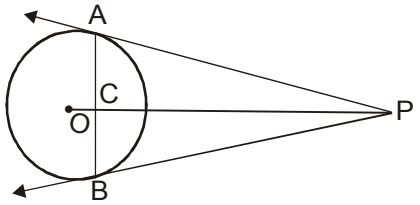
Hence, $||^{gm}$ ABCD is a rhombus.

Example.12

Prove that the tangents at the extremities of any chord make equal angles with the chord.

Solution

Let AB be a chord of a circle with centre O, and let AP and BP be the tangents at A and B respectively. Suppose, the tangents meet at point P. Join OP. Suppose OP meets AB at C.



We have to prove that $\angle PAC = \angle PBC$

In ΔPCA and ΔPCB,

PA = PB [$\therefore$ Tangent from an external point are equal]

$\angle APC = \angle BPC$ [$\because$ PA and PB are equally inclined to OP]

And PC= PC [Common]

So, by SAS criteria of congruence

$\Delta PAC \cong \Delta PBC$

$\Rightarrow$ $\angle PAC = \angle PBC$ [By CPCT]

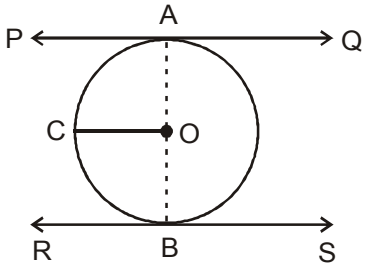
Example.13

Prove that the segment joining the points of contact of two parallel tangents passes through the centre.

Solution

Let PAQ and RBS be two parallel tangents to a circle with centre O. Join OA and OB. Draw OC||PQ.

Now, PA || CO



$\Rightarrow$ $\angle PAO + \angle COA = 180^o$ [Sum of co-interior angle is 180°]

$\Rightarrow$ $90^o + \angle COA = 180^o$ [$\therefore$ $\angle PAO = 90^o$]

$\Rightarrow$ $\angle COA = 90^o$

Similarly, $\angle COB = 90^o$

$\therefore$ $\angle COA + \angle COB = 90^o + 90^o = 180^o$

Hence, AOB is a straight line passing through O.

Solved Examples

D. SEGMENTS OF A CHORD

Let AB be a chord of a circle, and let P be a point on AB inside the circle. Then, P is said to divide AB internally into two segments PA and PB.

Theorem : If two chords of a circle intersect inside or outside the circle when produced, the rectangle fromed by two segments of one chord is equal in area to the rectangle formed by the two segments of another chord.

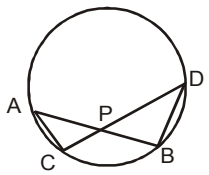


Fig. (i)

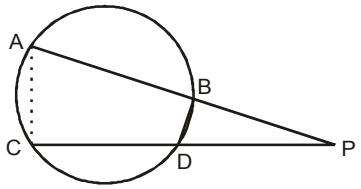


Fig. (ii)

Given : Two chords AB and CD of a circle C(O, r) intersecting at P, inside in fig. (i) and outside in fig.(ii).

To prove : PA . PB = PC. PD

Construction : Join AC and BD.

Proof :

Case 1. When AB and CD intersect at P inside the circle.

In Δ's APC and BPD,

$\angle APC = \angle BPD$ [Vertically opposite angles]

$\angle PAC = \angle PDB$ [Angles in the same segment]

$\therefore$ $\Delta APC \sim \Delta BPD$

Hence, $\frac{PA}{PD} = \frac{PC}{PB}$ ` $\Rightarrow$ $PA \times PB = PC \times PD$.

Case 2. When AB and CD when produced intersect at P outside the circle.

In ΔPBD and ΔPAC, $\angle PBD = \angle ACP$

[ABCD is a cyclic quadrilateral and exterior angle is equal to the interior opposite angle]

$\angle PDB = \angle CAP$

[ABCD is a cyclic quadrilateral and exterior angle is equal to the interior opposite angle]

$\therefore$ $\Delta PBD \sim \Delta PCA$

Hence, $\frac{PB}{PC} = \frac{PD}{PA}$ $\Rightarrow$ $PA \times PB = PC \times PD$.

Theorem : If PAB is a secant to a circle intersecting the circle at A and B and PT is a tangent segment, then $PA \times PB = PT^2$.

Given : A circle with centre O; PAB is a secant intersecting the circle at A and B and PT is a tangent segment to the circle.

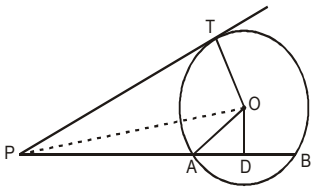
To prove : $PA \times PB = PT^2$.

Construction : Join OA, OP and OT. Draw $OD \perp AB$.

Proof : Since $OD \perp$ chord AB

$\therefore$ AD = DB.

Now $PA \times PB = (PD - AD)(PD + DB)$



$= (PD - AD)(PD + AD)$ [$\because$ AD = DB]

$= PD^2 - AD^2$

$= (OP^2 - OD^2) - AD^2$ [From right-angled ΔODP]

$= OP^2 - (OD^2 + AD^2)$

$= OP^2 - OA^2$ [From right-angled ΔODA]

$= OP^2 - OT^2$ $\because$ OA = OT [Radii]

$= PT^2$ [From right-angled ΔOTP]

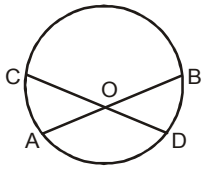
Hence, $PA \times PB = PT^2$.

Solved Examples

Example.14

Two chords AB and CD of a circle intersect each other at O internally. If AO = 3.5 cm, CO = 5 cm and DO = 7 cm, find OB.

Solution



We have, $AO \times OB = CO \times OD \Rightarrow (3.5) \times OB = 5 \times 7 = 35$

$OB = \frac{35}{3.5} = 10$ cm.

Example.15

In ΔABC, $AP \perp BC$ and $BQ \perp AC$ which intersect each other at O. Prove that $AO \times OP = BO \times OQ$.

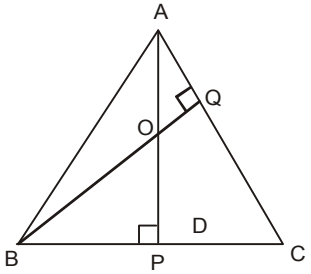
Solution

We have,

$\angle APB = 90^o$ and $\angle AQB = 90^o$ [$\because$ $AP \perp BC$ and $BQ \perp AC$]

In Δ AOQ & Δ BOP

$\angle AOQ = \angle BOP$ (Vertically opp. angles)



OQA = OPB = 90º

By, AA Similarity

$\Delta AOQ \sim \Delta BOP$

$\frac{AO}{OB} = \frac{OQ}{OP} \Rightarrow AO \times OP = OB \times OQ$

Example.16

In ΔABC, the bisector of $\angle A$ intersects the base BC at the point D.

Prove that $AB \times AC = BD \times DC + AD^2$.

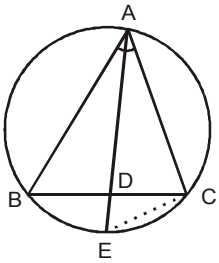
Solution

Draw a circumcircle of ΔABC. Produce AD to meet the circle at E. Join EC.

In Δs ABD and AEC, $\angle BAD = \angle EAC$ [$\because$ AD is the bisector of $\angle BAC$]

$\angle ABD = \angle AEC$ [$\angle$s in the same segment]

$\therefore$ $\Delta ABD \sim \Delta AEC$.



$\therefore \frac{AD}{AC} = \frac{AB}{AE}$

$\Rightarrow$ $AB \times AC = AD \times AE = AD \times (AD + DE)$

$AB \times AC = AD^2 + AD \times DE = AD^2 + BD \times BC$ [$\because$ $AD \times DE = BD \times DC$]

E. ANGLES IN THE ALTERNATE SEGMENTS

Let **PAQ** be a tangent to a circle at point **A** and **AB** be a chord. Then, the segment opposite to the angle formed by the chord of a circle with the tangent at a point is called the **alternate segment** for that angle.

Theorem : A line touches a circle and from the point of contact a chord is drawn. Prove that the angles which the chord makes with the given line are equal respectively to angles formed in the corresponding alternate segments.

Given : A circle with centre O and PAT is a tangent to the circle at A. A chord AB is drawn. Let C and D be the points on the circumference on opposite sides of AB.

To prove : $\angle BAT = \angle ACB$ and $\angle BAP = \angle ADB$.

Construction : Draw the diameter AOE and Join EB.

Proof : Since AE is the diameter,

$\therefore$ $\angle ABE = 90^o$ [Angle in semi circle]

$\therefore$ $\angle BEA + \angle EAB = 90^o$ ...(i) [Remaining $\angle$s of ΔBEA]

But since $EA \perp AT$ [$\because$ Radius is $\perp$ to the tangent]

$\therefore$ $\angle EAB + \angle BAT = 90^o$ ...(ii)

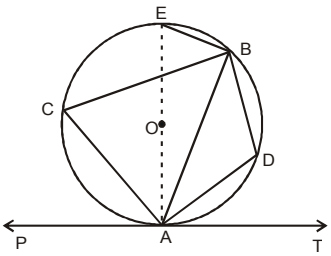
Hence from (i) and (ii)

$\angle BEA + \angle EAB = \angle EAB + \angle BAT$ ...(iii)

$\Rightarrow$ $\angle BEA = \angle BAT$

But $\angle BEA = \angle ACB$ [Angles in the same segment]

$\angle ACB = \angle BAT$



i.e. $\angle BAT = \angle ACB$.

This proves the first part.

Again, since ADBC is a cyclic quadrilateral.

$\therefore$ $\angle ACB + \angle ADB = 180^o$

Also $\angle BAT + \angle BAP = 180^o$ [Linear pair]

$\therefore$ $\angle ACB + \angle ADB = \angle BAT + \angle BAP$

$\Rightarrow$ $\angle BAT + \angle ADB = \angle BAT + \angle BAP$ [$\because \angle ACB = \angle BAT$]

$\Rightarrow$ $\angle ADB = \angle BAP$

$\therefore$ $\angle BAP = \angle ADB$

Hence, $\angle BAT = \angle ACB$ and $\angle BAP = \angle ADB$.

Solved Examples

Example.17

Two figures are given below :

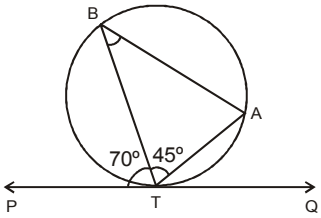


Fig (i)

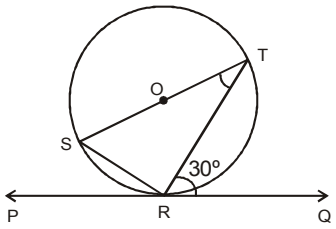


Fig (ii)

(a) In fig (i) PQ is a tangent to the circle at T, $\angle BTA = 45^o$ and $\angle PTB = 70^o$. Find $\angle ABT$.

(b) In fig (ii) O is the centre and PQ is a tangent to circle at R. Calculate $\angle STR$.

Solution

(a) $\angle ATQ = 180^o - (70^o + 45^o)$ [$\because$ Sum of the adjacent $\angle$s = 180º]

$= 180^o - 115^o = 65^o$.

$\therefore \angle ABT = \angle ATQ = 65^o$ [$\angle$s in the alternate segment]

(b) $\angle SRT = 90^o$ [Angle in a semi circle]

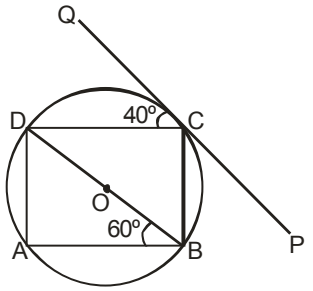
Now $\angle RST = \angle TRQ = 30^o$ [$\angle$s in the alternate segments]

$\therefore$ $\angle RTS = 90^o - 30^o = 60^o$.

Example.18 ABCD is a cyclic quadrilateral and PQ is a tangent to the circle at C. If BD is a diameter and $\angle DCQ = 40^o$ and $\angle ABD = 60^o$; find the measure of the following angles :

(i) $\angle DBC$ **(ii)** $\angle BCP$ **(iii)** $\angle ADB$.

Solution Since BD is a diameter of the circle.



$\therefore$ $\angle BAD = 90^o$ and also $\angle BCD = 90^o$

(i) $\angle DBC = \angle DCQ = 40^o$ [$\angle$s is the alternate segments]

(ii) $\angle BCP + \angle BCD + \angle DCQ = 180^o$

$\Rightarrow \angle BCP + 90^o + 40^o = 180^o$

$\therefore \angle BCP = 50^o$

(iii) Similarly from ΔBAD, $60^o + \angle BAD + \angle ADB = 180^o \Rightarrow 60^o + 90^o + \angle ADB = 180^o$

$\angle ADB = 30^o$.

Example.19

AB is diameter and AC is a chord of a circle such that $\angle BAC = 30^o$. The tangent at C intersects AB produced at D. Prove that BC = BD.

Solution

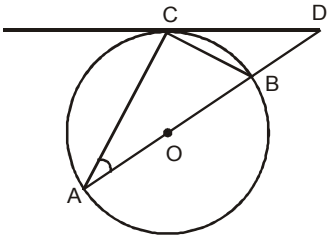
Since AB is the diameter of the circle

$\therefore \angle ACB = 90^o$ [Angle in a semi circle]

$\therefore \angle ABC = 90^o - 30^o = 60^o$ ...(i)

Again, $\angle BCD = \angle CAB = 30^o$ [$\angle$s in the alternate segments]

Now in ΔCBD,



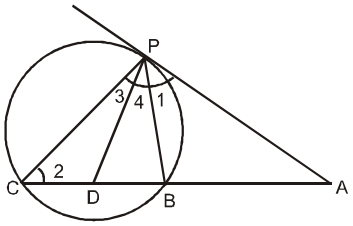
ext. $\angle CBA = \angle BCD + \angle BDC$ $\Rightarrow$ $60^o = 30^o + \angle BDC$

$\therefore \angle BDC = 30^o$

Hence, $\angle BCD = \angle BDC$ $\Rightarrow$ BC = AD.

Example.20

In the adjoining figure, AP is a tangent to the circle at P. ABC is a secant such that PD is the bisector of $\angle BPC$. Prove that



(i) $\angle ADP = \angle APD$ **(ii)** AP = AD **(iii)** $\angle BPD = \frac{1}{2}(\angle ABP - \angle APB)$

Solution

AP is a tangent to the circle and PB is a chord through the point of contact P.

$\therefore$ $\angle 1 = \angle 2$ [$\angle$s in the alternate segments]

$\Rightarrow$ $\angle 1 + \angle 4 = \angle 2 + \angle 3$ [$\because$ $\angle 3 = \angle 4$ since PD bisects $\angle CPB$]

$\Rightarrow$ $\angle APD = \angle 2 + \angle 3$ .....(i)

But from ΔPCD, ext. $\angle PDA = \angle 2 + \angle 3$

Hence from (i), $\angle APD = \angle PDA$

Thus (i) is proved.

Again $\angle APD = \angle PDA$ $\Rightarrow$ AP = AD.

Further, from ΔPCB, ext. $\angle PBA$ = sum of the two opposite interior angles viz.$\angle 2 + (\angle 3 + \angle 4)$

$= \angle 1 + (\angle 4 + \angle 4)$ [$\because$ $\angle 2 = \angle 1$ & $\angle 3 = \angle 4$]

$= \angle APB + 2\angle BPD$

$\Rightarrow 2\angle BPD = \angle PBA - \angle APB$

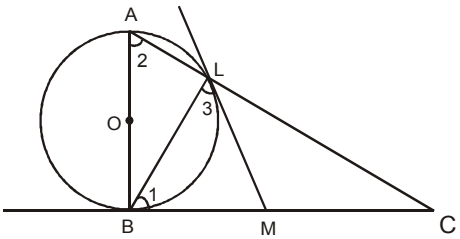
$\therefore \angle BPD = \frac{1}{2}(\angle PBA - \angle APB)$.

Example.21

In a right-angled ΔABC, a circle with side AB as diameter is drawn to intersect the hypotenuse AC in L. Prove that the tangent to the circle at L bisects the side BC.

Solution

Let the tangent at L meet BC in M. Join BL.



Firstly, $\angle ALB = 90^o$ [$\angle$ in a semi circle]

$\therefore$ $\angle BLC = 90^o$

Since the lengths of tangents from an external point are equal.

$\therefore$ MB = ML

Now LM is a tangent to the circle at L and LB is a chord through the point of contact L

$\therefore$ $\angle 3 = \angle 2$

Further, $\angle MLC = \angle BLC - \angle 3 = 90^o - \angle 3$

and $\angle LCM = \angle ACB = 90^o - \angle 2 = 90^o - \angle 3$ [$\angle 2 = \angle 3$]

$\angle MLC = \angle LCM$

$\therefore$ ML = MC

Thus BM = LM = MC.

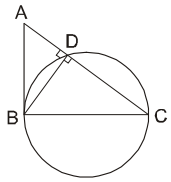
Hence, M is the mid-point of BC.

Example.22

In a right triangle ABC, the perpendicular BD on the hypotenuse AC is drawn. Prove that

(i) $AC \times AD = AB^2$ **(ii)** $AC \times CD = BC^2$

Solution



We draw a circle with BC as diameter. Since $\angle$ BDC = 90º.

$\therefore$ The circle on BC as diameter will pass through D. Again

$\therefore$ BC is a diameter and AB $\perp$ BC.

$\therefore$ AB is a tangent to the circle at B.

Since AB is a tangent and ADC is a secant to the circle.

$\therefore$ $AC \times AD = AB^2$ This proves (i)

Again

$AC \times CD = AC \times (AC - AD) = AC^2 - AC \times AD$

$= AC^2 - AB^2$ [Using (i)]

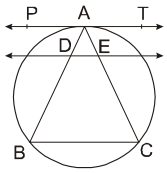
$= BC^2$ [ΔABC is a right triangle]

Hence, $AC \times CD = BC^2$. This proves (ii).

Example.23

In the given figure, PAT is a tangent to the circumcircle of a Δ ABC at the vertex A. A line parallel to PAT intersects the sides AB and AC at the points D and E respectively.

Prove that (a) $\Delta ABC \sim \Delta AED$ (b) $AB \cdot AD = AC \cdot AE$

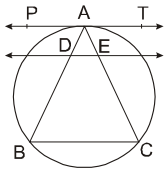


Solution

$\angle TAC = \angle TAE = \angle ABC$ … (i) [$\therefore$Angles in the alternate segment are equal

Similarly, $\angle PAB = \angle PAD = \angle ACB$ … (ii) [$\therefore$Angles in the alternate segment are equal]

Also $\angle TAE = \angle AED$ (alternate angles) … (iii)



$\angle PAD = \angle ADE$ [alternate angles] … (iv)

From (i) to (iv), we get

$\angle ABC = \angle AED$

$\angle ACB = \angle ADE$

$\therefore$ $\Delta ABC \sim \angle AED$ [By AA similarity]

Since corresponding sides of similar triangles are proportional.

$\frac{AB}{AE} = \frac{AC}{AD}$ $\Rightarrow$ $AB \times AD = AC \times AE$

Hence, the required results are proved.

Example.24

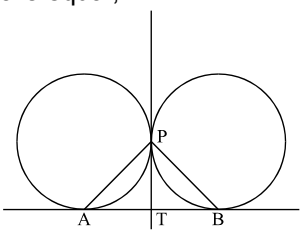
Two circles touch externally at P and a common tangent touches them at A and B. Prove that

(i) the common tangent at P bisects AB.

(ii) AB subtends a right angle at P.

Solution

Let PT be the common tangent at any point P. Since the tangent to a circle from an external point are equal,



$\therefore$ TA = TP, TB = TP

$\therefore$ TA = TB

i.e. PT bisects AB at T

TA = TP gives $\angle$ TAP = $\angle$ TPA (from ΔPAT)

TB = TP gives $\angle$ TBP = $\angle$ TPB [from ΔPBT]

$\therefore$ $\angle$ TAP + $\angle$ TBP = $\angle$TPA + $\angle$TPB = $\angle$APB

$\Rightarrow$ $\angle$ TAP + $\angle$ TBP + $\angle$APB = 2 $\angle$ APB

$\Rightarrow$ 2 $\angle$ APB = 180º [sum of $\angle$s of a Δ = 180º]

$\Rightarrow$ $\angle$ APB = 90º **Hence proved**

F. COMMON TANGENTS TO TWO CIRCLES

Definition : A line which touches the two given circles is **called common tangent** to the two circles. Let **C(O_1, r_1), C(O_2, r_2)** be two given circles. Let the distance between centres **O_1** and **O_2** be **d** i.e., **O_1O_2 = d**.

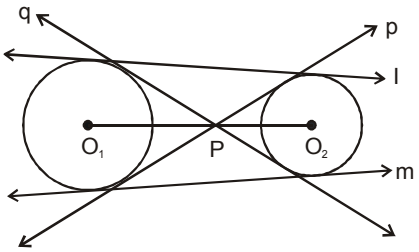


Fig.(i)

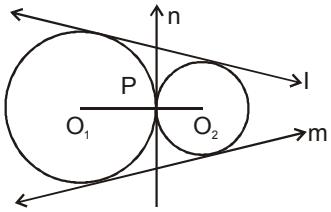


Fig.(ii)

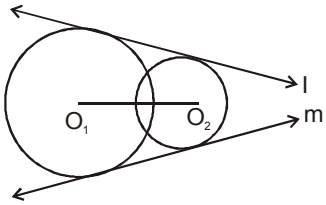


Fig.(iii)

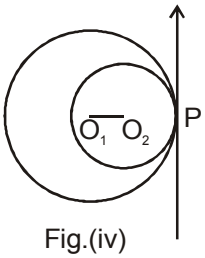


Fig.(iv)

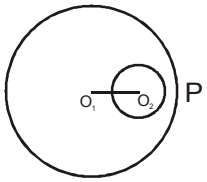


Fig.(v)

(a) In fig. (i) **d > r_1 + r_2** i.e. two circles do not intersect.

In this case, **four common tangents** are possible.

The tangent lines **l** and **m** are called **direct common tangents** and the tangent lines **p** and **q** are called **indirect (transverse) common tangents**.

(b) In fig. (ii), **d = r_1 + r_2.** In this case, two circles touch externally and there are three common tangents.

(c) In fig.(iii) **d < r_1 + r_2.** In this case two circles intersect in two distinct points and there are only two common tangents.

(d) In fig. (iv), **d = r_1 – r_2 (r_1 > r_2),** in this case, two circles touch internally and there is only one common tangent.

(e) In fig. (v), the circle **C(O_2, r_2)** lies wholly in the circle **C(O_1, r_1)** and there is no common tangent.

Solved Examples

Example.25

Two circles touch each other externally. Their radii are 9 cm and 4 cm and the distance between their centres is 13 cm. Find the length of their common tangent segment.

Solution

Given : Two circles C(O, 9) and C(O', 4) touch externally at S.

$\therefore$ OO' = 9 cm + 4 cm = 13 cm.

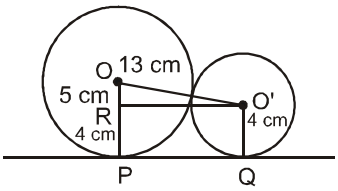
PQ is the common tangent segment.

Construction : Join OP and O'Q. Then OP $\perp$ PQ and O'Q $\perp$ PQ. From O' draw O'R $\perp$ OP.

Proof : Clearly PQ = O'R

Also, RP = O'Q = 4 cm.

$\therefore$ RP = OP – RP = 9 – 4 = 5 cm.



Now from right-angled ΔORO', $OO'^2 = OR^2 + O'R^2$

$\Rightarrow$ $13^2 = 5^2 + PQ^2$

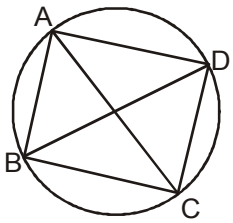
$\Rightarrow$ $PQ^2 = 13^2 - 5^2 = (13 - 5)(13 + 5)$

$\Rightarrow$ $PQ^2 = 8 \times 18 = 9 \times 16$

$\Rightarrow$ PQ = 3 × 4 = 12 cm.

G. PTOLEMY'S THEOREM

In cyclic quadrilateral, the product of the diagonals is equal to the sum of the product of the opposite sides.



AC × BD = AB × DC + BC × AD.

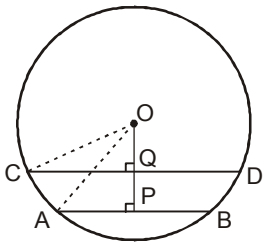
Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Problems based on chord

A-1 The radius of a circle is 13 cm and the length of one of its chords is 10 cm. Find the distance of the chord from the centre.

A-2 In figure, O is the centre of the circle of radius 5 cm OP ⊥ AB, OQ ⊥ CD, AB || CD, AB = 6 cm and CD = 8 cm. Determine PQ.



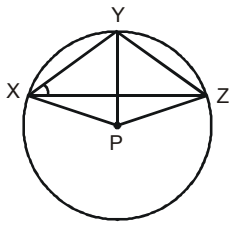
A-3 A circular park of radius 20 m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distances on its boundary each having a toy telephone in his hands to talk to each other. Find the length of the string of each phone.

A-4 AB and CD are two parallel chords of a circle such that AB = 10 cm and CD = 24 cm. If the chords are on the opposite side of the centre and the distance between them is 17 cm, find the radius of the circle.

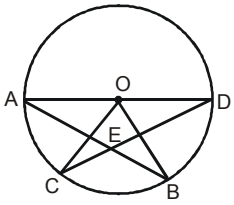
A-5 In a circle of radius 5 cm, AB and AC are two chords such that AB = AC = 6 cm. Find the length of the chord BC.

Section (B) : Problems based on angles

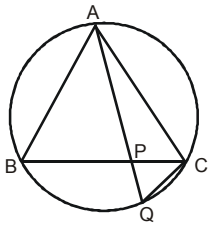
B-1 In the given figure, P is the centre of the circle. Prove that : ∠XPZ = 2 (∠XZY + ∠YXZ).



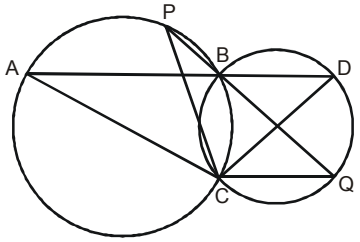
B-2 In a circle with centre O, chords AB and CD intersects inside the circumference at E. Prove that ∠AOC + ∠BOD = 2∠AEC.



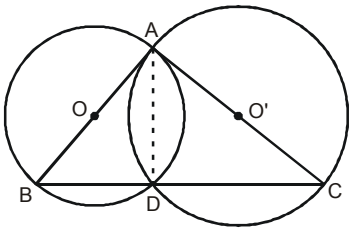
B-3 In figure, P is any point on the chord BC of a circle such that AB = AP. Prove that CP = CQ.



B-4 Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively as shown in figure. Prove that $\angle ACP = \angle QCD$.

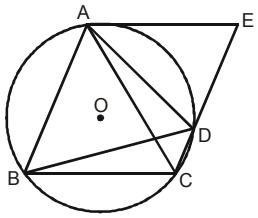


B-5 In the figure given below, two circles intersect at A and D, and AC, AB are respectively the diameters of the circles. Prove that the points C, D, B are collinear.

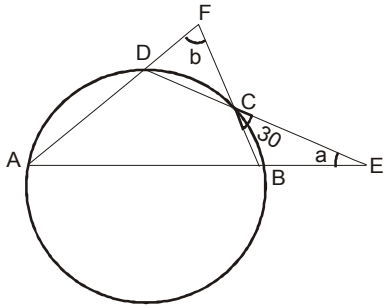


B-6 Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively. Prove that the angles of the triangle DEF are $90^\circ - \frac{1}{2}\angle A$, $90^\circ - \frac{1}{2}\angle B$ and $90^\circ - \frac{\angle C}{2}$

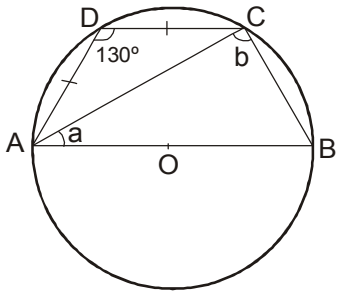
B-7 D is a point on the circumcircle of ΔABC in which AB = AC such that B and D are on the opposite side of line AC. If CD is produced to a point E such that CE = BD, prove that AD = AE.



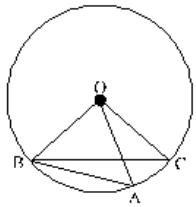
B-8 Find the value of a and b, if b = 2a.



B-9 Find the value of a & b.



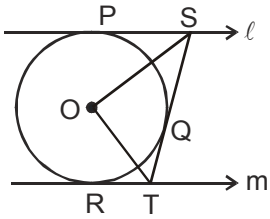
B-10. O is the centre of a circle, show that $\angle BOC = 2[\angle ABC + \angle ACB]$.



Section (C) : Problem based on secants and tangents

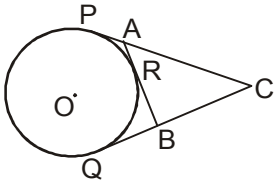
C-1 Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2\angle OPQ$.

C-2 In figure, ℓ and m are two parallel tangents at P and R. The tangent at Q makes an intercept ST between ℓ and m. Prove that $\angle SOT = 90^o$.



C-3 The radius of the incircle of a triangle is 4 cm and the segments into which one side is divided by the point of contact are 6 cm and 8 cm. Determine the other two sides of the triangle.

C-4 In figure, CP and CQ are tangents from an external point C to a circle with centre O. AB is another tangent which touches the circle at R. If CP = 11cm and BR = 4 cm, find the length of BC.



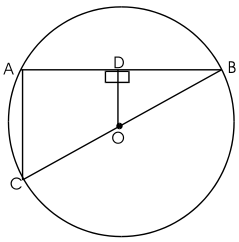
C-5 Prove that the lengths of the tangents drawn from an external point to a circle are equal. Using the above theorem prove that : If quadrilateral ABCD is circumscribing a circle, then AB + CD = AD + BC.

OBJECTIVE QUESTIONS

Section (A) : Problems based on chord

A-1. If a diameter of a circle bisects each of the two chords of a circle then the chords are :
(A) intersecting (B) concurrent (C) collinear (D) parallel

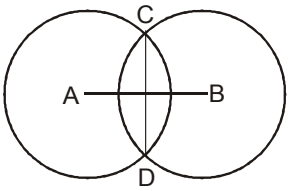
A-2. OD is perpendicular to a chord AB of a circle whose centre is O. If BC is a diameter, if CA = k × OD then k equals to



(A) 2 (B) 2.5 (C) 3 (D) 3.5

A-3. Two chords AB & CD of lengths 6 cm & 8 cm of a circle are parallel and are on the same side of its centre. If the distance between them is 1 cm. Find the radius of the circle.
(A) 3 cm (B) 4 cm (C) 5 cm (D) 2 cm

A-4. In the figure given above, A and B are the centers of the two congruent circles with radius 17 units. If AB = 30 units, the length of the common chord DC is :



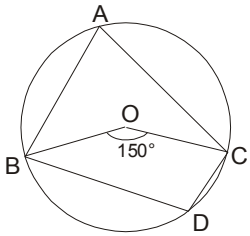
(A) 25 units (B) 18 units (C) 10 units (D) 16 units

A-5. AB and CD are the chords of a circle whose centre is O. They intersect each other at P. If PO be the bisector of ∠APD, then which one of the following option is not true
(A) AB = CD (B) OR = OQ (C) PR = PQ (D) AP = BP

Section (B) : Problems based on angles

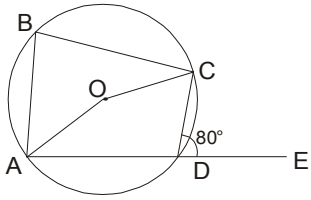
B-1. A, B, C are three points on the circumference of a circle with centre O such that ∠ OAC = 53° and ∠ CBO = 32°, then ∠ AOB =
(A) 100° (B) 120° (C) 150° (D) 170°

B-2. In figure, ∠ BDC =



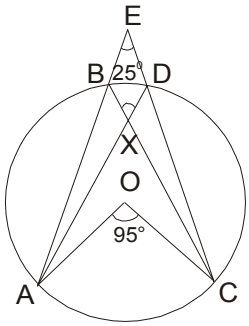
(A) 95° (B) 105° (C) 100° (D) 110°

B-3. In figure, O is centre, then reflex ∠ AOC =



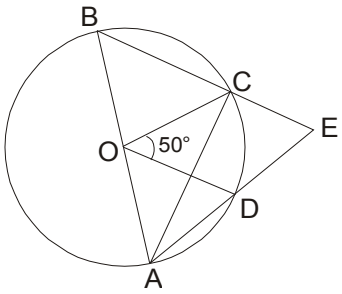
(A) 190° (B) 160º (C) 200° (D) 185°

B-4. In figure, O is centre, then ∠ BXD =



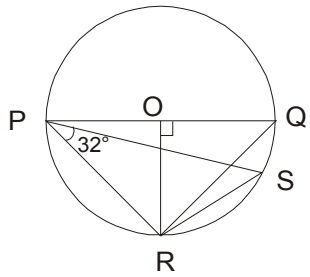
(A) 65° (B) 60° (C) 70° (D) 55°

B-5. In figure, O is centre, then $\angle$ AEB =



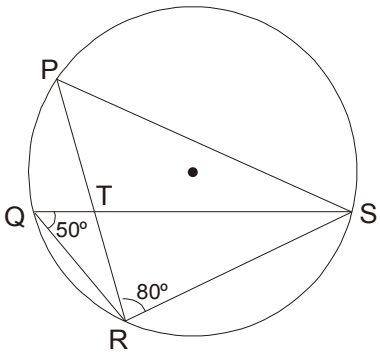
(A) 60° (B) 65° (C) 70° (D) 75°

B-6. In figure, PQ is a diameter of a circle with centre at O and OR $\perp$ PQ, where R is a point on the circle. If S is another point on the circle such that $\angle$ RPS = 32°, then $\angle$ QRS is :



(A) 13° (B) 26° (C) 45° (D) None of these

B-7. In figure, PQRS is a cyclic quadrilateral. Its diagonals PR and QS intersect each other at T. If $\angle$ PRS = 80º and $\angle$RQS = 50º, calculate $\angle$ PSR.

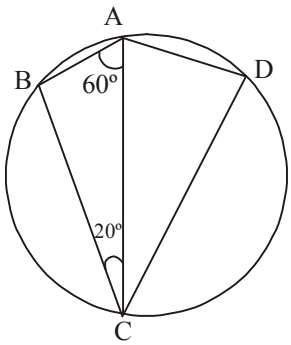


(A) 30° (B) 50° (C) 70° (D) 90°

B-8. C is the centre of a circle passing through the points P, Q, R and S taken in order. If $\angle$ PSR = 120º and PQ is a diameter, then $\angle$ QPR =

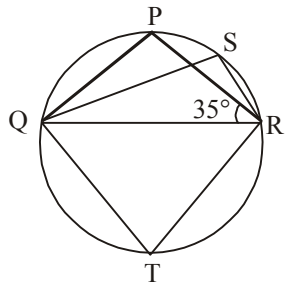
(A) 30º (B) 40º (C) 45º (D) 50º

B-9. In figure, $\angle$ ADC =



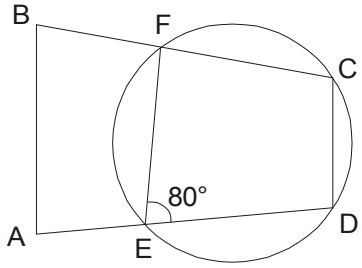
(A) 60º (B) 70º (C) 75º (D) 80º

B-10. In figure, PQ = PR, find $\angle$ QTR.



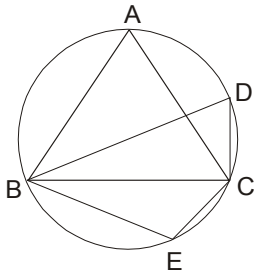
(A) 35º (B) 70º (C) 110º (D) None of these

B-11. In figure, CDEF is a cyclic quadrilateral, DE and CF are produced to A and B respectively such that AB || CD. If $\angle$ FED = 80°, find $\angle$ FBA.



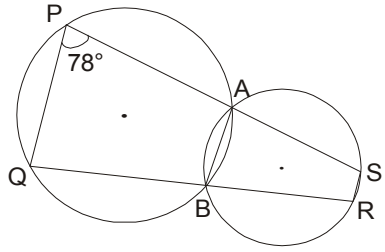
(A) 30º (B) 60º (C) 80º (D) None of these

B-12. ABC is an equilateral triangle. Find $\angle$ BEC.



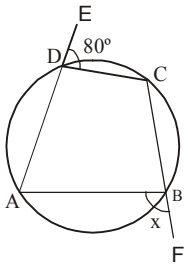
(A) 30º (B) 60º (C) 90º (D) 120º

B-13. Find $\angle$ ASR.



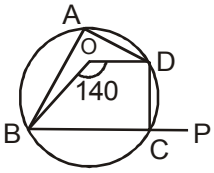
(A) 52º (B) 78º (C) 102º (D) None of these

B-14. ABCD is a cyclic quadrilateral. Then, find $\angle x^o$ as given in the **figure,** below.



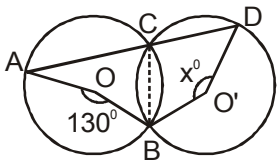
(A) 50º (B) 80º (C) 90º (D) 100º

B-15. In the given figure O is the centre of the circle. The angle subtended by the arc BCD at the centre is 140^0 . Find $\angle BAD$.



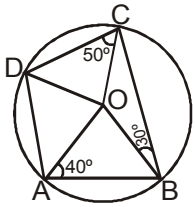
(A) 40º (B) 50º (C) 20º (D) 70°

B-16. In the given figure, O, O' are centres of two circles intersecting at B and C. ACD is a straight line. Find x.



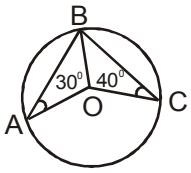
(A) 130º (B) 50º (C) 40º (D) None of these

B-17. ABCD is a cyclic quadrilateral inscribed in a circle with the centre O. Then $\angle OAD$ is equal to



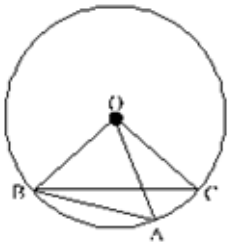
(A) 30º (B) 40º (C) 50º (D) 60º

B-18. In the given figure find the value of $\angle AOC$.



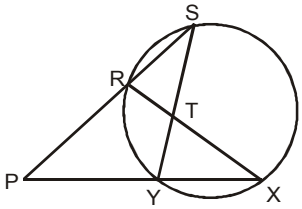
(A) 130º (B) 140º (C) 150º (D) 160º

B-19. In the given figure, O is the centre of a circle, if $\angle ABC = 10°$ and $\angle ACB = 50°$, then $\angle BOC$ is



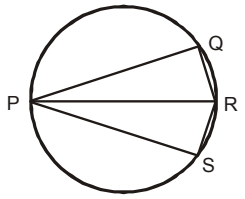
(A) 130º (B) 120º (C) 110º (D) 100º

B-20. In the adjoining figure, the chords XY and SR are produced to meet outside the circle at P. The chords XR and YS meet inside the circle at T. If X = xº and $\angle XTS$ = yº, then $\angle P$ is equal to :



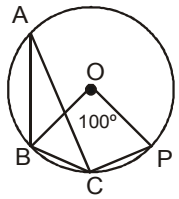
(A) (x + y)º (B) (y – x)º (C) 2(y – x)º (D) (y – 2x)º

B-21. If PR is a diameter of the circle and the angles RPS, QPR, QRP and PRS forms an AP with RPS = 15º, then the value of the angle QRP is :



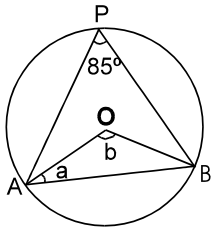
(A) 55º (B) 75º (C) 60º (D) 35º

B-22. In the given figure, ∠BCP is equal to :



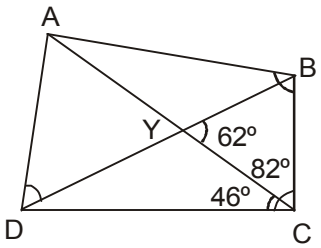
(A) 60º (B) 100º (C) 50º (D) 130º

B-23. In the given figure, find the value of 'a' is :



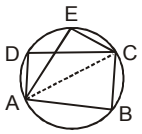
(A) 170º (B) 20º (C) 10º (D) 5º

B-24. In the given figure, find the measure of ∠ABC, if ABCD is a cyclic quadrilateral.



(A) 120º (B) 118º (C) 138º (D) 128º

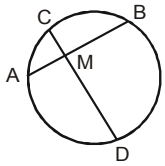
B-25. Given, a quadrilateral ABCD is inscribed in a circle as shown in the figure below.
If ∠B = 125º, then ∠E is equal to _____



(A) 55º (B) 125º (C) 130º (D) 62.5º

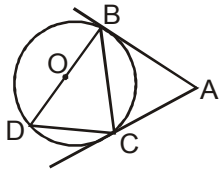
Section (C) : Problem based on secants and tangents

C-1. In the circle shown AB = 24, and the perpendicular chord CD bisects AB. If DM is 4 times as long as CM then the length of BD, is



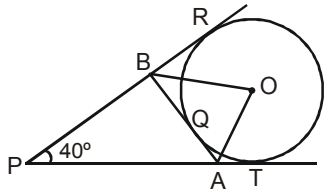
(A) 8 (B) $12\sqrt{5}$ (C) $16\sqrt{5}$ (D) $20\sqrt{5}$

C-2. In the given figure, O is the centre of a circle and BD is a diameter. AB and AC are tangents touching the circle at B & C respectively. If $\angle BAC = 70^0$ then $\angle OBC$ is :



(A) 30^0 (B) 35^0 (C) 40^0 (D) 45^0

C-3. Triangle PAB is formed by three tangents to circle O and $\angle APB = 40^o$, then angle AOB

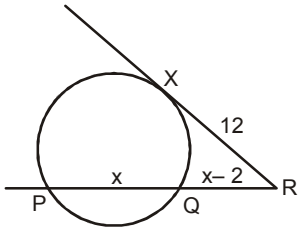


(A) 45º (B) 50º (C) 60º (D) 70º

C-4. ABC is a right angled triangle, right angled at B such that BC = 6 cm and AB = 8 cm. A circle with centre O is inscribed in ΔABC. The radius of the circle is :

(A) 1 cm (B) 2 cm (C) 3 cm (D) 4 cm

C-5. PQ is a chord of a circle. The tangent XR at X on the circle cuts PQ produced at R. If XR = 12 cm, PQ = x cm, QR = x – 2 cm, then x in cm is :



(A) 6 (B) 7 (C) 10 (D) 14

Exercise-2

OBJECTIVE QUESTIONS

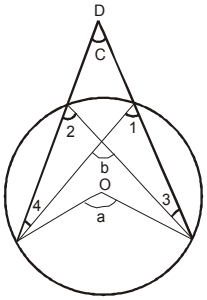
1. AB and AC are two chords of the circle of radius r. If AB = 2AC and the perpendicular drawn from the centre on those chords are of lengths a and b. If $b^2 = k(a^2 + 3r^2)$ then k =

(A) 1 (B) 2 (C) 3 (D) 1/4

2. Let P be a point on the circumference of a circle. Perpendiculars PA and PB are drawn to points A and B on two mutually perpendicular diameters. If AB = 36 cm, the diameter of the circle is :

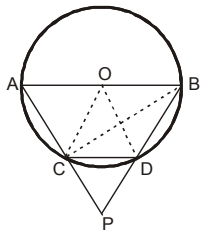
(A) 16 cm (B) 24 cm (C) 36 cm (D) 72 cm

3. O is the centre of the circle in the figure given below. The correct relationship between a, b and c is :



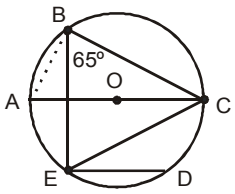
(A) a + b = c (B) a = b + c (C) a + c = b (D) None of these

4. In the given figure, AB is the diameter of a circle C (O, r). Chord CD is equal to radius OC. If AC and BD intersect at P, find $\angle APB$.



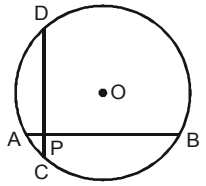
(A) 30º (B) 45º (C) 60º (D) 90º

5. In the adjoining figure, chord ED is parallel to the diameter AC of the circle. If $\angle CBE = 65º$, then what is the value of $\angle DEC$?



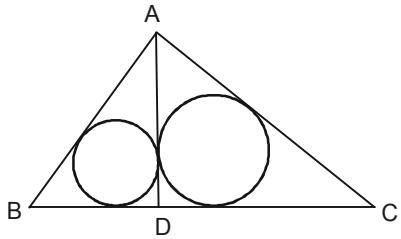
(A) 35º (B) 55º (C) 45º (D) 25º

6. In the circle with centre 'O' as shown, chord AB and CD intersect at P and are perpendicular to each other. If AP = 4, PB = 6 and PC = 2, then the area of the circle is



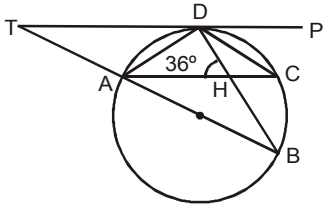
(A) 45π (B) 49π (C) 50π (D) $41\pi 4$

7. In a triangle ABC, AB = 130, AC = 200 and BC = 260. Point D is chosen on BC so that the circles inscribed in triangle ABD and ADC are tangent to AD at the same point. Length of BD is equal to :



(A) 105 (B) 95 (C) 90 (D) 85

8. In the figure, AB is a diameter of the circle. TD is a tangent. If $\angle AHD = 36º$, $\angle CDT$ is :

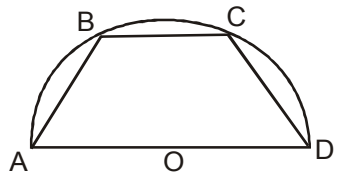


(A) 120º (B) 116º (C) 106º (D) 126º

9. Let ABCD be a square of side length l, and a circle passing through B and C, and touching AD. The radius of is

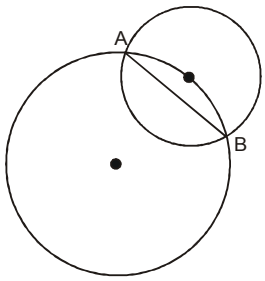
(A) $\frac{3}{8}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{5}{8}$

10. In the figure, the semicirlce centered at O has a diameter 6 cm. The chord BC is parallel to AD and BC = $\frac{1}{3}$ AD. The area of the trapezium ABCD in cm^2, is :



(A) 4 (B) $4\sqrt{2}$ (C) 8 (D) $8\sqrt{2}$

11. In the figure, the radius of the larger circle is 2 cm and the radius of the smaller circle is 1 cm and the larger circle passes through the centre of the smaller circle. The length (in cm) of the chord AB is



(A) $\frac{\sqrt{15}}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) $\frac{5\sqrt{34}}{17}$

12. In an acute-angled triangle ABC, the altitudes from A, B, C when extended intersect the circumcircle again at points A_1, B_1, C_1, respectively. If $\angle ABC = 45°$ then $\angle A_1B_1C_1$ equals

(A) 45° (B) 60° (C) 90° (D) 135°

13. In triangle ABC, with $\angle A = 90°$, the bisectors of the angles B and C meet at P. The distance from P to the hypotenuse is $4\sqrt{2}$. The distance AP is :

(A) 8 (B) 4 (C) $8\sqrt{2}$ (D) $4\sqrt{2}$

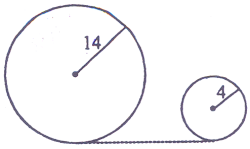
14. In a right triangle ABC. the in circle touches the hypotenuse AC at D. If AD = 10 and DC = 3, the inradius of ABC is -

(A) 5 (B) 4 (C) 3 (D) 2

15. Three circles each of radius 1 touch one another externally and they lie between two parallel lines. The minimum possible distance between the lines is

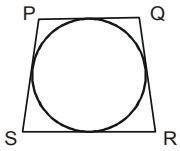
(A) $2+\sqrt{3}$ (B) $3+\sqrt{3}$ (C) 4 (D) $2+\frac{1}{\sqrt{3}}$

16. Two circles of radii 4 cm and 14 cm have a common external tangent of length 24 cm. The distance between the centres of these circles (in cm) is :



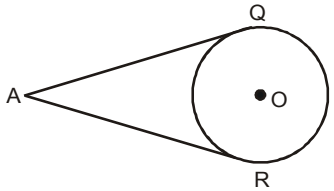
(A) 24 (B) 25 (C) 26 (D) 27

17. A circle is inscribe in trapezoid PQRS. If PS = QR = 25 cm, PQ = 18 cm what is the length of the diameter of the circle ?



(A) 14 cm (B) 25 cm (C) 24 cm (D) $\sqrt{674}$ cm

18. In the given figure AQ and AR are the tangents drawn from A to the circle with centre O. If the measure of $\angle QOR$ is 120º, then $\angle QAR$ is equal to :



(A) 45º (B) 50º (C) 65º (D) 60º

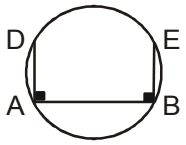
Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. A circle of radius 25 units has a chord going through a point that is located 10 units from the centre. What is the shortest possible length that chord could have ? **[NSTSE-2009]**

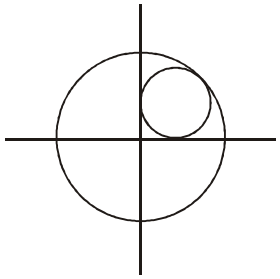
(A) 25 units (B) $\sqrt{525}$ units (C) 40 units (D) $\sqrt{2100}$ units

2. Given a chord AB in a circle as shown. If two more chords AD and BE are drawn perpendicular to AB as shown in the figure, then : **[NSTSE-2009]**



(A) AD = BE (B) AD = 2BE (C) 2AD = BE (D) AD $\neq$ BE

3. In the figure shown, the bigger circle has radius 1 unit. Therefore, the radius of smaller circle must be : **[IJSO-2009]**

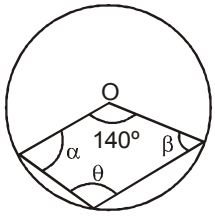


(A) $\sqrt{2}+1$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{\sqrt{2}+1}$

4. Two circles of radii 2 and 3 cm touch each other externally. The length of direct common tangent to the two circles will be : **[IJSO-2009]**

(A) $2\sqrt{6}$ cm (B) $\sqrt{26}$ cm (C) 5 cm (D) 2.4 cm

5. In the figure shown **[IJSO-2009]**

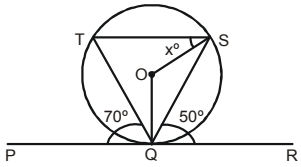


(A) $\alpha + \beta = 110º$ (B) $\alpha = \beta = 55º$ (C) $\theta = 110º$ (D) Both A & C

6. Two circles with centres A and B have radii 8 cm and 1 cm respectively. The distance AB is 13 cm . A third circle with centre C and radius r cm touches both circles externally. If angle ACB is 90º, then value of r is : **[IJSO-2009]**

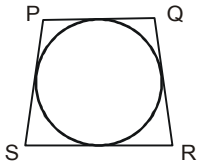
(A) 11 (B) 4 (C) 5 (D) 2

7. In the diagram, PQR is a tangent to the circle at Q and "O" is the centre of the circle. Find the value of "X". **[NSTSE 2009]**



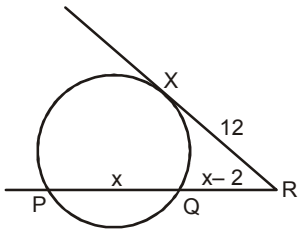
(A) 25º (B) 30º (C) 35º (D) 40º

8. A circle is inscribe in trapezoid PQRS. If PS = QR = 25 cm, PQ = 18 cm and SR = 32 cm, what is the length of the diameter of the circle ? **[NSTSE 2009]**



(A) 14 cm (B) 25 cm (C) 24 cm (D) cm

9. PQ is a chord of a circle. The tangent XR at X on the circle cuts PQ produced at R. If XR = 12 cm, PQ = x cm, QR = x – 2 cm, then x in cm is : **[NSTSE 2010]**



(A) 6 (B) 7 (C) 10 (D) 14

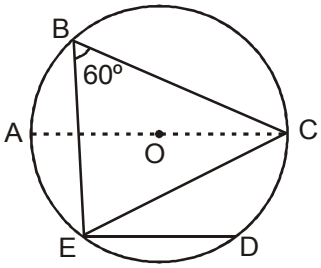
10. A point P is outside a circle at a distance of 13 cm from its centre. A secant from P cuts the circle in Q and R such that QR = 7 cm and the segment PQ of the secant exterior to the circle is 9 cm. Therefore, the radius of circle is : **[IJSO-2010]**

(A) 3 cm (B) 4 cm (C) 5 cm (D) 6 cm

11. The hypotenuse of a right angled triangle is 10 cm and the radius of its inscribed circle is 1cm. Therefore, perimeter of the triangle is : **[IJSO-2010]**

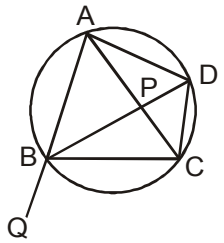
(A) 22 cm (B) 24 cm (C) 26 cm (D) 30 cm

12. In the figure shown, chord ED is parallel to diameter AC of a circle. If ∠CBE = 60º, then ∠DEC must be : **[IJSO-2010]**



(A) 15º (B) 30º (C) 10º (D) 20º

13. In the figure shown below, m $\angle ABD = 2x$, m $\angle ADB = 3x$, m $\angle APB = 4x$ and m $\angle CBQ = 7x$. Therefore, m $\angle BCD$ is **[IJSO-2011]**



(A) 90º (B) 85º (C) 72º (D) 96º

14. If ABCD is a cyclic quadrilateral. AB = 204, BC = 104, CD = 195, DA = 85 and BD = 221. Then find AC **[IJSO-2012]**

(A) 205 (B) 210 (C) 220 (D) 225

15. A circle is inscribed in an isosceles trapezium ABCD in which AB is parallel DC. If AB = 10 and DC = 30. Find the area of the circle. **[IJSO-2012]**

(A) 45π (B) 50π (C) 60π (D) 75π

16. Two circles each of radius 3 touch each other externally in the plane. Inhow many ways can a circle of radius 8 be placed inthe plane touching each of these two circles ? **[IJSO-2013]**

(A) 2 (B) 4 (C) 6 (D) 8

17. Two parallel chords 96 cm and 28 cm long are on the opposite side of the centre of the circle with radius 50 cm. Find the area of the quadrilateral whose vertices are the end points of the chords. **[IJSO-2015]**

(A) 3488 (B) 3848 (C) 3844 (D) 3484

18. In a n-sided regular polygon, the radius of the circum-circle is equal in length to the shortest diagonal. The number of values of n < 60 for which this can happen is **[IJSO-2016]**

(A) 0 (B) 1 (C) 10 (D) 2

19. Let AB be a diameter of a circle C_1 of radius 30 cm and with center O. Two circles C_2 and C_3 of radii 15 cm and 10 cm touch C_1 internally at A and B respectively. A fourth circle C_4 touches C_1, C_2 and C_3. What is the largest possible radius of C_4 **[IJSO-2018]**

(A) 12 cm (B) 15 cm (C) 20 cm (D) 30 cm

20. Let ABC be an equilateral triangle. The bisector of $\angle$ BAC meets the circumcircle of ABC in D. Suppose DB + DC = 4. The diameter of the circumcircle of ABC is **[IJSO-2018]**

(A) 4 (B) $3\sqrt{3}$ (C) $2\sqrt{3}$ (D) 2

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 12 cm. **A-2** 1 cm. **A-3** $20\sqrt{3}$ **A-4** 13 cm.

A-5 9.6 cm.

Section (B)

B-8 a = 40º, b = 80º. **B-9** a = 40º and b = 90º.

Section (C)

C-3 AB = 13 cm, AC = 15 cm. **C-4** 7 cm.

OBJECTIVE QUESTIONS

Section (A)

A-1. (D) **A-2.** (A) **A-3.** (C) **A-4.** (D) **A-5.** (D)

Section (B)

B-1. (D) **B-2.** (B) **B-3.** (C) **B-4.** (C) **B-5.** (B)

B-6. (A) **B-7.** (B) **B-8.** (A) **B-9.** (D) **B-10.** (B)

B-11. (C) **B-12.** (D) **B-13.** (C) **B-14.** (D) **B-15.** (D)

B-16. (A) **B-17.** (D) **B-18.** (B) **B-19.** (B) **B-20.** (D)

B-21. (A) **B-22.** (D) **B-23.** (D) **B-24.** (D) **B-25.** (A)

Section (C)

C-1. (B) **C-2.** (B) **C-3.** (D) **C-4.** (B) **C-5.** (C)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ans.	D	D	B	C	D	C	B	D	D	D	A	C	A	D	A	C	C	D

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

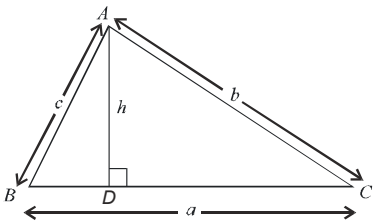
Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	D	A	D	B	B	C	C	C	A	B	A	C	D	C	C	B	C	A

MENSURATION

A. INTRODUCTION

Perimeter : The perimeter of a plane figure is the length of its boundary. The unit of perimeter is same as the unit of length i.e cm, m, etc.
Area : The area of plane figure is the measure of the surface enclosed by its boundary. The unit of area is cm^2, m^2, etc.

B. AREA AND PERIMETER OF TRIANGLES



(i) Scalene triangle
Perimeter = a + b + c

Area = $\frac{1}{2}$ × Base × Height = $\frac{1}{2}$ ah

Heron's formula :

Area = $\sqrt{s(s-a)(s-b)(s-c)}$

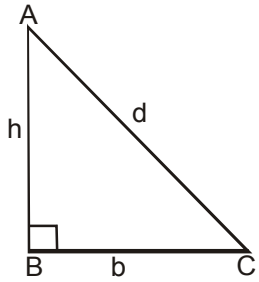
Where, s = semi-perimeter = $\frac{a+b+c}{2}$

(ii) Isosceles triangle

Area = $\frac{1}{2}$ × base × $\sqrt{(\text{equal side})^2 - \frac{1}{4}(\text{base})^2}$

(iii) Right-angled triangle.
For an right-angled triangle, let **b** be the base, **h** be the perpendicular and **d** be the hypotenuse. Then
(A) Perimeter = b + h + d

(B) Area = $\frac{1}{2}$ (Base × Height) = $\frac{1}{2}$ bh

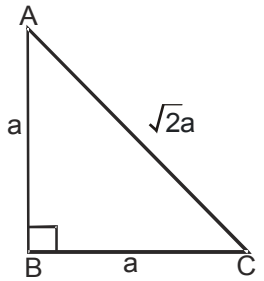


(C) Hypotenuse, d = $\sqrt{b^2+h^2}$ [Pythagoras theorem]

(iv) Isosceles right-angled triangle
For an isosceles right-angled triangle, let **a** be the equal sides, then

(A) Hypotenuse = $\sqrt{a^2+a^2}$ = $\sqrt{2}$a

(B) Perimeter = $2a + \sqrt{2}a$



(C) Area = $\frac{1}{2}$ (Base × Height) = $\frac{1}{2}$ (a × a) = $\frac{1}{2}$ a^2.

(v) Equilateral triangle

Area = $\frac{\sqrt{3}}{4}$ $(\text{side})^2$, Perimeter = 3(side)

Solved Examples

Example.1

The sides of a triangle containing the right triangle are 5x cm and (3x – 1) cm. If the area of the triangle is 60 cm^2, calculate the lengths of the sides of the triangle.

Solution

Since, area of a right angled triangle = $\frac{1}{2}$ × product of its sides containing the right triangle

$\therefore$ $60 = \frac{1}{2} \times 5x \times (3x - 1)$ $\Rightarrow$ $120 = 15x^2 - 5x$

$\Rightarrow$ $15x^2 - 5x - 120 = 0$ $\Rightarrow$ $3x^2 - x - 24 = 0$

On solving we get x = 3 and x = $-\frac{8}{3}$.

Since, x = $-\frac{8}{3}$ will give negative values of the sides of the triangle, which is impossible; therefore, x = 3.

$\therefore$ Length of the sides = 5 (3) = 15 cm and (3 × 3 – 1) = 8 cm.

Example.2

Find the perimeter of an equilateral triangle whose area is equal to that of a triangle with sides 21 cm, 16 cm and 13 cm. Answer correct to 2 decimal places.

Solution

For the given triangle a = 21 cm, b = 16 cm and c = 13 cm.

$\therefore s = \frac{a+b+c}{2} = \frac{21+16+13}{2} = 25$ cm.

So, Area = $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{25(25-21)(25-16)(25-13)} = 60\sqrt{3}$ cm^2

Given :

Area of equilateral triangle = Area of this triangle

$\Rightarrow$ $\frac{\sqrt{3}}{4}(\text{side})^2 = 60\sqrt{3}$

$\Rightarrow$ $(\text{side})^2 = 240$

$\Rightarrow$ side = $\sqrt{240} = 4\sqrt{15}$ cm

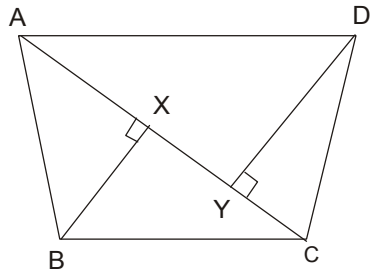
Perimeter of the equilateral triangle = 3 (side)

= $3 \times 4\sqrt{15}$

= 12 × 3.873 = 46.48 cm (Approx.)

C. AREA AND PERIMETER OF QUADRILATERALS

1. When one diagonal and perpendiculars to this diagonal from the remaining vertices are given.



In quadrilateral ABCD, the diagonal AC and perpendiculars BX and DY to AC from the remaining vertices B and D respectively are given, then the

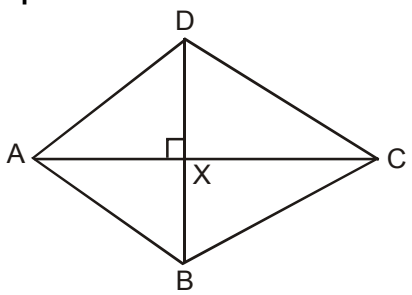
Area of quadrilateral ABCD = Ar. of ΔABC + Ar. of ΔADC

$$= \frac{1}{2}\ AC \times BX + \frac{1}{2}\ AC \times DY$$

$$= \frac{1}{2}\ AC\ (\ BX + DY\ )$$

$\therefore$ Area of quadrilateral = $\frac{1}{2}$ × one diagonal × sum of the lengths of the perpendiculars drawn on it from the remaining two vertices

2. When two diagonals of a quadrilateral cut each other at right angles.



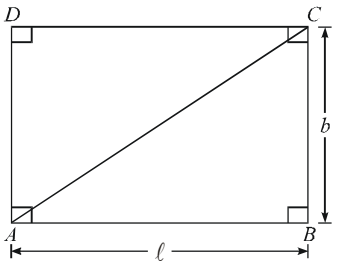
Area of Quadrilateral ABCD

= Ar. of ΔABC + Ar. of ΔADC = $\frac{1}{2}$ AC × BX + $\frac{1}{2}$ AC × DX

= $\frac{1}{2}$ AC (BX + DX) = $\frac{1}{2}$ AC × BD = $\frac{1}{2}$ × The product of the diagonals

D. SOME SPECIAL TYPES OF QUADRILATERALS

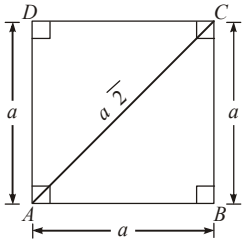
1. Rectangle :



Perimeter = 2 (ℓ + b)

Area = $\ell \times b$ Length of diagonal (d) = $\sqrt{\ell^2 + b^2}$

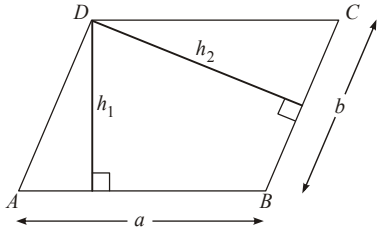
2. Square :



Perimeter = 4*a* Area = a^2

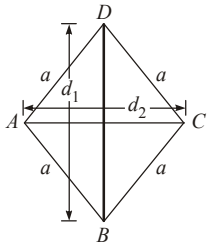
Length of diagonal = $a\sqrt{2}$

3. Parallelogram :



Perimeter = 2 (a + b) Area = $ah_1 = bh_2$

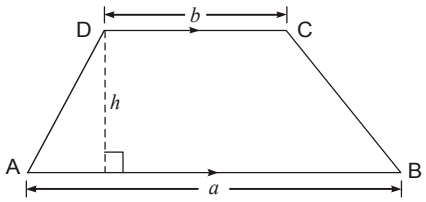
4. Rhombus :



Perimeter = 4a = $2\sqrt{d_1^2 + d_2^2}$

Area = $\frac{1}{2}\ d_1 d_2$

5. Trapezium :



Area = $\frac{1}{2}$ h (a + b)

Solved Examples

Example.3

The length and breadth of a rectangular field are in the ratio 3 : 2. If the area of the field is 3456 m^2, find the cost of fencing the field at Rs 3.50 per metre.

Solution

Let the length and breadth of the rectangular field be 3x and 2x metres respectively. Then,

Area of the rectangular field = $(3x2x)m^2 = 6x^2\ m^2$.

Also, area of the rectangular field = 3456 m^2

$\therefore$ $6x^2 = 3456$ $x^2 = \frac{3456}{6}$

$\Rightarrow$ $x^2 = 576$ $x = \sqrt{576} = 24$

length = (324) m = 72 m, breadth = (224) m = 48 m

Perimeter of the field = 2 (length + breadth) = [2 × (72 + 48)] m = 240 m

Rate of fencing = Rs 3.50 per metre

Cost of fencing = Rs (2403.50) = Rs 840.

Example.4

A grassy plot is 80 m × 60 m . Two cross paths each 4 m wide are constructed at right angles through the centre of the field , such that each path is parallel to one of the sides of rectangle. Find the total area used as path. Also, find the cost of gravelling them at Rs 5 per square metre.

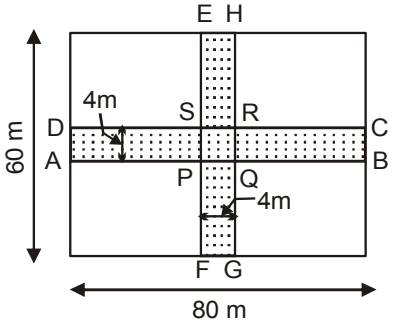
Solution

Let ABCD and EFGH be the cross paths.

We have , AB = 80 m and BC = 4 m.

∴ Area of path ABCD = (80 × 4) m^2 = 320 m^2

Again, EF = 60 m and FG = 4 m



∴ Area of path EFGH = (60 × 4) m^2 = 240 m^2.

Clearly, area PQRS is common to both the paths.

We have,

Area PQRS = (4 × 4) m^2 = 16 m^2

∴ Total area used as path = Area of path ABCD + Area of path EFGH – Area PQRS

= (320 + 240 – 16) m^2 = 544 m^2

Rate of gravelling the path = Rs 5 per square metre

∴ Total cost of gravelling the path = Rs (5 × 544) = Rs 2720

Example.5

The base of a parallelogram is thrice its height. If the area is 876 cm^2, find the base and height of the parallelogram.

Solution

Let the height of the parallelogram be x cm. Then, base = 3x cm.

∴ Area of the parallelogram= (x × 3x) cm^2 = $3x^2$ cm^2

But, area of the parallelogram is given as 867 cm^2

∴ $3x^2 = 867 \Rightarrow x^2 = 289 \Rightarrow x^2 = 17^2 \Rightarrow x = 17$

Thus, height = 17 cm and base = (3 × 17) cm = 51 cm.

Example.6

The area of a rhombus is 72 cm^2. If its perimeter is 32 cm, find its altitude.

Solution

We have, perimeter of the rhombus = 32 cm

∴ 4 (side) = 32 cm [∵ Perimeter = 4 (side)]

side = $\frac{32}{4}$ cm = 8 cm

Now, area of the rhombus = 72 cm^2

(Side × Altitude) = 72

8 × Altitude = 72

Altitude = $\frac{72}{8}$ cm = 9 cm

Example.7

If the area of a rhombus be 48 cm^2 and one of its diagonal is 12 cm, find its altitude.

Solution

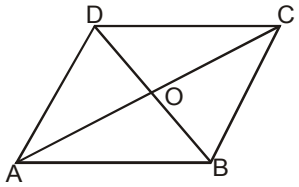
Let ABCD be a rhombus of area 48 cm^2 and diagonal BD = 12 cm.

Now, Area = 48 cm^2

$\Rightarrow \frac{1}{2} \times AC \times BD = 48$

$\Rightarrow \frac{1}{2} \times AC \times 12 = 48$

$\Rightarrow 6 \times AC = 48$



$\Rightarrow AC = \frac{48}{6}$ cm = 8 cm

Since the diagonal of a rhombus bisect each other at right angles.

$\therefore OA = \frac{1}{2} AC = 4$ cm, $OB = \frac{1}{2} BD = 6$ cm.

Also, $AB^2 = OA^2 + OB^2$ [Using pythagoras theorem]

$\Rightarrow AB^2 = 4^2 + 6^2 \Rightarrow AB^2 = 16 + 36$

$\Rightarrow AB = \sqrt{52}$

Since a rhombus is a parallelogram also, therefore,

Area of rhombus = AB × Altitude

$\Rightarrow 48 = \sqrt{52} \times$ Altitude $\Rightarrow$ Altitude $= \frac{48}{\sqrt{52}}$ cm .

Example.8

Find the area of a trapezium whose parallel sides are 25 cm, 13 cm and other sides are 15 cm and 15 cm.

Solution

Let ABCD be the given trapezium in which AB = 25 cm, CD = 13 cm, BC = 15 cm and AD = 15 cm.

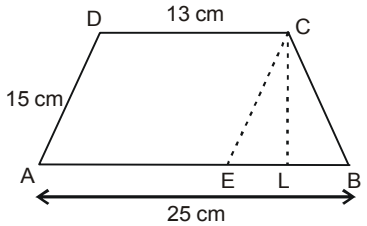
Draw CE || AD.

Now, ADCE is a parallelogram is which AD || CE and AE || CD.

$\therefore$ AE = DC = 13 cm and BE = AB – AE = 25 – 13 = 12 cm.

In ΔBCE, we have

$s = \frac{15+15+12}{2} = 21$



$\therefore$ Area of $\Delta BCE = \sqrt{s(s-a)(s-b)(s-c)}$

$\Rightarrow$ Area of $\Delta BCE = \sqrt{21(21-15)(21-15)(21-12)}$

$\Rightarrow$ Area of $\Delta BCE = \sqrt{21 \times 6 \times 6 \times 9}$

$= 18\sqrt{21}$ cm^2 ...(i)

Let h be the height of ΔBCE, then

Area of ΔBCE = $\frac{1}{2}$ (Base × Height)

$= \frac{1}{2} \times 12 \times h = 6h$ ..(ii)

From (i) and (ii), we have,

$6h = 18\sqrt{21} \Rightarrow h = 3\sqrt{21}$ cm

Clearly, the height of trapezium ABCD is same as that of ΔBCE

$\therefore$ Area of trapezium = $\frac{1}{2}$ (AB + CD) × h

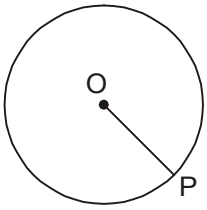
$\Rightarrow$ Area of trapezium = $\frac{1}{2}$ (25 + 13) × $3\sqrt{21}$ cm^2 = $57\sqrt{21}$ cm^2.

E. PERIMETER AND AREA OF A CIRCLE

Circle : The collection of all points in a plane, which are at a fixed distance from a fixed point in the plane, is called a circle.

The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle. The diameter of a circle is twice its radius.

In figure, O is the centre and the length OP is the radius of the circle.



The length of the boundary of a circle is called its circumference or its perimeter. We know that the ratio of the circumference of a circle to its diameter is always a constant. This constant ratio is denoted by the greek letter π.

$\frac{\text{Circumference}}{\text{Diameter}} = \pi$

Circumference = $\pi \times 2r = 2\pi r$ [$\because$ Diameter = $2r$]

The exact value of π is not known, because π is an irrational number. For all practical purposes, the value of π is approximately taken as $\frac{22}{7}$ or 3.14

If r is the radius of a circle, then

(i) Circumference = $2\pi r$ or πd, where d = 2r is the diameter of the circle.

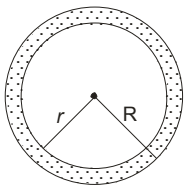
(ii) Area = πr^2.

(iii) Area of semi-circle = $\frac{\pi r^2}{2}$.

(iv) Perimeter of the semi-circle = $\pi r + 2r$.

(v) Area enclosed by two concentric circles = $\pi R^2 - \pi r^2 = \pi (R^2 - r^2) = \pi(R + r)(R - r)$

Where R and r and radii of two concentric circles.



NOTE :

1. If two circles touch each other externally, then the distance between their centres is equal to sum of their radii.

2. If two circles touch each other internally, then the distance between their centres is equal to difference of their radii.

3. The distance moved by a rotating wheel in one revolution is equal to the circumference of the wheel.

Solved Examples

Example.9

If the perimeter of a semi-circular protractor is 66 cm, find the diameter of the protractor (Take $\pi = \frac{22}{7}$).

Solution

Let the radius of the protractor be r cm.

Then, Perimeter = 66 cm

$\Rightarrow \quad \pi r + 2r = 66 \qquad \Rightarrow \quad r\left[\frac{22}{7} + 2\right] = 66$

$\Rightarrow \quad r\left(\frac{36}{7}\right) = 66 \qquad \Rightarrow \quad r = \frac{66 \times 7}{36}$

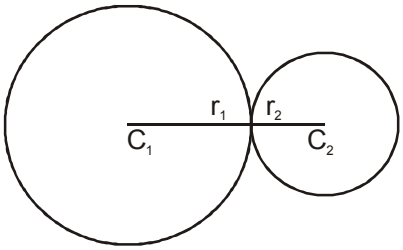
$\Rightarrow \quad r = \frac{77}{6}$

$\therefore$ Diameter = 2r = $\frac{77}{3}$.

Example.10

Two circles touch externally. The sum of their areas is 130 π sq. cm. and the distance between their centres is 14 cm. Find the radii of the circles.

Solution



If two circles touch externally, then the distance between their centres is equal to the sum of their radii.

Let the radii of the two circles be r_1 cm and r_2 cm respectively.

Let C_1 and C_2 be the centres of the given circles. Then,

$C_1C_2 = r_1 + r_2 \qquad [\because C_lC_2 = 14\ cm\ (given)]$

$\Rightarrow \quad 14 = r_1 + r_2$

$\Rightarrow \quad r_1 + r_2 = 14 \qquad ...(i)$

It is given that the sum of the areas of two circles is equal to 130 π cm^2.

$\therefore \quad \pi r_1^2 + \pi r_2^2 = 130\ \pi$

$\Rightarrow \quad r_1^2 + r_2^2 = 130 \qquad ...(ii)$

Now, $(r_1 + r_2)^2 = r_1^2 + r_1^2 + 2r_1r_2$

$\Rightarrow \quad 14^2 = 130 + 2\ r_1r_2 \qquad$ [Using (i) and (ii)]

$\Rightarrow \quad 196 - 130 = 2r_1r_2$

$\Rightarrow \quad r_1r_2 = 33 \qquad ...(iii)$

Now, $(r_1 - r_2)^2 = r_1^2 + r_1^2 - 2r_1r_2$

$\Rightarrow \quad (r_1 - r_2)^2 = 130 - 2 \times 33$ [Using (ii) and (iii)]

$\Rightarrow \quad (r_1 - r_2)^2 = 64$

$\Rightarrow \quad r_1 - r_2 = 8$ ...(iv)

Solving (i) and (iv), we get r_1 = 11 cm and r_2 = 3 cm.

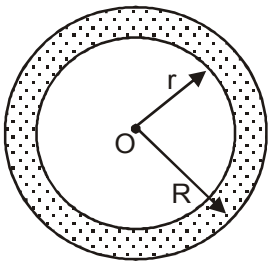
Hence, the radii of the two circles are 11 cm and 3 cm.

Example.11

A race track is in the form of a ring whose inner circumference is 352 m, and the outer circumference is 396 m. Find the width of the track.

Solution

Let the outer and inner radii of the ring be R metres and r metres respectively.



Then, $2\pi R = 396$ and $2\pi r = 352$ $\Rightarrow$ $2 \times \frac{22}{7} \times R = 396$ and $2 \times \frac{22}{7} \times r = 352$

$R = 396 \times \frac{7}{22} \times \frac{1}{2}$ and $r = 352 \times \frac{7}{22} \times \frac{1}{2}$ $\Rightarrow$ R = 63 m and r = 56 m

Hence, width of the track = (R – r) m = (63 – 56) m = 7 m.

Example.12

The diameter of a cycle wheel is 28 cm. How many revolution will it make in moving 13.2 km ?

Solution

Distance travelled by the wheel in one revolution $= 2\pi r = 2 \times \frac{22}{7} \times \frac{28}{2}$ = 88 cm and the total distance covered by the wheel = 13.2 × 1000 × 100 cm = 1320000 cm

$\therefore$ Number of revolutions made by the wheel = $\frac{1320000}{88}$ = 15000.

Example.13

A wire is looped in the form of a circle of radius 28 cm. It is re-bent into a square form. Determine the length of the side of the square.

Solution

We have,

Length of the wire = Circumference of the circle

Length of the wire = $\left\{2 \times \frac{22}{7} \times 28\right\}$cm

Length of the wire = 176 cm ....(i)

Let the side of the square be x cm.

Then,

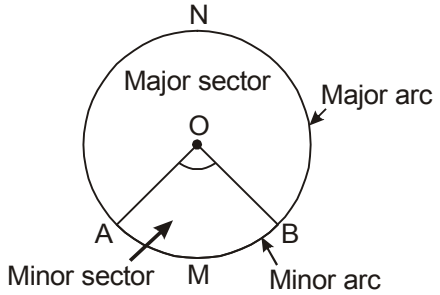
Perimeter of the square = Length of the wire

$\Rightarrow \quad 4x = 176 \quad \Rightarrow \quad x = 44$ cm

Hence, the length of the side of the square is 44 cm.

F. SECTOR OF A CIRCLE AND ITS AREA

The region bounded by an arc of a circle and its two bounding radii is called a sector of the circle.

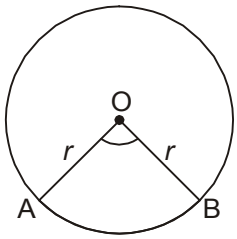


If the arc is a minor arc then the corresponding sector is called the minor sector and the remaining part bounded by the major arc is called the major sector.

In figure shaded region OAMB is the minor sector and the remaining portion OANB is the major sector.

Length of Arc and Area of sector :

Let r be the radius of the circle with centre O and AOB be a sector of the circle such that $\angle AOB = \theta$. $0 < \theta < 180°$, then the arc AB is a minor arc of the circle.



Now, if θ increases, the length of arc AB is also increases and if θ becomes 180°, then arc AB becomes the circumference of a semi-circle.

When an arc subtends an angle 180° at the centre, then the length of the arc of the semi-circle = πr.

When an arc subtends angle θ at the centre, then length of the arc = $\frac{\pi r}{180°} \times \theta = \frac{\pi r \theta}{180°}$.

If L be the length of the arc AB, then L= $\frac{\pi r \theta}{180°}$

Again, when an arc subtends angle 180° at the centre, the corresponding sector is a semi-circular region of area $\frac{1}{2}\pi r^2$.

$\therefore$ When an arc subtends an angle 180° at the centre, then the area of the corresponding sector is $\frac{\pi r^2}{2}$.

$\therefore$ When an arc subtends an θ at the centre, then area of the sector = $\frac{\pi r^2}{2} \times \frac{1}{180°} \times \theta = \frac{\pi r^2 \theta}{360°}$

If A be the area of the sector, then A = $\frac{\pi r^2 \theta}{360°}$ and length of an arc L = $\frac{\pi r \theta}{180°}$

$\Rightarrow \frac{A}{L} = \frac{\pi r^2 \theta}{360°} \times \frac{180°}{\pi r \theta} = \frac{r}{2}$

Hence, A = $\frac{Lr}{2}$.

NOTE :

(i) Angle described by minute hand in 60 minutes = 360°.

$\therefore$ Angle described by minute hand in one minute = $\left(\frac{360°}{60}\right)^{\circ}$ = 6°.

Thus, minute hand rotates through an angle of 6° in one minute.

(ii) Angle described by hour-hand in -12 hours. = 360°.

$\therefore$ Angle described by hour hand in one hour = $\left(\frac{360°}{12}\right)^{\circ}$ = 30°.

$\therefore$ Angle described by hour hand in one minute = $\left(\frac{30}{60}\right)^{\circ} = \left(\frac{1}{2}\right)^{\circ}$.

Thus, hour hand rotates through an angle of $\left(\frac{1}{2}\right)^{\circ}$ in one minute.

Solved Examples

Example.14

A sector is cut from a circle of radius 21 cm. The angle of the sector is 150°. Find the length of its arc and area.

Solution

The length or arc ℓ and area A of a sector of angle θ in a circle of radius *r* are given by

$\ell = \frac{\theta}{360°} \times 2\pi r$ and $A = \frac{\theta}{360°} \times \pi r^2$ respectively.

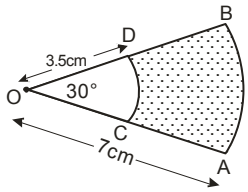
Here, r = 21 cm and θ = 150

$\therefore \ \ell = \left\{\frac{150}{360} \times 2 \times \frac{22}{7} \times 21\right\}$ cm = 55 cm and $A = \left\{\frac{150}{360} \times \frac{22}{7} \times (21)^2\right\}$ cm^2

$= \frac{1155}{2}$ cm^2 = 577.5 cm^2.

Example.15

In figure, there are shown sector of two concentric circles of radii 7 cm and 3.5 cm. Find the area of the shaded region. (Use $\pi = \frac{22}{7}$).



Solution.

Let A_1 and A_2 be the areas of sectors OAB and OCD respectively. Then,

A_1 = Area of a sector of angle 30° in a circle of radius 7 cm

$\Rightarrow \quad A_1 = \left\{\frac{30}{360} \times \frac{22}{7} \times 7^2\right\}$ [Using : $A = \frac{\theta}{360} \times \pi r^2$]

$\Rightarrow \quad A_1 = \frac{77}{6}$ cm^2

A_2 = Area of a sector of angle 30° in a circle of radius 3.5 cm.

$\Rightarrow \quad A_2 = \left\{\frac{30}{360} \times \frac{22}{7} \times (3.5)^2\right\} \Rightarrow \quad A_2 = \left\{\frac{1}{12} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right\} = \frac{77}{24}$ cm^2.

$\therefore$ Area of the shaded region = $A_1 - A_2 = \left(\frac{77}{6} - \frac{77}{24}\right)$ cm^2 = $\frac{77}{24} \times (4 - 1)$ cm^2 = $\frac{77}{8}$ cm^2 = 9.625 cm^2.

Example.16

The minute hand of a clock is 10 cm long. Find the area of the face of the clock described by the minute hand between 9 A.M. and 9.35 A.M.

Solution

We have,

Angle described by the minute hand in one minute = 6°.

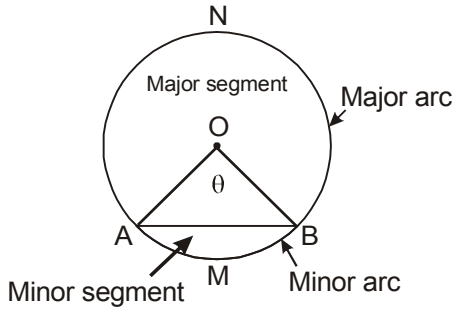
∴ Angle described by the minute hand in 35 minutes = (6 × 35)° = 210°

∴ Area swept by the minute hand in 35 minutes.

= Area of a sector of angle 210° in a circle of radius 10 cm = $\left\{\frac{210}{360}\times\frac{22}{7}\times(10)^2\right\}$ cm^2 = 183.3 cm^2.

G. SEGMENT OF A CIRCLE AND ITS AREA

The region enclosed by an arc and a chord is called the segment of the circle.

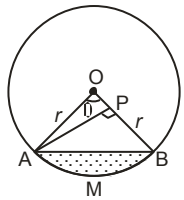


The segment containing the minor arc is called a minor segment and the remaining segment containing the major arc is called the major segment.

In the figure, the shaded region is the minor segment and the remaining part of the circle is major segment.

Area of a Segment of a Circle :

Let r be the radius of a circle with centre O and let AB be an arc subtending an angle θ at the centre O. we shall find the area of the shaded segment AMB.



Let AP ⊥ OB.

Now, area of the segment AMB = Area of the sector OAMB – area of ΔOAB

= $\frac{\pi r^2\theta}{360°}-\frac{1}{2}$ × OB × AP = $\frac{\pi r^2\theta}{360°}-\frac{1}{2}$ r × OA sin θ [∵ From ΔAOP, sin θ = $\frac{AP}{OA}$ ⇒ AP = OA sin θ]

= $\frac{\pi r^2\theta}{360°}-\frac{1}{2}$ r^2 sin θ [∵ OA = OB = r]

Hence, area of the segment = $\frac{\pi r^2\theta}{360°}-\frac{1}{2}$ r^2 sin θ.

Solved Examples

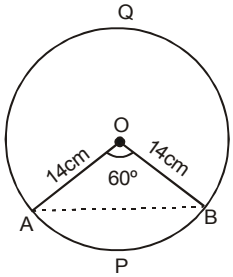
Example.17

A chord of circle 14 cm makes an angle of 60º at the center of the circle. Find :

(i) area of minor sector **(ii)** area of the minor segment

(iii) area of the major sector **(iv)** area of the major segment

Solution. Given, r = 14 cm, θ = 60º



(i) Area of minor sector OAPB = $\frac{\theta}{360^\circ}\pi r^2 = \frac{60^\circ}{360^\circ} \times 3.14 \times 14 \times 14 = 102.57$ cm²

(ii) Area of minor segment APB = $\frac{\pi r^2\theta}{360^\circ} - \frac{r^2}{2}\sin\theta = 102.57 - \frac{14\times14}{2}\sin 60^\circ$

= 102.57 – 98 × = 17.80 cm².

(iii) Area of major sector = Area of circle – Area of minor sector OAPB = $\pi(14)^2$ – 102.57

= 615.44 – 102.57 = 512.87 cm²

(iv) Area of major segment AQB = Area of circle – Area of minor segment APB

= 615.44 – 17.80 = 597.64 cm²

H. AREAS OF COMBINATIONS OF PLANE FIGURES

In our daily life we come across various plane figures, which are combinations of two or more plane figures. For example, window designs, flower beds, circular paths etc. In this section, we shall discuss problems of combinations of plane figures.

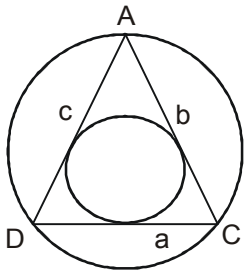
Important Formulae :

In any triangle ABC, the circle which pass through the vertex of triangle is known Circumcircle and its radius is called circumradius.

In any triangle ABC, the circle which touches the sides of triangle is known incircle and its radius is called inradius.

Area of triangle = r s,

Area of triangle = $\frac{abc}{4R}$

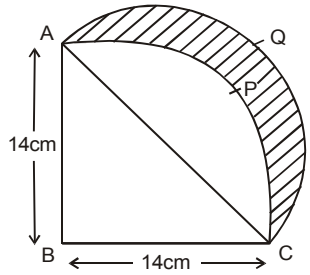


Where a,b and c represent the length of side of Triangle,s represent the semiperimeter of triangleand r, R represent the inradius and circumradius respectively.

Solved Examples

Example.18

ABCP is a quadrant of a circle of radius 14 cm. With AC as diameter, a semicircle is drawn. Find the area of the shaded portion. **(figure)**



Solution

In right angled triangle ABC,
we have

$AC^2 = AB^2 + BC^2$

$AC^2 = 14^2 + 14^2$

$AC = \sqrt{2\times14^2} = 14\sqrt{2}$ cm

Now required Area

= Area APCQA

= Area ACQA – Area ACPA

= Area ACQA – (Area ABCPA – Area of ΔABC)

$$= \frac{1}{2} \times \pi \times \left(\frac{14\sqrt{2}}{2}\right)^2 - \left[\frac{1}{4}\times\pi(14)^2 + \frac{1}{2}\times14\times14\right]$$

$$= \frac{1}{2} \times \frac{22}{7} \times 7\sqrt{2} \times 7\sqrt{2} - \frac{1}{4} \times \frac{22}{7} \times 14 \times 14 + 7 \times 14$$

$= 154 - 154 + 98 = 98$ cm^2.

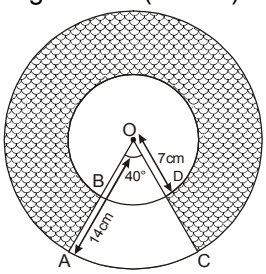
Example.19

Find the area of the shaded region in figure, where radii of the two concentric circles with centre O are 7 cm and 14 cm respectively and $\angle$AOC = 40°.

Solution

We have,

Area of ring $= \pi (R^2 - r^2) = \pi \times (14^2 - 7^2) = 462$ cm^2



Area of the region ABDC

= Area of sector AOC – Area of sector BOD

$$= \left(\frac{40}{360} \times \frac{22}{7} \times 14 \times 14 - \frac{40}{360} \times \frac{22}{7} \times 7 \times 7\right) \text{ cm}^2$$

$$= \left(\frac{1}{9} \times 22 \times 14 \times 2 - \frac{1}{9} \times 22 \times 7 \times 1\right) \text{ cm}^2 = \frac{22}{9} \times (28-7) \text{ cm}^2 = \frac{154}{3} \text{ cm}^2$$

Hence, Required shaded area

$$= \left(462 - \frac{154}{3}\right) \text{ cm}^2 = \frac{1232}{3} \text{ cm}^2 = 410.67 \text{ cm}^2$$

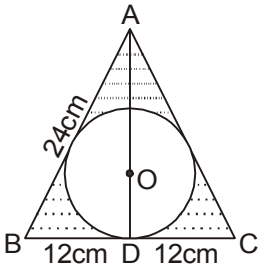
Example.20

In an equilateral triangle of side 24 cm, a circle is inscribed touching its sides. Find the area of the remaining portion of the triangle [Take $\sqrt{3}$ = 1.732].

Solution

Let ABC be an equilateral triangle of side 24 cm, and let AD be perpendicular from A on BC. Since the triangle is equilateral, so AD bisects BC.

$\therefore$ BD = CD = 12 cm



The centre of the inscribed circle will coincide with the centroid of ΔABC.

$\therefore$ OD = $\frac{AD}{3}$

In Δ ABD, we have

$AB^2 = AD^2 + BD^2$ [Using Pythagoras Theorem]

$\Rightarrow$ $24^2 = AD^2 + 12^2$

$\Rightarrow$ AD = $\sqrt{24^2 - 12^2} = \sqrt{(24-12)(24+12)} = \sqrt{36 \times 12} = 12\sqrt{3}$ cm.

$\therefore$ OD = $\frac{1}{3}$ AD = $\left(\frac{1}{3} \times 12\sqrt{3}\right)$ cm = $4\sqrt{3}$ cm

Area of the incircle = π $(OD)^2$

$= \left\{\frac{22}{7} \times \left(4\sqrt{3}\right)^2\right\} cm^2 = \left\{\frac{22}{7} \times 48\right\} cm^2 = 150.85\ cm^2$

Area of the triangle ABC = $\frac{\sqrt{3}}{4}$ $(Side)^2$ = $\frac{\sqrt{3}}{4}$ $(24)^4$ = 249.4 cm^2

$\therefore$ Area of the remaining portion of the triangle = (249.4 – 150.85) cm^2 = 98.55 cm^2.

Example.21

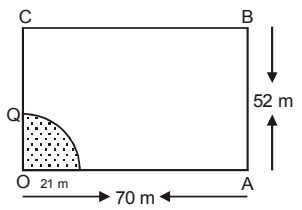
A horse is placed for grazing inside a rectangular field 70 m by 52 m and is tethered to one corner by a rope 21 m long. On how much area can it graze ?

Solution

Shaded portion indicates the area which the horse can graze. Clearly, shaded area is the area of a quadrant of a circle of radius r = 21 m.

$\therefore$ Required area = $\frac{1}{4}\pi r^2$

$\Rightarrow$ Required area = $\left\{\frac{1}{4} \times \frac{22}{7} \times (21)^2\right\} cm^2$

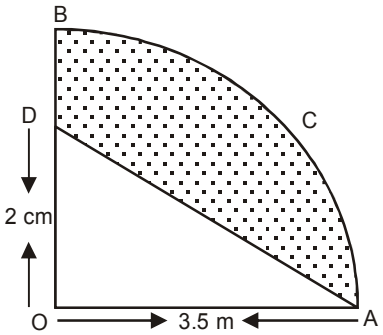


$\Rightarrow$ Required area = $\frac{693}{2} cm^2$=346.5 cm^2

Example.22

In figure, AOBCA represents a quadrant of a circle of radius 3.5 cm with centre O. Calculate the area of the shaded portion (Take $\pi = \frac{22}{7}$).

Solution



We have,

Area of quadrant AOBCA $= \frac{1}{4}\pi r^2 = \frac{1}{4} \times \frac{22}{7} \times (3.5)^2 = 9.625$ cm^2

Area of ΔAOD = $\frac{1}{2}$ × Base × Height = $\frac{1}{2}(3.5)(2) = 3.5$ cm^2

Hence, Area of the shaded portion

= Area of quadrant – Area of ΔAOD

= (9.625 – 3.5) cm^2 = 6.125 cm^2

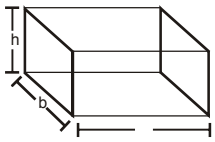
I. 3-D FIGURES OR SOLID FIGURES

If any figure such as cuboid, which has three dimensions length, width and height are known as three dimensional figures. Where as rectangle has only two dimensions i.e. length and width. Three dimensional figures have volume in addition to areas of surface from which these solid figures are formed.

Some of the main solid figures are :

(a) Cuboid :

Total Surface Area (T.S.A.) : The area of surface from which cuboid is formed. There are six faces (rectangular), eight vertices and twelve edges in a cuboid.



(i) Total Surface Area (T.S.A.) = 2 [ℓ × b + b × h + h × ℓ]

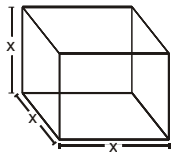
(ii) Lateral Surface Area (L.S.A.) = 2 [b × h + h × ℓ] (or **Area of 4 walls)** = 2 h [ℓ + b]

(iii) Volume of Cuboid = (Area of base) × height = (ℓ × b) × h

(iv) Length of diagonal = $\sqrt{\ell^2 + b^2 + h^2}$

(b) Cube :

Cube has six faces. Each face is a square.



(i) **T.S.A.** $= 2\,[x \cdot x + x \cdot x + x \cdot x]$
$= 2\,[x^2 + x^2 + x^2] = 2\,(3x^2) = 6x^2$

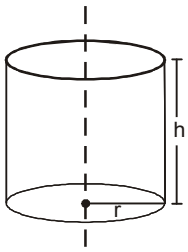
(ii) **L.S.A.** $= 2\,[x^2 + x^2] = 4x^2$

(iii) **Volume** = (Area of base × Height) = $(x^2) \cdot x = x^3$

(iv) **Length of diagonal** = $x\sqrt{3}$

(c) Cylinder :

Curved surface area of cylinder (C.S.A.) : It is the area of surface from which the cylinder is formed. When we cut this cylinder, we will find a rectangle with length $2\pi r$ and height h units.



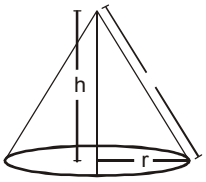
(i) **C.S.A. of cylinder** = $(2\pi r) \times h = 2\pi rh$.

(ii) **Total Surface Area (T.S.A.) :**

T.S.A. = C.S.A. + circular top & bottom
$= 2\pi rh + 2\pi r^2 = 2\pi r\,(h + r)$ sq. units.

(iii) **Volume of cylinder :**
Volume = Area of base × height
$= (\pi r^2) \times h = \pi r^2 h$ cubic units

(d) Cone :



(i) **C.S.A.** $= \pi r\ell$

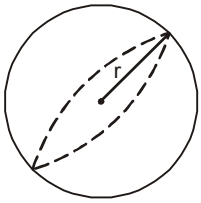
(ii) **T.S.A.** = C.S.A. + other area
$= \pi r\ell + r^2$
$= \pi r\,(\ell + r)$

(iii) **Volume** = $\frac{1}{3}\pi r^2 h$

Where, h = height
r = radius of base
ℓ = slant height

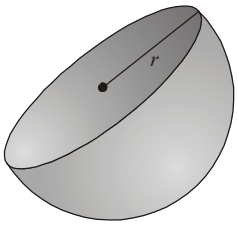
(e) Sphere :

T.S.A. = S.A. = $4\pi r^2$



Volume $= \frac{4}{3}\pi r^3$

(f) Hemisphere :

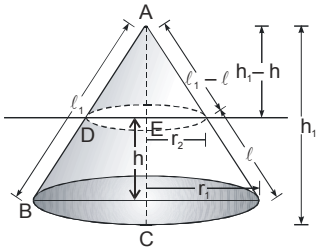


C.S.A. = $2\pi r^2$
T.S.A. = C.S.A. + other area
$= 2\pi r^2 + \pi r^2$
$= 3\pi r^2$

Volume = $\frac{2}{3}\pi r^3$

(a) Frustum of a Cone :

When a cone is cut by a plane parallel to base, a small cone is obtained at top and other part is obtained at bottom. That other part is known as **'Frustum of Cone'.**



Δ ABC ~ Δ ADE

$\therefore \quad \frac{AC}{AE} = \frac{AB}{AD} = \frac{BC}{DE}$

$$\frac{h_1}{h_1 - h} = \frac{\ell_1}{\ell_1 - \ell} = \frac{r_1}{r_2}$$

Or $\quad \frac{h_1}{h} = \frac{\ell_1}{\ell} = \frac{r_1}{r_1 - r_2}$

Volume of Frustum

$$= \frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 (h_1 - h) = \frac{1}{3}\pi [r_1^2 h_1 - r_2^2 (h_1 - h)]$$

$$= \frac{1}{3}\pi\left[r_1^2\left(\frac{r_1 h}{r_1 - r_2}\right) - r_2^2\left(\frac{r_1 h}{r_1 - r_2} - h\right)\right] = \frac{1}{3}\pi h\left[\frac{r_1^3 - r_2^3}{r_1 - r_2}\right] = \frac{1}{3}\pi h\left[r_1^2 + r_2^2 + r_1 r_2\right]$$

Curved Surface Area of Frustum

$$= \pi r_1 \ell_1 - \pi r_2(\ell_1 - \ell) = \pi\left[r_1\left(\frac{r_1 \ell}{r_1 - r_2}\right) - r_2\left(\frac{r_1 \ell}{r_1 - r_2} - \ell\right)\right]$$

$$= \pi\ell\left[\frac{r_1^2}{r_1 - r_2} - \frac{r_2^2}{r_1 - r_2}\right] = \pi\ell\,(r_1 + r_2)$$

Total Surface Area of a Frustum

$= \text{CSA of frustum} + \pi r_1^2 + \pi r_2^2$

$= \pi\ell\,(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$

Slant height of a Frustum = $\sqrt{h^2 + (r_1 - r_2)^2}$

where, h = height of the frustum
r_1 = radius of larger circular end
r_2 = radius of smaller circular end

Example.23

If v is the volume of a cuboid of dimensions a, b, and c and s is its surface area, then prove that $\frac{1}{v} = \frac{2}{s}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

Solution

Volume of cuboid (v) = abc cubic unit.

Surface area of cuboid (s) = 2 (ab + bc + ca) sq. units

L.H.S. = $\frac{1}{v} = \frac{1}{abc}$ ... (i)

R.H.S. = $\frac{2}{s}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{2}{2(ab + bc + ca)}\left(\frac{bc + ca + ab}{abc}\right)$

$= \frac{1}{abc}$ .. (ii)

From (i) and (ii) $\frac{1}{v} = \frac{2}{s}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$. **Hence proved**

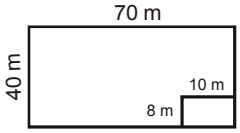
Example.24

A field is 70 m long and 40 m broad. In one corner of the field, a pit which is 10 m long, 8 m broad and 5 m deep, has been dug out. The earth taken out of it is evenly spread over the remaining part of the field. Find the rise in the level of the field.

Solution

Area of the field = (70 × 40) m^2 = 2800 m^2.

Area of the pit = (10 × 8) m^2 = 80 m^2



Area over which the earth is spread over = (2800 – 80) m^2 = 2720 m^2

Volume of the earth dug out = (10 × 8 × 5) m^3 = 400 m^3.

Rise in level of the field = $\left(\frac{\text{Volume of the earth dug out}}{\text{Area on which the earth is spread}}\right) = \left(\frac{400}{2720}\right)$ m

$= \left(\frac{400 \times 100}{2720}\right)$ cm $= \left(\frac{250}{17}\right)$ cm = 14.70 cm

Example. 25

A well of diameter 2 m is dug 14 m deep. The earth taken out of it is spread evenly all around it to a width of 5 m to form an embankment. Find the height of the embankment.

Solution

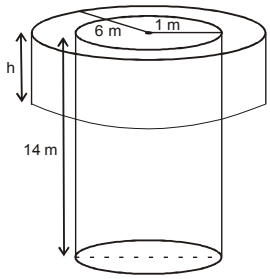
Let **h** be the required height of the embankment.

The shape of the embankment will be like the shape of a cylinder of internal radius 1 m and external radius (5 + 1) m = 6 m.

The volume of the embankment will be equal to the volume of the earth dug out from the well. Now, the volume of the earth dug out

= volume of the cylindrical well

= $\pi \times 1^2 \times 14\ m^3 = 14\ \pi m^3$



Also, the volume of the embankment

$= \pi(6^2 - 1^2)\, h \text{ cm}^3 = 35\, \pi h \text{ m}^3$

Hence, we have 35h = 14

$\Rightarrow \quad h = \frac{14}{35} = \frac{2}{5} = 0.4$

Hence, the required height of the embankment = 0.4 m.

Example.26

The ratio of the volumes of the two cones is 4 : 5 and the ratio of the radii of their bases is 2 : 3. Find the ratio of their vertical heights.

Solution Let the radii of bases, vertical heights and volumes of the two cones be r_1, h_1, v_1 and r_2, h_2, v_2 respectively.

According to the question,

$$\frac{v_1}{v_2} = \frac{4}{5} \qquad \text{... (i)}$$

$$\frac{r_1}{r_2} = \frac{2}{3} \qquad \text{... (ii)}$$

From (i), we have $\dfrac{\frac{1}{3}\pi r_1^2 h_1}{\frac{1}{3}\pi r_2^2 h_2} = \dfrac{4}{5}$

$\Rightarrow \quad \dfrac{r_1^2 h_1}{r_2^2 h_2} = \dfrac{4}{5} \quad \Rightarrow \quad \left(\dfrac{r_1}{r_2}\right)^2 \dfrac{h_1}{h_2} = \dfrac{4}{5}$

$\Rightarrow \quad \left(\dfrac{2}{3}\right)^2 \dfrac{h_1}{h_2} = \dfrac{4}{5} \quad \Rightarrow \quad \dfrac{h_1}{h_2} = \dfrac{4}{5}\left(\dfrac{3}{2}\right)^2$ [Using (ii)]

$\Rightarrow \quad \dfrac{h_1}{h_2} = \dfrac{9}{5}$

Hence the ratio of their vertical height is 9 : 5.

Example.27

The surface area of a sphere of radius 5 cm is five times the area of the curved surface of a cone of radius 4 cm. Find the height of the cone.

Solution

Surface area of sphere of radius 5 cm = $4\pi\,(5)^2$ cm^2

Area of the curved surface of cone of radius 4 cm = $\pi(4)\,\ell$ cm^2

where ℓ cm is the slant height of the cone.

According to the question,

$4\pi\,(5)^2 = 5[\pi(4)\ell] \quad \Rightarrow \quad \ell = 5$ cm

$\Rightarrow \quad \sqrt{r^2 + h^2} = 5 \quad \Rightarrow \quad r^2 + h^2 = 25$

$\Rightarrow \quad (4)^2 + h^2 = 25 \quad \Rightarrow \quad 16 + h^2 = 25$

$\Rightarrow \quad h^2 = 9 \quad \Rightarrow \quad h = 3.$

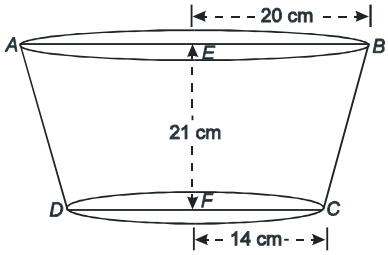
Hence, the height of the cone is 3 cm.

Example.28

A bucket is 40 cm in diameter at the top and 28 cm in diameter at the bottom. Find the capacity of the bucket in litres, if it is 21 cm deep. Also, find the cost of tin sheet used in making the bucket, if the cost of tin is Rs 1.50 per sq dm.

Solution

Given : r_1 = 20 cm, r_2 = 14 cm and h = 21 cm



Now, the required capacity (i.e., volume) of the bucket $= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$

$= \frac{22\times21}{7\times3}(20^2 + 20 \times 14 + 14^2)\ cm^3 = 22 \times 876\ cm^3 = 19272\ cm^3$

$= \frac{19272}{1000}$ litres = 19.272 litres.

Now,

$\ell = \sqrt{(r_1-r_2)^2+h^2}\ \sqrt{(20-14)^2+21^2} = cm$

$= \sqrt{6^2+21^2}\ cm = \sqrt{36+441}\ cm = \sqrt{477}\ cm = 21.84\ cm.$

Total surface area of the bucket (which is open at the top)

$= \pi\ell(r_1 + r_2) + r_2^2 = \pi[(r_1+ r_2)\ell + r_2^2]$

$= \frac{22}{7}\left[(20+14)\ 21.84+14^2\right] = 2949.76\ cm^2.$

Required cost of the tin sheet at the rate of Rs 1.50 per dm^2 i.e., per 100 cm^2

$= Rs\ \frac{1.50\times2949.76}{100} \cong$ Rs 44.25.

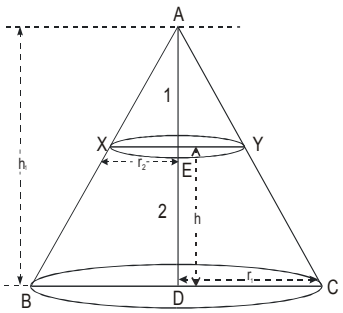
Example.29

A cone is divided into two parts by drawing a plane through a point which divides its height in the ratio 1 : 2 starting from the vertex and the plane is parallel to the base. Compare the volume of the two parts.

SolutionLet the plane XY divide the height AD of cone ABC such that AE : ED = 1 : 2, where AED is the axis of the cone. Let r_2 and r_1 be the radii of the circular section XY and the base BC of the cone respectively and let $h_1 - h$ and h_1 be their heights **[figure]**.

Then, $\frac{h_1}{h} = \frac{3}{2} \Rightarrow h_1 = \frac{3}{2}h$

And $\frac{r_1}{r_2} = \frac{h_1}{h_1-h} = \frac{\frac{3}{2}h}{\frac{1}{2}h} = 3$



$r_1 = 3r_2$

Volume of cone AXY

$= \frac{1}{3}\pi r_2^2 (h_1 - h) = \frac{1}{3}\pi r_2^2 (h - h) = \frac{1}{6}\pi r_2^2 h$

Volume of frustum XYBC

$$= \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$$

$$= \frac{1}{3}\pi h(9r_2^2 + r_2^2 + 3r_2^2)$$

$$= \frac{1}{3}\pi h(13r_2^2)$$

So, $$\frac{Volume\ of\ cone\ AXY}{Volume\ of\ frustum\ XYBC} = \frac{\frac{1}{6}\pi r_2^2 h}{\frac{13}{3}\pi r_2^2 h}$$

$$\frac{\text{Volume of cone AXY}}{\text{Volume of frustum XYBC}} = \frac{1}{26}.$$

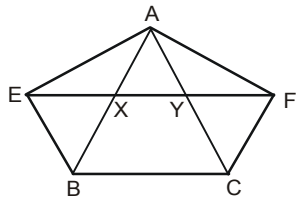
i.e., the ratio between the volume of the cone AXY and the remaining portion BCYX is 1 : 26.

Exercise-1

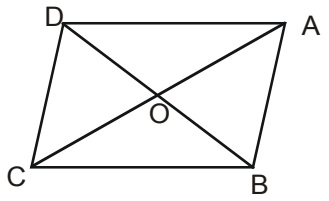
SUBJECTIVE QUESTIONS

Section (A) : Perimeter and area of plane figures

A-1 In the given figure, XY is a line parallel to side BC of a ΔABC. BE || AC and CF || AB meet XY in E and F respectively. Show that ar(ΔABE) = ar(ΔACF).



A-2 Diagonals AC and BD of a quadrilateral ABCD intersect at O, such that OB = OD. If AB = CD, then show that :

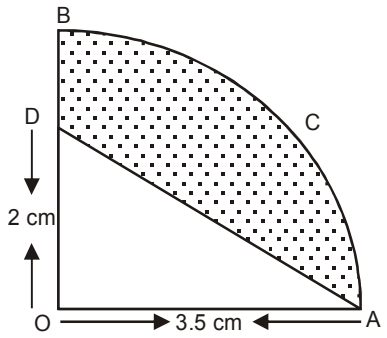


(i) ar (DOC) = ar (AOB)
(ii) ar (DCB) = ar (ACB)
(iii) DA || CB or ABCD is a parallelogram.

A-3 In an equilateral triangle of side 24 cm, a circle is inscribed touching its sides. Find the area of the remaining portion of the triangle [Take $\sqrt{3}$ = 1.732].

A-4 A horse is placed for grazing inside a rectangular field 70 m by 52 m and is tethered to one corner by a rope 21 m long. On how much area can it graze ?

A-5 In figure , AOBCA represents a quadrant of a circle of radius 3.5 cm with centre O. Calculate the area of the shaded portion (Take $\pi = \frac{22}{7}$).



Section (B) : Cube and cuboid

B-1 If v is the volume of a cuboid of dimensions a, b, and c and s is its surface area, then prove that $\frac{1}{v} = \frac{2}{s}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

B-2 A class room is 7 m long, 6.5 m wide and 4 m high. It has one door 3 m × 1.4 m and three windows each measuring 2 m × 1 m The interior walls are to be colour-washed. The contractor charge Rs. 15 per sq.m. Find the cost of colour washing.

B-3 The dimensions of a cinema hall are 100 m, 50 m and 18 m. How many persons can sit in the hall, if each required 150 m^3 of air ?

B-4 Water in a canal, 30 dm wide and 12 dm deep, is flowing with a speed of 10 km/hr. How much area will it rrigate in 30 minutes, if 8 cm of standing water is required for irrigation.

Section (C) : Cylinder

C-1 A cylindrical vessel, without lid, has to be tin coated both of its surface (inner and outer). If the radius of its base is $\frac{1}{2}$ m and its height is 1.4 m, calculate the cost of tin-coating at the rate of Rs 50 per 1000 cm^2.

C-2 The diameter of a roller 120 cm long is 84 cm. If it takes 500 complete revolutions to level a playground, determine the cost of levelling at the rate of Rs 25 per square metre. [Use $\pi = \frac{22}{7}$]

C-3 A cylindrical metallic pipe is 14 cm long. The difference between the outside and inside surfaces is 44 cm^2. If the pipe is made up of 99 cubic cm of metal, find the outer and inner radii of the pipe.

C-4 An iron rod of length 1 m and diameter 4 cm is melted and cast into thin wires of length 20 cm each. If the number of such wires be 2000, find the radius of each thin wire.

C-5 A well of diameter 2 m is dug 14 m deep. The earth taken out of it is spread evenly all around it to a width of 5 m to form an embankment. Find the height of the embankment.

Section (D) : Cone and frustum

D-1 How many metres of cloth of 1.1 m width will be required to make a conical tent whose vertical height is 12 m and base radius is 16 m ? Find also the cost of the cloth used at the rate of Rs 14 per metre.

D-2 A bus stop is barricated from the remaining part of the road, by using 50 hollow cones made of recycled card - board. Each cone has a base diameter of 40 cm and height 1 m. If the outer side of each of the cones is to be painted is Rs. 12 per m^2, what will be the cost of painting all these cones. [Use π = 3.14 and $\sqrt{1.04}$ = 1.02]

D-3 A right triangle ABC with its sides 5 cm, 12 cm and 13 cm is revolved about side 12 cm. Find the volume of the solid so formed. If the triangle ABC is revolved about side 5 cm, then find the volume of the solid so obtained. Find also the ratio of the two volumes obtained.

D-4 A cone is divided into two parts by drawing a plane through a point which divides its height in the ratio 1 : 2 starting from the vertex and the plane is parallel to the base. Compare the volume of the two parts.

D-5 The height of a cone is 30 cm. A small cone is cut off at the top by a plane parallel to the base. If its volume be $\frac{1}{27}$ of the volume of the given cone, at what height above the base is the section made?

Section (E) : Sphere and hemisphere

E-1 The diameters of external and internal surfaces of a hollow spherical shell are 10 cm and 6 cm respectively. If it is melted and recast into a solid cylinder of length of $2\frac{2}{3}$ cm, find the diameter of the cylinder.

E-2 The internal and external diameters of a hollow hemispherical Vessal are 24 cm and 25 cm respectively. The cost to paint 1 cm^2 the surface is Rs. 0.05. Find the total cost to paint the Vessal all over. [Use $\pi = \frac{22}{7}$]

E-3 How many balls, each of radius 1 cm, can be made from a solid sphere of lead of radius 8 cm ?

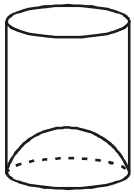
E-4 A hemispherical tank of radius $1\frac{3}{4}$ m is full of water. It is connected with a pipe which empties it at the rate of 7 liters per second. How much time will it take to empty the tank completely?

E-5 A sphere of radius 6 cm is dropped into a cylindrical vessel partly filled with water. The radius of the vessel is 8 cm. If the sphere is submerged completely, then find rise in the water level.

Section (F) : Miscellaneous

F-1 Water flows at the rate of 10 m per minute through a cylindrical pipe having its diameter as 5 mm. How much time will it take to fill a conical vessel whose diameter of the base is 40 cm and depth 24 cm?

F-2 A juice seller serves his customers using a glass as shown in figure. The inner diameter of the cylindrical glass is 5 cm, but the bottom of the glass has a hemispherical portion raised which reduces the capacity of the glass. If the height of the glass is 10 cm, find the apparent capacity of the glass and its actual capacity.



F-3 A solid cylinder of diameter 12 cm and height 15 cm is melted and recast into 12 toys in the shape of a right circular cone mounted on a hemisphere. Find the radius of the hemisphere and the total height of the toy if height of the conical part is 3 times its radius.

F-4 A cylindrical vessel with internal diameter 10 cm and height 10.5 cm is full of water. A solid cone of base diameter 7 cm and height 6 cm is completely immersed in water. Find the volume of
(i) water displaced out of the cylindrical vessel.
(ii) water left in the cylindrical vessel. [Take $\pi = \frac{22}{7}$]

F-5 A hemispherical bowl of internal radius 36 cm is full of liquid. The liquid is to be filled into cylindrical shaped small bottles each of diameter 3 cm and height 6 cm. How many bottles are need to empty the bowl ?

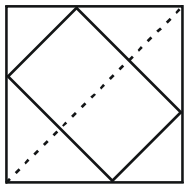
OBJECTIVE QUESTIONS

Section (A) : Perimeter and area of plane figures

A-1. ΔABC is an isosceles right triangle with area P. The radius of the circle that passes through the point A, B and C is

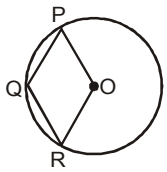
(A) $\sqrt{P}$ (B) $\sqrt{\frac{P}{2}}$ (C) $\frac{\sqrt{P}}{2}$ (D) $\sqrt{2P}$

A-2. A rectangle is inscribed in a square creating four isosceles right triangle. If the total area of these four triangles is 200. The length of the diagonal of the rectangle is :



(A) 10 (B) 15 (C) 20 (D) 25

A-3. In the given figure, OPQR is a rhombus, three of whose vertices lie on a circle with centre O. If the area of rhombus is $32\sqrt{3}$ cm^2. The radius of circle is :



(A) 8 (B) 9 (C) 10 (D) 11

A-4. A wire in the shape of an equilateral triangle encloses an area of S sq. cm. If the same wire is bent to form of a circle. The area of the circle will be :

(A) $\frac{\pi S^2}{9}$ (B) $\frac{3S^2}{\pi}$ (C) $\frac{3S}{\pi}$ (D) $\frac{3\sqrt{3}S}{\pi}$

A-5. In a rhombus one of the diagonals is twice the other diagonal. Let A be the area of the rhombus in square units. Then each side of the rhombus is :

(A) $\sqrt{A}$ (B) $\frac{1}{2}\sqrt{2A}$ (C) $\frac{1}{2}\sqrt{5A}$ (D) $\frac{1}{4}\sqrt{4A}$

Section (B) : Cube and cuboid

B-1. The cost of white washing the four walls of a room is Rs. 25. The cost of white-washing a room twice in length, breadth and height will be :

(A) Rs. 50 (B) Rs. 75 (C) Rs. 100 (D) Rs. 200

B-2. Three equal cubes are placed adjacently in a row. Find the ratio of the total surface area of the new cuboid to that of the sum of the surface areas of three cubes.

(A) 5 : 9 (B) 3 : 5 (C) 2 : 5 (D) 7 : 9

B-3. The largest sphere is cut off from a cube of side 5 cm. The volume of the sphere will be :

(A) 27 π cm^3 (B) $\frac{125}{6}\pi$ cm^3 (C) 108 π cm^3 (D) 30 π cm^3

B-4. Find the volume of a Cube whose surface area is 600 cm^2.
(A) 900 cm^3 (B) 1000 cm^3 (C) 1500 cm^3 (D) 810 cm^3

B-5. The three different face diagonals of a cuboid (rectangular parallelopiped) have lengths 39, 40, 41. The length of the main diagonal of the cuboid which joins a pair of opposite corners :
(A) 49 (B) $49\sqrt{2}$ (C) 60 (D) $60\sqrt{2}$

Section (C) : Cylinder

C-1 The ratio of the height of a circular cylinder is to the diameter of its base is 1 : 2. Then the ratio of the areas of its curved surface to the sum of the areas of its two ends is :
(A) 1 : 1 (B) 1 : 2 (C) 2 : 1 (D) 1 : 3

C-2. The diameter of a garden roller is 1.4 m and it is 2 m long. How much area will it cover in 5 revolutions? (Use π = 22/7)
(A) 40 m^2 (B) 41.40 m^2 (C) 44 m^2 (D) 42.40 m^2

C-3. An iron pipe 20 cm long has exterior diameter equal to 25 cm. If the thickness of the pipe is 1 cm, find the whole surface area of the pipe.
(A) 3168 cm^2 (B) 3160 cm^2 (C) 3068 cm^2 (D) 3268 cm^2

C-4. Savitri had to make a model of a cylindrical kaleidoscope for her science project. She wanted to use chart paper to make the curved surface of the kaleidoscope. What would be the area of chart paper required by her, if she wanted to make a kaleidoscope of length 25 cm with a 3.5 cm radius ? You may take (π = 22/7).
(A) 540 cm^2 (B) 520 cm^2 (C) 550 cm^2 (D) 560 cm^2

C-5. Radius and height of a cylinder are 7 cm and 14 cm respectively. It is divided along its axis vertically into equal parts. The increase in its total surface area is :
(A) 392 sq. cm (B) 308 sq. cm (C) 196 sq. cm (D) 154 sq. cm

Section (D) : Cone and frustum

D-1. The radius and height of a cone are in the ratio 4 : 3. The area of the base is 154 cm^2. Find the area of the curved surface.
(A) 192.5 cm^2 (B) 195 cm^2 (C) 190.5 cm^2 (D) 185.5 cm^2

D-2. A corn cob, shaped somewhat like a cone, has the radius of its broadest end as 2.1 cm and length (height) as 20 cm. If each 1 cm^2 of the surface of the cob carries an average of four grains, find how many grains (approximately) you would find on the entire cob.
(A) 511 (B) 531 (C) 551 (D) 571

D-3. The radius of the base of a conical tent is 7 m. The tent is 24 m high. Find the cost of the canvas required to make the tent, if one square meter of canvas costs Rs. 180 (Take π = 22/7).
(A) Rs 99000 (B) Rs 98000 (C) Rs 95000 (D) Rs 97000

D-4. A right triangle with its sides 9 cm, 12 cm and 15 cm is revolved about the side 12 cm. Find the volume of the solid so formed.
(A) 1018.28 cm^3 (B) 702.57 cm^3 (C) 1200 cm^3 (D) 1218.28 cm^3

D-5. A cone is divided into two parts by drawing a plane through the mid point of its axis parallel to its base then the ratio of the volume of two parts is :
(A) 1 : 3 (B) 1 : 7 (C) 1 : 8 (D) 1 : 9

Section (E) : Sphere and hemisphere

E-1. Find the difference between total surface area &curved surface area of a hemisphere of radius 21 cm.
(A) 1376 cm^2 (B) 1386 cm^2 (C) 1396 cm^2 (D) 1404 cm^2

E-2. The hollow sphere, in which the circus motorcyclist performs his stunts, has diameter of 7 m. Find the area available to motorcyclist for riding.
(A) 154 m^2 (B) 144 m^2 (C) 38.5 m^2 (D) 176 m^2

E-3. A hemispherical dome of a building needs to be painted. If circumference of the base of the dome is 44 m, find the cost of painting it, given the cost of painting is Rs 2 per 100 cm^2.
(A) Rs 62600 (B) Rs 63000 (C) Rs 61000 (D) Rs 61600

E-4. How many spherical bullets can be made out of a solid cube of lead whose edge measures 44 cm, each bullet being 4 cm in diameter.
(A) 2451 (B) 2541 (C) 2304 (D) 2536

E-5. A hemispherical tank of radius $1\frac{3}{4}$ m is full of water. It is connected with a pipe which empties it at the rate of 7 liters per second. How much time will it take to empty the tank completely?
(A) 26.74 min. (B) 26.54 min. (C) 26.4 min. (D) 26 min.

Section (F) : Miscellaneous

F-1. A sphere, a cylinder and a cone are of the same radius and same height. Find the ratio of their curved surfaces.
(A) $4 : 4 : \sqrt{5}$ (B) $4 : 5 : \sqrt{5}$ (C) $3 : 4 : \sqrt{5}$ (D) $2 : 3 : 2\sqrt{5}$

F-2. The surface area of a sphere of radius 5 cm is five times the area of the curved surface of a cone of radius 4 cm. Find the height of the cone.
(A) 3 cm (B) 2 cm (C) 4 cm (D) 5 cm

F-3. The shape of a solid is a cylinder surmounted by a cone. If the volume of the solid is 40656 cm^3, the diameter of the base is 42 cm and the height of the cylinder is 20 cm, find the slant height of the conical portion.
(A) 45 cm (B) 35 cm (C) 40 cm (D) 50 cm

F-4. A piece of wire in the form of a rectangle 15 cm long and 7 cm broad is reshaped and bent into the form of a circle. Find the radius of the circle.
(A) 5 cm (B) 7 cm (C) 8 cm (D) 9 cm

F-5. A cone, a right cylinder and a hemi shpere stand on equal bases and have the same height. Their volumes are in the ratio :
(A) 1 : 2 : 3 (B) 1 : 3 : 2 (C) 2 : 3 : 1 (D) 2 : 1 : 3

Exercise-2

OBJECTIVE QUESTIONS

1. All the 3 sides of a right triangle are integers and one side has a length 11 units. Area of the triangle in square units lies between
(A) 1 and 100 (B) 100 and 200 (C) 200 and 300 (D) More than 300

2. A triangle EFG is inscribed in a unit square ABCD with E on AB, F on DA, G on CD such that AE = DF = CG = 1/3. The area of the triangle EFG is :

(A) $\frac{5}{18}$ (B) $\frac{1}{3}$ (C) $\frac{5}{9}$ (D) $\frac{4}{9}$

3. ABCD is a rectangle, P lies on AD and Q on AB. The triangles PAQ, QBC and PCD all have the same area, and BQ = 2. The length of AQ, is :

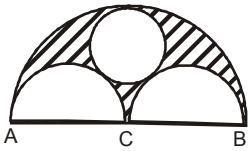
(A) $3+\sqrt{5}$ (B) $2\sqrt{3}$

(C) $\sqrt{5}+1$ (D) not uniquely determined

4. Three parallel lines ℓ_1, ℓ_2 and ℓ_3 are drawn through the vertices A, B and C of a square ABCD. If the distance between ℓ_1 and ℓ_2 is 7 and between ℓ_2 and ℓ_3 is 12, then the area of the square ABCD is :

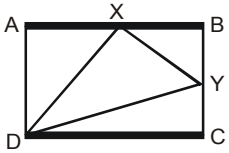
(A) 193 (B) 169 (C) 196 (D) 225

5. In the given figure, the diameter of the biggest semi-circle is 56 cm and the radius of the smallest circle is 7 cms. The area of the shaded portion is :



(A) 482 cm^2 (B) 462 cm^2 (C) 654 cm^2 (D) 804 cm^2

6. ABCD is a rectangle and lines DX, DY and XY are drawn as shown. Area of ΔAXD is 5, Area of ΔBXY is 4 and area of ΔCYD is 3. If the area of ΔDXY can be expressed as $\sqrt{x}$ where $x \in N$ then x is equal to



(A) 72 (B) 75 (C) 84 (D) 96

7. Water in a canal, 30 dm wide and 12 dm deep, is flowing with a speed of 10 km/hour. How much area will it irrigate in 30 minutes, if 8 cm of standing water is required for irrigation.

(A) 220500 m^2 (B) 22500 m^2 (C) 220000 m^2 (D) 225000 m^2

8. If a right circular cone, with slant height ℓ , and a right circular cylinder have the same radius r, same total surface area and heights h and h' respectivley, then $\sqrt{\frac{\ell - r}{\ell + r}}$ =

(A) h/h' (B) 2h/h' (C) h/2h' (D) 2h'/h

9. A solid metallic sphere of radius r is converted into a solid right circular cylinder of radius R. If the height of the cylinder is twice the radius of the sphere then :

(A) R = r (B) R = $\frac{r\sqrt{2}}{\sqrt{3}}$ (C) R = $\sqrt{\frac{2r}{3}}$ (D) R = $\frac{\sqrt{3}\,r}{\sqrt{2}}$

10. A closed conical vessel is filled with water fully and is placed with its vertex down. The water is let out at a constant speed. After 21 minutes, it was found that the height of the water column is half of the original height. How much more time in minutes does it empty the vessel ?

(A) 21 (B) 14 (C) 7 (D) 3

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. A cylinder of radius 6 cm and height h cm is filled with ice cream. The ice cream is then distributed among 10 children in identical cones having hemispherical tops. The radius of the base of the cone is 3 cm and its height is 12cm. Then the height h of the cylinder must be : **[IJSO-2008]**
(A) 100/7 cm (B) 18 cm (C) 15 cm (D) 200 / 11 cm

2. A regular hexagon of maximum possible area is cut off from an equilateral triangle. The ratio of area of triangle to the area of hexagon will be : **[IJSO-2009]**
(A) $\sqrt{\frac{3}{2}}$ (B) $\frac{\sqrt{6}}{2}$ (C) $\frac{3}{\sqrt{2}}$ (D) $\frac{3}{2}$

3. Three cubes with sides in the ratio 3 : 4 : 5 are melted to form a single cube whose diagonal is $12\sqrt{3}$ cm. The sides of the cubes are : **[NSTSE-2010]**
(A) 3 cm, 4 cm, 5 cm (B) 6 cm, 8 cm, 10 cm
(C) 9 cm, 12 cm, 15 cm (D) None of these

4. The radius of a cone is $\sqrt{2}$ times the height of the cone. A cube of maximum possible volume is cut from the same cone. What is the ratio of the volume of the cone to the volume of the cube ? **[NSTSE-2010]**
(A) 3.18 π (B) 2.25 π (C) 2.35 (D) can't be determined

5. A right circular cone has for its base a circle having the same radius as a given sphere. The volume of the cone is one-half that of the sphere. The ratio of the altitude of the cone to the radius of its base is : **[NSTSE-2010]**
(A) $\frac{1}{1}$ (B) $\frac{1}{2}$ (C) $\frac{2}{1}$ (D) $\frac{2}{3}$

6. The perimeter of an isosceles right angled triangle is 2p. Its area is **[IJSO-2010]**
(A) $\left(2+\sqrt{2}\right)$ p^2 (B) $\left(2-\sqrt{2}\right)$ p^2 (C) $\left(3-2\sqrt{2}\right)$ p^2 (D) $\left(3+2\sqrt{2}\right)$ p^2

7. The area of a circle is doubled when its radius r is increased by a. Therefore, radius r equals. **[IJSO-2010]**
(A) $\left(\sqrt{2}+1\right)$ a (B) $\left(\sqrt{2}-1\right)$ a (C) a (D) $\left(2-\sqrt{2}\right)$ a

8. In case of a right circular cylinder the radius of base and height are in the ratio 2 : 3. Therefore, the ratio of lateral surface area to the total surface area is : **[IJSO-2010]**
(A) 5 : 3 (B) 3 : 5 (C) 2 : 5 (D) 2 : 3

9 A sphere and cube have equal surface areas.The ratio of their volumes is **[IJSO-2011]**
(A) $\frac{\pi}{6}$ (B) $\frac{6}{\pi}$ (C) $\sqrt{\frac{6}{\pi}}$ (D) $\sqrt{\frac{\pi}{3}}$

10. An equilateral triangle has area A$\sqrt{3}$ .Three circles are drawn with their centres at the vertices of the triangle.Diameter of each circle is equal to the length of each side of the triangle.The area of the triangle NOT included in any of the three circles is **[IJSO-2011]**
(A) A ($\sqrt{3}-\frac{\pi}{6}$). (B) A($\pi-\sqrt{3}$). (C) A($3\pi-\sqrt{3}$) (D) A($\sqrt{3}-\frac{\pi}{2}$).

11. A conical vessel of radius 6 cm and height 8 cm is completely filled with water. A metal sphere is now lowered into the water. The size of the sphere is such that when it touches the inner surface, it just gets immersed.The fraction of water that overflows from the conical vessel is **[IJSO-2011]**

(A) $\frac{3}{8}$ (B) $\frac{5}{8}$ (C) $\frac{7}{8}$ (D) $\frac{5}{16}$

12. An equilateral triangle has area A cm^2. A regular hexagon of maximum area is cut off from the triangle. If the area of the hexagon is 320 cm^2, the area A is **[IJSO-2011]**

(A) 640 cm^2 (B) 480 cm^2 (C) 600 cm^2 (D) 400 cm^2

13. What is the radius of the circumcircle of a triangle whose sides are 30cm, 36cm and 30cm in length **[IJSO-2012]**

(A) 18 cm (B) 18.25cm (C) 18.50cm (D) 18.75cm

14. If the radius of the base of a cone is doubled then the slant area becomes 3 times the original slant area. Suppose when the radius of the base of the cone is quadrupled (that is increased to 4 times), the slant area becomes k times the original slant area. Then the integer closest to k is **[IJSO-2013]**

(A) 6 (B) 8 (C) 10 (D) 11

15. Which is of semicircular shape a pavement of uniform width is laid. If the ratio of the area of the lawn to the area of the pavement is 25:24, then the ratio of the outer and inner perimeters of the pavement is **[IJSO-2013]**

(A) $\frac{7}{5}$ (B) $\frac{6}{5}$ (C) $\frac{5}{4}$ (D) $\frac{5}{2\sqrt{6}}$

16. The adjacent figure is a modification of the Switzerland flag to suit the problem ! Five identical small squares from the central cross. The length of each side of the big square is 10 m. If the area of the white cross is 20% of the area of the flag, then the length of the side of the small square is : **[IJSO-2014]**

(A) 2m (B) 2.25 (C) 1.6 m (D) 1.75 m

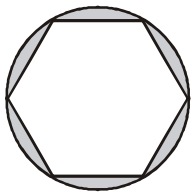
17. A piece of wire 60 cm long is cut into two parts, one of them being 24 cm long. Each part is then bent to form a square. The ratio of the area of the larger to the smaller square is : **[IJSO-2014]**

(A) 9/4 (B) 7/4 (C) 3/2 (D) 11/3

18. If set of marbles, of radius 5 cm, is poured into a cube of side 1 m. The maximum number of marbles that can be filled into the box are **[IJSO-2014]**

(A) 2000 (B) 1000 (C) 1500 (D) 3000

19. A round table cover has six equal designs as shown in the adjacent figure. If the radius of the cover is 4 cm, then cost of making the designs at the rate of Rs 10.00 per cm^2 (round off your answer to a nearest rupee) is **[IJSO-2014]**



(A) 85 (B) 86 (C) 90 (D) 87

20. Total surface area of a sphere S with radius $\sqrt{2}$ + $\sqrt{3}$ cm is **[IJSO-2016]**

(A) 400 π (5 + 2$\sqrt{6}$) sqmm (B) π ($\sqrt{2}$ + $\sqrt{3}$)^2 sqcm

(C) 2π ($\sqrt{2}$ + $\sqrt{3}$)^2 sqcm (D) 40π (5 + 2$\sqrt{6}$) sqmm

21. The probability of a point within an equilateral triangle with side 1- unit lying outside its in-circle (inscribed circle) is **[IJSO-2016]**

(A) $1 - \frac{1}{2\left(\sqrt{3}\right)}$ (B) $1 - \frac{\pi}{3\left(\sqrt{3}\right)}$ (C) $1 - \frac{\pi}{2\sqrt{3}}$ (D) $1 - \frac{2\pi}{3\sqrt{3}}$

22. A craft teacher reshapes the wax from a cylinder of candle with section diameter 6 cm and the height 6 cm into a sphere. The radius of this sphere will be **[IJSO-2016]**

(A) r = 6 $\sqrt{3/2}$ cm (B) r = 6 cm (C) r = 3$\sqrt[3]{3/2}$ cm (D) r = 6$\sqrt[3]{2}$ cm

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-3 98.55 cm^2. **A-4** 346.5 cm^2 **A-5** 6.125 cm^2

Section (B)

B-2 Rs. 1467. **B-3** 600 persons can sit in the hall. **B-4** 225000 m^2.

Section (C)

C-1 Rs 5181. **C-2** Rs. 39600.

C-3 Outer radius = 2.5 cm, Inner radius = 2 cm. **C-4** $\frac{1}{10}$ cm or 0.1 cm.

C-5 0.4 m

Section (D)

D-1 Rs 12800. **D-2** Rs. 384.34 **D-3** 5 : 12. **D-4** 1 : 26.

D-5 20 cm.

Section (E)

E-1 14 cm. **E-2** Rs. 96.29 **E-3** 512. **E-4** 26.74 min. nearly.

E-5 4.5 cm

Section (F)

F-1 51.2 minutes. **F-2** 163.54 cm^3. **F-3** 9 cm. **F-4** 77 cm^3, 784 cm^3.

F-5 20 cm.

OBJECTIVE QUESTIONS

Section (A)

A-1. (A) **A-2.** (C) **A-3.** (A) **A-4.** (D) **A-5.** (C)

Section (B)

B-1. (C) **B-2.** (D) **B-3.** (B) **B-4.** (B) **B-5.** (A)

Section (C)

C-1 (A) **C-2**. (C) **C-3**. (A) **C-4**. (C) **C-5.** (A)

Section (D)

D-1. (A) **D-2**. (B) **D-3.** (A) **D-4.** (A) **D-5.** (B)

Section (E)

E-1. (B) **E-2.** (A) **E-3.** (D) **E-4.** (B) **E-5.** (A)

Section (F)

F-1. (A) **F-2.** (A) **F-3.** (B) **F-4.** (B) **F-5.** (B)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10
Ans.	D	A	C	A	B	C	D	D	B	D

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	C	D	B	B	C	C	A	B	C	D	A	B	D	D	A	A	A	B	D	A
Ques.	21	22																		
Ans.	B	C																		

PERMUTATION AND COMBINATION

A. FUNDAMENTAL PRINCIPLE OF COUNTING

(a) Factorial

The continued product of first **n** natural numbers is called the **" n factorial'** and is denoted by **n!**.

i.e. $n! = 1 \times 2 \times 3 \times 4 \times ....\times (n-1) \times n$.

Thus, $3! = 1 \times 2 \times 3 = 6$; $4! = 1 \times 2 \times 3 \times 4 = 24$,

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ etc.

Clearly, **n!** is defined for positive integers only.

❖ **NOTE :**

(i) Factorials of negative integers are not defined.

(ii) $0! = 1! = 1$

(iii) ${}^nP_n = n! = n.(n-1)!$

(iv) $(2n)! = 2^n.\, n!\,[1.\,3.\,5.\,7...(2n-1)]$

Solved Examples

Example. 1

If $\frac{1}{9!}+\frac{1}{10!}=\frac{x}{11!}$, find x.

Sol. We have,

$$\frac{1}{9!}+\frac{1}{10!}=\frac{x}{11!}$$

$$\Rightarrow \frac{1}{9!}+\frac{1}{10\times 9!} = \frac{x}{11\times 10\times 9!}$$

$$\Rightarrow \frac{1}{9!}\left[1+\frac{1}{10}\right] = \left(\frac{x}{11\times 10}\right)\times\frac{1}{9!}$$

$$\Rightarrow 1+\frac{1}{10} = \frac{x}{11\times 10} \quad \Rightarrow \quad \frac{11}{10} = \frac{x}{11\times 10}$$

$$\Rightarrow x = 11 \times 11 = 121$$

Example. 2

If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2 : 1, find the value of n.

Sol. We have,

$$\frac{n!}{2!(n-2)!} : \frac{n!}{4!(n-4)!} = 2 : 1$$

$$\Rightarrow \frac{n!}{2!(n-2)!} \times \frac{4!(n-4)!}{n!} = \frac{2}{1} \quad \Rightarrow \quad \frac{4!(n-4)!}{2!(n-2)\times(n-3)\times(n-4)!} = \frac{2}{1}$$

$$\Rightarrow \frac{4\times 3\times 2!}{2!(n-2)(n-3)} = \frac{2}{1} \quad \Rightarrow \quad (n-2)(n-3) = 6 \Rightarrow n^2 - 5n = 0$$

$$\Rightarrow n(n-5) = 0 \quad \Rightarrow \quad n = 0, 5.$$

But, for n = 0, (n – 2) ! and (n – 4) ! are not meaningful. Therefore, n = 5.

(b) Fundamental principle of counting

Principle of Multiplication:

If an event can occur in **'m'** different ways, following which another event can occur in **'n'** different ways, then total number of different ways of simultaneous occurrence of both the events in a definite order is **m ×n**.

Principle of Addition:
If an event can occur in **'m'** different ways, and another event can occur in **'n'** different ways, then exactly one of the events can happen in **m + n** ways.

Solved Examples

Example. 3

There are 8 buses running from Kota to Jaipur and 10 buses running from Jaipur to Delhi. In how many ways a person can travel from Kota to Delhi via Jaipur by bus.

Sol. Let E_1 be the event of travelling from Kota to Jaipur & E_2 be the event of travelling from Jaipur to Delhi by the person.
E_1 can happen in 8 ways and E_2 can happen in 10 ways.
Since both the events E_1 and E_2 are to be happened in order, simultaneously, the number of ways = 8 × 10 = 80.

Example. 4

How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if
(i) No digit is repeated in any number.
(ii) Digits can be repeated.

Sol. **(i)** Number of two digit numbers = 5 × 4 = 20
Number of three digit numbers = 5 × 4 × 3 = 60
Number of four digit numbers = 5 × 4 × 3 × 2 = 120
Total = 200

(ii) Number of two digit numbers = 5 × 5 = 25
Number of three digit numbers = 5 × 5 × 5 = 125
Number of four digit numbers = 5 × 5 × 5 × 5 = 625
Total = 775

Example. 5

How many numbers are there between 100 and 1000 in which all the digits are distinct ?

Solution

A number between 100 and 1000 has three digits. So, we have to form all possible 3-digit numbers with distinct digits.
We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1, 2, 3,...., 9. So, there are 9 ways of filling the hundred's place.
Now, 9 digits are left including 0. So, ten's place can be filled with any of the remaining 9 digits in 9 ways. Now, the unit's place can be filled with in any of the remaining digits. So, there are 8 ways of filling the unit's place.
Hence, the total number of required numbers = 9 × 9 × 8 = 648.

B. PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of things is called a **permutation**.
^nP_r denotes the number of permutations of **n** different things, taking **r** at a time, then

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}.$$

For example if there are three objects a, b & c, then the permutations of these objects, taking two at a time, are ab, ba, bc, cb, ac, ca

So, the number of permutations of three different things taken two at a time is 6.

❖ **NOTE :**

It should be noted that in permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

Solved Examples

Example. 6

If ${}^{10}P_r = 5040$, find the value of r.

Sol. ${}^{10}P_r = 5040$ $\Rightarrow$ $\frac{10!}{(10-r)!} = 10 \times 504$

$\Rightarrow$ $\frac{10!}{(10-r)!} = 10 \times 9 \times 8 \times 7$ $\Rightarrow$ $\frac{10!}{(10-r)!} = \frac{10\times9\times8\times7\times6!}{6!}$

$\Rightarrow$ $\frac{10!}{(10-r)!} = \frac{10!}{6!}$ $\Rightarrow$ $(10 - r)! = 6!$ $\Rightarrow 10 - r = 6$

$\Rightarrow$ $r = 4$.

Example. 7

If ${}^{9}P_5 + 5.\ {}^{9}P_4 = {}^{10}P_r$, find r.

Sol. We have

${}^{9}P_5 + 5.\ {}^{9}P_4 = {}^{10}P_r$

$\Rightarrow \frac{9!}{(9-5)!} + 5.\frac{9!}{(9-4)!} = \frac{10!}{(10-r)!}$ $\Rightarrow$ $\frac{9!}{4!} + 5.\frac{9!}{5!} = \frac{10!}{(10-r)!}$

$\Rightarrow \frac{9!}{4!} + \frac{9!}{4!} = \frac{10!}{(10-r)!}$ $\Rightarrow$ $2 \times \frac{9!}{4!} = \frac{10!}{(10-r)!}$

$\Rightarrow \frac{5\times2\times9!}{5\times4!} = \frac{10!}{(10-r)!}$ $\Rightarrow$ $\frac{10\times9!}{5!} = \frac{10!}{(10-r)!}$

$\Rightarrow \frac{10!}{5!} = \frac{10!}{(10-r)!}$ $\Rightarrow$ $(10 - r)! = 5!$ $\Rightarrow 10 - r = 5$ $\Rightarrow r = 5$.

Example. 8

In how many ways three different rings can be worn in four fingers with at most one in each finger ?

Sol. The total number of ways is same as the number of arrangements of 4 fingers, taken 3 at a time.

So, required number of ways = ${}^{4}P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 24$.

Example. 9

How many different signals can be made by 5 flags from 8 flags of different colours ?

Sol. The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, required number of signals = ${}^{8}P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8\times7\times6\times5\times4\times3!}{3!} = 6720$.

Example.10

How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once.

Sol. There are eight letters in the word 'EQUATION'. So, the total number of words is equal to the number of arrangement of these letters, taken all at a time. The number of such arrangement is ${}^{8}P_8 = 8!$. Hence, the total number of words = 8!.

Example.11

How many numbers of three digits can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits. How many of these are even.

Sol. Three places are to be filled with 5 different objects.

$\therefore$ Number of ways = ${}^5P_3 = 5 \times 4 \times 3 = 60$

For even numbers unit digit can be filled in two ways & the remaining two digits can be filled in ${}^4P_2$ ways.

$\therefore$ Number of even numbers = $2 \times {}^4P_2 = 24$.

(a) Permutations of Objects not all Distinct :

So, far we were discussing permutations of distinct objects (things) by taking some or all at a time. In this article we intend to discuss the permutations of a given number of objects when objects are not all different. **For example**, the number of arrangements of the letters of the word **MISSISSIPPI**, the number of six digit numbers formed by using the digits 1, 1, 2, 3, 3, 4 etc. The following theorem is very helpful to determine the number of such arrangements.

Theorem : The number of mutually distinguishable permutations of **n** things, taken all at a time, of which **p** are alike of one kind, **q** alike of second such that **p + q = n**, is $\frac{n!}{p!q!}$.

NOTE : The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind....p_r are alike of r th kind such that $p_1 + p_2 + ... p_r = n$, is $\frac{n!}{p_1!p_2!p_3!.....p_r!}$

The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\frac{n!}{p!q!}$.

Suppose, there are r things to be arranged, allowing repetitions. Let further p_1, p_2, ... p_r be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\frac{(p_1+p_2.....p_r)!}{p_1!p_2!p_3!.....p_r!}$.

Solved Examples

Example.12

How many different words can be formed with the letters of the word 'MISSISSIPPI'?

Sol. There are 11 letters in the given word, of which 4 are S's, 4 are I's and 2 are P's. So, total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind is $\frac{11!}{4!4!2!}$

Hence, the total number of words = $\frac{11!}{4!4!2!}$ = 34650.

Example. 13

(i) How many different words can be formed with the letters of the word HARYANA ?

(ii) How many of these begin with H and end with N ?

(iii) In how many of these H and N are together ?

Solution

(i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words = $\frac{7!}{3!1!1!1!1!} = \frac{7!}{3!}$ = 840.

(ii) After fixing H in first place and N in last place, we have 5 letters out of which three are alike i.e. A's and remaining all are each of its own kind.

So, total number of words = $\frac{5!}{3!}$ = 20.

(iii) Considering H and N together we have 7 – 2 + 1 = 6 letters out of which three are alike i.e. A's and others are each of its own kind. These six letters can be arranged in $\frac{6!}{3!}$ ways. But H and N can be arranged amongst themselves in 2! ways.

Hence, the required number of words = $\frac{6!}{3!} \times 2!$ = 120 × 2 = 240.

Example.14

How many different words can be formed by using all the letters of the word 'ALLAHABAD'?

(i) In how many of them vowels occupy the even positions ?

(ii) In how many of them both L do not come together ?

Sol. There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the required number of words = $\frac{9!}{4!2!}$ = 7560.

(i) There are 4 vowels and all are alike i.e. A's. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So these 4 even places can be occupied by 4 vowels in $\frac{4!}{4!}$ = 1 way. Now, we are left with 5 places in which 5 letters, of which two are alike (2L's) and other distinct, can be arranged in $\frac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupy the even places = $\frac{5!}{2!} \times \frac{4!}{4!} = \frac{5!}{2!}$ = 60.

(ii) Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

So, the number of words in which both L come together = $\frac{8!}{4!}$ = 1680.

Hence, the number of words in which both L do not come together
= Total number of words – Number of words in which both L come together
= 7560 – 1680 = 5880.

(b) Circular Permutation :

The number of circular permutations of **n** different things taken all at a time is **(n – 1) !** .

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

NOTE : Number of circular permutations of **n** things when **p** are alike and the rest are different, taken all at a time, distinguishing clockwise and anticlockwise arrangement is $\frac{(n-1)!}{p!}$.

Solved Examples

Example.15

In how many ways can we arrange 6 different flowers in a circle. In how many ways we can form a garland using these flowers.

Sol. The number of circular arrangements of 6 different flowers = (6 – 1)! = 120.

When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the number of ways of forming garland = $\frac{1}{2}$ (6 – 1) ! = 60.

Example.16

In how many ways 6 persons can sit at a round table, if two of them prefer to sit together.

Sol. Let P_1, P_2, P_3, P_4, P_5, P_6 be the persons, where P_1, P_2 want to sit together.
Regard these person as 5 objects. They can be arranged in a circle in (5 – 1)! = 24. Now P_1, P_2 can be arranged in 2! ways. Thus the total number of ways = 24 × 2 = 48.

C. COMBINATIONS

Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.

Theorem : The number of all combination of **n** distinct objects, taken **r** at a time is given by

$${}^{n}C_r = \frac{n!}{(n-r)!\,r!}.$$

NOTE : (i) ${}^{n}C_r = {}^{n}C_{n-r}$

(ii) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

(iii) ${}^{n}C_r = 0$ if $r = \{0, 1, 2, 3....., n\}$

Solved Examples

Example.17

If the ratio ${}^{2n}C_3 : {}^{n}C_3$ is equal to 11 : 1, find n.

Sol. We have, ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$

$$\Rightarrow \frac{{}^{2n}C_3}{{}^{n}C_3} = \frac{11}{1} \qquad \Rightarrow \frac{\dfrac{(2n)!}{(2n-3)!\,3!}}{\dfrac{n!}{(n-3)!(3!)}} = \frac{11}{1}$$

$$\Rightarrow \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1} \qquad \Rightarrow \frac{(2n)(2n-1)(2n-2)(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n(n-1)(n-2)(n-3)!} = \frac{11}{1}$$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1} \Rightarrow \frac{4(2n-1)}{n-2} = \frac{11}{1}$$

$$\Rightarrow 8n - 4 = 11n - 22 \qquad \Rightarrow 3n = 18$$

$$\Rightarrow n = 6.$$

Example.18

If ${}^{49}C_{3r-2} = {}^{49}C_{2r+1}$, find 'r'.

Sol. ${}^{n}C_r = {}^{n}C_s$ if either r = s or r + s = n.

Thus, 3r – 2 = 2r + 1

$\Rightarrow$ r = 3

or 3r – 2 + 2r + 1 = 49

$\Rightarrow$ 5r – 1 = 49

$\Rightarrow$ r = 10

$\therefore$ r = 3, 10.

Example.19

A committee of 5 is to be formed out of 6 gents and 4 ladies. in how many ways this can be done, when

(i) at least two ladies are included

(ii) at most two ladies are included ?

Sol. **(i)** A committee of person, consisting of at least 2 ladies, can be formed in the following ways :

(a) Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in ${}^4C_2 \times {}^6C_3$ ways.

(b) Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in ${}^4C_3 \times {}^6C_2$ ways.

(c) Selecting 4 ladies out of 4 and 1 gents out of 6. This can be done in ${}^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case, therefore by the fundamental principle of addition, the total number of ways of forming the committee

$= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1 = 120 + 60 + 6 = 186$

(ii) A committee of 5 persons, consisting of at most two ladies, can be done in the following ways :

(a) Selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

(b) Selecting 4 gents only out of 6 and one lady out of 4. This can be done in ${}^6C_4 \times {}^4C_1$ ways.

(c) Selecting 3 gents only out of 6 and two ladies out of 4. This can be done in ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case, so, the total number of ways of forming the committee

$= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186.$

Example.20

How many diagonals are there in a polygon with n sides ?

Sol. A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a n sided polygon taken two at a time.

$= \text{Number of ways of selecting 2 out of n} = {}^nC_2 = \dfrac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon

$= \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$.

Example.21

Fifteen players are selected for a cricket match.

(i) In how many ways the playing 11 can be selected

(ii) In how many ways the playing 11 can be selected including a particular player.

(iii) In how many ways the playing 11 can be selected excluding two particular players.

Sol. **(i)** 11 players are to be selected from 15

Number of ways = ${}^{15}C_{11} = 1365$.

(ii) Since one player is already included, we have to select 10 from the remaining 14

Number of ways = ${}^{14}C_{10} = 1001$.

(iii) Since two players are to be excluded, we have to select 11 from the remaining 13.

Number of ways = ${}^{13}C_{11} = 78$.

Example.22

In how many ways we can select 4 letters from the letters of the word MISSISSIPPI.

Sol. M

IIII

SSSS

PP

Number of ways of selecting 4 alike letters = ${}^2C_1 = 2$.

Number of ways of selecting 3 alike and 1 different letters = ${}^2C_1 \times {}^3C_1 = 6$.

Number of ways of selecting 2 alike and 2 alike letters = ${}^3C_2 = 3$.

Number of ways of selecting 2 alike & 2 different = ${}^3C_1 \times {}^3C_2 = 9$.

Number of ways of selecting 4 different = ${}^4C_4 = 1$

Total = 2 + 6 + 3 + 9 +1 = 21.

Example.23

In how many ways we can arrange 3 red flowers, 4 yellow flowers and 5 white flowers in a row. In how many ways this is possible if the white flowers are to be separated in any arrangement (Flowers of same colour are identical).

Sol. Total we have 12 flowers 3 red, 4 yellow and 5 white.

Number of arrangements = $\frac{12!}{3!4!5!}$ = 27720.

For the second part, first arrange 3 red & 4 yellow

This can be done in $\frac{7!}{3!4!}$ = 35 ways.

Now select 5 places from among 8 places (including extremes) & put the white flowers there.
This can be done in ${}^{8}C_{5}$ = 56.

$\therefore$ The number of ways for the 2nd part = 35 × 56 = 1960.

Example.24

In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative positions of vowels & consonants.

Sol. The consonants in their positions can be arranged in $\frac{4!}{2!}$ = 12 ways.

The vowels in their positions can be arranged in $\frac{3!}{2!}$ = 3 ways

$\therefore$ Total number of arrangements = 12 × 3 = 36.

Example.25

12 different toys are to be distributed to three children equally. In how many ways this can be done.

Sol. The problem is to divide 12 different things into three different groups.

Number of ways = $\frac{12!}{4!4!4!}$ = 34650.

Example.26

In how many ways 10 persons can be divided into 5 pairs.

Sol. We have each group having 2 persons and the qualitative characteristic are same (Since there is no purpose mentioned or names for each pair).

Thus the number of ways = $\frac{10!}{(2!)^5\, 5!}$ = 945.

(a) Selection of one or more objects

Number of ways in which atleast one object to be selected out of **'n'** distinct objects is

$$\mathbf{{}^{n}C_{1} + {}^{n}C_{2} + {}^{n}C_{3} + \ldots\ldots\ldots\ldots + {}^{n}C_{n} = 2^{n} - 1}$$

Number of ways in which atleast one object may be selected out of **'p'** alike objects of one type **'q'** alike objects of second type and **'r'** alike of third type is

$$\mathbf{(p + 1)\,(q + 1)\,(r + 1) - 1}$$

Number of ways in which atleast one object may be selected from **'n' o**bjects where **'p'** alike of one type **'q'** alike of second type and **'r'** alike of third type and rest **n – (p + q + r)** are different, is

$$\mathbf{(p + 1)\,(q + 1)\,(r + 1)\,2^{n-(p+q+r)} - 1}$$

Solved Examples

Example.27

There are 12 different books on a shelf. In how many ways we can select atleast one of them.

Sol. We may select 1 book, 2 books,....., 12 books.

$\therefore$ The number of ways = ${}^{12}C_1 + {}^{12}C_2 + ... + {}^{12}C_{12} = 2^{12} - 1 = 4095$.

Example.28

There are 12 fruits in a basket of which 5 are apples, 4 mangoes and 3 bananas (fruits of same species are identical). How many ways are there to select atleast one fruit.

Sol. Let x be the number of apples being selected
y be the number of mangoes being selected and
z be the number of bananas being selected.
Then x = 0, 1, 2, 3, 4, 5
y = 0, 1, 2, 3, 4
z = 0, 1, 2, 3
Total number of triplets (x, y, z) is $6 \times 5 \times 4 = 120$
Exclude (0, 0, 0)
$\therefore$ Number of combinations = 120 – 1 = 119.

Exercise-1

SUBJECTIVE QUESTIONS

Section (A) : Fundamental Principle of Counting

A-1 Find the number of 9 digit numbers that can be formed by using the digits 1, 2, 3, 4 and 5.

A-2 How many 3–digit odd numbers can be formed using the digits 1, 2, 3, 4, 5, 6 if
(i) The repetition of digits is not allowed ?
(ii) The repetition of digits is allowed ?

A-3 How many numbers lying between 999 and 10000 can be formed with the help of the digit 0,2,3,6,7,8 when the digits are not be repeated.

COMPREHENSION (Q. No. A-4 to A-5)

There are 6 roads between A & B and 4 roads between B & C.

A-4. In how many ways can one drive from A to C by way of B.

A-5. In how many ways can one drive from A to C and back to A, passing through B on both trips?

Section (B) : Permutation

B-1 Find the number of all five-digit numbers with distinct digits.

B-2 In how many different ways can the letters of the word 'HEXAGON' be permuted ?

B-3 Find the number of signals which can be given with 5 flags of different colours hoisted one above the other, by using any number of flags.

B-4 7 candidates are to be examined, 2 in mathematics and 5 in different subjects. In how many ways can they be seated in a row so that the two examinees in Mathematics may not sit together ?

B-5 How many different numbers with distinct digits can be formed from the digits 2, 3, 5, 7, 9 ? How many of them are odd ?

Section (C) : Combination

C-1 In how many ways 6 passengers can board in 3 vans if no van remains empty?

C-2 To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If 3 of the vacancies are reserved for scheduled caste candidates while the rest are open to all, then find the number of ways in which the selection can be made.

C-3 Find the maximum number of points of intersection of 4 parabolas and 3 straight lines.

C-4 In how many ways two different 3 x 1 rectangle can be formed from a chess board of 8 x 8 size?

C-5 In how many ways letters from the word of the "SAMSUNG" can be arranged so that it never end and starts with S?

OBJECTIVE QUESTIONS

Section (A) : Fundamental Principle of Counting

A-1. Different 7–digit telephone numbers that can be formed from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is :
(A) 10^7 (B) 9×10^6 (C) 9! (D) None of these

A-2. Number of ways in which 5 letters can be posted in 4 letter boxes is :
(A) 5^4 (B) 20 (C) 25 (D) 4^5

A-3. Total numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8 is :
(A) 125 (B) 120 (C) 25 (D) 625

A-4. Numbers of different signals can be given using any number of flags from 5 flags of different colours is :
(A) 325 (B) 240 (C) 60 (D) None of these

A-5. How many four digit numbers will have all the four digits distinct ?
(A) 4536 (B) 3024 (C) 5040 (D) 9000

Section (B) : Permutation

B-1. Value of r when ${}^{10}P_r = 720$ is :
(A) 3 (B) 5 (C) 7 (D)None of these

B-2. Number of ways that can 3 scholarships of unequal value be awarded to 17 candidates, such that no candidates gets more than one scholarship is :
(A) ${}^{17}P_1$ (B) ${}^{17}P_{17}$ (C) ${}^{17}P_3$ (D) None of these

B-3. Numbers of ways to arrange the letters of the word 'FIGMENT' when it starts with F is :
(A) 360 (B) 720 (C) 1080 (D) 1440

Section (C) : Combination

C-1. Out of 7 consonants and 4 vowels, number of words of 3 consonants and 2 vowels is :
(A) 462 (B) 2800 (C) 25200 (D) 24540

C-2. For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made.
(A) 11760 (B) 1170 (C) 16710 (D)None of these

Directions Q. C-3. – C-4.

There are 10 points in a plane. Except for 4 points which are collinear no three points are in a straight line.

C-3. The number of straight line obtained by joining these points is :
(A) 40 (B) 50 (C) 60 (D) 70

C-4. Number of triangles that can be formed with the vertices as these points is :
(A) 120 (B) 117 (C) 116 (D) 110

C-5. A group of 6 students comprised of 3 boys and 3 girls. Number of ways could they be arranged in a straight line such that no two boys were sitting together is :
(A) 36 (B) 72 (C) 108 (D) 144

Exercise-2

OBJECTIVE QUESTIONS

1. Number of ways in which these scholarships can be awarded is :
(A) 12 (B) 60 (C) 720 (D) None of these

2. Number of ways in which one of these scholarships can be awarded is :
(A) 12 (B) 60 (C) 720 (D) None of these

3. Number of different ways can a person reach Mumbai from Ahmedabad if there are 4 different routes from Ahmedabad to Surat and 3 different routes from Surat to Mumbai is :
(A) 7 (B) 12 (C) 24 (D) 36

4. A letter lock contains 4 rings, each ring containing 5 letters. If the lock opens in only one arrangements of 4 letters, then number of unsuccessful events possible is :
(A) 120 (B) 625 (C) 624 (D) None of these

5. There are 10 digits from 0 to 9 in the decimal system. How many 5 – digit numbers can be formed, such that no 2 digits are the same.
(A) 12216 (B) 26127 (C) 62716 (D) 27216

6. There are 10 digits from 0 to 9 in the decimal system. How many 4 – digit numbers can be formed using these 10 digits?
(A) 4500 (B) 9000 (C) 10500 (D) 12000

Directions Q.7 – Q. 9

There are 5 friends : A, B, C, D and E. They wanted to take a group photograph of all of them sitting in a single row.

7. How many distinctly different photographs can be clicked.
(A) 60 (B) 120 (C) 240 (D) None of these

8. In how many of these photographs would A be sitting in the middle.
(A) 12 (B) 18 (C) 24 (D) None of these

9. In how many of these would A and B be sitting next to each other ?
(A) 24 (B) 30 (C) 36 (D) 48

10. Numbers of ways to arrange the letters of the word 'FIGMENT' when vowels at first and last positions is :
(A)120 (B) 240 (C) 360 (D) 720

11. In an examination, a student has to answer 4 questions out of 5 questions, questions 1 and 2 are however compulsory. Number of ways in which the student can make a choice is :
(A) 6 (B) 3 (C) 2 (D) 60

12. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, then the value of n is :
(A) 16 (B) 17 (C) 18 (D) 19

13. Number of ways in which 7 boys can be seated at a round table so that 2 particular boys are separated is :
(A) 120 (B) 240 (C) 360 (D) 480

14. Number of different ways that can 3 travellers stay in 4 hotels when each one should stay in different hotel is :
(A) 24 (B) 30 (C) 36 (D) 48

15. Number of different ways in which 5 different balls be distributed to 3 different boxes, when each box can hold any number of balls is :
(A) 120 (B) 240 (C) 243 (D) 486

16. Number of ways in which 17 billiard balls can be arranged in a row if 7 are black, 6 red and 4 white is :

(A) ${}^{17}P_7 \times {}^{17}P_6 \times {}^{17}P_4$ (B) ${}^{17}P_3$ (C) $\frac{17!}{7!\,6!\,4!}$ (D) ${}^{17}C_7 \times {}^{17}C_6 \times {}^{17}C_4$

17. A group of 6 students comprised of 3 boys and 3 girls. Number of ways could they be arranged in a straight line such that no two boys were sitting together is :
(A) 36 (B) 72 (C) 108 (D) 144

18. Number of ways in which 7 boys can be seated at a round table so that 2 particular boys are next to each other is :
(A) 120 (B) 240 (C) 360 (D) 480

19. Let *m* be the number of ways in which two couples can be seated on 4 chairs in a row so that no wife is next to her husband and n be the number of ways in which they can be seated in 4 chairs in a circle. In the latter case, rotations are considered different configurations. Then :
(A) m = n (B) m = 2n (C) m = 4n (D) m = 8n

20. How many six-digit numbers are there in which no digit is repeated, even digits appear at even places, odd digits appear at odd places and the number is divisible by 4?
(A) 3600 (B) 2700 (C) 2160 (D) 1440

21. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$,then the values of n and r is :
(A) n = 4, r = 2 (B) n = 4, r = 3 (C) n = 3, r = 2 (D) n = 3, r = 1

22. Different words can be formed using the letter 'HALLUCINATION' if all vowels occupy odd places is

(A) ${}^7C_6 \times \frac{6!}{2!\,2!} \times \frac{7!}{2!\,2!}$ (B) ${}^7C_6 \times 6! \times \frac{7!}{2!\,2!}$

(C) ${}^7C_6 \times \frac{6!}{2!\,2!} \times 7!$ (D) ${}^7C_6 \times 6! \times 7!$

23. Total number of triangles that can be formed by joining the vertices of the polygon that has the same number of diagonals as its sides is :
(A) 5 (B) 9 (C) 8 (D) 10

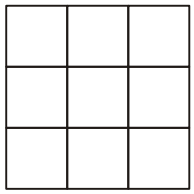
24. Four gentlemen and four ladies are to sit at a round table so that no two gentlemen sit together. Then, the number of ways in which this can be done is :
(A) (4!) (3!) (B) $(4!)^2$ (C) 7! (D) 8!

25. The number of rectangles that can be obtained by joining four of the twelve vertices of a 12-sided regular polygon is :
(A) 66 (B) 30 (C) 24 (D) 15

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

1. Figure shows a square gird of order 3, which of the following is correct formula for the total number of squares in a similar grid of order n. **[IJSO-2014]**



(A) $\frac{n(n+1)}{2}$ (B) $\frac{n^2(n+1)^2}{4}$ (C) $\frac{n(n+1)(2n+1)}{6}$ (D) $\frac{n(n+1)(n+2)}{6}$

2. How many triangles are there in this figure? **[IJSO-2015]**

(A) 50 (B) 70 (C) 84 (D) 91

3. let the number of rectangles formed by 6 horizontal and 4 vertical lines be n and those formed by 5 vertical and 5 horizontal lines be m then we have **[IJSO-2016]**

(A) $n = m$ (B) $n \geq m + 1$ (C) $m \leq n$ (D) $m > n + 5$

4. The number of squares formed by 5 vertical and 4 horizontal lines (all are equispaced) is **[IJSO-2016]**

(A) 60 (B) 20 (C) 40 (D) 46

5. A 5 × 5 × 5 cube is built using unit cubes. How many different cuboids (that differ in at least one unit cube) can be formed using the same number of unit cubes ? **[IJSO-2018]**

(A) 1000 (B) 1728 (C) 2730 (D) 3375

6. A person kept rolling a regular (six faced) die until one of the numbers appeared third time on the top. This happed in 12th throw and the sum of all the numbers in 12 throws was 46. Which number appeared least number of times ? **[IJSO-2018]**

(A) 6 (B) 4 (C) 2 (D) 1

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1 5^9 **A-2** **(i)** 60 **(ii)** 108 **A-3** 300 **A-4.** 24 **A-5.** 576

Section (B)

B-1 27216. **B-2** 7! **B-3** 325 **B-4** 3600. **B-5** 260.

Section (C)

C-1 $3^6 - {}^3C_1.2^6 + {}^3C_2$ **C-2** ${}^5C_3 \times {}^{22}C_9$ **C-3** 51 **C-4** ${}^{96}C_2$ ways

C-5 1200

OBJECTIVE QUESTIONS

Section (A)

A-1. (B) **A-2.** (D) **A-3.** (C) **A-4.** (A) **A-5.** (A)

Section (B)

B-1. (A) **B-2.** (C) **B-3.** (B)

Section (C)

C-1. (C) **C-2.** (A) **C-3.** (A) **C-4.** (C) **C-5.** (D)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	B	A	B	C	D	B	B	C	D	B	B	D	D	A	C
Ques.	16	17	18	19	20	21	22	23	24	25					
Ans.	C	D	B	D	D	C	A	D	A	D					

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1	2	3	4	5	6
Ans.	C	D	D	B	D	CD

STATISTICS

A. INTRODUCTION AND ARITHMETIC MEAN

Statistics is the study of the collection, organisation, analysis, interpretation and presentation of data.

(a) Measures of central tendency

A certain value representative of the whole data and signifying its characteristics is called measure of central tendency or average. Mean, median and mode are the measures of central tendency.

(b) Arithmetic mean

The arithmetic mean (or simple mean) of a set of observations is obtained by dividing the sum of the values of observations by the number of observations.

(i) Mean of raw data

The mean of n observations $\mathbf{x_1, x_2, x_3, \ldots, x_n}$ is

$$\bar{x} = \frac{x_1 + x_1 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$n\bar{x}$ = sum of obersvations = $\sum_{i=1}^{n} x_i$

i.e. product of mean & no. of items gives sum of observations.

(ii) Mean for an ungrouped frequency distribution

Let $x_1, x_2, \ldots, x_n$ be n observations with respective frequencies $f_1, f_2, \ldots, f_n$. This means observation

x_1 occurs f_1 times, x_2 occurs f_2 times and so on.

Then, sum of the values of all the observations

$= f_1x_1 + f_2x_2 + \ldots + f_nx_n$

$= \sum_{i=1}^{n} f_i\, x_i$

and number of observations

$= f_1 + f_2 + \ldots + f_n = \sum_{i=1}^{n} f_i$

$$\therefore \quad \text{Mean } (\bar{x}) = \frac{\sum_{i=1}^{n} f_i\, x_i}{\sum_{i=1}^{n} f_i}$$

It can also be written as

$$(\bar{x}) = \frac{\sum f_i\, x_i}{\sum f_i}$$

It is understood that i varies from 1 to n.

(iii) Mean of Grouped Data

To calculate the mean of grouped data, we have three methods, which are

(I) Direct method

(II) Assumed mean method

(III) Step deviation method

(I) Direct Method :

Firstly, find the class marks of each class interval. These class marks are represent by x_i. In general, for the ith class interval, we have the frequency f_i corresponding to the class marks x_i.
Different steps used in direct method are given below
Step : 1 Find the class mark (mid value), x_i of each class.

$$\text{Class mark of a class} = \frac{\text{Lower limit} + \text{Upper limit}}{2}$$

Step : 2 Calculate f_ix_i for each x_i
Step : 3 Find Σf_ix_i and Σf_i
Step : 4 By using formula, mean, $(\bar{x}) = \dfrac{\sum f_ix_i}{\sum f_i}$, calculate the mean

(II) Assumed Mean Method (Shortcut Method)

The cases, in which numerical values of x_i and f_i are large, become tedious and time consuming to find product of x_i and f_i. In these cases, assumed mean method is used. In this method, first of all one among x_i's is chosen as the assumed mean and denoted by a.
We may take 'a' to be that x_i which lies in the centre of $x_1, x_2, \ldots, x_n$ to reduce our calculation work.

In this method, we subtract an arbitrary constant 'a', called the assumed mean from each value of x_i. The reduced value is $x_i - a$. We denote it by d_i and write

$$d_i = x_i - a.$$

d_i is called the deviation of x_i from the assumed mean a.
Hence, $x_i = a + d_i$

$\Rightarrow \quad f_ix_i = a f_i + d_i f_i$

$\Rightarrow \quad \sum_{i=1}^{n} f_ix_i = \sum_{i=1}^{n} af_i + \sum_{i=1}^{n} d_if_i$

$\Rightarrow \quad \sum_{i=1}^{n} f_ix_i = a\sum_{i=1}^{n} f_i + \sum_{i=1}^{n} d_if_i = aN + \sum_{i=1}^{n} d_if_i \quad [\because \sum_{i=1}^{n} f_i = N]$

$\Rightarrow \quad \frac{1}{N}\sum_{i=1}^{n} f_ix_i = a + \frac{1}{N}\sum_{i=1}^{n} d_if_i$ [Dividing both sides by N]

$\Rightarrow \quad \bar{x} = a + \frac{1}{N}\sum_{i=1}^{n} d_if_i$

So, $\quad \bar{x} = a + \dfrac{\sum f_id_i}{\sum f_i}$

where, a = Assumed mean
d_i = Deviation from mean $(x_i - a)$
Different steps used in assumed mean method are given below
Step - 1 : for each class interval, calculate the class marks x_i, by using the formula.

$$x_i = \frac{1}{2}\text{(Lower class limit + Upper class limit)}$$

Step - 2 : Choose a suitable value of x_i in the middle as the assumed mean and denote it by a.
Step - 3 : Calculate the deviation $d_i = (x_i - a)$ for each i.
Step - 4 : Calculate the product f_id_i for each i.
Step - 5 : Find $N = \Sigma f_i$.
Step - 6 : Calculate the mean $(\bar{x})$, by using the formula

Mean $\bar{x} = a + \dfrac{\sum f_id_i}{\sum f_i}$

(III) Step-Deviation Method

The main objective of subtracting an arbitrary number from each observation x_i for the calculation of arithmetic mean by short-cut method is to reduce these numbers x_i. In this case, the calculation becomes simpler. We can make these differences still smaller, if these are divisible by a common non-zero number, say h. In this case, the calculation becomes much simpler. We can, therefore, modify the formula of short-cut method as follows:

Let $u_i = \frac{x_i - a}{h}$, i = 1, 2, 3, ..., n.

$\therefore \quad x_i = a + h\,u_i \qquad \Rightarrow \quad f_i x_i = af_i + h\,u_i f_i$

$\Rightarrow \quad \sum_{i=1}^{n} f_i x_i = a\sum_{i=1}^{n} f_i + h\sum_{i=1}^{n} u_i f_i = aN + h\sum_{i=1}^{n} u_i f_i \qquad \Rightarrow \quad \frac{1}{N}\sum_{i=1}^{n} f_i x_i = a + \frac{h}{N}\sum_{i=1}^{n} u_i f_i$

Divide the above equation by $[\sum_{i=1}^{n} f_i = N]$

$\Rightarrow \quad \bar{x} = a + \frac{h}{N}\sum u_i f_i$

Here, h is any arbitrary non-zero constant and a is an assumed mean which is also arbitrary.

So, $\quad \bar{x} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right)h$

where, a = Assumed mean, $u_i = \frac{x_i - a}{h}$, h = Width of class interval

Different steps used in step deviation method are given below

Step - 1 : For each class interval, calculate the class marks x_i, by using the formula.

$x_i = \frac{1}{2}$ (Lower class limit + Upper class limit)

Step - 2 : Choose a suitable value of x_i in the middle as the assumed mean and denote it by a.

Step - 3 : Calculate the deviation $d_i = (x_i - a)$ for each i.

Step - 4 : Calculate $u_i = \frac{x_i - a}{h}$ for each $d_i = x_i - a$, where h is the class size or common factor of all d_i.

Step - 5 : Find $f_i u_i$ for each u_i.

Step - 6 : Find $\Sigma f_i u_i$ and Σf_i.

Step - 7 : Calculate mean by using formula

$(\bar{x}) = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right)h$

(c) Properties of mean

(i) Sum of deviations from mean is zero i.e. $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$.

(ii) If a constant real number 'a' is added to each of the observation than new mean will be $\bar{x} + a$.

(iii) If a constant real number 'a' is subtracted from each of the observation then new mean be $\bar{x} - a$.

(iv) If a constant real number 'a' is multiplied with each of the observation then new mean will be $a\bar{x}$.

(v) If each of the observation is divided by a constant no 'a', then new mean will be $\frac{\bar{x}}{a}$.

(d) Merits of arithmetic mean

(i) It is rigidly defined, simple, easy to understand and easy to calculate.

(ii) It is based upon all the observations.

(iii) Its value being unique, we can use it to compare different sets of data.

(iv) It is least affected by sampling fluctuations.

(v) Mathematical analysis of mean is possible. So, It is relatively reliable.

(e) Demerits of arithmetic mean

(i) It can not be determined by inspection nor it can be located graphically.

(ii) Arithmetic mean cannot be used for qualitative characteristics such as intelligence, honesty, beauty etc.

(iii) It cannot be obtained if a single observation is missing.

(iv) It is affected very much by extreme values. In case of extreme items, A.M. gives a distorted picture of the distribution and no longer remains representative of the distribution.

(v) It may lead to wrong conclusions if the details of the data from which it is computed are not given.

(vi) It can not be calculated if the extreme class is open, e.g. below 10 or above 90.

(vii) It cannot be used in the study of ratios, rates etc.

(f) Uses of arithmetic mean

(i) It is used for calculating average marks obtained by a student.

(ii) It is extensively used in practical statistics and to obtain estimates.

(iii) It is used by businessman to find out profit per unit article, output per machine, average monthly income and expenditure etc.

Solved Examples

Example. 1

Find the mean of the factors of 10.

Sol. Factors of 10 are 1, 2, 5 & 10.

$$\bar{x} = \frac{1+2+5+10}{4} = \frac{18}{4} = 4.5$$

Example. 2

The mean of marks scored by 100 students was found to be 40. Later on it was discovered that a score of 53 was misread as 83. Find the correct mean.

Sol. n = 100, $\bar{x} = 40$

$$\bar{x} = \frac{1}{n}\left(\sum x_i\right) \quad \Rightarrow \quad 40 = \frac{1}{100}\left(\sum x_i\right)$$

$\therefore$ Incorrect value of $\sum x_i$ = 4000.

Now, Correct value of $\sum x_i$ = 4000 – 83 + 53 = 3970

$\therefore$ Correct mean = $\dfrac{\text{correct value of } \sum x_i}{n} = \dfrac{3970}{100} = 39.7$

So, the correct mean is 39.7. **Ans.**

Example. 3

Find the missing value of P for the following distribution whose mean is 12.58

x	5	8	10	12	P	20	25
f	2	5	8	22	7	4	2

Sol. Given $\bar{x}$ = 12.58

Calculation of Mean :

x_i	f_i	f_ix_i
5	2	10
8	5	40
10	8	80
12	22	264
P	7	7P
20	4	80
25	2	50
	$\sum f_i = 50$	$\sum f_ix_i = 524 + 7P$

$$\bar{x} = \frac{\sum f_ix_i}{\sum f_i} \Rightarrow \quad 12.58 = \frac{524+7P}{50}$$

629 = 524 + 7P

7P = 105

P = 15.

Example. 4

Find the mean for the following distribution :

Marks	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
Frequency	6	8	13	7	3	2	1

Sol.

Marks	Mid values x_i	No. of students f_i	f_ix_i
10 - 20	15	6	90
20 - 30	25	8	200
30 - 40	35	13	455
40 - 50	45	7	315
50 - 60	55	3	165
60 - 70	65	2	130
70 - 80	75	1	75
		$\sum f_i = 40$	$\sum f_ix_i = 1430$

$$\bar{x} = \frac{\sum f_ix_i}{\sum f_i} = \frac{1430}{40} = \frac{143}{4} = 35.75$$

Example. 5

Find the mean for the following distribution by using deviation method

x_i	15	20	22	24	25	30	33	38
Frequency	5	8	11	20	23	18	13	2

Sol.

x_i	f_i	Let a = 25 $d_i = x_i - 25$	f_id_i
15	5	–10	– 50
20	8	– 5	– 40
22	11	– 3	– 33
24	20	– 1	– 20
25	23	0	0
30	18	5	90
33	13	8	104
38	2	13	26
	$\sum f_i = 100$		$\sum f_id_i = 77$

$$\bar{x} = a + \frac{\sum f_id_i}{\sum f_i} = 25 + \frac{77}{100} = 25.77$$

Example. 6

Find the mean of following distribution with step-deviation method

Class	10 -15	15 - 20	20 - 25	25 - 30	30 - 35	35 - 40
Frequency	5	6	8	12	6	3

Sol. **Calculation of mean :**

Class	x_i	f_i	Let a = 27.5 $u_i = \frac{x_i - 27.5}{5}$	f_iu_i
10 - 15	12.5	5	–3	– 15
15 - 20	17.5	6	– 2	– 12
20 - 25	22.5	8	– 1	– 8
25 - 30	27.5	12	0	0
30 - 35	32.5	6	1	6
35 - 40	37.5	3	2	6
		$\sum f_i = 40$		$\sum f_iu_i = -23$

$$\Rightarrow \bar{x} = a + \left(\frac{\sum f_iu_i}{\sum f_i}\right)h$$

$$\Rightarrow \bar{x} = 27.5 + 5 \times \left(\frac{-23}{40}\right) = 24.625 \quad .$$

Example. 7

The mean of the following frequency distribution is 62.8 and the sum of all frequencies is 50. Compute the missing frequency f_1 and f_2

Class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100	100 – 120
Frequency	5	f_1	10	f_2	7	8

Sol. Let a = 30, h = 20

Class	x_i	f_i	$u_i = \frac{x_i - a}{h}$	f_iu_i
0 – 20	10	5	–1	– 5
20 – 40	30	f_1	0	0
40 – 60	50	10	+1	10
60 – 80	70	f_2	+ 2	$2f_2$
80 – 100	90	7	+ 3	21
100 –1 20	110	8	+ 4	32
		$\sum f_i = 30 + f_1 + f_2$		$\sum f_iu_i = 58 + 2f_2$

Given: $30 + f_1 + f_2 = 50$

$f_1 + f_2 = 20$ ..... (i)

$\bar{x} = a + \left(\frac{\sum f_iu_i}{\sum f_i} \times h\right)$ $\Rightarrow$ $62.8 = 30 + \left(\frac{58+2f_2}{50} \times 20\right)$

$\Rightarrow$ $32.8 = (58 + 2f_2) \times \frac{2}{5}$ $\Rightarrow$ $32.8 \times 5 = 116 + 4f_2$ $\Rightarrow$ $164 = 116 + 4f_2$

$\Rightarrow$ $4f_2 = 164 - 116$ $\Rightarrow$ $4f_2 = 48$ $\Rightarrow$ $f_2 = 12$

Now, $f_1 + f_2 = 20$

$f_1 + 12 = 20$

$f_1 = 8$

So, the missing frequencies are $f_1 = 8$ and $f_2 = 12$.

Example. 8

Find the mean marks from the following data

Marks	No. of Students
Below 10	5
Below 20	9
Below 30	17
Below 40	29
Below 50	45
Below 60	60
Below 70	70
Below 80	78
Below 90	83
Below 100	85

Sol. Changing less than type frequency distribution in general frequency distribution.

Marks	x_i	f_i	$a = 45, h = 10$ $u_i = \frac{x_i - a}{h}$	f_iu_i
0 –10	5	5	– 4	– 20
10 – 20	15	4	– 3	– 12
20 – 30	25	8	– 2	– 16
30 –40	35	12	– 1	– 12
40 – 50	45	16	0	0
50 – 60	55	15	+ 1	15
60 –70	65	10	+ 2	20
70 – 80	75	8	+ 3	24
80 – 90	85	5	+ 4	20
90 – 100	95	2	+ 5	10
		$\sum f_i = 85$		$\sum f_iu_i = 29$

According to step deviation formula for mean

$$\bar{x} = a + \left(\frac{\sum f_iu_i}{\sum f_i} \times h\right) \quad \Rightarrow \quad \bar{x} = 45 + \left(\frac{29}{85} \times 10\right)$$

$$\Rightarrow \quad \bar{x} = 45 + 3.41 \quad \Rightarrow \quad \bar{x} = 48.41$$

So, the mean marks is 48.41

Check Your Progress

1. The mean of the data 16, 20, 26, 40, 50, 60, 70, 30.

2. The weight in kg of 9 members in a school boxing team are 54, 59, x, 53, 73, 49, 50, 58, 45. If the average is 56 then x is

3. The mean of $x_1 + x_2 + .. + x_{10}$ is M. When x_i, i = 1, 2,….10 is replaced by $x_i + 10$, then new mean

4. Find the arithmetic mean age of the below data

Age in years	15	16	17	18	19	20
No. of students	2	5	9	4	3	2

5. The following set of raw data shows the lengths, in millimeters, measured to the nearest mm, of 40 leaves taken from plants of a certain species. This is the table of frequency distribution:

Lengths (mm)	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49	50 – 54	55 – 59
Frequency	2	4	7	10	8	6	3

Choosing a suitable assumed mean compute the mean by step-deviation method.

Answers

1. 39 **2.** 63 **3.** M + 10 **4.** 17.28 **5.** 43

B MEDIAN

Median is the middle value of the distribution. It is the value of variable such that the number of observations above it is equal to the number of observations below it.

(a) Median of raw data

Step - 1 : Arrange the data in ascending order.
Step - 2 : Count the no. of observation (Let there be 'n' observation)

(i) if n be odd then Median = value of $\left(\frac{n+1}{2}\right)^{th}$ observation.

(ii) if n is even then median is the arithmetic mean of $\left(\frac{n}{2}\right)^{th}$ observation and $\left(\frac{n}{2}+1\right)^{th}$ observation.

(b) Median of discrete frequency distribution

In case of a discrete frequency distribution, we calculate the median by using the following steps.

Step-1 : Find the cumulative frequencies (c.f.).

Step-2 : Find N/2, where N = $\sum_{i=1}^{n} f_i$.

Step-3 : See the cumulative frequency (c.f.) just greater than N/2 and determine the corresponding value of the variable. This value is the required median.

(c) Median of class interval data (grouped)

In a grouped data, we may not find the middle observation by looking at the cumulative freqeuncies, since the middle observation will be some value in a class interval, so it is necessary to find the value inside a class that divides the whole distribution into two halves. For this, we find the cumulative frequencies of all the classes and $\frac{N}{2}$, where N = number of observations. Now, locate the class whose cumulative frequency is greater than (and nearest to) $\frac{N}{2}$. This class is called median class. After finding the median class, use the following formula for calculating the median.

Median = $\ell + \frac{\frac{N}{2} - C}{f} \times h$

ℓ = lower limit of median class,
N = total no of observation
C = cumulative frequency of the class preceding the median class
h = size of the median class
f = frequency of the median class.

(d) Merits of median

(i) It is rigidly defined, easily understood and calculate.

(ii) It is not at all affected by extreme values.

(iii) It can be located graphically, even if the class-intervals are unequal.

(iv) It can be determined even by inspection is some cases.

(e) Demerits of median

(i) In case of even numbers of observations median cannot be determined exactly.

(ii) It is not based on all the observations.

(iii) It is not subject to algebraic treatment.

(iv) It is much affected by fluctuations of sampling.

(f) Uses of median

(i) Median is the only average to be used while dealing with qualitative data which cannot be measured quantitatively but can be arranged in ascending or descending order of magnitude.

(ii) It is used for determining the typical value in problems concerning wages, distribution of wealth etc.

Solved Examples

Example. 9

Following are the lives in hours of 15 pieces of the components of air craft engine.
Find the median : 715, 724, 725, 710, 729, 745, 649, 699, 696, 712, 734, 728, 716, 705, 719

Sol. Arranging the data in ascending order
649, 696, 699, 705, 710, 712, 715, 716, 719, 724, 725, 728, 729, 734, 745
N = 15

So, Median = $\left(\frac{N+1}{2}\right)^{th}$ observation = $\left(\frac{15+1}{2}\right)^{th}$ observation = 8th observation = 716.

Example. 10

The daily wages (in rupees) of 100 workers in a factory are given below

Daily wages (in Rs.)	125	130	135	140	145	150	160	180
No. of workes	6	20	24	28	15	4	2	1

Find the median wage of a worker for the above data.

Sol.

Daily wages (in Rs.)	No. of workes	Cumulative frequency
125	6	6
130	20	26
135	24	50
140	28	78
145	15	93
150	4	97
160	2	99
180	1	100

N = 100 (even)

$$\text{Median} = \frac{\left(\frac{N}{2}\right)^{th}\text{observation} + \left(\frac{N}{2}+1\right)^{th}\text{observation}}{2}$$

$$\text{Median} = \frac{50^{th}\text{observation} + 51^{th}\text{observation}}{2} = \frac{135+140}{2} = 137.50$$

$\therefore$ Median wage of a workers in the factory is Rs 137.50.

Example. 11

Calculate the median for the following distribution class

Class	0 -10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
Frequency	5	10	20	7	8	5

Sol. First we find $\left(\frac{N}{2}\right)^{th}$ value i.e. $\left(\frac{55}{2}\right)^{th} = 27.5^{th}$

which lies in 20-30.

$\therefore$ 20-30 class in median class

Class	f	c.f.
0 - 10	5	5
10 - 20	10	15
20 - 30	20	35
30 - 40	7	42
40 - 50	8	50
50 - 60	5	55

Here : $\ell = 20$, $\frac{N}{2} = 27.5$, C = 15,

f = 20, h = 10

$\therefore$ median $= 20 + \frac{27.5-15}{20} \times 10$

Median = 26.25

Example. 12

If the median of the following frequency distribution is 46, find the missing frequencies

Variable	10-20	20-30	30 - 40	40 - 50	50 -60	60 -70	70 -80	Total
Frequency	12	30	f_1	65	f_2	25	18	229

Sol.

Class Interval	Frequency	C.F.	
10 - 20	12	12	
20 - 30	30	42	
30 - 40	f_1	$42 + f_1$	
40 - 50	65	$107 + f_1$	(Median Class)
50 - 60	f_2	$107 + f_1+f_2$	
60 - 70	25	$132 + f_1+f_2$	
70 - 80	18	$150+ f_1+f_2$	

Let the frequency of the class 30 – 40 be f_1 and that of the class 50 – 60 be f_2 .

The total frequency is 229.

$12 + 30 + f_1 + 65 + f_2 + 25 + 18 = 229 \quad \Rightarrow \quad f_1 + f_2 = 79$

It is given that median is 46., clearly, 46 lies in the class 40 – 50. So, 40 – 50 is the median class

$\therefore \quad \ell = 40$, h = 10, f = 65 and C = 42 + f_1 , N = 229

Median = $\ell + \dfrac{\frac{N}{2} - C}{f} \times h \quad \Rightarrow \quad 46 = 40 + \dfrac{\frac{229}{2} - (42 + f_1)}{65} \times 10$

$\Rightarrow \quad 46 = 40 + \dfrac{145 - 2f_1}{13} \quad \Rightarrow \quad 6 = \dfrac{145 - 2f_1}{13}$

$\Rightarrow \quad 2f_1 = 67 \quad \Rightarrow \quad f_1 = 33.5$ or 34

Since, $f_1 + f_2 = 79$

$\therefore \quad f_2 = 45$

Hence, $f_1 = 34$, and $f_2 = 45$.

Check Your Progress

1. Find the median of 9, 5,7, 11, 13, 3.

2. The shirt sizes worn by a group of 200 persons are as follows:

Shirt size	37	38	39	40	41	42	43	44
No. of persons	15	25	39	41	36	17	13	12

Find the median shirt size .

3. The number of words in each of the first eighty sentences of a book were counted.
The results are recorded in groups as follows:

No. of words	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20	21 – 24	25 – 28	29 – 32
No. of sentences	2	5	11	23	21	13	4	1

Estimate the median length of sentence correct to one decimal place.

4. The weights of 80 parcels were each measured to the nearest tenth of a kilogram, and the results recorded in groups as follows:

Weight in kg	20 – 24	24 – 28	28 – 32	32 – 36	36 – 40	40 – 44
No of parcels	2	12	17	25	17	7

Calculate the median weight.

Answers

1. 8 **2.** 40 **3.** 16.3 **4.** 33.4

(C) MODE

Mode or modal value of the distribution is that value of variable for which the frequency is maximum.

(a) Mode of ungrouped data (By inspection only)

Arrange the data in an array and then count the frequencies of each variate.
The variate having maximum frequency is the mode.

(b) Mode of continuous frequency distribution

In case of grouped frequency distribution, it is not possible to determine the mode by only looking at the frequencies of observations. In such situation, we can only locate a class having maximum frequency. This class is called 'modal class'.
The mode is obtained by using the value inside the modal class. Mode is calculated by using the formula.

Mode = $\ell + \dfrac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$

Where ℓ = lower limit of the modal class

f_1 = frequency of the modal class i.e. the largest frequency.

f_0 = frequency of the class preceding the modal class.

f_2 = frequency of the class succeeding the modal class.

h = width of the modal class

(c) Merits of mode

(i) It can be easily understood and is easy to calculate.

(ii) It is not affected by extreme values and can be found by inspection in some cases.

(iii) It can be measured even if open-end classes and can be represented graphically.

(d) Demerits of mode

(i) It is ill-fined. It is not always possible to find a clearly defined mode.

(ii) It is not based upon all the observation.

(iii) It is not capable of further mathematical treatment. it is after indeterminate.

(iv) It is affected to a greater extent by fluctuations of sampling.

(e) Uses of mode

Mode is the average to be used to find the ideal size, e.g., in business forecasting, in manufacture of ready-made garments, shoes etc.

(f) Empirical Relationship between the three Measures of Central Tendency

The Empirical relationship between mean, median and mode is
Mode = 3 median - 2 mean.
This relationship can be used to find a central tendency when other two central tendencies are given.

Solved Examples

Example. 13

Find the mode of the following data :
25, 16, 19, 48, 19, 20, 34, 15, 19, 20, 21, 24, 19, 16, 22, 16, 18, 20, 16, 19.

Sol. Frequency table for the given data as given below

Value x_i	15	16	18	19	20	21	22	24	25	34	48
Frequency f_i	1	4	1	5	3	1	1	1	1	1	1

19 has the maximum frequency of 5. So, Mode = 19.

Example. 14

The following table shows the age distribution of cases of a certain disease admitted during a year in a particular hospital. Find the modal age.

Age (in Years)	5 - 14	15 - 24	25 - 34	35 - 44	45 - 54	55 - 64
No. of Cases	6	11	21	23	14	5

Sol. Here class intervals are not in inclusive form. So, Converting the above frequency table in inclusive form.

Age (in Years)	4.5 - 14.5	14.5 - 24.5	24.5 - 34.5	34.5 - 44.5	44.5 - 54.5	54.5 - 64.5
No. of Cases	6	11	21	23	14	5

Class 34.5 - 44.5 has maximum frequency. So it is the modal class.

ℓ = 34.5, h = 10, f_0 = 21, f_1 = 23 and f_2 = 14.

$\therefore$ Mode = $\ell + \dfrac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$.

Mode = $34.5 + \dfrac{23-21}{46-21-14} \times 10$ = $34.5 + \dfrac{2}{11} \times 10$ = 36.31

Example. 15

Find the mode of the following distribution

Daily Wages	31 - 36	37 - 42	43 - 48	49 - 54	55 - 60	61 - 66
No. of workers	6	12	20	15	9	4

Sol.

Daily Wages	No. of workers	Daily wages	No of workers
31 - 36	6	30.5 - 36.5	6
37 - 42	12	36.5 - 42.5	12
43 - 48	20	42.5 - 48.5	20
49 - 54	15	48.5 - 54.5	15
55 - 60	9	54.5 - 60.6	9
61 - 66	4	60.5 - 66.5	4
			$\boxed{\sum f_i = 66}$

Modal class frequency is 42.5 – 48.5.

ℓ = 42.5 , f_1 = 20, f_0 = 12, f_2 = 15, h = 6

$\therefore$ Mode = 42.5 + $\dfrac{20-12}{2(20)-12-15}$ × 6

$\therefore$ Mode = 46.2

Example. 16

If the mean and the median of the data is 54 and 51 respectively, then find the mode of the data.

Sol. Mode = 3 median – 2 mean = 3 × 51 – 2 × 54 = 153 – 108 = 45.

Check Your Progress

1. Find the mode of 0, 0, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 6, 7, 8

2. Write the emprical relation between mean,mode and median.

3. In an examination taken by 400 students, the scores were as shown in the following distribution table:

Marks	1 - 10	11 - 20	21 - 30	31 - 40	41 - 50	51 - 60	61 - 70	71 - 80	81 – 90	91 -100
Frequency	8	14	32	56	102	80	54	30	16	8

Find the mode scores.

4 Find the mode of the following distribution.

Below 20	Below 25	Below 30	Below 35	Below 40	Below 45	Below 50	Below 55	Below 60
2	6	24	45	78	89	92	98	100

5. If median and mode are 9.98 and 10.09 respectively, then find mean.

Answers

1. 5 **2.** Mode = 3 Median – 2 Mean **3.** 47.26

4. 36.76 **5.** 9.925

D GRAPHICAL REPRESENTATION OF CUMULATIVE FREQUENCY DISTRIBUTION

The curve obtained by graphically representing cumulative frequency distribution is called a cumulative frequency curve or ogive.

(i) less than type ogives **(ii)** more than type ogives

(a) Less than type ogives

To construct an ogive by less than method, we follow the procedure given below :

Step I : Construct the cumulative frequency table by adding class frequencies.

Step II : Mark upper class limits along *x*-axis as a suitable scale.

Step III : Mark cumulative frequencies along *y*-axis on a suitable scale.

Step IV : Plot the points (x_i, f_i), where x_i is the upper limit of a class and f_i is corresponding cumulative frequency.

Step V : Join the points obtained in step IV by a free hand smooth curve to get the ogive and to get the cumulative frequency polygon join the points obtained in step IV by line segments.

(b) More than type ogives

To construct a cumulative frequency polygon and an ogive by more than method, we follow the procedure given below :

Step I : Construct the cumulative frequency table by subtracting the frequency of each class from the total frequency.

Step II : Mark lower class limits along *x*-axis on a suitable scale.

Step III : Mark cumulative frequencies along *y*-axis on a suitable scale.

Step IV : Plot the points (x_i, f_i), where x_i is the lower limit of a class and f_i is corresponding cumulative frequency.

Step V : Join the points obtained in step IV by a free hand smooth curve to get the ogive and to get the cumulative frequency polygon join the points obtained in step IV by line segments.

Note : When nothing is mentioned, then we generally construct the 'less than type ogive.

(C) Determining the median from the ogive

Ogive can be used to find the median of a frequency distribution.
To find the median, when we have to draw any one of the types of ogives, we follow the procedure given below.

Step I : Draw any one of the two types of ogives on the graph paper.

Step II : Compute N/2 (N = Σfi) and mark the corresponding part on y-axis.

Step III : Draw a line parallel to *x*-axis, from the point marked in step II, meeting the ogive at point A(say).

Step IV : Draw perpendicular AB from A on *x*-axis. The *x*-coordinate of point B gives the median.
To find the median, when we have to draw both the types of ogives on graph paper, we follow the procedure given below.

Step I : Draw less than type and more than type ogives on the graph paper.

Step II : Mark the point of intersection of the two curves drawn in step I. Let this point be- A.

Step III : Draw perpendicular AB from A on the *x*-axis.

The *x*-coordinate of point A gives the median.

Solved Examples

Example. 17

The marks obtained by 400 students in medical entrance exam are given in the following table.

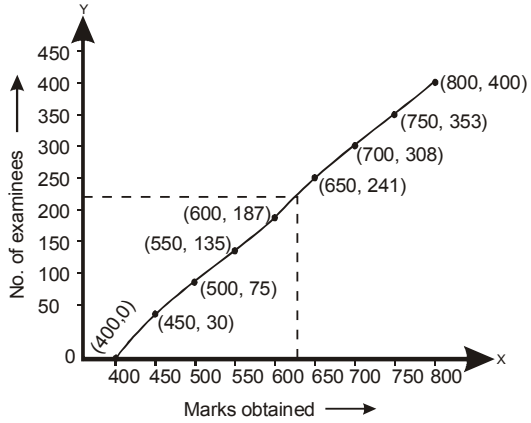
Marks Obtained	400 - 450	450 - 500	500 - 550	550 - 600	600 - 650	650 - 700	700 - 750	750 - 800
No. of Examinees	30	45	60	52	54	67	45	47

(i) Draw Ogive by less than method.
(ii) Draw Ogive by more than method.
(iii) Find the number of examinees, who have obtained the marks less than 625.
(iv) Find the number of examinees, who have obtained 625 and more than marks.

Sol. **(i)** Cumulative frequency table for less than Ogive method is as following

Marks Obtained	No. of Examinees
Less than 450	30
Less than 500	75
Less than 550	135
Less than 600	187
Less than 650	241
Less than 700	308
Less than 750	353
Less than 800	400

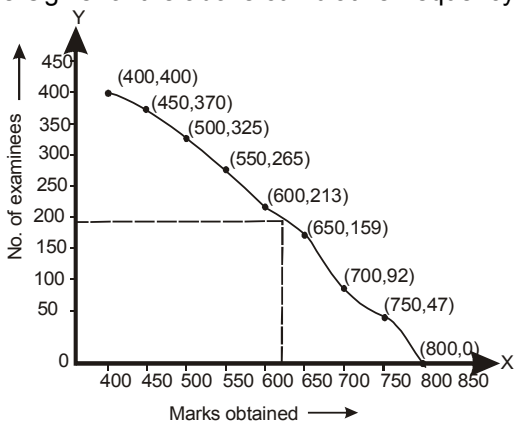
Following are the Ogive for the above cumulative frequency table by applying the given method
and the assumed scale.



(ii) Cumulative frequency table for more than Ogive method is as following :

Marks Obtained	No. of Examinees
400 and more	400
450 and more	370
500 and more	325
550 and more	265
600 and more	213
650 and more	159
700 and more	92
750 and more	47

Following are the Ogive for the above cumulative frequency table.



(iii) So, the number of examinees, scoring marks less then 625 are approximately 220.

(iv) So, the number of examinees, scoring marks 625 and more will be approximately 190.

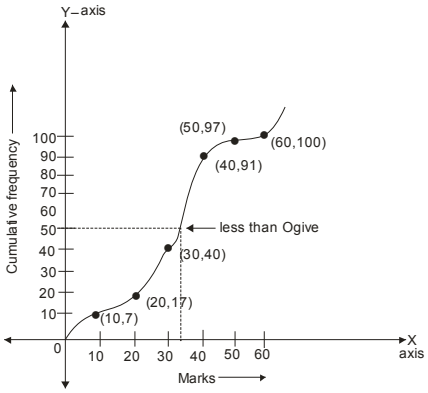
Example. 18

Draw an Ogive for the following frequency distribution by less than method and also find its median from the graph

Marks	0-10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
Number of students	7	10	23	51	6	3

Sol. Converting the frequency distribution into less than cumulative frequency distribution

Marks	No. of Students
less than 10	7
less than 20	17
less than 30	40
less than 40	91
less than 50	97
less than 60	100



According to graph median = 34 marks.

Check Your Progress

1. Which measure of the central tendency is obtained as the x coordinate of the point of intersection of the two ogives for the data?

2. The point of intersection of the ogive is given by (10,20). What is the median

3. The point of intersection of the ogive is given by (12,35), then find the sum of frequency.

4. From a list of names in an office, number of employees and their number of years of service were recorded as follows.

Experience in years	1 – 4	4 – 7	7 – 10	10 – 13	13 – 16	16 – 19
No. of employees	6	30	40	16	4	4

Construct a cumulative frequency distribution table of both the types and sketch the ogives. From the graphs estimate the median of the data. Calculate the median by using the formula and compare the values.

5. Construct a cumulative frequency distribution table for the data given below and draw the ogives of both the types. Find the median value from the graphs.

Class	101 – 150	151 – 200	201 – 250	251 – 300	301 – 350	351 – 400	401 – 450	451 – 500
Frequency	15	18	25	35	32	28	24	20

Answers

1. Median **2.** 10 **3.** 70

Exercise-1

SUBJECTIVE QUESTIONS

Subjective Easy, only learning value problems

Section (A) : Introduction and Arithmetic mean

A-1. Find the mean of the first five multiple of 6.

A-2. If the average of 11, 12, 13, 14 and x is 13, then find the value of x.

A-3. The mean of six numbers is 15. If 2 is taken away from every number, then find the new mean.

A-4. If the mean of the following data is 18.75, find the value of p :

x	10	15	p	25	30
f	5	10	7	8	2

A-5. Find the mean of following distribution by step deviation method :

Class interval	50 - 70	70 - 90	90 - 110	110 - 130	130 - 150	150 - 170
No. of workers	18	12	13	27	8	22

A-6. If the mean of following distribution frequency distribution is 188, find the missing frequencies f_1 and f_2.

C.I.	0– 80	80 – 160	160 – 240	240 – 320	320 – 400	Total
Freq.	20	25	f_1	f_2	10	100

Section (B) : Median

B-1. If the mean and median of a set of numbers are 11 and 10 respectively, then find the mode

B-2. Calculate the median from the following data :

Rent (in Rs.)	**15 - 25**	**25 - 35**	**35 - 45**	**45 - 55**	**55 - 65**	**65 - 75**	**75 - 85**	**85 - 95**
No. of Houses	8	10	15	25	40	20	15	7

B-3. Find the median class of the following data

Marks obtained	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Frequencey	8	10	12	22	30	18

B-4. Find the missing frequencies and the median for the following distribution if the mean is 1.46 and the sum of all frequencies is 200.

No. of accidents	0	1	2	3	4	5
No. of days	46	f_1	f_2	25	10	5

Section (C) : Mode

C-1. The marks in science of 80 students of class X are given below . Find the mode of the marks obtained by the students in science.

C.I.	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 -70	70 - 80	80 - 90	90 - 100
Freq.	3	5	16	12	13	20	5	4	1	1

C-2. Find the mode of following distribution

C.I.	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 -70	70 - 80
Freq.	5	8	7	12	28	20	10	10

C-3. Find the mean, mode and median of the following data

Classes	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
Frequencey	5	10	18	30	20	12	5

C-4 If the ratio of mean and mode is 5 : 2 then find the ratio of median : mode.

C-5 The mode of the following distribution is 42. Find the missing frequency x.

Class interval	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	12	20	x	26	6	7

Section (D) : Graphical representation of cumulative frequency distribution

D-1. The following table gives the height of trees

Height	Less than 7	Less than 14	Less than 21	Less than 28	Less than 35	Less than 42	Less than 49	Less than 56
No. of trees	26	57	92	134	216	287	341	360

Draw “less than” ogive and “more than” ogive.

D-2. During the medical check-up of 35 students of a class , their weights were recorded as follows :

Weight (in kg)	Number of students
38 - 40	3
40 - 42	2
42 - 44	4
44 - 46	5
46 - 48	14
48 - 50	4
50 - 52	3

Draw a less than & more than type ogive for the given data. Hence, obtain median weight from the graph.

D-3. Draw ‘less than ogive’ and ‘more than ogive’ for the following distribution and hence find its median.

Class	20–30	30–40	40–50	50–60	60–70	70–80	80–90
Frequency	8	12	24	6	10	15	25

D-4 The annual profits earned by 40 shops of a shopping complex in a locality give rise to the following distribution

Profit (in lakh)	Number of shops (frequency)
More than 5	40
More than 10	36
More than 15	22
More than 20	16
More than 25	12
More than 30	6
More than 35	2

Find the median from the graph.

D-5. Following distribution shows the marks obtained by 100 students.

Marks	10–20	20–30	30–40	40–50	50–60	60–70
No. of students	10	15	30	32	8	5

Draw 'less than ogive' for the above data. Find median graphically and verify the result by actual method.

OBJECTIVE QUESTIONS

Single Choice Objective, straight concept/formula oriented

Section (A) : Introduction and Arithmetic mean

A-1. If the arithmetic mean of 5, 7, 9, x is 9 then the value of x is :
(A) 11 (B) 15 (C) 18 (D) 16

A-2. If the first five elements of the set $x_1, x_2, ...x_{10}$ are replaced by $x_i + 5$, i = 1, 2, 3, 4, 5 and next five elements are replaced by $x_j - 5$, j = 6, 7, ... 10, then the mean will change by :
(A) 0 (B) $\frac{n+1}{2}$ (C) 10 (D) 25

A-3 Arun scored 36 marks in English, 44 marks in Hindi, 75 marks in Mathematics and x marks in Science. If he has scored an average of 50 marks, then find the value of x.
(A) 45 (B) 55 (C) 35 (D) 40

A-4. If $u_i = \frac{x_i - 20}{10}$, $\Sigma f_i u_i = 30$ and $\Sigma f_i = 40$, then find the value of .
(A) 30 (B) 25 (C) 28.5 (D) 27.5

A-5. The mean of 20 numbers is 13. The new mean if each observation is increased by 5, is
(A) 13 (B) 18 (C) 65 (D) 8

Section (B) : Median

B-1. The median of following series is 520, 20, 340, 190, 35, 800, 1210, 50, 80.
(A) 1210 (B) 520 (C) 190 (D) 35

B-2. The upper limit of the median class of the following distribution is :

Class	0 - 5	6 - 11	12 - 17	18 - 23	24 - 29
Frequency	13	10	15	8	11

(A) 17 (B) 17.5 (C) 18 (D) 18.5

B-3. A data has 25 observations (arranged in descending order). Which observation represents the median ?
(A) 12th (B) 13th (C) 14th (D) 15th

B-4. The median of the observation 11, 12, 14, 18, x + 2, x + 4, 30, 32, 35, 41 arranged in ascending order is 24. Find the value of x.
(A) 20 (B) 21 (C) 22 (D) 23

B-5. If in a data, 10 numbers are arranged in increasing order. If the 7th entry is increased by 4, then the median increases by -
(A) zero (B) 4 (C) 6 (D) 5

Section (C) : Mode

C-1. The mode of the distribution 3, 5, 7, 4, 2, 1, 4, 3, 4 is :
(A) 7 (B) 4 (C) 3 (D) 1

C-2. If the mean and median of a set of numbers are 8.9 and 9 respectively, then the mode will be :
(A) 7.2 (B) 8.2 (C) 9.2 (D) 10.2

C-3. If mode = 80 and mean = 110, then the median is :
(A) 110 (B) 120 (C) 100 (D) 90

C-4. If the mode of the following data is 7, then value of k in 2, 4, 6, 7, 5, 6, 10, 6, 7, 2k + 1, 9, 7, 13 is :
(A) 3 (B) 7 (C) 4 (D) 2

C-5. For a symmetrical distribution, which is correct :
(A) Mean > Mode > Median (B) Mean < Mode < Median
(C) Mode = $\frac{\text{Mean}-\text{Median}}{2}$ (D) Mean = Median = Mode

Section (D) : Graphical representation of cumulative frequency distribution

D-1. Which measure of central tendancy is given by the x co-ordinate of the point of intersection of the more than Ogive and less than Ogive.
(A) Mean (B) Median (C) Mode (D) All the above

D-2. For a given data with 50 observations the 'less than ogive' and the 'more than ogive' intersect at (15.5, 20). The median of the data is :
(A) 4.5 (B) 20 (C) 50 (D) 15.5

D-3. The classmark of a class interval is :
(A) Lower limit + Upper limit (B) Upper limit – Lower limit
(C) $\frac{1}{2}$ (Lower limit + Upper limit) (D) $\frac{1}{4}$ (Lower limit + Upper limit)

D-4. Construction of a cumulative frequency table is useful in determining the :
(A) Mean (B) Median
(C) Mode (D) All the above three measures

D-5. Which of the following is not a measure of central tendency :
(A) Mean (B) Median (C) Range (D) Mode

Exercise-2

OBJECTIVE QUESTIONS

1. Range of 14, 12, 17, 18, 16 and x is 20. Find x (x > 0)
(A) 2 (B) 28 (C) 32 (D) Cannot be determined

2. The range of 15, 14, x, 25, 30, 35 is 23. Find the least possible value of x.
(A) 14 (B) 12 (C) 13 (D) 11

3. The width of each of nine classes in a frequency distribution is 2.5 and the lower class boundary of the lowest class is 10.6. Which one of the following is the upper class boundary of the highest class ?
(A) 35.6 (B) 33.1 (C) 30.6 (D) 28.1

4. Let L be the lower class boundary of a class in a frequency distribution and m be the mid point of the class. Which one of the following is the upper class boundary of the class ?
(A) $m + \frac{m+L}{2}$ (B) $L + \frac{m+L}{2}$ (C) 2m – L (D) m – 2L

5. The arithmetic mean of the set of observations 1, 2, 3....n is :
(A) $\frac{n+1}{2}$ (B) $\left(\frac{n}{2}+1\right)$ (C) $\frac{n}{2}$ (D) $\frac{1}{2}(n-1)$

6. In a monthly test, the marks obtained in mathematics by 16 students of a class are as follows :
0, 0, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 6, 7, 8
The arithmetic mean of the marks obtained is :
(A) 3 (B) 4 (C) 5 (D) 6

7. Mean of a certain number of observation is m. If each observation is divided by x ($x \neq 0$) and increased by y, then the mean of new observation is :
(A) mx + y (B) $\frac{mx+y}{x}$ (C) $\frac{m+xy}{x}$ (D) m + xy

8. The mean of a variable x having 50 observations is 45. If a new variable is defined as U = x + 45, the mean of the new variable is :
(A) 45 (B) 0 (C) 95 (D) 90

9. If the arithmetic mean of the observations x_1, x_2, x_3 ..... x_n is 1, then the arithmetic mean of $\frac{x_1}{k}, \frac{x_2}{k}, \frac{x_3}{k}, \ldots \frac{x_n}{k}$ (k > 0) is :
(A) greater than 1 (B) less than 1 (C) equal to 1 (D) Both (A) or (B)

10. The mean of a set of observation is a. If each observation is multiplied by b and each product is decreased by c, then the mean of new set of observation is.
(A) $\frac{a}{b} + c$ (B) ab – c (C) $\frac{a}{b} - c$ (D) ab + c

11. The arithmetic mean of 5 numbers is 27. If one of the numbers be excluded, their mean is 25. The excluded number is :
(A) 28 (B) 26 (C) 25 (D) 35

12. The combined mean of three groups is 12 and the combined mean of first two groups is 3. If the first, second and third groups have 2, 3 and 5 times respectively, then mean of third group is :
(A) 10 (B) 21 (C) 12 (D) 13

13. Out of 100 numbers, 20 were 5s, 30 were 6s and the remaining were 7s. The arithmetic mean of the numbers is :
(A) 6.3 (B) 5.4 (C) 6.1 (D) 6.5

14. While dividing each entry in a data by a non-zero number a, the arithmetic mean of the new data :
(A) is multiplied by a (B) does not change (C) is divided by a (D) is diminished by a

15. In order to make the computation of the arithmetic mean of a set of 50 numbers simpler, each observation is subtracted from 53 and the arithmetic mean of the set of differences is found to be – 3.5 . The arithmetic mean of the set of given numbers is :
(A) 53.07 (B) 52.93 (C) 56.50 (D) 49.50

16. The mean of first n odd natural numbers is $\frac{n^2}{81}$. Find n.
(A) 9 (B) 81 (C) 27 (D) None of these

17. The arithmetic mean of 12 observations is 15. If two observations 20 and 25 are removed, then the arithmetic mean of remaining observations is :
(A) 14.5 (B) 13.5 (C) 12.5 (D) 13

18. Consider the table given below :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	12	18	27	20	17	6

The arithmetic mean of the marks given above, is :
(A) 18 (B) 28 (C) 27 (D) 6

19. The median of 21 observations is 18. If two observations 15 and 24 are included to the observation, then the median of new series is :
(A) 15 (B) 18 (C) 24 (D) 16

20. The median of the following distribution is :

Class interval	35-45	45-55	55-65	65-75
Frequency	8	12	20	10

(A) 56.5 (B) 57.5 (C) 58.7 (D) 59

21. Find the median of the following data.

C.I.	0–10	10–20	20–30	30–40	40–50
f	12	13	25	20	10

(A) 25 (B) 23 (C) 24 (D) 26

22. If the observations 20, 22, 23, 25, (x + 1), (x + 3), 36,38, 39 and 41 have median 30, then the value of x is :
(A) 28 (B) 29 (C) 30 (D) 32

23. If the difference of mode and median of a data is 24, then the difference of median and mean is :
(A) 12 (B) 24 (C) 8 (D) 36

24. The arithmetic mean and mode of a data is 24 and 12 respectively, then the median of the data is .
(A) 25 (B) 18 (C) 20 (D) 22

25. Which of the following does not change for the observation 23, 50, 27, 2x, 48, 59, 72, 89, 5x, 100, 120, when x lies between 15 and 20 ?
(A) Arithmetic mean (B) Range (C) Median (D) Average

Exercise-3

IJSO STAGE – I (PREVIOUS YEARS)

1. The mean of the following frequency distribution is _____ **[IJSO 2017]**

Class interval	0-10	10-20	20-30	30-40	40-50
Frequency	4	6	8	10	12

(A) 25 (B) 28 (C) 30 (D) 32

Answers

Exercise-1

SUBJECTIVE QUESTIONS

Section (A)

A-1. 18 **A-2.** 15 **A-3.** 13 **A-4.** 20 **A-5.** 112.20

A-6. $f_1 = 15$ and $f_2 = 30$.

Section (B)

B-1. 8 **B-2.** 58. **B-3.** 30 – 40 **B-4.** $f_1 = 76$, $f_2 = 38$ and Median = 1

Section (C)

C-1. 53.17 **C-2.** 46.67 **C-3.** Mean = 35.6, Medain = 35.67 and Mode = 35.45

C-4 2 : 1. **C-5** 21

Section (D)

D-2. 46.5 kg **D-3.** 60. **D-4.** 8.75 **D-5.** 38.33

OBJECTIVE QUESTIONS

Section (A)

A-1. (B) **A-2.** (A) **A-3** (A) **A-4.** (D) **A-5.** (B)

Section (B)

B-1. (C) **B-2.** (B) **B-3.** (B) **B-4.** (B) **B-5.** (A)

Section (C)

C-1. (B) **C-2.** (C) **C-3.** (C) **C-4.** (A) **C-5.** (D)

Section (D)

D-1. (B) **D-2.** (D) **D-3.** (C) **D-4.** (B) **D-5.** (C)

Exercise-2

OBJECTIVE QUESTIONS

Ques.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	B	C	A	B	C	D	D	B	D	B	A	C	C	B	B	B	B	B
Ques.	21	22	23	24	25															
Ans.	D	A	A	C	B															

Exercise-3

IJSO STAGE-I (PREVIOUS YEARS)

Ques.	1
Ans.	C