Resonance Guide to Pre-RMO for class 8 9 and 10 DLPD Distance Learning Programmes Division Regional Mathematics Olympiad

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Distance Learning Programmes Division (DLPD)

A Guide to

Pre-RMO

(Pre-Regional Mathematical Olympiad)

Class VIII | IX | X

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PRE RMO
CLASS VIII TO X

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PREFACE

An Important Note to Parents and Students

Dear Student,

If you are reading this, it means you are serious about performing in your **PRMO Exam.** To help you achieve your potential, this booklet is designed in a way which is highly beneficial for students. Let's see how to use the different components of the booklet :

- **Theory Part :** The theory part has been designed with perfect blend of solved examples, text and important notes. At relevant checkpoints, in theory, exercises have been inserted to enhance the reading experience of a student.

- **Exercise 1 :** This exercise is **Competitive Level Exercise (CLE).** After going through the theory and thus absorbing the important concepts, students are ready to implement their learning in competitive level questions. These questions are in accordance to the level being asked in the National Level Competitive Examinations especially **PRMO**. These questions are must for all the students to strengthen their concepts.

- **Exercise 2 :** This exercise consists of questions from previous year papers of **PRMO and RMO**. This exercise is included for students to determine their current proficiency level of that chapter with respect to the competitive examination.

After reading the theory and completing the exercises, a student should be able to have a conceptual framework and problem solving aptitude in that particular chapter. For best results, **all exercises should be solved in a fair notebook and all the solutions should be maintained so that when time of revision comes, this notebook proves to be a handy one.**

"You have to dream before your dreams come true"

Pre-foundation Career Care Programmes (PCCP) Division

Every effort has been taken to make our study material error free, however any suggestion to improve is welcome in this regard.

Number Theory

DIVISIBILITY :

An integer $a \neq 0$ divides b, if there exists an integer x such that $b = ax$, and thus, we write as $a \mid b$ (read a divides b). This can also be stated as b is divisible by a or a is a divisor of b or b is a multiple of a. If a does not divide b we write as $a \mid b$.

Properties of Divisibility

1. $a \mid b$ and $b \mid c \Rightarrow a \mid c$
2. $a \mid b, a \mid c \Rightarrow a \mid (b + c)$, and $a \mid (b - c)$
3. $a \mid b, a \mid (b + c) \Rightarrow a \mid c$
4. $a \mid b, a \mid (b - c) \Rightarrow a \mid c$
5. $a \mid b$ and $a \mid c \Rightarrow a \mid (kb \pm lc)$ for all $k, l \in Z$
6. $a \mid b$ and $b \mid a \Rightarrow a = \pm b$
7. $a \mid b \Rightarrow b = 0$ or $|a| \leq |b|$. In particular if $a \mid b$ where $a > 0, b > 0$, then $a < b$
8. $a \mid b \Rightarrow a \mid bc$ for any integer c
9. $a \mid b$ iff $ma \mid mb$ where $m \neq 0$

Notes:

1. $(x + y) \mid (x^{2n+1} + y^{2n+1}) \ \forall n \in N_0$
   Proof:
   For $n = 0$ it is obvious, for $n \geq 1$, we have
   $(x^{2n+1} + y^{2n+1}) = (x + y)(x^{2n} - x^{2n-1}y + x^{2n-2}y2 - ... + y^{2n})$

2. $(x - y) \mid (x^n - y^n) \ \forall n \in N$
   Proof:
   For $n = 1$ it is obvious, for $n \geq 2$, we have
   $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + ... + y^{n-1})$

GCD and LCM :

The greatest common divisor of two positive integers a and b is the great- est positive integer that divides both a and b, which we denote by gcd(a, b), and similarly, the lowest common multiple of a and b is the least positive say that a and b are relatively prime if gcd(a, b) = 1. For integers $a_1, a_2, \ldots, a_n$, $\gcd(a_1, a_2, \ldots, a_n)$ is the greatest positive integer that divides all of $a_1, a_1, \ldots, a_n$, and $\text{lcm}(a_1, a_2, \ldots, a_n)$ is defined similarly.

Useful Facts

- For all a, b, $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$.
- For all a, b, and m, $\gcd(ma, mb) = m \gcd(a, b)$ and $\text{lcm}(ma, mb) = m\text{lcm}(a, b)$.
- If $d \mid \gcd(a, b)$, then $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{\gcd(a,b)}{d}$

In particular, if $d = \gcd(a, b)$, then $\gcd(a/d, b/d) = 1$; that is, a/d and b/d are relatively prime.

- If $a \mid bc$ and $\gcd(a, c) = 1$, then $a \mid b$.
- For positive integers a and b, if d is a positive integer such that $d \mid a$,

  $d \mid b$, and for any d', $d' \mid a$ and $d' \mid b$ implies that $d' \mid d$, then d = gce (a,b). This is meraly the assertion that any common divisor of a and b divides gcd (a, b)

- If $a_1 a_2 \cdots a_n$ is a perfect k^{th} power and the a_i are pairwise relatively prime, then each a_i is a perfect k^{th} power.
- Any two consecutive integers are relatively prime.

EUCLID DIVISION LEMMA (EDL) :

For any positive integer a and integer b, there exist unique integers q and r such that

$b = qa + r$ and $0 \le r < a$, with $r = 0$ iff $a \mid b$.

EUCLID DIVISION ALGORITHM (EDA) :

If a, b, q, r are integers such that $a = bq + r$ then common factor of a and b is also common factor of b and r

Useful Facts :

(i) For any positive integers a and b there exist integers x and y such that $ax + by = \gcd(a, b)$.Furthermore, as x and y vary over all integers, ax + by attains all ultiples and only multiples of gcd(a,b).

Proof. Let S be the set of all integers of the form ax+by, and let d be the least positive element of S. By the division algorithm, there exist integers q and r such that $a = qd + r$, $0 \le r < d$. Then $r = a - qd = a - q(ax + by) = (1 - qx)a - (qy)b$, so r is also in S. But $r < d$, so $r = 0 \Rightarrow d \mid a$, and similarly, $d \mid b$, so $d \mid \gcd(a, b)$. However, gcd(a, b) divides all elements of S, so in particular $\gcd(a,b) \mid d \Rightarrow d = \gcd(a,b)$.

(ii). The positive integers a and b are relatively prime iff there exist integers x and y such that $ax+by = 1$.

(iii) For any positive integers $a_1, a_2, \ldots, a_n$, there exist integers $x_1, x_2, \ldots, x_n$, such that $a_1 x + a_2x_2 + \cdots + a_n x_n = \gcd(a_1, a_2, \ldots, a_n)$.

(iv). Let a and b be positive integers, and let n be an integer. Then the equation $ax + by = n$ has a solution in integers x and y iff $\gcd(a,b) \mid n$. If this is the case, then all solutions are of the form $(x,y) = \left(x_0 + t.\frac{b}{d},\ y_0 - t.\frac{a}{d}\right)$ where $d = \gcd(a, b)$, (x_0, y_0) is a specific solution of $ax + by = n$, and t is an integer.

Proof. The first part follows from (i) For the second part, as stated, let $d = \gcd(a, b)$, and let (x_0, y_0) be a specific solution of $ax + by = n$, So that $ax_0 + by_0 = n$. If $ax + by = n$, then $ax + by - ax_0 - by_0 = a(x - x_0) + b(y - y_0) = 0$, or $a(x - x_0) = b(y_0 - y)$, and hence $(x - x_0)\frac{a}{d} = (y_0 - y)\frac{b}{d}$ Since a/d and b/d are relatively prime, b/d must divide $x - x_0$, and a/d must divide $y_0 - y$.Let $x - x_0 = tb/d$ and $y_0 - y = ta/d$. This gives the solutions described above.

Example 1.

Let a and b be natural numbers and let q and r be the quotient and remainder respectively when $a^2 + b^2$ is divided by a + b. Determine the number q and r if $q^2 + r = 2000$.

Sol. $a^2 + b^2 = q(a + b) + r$ .... (i)

as $0 \le r < (a + b)$

$q^2 + r = 2000$

$q^2 = 2000 - r$

$q^2 \le 2000$

$q \le 44$ ... (ii)

as $\frac{(a+b)^2}{2} \le (a^2 + b^2)$

$\Rightarrow$ $(a + b)^2 \le 2(a^2 + b^2)$

from (i) and (ii)

$a^2 + b^2 < 45(a + b)$

$\Rightarrow$ $2(a^2 + b^2) < 90(a + b)$ $\Rightarrow$ $(a + b)^2 < 90 (a + b)$

$\Rightarrow$ $(a + b) < 90$

so $r < 90$

$q^2 = 2000 - r > 2000 - 90 = 1910$

$q^2 > 1910$

$q > 43$ ... (iii)

From (ii) & (iii) q = 44

r = 64

Example 2.

A number n is called multiplicatively perfect if the product of all the positive divisors of n is n^2. Determine the number of positive multiplicatively perfect numbers less than 100.

Sol. All multiplicatively perfect numbers have exactly 4 distinct positive divisors, or 1. So, we must look for numbers that are either – 1

– a product of two distinct primes

– a cube of a prime

Numbers satisfying one of these condition less than 100 are : 1, 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 86, 87, 91, 93, 94, 95. There are **33** of these

Self Practice Problems :

1. Find all 3-digit numbers which are the sums of the cubes of their digits.

Ans. 153, 370, 371, 407

The Fundamental Theorem of Arithmetic.

Every integer greater than 1 can be written uniquely in the form $p_1^{e_1} p_2^{e_2} .... p_k^{e_k}$ where the p_i are distinct primes and the e_i are positive integers.

Note : There exist an infinite number of primes.

Proof. Suppose that there are a finite number of primes, say $p_1, p_2, \ldots, p_n$. let $N = p_1 p_2 \cdots p_n + 1$. By the fundamental theorem of arithmetic, N is divisible by some prime p. This prime p must be among the pi, since by assumption these are all the primes, but N is seen not to be divisible by any of the p_i contradiction.

Greatest integer function or step up function :

The function y = f (x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. For example :

[3.2] = 3; [– 3.2] = – 4

for $-1 \le x < 0$; [x] = – 1 ; for $0 \le x < 1$; [x] = 0

for $1 \le x < 2$; [x] = 1 ; for $2 \le x < 3$; [x] = 2 and so on.

Properties of greatest integer function :

(a) $x - 1 < [x] \le x$ (b) $[x \pm m] = [x] \pm m$ iff m is an integer.

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$ (d) $[x] + [-x] = \begin{cases} 0; & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

Exponent of prime p in n! (Legender's theorem):

Let p be a prime number, n be a positive integer and Let $E_p(n)$ denote the exponent of the prime p in the positive integer n. Then,

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots + \left[\frac{n}{p^s}\right]$$

where s is the largest positive integer such that $p^s \le n < p^{s+1}$

Example 3.

If $(2^{200} - 2^{192}.31 + 2^n)$ is the perfect square of a natural number, then find the sum of digits of 'n'

Ans. 18

Sol. $2^{192}(2^8 - 31) + 2^n \Rightarrow 2^{192}.225 + 2^n = m^2$

$2^n = (m - 2^{96}.15)(m + 2^{96}.15)$

now let $m - 2^{96}.15 = 2^{\alpha}$ and $m + 2^{96}.15 = 2^{\alpha+\beta}$

Hence : $2^{97}.15 = 2^{\alpha+\beta} - 2^{\alpha}$

$2^{\alpha}(2^{\beta} - 1) = 2^{97}(2^4 - 1)$

$\alpha = 97 \quad \beta = 4$

$2^n = 2^{2\alpha+\beta}$

$n = 2\alpha + \beta = 198$

$p = \dfrac{-1-2q}{2-5q} = \dfrac{2q+1}{5q-2}$ not possible

Example4.

Find the number of quadruplets of positive integers (a, b, c, d) satisfying the following relations:
$1 \le a \le b \le c \le d$ and $ab + cd = a + b + c + d + 3$

Ans. 4

Sol. $ab - a - b + 1 + cd - c - d + 1 = 5$

$(a - 1)(b - 1) + (c - 1)(d - 1) = 5$

case-1

$a - 1 \ge 2 \Rightarrow b - 1, c - 1, d - 1 \ge 2$

Hence LHS ≥ 8

If follows that

$a - 1 = 0, 1$

Case-1 $a - 1 = 0$

$(c - 1)(d - 1) = 5$

$c - 1 = 1$ and $d - 1 = 5$ hence $c = 2$, $d = 6$

Now $b - 1$ can be 0, 1 $(c \geq b)$

$b = 1, c = 2, d = 6, a = 1$

$b = 2, c = 2\ d = 6, a = 1$

$(a, b, c, d) = (1,1,2,6)$ and $(1,2,2,6)$

Case-2 $a - 1 = 1$

Now $b - 1$ can be 1, 2

but $b - 1 = 2$ not possible

$b - 1 = 1 \Rightarrow b = 2$

$(c - 1)(d - 1) = 4$

$c - 1 = 2$ and $d - 1 = 2$ or $c - 1 = 1$ and $d - 1 = 4$

$c = 3$ and $d = 3$ $\quad c = 2 \quad d = 5$

$(a, b, c, d) = (2, 2, 3, 3)\ (2, 2, 2, 5)$

Self Practice problems:

1. A three digit number is equal to the sum of the factorial of their digits. If the sum of all such three digit numbers is λ then find the sum of digit of λ.

2. $\dfrac{(2007)!}{(2007)^n}$ is an integer & $n \in N$, then find maximum value of n.

Answers.

1. 10 **2.** 9

Modular Arithmetic

For a positive integer m and integers a and b, we say that a is congruent to b modulo m if $m \mid (a - b)$, and we denote this by $a \equiv b$ modulo m, or more commonly $a \equiv b \pmod{m}$. Otherwise, a is not congruent to b modulo m, and we denote this by $a \equiv b \pmod{m}$ (although this notation is not used often). In the above notation, m is called the modulus, and we consider the integers modulo m.

Useful Facts :

(i) If $a \equiv b$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$.

Proof. If $a \equiv b$ and $c \equiv d \pmod{m}$, then there exist integers k and ℓ such that $a = b + km$ and $c = d + lm$.

Hence, $a + c = b + d + (k + \ell)m$, so

$a + c \equiv b + d \pmod{m}$. Also,

$ac = bd + dkm + b\ell m + k\ell m^2$

$= bd + (dk + bl + k\ell m)m,$

so $ac \equiv bd \pmod{m}$

(ii) For all integers n, $n^2 \equiv \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \pmod{4} \begin{cases} \text{if n is even,} \\ \text{if n is od.} \end{cases}$

(iii) For all integers n, $n^2 \equiv \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix} \pmod{8} \begin{cases} \text{if } n \equiv 0 \pmod{4} \\ \text{if } n \equiv 2 \pmod{4} \\ \text{if } n \equiv 1 \pmod{2} \end{cases}$

(iv) If f is a polynomial with integer coefficients and $a \equiv b \pmod{m}$, then $f(a) \equiv f(b) \pmod{m}$.

Example 5.

Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split evenly. What is the least number of coins there could have been?

Sol. Let y be the number of coins in the chest. From the problem, we know that $y \equiv 5 \pmod{11}$, $y \equiv 3 \pmod{10}$, and $y \equiv 0 \pmod 9$. Combining these gives us that $y \equiv 423 \pmod{990}$, so the answer is 423

Example 6.

Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that
$a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set
$T = \{abcdef : (a, b, c, d, e, f) \in S\}$.
Find the greatest common divisor of all the members of T.

Ans. 24

Sol. We show that the required gcd is 24. Consider an element $(a,d,c,d,e,f) \in S$. We have
$a^2 + b^2 + c^2 + d^2 + e^2 = f^2$.

We first observe that not all a, b, c, d, e can be odd. Otherwise, we have $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 8$ and hence $f^2 \equiv 5 \pmod 3$ which is impossible because no square can be congruent to 5 modulo 8. Thus at least one of a, b, c, d, e is even.

Similarly if none of a, b, c, d, e is divisible by 3, then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 3$ and hence $f^2 \equiv 2 \pmod 3$ which again is impossible because no square is congruent to 2 modulo 3. Thus 3 divides abcdef.

There are several possibilities for a, b, c, d, e.

Case -1 Suppose one of them is even and the other four are odd; say a is even b, c, d, e are odd. Then $b^2 + c^2 + d^2 + e^2 \equiv 4 \pmod 8$ which again gives that $4|a$ and $2|f$ so that $8|af$. It follows that $8|abcdef$ and hence $24|abcdef$.

Case-2 Suppose a, b, are even and c, d, e are odd. Then $c^2 + d^2 + e^2 \equiv 3 \pmod 8$. Since $a^2 + b^2 \equiv 0$ or 4 modulo 8, it follows that $f^2 \equiv 3$ or $7 \pmod 8$ which impossible. Hence this case does not arise.

Case-3 If three of a, b, c, d, e are even and two odd, then $8|abcde f$ and hence $24|abcdef$.

Case-4 If four of a,b,c,d,e are even, then again $8|abcdef$ and $24|abcdef$. Hence again for any six tuple (a, b, c, d, e, f) in S, we observe that $24|abcde f$. Since

$$1^2 + 1^2 + 1^2 + 2^2 + 3^2 = 4^2.$$

We see that $(1, 1, 1, 2, 3, 4) \in S$ and hence $24 \in T$. Thus 24 is the gcd of T.

Self Practice problems:

1. What is the smallest positive integer t such that there exist integer $n_1, n_2, \ldots\ldots n_t$ with $(n_1^3 + n_2^3 + n_3^3 + \ldots\ldots + n_t^3 = 4000^{4000})$.

2. What are the last two digits of $7^{7^{7^7}}$?

Ans. 1. 4 2. 43

Fermat's theorem, Euler's theorem

Fermat's Little Theorem (FLT). If p is a prime, and p does not divide a, then $a^{p-1} \equiv 1 \pmod p$.

Euler's Theorem. If a is relatively prime to m, then $a^{\phi(m)} \equiv 1 \pmod m$.

Proof. Let $a_1, a_2, \ldots, a_{\phi(m)}$ be the positive integers less than m that are relatively prime to m. Consider the integers $aa_1, aa_2, \ldots, aa_{\phi(m)}$. We claim that they are a permutation of the original $\phi(m)$ integers a_i, modulo m. For each i, aa_i is also relatively prime to m, so $aa_i \equiv a_k$ for some k. Since $aa_i \equiv aa_j \Leftrightarrow a_i \equiv a_j \pmod m$, each a_i gets taken to a different a_k under multiplication by a, so indeed they are permuted. Hence,

$$a_1a_2 \cdots a_{\phi(m)} \equiv (aa_1)(aa_2) \cdots (aa_{\phi(m)}) \equiv a^{\phi(m)}a_1a_2 \cdots a_{\phi(m)} \Rightarrow 1 \equiv a^{\phi(m)} \pmod m$$

Wilson's Theorem : $(p-1)! \equiv -1 \pmod p$

Useful Facts :

Note that if $(p-1)! \not\equiv -1 \pmod p$ then p is not a prime, hence it is a test to determine whether p is prime or not

Example 7.

Show that if a and b are relatively prime positive integers, then there exist integers m and n such that $a^m + b^n \equiv 1 \pmod{ab}$.

Sol. Let $S = a^m + b^n$ where $m = \varphi(b)$ and $n = \phi(a)$. Then by Euler's Theorem, $S \equiv b^{\phi(a)} \equiv 1 \pmod a$,

or $S - 1 \equiv 0 \pmod a$, and $S \equiv a^{\phi(b)} \equiv 1 \pmod b$,

or $S - 1 \equiv 0 \pmod b$. Therefore, $S - 1 \equiv 0$, or $S \equiv 1 \pmod{ab}$.

Example 8.

For all positive integers i, let S_i be the sum of the products of 1, 2, . . . , p–1 taken i at a time, where p is an odd prime. Show that $S_1 \equiv S_2 \equiv \cdots \equiv S_{p-2} \equiv 0 \pmod p$.

Sol. First, observe that

$(x-1)(x-2)\cdots(x-(p-1)) = x^{p-1} - S_1x^{p-2} + S_2x^{p-3} - \cdots - S_{p-2}x + S_{p-1}$

This polynomial vanishes for $x = 1, 2, \ldots, p-1$. But by Fermat's Little Theorem, so does $x^{p-1} - 1$ modulo p. Taking the difference of these two polynomials, we obtain another polynomial of degree $p-2$ with $p-1$ roots modulo p, so it must be the zero polynomial, and the result follows from comparing coefficients.

Self Practice Problems:

1. Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.

Sol. Let p be the least prime divisor of n. Then $\gcd(n, p-1) = 1$, and by Corollary 2.2, there exist integers x and y such that $nx + (p-1)y = 1$. If $p \mid (2^n - 1)$,

then $2 \equiv 2^{nx+(p-1)y} \equiv (2^n)^x(2^{p-1})^y \equiv 1 \pmod p$ by Fermat's Little Theorem, contradiction. Therefore,

$p \nmid (2^n - 1) \quad \Rightarrow n \nmid (2^n - 1)$.

Arithmetic Functions

There are several important arithmetic functions, of which three are pre-sented here. If the prime factorization of $n > 1$ is $p_1^{e_1}p_2^{e_2}\ldots\ldots p_k^{e_k}$, then the number of positive integers less than n, relatively prime to n, the number of divisors of n is

$$\phi(n) = \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)n$$

$$p_1^{e_1-1}p_2^{e_2-1}\ldots p_k^{e_k-1}(p_1-1)(p_2-1)\ldots(p_k-1)$$

the number of divisors of n is

$\tau(n) = (e_1+1)(e_2+1)\ldots.(e_k+1)$,

and the sum of the divisors of n is

$$\sigma(n)=\left(p_1^{e_1}+p_1^{e_1-1}+....+1\right)\left(p_2^{e_2}+p_2^{e_2-1}+....+1\right)....\left(p_k^{e_k}+p_k^{e_k-1}+....+1\right)$$

$$=\left(\frac{p_1^{e_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{e_2+1}-1}{p_2-1}\right).....\left(\frac{p_k^{ek+1}-1}{p_k-1}\right)$$

Also, $\phi(1)$, $\tau(1)$, and $\sigma(1)$ are defined to be 1. We say that a function f is multiplicative if f (mn) = f (m) f (n) for all relatively prime positive integers m and n, and f (1) =1 (otherwise, f (1) = 0, which implies that f (n) = 0 for all n).

(I) The functions ϕ, τ, and σ are multiplicative.
Hence, by taking the prime factorization and evaluating at each prime power, the formula above are found easily.

(II) If (m,n)=1 then $\phi(mn)=\phi(m)\,\phi(n)$

Proof : We have to choose elements from 1 to mn which are coprime to m as well as to n

$$\text{Let } A=\begin{pmatrix} 1 & m+1 & 2m+1 & ....(n-1)m+1 \\ 2 & m+2 & 2m+2 & ....(n-1)m+2 \\ \vdots & \vdots & \vdots & \vdots \\ k & m+k & 2m+k & ....(n-1)m+k \\ \vdots & \vdots & \vdots & \vdots \\ m & 2m & 3m & ....nm \end{pmatrix}$$

No of element in k^{th} sow which are coprime to n are $\phi(n)$ as (m,n)=1
We have choosen k such that (k,m)=1 as we (By proposition(ii))
are trying to find those element in k^{th} row which are also prime to m
As (k,m)=1 so all elements of k^{th} row are coprime to m. As (k,m) = 1 so all elements of k^{th} row are coprime to m. This means all the $\phi(n)$ elements taken from k^{th} row are coprime to n & so in each row starting with
$k_1, k_2, ..., k_{\phi(m)}$ we have $\phi(n)$ elements coprime to mn
$\Rightarrow$ total $=\phi(m)\,\phi(n) \Rightarrow \phi(mn)=\phi(m)\,\phi(n)$

(iii) If $n_1, n_2, ...n_k$ are mutually prime then $\phi(n_1n_2....n_k)=\phi(n_1)\,\phi(n_2)\,....\,\phi(n_k)$

(iv) If $n=p_1^{a_1}p_2^{a_2}.....p_k^{a_k}$ is unique prime factorisation of n then $\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)....\left(1-\frac{1}{p_k}\right)$

Proof : $\phi(n)=\phi\left(p_1^a\ \ p_2^b\right)=\phi\left(p_1^a\right)\phi\left(p_2^b\right)=\left(p_1^a-\ p_1^{a-1}\right)\left(p_2^b-\ p_2^{b-1}\right)=p_1^ap_2^b\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)$

(v) If p is prime then $\phi(p^k)=p^k-p^{k-1}$ where $k\in N$
Proof : It is true for k=1
For k>1, numbers from 1 to p^k which are not prime to p^k are p(1), p(2), ...,$p(p^{k-1})$. These are p^{k-1} in number

$\therefore \phi(p^k)=p^k-p^{k-1}=p^k\left(1-\frac{1}{p}\right)$

Example 9.
Find number of natural numbers less than 10^7 which have exactly 77 divisors.

Ans. 2

Sol. $N=2^{10}3^6$ has 77 divisors and it is less than 10^7
$N=2^63^{10}$ has 77 divisors and it is less than 10^7

Exercise-1

PART - I

1. When the tens digit of a three digit number abc is deleted, a two digit number ac is formed. How many numbers abc are there such that abc=9ac + 4c.

2. Consider two positive integer a and b. Find the least possible value of the product ab if $a^b b^a$ is divisible by 2000

3. Find all solutions to $aabb = n^4 - 6n^3$, where a and b are non-zero digits, and n is an integer (a and b are not necessarily distinct).

4. Find the least possible value of a + b, where a, b are positive integers such that 11 divides a + 13b and 13 divides a + 11b.

5. The sum of all three digit numbers each of which is equal to 11 times the sum of the squares of its digits is λ. Find the sum of digits of λ.

6. N is 50 digit number (in decimal form). All digits except the 26th digit (from left) are 1. If N is divisible by 13, find the 26th digit.

7. Find remainder when 4444^{4444} is divided by 9

8. How many positive integers appear in the list $\left\lfloor \frac{2006}{1} \right\rfloor, \left\lfloor \frac{2006}{2} \right\rfloor, \ldots, \left\lfloor \frac{2006}{2006} \right\rfloor$ where $\lfloor \chi \rfloor$ represents the greatest integer that does not exceed χ?

9. Find number of positive integer less than 2431 and prime to 2431.

10. Find the smallest natural number n which has last digit 6 & if this last is moved to the front of the number, the number becomes 4 times larger.

11. Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, $n \in I$?

12. Let A be the sum of the digits of the number $(4444)^{4444}$ and B be the sum of the digits of the number A. Find the sum of the digits of the number B.

13. For how many integers n is $\sqrt{9-(n+2)^2}$ a real number?

14. The number of prime numbers less than 1 million whose digital sum is 2 is:

15. An eight digit number is a multiple of 73 and 137. If the second digit from left is 7, what is the 6th digit from the left of the number?

16. The number of natural numbers n for which $\frac{15n^2+8n+6}{n}$ is a natural number is :

17. Let A be the least number such that 10A is a perfect square and 35 A is perfect cube. Then the number of positive divisors of A is :

18. The number of 2 digit numbers having exactly 6 factors is :

19. The number of positive integers 'n' for which $3n-4$, $4n-5$ and $5n-3$ are all primes is :

20. The number of positive integral values of n for which $(n^3 - 8n^2 + 20n - 13)$ is a prime number is :

21. a, b, c are digits of a 3-digit number such that $64a + 8b + c = 403$, then the value of $a + b + c + 2013$ is

22. N is a five digit number. 1 is written after the 5 digit of N to make it a six digit number, which is three times the same number with 1 written before N. (If N = 23456 it means 234561 and 123456) . Then the middle digit of the number N is

23. The sum of all values of integers n for which $\frac{n^2-9}{n-1}$ is also an integer is

24. The number of natural number pairs (x, y) in which $x > y$ and $\frac{5}{x}+\frac{6}{y} = 1$ is :

25. The number of positive integer pairs (a, b) such that $ab - 24 = 2b$ is

26. $A = (2 + 1)(2^2 + 1)(2^4 + 1) \ldots\ldots (2^{2048} + 1)$. The value of $(A + 1)^{1/2048}$ is

27. The least positive integer n such that $2015^n + 2016^n + 2017^n$ is divisible by 10 is

28. The number of pairs of relatively prime positive integers (a, b) such that $\frac{a}{b}+\frac{15b}{4a}$ is an integer is

29. The four digit number 8ab9 is a perfect square. The value of $a^2 + b^2$ is

30. a, b are positive real numbers such that $\frac{1}{a}+\frac{9}{b}=1$. The smallest value of a + b is

PART - II

1. Prove that there exist infinitely many natural numbers 'a' with the property that the number $p = n^4 + a$ is not prime for any natural number n.

2. For what values of natural numbers n can the product of the numbers n, n + 1, n + 2, n + 4, n + 5 be equal to the product of remaining ones?

3. Determine all pairs (x, y) of positive integers satisfying the equation $1 + 2^x + 2^{2x+1} = y^2$.

4. Let a and b be two positive rational numbers such that $(a)^{1/3} + (b)^{1/3}$ is also a rational number. Prove that $(a)^{1/3}$, $(b)^{1/3}$ themselves are rational numbers.

5. Prove that the only solution in rational numbers of the equation $x^3 + 3y^3 + 9z^3 - 9xyz = 0$ is $x = y = z = 0$.

6. If a, b, x, y are integers greater than 1 such that a and b have no common factor except 1 and $x^a = y^b$ show that $x = n^b$, $y = n^a$ for some integer $x > 1$.

7. Show that there are an infinite number of primes of the form 4k + 1 and of the form 4k + 3.

8. A natural number n is said to have the property P if whenever n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$.
(a) Show that every prime number has property P.
(b) Show that there are infinitely many composite numbers 'n' that possess property P.

9. Find all natural numbers n for which every natural number whose decimal representation has n – 1 digits 1 and one digit 7 is prime.

10. Prove that there are infinitely many positive integers n such that n(n + 1) can be expressed as a sum of two positive squares in at least two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

11. Let $0 < a_1 < a_2 < \cdots < a_{mn+1}$ be mn + 1 integers. Prove that you can select either m + 1 of them no one of which divides any other, or n + 1 of them each dividing the following one.

12. Let m and b be non-negative integers. Prove that m! n!(m + n)! divides (2m)!(2n)!

13. For every integer n prove that the fraction $\frac{25n+3}{15n+2}$ cannot be further reduced.

14. Let n be a positive integer and $p_1, p_2 \ldots\ldots p_n$ be n prime numbers all larger than 5 such that 6 divides $(p_1^2 + p_2^2 + p_3^2 \ldots\ldots + p_n^2)$. Prove that 6 divides n.

15. Prove that $2^{20} - 1$ is divisible by 41

16. Prove that $53^{103} + 103^{53}$ is divisible by 39.

17. Let $< p_1, p_2, \ldots p_n, \ldots >$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1$, p_{n+1} is the largest prime factor of $p_1 p_2 \ldots p_n + 1$ (Thus $p_2 = 3$, $p_3 = 7$). Prove that $p_n \neq 5$ for any n.

18. Find the number of solutions in ordered pairs of positive integers (x, y) of the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$, where n is a positive integer.

19. Prove that $19^{93} - 13^{99}$ is positive integer divisible by 162.

20. Show that product of any n consecutive integers is always divisible by n!

Exercise-2

PART - I

1. If $\frac{1}{\sqrt{2011+\sqrt{2011^2-1}}} = \sqrt{m} - \sqrt{n}$ where m and n are positive integers, what is the value of m + n.

[PRMO 2012]

2. Let p(n) = (n+1) (n+3) (n+5) (n+7) (n+9). What is the largest integer that is a divisor of p(n) for all positive even integers n ? **[PRMO 2012]**

3. How many non-negative integral values of x satisfy the equation

$\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]$? (Here [x] denotes the greatest integer less than or equal to x.

(For example [3.4] = 3 and [–2.3] = – 3). **[PRMO 2012]**

4. What is the sum of the squares of the roots of the equation $x^2 - 7[x] + 5 = 0$?
(Here [x] denotes the greatest integer less than or equal to x. For example [3.4] = 3 and [–2.3] = – 3). **[PRMO 2012]**

5. Let S(M) denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that S(N) = 2017. Find the value of S(5N + 2017) **[PRMO 2013]**

6. Let Akbar and Birbal together have n marbles where n > 0. Akbar says to Birbal "If I give you some marbles then you will have twice as many marbles as I will have".Birbal says to Akbar" If I give you some marbles then you will have thrice as many marbles as I will have" What is the minimum possible value of n for which the above statements are true ? **[PRMO 2013]**

7. To each element of the set S = {1, 2, 3.....1000}, a colour is assigned. Suppose that for any two elements a, b of S, if 15 divides a + b they are both assigned the same colour. What is the maximum possible number of distinct colours used **[PRMO 2013]**

8. What is the smallest positive integer k such that $k(3^3 + 4^3 + 5^3 + 6^3) = a^n$ for some positive integer a and n with n > 1 ? **[PRMO 2013]**

9. Let f be a one-to-one function from the set of natural number to itself such that f(mn) = f(m) f(n) for all natural numbers m and n. What is the least possible value of f(999) ? **[PRMO 2014]**

10. One morning, each member of manjul's family drank an 8-ounce mixture of coffee and milk. The amount of coffee and milk varied from cup to cup but never zero. Manjul drank $\left(\frac{1}{7}\right)^{th}$ of the total amount of milk and $\left(\frac{2}{7}\right)^{th}$ of the total amount of coffee. How many people are there in manjul's family ?

[PRMO 2014]

11. For how many natural numbers n between 1 and 2014 (both inclusive) is $\frac{8n}{9999-n}$ an integer ?

[PRMO 2014]

12. For natural numbers x and y, let (x,y) denote the gcd of x and y. How many pairs of natural numbers x and y with $x \le y$ satisfy equation xy = x + y + (x,y) ? **[PRMO 2014]**

13. Suppose f is a quadratic polynomial ie. a polynomial of degree 2 with leading coefficient 1 such that $f(f(x)+x) = f(x)\,(x^2 + 786x + 439)$ for all real numbers x. What is the value of f(3) ? **[PRMO 2015]**

14. What is the greatest possible perimeter of a right angled triangle with integer side lengths if one of the sides has length 12? **[PRMO 2015]**

15. Positive integers a and b are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$? [PRMO 2015]

16. Let n be the largest integer that is the product of exactly 3 distinct prime numbers x, y and 10x + y where x and y are digits. What is the sum of digits of n ? [PRMO 2015]

17. The digits of a positive integer n are four consecutive integers in decreasing order when read from left to right . What is the sum of the possible remainders when n is divided by 37 ? [PRMO 2015]

18. For positive integers m and n, let gcd (m,n) denote the largest integer that is a factor of both m and n. Find gcd (2015! + 1, 2016! + 1) where n! denotes the factorial of a positive integer n. [PRMO 2015]

19. Find the total number of solutions to the equation $x^2 + y^2 = 2015$ where both x and y are integers. [PRMO 2015]

20. For positive integers m and n, let gcd(m, n) denote the largest integer that is a factor of both m and n. Find the sum of all possible values of $\gcd(a - 1, a^2 + a + 1)$ where a is a positive integer. [PRMO 2015]

21. Let n!, the factorial of a positive integer n, be defined as the product of the integers 1, 2 ...., n. In words, $n! = 1\times2\times...\times n$. What is the number of zeros at the end of the integer $10^2! + 11^2! + 12^2! + ...+99^2!$ [PRMO 2015]

22. a, b, c, d are integers such that ad + bc divides each of a, b, c and d.Prove that ad + bc = ± 1 [PRMO 2016]

23. At some integer points a polynomial with integer coefficients take values 1, 2 and 3. Prove that there exist not more than one integer at which the polynomial is equal to 5. [PRMO 2016]

24. The five digit number 2a9b1 is a perfect square. Find the value of $a^{b-1} + b^{a-1}$. [PRMO 2016]

25. Find the number of integer solutions of $\left[\frac{x}{100}\left[\frac{x}{100}\right]\right] = 5$ (Here [x] denotes the greatest integer less than or equal to x. (For example [3.4] = 3 and [–2.3] = – 3). [PRMO 2016]

26. Integers 1,2,3........n where n > 2, are written on a board. Two numbers m, k such that 1 < m < n, 1 < k < n are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers ? [PRMO 2017]

27. Suppose an integer r, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p. [PRMO 2017]

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of 3pq for all positive integers n. Find the least possible value of p + q. [PRMO 2017]

29. For each positive integer n, consider the highest common factor h_n of the two numbers n! + 1 and (n + 1)!. For n < 100, find the largest value of h_n. [PRMO 2017]

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium ABCD. The product of these areas, taken two at time, are computed. If among the six products so obtained, two product are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer. [PRMO 2017]

31. If a,b,c $\geq$ 4 are integers, not all equal, and 4abc = (a + 3) (b + 3) (c + 3), then what is the value of a + b + c ? **[PRMO 2018]**

32. Let a and b natural numbers such that 2a – b, a – 2b and a + b are all distinct squares. What is the smallest possible value of b ? **[PRMO 2018]**

33. Consider all 6-digit numbers of the form **abccba** where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7. **[PRMO 2018]**

34. The equation 166 × 56 = 8590 is valid in some base b $\geq$ 10 (that is 1,6,5,8,9,0 are digits in base b in the above equation). Find the sum of all possible values of b $\geq$ 10 satisfying the equation. **[PRMO 2018]**

35. What is the value of $\sum\limits_{\substack{1\le i<j\le 10\\ i+j=\text{odd}}} (i+j) - \sum\limits_{\substack{1\le i<j\le 10\\ i+j=\text{even}}} (i+j)$? **[PRMO 2018]**

36. Let N = 6 + 66 + 666 + ... + 666....66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N ? **[PRMO 2018]**

37. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$. **[PRMO 2018]**

PART - II

1. Show that there is no integer 'a' such that $a^2 - 3a - 19$ is divisible by 289. **[RMO 2009]**

2. A natural number n is chosen strictly between two consecutive perfect square. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding ℓ to n. Prove that $n - k\ell$ is a perfect square. **[RMO-2011]**

3. Prove that for all positive integers n, 169 divides $21n^2 + 89n + 44$ if 13 divides $n^2 + 3n + 51$. **[RMO-2012]**

4. Determine with proof all triples (a, b, c) of positive integers satisfying $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$, where a is a prime number and $a \leq b \leq c$. **[RMO-2012]**

5. Let a,b,c be positive integer such that a divides b^3, b divides c^3 and c divides a^3. Prove that abc divides $(a + b + c)^{13}$. **[RMO-2012]**

6. Find all triples (p, q, r) of primes such that pq = r + 1 and $2(p^2 + q^2) = r^2 + 1$. **[RMO-2013]**

7. Let a_1, b_1, c_1 be natural numbers. We define **[RMO-2013]**
$a_2 = \gcd(b_1, c_1)$, $b_2 = \gcd(c_1, a_1)$, $c_2 = \gcd(a_1, b_1)$,
and
$a_3 = \text{lcm}(b_2, c_2)$, $b_3 = \text{lcm}(c_2, a_2)$, $c_3 = \text{lcm}(a_2, b_2)$.
Show that $\gcd(b_3, c_3) = a_2$.

8. For any natural number n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$. **[RMO-2016]**

9. For any positive integer n, let d(n) denotes the number of positive divisors of n; and let $\phi(n)$ denote the number of elements from the set {1, 2,..... n} that are coprime to n. **[RMO-2017]**
(For example, d(12) =6 and $\phi(12) = 4$.)
Find the smallest positive integer n such that $d(\phi(n)) = 2017$.

10. Show that there are infinitely many tuples (a, b, c, d) of natural numbers such that $a^3 + b^4 + c^5 = d^7$ **[RMO-2018]**

11. Find all natural numbers n such that $1+\left[\sqrt{2n}\right]$ divides 2n. **[RMO-2018]**

12. For a rational number r, its *period* is the length of the smallest repeating block in its decimal expansion. for example, the number r = 0.123123123.... has period 3. If S denotes the set of all rational numbers of the form r = abcdefgh having period 8, find the sum of all elements in S. **[RMO-2018]**

Answers

Exercise-1

PART - I

1.	6	**2.**	20	**3.**	a = 6, b = 5	**4.**	28	**5.**	12
6.	3	**7.**	7	**8.**	88	**9.**	1920		
10.	153846	**11.**	No	**12.**	7	**13.**	7	**14.**	3
15.	7	**16.**	4	**17.**	72	**18.**	16	**19.**	1
20.	3	**21.**	2024	**22.**	8	**23.**	8	**24.**	3
25.	8	**26.**	4	**27.**	2	**28.**	4	**29.**	52
30.	16								

PART - II

2. No value **3.** (4, 23) **9.** n = 1, 2

18. $(2e_1 + 1)(2e_2 + 1) \cdots (2e_k + 1)$, where $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$

Exercise-2

PART - I

1.	2011	**2.**	15	**3.**	9	**4.**	69	**5.**	6
6.	12	**7.**	8	**8.**	3	**9.**	24	**10.**	8
11.	1	**12.**	3	**13.**	2015	**14.**	84	**15.**	2
16.	12	**17.**	217	**18.**	1	**19.**	0	**20.**	4
21.	24	**23.**	15	**24.**	21	**25.**	50	**26.**	51
27.	47	**28.**	28	**29.**	97	**30.**	13	**31.**	16
32.	21	**33.**	70	**34.**	12	**35.**	55	**36.**	33
37.	17								

PART - II

4. (2,5,30) (2,6,18)(2,7,14) (2,8,12) (2,10,10) (3,4,6) (3,6,9) **6.** (2, 3, 5), (3, 2, 5)

8. 50, 162, 392, 648 **9.** 2^{2017}

Equations

Equations

Intervals :

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers a, b $\in$ R such that a < b, we can define four types of intervals as follows :

Name	Representation	Discription
Open Interval	(a, b)	{x : a < x < b} i.e. end points are not included.
Close Interval	[a, b]	{x : a ≤ x ≤ b} i.e. end points are also included. This is possible only when both a and b are finite.
Open - Closed Interval	(a, b]	{x : a < x ≤ b} i.e. a is excluded and b is included.
Close - Open Interval	[a, b)	{x : a ≤ x < b} i.e. a is included and b is excluded.

Note : **(1)** The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$ **(ii)** $[a, \infty) = \{x : x \geq a\}$

(iii) $(-\infty, b) = \{x : x < b\}$ **(iv)** $(-\infty, b] = \{x : x \leq b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

(2) $x \in \{1, 2\}$ denotes some particular values of x, i.e. x = 1, 2

(3) If there is no value of x, then we say $x \in \phi$ (null set)

General Method to solve Inequalities :

(Method of intervals (Wavy curve method)

Let $$g(x) = \left(\frac{(x-b_1)^{k_1}(x-b_2)^{k_2} ---(x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2} ---(x-a_n)^{r_n}}\right) \quad \text{... (i)}$$

Where k_1, k_2 .....k_n and r_1, r_2 ....r_n $\in$ N and b_1, b_2.....b_n and a_1, a_2 ......a_n are real numbers.

Then to solve the inequality following steps are taken.

Steps : -

Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(i) First we find the zeros and poles of the function.

(ii) Then we mark all the zeros and poles on the real line and put a vertical bar there dividing the real line in many intervals.

(iii) Determine sign of the function in any of the interval and then alternates the sign in the neighboring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

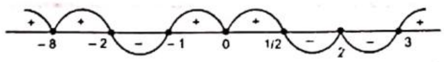
(iv) Thus we consider all the intervals. The solution of the g(x) > 0 is the union of the intervals in which we have put the plus sign and the solution of g(x) < 0 is the union of all intervals in which we have put the minus sign.

Example# 1 :

Solve the inequality if $f(x) = \dfrac{(x-2)^{10}(x+1)^3\left(x-\frac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ is > 0 or < 0.

Solution.

Let $f(x) = \dfrac{(x-2)^{10}(x+1)^3\left(x-\frac{1}{2}\right)^5(x+8)^2}{x^{24}(x-3)^3(x+2)^5}$ the poles and zeros are 0, 3, – 2, – 1, $\frac{1}{2}$, – 8, 2



If $f(x) > 0$, then $x \in (-\infty, -8) \cup (-8, -2) \cup (-1, 0) \cup \left(0, \frac{1}{2}\right) \cup (3, \infty)$

and if $f(x) < 0$, then $x \in (-2, -1) \cup \left(\frac{1}{2}, 2\right) \cup (2, 3)$ **Ans.**

1. Polynomial :

A function f defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

where $a_0, a_1, a_2, ..., a_n \in R$ is called a polynomial of degree n with real coefficients ($a_n \neq 0$, $n \in W$).

If $a_0, a_1, a_2, ..., a_n \in C$, it is called a polynomial with complex coefficients.

2. Quadratic polynomial & Quadratic equation :

A polynomial of degree 2 is known as quadratic polynomial. Any equation $f(x) = 0$, where f is a quadratic polynomial, is called a quadratic equation. The general form of a quadratic equation is

$ax^2 + bx + c = 0$ ....(i)

Where a, b, c are real numbers, $a \neq 0$.

If $a = 0$, then equation (i) becomes linear equation.

3. Difference between equation & identity :

If a statement is true for all the values of the variable, such statements are called as identities. If the statement is true for some or no values of the variable, such statements are called as equations.

Example : (i) $(x + 3)^2 = x^2 + 6x + 9$ is an identity

(ii) $(x + 3)^2 = x^2 + 6x + 8$, is an equation having no root.

(iii) $(x + 3)^2 = x^2 + 5x + 8$, is an equation having -1 as its root.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

$ax^2 + bx + c = 0$ is:

★	a quadratic equation if	$a \neq 0$	Two Roots
★	a linear equation if	$a = 0, b \neq 0$	One Root
★	a contradiction if	$a = b = 0, c \neq 0$	No Root
★	an identity if	$a = b = c = 0$	Infinite Roots

If $ax^2 + bx + c = 0$ is satisfied by three distinct values of 'x', then it is an identity.

Example # 2 :

$(a^2 - 1)x^2 + (a^2 + 5a + 4)x + (a^2 + 4a + 3) = 0$. Find the value of a for which

(i) equation has two roots

(ii) Equation has one real root

(iii) It is an identity

Solution :

(i) $a \neq \pm 1$ **(ii)** $a = 1$ **(iii)** $a = -1$

4. Relation Between Roots & Coefficients:

(i) The solutions of quadratic equation, $ax^2 + bx + c = 0$, $(a \neq 0)$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression, $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

(ii) If α, β are the roots of quadratic equation,

$ax^2 + bx + c = 0$ ....(i)

then equation (i) can be written as

$a(x - \alpha)(x - \beta) = 0$

or $\quad ax^2 - a(\alpha + \beta)x + a\,\alpha\beta = 0 \quad$ ......(ii)

equations (i) and (ii) are identical,

$\therefore$ by comparing the coefficients sum of the roots, $\alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

and product of the roots, $\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

(iii) Dividing the equation (i) by a, $\quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\Rightarrow \quad x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} = 0 \quad \Rightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow \quad x^2$ – (sum of the roots) x + (product of the roots) = 0

Hence we conclude that the quadratic equation whose roots are α & β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Example # 3 :

If α and β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha+2$ and $\beta+2$.

Solution :

Replacing x by x – 2 in the given equation, the required equation is

$a(x-2)^2 + b(x-2) + c = 0 \quad$ i.e., $\quad ax^2 - (4a - b)x + (4a - 2b + c) = 0$.

Example # 4 :

The coefficient of x in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be – 2 and – 15. Find the roots of the original equation.

Solution :

Here $q = (-2) \times (-15) = 30$, correct value of p = 13. Hence original equation is

$x^2 + 13x + 30 = 0$ as $(x + 10)(x + 3) = 0$

$\therefore \quad$ roots are $-10, -3$

Example # 5 :

If a is a root of $x^2 - 3x - 5 = 0$ find the value of $a^4 - 2a^3 - 7a^2 - 8a$

Solution :

Note that given expression is $(a^2 - 3a - 5)(a^2 + a + 1) + 5$ hence value of the expression is 5.

Self practice problems :

(1) If α, β are the roots of the quadratic equation $cx^2 - 2bx + 4a = 0$ then find the quadratic equation whose roots are

(i) $\frac{\alpha}{2}, \frac{\beta}{2}$ (ii) α^2, β^2 (iii) $\alpha + 1, \beta + 1$ (iv) $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}$ (v) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

(2) If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that $\frac{(r+1)^2}{r} = \frac{b^2}{ac}$.

(3) If roots of the equation $x^2 - 10ax - 11b = 0$ are c and d and those of $x^2 - 10cx - 11d = 0$ are a and b, then find the value of a + b + c + d. (where a, b, c, d are all distinct numbers)

Answers :

(1) (i) $cx^2 - bx + a = 0$ (ii) $c^2x^2 + 4(b^2 - 2ac)x + 16a^2 = 0$

(iii) $cx^2 - 2x(b + c) + (4a + 2b + c) = 0$ (iv) $(c - 2b + 4a)x^2 + 2(4a - c)x + (c + 2b + 4a) = 0$

(v) $4acx^2 + 4(b^2 - 2ac)x + 4ac = 0$

(3) 1210

5. Nature of Roots:

Consider the quadratic equation, $ax^2 + bx + c = 0$ having α, β as its roots;

$D \equiv b^2 - 4ac$

$D = 0$
Roots are equal i.e. $\alpha = \beta = -b/2a$
& the quadratic expression can be expressed
as a perfect square of a linear polynomial

$D \neq 0$
Roots are unequal

$a, b, c \in R$ & $D > 0$
Roots are real

$a, b, c \in R$ & $D < 0$
Roots are imaginary $\alpha = p + iq$, $\beta = p - iq$

$a, b, c \in Q$ &
D is square of a rational number
$\Rightarrow$ Roots are rational
$\downarrow$
$a = 1$, $b, c \in I$ & D is square of an integer
$\Rightarrow$ Roots are integral.

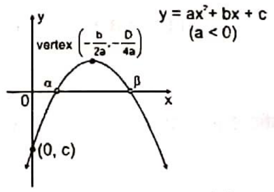
$a, b, c \in Q$ &
D is not square of a rational number
$\Rightarrow$ Roots are irrational
i.e. $\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$

6. Graph of Quadratic Expression :

★ the graph between x, y is always a parabola.

★ the co–ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

★ If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.



★ the parabola intersect the y-axis at point (0, c).

★ the x–coordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation f (x) = 0. Hence the parabola may or may not intersect the x–axis.

Example # 6 :

For what values of m the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Solution :

Given equation is $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ ....(i)

Let D be the discriminant of equation (i).

Roots of equation (i) will be equal if D = 0.

or $4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$

or $4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

or $m^2 - 3m = 0$ or, $m(m - 3) = 0$

$\therefore$ $m = 0, 3$.

Example # 7 : Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution : Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \quad \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$

Example # 8 : If the roots of the equation $(x - a)(x - b) - k = 0$ be c and d, then prove that the roots of the equation $(x - c)(x - d) + k = 0$, are a and b.

Solution : By given condition $(x - a)(x - b) - k \equiv (x - c)(x - d)$

or $(x - c)(x - d) + k \equiv (x - a)(x - b)$

Above shows that the roots of $(x - c)(x - d) + k = 0$ are a and b.

Example # 9 : Determine 'a' such that $x^2 - 11x + a$ and $x^2 - 14x + 2a$ may have a common factor.

Solution : Let $x - \alpha$ be a common factor of $x^2 - 11x + a$ and $x^2 - 14x + 2a$.

Then $x = \alpha$ will satisfy the equations $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$.

$\therefore \quad \alpha^2 - 11\alpha + a = 0$ and $\quad \alpha^2 - 14\alpha + 2a = 0$

Solving (i) and (ii) by cross multiplication method, we get a = 0, 24.

Example # 10 : Show that the expression $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$ will be a perfect square if a = b = c.

Solution : Given quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero.

i.e. $\quad 4(a + b + c)^2 - 4.3(bc + ca + ab) = 0$

or $\quad (a + b + c)^2 - 3(bc + ca + ab) = 0$

or $\quad \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2) = 0$

which is possible only when a = b = c.

Self practice problems :

(1) For what values of 'k' the expression $(4 - k)x^2 + 2(k + 2)x + 8k + 1$ will be a perfect square ?

(2) If $(x - \alpha)$ be a factor common to $a_1x^2 + b_1x + c$ and $a_2x^2 + b_2x + c$, then prove that $\alpha(a_1 - a_2) = b_2 - b_1$.

(3) If $3x^2 + 2\alpha xy + 2y^2 + 2ax - 4y + 1$ can be resolved into two linear factors, Prove that 'α' is a root of the equation $x^2 + 4ax + 2a^2 + 6 = 0$.

(4) Let $4x^2 - 4(\alpha - 2)x + \alpha - 2 = 0$ ($\alpha \in R$) be a quadratic equation. Find the values of 'α' for which

(i) Both roots are real and distinct.
(ii) Both roots are equal.
(iii) Both roots are imaginary
(iv) Both roots are opposite in sign.
(v) Both roots are equal in magnitude but opposite in sign.

(5) If $P(x) = ax^2 + bx + c$, and $Q(x) = -ax^2 + dx + c$, $ac \neq 0$ then prove that $P(x) \cdot Q(x) = 0$ has atleast two real roots.

Answers.

(1) 0, 3

(4) **(i)** $(-\infty, 2) \cup (3, \infty)$ **(ii)** $\alpha \in \{2, 3\}$ **(iii)** $(2, 3)$ **(iv)** $(-\infty, 2)$ **(v)** ϕ

7. Common Roots:

Consider two quadratic equations, $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

(i) If two quadratic equations have both roots common, then the equations are identical and their co-efficient are in proportion.

i.e. $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

(ii) If only one root is common, then the common root 'α' will be :

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

Hence the condition for one common root is :

$$\Rightarrow \quad (c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$$

Note : If $f(x) = 0$ & $g(x) = 0$ are two polynomial equation having some common root(s) then those common root(s) is/are also the root(s) of $h(x) \equiv a\,f(x) + bg\,(x) = 0$.

Example # 11 :

If $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a root in common and the second equation has equal roots, show that $b + q = \dfrac{ap}{2}$.

Solution :

Given equations are : $x^2 - ax + b = 0$ .... (i)

and $x^2 - px + q = 0.$ .... (ii)

Let α be the common root. Then roots of equation (ii) will be α and α. Let β be the other root of equation (i). Thus roots of equation (i) are α, β and those of equation (ii) are α, α.

Now $\alpha + \beta = a$ .... (iii)

$\alpha\beta = b$ .... (iv)

$2\alpha = p$ .... (v)

$\alpha^2 = q$ .... (vi)

L.H.S. $= b + q = \alpha\beta + \alpha^2 = \alpha(\alpha + \beta)$ .... (vii)

and R.H.S. $= \dfrac{ap}{2} = \dfrac{(\alpha+\beta)\,2\alpha}{2} = \alpha\,(\alpha + \beta)$ .... (viii)

from (vii) and (viii), L.H.S. = R.H.S.

Example # 12 :

If a, b, c $\in$ R and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, show that $a : b : c = 1 : 2 : 9$.

Solution :

Given equations are : $x^2 + 2x + 9 = 0$ ....(i)

and $ax^2 + bx + c = 0$ ....(ii)

Clearly roots of equation (i) are imaginary since equation (i) and (ii) have a common root, therefore common root must be imaginary and hence both roots will be common.

Therefore equations (i) and (ii) are identical

$\therefore \quad \dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{9}$

$\therefore \quad a : b : c = 1 : 2 : 9$

Self practice problems:

(1) If the equations $ax^2 + bx + c = 0$ and $x^3 + x - 2 = 0$ have two common roots then show that $2a = 2b = c$.

(2) If $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have a common root and $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in A.P. show that a_1, b_1, c_1 are in G.P.

8. Theory Of Equations :

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\alpha_n$ are the roots of the equation;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots. + a_{n-1}x + a_n = 0$ where $a_0, a_1, \ldots., a_n$ are all real & $a_0 \neq 0$ then,

$\Sigma\alpha_1 = -\frac{a_1}{a_0}, \Sigma\alpha_1\alpha_2 = +\frac{a_2}{a_0}, \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \ldots\ldots, \alpha_1\alpha_2\alpha_3\ldots\ldots.\alpha_n = (-1)^n\frac{a_n}{a_0}$

Note :

(i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.

(ii) Every equation of n^{th} degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

(iv) An equation of odd degree will have odd number of real roots and an equation of even degree will have even numbers of real roots.

(v) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not square of a rational number.

(vi) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have odd number of real roots (also atleast one real root) between 'a' and 'b'.

(vii) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term. (If coefficient of highest degree term is positive).

Example # 13 :

If $2x^3 + 3x^2 + 5x + 6 = 0$ has roots α, β, γ then find $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

Solution :

Using relation between roots and coefficients, we get

$\alpha + \beta + \gamma = = -\frac{3}{2}$, $\quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{5}{2}$, $\quad \alpha\beta\gamma = -\frac{6}{2} = -3$.

Self practice problems :

(1) If $2p^3 - 9pq + 27r = 0$ then prove that the roots of the equations $rx^3 - qx^2 + px - 1 = 0$ are in H.P.

(2) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ then find the equation whose roots are

(a) $2\alpha + 2\beta + \gamma, \alpha + 2\beta + 2\gamma, 2\alpha + \beta + 2\gamma$ (b) $-\frac{r}{\alpha}, -\frac{r}{\beta}, -\frac{r}{\gamma}$

(c) $(\alpha + \beta)^2, (\beta + \gamma)^2, (\gamma + \alpha)^2$ (d) $-\alpha^3, -\beta^3, -\gamma^3$

Answers :

(2) (a) $x^3 + qx - r = 0$ (b) $x^3 - qx^2 - r^2 = 0$

(c) $x^3 + 2qx^2 + q^2x - r^2 = 0$ (d) $x^3 - 3x^2r + (3r^2 + q^3)x - r^3 = 0$

Example 14 :

Let α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 1 = 0$. If the value of

$$\left(\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{1}{\alpha^3}\right)\left(\frac{1}{\alpha^3}+\frac{1}{\gamma^3}-\frac{1}{\beta^3}\right)+\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}\right)\left(\frac{1}{\alpha^3}+\frac{1}{\gamma^3}-\frac{1}{\beta^3}\right)+\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}\right)\left(\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{1}{\alpha^3}\right)$$ is

λ. Then find the sum of digit of $|\lambda|$

Solution :

$x^3 + 2x^2 + 3x + 1 = 0$ with roots α, β, γ

find the equation whose roots are $\alpha^3, \beta^3, \gamma^3$

$x^3 - 7x^2 + 12x + 1 = 0$

change $x \rightarrow \dfrac{1}{x}$

$x^3 + 12x^2 - 7x + 1 = 0$ with roots $\dfrac{1}{\alpha^3} = a$, $\dfrac{1}{\beta^3} = b$, $\dfrac{1}{\gamma^3} = c$

$a + b + c = -12$

roots are $b + c - a,\ a + c - b,\ a + b - c$

$= -12 - 2a,\ -12 - 2b,\ -12 - 2c$

equation is

$$-\frac{(12+x)^3}{8} + 12\frac{(12+x)^2}{4} + \frac{7(12+x)}{2} + 11 = 0$$

$x^3 + 12x^2 - 172x - 2152 = 0$

$|\lambda| = 172$

9. Range of Quadratic Expression $f(x) = ax^2 + bx + c$.

(i) Range :

If $a > 0 \quad \Rightarrow \quad f(x) \in \left[-\dfrac{D}{4a}, \infty\right)$

If $a < 0 \quad \Rightarrow \quad f(x) \in \left(-\infty, -\dfrac{D}{4a}\right]$

Hence maximum and minimum values of the expression f (x) is $-\dfrac{D}{4a}$ in respective cases and it occurs at $x = -\dfrac{b}{2a}$ (at vertex).

(ii) Range in restricted domain:

Given $x \in [x_1, x_2]$

(a) If $-\dfrac{b}{2a} \notin [x_1, x_2]$ then,

$f(x) \in [\min\{f(x_1), f(x_2)\},\ \max\{f(x_1), f(x_2)\}]$

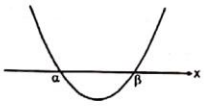
(b) If $-\dfrac{b}{2a} \in [x_1, x_2]$ then,

$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\},\ \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

10. Sign of Quadratic Expressions :

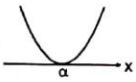
The value of expression $f(x) = ax^2 + bx + c$ at $x = x_0$ is equal to y–co–ordinate of the point on parabola $y = ax^2 + bx + c$ whose x–co–ordinate is x_0. Hence if the point lies above the x–axis for some $x = x_0$, then $f(x_0) > 0$ and vice–versa.

We get six different positions of the graph with respect to x–axis as shown.

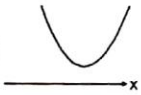
(i) 

Conclusions :

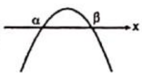
(a) $a > 0$
(b) $D > 0$
(c) Roots are real & distinct.
(d) $f(x) > 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$
(e) $f(x) < 0$ in $x \in (\alpha, \beta)$

(ii) 

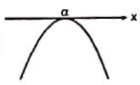
(a) $a > 0$
(b) $D = 0$
(c) Roots are real & equal.
(d) $f(x) > 0$ in $x \in R - \{\alpha\}$

(iii) 

(a) $a > 0$
(b) $D < 0$
(c) Roots are imaginary.
(d) $f(x) > 0 \ \forall \ x \in R$.

(iv) 

(a) $a < 0$
(b) $D > 0$
(c) Roots are real & distinct.
(d) $f(x) < 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$
(e) $f(x) > 0$ in $x \in (\alpha, \beta)$

(v) 

(a) $a < 0$
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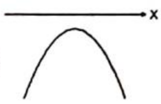
(c) Roots are real & equal.
(d) $f(x) < 0$ in $x \in R - \{\alpha\}$

(vi) 

(a) $a < 0$
(b) $D < 0$
(c) Roots are imaginary.
(d) $f(x) < 0 \ \forall \ x \in R$.

Example # 15: If $c < 0$ and $ax^2 + bx + c = 0$ does not have any real roots then prove that
(i) $a - b + c < 0$ **(ii)** $9a + 3b + c < 0$.

Solution : $c < 0$ and $D < 0$ $\Rightarrow$ $f(x) = ax^2 + bx + c < 0$ for all $x \in R$
$\Rightarrow$ $f(-1) = a - b + c < 0$
and $f(3) = 9a + 3b + c < 0$

Example # 16: Find the range of $f(x) = x^2 - 5x + 6$.

Solution : minimum of $f(x) = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$ $= -\left(\frac{25-24}{4}\right)$ at $x = \frac{5}{2}$ $= -\frac{1}{4}$
maximum of $f(x) \to \infty$
Hence range is $\left[-\frac{1}{4}, \infty\right)$

Example#17 : Find the range of rational expression $y = \frac{x^2 - x + 4}{x^2 + x + 4}$ if x is real.

Solution : $y = \frac{x^2 - x + 4}{x^2 + x + 4}$ $\Rightarrow$ $(y - 1)x^2 + (y + 1)x + 4(y - 1) = 0$ ....(i)
<u>case-I</u> : if $y \neq 1$, then equation (i) is quadratic in x
and $\because$ x is real
$\therefore$ $D \geq 0$ $\Rightarrow$ $(y + 1)^2 - 16(y - 1)^2 \geq 0$ $\Rightarrow$ $(5y - 3)(3y - 5) \leq 0$
$\therefore$ $y \in \left[\frac{3}{5}, \frac{5}{3}\right] - \{1\}$
<u>case-II</u> : if $y = 1$, then equation becomes
$2x = 0 \Rightarrow$ $x = 0$ which is possible as x is real.
$\therefore$ Range $\left[\frac{3}{5}, \frac{5}{3}\right]$

Example#18 : Find the range of $y = \frac{x+3}{2x^2 + 3x + 9}$, if x is real.

Solution : $y = \frac{x+3}{2x^2 + 3x + 9}$
$\Rightarrow$ $2yx^2 + (3y - 1)x + 3(3y - 1) = 0$ ....(i)
<u>Case-I</u> : if $y \neq 0$, then equation (i) is quadratic in x
$\because$ x is real
$\therefore$ $D \geq 0$
$\Rightarrow$ $(3y - 1)^2 - 24y(3y - 1) \geq 0$
$\Rightarrow$ $(3y - 1)(21y + 1) \leq 0$
$y \in \left[-\frac{1}{21}, \frac{1}{3}\right] - \{0\}$
<u>Case-II</u> : if $y = 0$, then equation becomes
$x = -3$ which is possible as x is real
$\therefore$ Range $y \in \left[-\frac{1}{21}, \frac{1}{3}\right]$

Self practice problems :

(1) If c > 0 and $ax^2 + 2bx + 3c = 0$ does not have any real roots then prove that
(i) $4a - 4b + 3c > 0$ **(ii)** $a + 6b + 27c > 0$ **(iii)** $a + 2b + 6c > 0$

(2) If $f(x) = (x - a)(x - b)$, then show that $f(x) \geq -\frac{(a-b)^2}{4}$.

(3) Find the least integral value of 'k' for which the quadratic polynomial
$(k - 1)x^2 + 8x + k + 5 > 0 \ \forall \ x \in R$.

(4) Find the range of the expression $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$, if x is a real.

(5) Find the interval in which 'm' lies so that the expression $\frac{mx^2 + 3x - 4}{-4x^2 + 3x + m}$ can take all real values, $x \in R$.

(6) Find the value of b for which difference between maximum and minimum value of $x^2 - 2bx - 1$ in [0, 1] is 1.

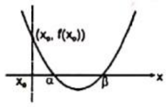
(7) Find all numbers a for each of which the least value of the quadratic trinomial $4x^2 - 4ax + a^2 - 2a + 2$ on the interval $0 \leq x \leq 2$ is equal to 3.

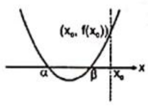
Answers :

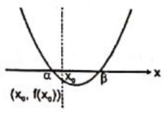
(3)	k = 4	**(4)**	$(-\infty, 5] \cup [9, \infty)$	**(5)**	$m \in (1, 7)$
(6)	b = 0	**(7)**	$a = 1 - \sqrt{2}$ or $5 + \sqrt{10}$		

11. Location of Roots :

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) 

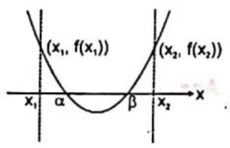
(ii) 

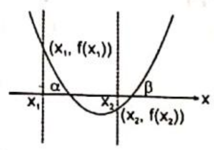
(iii) 

(i) Conditions for both the roots of f(x) = 0 to be greater than a specified number 'x_0' are $b^2 - 4ac \geq 0$ & $f(x_0) > 0$ & $(-b/2a) > x_0$.

(ii) Conditions for both the roots of f(x) = 0 to be smaller than a specified number 'x_0' are $b^2 - 4ac \geq 0$ & $f(x_0) > 0$ & $(-b/2a) < x_0$.

(iii) Conditions for a number 'x_0' to lie between the roots of f(x) = 0 is $f(x_0) < 0$.

(iv) 

(v) 

(iv) Conditions that both roots of f(x) = 0 to be confined between the numbers x_1 and x_2, $(x_1 < x_2)$ are $b^2 - 4ac \geq 0$ & $f(x_1) > 0$ & $f(x_2) > 0$ & $x_1 < (-b/2a) < x_2$.

(v) Conditions for exactly one root of f(x) = 0 to lie in the interval (x_1, x_2) i.e.
$x_1 < x < x_2$ is $f(x_1).f(x_2) < 0$.

Example # 19 :

Let $x^2 - (m - 3)x + m = 0$ $(m \in R)$ be a quadratic equation, then find the values of 'm' for which

(a) both the roots are greater than 2.

(b) both roots are positive.

(c) one root is positive and other is negative.

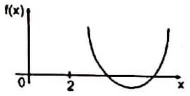
(d) One root is greater than 2 and other smaller than 1

(e) Roots are equal in magnitude and opposite in sign.

(f) both roots lie in the interval (1, 2)

Solution :

(a)



Condition - I : $D \ge 0$ $\Rightarrow$ $(m - 3)^2 - 4m \ge 0$ $\Rightarrow$ $m^2 - 10m + 9 \ge 0$

$\Rightarrow$ $(m - 1)(m - 9) \ge 0$

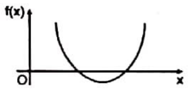
$\Rightarrow$ $m \in (-\infty, 1] \cup [9, \infty)$ ......(i)

Condition - II : $f(2) > 0$ $\Rightarrow$ $4 - (m - 3)2 + m > 0 \Rightarrow$ $m < 10$ ......(ii)

Condition - III : $-\dfrac{b}{2a} > 2$ $\Rightarrow$ $\dfrac{m-3}{2} > 2 \Rightarrow$ $m > 7$ ......(iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

(b)



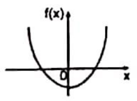
Condition - I $D \ge 0$ $\Rightarrow$ $m \in (-\infty, 1] \cup [9, \infty)$

Condition - II $f(0) > 0$ $\Rightarrow$ $m > 0$

Condition - III $-\dfrac{b}{2a} > 0$ $\Rightarrow$ $\dfrac{m-3}{2} > 0$ $\Rightarrow$ $m > 3$

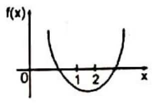
intersection gives $m \in [9, \infty)$ Ans.

(c)



Condition - I $f(0) < 0$ $\Rightarrow$ $m < 0$ Ans.

(d)



Condition - I $f(1) < 0$ $\Rightarrow$ $4 < 0$ $\Rightarrow$ $m \in \phi$

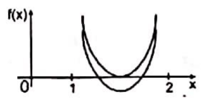
Condition - II $f(2) < 0$ $\Rightarrow$ $m > 10$

Intersection gives $m \in \phi$ Ans.

(e) sum of roots = 0 $\Rightarrow$ $m = 3$

and $f(0) < 0$ $\Rightarrow$ $m < 0$ $\therefore$ $m \in \phi$ **Ans.**

(f)



Condition - I $D \ge 0$ $\Rightarrow$ $m \in (-\infty, 1] \cup [9, \infty)$

Condition - II $f(1) > 0$ $\Rightarrow$ $1 - (m - 3) + m > 0$ $\Rightarrow$ $4 > 0$ which is true $\forall\, m \in R$

Condition - III $f(2) > 0$ $\Rightarrow$ $m < 10$

Condition - IV $1 < -\frac{b}{2a} < 2 \Rightarrow 1 < \frac{m-3}{2} < 2 \Rightarrow 5 < m < 7$

intersection gives $m \in \phi$ **Ans.**

Example#20:

Find all the values of 'a' for which both the roots of the equation $(a - 2)x^2 - 2ax + a = 0$ lies in the interval $(-2, 1)$.

Solution :

Case-I : $f(-2) > 0 \Rightarrow 4(a-2) + 4a + a > 0$

$9a - 8 > 0 \Rightarrow a > \frac{8}{9}$

$f(1) > 0 \Rightarrow a - 2 - 2a + a > 0$

$-2 > 0$ not possible $\therefore a \in \phi$

Case-II : $a - 2 < 0 \Rightarrow a < 2$

$f(-2) < 0 \Rightarrow a < \frac{8}{9}$

$f(1) < 0 \Rightarrow a \in R$

$-2 < \frac{b}{2a} - < 1 \Rightarrow a < \frac{4}{3}$

$D \geq 0 \Rightarrow a \geq 0$

intersection gives $a \in \left[0, \frac{8}{9}\right)$ complete solution $a \in \left[0, \frac{8}{9}\right) \cup \{2\}$

Self practice problems :

(1) Let $x^2 - 2(a-1)x + a - 1 = 0$ $(a \in R)$ be a quadratic equation, then find the value of 'a' for which
- **(a)** Both the roots are positive
- **(b)** Both the roots are negative
- **(c)** Both the roots are opposite in sign.
- **(d)** Both the roots are greater than 1.
- **(e)** Both the roots are smaller than 1.
- **(f)** One root is small than 1 and the other root is greater than 1.

(2) Find the values of p for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$ are less than 2.

(3) Find the values of 'α' for which 6 lies between the roots of the equation $x^2 + 2(\alpha - 3)x + 9 = 0$.

(4) Let $x^2 - 2(a-1)x + a - 1 = 0$ $(a \in R)$ be a quadratic equation, then find the values of 'a' for which
- **(i)** Exactly one root lies in $(0, 1)$.
- **(ii)** Both roots lies in $(0, 1)$.
- **(iii)** Atleast one root lies in $(0, 1)$.
- **(iv)** One root is greater than 1 and other root is smaller than 0.

(5) Find the values of a, for which the quadratic expression $ax^2 + (a-2)x - 2$ is negative for exactly two integral values of x.

Answers :

(1) **(a)** $[2, \infty)$ **(b)** ϕ **(c)** $(-\infty, 1)$ **(d)** ϕ **(e)** $(-\infty, 1]$ **(f)** $(2, \infty)$

(2) $(-\infty, -1)$ **(3)** $\left(-\infty, -\frac{3}{4}\right)$

(4) **(i)** $(-\infty, 1) \cup (2, \infty)$ **(ii)** ϕ **(iii)** $(-\infty, 1) \cup (2, \infty)$ **(iv)** ϕ

(5) $[1, 2)$

12. Solving Equations :

(i) **Type – I Relation ship between root and coefficient :**
(ii) **Type-II Solve for x, y, z**
(iii) **Type-III Miscellaneous**

Example# 21 :

Find all real numbers 'r' which there is atleast one triplet (x, y,z) of nonzero real numbers such that $x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 = rxyz$

Ans. $r \in (-\infty, -1] \cup [3, \infty)$

Sol. Divide by xyz as x,y,z are non zero

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} = \frac{y}{z} + \frac{z}{x} + \frac{x}{y} = r$$

now assume $\frac{x}{y} = a,\ \frac{y}{z} = b$ and $\frac{z}{x} = c$

$\Rightarrow a + b + c = r$, $abc = 1$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = r$

$\Rightarrow ab + bc + ca = r$ .... (ii)

now we can write that a, b,c are the roots of the cubic polynomial $t^3 - rt^2 + rt - 1 = 0$

$\Rightarrow (t^3 - 1) - rt(t - 1) = 0 \Rightarrow (t - 1)(t^2 + t + 1 - rt) = 0 \Rightarrow (t - 1)(t^2 - (r - 1)t + 1) = 0$

all solutions should be real so discriminant of quadratic should be ≥ 0

$\Rightarrow (r - 1)^2 - 4 \geq 0 \quad \Rightarrow \quad (r - 3)(r + 1) \geq 0$

$\Rightarrow r \in (-\infty, -1] \cup [3,\infty]$ Ans.

Example# 22 :

Find all polynomials whose coefficients are equal to 1 or – 1 and whose all roots are real.

Ans. $\pm(x + 1)$, $\pm(x - 1)$
$\pm(x^2 + x - 1)$, $\pm(x^2 - x - 1)$
$\pm(x^3 + x^2 - x - 1)$
$\pm(x^3 - x^2 - x + 1)$
$\pm(x^3 + x^2 - x + 1)$

Sol. Let the polynomial is $p(x) = a_nx^n + a_{n-1}x^{n-1} \ldots\ldots + a_1x + a_0$

let the roots are $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$

$$\Sigma\alpha_1 = -\left(\frac{a_{n-1}}{a_n}\right),\ \Sigma\alpha_1\alpha_2 = \frac{a_{n-2}}{a_n} \Rightarrow \Sigma\alpha_1^2 = \left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right) = 1 - 2\left(\frac{a_{n-2}}{a_n}\right)$$

L.H.S. is always positive so $\Rightarrow \frac{a_{n-2}}{a_n}$ has to be – 1 $\quad \Rightarrow \Sigma\alpha_1^2 = 3$

and $\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = \frac{a_0}{a_n}$ or $-\frac{a_0}{a_n} \Rightarrow \alpha_1^2\alpha_2^2 \ldots\ldots \alpha_n^2 = \left(\frac{a_0}{a_n}\right)^2 = 1$

By A.M $\geq$ G.M

$$\frac{\alpha_1^2 + \alpha_2^2 \ldots\ldots \alpha_n^2}{n} \geq (\alpha_1^2\alpha_2^2 \ldots\ldots \alpha_n^2)^{1/n} \quad \Rightarrow \quad \frac{3}{n} \geq 1 \quad \Rightarrow \quad n \leq 3$$

Case-I n = 1

or $p(x) = \pm(x+1), \pm(x - 1)$

Case-II

$n = 2 \Rightarrow p(x) = ax^2 + bx + c$

$D = b^2 - 4ac \geq 0 \quad \Rightarrow \quad 1 - 4ac \geq 0 \quad \Rightarrow \quad ac < 0$

$\Rightarrow p(x) = \pm(x^2 + x - 1)$ or $p(x) = \pm(x^2 - x - 1)$

Case-III

$n = 3 \Rightarrow$ let $P(x) = x^3 + bx^2 + cx + d = 0$

$(\alpha + \beta + \gamma)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta = (-b)^2$

$\Rightarrow 1 - 2\,\Sigma\alpha\beta = \Sigma\alpha^2 \Rightarrow 1 - 2\Sigma\alpha\beta = 3 \qquad \Rightarrow \Sigma\alpha\beta = -1$

$\alpha\beta + \beta\gamma + \gamma\alpha = -1 \qquad$ so $c = -1$

Now by choosing b, d from $\{-1,1\}$ we can conclude only two cubic polynomials with their negative sign have real roots

$\Rightarrow \qquad P(x) = \pm(x^3 + x^2 - x - 1) \quad$ or $\quad P(x) = \pm(x^3 - x^2 - x + 1) \qquad P(x) = \pm(x^3 + x^2 - x + 1)^3$

Example# 23 :

If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a, we denote by [a] the greatest integer not exceeding a.)

Sol. Suppose α is a real root of the given equation. Then

$\alpha^5 - \alpha^3 + \alpha - 2 = 0$.

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) = 1$. Observe that $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha+2)$. If $-1 \leq \alpha < 0$, then $\alpha + 2 > 0$, giving $\alpha^2(\alpha + 2) > 0$ and hence $(\alpha - 1)\,\alpha^4 + \alpha^3 + 1 > 0$. This again gives $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$.

The above reasoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 = 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get

$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$

The statement $[\alpha^6] = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (1), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $< \alpha < 2$. Now observe that if $a \geq 2$ then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) \geq 25$ which is the impossible.

If $0\ \frac{1}{2} < \alpha \leq \frac{1}{2}$, then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ which again is impossible. We conclude that $\frac{1}{2} < \alpha < 2$.

Similarly $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha - 1)^2 \geq 0$ which is valid. Hence $3 \leq \alpha^6 < 4$ and we get $[\alpha^6] = 3$.

Example# 24 :

Find the number of ordered pair of real numbers (x, y) satisfying the equation

$$5x\left(1+\frac{1}{x^2+y^2}\right) = 12 \ \&\ 5y\left(1-\frac{1}{x^2+y^2}\right) = 4.$$

Solution :

$$\text{Now } (5x)^2 + (5y)^2 = \frac{12^2}{\left(1+\frac{1}{x^2+y^2}\right)^2} + \frac{4^2}{\left(1-\frac{1}{x^2+y^2}\right)^2}$$

put $x^2 + y^2 = \frac{1}{t}$

$$\frac{25}{t} = \frac{144}{(1+t)^2} + \frac{16}{(1-t)^2}$$

$$25t^4 - 160t^3 + 206t^2 - 160t + 25 = 0$$

$$25\left(t^2 + \frac{1}{t^2}\right) - 160\left(t + \frac{1}{t}\right) + 206 = 0$$

$$t + \frac{1}{t} = u \Rightarrow 25u^2 - 160u + 156 = 0$$

$u = \frac{6}{5}, \frac{26}{5}$

$t + \frac{1}{t} = \frac{6}{5}$ and $t + \frac{1}{t} = \frac{26}{5}$

$t \in \phi \qquad t = 5, \frac{1}{5}$

If $x^2 + y^2 = \frac{1}{5} \Rightarrow (x, y) = \left(\frac{2}{5}, -\frac{1}{5}\right)$

and $x^2 + y^2 = 5 \Rightarrow (x, y) = (2, 1)$

Example# 25 :

Find the number of triplets (x, y, z) of integers satisfying the equations $x + y = 1 - z$ and $x^3 + y^3 = 1 - z^2$ (where $z \neq 1$)

Solution:

$x^3 + y^3 + [1-(x + y)]^2 = 1$

$(x + y)[x^2 - xy + y^2 + x + y - 2] = 0$

If $x + y = 0$ then $z = 1$ so rejected

$(x, y, z) = (m, -m, 1)$

and $x + y \neq 0$

$\Rightarrow x^2 - xy + y^2 + x + y - 2 = 0$

$\Rightarrow (2x - y + 1)^2 + 3(y + 1)^2 = 12$

$\Rightarrow 2x - y + 1 = 0, y + 1 = \pm 2$

or $2x - y + 1 = \pm 3, y + 1 = \pm 1$

$(x, y, z) = (0, 1, 0), (-2, -3, 6), (1, 0, 0), (0, -2, 3), (-2, 0, 3), (-3, -2, 6)$

Example : 26

Let $(x + y - z)(x - y + z) = ayz$

$(y + z - x)(y - z + x) = bzx$

$(z + x - y)(z - x + y) = cxy.$

If $abc = (q - a - b - c)^r$ (where $q, r \in N$) find the value of $(q + r)$

Solution :

Multiply $(x + y - z)^2 (x - y + z)^2 (y + z - x)^2 = abc\, x^2y^2z^2$

$(-\Sigma x^3 + y^2z + yz^2 + z^2x + zx^2 + x^2y + xy^2 - 2xyz)^2 = abc\, x^2y^2z^2.$

$$abc = \left(-\frac{x^2}{yz} - \frac{y^2}{zx} - \frac{z^2}{xy} + \sum\left(\frac{x}{y} + \frac{y}{x}\right) - 2\right) \qquad \text{............(1)}$$

Equation (1)

$$a = \frac{x^2 - y^2 - z^2 + 2yz}{yz} = \frac{x^2}{yz} - \frac{y}{z} - \frac{z}{y} + 2$$

$$a - 2 = \frac{x^2}{yz} - \frac{y}{z} - \frac{z}{y}$$

$$b - 2 = \frac{y^2}{xz} - \frac{z}{x} - \frac{x}{z}$$

$$c - 2 = \frac{z^2}{xy} - \frac{x}{y} - \frac{y}{x}$$

$abc = ((2-a) + (2-b) + (2-c) - 2)^2$

$abc = (4 - a - b - c)^2.$

Example# 27 :

Let $P(x) = 0$ be a fifth degree polynomial equation with integer coefficients that has atleast one integral root. If $P(2) = 13$ and $P(10) = 5$, then find the integral value of 'x' that must satisfy $P(x) = 0$.

Sol. Let r be an integer such that $p(r) = 0$

$P(x) = (x - 2)\, q(x) + P(2)$

$P(r) = (r - 2)\, q(r) + 13$

$q(r) = \dfrac{-13}{r-2}$

Now, $r - 2 = \pm 1, \pm 13$

$r = 3, 1, 15, -11$

$P(x) = (x - 10)\, q(x) + P(10)$

$P(r) = 0 \Rightarrow q(r) = \dfrac{-5}{r-10}$

$r - 10 = 1, -1, 5, -5$

$r = 11, 9, 15, 5$

Thus $r = 15$.

Example# 28 :

Find the number of ordered pairs of natural numbers (x, y) satisfying the equation

$(xy - 1)^2 = (x + 1)^2 + (y + 1)^2$

Solution :

$(xy - 1)^2 - (x + 1)^2 + (y + 1)^2$

$(xy - x - 2)(xy + x) = (y + 1)^2$

$(y + 1)[x(xy - x - 2) - (y + 1)] = 0$

If $y = -1$ then $x \in R$

Similarly $x = -1$ then $y \in R \Rightarrow (x, y) = (-1, y), (x, -1)$

Case-1 $x \neq -1\ y \neq -1$

$x(xy - x - 2)(y + 1) = (y + 1)^2$

$x(xy - x - 2) = y + 1$

$x^2y - x^2 - 2x - y - 1 = 0$

$y(x - 1)(x + 1) = (x + 1)^2$

Since $x \neq -1$

$y = \dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1}$

Now $x - 1 = -1, 2, -2, 1$

$x = 0, 3, -1, 2\ (x \neq -1)$

$(x, y) = (3, 2)\ (2, 3), (0, -1)$

Exercise-1

PART – I : PRE RMO

1. If the two equations $x^2 - cx + d = 0$ and $x^2 - ax + b = 0$ have one common root and the second equation has equal roots, then $2(b + d) =$

2. If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and $a, b, c \in N$, then the minimum value of $(a + b + c)$ is

3. $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d = 0$, then the real root of above equation is $(a, b, c, d \in R)$

4. If roots of equation $2x^4 - 3x^3 + 2x^2 - 7x - 1 = 0$ are α, β, γ and δ then value of $\sum \frac{\alpha+1}{\alpha}$ is equal to

5. If two roots of the equation $x^3 - px^2 + qx - r = 0$ are equal in magnitude but opposite in sign, then:

6. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ then the value of $\left(\alpha - \frac{1}{\beta\gamma}\right)\left(\beta - \frac{1}{\gamma\alpha}\right)\left(\gamma - \frac{1}{\alpha\beta}\right)$ is

7. If the inequality $(m - 2)x^2 + 8x + m + 4 > 0$ is satisfied for all $x \in R$, then the least integral value of m is:

8. The real values of 'a' for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possess roots of opposite sign is given by:

9. If α, β are the roots of the quadratic equation $x^2 - 2p(x - 4) - 15 = 0$, then the set of values of p for which one root is less than 1 & the other root is greater than 2 is:

10. If α, β are the roots of quadratic equation $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + px - r = 0$, then $(\alpha - \gamma) . (\alpha - \delta)$ is equal to :

11. Each root of the equation $ax^2 + bx + c = 0$ is decreased by 1. The quadratic equation with these roots is $x^2 + 4x + 1 = 0$. The numerical value of $b + c$ is ________.

12. x, y, z are distinct real numbers such that $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$. The value of $x^2y^2z^2$ is ________.

13. A hare sees a hound 100 m away from her and runs off in the opposite direction at a speed of 12 KM an hour. A minute later the hound perceives her and gives a chase at a speed of 16 KM an hour. The distance at which the hound catches the hare (in meters) is ________

14. a and b are positive integers such that $a^2 + 2b = b^2 + 2a + 5$. The value of b is ________.

15. $a \neq 0, b \neq 0$ The number of real number pair (a, b) which satisfy the equation $a^4 + b^4 = (a + b)^4$ is

16. The value of $\sqrt[3]{5+2\sqrt{13}} + \sqrt[3]{5-2\sqrt{13}}$ is = ______ .

17. If a, b, c, d satisfy the equation $a + 7b + 3c + 5d = 0$, $8a + 4b + 6c + 2d = -16$, $2a + 6b + 4c + 8d = 16$, $5a + 3b + 7c + d = -16$ then the value of $(a + d)(b + c) =$ ______ .

18. The combined age of a man and his wife is six times the combined ages of their children. Two years ago their united ages were ten times the combined ages of their children. Six years hence their combined age will be three times the combined age of the children. The number of children they have is ____.

19. If $(x + 1)^2 = x$, the value of $11x^3 + 8x^2 + 8x - 2$ is

20. If a = 2012, b = –1005, c = – 1007, then the value of $\frac{a^4}{b+c}+\frac{b^4}{c+a}+\frac{c^4}{a+b}+3abc$ is

21. If one root of $\sqrt{a-x}+\sqrt{b+x}=\sqrt{a}+\sqrt{b}$ is 2012, then a possible value of (a, b) is

22. a and b are the roots of the quadratic equation $x^2+\lambda x-\frac{1}{2\lambda^2}=0$ where x is the unknown and λ is a real parameter. The minimum value of a^4+b^4 is :

23. The remainder when the polynomial $x+x^3+x^9+x^{27}+x^{81}+x^{243}$ is divided by x^2-1

24. If x, y are positive real numbers satisfying the system of equations $x^2+y\sqrt{xy}=336$, $y^2+x\sqrt{xy}=112$, then x + y equals

25. If a, b, c are positive integers such that $a^2+2b^2-2ab=169$ and $2bc-c^2=169$ then a + b + c is :

26. $P=2008^{2007}-2008$; $Q=2008^2+2009$. The remainder when P is divided by Q is

27. The number of integer values of a for which $x^2+3ax+2009=0$ has two integer roots is :

28. The sum of the fourth powers of the roots of the equation $x^3-x^2-2x+2=0$ is

29. If a, b, c are real ; a ≠ 0, b ≠ 0, c ≠ 0 and a + b + c ≠ 0 and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ then (a + b) (b + c) (c + a) =

30. The roots of the equation $x^5-40x^4+Px^3+Qx^2+Rx+S=0$ are in geometric progression. The sum of their reciprocals is 10. Then |S| is equal to :

31. The number of solutions (x, y) where x and y are integers, satisfying $2x^2+3y^2+2x+3y=10$ is :

32. If $\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1$ and $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=0$ then the value of $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}$ is :

33. A cubic polynomial P is such that P(1) = 1, P(2) = 2, P(3) = 3 and P(4) = 5. Then P(6) is

34. Which of the following is the best approximation to $\frac{(2^3-1)(3^3-1).....(1000^3-1)}{(2^3+1)(3^3+1).....(1000^3+1)}$

35. Given that $(1-x)(1+x+x^2+x^3+x^4)=\frac{31}{32}$ and x is a rational number. Then $1+x+x^2+x^3+x^4+x^5$ is :

36. Solve the equation $3x^4-10x^3+4x^2-x-6=0$ one root being $\frac{1+\sqrt{-3}}{2}$

37. Find the smallest integral x satisfying the inequality $\frac{x-5}{x^2+5x-14}>0$.

38. Find integral x's which satisfy the inequality $x^4-3x^3-x+3<0$.

39. Find the largest integral x which satisfies the following inequality $(x+1)(x-3)^2(x-5)(x-4)^2(x-2)<0$

40. Given $3x^2+x=1$, find the value of $6x^3-x^2-3x+2010$.

41. If $\frac{1}{x}-\frac{1}{y}=4$, find the value of $\frac{2x+4xy-2y}{x-y-2xy}$.

42. Let $P(x)=ax^7+bx^3+cx-5$, where a, b, c are constants. Given P(–7) = 7, find the value of P(7).

43. If xy = a, xz = b, yz = c and $abc \neq 0$, find the value of $x^2 + y^2 + z^2$ in terms of a, b, c.

44. Find the number of positive integers x satisfying the equation $\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}=\frac{13}{12}$

45. Solve the following equation : $(x-1)(x-2)(x-3)(x-4) = 15$

46. Solve the following equation : $\frac{x^2-3.5x+1.5}{x^2-x-6}=0$

PART – II : RMO

1. Show that the expression $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when x is real, if $a^2 - b^2$ and $c^2 - d^2$ have the same sign.

2. If by eliminating x between the equations $x^2 + ax + b = 0$ and $xy + \ell(x + y) + m = 0$, a quadratic in y is formed whose roots are the same as those of the original quadratic in x, then prove that either $a = 2\ell$ and $b = m$ or $b + m = a\ell$.

3. If the roots of the equation $\left(1-q+\frac{p^2}{2}\right)x^2 + p(1 + q)x + q(q-1) + \frac{p^2}{2} = 0$ are equal, then show that $p^2 = 4q$

4. If each pair of the three equations $x^2 - p_1x + q_1 = 0$, $x^2 - p_2c + q_2 = 0$, $x^2 - p_3x + q_3 = 0$, have common root, prove that $p_1^2+p_2^2+p_3^2+4(q_1+q_2+q_3) = 2(p_2p_3 + p_3p_1 + p_1p_2)$

5. Solve the system
$x \sin a + y \sin 2a + z \sin 3a = \sin 4a$
$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$
$x \sin c + y \sin 2c + z \sin 3c = \sin 4c$

6. If α, β, γ are the roots of the equation $x^3 - x^2 + 4 = 0$, then form an equation where roots are $\alpha + \beta^2 + \gamma^2$, $\beta + \alpha^2 + \gamma^2$, $\gamma + \alpha^2 + \beta^2$.

7. If α, β, γ are the roots of the equation $x^3 + cx + d = 0$ then form an equation whose roots are $\frac{\beta\gamma+\beta+\gamma}{\beta+\gamma-\alpha}, \frac{\alpha\gamma+\alpha+\gamma}{\alpha+\gamma-\beta}, \frac{\beta\alpha+\beta+\alpha}{\beta+\alpha-\gamma}$

8. Determine all real values of the parameter 'a' for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four distinct real roots that form a geometric progression

9. Consider the equation $x^5 + 5\lambda x^4 - x^3 + (\lambda\alpha - 4)x^2 - (8\lambda + 3)x + (\lambda\alpha - 2) = 0$ where $\lambda, \alpha \in R$
(i) Determine α such that the given equation has exactly one root independent of λ.
(ii) Determine α such that the given equation has exactly two roots independent of λ.

10. Find all real numbers 'r' for which there is atleast one triplet (x, y,z) of nonzero real numbers such that $x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 = rxyz$

11. Find all polynomials whose coefficients are equal to 1 or – 1 and whose all roots are real.

12. If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 1 = 0$, then form an equation whose roots are $\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{1}{\alpha^3}$; $\frac{1}{\alpha^3}+\frac{1}{\gamma^3}-\frac{1}{\beta^3}$; $\frac{1}{\alpha^3}+\frac{1}{\beta^3}-\frac{1}{\gamma^3}$.

13. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, then prove that $[\alpha^6] = 3$. (For any real number a, we denote by [a] the greatest integer not exceeding a.)

14. The three equations
$x + y + z = 3$,
$x^3 + y^3 + z^3 = 15$ and
$x^4 + y^4 + z^4 = 35$
has a real solution x, y, z for which $x^2 + y^2 + z^2 < 10$. Find the value of $(x^5 + y^5 + z^5)$

15. Determine all real solutions of the given equation where p is real number $\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$

16. Find the real numbers (x, y) that satisfy the equation
$xy^2 = 15x^2 + 17xy + 15y^2$
$x^2y = 20x^2 + 3y^2$

17. Find the solution of equations
$(3x + y)(x + 3y)\sqrt{xy} = 14$ and
$(x + y)(x^2 + 14xy + y^2) = 36$ where $x, y \in R$

18. Find all solutions $(x_1, x_2, x_3 \ldots\ldots\ldots x_n)$ of the equation

$$1 + \frac{1}{x_1} + \frac{x_1 + 1}{x_1x_2} + \frac{(x_1+1)(x_2+1)}{x_1x_2x_3} + \ldots\ldots\ldots\ldots + \frac{(x_1+1)(x_2+1)\ldots\ldots(x_{n-1}+1)}{x_1x_2\ldots\ldots x_n} = 0$$

19. The roots x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$, where a is a non-zero real, satisfy $\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \frac{x_3^2}{x_1} = -8$. Find x_1, x_2, x_3.

20. Solve the given equations :
$x + y + z = 0$
$x^3 + y^3 + z^3 = 18$
$x^7 + y^7 + z^7 = 2058$ where $x, y, z \in R$

Exercise-2

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

1. Let $S_n = n^2 + 20n + 12$ where n is a positive integer. What is the sum of all possible values of n for which S_n is a perfect square ? **[PRMO-2012]**

2. Let x_1, x_2, x_3 be roots of equation $x^3 + 3x + 5 = 0$. What is the value of the expression $\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)$? **[PRMO-2012]**

3. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$? **[PRMO-2012]**

4. It is given that the equation $x^2 + ax + 20 = 0$ has integer roots. What is the sum of all possible values of a ? **[PRMO-2013]**

5. Three real numbers x, y, z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$ and $z^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$ **[PRMO-2013]**

6. Let $f(x) = x^3 - 3x + b$ and $g(x) = x^2 + bx - 3$ where b is a real number. What is the sum of all possible values of b for which the equations f(x)= 0 and g(x) = 0 have a common root ? **[PRMO-2013]**

7. What is the smallest possible natural number 'n' for which the equation $x^2 - nx + 2014 = 0$ has integer roots. **[PRMO-2014]**

8. Natural numbers k, ℓ, p and q are such that if a and b are roots of $x^2 - kx + \ell = 0$ then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are roots of $x^2 - px + q = 0$. What is the sum of all possible values of q ? **[PRMO-2014]**

9. Let $x_1, x_2, x_3, \ldots\ldots x_{2014}$ be real numbers different from 1 such that $x_1 + x_2 + \ldots\ldots + x_{2014} = 1$ and $\frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \ldots\ldots\ldots \frac{x_{2014}}{1-x_{2014}} = 1$. What is the value of $\frac{x_1^2}{1-x_1} + \frac{x_2^2}{1-x_2} + \ldots\ldots\ldots \frac{x_{2014}^2}{1-x_{2014}}$ **[PRMO-2014]**

10. If real numbers a, b, c, d, e satisfy $a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3$ then find the value of $a^2 + b^2 + c^2 + d^2 + e^2$ **[PRMO-2014]**

11. The equations $x^2 - 4x + k = 0$ and $x^2 + kx - 4 = 0$ where k is a real number have exactly one common root. What is the value of k. **[PRMO-2015]**

12. Let a, b and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. What is the value of $a^2 - b^2 + c^2$. **[PRMO-2015]**

13. Let a, b and c be such that $a + b + c = 0$ and $P = \frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ca} + \frac{c^2}{2c^2+ab}$ is defined. What is the value of P . **[PRMO-2015]**

14. Suppose $x^2 - x + 1$ is factor of $2x^6 - x^5 + ax^4 + x^3 + bx^2 - 4x - 3$. Find $a - 4b$ **[PRMO-2015]**

15. Let R(x) be the remainder upon dividing $x^{44} + x^{33} + x^{22} + x^{11} + 1$ by the polynomial $x^4 + x^3 + x^2 + x + 1$. Find $R(1) + 2R(2) + 3R(3)$. **[PRMO-2015]**

16. Let $P(x) = (x - 3)(x - 4)(x - 5)$. For how many polynomials Q(x), does there exist a polynomial R(x) of degree 3 such that $P(Q(x)) = P(x)R(x)$? **[PRMO-2015]**

17. For how many pairs of odd positive integers (a, b), both a, b less than 100, does the equation $x^2 + ax + b = 0$ have integer roots? **[PRMO-2015]**

18. Find the sum of all those integers n for which $n^2+20n+15$ is the square of an integer. **[PRMO-2015]**

19. Let α and β be the roots of equation $x^2 + x - 3 = 0$. Find the value of the expression $4\beta^2 - \alpha^3$. **[PRMO-2016]**

20. Let $x^3 + ax + 10 = 0$ and $x^3 + bx^2 + 50 = 0$ have two roots in common. Let P be the product of these common roots. Find the numerical value of P^3, not involving a, b. **[PRMO-2016]**

21. For real numbers x and y, let M be the maximum value of expression **[PRMO-2016]** $x^4y + x^3y + x^2y + xy + xy^2 + xy^3 + xy^4$, subject to $x + y = 3$. Find [M] where [.] = G.I.F.

22. Between 5pm and 6pm, I looked at my watch mistaking the hour hand for the minute hand and the minute hand for the hour hand, I mistook the time to be 57 minutes earlier than the actual time. Find the number of minutes past 5 when I looked at my watch. **[PRMO-2016 (Delhi)]**

23. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183$, $a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a+b)$. **[PRMO-2017]**

24. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of a + b ? **[PRMO-2017]**

25. In a class, the total numbers of boys and girls are in the ratio 4 : 3. On one day it was found that 8 boys and 14 girls were absent from the class and that the number of boys was the square of the number of girls. What is the total number of students in the class ? **[PRMO-2017]**

26. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yz + 2zx = 24$. what is the value of $x^2 + y^2 + z^2$? **[PRMO-2017]**

27. Suppose 1,2,3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c. **[PRMO-2017]**

28. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$. **[PRMO-2017]**

29. Suppose a, b are integers and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible values of b^2 ? **[PRMO-2017]**

30. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$? **[PRMO-2018]**

PART – II : PREVIOUSLY ASKED QUESTION OF RMO

1. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer. **[RMO-2010]**

2. For any natural number n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$. **[RMO-2010]**

3. Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with c 6= 0. Suppose that the following conditions hold: **[RMO-2013]**
   (a) $f(1) = 0$;
   (b) the roots of $g(x) = 0$ are the squares of the roots of $f(x) = 0$.
   Find the value of $a^{2013} + b^{2013} + c^{2013}$.

4. Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6. **[RMO-2013]**

5. Find all positive real numbers x, y, z such that **[RMO-2014]**
   $$2x - 2y + \frac{1}{z} = \frac{1}{2014}, \qquad 2y - 2z + \frac{1}{x} = \frac{1}{2014} \qquad 2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

6. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numbers $s \neq t$ such that $P(s) = t$ and $P(t) = s$. Prove that $b - st$ is a root of the equation $x^2 + ax + b - st = 0$. **[RMO-2015]**

7. Find all integers a,b,c such that $a^2 = bc + 1$, $b^2 = ca + 1$. **[RMO-2015]**

8. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)\,Q(x) = Q(P(x))$ for all real x. Find all the real roots of $P(Q(x)) = 0$. **[RMO-2017]**

Answers

Exercise-1

PART – I : PRE RMO

1.	ac	**2.**	9	**3.**	(a – b)/a	**4.**	–3	**5.**	pq = r
6.	$-\frac{(r+1)^3}{r^2}$	**7.**	5	**8.**	0 < a < 4	**9.**	$\left(-\infty, \frac{7}{3}\right)$		
10.	– (q + r)	**11.**	0	**12.**	1	**13.**	1200 meter.	**14.**	3
15.	no real value	**16.**	1	**17.**	–16	**18.**	3	**19.**	1
20.	0	**21.**	(4024, 2012)	**22.**	$2+\sqrt{2}$	**23.**	6x	**24.**	20
25.	39	**26.**	4032066	**27.**	6	**28.**	9	**29.**	0
30.	32	**31.**	4	**32.**	1	**33**	16	**34.**	$\frac{333}{500}$
35.	$\frac{63}{32}$	**36.**	$3, \frac{2}{3}, \frac{1-\sqrt{-3}}{2}$	**37.**	–6	**38.**	2	**39.**	–2
40.	2009	**41.**	2/3	**42.**	–17				

43. $\left(\sqrt{\frac{ab}{c}},\sqrt{\frac{ca}{b}},\sqrt{\frac{bc}{a}}\right)$ *or* $\left(-\sqrt{\frac{ab}{c}},-\sqrt{\frac{ac}{b}},-\sqrt{\frac{bc}{a}}\right)$ **44.** 1

45. $\frac{5\pm\sqrt{21}}{2}$ **46.** 1/2

PART – II : RMO

5. $x = 2(\cos a + \cos b + \cos c) + 8(\cos a \cos b \cos c)$
$y = -2 - 4(\cos a \cos b + \cos c + \cos b \cos c)$
$z = 2(\cos a + \cos b + \cos c)$

6. $x^3 - 3x^2 - x + 19$ **7.** $8dx^3 - 4x^2(c^2 + 3d) + 2x(3d + 2c^2 - 2cd) - (d + c^2 + d^2 - 2cd) = 0$

8. 170 **9.** **(i)** $\alpha = -\frac{64}{5}$ **(ii)** $\alpha = -3$

10. $r \in (-\infty, -1] \cup [3, \infty)$

11. $\pm(x + 1)$, $\pm(x - 1)$
$\pm(x^2 + x - 1)$, $\pm(x^2 - x - 1)$
$\pm(x^3 + x^2 - x - 1)$
$\pm(x^3 - x^2 - x + 1)$
$\pm(x^3 + x^2 - x + 1)$

12. $z^3 + 12z^2 - 172z - 2152 = 0$ **14.** 83 **15.** $x = \frac{4-p}{2\sqrt{4-2p}}$ $\left(0 \le p \le \frac{4}{3}\right)$

16. $(x, y) = (19, 45)$ & $(0, 0)$ **17.** $(x, y) = \left(\frac{3}{2}+\sqrt{2}, \frac{3}{2}-\sqrt{2}\right)$ or $\left(\frac{3}{2}-\sqrt{2}, \frac{3}{2}+\sqrt{2}\right)$

18. $x_i = -1$ for atleast one i is the solution where $i \in \{1,2,3,......n\}$

19. $\{x_1, x_2, x_3\} = \{-2, 1 - \sqrt{5}, 1 + \sqrt{5}\}$ all its permutation

20. $(x, y, z) = (2, -1, -3), (2, -3, -1) (1-3,-2) (1,-2,-3), (3,-1,-2), (3,-2,-1)$

Exercise-2

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

1.	16	**2.**	$\frac{-29}{5}$	**3.**	6	**4.**	0	**5.**	14
6.	0	**7.**	91	**8.**	4	**9.**	0	**10.**	10
11.	3	**12.**	1	**13.**	1	**14.**	6	**15.**	0
16.	22	**17.**	0	**18.**	– 40	**19.**	19	**20.**	500
21.	36	**22.**	12 min.	**23.**	73	**24.**	25	**25.**	42
26.	21	**27.**	36	**28.**	17	**29.**	81	**30.**	18

PART – II : PREVIOUSLY ASKED QUESTION OF RMO

1. 2, 3, 6 **2.** 17 **3.** – 1 **5.** $x = y = z = 2014$

7. $(1,1,0)\ (-1,-1,0), (-1,1,0), (1,-1,0), (-1,0,+1), (+1,0,-1), (0,1,-1), (0,-1,1)$

8. $\frac{1}{2}, -1$

Sequence and Series

Sequence & Series

"1729 is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." ...... S.Ramanujan

Sequence :

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. If $f : N \to R$, then $f(n) = t_n$, $n \in N$ is called a sequence and is denoted by

$\{f(1), f(2), f(3), \ldots\ldots\} = \{t_1, t_2, t_3, \ldots\ldots\} = \{t_n\}$

Real sequence :

A sequence whose range is a subset of R is called a real sequence.

e.g. (i) 2, 5, 8, 11, ...

(ii) 4, 1, – 2, – 5, ...

Types of sequence :

On the basis of the number of terms there are two types of sequence.

(i) Finite sequences : A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences : A sequence is said to be infinite if it has infinitely many terms.

Example # 1 :

Write down the sequence whose n^{th} term is $\dfrac{(-2)^n}{(-1)^n + 2}$

Solution :

Let $t_n = \dfrac{(-2)^n}{(-1)^n + 2}$

put n = 1, 2, 3, 4, ... we get

$t_1 = -2,\ t_2 = \dfrac{4}{3},\ t_3 = -8,\ t_4 = \dfrac{16}{3}$

so the sequence is $-2, , -8, \dfrac{16}{3}$, ...

Series :

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, \ldots\ldots a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots\ldots + a_n$ is a series.

e.g. (i) $1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots + n$

(ii) $2 + 4 + 8 + 16 + \ldots\ldots\ldots$

(iii) $-1 + 3 - 9 + 27 - \ldots\ldots\ldots$

Progression :

The word progression refers to sequence or series – finite or infinite

Arithmetic progression (A.P.) :

A.P. is a sequence whose successive terms are obtained by adding a fixed number 'd' to the preceding terms. This fixed number 'd' is called the common difference. If a is the first term & d the common difference, then A.P. can be written as a, a + d, a + 2 d,....... ,a + (n – 1) d,........

e.g. – 4, – 1, 2, 5 ....

n^{th} term of an A.P. :

Let 'a' be the first term and 'd' be the common difference of an A.P., then
$t_n = a + (n - 1) d$, where $d = t_n - t_{n-1}$

Example # 2 :

Find the number of terms in the sequence 4, 7, 10, 13, ...,82.

Solution :

Let a be the first term and d be the common difference
$a = 4, d = 3$ so $82 = 4 + (n - 1)3$ $\Rightarrow$ $n = 27$

The sum of first n terms of an A.P. :

If a is first term and d is common difference, then sum of the first n terms of AP is

$$S_n = \frac{n}{2}[2a + (n - 1) d]$$

$= \frac{n}{2}[a + \ell] \equiv n t_{\left(\frac{n+1}{2}\right)}$, for n is odd. (Where ℓ is the last term and $t_{\left(\frac{n+1}{2}\right)}$ is the middle term.)

Note : For any sequence $\{t_n\}$, whose sum of first r terms is S_r, r^{th} term, $t_r = S_r - S_{r-1}$.

Example # 3:

If in an A.P., 3rd term is 18 and 7 term is 30, then find sum of its first 17 terms

Solution :

Let a be the first term and d be the common difference
$a + 2d = 18$
$a + 6d = 30$
$d = 3, a = 12$

$$s_{17} = \frac{17}{2}[2 \times 12 + 16 \times 3] = 612$$

Example # 4 :

Find the sum of all odd numbers between 1 and 1000 which are divisible by 3

Solution :

Odd numbers between 1 and 1000 are
3, 5, 7, 9, 11, 13, ----- 993, 995, 997, 999.
Those numbers which are divisible by 3 are
3, 9, 15, 21, ------- 993, 999
They form an A.P. of which $a = 3$, $d = 6$, $\ell = 999$ $\therefore$ $n = 167$

$$S = \frac{n}{2}[a + \ell] = 83667$$

Example # 5 :

The ratio between the sum of n term of two A.P.'s is 3n + 8 : 7n + 15. Then find the ratio between their 12 th term

Solution :

$$\frac{S_n}{S_n'} = \frac{(n/2)[2a + (n-1)d]}{(n/2)[2a' + (n-1)d']} = \frac{3n+8}{7n+15} \text{ or } \frac{a + \{(n-1)/2\}d}{a' + (n-1)/2d'} = \frac{3n+8}{7n+15} \quad \text{----- (i)}$$

we have to find $\frac{T_{12}}{T_{12}'} = \frac{a + 11d}{a' + 11d'}$

choosing $(n - 1)/2 = 11$ or $n = 23$ in (1),

we get $\frac{T_{12}}{T_{12}'} = \frac{a + 11d}{a' + 11d'} = \frac{3(23) + 8}{(23) \times 7 + 15} = \frac{77}{176} = \frac{7}{16}$

Example # 6 :

If sum of n terms of a sequence is given by $S_n = 3n^2 - 4n$, find its 50^{th} term.

Solution :

Let t_n is n^{th} term of the sequence so $t_n = S_n - S_{n-1}$.

$= 3n^2 - 4n - 3(n-1)^2 + 4(n-1) = 6n - 7$

so $t_{50} = 293$.

Self practice problems :

(1) Which term of the sequence 2005, 2000, 1995, 1990, 1985, ... is the first negative term

(2) For an A.P. show that $t_m + t_{2n+m} = 2\, t_{m+n}$

(3) Find the maximum sum of the A.P. 40 + 38 + 36 + 34 + 32 + ...

(4) Find the sum of first 16 terms of an A.P. a_1, a_2, a_3..........
If it is known that $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 147$

Answers.

(1) 403 (3) 420 (4) 392

Remarks :

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P. $\Rightarrow a + d = b + c$.

(iii) Three numbers in A.P. can be taken as $a - d, a, a + d$; four numbers in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$; five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$;
six terms in A.P. are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it.
$a_n = 1/2\,(a_{n-k} + a_{n+k})$, $k < n$. For k = 1, $a_n = (1/2)\,(a_{n-1} + a_{n+1})$; For k = 2, $a_n = (1/2)\,(a_{n-2} + a_{n+2})$ and so on.

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

(vii) The sum and difference of two AP's is an AP.

Example # 7 :

The numbers $t\,(t^2 + 1)$, t^2 and 6 are three consecutive terms of an A.P. If t be real, then find the the next two term of A.P.

Solution :

$2b = a + c \quad \Rightarrow \quad -t^2 = t^3 + t + 6$

or $\quad t^3 + t^2 + t + 6 = 0$

or $\quad (t + 2)(t^2 - t + 3) = 0$

$\therefore \quad t^2 - t + 3 \neq 0 \quad \Rightarrow \quad t = -2$

the given numbers are – 10, – 2, 6, which are in an A.P. with d = 8. The next two numbers are 14, 22

Example # 8 :

If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Solution :

As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10$.

Example # 9 :

If a(b + c), b(c+a), c(a + b) are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Solution :

$\because$ a(b + c), b(c+a), c(a + b) are in A.P.

$\Rightarrow$ subtract ab + bc + ca from each

– bc, – ca, – ab are in A.P.

divide by –abc

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Example # 10 :

If $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P. then prove that $\frac{1}{a}$, b $\frac{1}{c}$ are in A.P.

Solution :

$\therefore$ $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P.

$$b - \frac{a+b}{1-ab} = \frac{b+c}{1-bc} - b$$

$$\frac{-a(b^2+1)}{1-ab} = \frac{c(1+b^2)}{1-bc}$$

$\Rightarrow$ $-a + abc = c - abc$

$a + c = 2abc$

divide by ac

$\frac{1}{c}+\frac{1}{a} = 2b$ $\Rightarrow$ $\frac{1}{a}$, b, $\frac{1}{c}$ are in A.P.

Arithmetic mean (mean or average) (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

A.M. for any n numbers $a_1, a_2, \ldots, a_n$ is; $A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$.

n-Arithmetic means between two numbers :

If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in A.P., then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a & b.

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1}, \ldots\ldots, A_n = a + \frac{n(b-a)}{n+1}$$

Note : Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$, where A is the single A.M. between a & b i.e. $A = \frac{a+b}{2}$

Example # 11 :

If a, b, c,d,e, f are A. M's between 2 and 12, then find a + b + c + d + e + f.

Solution :

Sum of A.M.• = 6 single A.M. = $\frac{6(2+12)}{2} = 42$

Example # 12 :

Insert 10 A.M. between 3 and 80.

Solution :

Here 3 is the first term and 80 is the 12^{th} term of A.P. so $80 = 3 + (11)d \Rightarrow d = 7$

so the series is 3, 10, 17, 24, ..., 73, 80

$\therefore$ required means are 10, 17, 24, ..., 73.

Self practice problems :

(5) There are n A.M.'s between 3 and 29 such that 6th mean : (n – 1)th mean : : 3 : 5 then find the value of n.

(6) For what value of n, $\dfrac{a^{n+3}+b^{n+3}}{a^{n+2}+b^{n+2}}$, $a \neq b$ is the A.M. of a and b.

Answers.

(5) n = 12 **(6)** n = –2

Geometric progression (G.P.) :

G.P. is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a G.P. the ratio of successive terms is constant. This constant factor is called the **common ratio** of the series & is obtained by dividing any term by that which immediately preceeds it. Therefore a, ar, ar^2, ar^3, ar^4,...... is a G.P. with 'a' as the first term & 'r' as common ratio.

e.g. (i) 2, 4, 8, 16, ... (ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, ...

Results :

(i) n^{th} term of GP = $a r^{n-1}$

(ii) Sum of the first n terms of GP

$$S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1}, & r \neq 1 \\ na, & r = 1 \end{cases}$$

(iii) Sum of an infinite terms of GP when $|r| < 1$. When $n \to \infty$, $r^n \to 0$ if $|r| < 1$ therefore,

$$S_\infty = \frac{a}{1-r} \; (|r| < 1)$$

Example # 13 :

The n^{th} term of the series 3, $\sqrt{3}$, 1, ------ is $\frac{1}{243}$, then find n

Solution :

$$3.\left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{243} \Rightarrow n = 13$$

Example # 14 :

The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Solution :

Let the G.P. be 1, r, r^2, r^3, ...

given condition $\Rightarrow$ $r = \dfrac{r^2}{1-r}$ $\Rightarrow$ $r = \dfrac{1}{2}$,

Hence series is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots\ldots\ldots\infty$

Example # 15

In a G.P.,., $T_2 + T_5 = 216$ and $T_4 : T_6 = 1 : 4$ and all terms are integers, then find its first term :

Solution :

$ar(1 + r^3) = 216$ and $\dfrac{ar^3}{ar^5} = \dfrac{1}{4}$

$\Rightarrow$ $r^2 = 4 \Rightarrow r = \pm 2$

when $r = 2$ then $2a(9) = 216 \Rightarrow a = 12$

when $r = -2$, then $-2a(1-8) = 216$

$\therefore$ $a = \dfrac{216}{14} = \dfrac{108}{7}$, which is not an integer.

Self practice problems :

(7) Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.

(8) If x, 2y, 3z are in A.P. where the distinct numbers x, y, z are in G.P. Then find the common ratio of G.P.

(9) A G.P. consist of 2n terms. If the sum of the terms occupying the odd places is S_1 and that of the terms occupying the even places is S_2 , then find the common ratio of the progression.

(10) If continued product of three number in G.P. is 216 and sum of there product in pairs is 156. Find the numbers.

Answers.

(7) 2, 6, 18, 54, 162, 486 **(8)** $\dfrac{1}{3}$ **(9)** $\dfrac{S_2}{S_1}$ **(10)** 2, 6, 18

Remarks :

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if $a_1, a_2, a_3, a_4, \ldots\ldots\ldots a_{n-1}, a_n$ are in G.P., then $a_1a_n = a_2a_{n-1} = a_3\,a_{n-2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(ii) Any three consecutive terms of a G.P. can be taken as $\dfrac{a}{r}$, a , ar.

(iii) Any four consecutive terms of a G.P. can be taken as, $\dfrac{a}{r^3}, \dfrac{a}{r}$ ar, ar^3.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the same non–zero quantity, the resulting sequence is also a G.P..

(v) If $a_1, a_2, a_3, \ldots\ldots$ and $b_1, b_2, b_3, \ldots\ldots$ are two G.P's with common ratio r_1 and r_2 respectively, then the sequence $a_1b_1, a_2b_2, a_3b_3, \ldots..$ is also a G.P. with common ratio $r_1\,r_2$.

(vi) If $a_1, a_2, a_3, \ldots\ldots\ldots$are in G.P. where each $a_i > 0$, then log a_1, loga_2, loga_3,..........are in A.P. and its converse is also true.

Example # 16 :

Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of G.P. is :

Solution :

Three number in G.P.are $\dfrac{a}{r}$, a, ar then $\dfrac{a}{r}$, 2a ar are in A.P. as given.

$\therefore$ $2(2a) = a\left(r + \dfrac{1}{r}\right)$ or $r^2 - 4r + 1 = 0$

or $r = 2 \pm \sqrt{3}$

or $r = 2 + \sqrt{3}$ as $r > 1$ for an increasing G.P.

Example # 17

The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of this infinite series is 24. Then find its first term and common ratio :

Solution :

Let a be the first term and r be the common ratio of G.P.

$\frac{a}{1-r} = 2, \frac{a^3}{1-r^3} = 24 \ , -1 < r < 1$

Solving we get $a = 3, r = -\frac{1}{2}$

Example # 18

Express $0.4\dot{2}\dot{3}$ in the form of $\frac{p}{q}$, (where $p, q \in I, q \neq 0$)

Solution :

$$S = \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots\ldots \infty = \frac{4}{10} + \frac{a}{1-r} = \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$$

Example # 19 :

Evaluate 9 + 99 + 999 + ... upto n terms.

Solution :

Let S = 9 + 99 + 999 + ...upto n terms.

= [9 + 99 + 999 + ...]

= $[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots\ldots$ + upto n terms]

$= [10 + 10^2 + 10^3 + \ldots\ldots + 10^n - n] = \left(\frac{10(10^n - 1)}{9} - n\right)$

Geometric means (mean proportional) (G.M.):

If a, b, c are in G.P., b is called as the G.M. of a & c.

If a and c are both positive, then $b = \sqrt{ac}$ and if a and c are both negative, then $b = -\sqrt{ac}$.

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0, c > 0$.

n-Geometric means between a, b :

If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in G.P.. Then

$G_1, G_2, G_3, \ldots, G_n$ are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}, G_2 = a(b/a)^{2/n+1}, \ldots\ldots, G_n = a(b/a)^{n/n+1}$

Note : The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\prod_{r=1}^{n} G_r = \left(\sqrt{ab}\right)^n = G^n$, where G is the single G.M. between a & b.

Example # 20

Between 4 and 2916 are inserted odd number (2n + 1) G.M's. Then the (n + 1)th G.M. is

Solution :

$4, G_1, G_2, \ldots\ldots G_{n+1}, \ldots G_{2n}, G_{2n+1}, 2916$

G_{n+1} will be the middle mean of (2n +1) odd means and it will be equidistant from Ist and last term

$\therefore$ $4, G_{n+1}, 2916$ will also be in G.P.

$\therefore$ $G_{n+1}^2 = 4 \times 2916 = 4 \times 9 \times 324 = 4 \times 9 \times 4 \times 81$

$G_{n+1} = 2 \times 3 \times 2 \times 9 = 108.$

Self practice problems :

(11) Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the G.M. between a and b.

(12) If $a = \underbrace{111 \text{..........} 1}_{55}$, $b = 1 + 10 + 10^2 + 10^3 + 10^4$ and $c = 1 + 10^5 + 10^{10} + \ldots\ldots + 10^{50}$, then prove that

(i) 'a' is a composite number (ii) a = bc.

Answers.

(11) $n = -\frac{1}{2}$

Harmonic progression (H.P.)

A sequence is said to be in H.P if the reciprocals of its terms are in A.P.. If the sequence $a_1, a_2, a_3, \ldots\ldots, a_n$ is in H.P. then $1/a_1, 1/a_2, \ldots\ldots, 1/a_n$ is in A.P.

Note : (i) Here we do not have the formula for the sum of the n terms of an H.P.. For H.P. whose first term is a and second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$.

(ii) If a, b, c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

(iii) If a, b, c are in A.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{a}$

(iv) If a, b, c are in G.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{b}$

Harmonic mean (H.M.):

If a, b, c are in H.P., b is called as the H.M. between a & c, then $b = \frac{2ac}{a+c}$

If $a_1, a_2, \ldots\ldots\ldots a_n$ are 'n' non-zero numbers then H.M. 'H' of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n}\right]$$

Example # 21

The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.

Solution :

Let a be the first term and d be the common difference of corresponding A.P.

a + 6d = 10

a + 11d = 25

5d = 15

d = 3, a = – 8

$T_{20} = a + 19d = -8 + 19 \times 3 = 49$

20 term of H.P. = $\frac{1}{49}$

Example # 22

Insert 4 H.M between $\frac{3}{4}$ and $\frac{3}{19}$.

Solution :

Let 'd' be the common difference of corresponding A.P..

so $\quad d = \dfrac{\frac{19}{3}-\frac{4}{3}}{5} = 1.$

$\therefore \quad \dfrac{1}{H_1} = \dfrac{4}{3} + 1 = \dfrac{7}{3} \quad$ or $\quad H_1 = \dfrac{3}{7}$

$\dfrac{1}{H_2} = \dfrac{4}{3} + 2 = \dfrac{10}{3} \quad$ or $\quad H_2 = \dfrac{3}{10}$

$\dfrac{1}{H_3} = \dfrac{4}{3} + 3 = \dfrac{13}{3} \quad$ or $\quad H_3 = \dfrac{3}{13}$

$\dfrac{1}{H_4} = \dfrac{4}{3} + 4 = \dfrac{16}{3} \quad$ or $\quad H_4 = \dfrac{3}{16}.$

Example # 23

Find the largest positive term of the H.P., whose first two term are $\dfrac{2}{5}$ and $\dfrac{12}{23}$.

Solution.

The corresponding A.P. is $\dfrac{5}{2}, \dfrac{23}{12}$ ....or $\dfrac{30}{12}, \dfrac{23}{12}, \dfrac{16}{12}, \dfrac{9}{12}, \dfrac{2}{12}, \dfrac{-5}{12}$.......

The H.P. is $\dfrac{12}{30}, \dfrac{12}{23}, \dfrac{12}{16}, \dfrac{12}{9}, \dfrac{12}{2}, -\dfrac{12}{5}$, ...

Largest positive term = $\dfrac{12}{2} = 6$

Self practice problems :

(13) If a, b, c, d, e are five numbers such that a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. prove that a, c, e are in G.P.

(14) If the ratio of H.M. between two positive numbers 'a' and 'b' (a > b) is to their G.M. as 12 to 13, prove that a : b is 9 : 4.

(15) a, b, c are in H.P. then prove that $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c} = 2$

(16) If a, b, c, d are in H.P., then show that $ab + bc + cd = 3ad$

Arithmetico-geometric series :

A series, each term of which is formed by multiplying the corresponding terms of an A.P. & G.P. is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$
Here 1, 3, 5,.... are in A.P. & 1, x, x^2, x^3.... are in G.P..

Sum of n terms of an arithmetico-geometric series:
Let $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n-1)d]r^{n-1}$, then

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, \; r \neq 1.$$

Sum to Infinity: If $|r| < 1$ & $n \to \infty$, then $\underset{n\to\infty}{\text{Limit}}\; r^n = 0$ and $\underset{n\to\infty}{\text{Limit}}\; n.r^n = 0$

$$\therefore \quad S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

Example # 24 :

The sum to n terms of the series $1 + 5\left(\frac{4n+1}{4n-3}\right) + 9\left(\frac{4n+1}{4n-3}\right)^2 + 13\left(\frac{4n+1}{4n-3}\right)^3 + ...$ is.

Solution

Let $x = \frac{4n+1}{4n-3}$, then

$$1 - x = \frac{-4}{4n-3}, \frac{1}{1-x} = -\frac{(4n-3)}{4}$$

$$\frac{x}{1-x} = -\frac{(4n+1)}{4}$$

$$S = 1 + 5x + 9x^2 + ... + (4n-3)x^{n-1}$$

$$Sx = x + 5x^2 + ... (4n-3)x^n$$

$$S - Sx = 1 + 4x + 4x^2 + ... + 4x^{n-1} - (4n-3)x^n.$$

$$S(1-x) = 1 + \frac{4x}{1-x}[1 - x^{n-1}] - (4n-3)x^n$$

$$S = \frac{1}{1-x}\left[1 + \frac{4x}{1-x} - \frac{4x^n}{1-x} - (4n-3)x^n\right] = -\frac{(4n-3)}{4}[1-(4n+1)+(4n-3)x^n-(4n-3)x^n] = n(4n-3).$$

Example # 25 :

Find sum to infinite terms of the series $1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + 4\left(\frac{11}{10}\right)^3 + ...$

Solution.

Let $x = \left(\frac{11}{10}\right)$

Let $S = 1 + 2x + 3x^2 + 4x^3 + ...$ ....(i)

$xS = x + 2x^2 + 3x^3 + ...$ ....(ii)

(i) - (ii) $\Rightarrow (1-x)S = 1 + x + x^2 + x^3 + ...$

or $S = \frac{1}{(1-x)^2}$

$$S = \frac{1}{\left(1-\frac{11}{10}\right)^2} = 100$$

Example # 26

Evaluate : $1^2 + 2^2x + 3^2x^2 + 4^2x^3.......$ upto infinite terms for $|x| < 1$.

Solution :

Let $s = 1^2 + 2^2x + 3^2x^2 + 4^2x^3............ \infty$ ...(i)

$xs = 1^2x + 2^2x^2 + 3^2x^3.............. \infty$ ...(ii)

(i) – (ii)

$(1-x)s = 1 + 3x + 5x^2 + 7x^3 + ...$

$$(1-x)s = \frac{1}{1-x} + \frac{2x}{(1-x)^2}$$

$$s = \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}$$

$$s = \frac{1-x+2x}{(1-x)^3}$$

$$s = \frac{1+x}{(1-x)^3}$$

Self practice problems :

(17) If $4 + \frac{4+d}{5} + \frac{4+2d}{5^2}$ .... = 1, then find d.

(18) Evaluate : $1 + 3x + 6x^2 + 10x^3 +$ ...... upto infinite term, where $|x| < 1$.

(19) Sum to n terms of the series : $1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + \ldots\ldots$

Answers.

(17) $-\frac{64}{5}$ **(18)** $\frac{1}{(1-x)^3}$ **(19)** n^2

Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being positive, then $G^2 = AH$ (i.e. A, G, H are in G.P.) and $A \geq G \geq H$.

Example # 27

The A.M. of two numbers exceeds the G.M. by 2 and the G.M. exceeds the H.M. by $\frac{8}{5}$; find the numbers.

Solution.

Let the numbers be a and b, now using the relation

$G^2 = AH = (G + 2)\left(G - \frac{8}{5}\right) \Rightarrow G = 8$; $A = 10$

i.e. $ab = 64$ also $a + b = 20$

Hence the two numbers are 4 and 16.

Results :

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$.

(ii) $\sum_{r=1}^{n} k\,a_r = \sum_{r=1}^{n} k\,a_r$.

(iii) $\sum_{r=1}^{n} k = k + k + k + \ldots\ldots\ldots\ldots\ldots n$ times $= nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Example # 28

Find the sum of the series to n terms whose n^{th} term is $3n + 2$.

Solution.

$S_n = \Sigma T_n = \Sigma(3n + 2) = 3\Sigma n + \Sigma 2 \quad = \frac{3(n+1)\,n}{2} + 2n = \frac{n}{2}(3n + 7)$

Example # 29

$T_k = k^3 + 3^k$, then find $\sum_{k=1}^{n} T_k$.

Solution :

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3^k = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{3-1} = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3}{2}(3^n-1)$$

Method of difference for finding n^{th} term :

Let u_1, u_2, u_3 .... be a sequence, such that $u_2 - u_1, u_3 - u_2$, ... is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + \text{..........} + u_n$ ....(i)

$S = \quad u_1 + u_2 + \text{..........} + u_{n-1} + u_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + \text{..........} + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + \text{.........} + (u_n - u_{n-1})$ is

either in A.P. or in G.P. then we can find u_n. So sum of series $S = \sum_{r=1}^{n} u_r$

Note : The above method can be generalised as follows :

Let u_1, u_2, u_3, ... be a given sequence.

The first differences are $\Delta_1 u_1, \Delta_1 u_2, \Delta_1 u_3$, ... where $\Delta_1 u_1 = u_2 - u_1$, $\Delta_1 u_2 = u_3 - u_2$ etc.

The second differences are $\Delta_2 u_1, \Delta_2 u_2, \Delta_2 u_3$, ..., where $\Delta_2 u_1 = \Delta_1 u_2 - \Delta_1 u_1$, $\Delta_2 u_2 = \Delta_1 u_3 - \Delta_1 u_2$ etc.

This process is continued untill the k^{th} differences $\Delta_k u_1$, $\Delta_k u_2$, ... are obtained, where the k^{th} differences are all equal or they form a GP with common ratio different from 1.

<u>**Case - 1**</u> : The k^{th} differences are all equal.

In this case the n^{th} term, u_n is given by

$u_n = a_0 n^k + a_1 n^{k-1} + \text{.....} + a_k$, where $a_0, a_1, \text{....}, a_k$ are calculated by using first 'k + 1' terms of the sequence.

<u>**Case - 2**</u> : The k^{th} differences are in GP with common ratio r ($r \neq 1$)

The n^{th} term is given by $u_n = \lambda r^{n-1} + a_0 n^{k-1} + a_1 n^{k-2} + \text{.....} + a_{k-1}$

Example # 30

Find the n^{th} term of the series 1, 3, 8, 16, 27, 41, ...

Solution.

$s = 1 + 3 + 8 + 16 + 27 + 41 + \text{......} T_n$ .....(i)

$s = \quad 1 + 3 + 8 + 16 + 27 \text{.......} T_{n-1} + T_n$ .....(ii)

(i) – (ii)

$T_n = 1 + 2 + 5 + 8 + 11 + \text{.......} (T_n - T_{n-1})$

$$T_n = 1 + \left(\frac{n-1}{2}\right)[2 \times 2 + (n-2)3] = \frac{1}{2}[3n^2 - 5n + 4]$$

Example # 31

Find the sum to n terms of the series 5, 7, 13, 31, 85 + ...

Solution.

Successive difference of terms are in G.P. with common ratio 3.

$T_n = a(3)^{n-1} + b$

$a + b = 5$

$3a + b = 7 \quad \Rightarrow \quad a = 1, b = 4$

$T_n = 3^{n-1} + 4$

$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = (1 + 3 + 3^2 + \ldots\ldots + 3^{n-1}) + 4n$

$\frac{1}{2}[3^n + 8n - 1]$

Method of difference for finding s_n

If possible express r^{th} term as difference of two terms as $t_r = \pm (f(r) - f(r \pm 1))$. This can be explained with the help of examples given below.

$t_1 = f(1) - f(0)$,

$t_2 = f(2) - f(1)$,

$\vdots \quad \vdots \quad \vdots$

$t_n = f(n) - f(n-1)$

$\Rightarrow \quad S_n = f(n) - f(0)$

Example # 32

Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 + ...

Solution.

$T_r = (3r - 1)(3r + 2) = 9r^2 + 3r - 2$

$$S_n = \sum_{r=1}^{n} T_r = 9\sum_{r=1}^{n} T_r + 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 3\left(\frac{n(n+1)}{2}\right) - 2n$$

$= 3n(n + 1)^2 - 2n$

Example # 33

Sum to n terms of the series $\frac{1}{(1+x)(1+3x)} + \frac{1}{(1+3x)(1+5x)} + \frac{1}{(1+5x)(1+7x)} + \ldots\ldots\ldots$

Solution.

Let T_r be the general term of the series

$$T_r = \frac{1}{[1+(2r-1)x][1+(2r+1)x]}$$

So $$T_r = \frac{1}{2x}\left[\frac{(1+(2r+1)x)-(1+(2r-1)x)}{(1+(2r-1)x)(1+(2r+1)x)}\right] = \left[\frac{1}{(1+(2r-1)x)} - \frac{1}{(1+(2r+1)x)}\right]$$

$\therefore \quad S_n = \Sigma T_r = T_1 + T_2 + T_3 + \ldots\ldots\ldots + T_n$

$$= \frac{1}{2x}\left[\frac{1}{1+x} - \frac{1}{(1+(2n+1)x)}\right] = \frac{n}{(1+x)[1+(2n+1)x]}$$

Example # 34

Sum to n terms of the series $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \ldots\ldots\ldots\ldots$

Solution.

$$T_n = \frac{1}{(3n-2)(3n+1)(3n+4)} = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \frac{1}{6}\left[\left(\frac{1}{1.4}-\frac{1}{4.7}\right)+\left(\frac{1}{4.7}-\frac{1}{7.10}\right)+......+\frac{1}{(3n-2)(3n+1)}-\frac{1}{(3n+1)(3n+4)}\right]$$

$$= \frac{1}{6}\left[\frac{1}{4}-\frac{1}{(3n+1)(3n+4)}\right]$$

Example # 35

Find the general term and sum of n terms of the series
1 + 5 + 19 + 49 + 101 + 181 + 295 + ...

Solution :

The sequence of difference between successive term 4, 14, 30, 52, 80 .....
The sequence of the second order difference is 10, 16, 22, 28, ... clearly it is an A.P>
so let nth term

$T_n = an^3 + bn^2 + cn + d$

$a + b + c + d = 1$ ....(i)
$8a + 4b + 2c + d = 5$ ....(ii)
$27a + 9b + 3c + d = 19$ ....(iii)
$64a + 16b + 4c + d = 49$ ....(iv)

from (i), (ii), (iii) & (iv)

$a = 1, b = -1, c = 0, d = 1 \Rightarrow T_n = n^3 - n^2 + 1$

$$s_n = \Sigma(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$$

Self practice problems :

(25) Sum to n terms the following series

(i) $\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+...........$

(ii) $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4)$ ....

(iii) $4 + 14 + 30 + 52 + 82 + 114 +$ ....

(26) If $\sum_{r=1}^{n} T_r = (n + 1)(n + 2)(n + 3)$ then find $\sum_{r=1}^{n}\frac{1}{T_r}$

Answers.

(25) **(i)** $\frac{2n+n^2}{(n+1)^2}$ **(ii)** $\frac{n(n+1)(n+2)}{6}$ **(iii)** $n(n + 1)^2$

(26) $\frac{n^3+3n}{2(n+1)(n+2)}$

Exercise-1

PART – I : PRE RMO

1. In the following two A.P.'s how many terms are identical ?
2, 5, 8, 11 .... to 60 terms and 3, 5, 7, ... 50 terms

2. The value of $9^{1/3} . 9^{1/9} . 9^{1/27}$ ......upto ∞, is-

3. The sum of 10 terms of the series 0.7 + .77 + .777 + ...is-

4. The n^{th} terms of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ...$ is

5. The sum of infinite terms of the series $5 - \frac{7}{3} + \frac{9}{3^2} - \frac{11}{3^3} + ...\infty$ is

6. The sum of the series 1.2 + 2.3 + 3.4 + ...up to 20 terms is

7. $\sum_{r=2}^{\infty} \frac{1}{r^2 - 1}$ is equal to:

8. If $(1^2 - t_1) + (2^2 - t_2) + ... + (n^2 - t_n) = \frac{1}{3} n(n^2 - 1)$, then t_n is

9. If x > 0, then the expression $\frac{x^{100}}{1 + x + x^2 + x^3 + ... + x^{200}}$ is always less than or equal to

10. Given the sequence a, ab, aab, aabb, aaabb,aaabbb, ... upto 2004 terms, the total number of times a's and b's are used from 1 to 2004 terms are :

11. A sequence $a_0, a_1, a_2, a_3, ..., a_n$ ... is defined such that $a_0 = a_1 = 1$ and $a_{n+1} = (a_{n-1} a_n) + 1$ for $n \geq 1$. Which of the following is true ?

12. The first two terms of a sequence are 0 and 1. The n^{th} terms $T_n = 2T_{n-1} - T_{n-2}$, $n \geq 3$. For example the third term $T_3 = 2T_2 - T_1 = 2 - 0 = 2$. The sum of the first 2006 terms of this sequence is :

13. Consider the following sequence : $a_1 = a_2 = 1$, $a_i = 1 +$ minimum $\{a_{i-1}, a_{i-2}\}$ for i > 2. Then $a_{2006} =$

14. The sum of $\frac{1}{2\sqrt{1} + 1\sqrt{2}} + \frac{1}{3\sqrt{2} + 2\sqrt{3}} + \frac{1}{4\sqrt{3} + 3\sqrt{4}} + ... + \frac{1}{25\sqrt{24} + 24\sqrt{25}}$ is

15. If f(x) + f(1–x) is equal to 10 for all real numbers x then $f\left(\frac{1}{100}\right) + f\left(\frac{2}{100}\right) + f\left(\frac{3}{100}\right) + ... + f\left(\frac{99}{100}\right)$ equals

16. Consider the sequence 4,4,8,2,0,2,2,4,6,0,.... where the nth term is the units place of the sum of the previous two terms for $n \geq 3$. If S_n is the sum to n terms of this sequence then the smallest 'n' for which $S_n > 2010$ is :

17. For some natural number 'n', the sum of the fist 'n' natural numbers is 240 less than the sum of the first (n + 5) natural numbers. Then n itself is the sum of how many natural numbers starting with 1.

18. An arithmetical progression has positive terms. The ratio of the difference of the 4^{th} and 8^{th} term to 15^{th} term is $\frac{4}{15}$ and the square of the difference of the 4^{th} and the 1^{st} term is 225. Which term of the series is 2015 ?

19. The 12 numbers $a_1, a_2, ..., a_{12}$ are in arithmetical progression. The sum of all these numbers is 354. Let $P = a_2 + a_4 + ... + a_{12}$ and $Q = a_1 + a_3 + ... + a_{11}$. If the ratio P : Q is 32 : 27, the common difference of the progression is

20. Each term of a sequence is the sum of its preceding two terms from the third term onwards. The second term of the sequence is –1 and the 10th term is 29. The first term is ____ .

21. n is a natural number. It is given that (n + 20) + (n + 21) + ...+ (n + 100) is a perfect square. Then the least value of n is __________.

22. In a G.P. of real numbers, the sum of the first two terms is 7. The sum of the first six terms is 91. The sum of the first four terms is __________ .

23. In a potato race, a bucket is placed at the starting point, which is 7 meter from the first potato. The other potatoes are placed 4 m a part in a straight line from the first one. There are n potatoes in the line. Each competitor starts from the bucket, picks up the nearest potato, runs back with it, drops in the bucket, runs back to pick up the next potato, runs to the bucket and drops it and this process continues till all the potatoes are picked up and dropped in the bucket. Each competitor ran a total of 150 m. The number of potatoes is.________.

24. The coefficient of the quadratic equation $ax^2 + (a + d)x + (a + 2d) = 0$ are consecutive terms of a positively valued, increasing arithmetic sequence. Determine the least integral value of $\frac{d}{a}$ such that the equation has real solutions.

25. Find the sum of integers from 1 to 100 that are divisible by 2 or 5

26. The sum of three numbers in A.P. is 27, and their product is 504, find them.

27. Three friends whose ages form a G.P. divide a certain sum of money in proportion to their ages. If they do that three years later, when the youngest is half the age of the oldest, then he will receive 105 rupees more that he gets now and the middle friend will get 15 rupees more that he gets now, then find the ages of the friends

28. The roots of the equation $x^5 - 40x^4 + ax^3 + bx^2 + cx + d = 0$ are in GP. If sum of reciprocals of the roots is 10, then find $|c|$ and $|d|$

29. Let T_n denotes the n^{th} term of a G.P. with common ratio 2 and $(\log_2(\log_3(\log_{512} T_{100}))) = 1$. If three sides of a triangle ABC are the values of $(T_1 + T_2)$, T_2 and T_3 then area of the triangle is $\frac{\sqrt{2160}}{N}$, where N is a positive integer. Find the remainder when N is divided by 2^{10}.

30. If a, b, c are in A.P. and if $(b - c) x^2 + (c - a) x + a - b = 0$ and $2(c + a) x^2 + (b + c)x = 0$ have a common root, then show that a^2, c^2, b^2 are in A.P.

31. Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried out the job starting with the stone in the middle, carrying stones in succession, thereby covering a distance of 4.8 km. find the number of stones

32. a, b, c are positive real numbers forming a. G.P. If $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that d/a, e/b, f/c are in A.P.

33. Determine all pairs (a, b) of real numbers such that 10, a, b, ab are in arithmetic progression.

34. If $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+....+\sqrt{1+\frac{1}{(1999)^2}+\frac{1}{(2000)^2}} = x-\frac{1}{x}$, then find the value of x

35. If n is any positive integer, then find the number whose square is $\underbrace{111........1}_{2n\text{ times}} - \underbrace{222........2}_{n\text{ times}}$

36. Find the sum of infinite terms of the series : $\frac{3}{2.4} + \frac{5}{2.4.6} + \frac{7}{2.4.6.8} + \frac{9}{2.4.6.8.10} + ...$

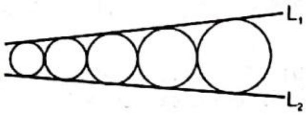
37. If $S_1, S_2, S_3, ..., S_{2n}$ are the sums of infinite geometric series whose first terms are respectively $1, 2, 3, ..., 2n$ and common ratio are respectively, $\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{2n+1}$, find the value of, $S_1^2 + S_2^2 + ... + S^2_{2n-1}$.

38. Find the Sum of series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots\ldots$ upto n terms

39. Let $a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$, $j \in N$. If $b_j = a_j^2 + a_j$, sum of the infinite series formed by b_j's is $(10 + \alpha)$ where $\left[\frac{1}{\alpha}\right]$ is equal to ([ ] represent greatest integer function)

40. The sequence 9, 18, 27, 36, 45, 54,..... consists of successive multiples of 9. This sequence is then altered by multiplying every other term by –1, starting with the first term, to produce the new sequence –9, 18, –27, 36, –45, 54,......... If the sum of the first n terms of this new sequence is 180, determine n.

41. As shown in the figure, the five circles are tangent to one another consecutively and to the lines L_1 and L_2. If the radius of the largest circle is 18 and that of the smaller one is 8, If radius of the middle circle is r, then find the value of r/2 .



42. The arithmetic mean of the nine numbers in the given set {9, 99, 999, ... 999999999} is a 9 digit number N, all whose digits are distinct. Then which digit does not appear in number

43. Let a sequence whose n^{th} term is $\{a_n\}$ be defined as

$a_1 = \frac{1}{2}$ and $(n-1)a_{n-1} = (n+1)a_n$ for $n \geq 2$ then find $\lim_{n\to\infty} S_n$

44. If $x = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \ldots\ldots + \frac{1001^2}{2001}$, $y = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \ldots\ldots + \frac{1001^2}{2003}$, then $\frac{[x-y]}{10}$ is equal to (where [.] denotes greatest integer function)

45. Let K is a positive integer such that 36 + K, 300 + K, 596 + K are the squares of three consecutive terms of an arithmetic progression. Find K.

PART – II : RMO

1. Show that all the terms of a strictly increasing A.P. of positive integers cannot be prime numbers.

2. The first term of an A.P. is equal to first term of a G.P. and first term of an H.P.. Further the last terms of all these three sequences are also equal, moreover these three sequences also ghave equal number of terms. If A, G, H are the r^{th} term of A.P., G.P., H.P., respectively then show that $A \geq G \geq H$

3. Let sequence $\{a_n\}$ be defined as below, $a_1 = 1$, when $n \geq 1$, $a_{n+1} = a_n + \frac{1}{a_n}$, then show that $12 < a_{75} < 15$

4. Show that for all $\alpha > 1$ there exists an infinite sequence of positive real numbers $x_1, x_2, \ldots,$ such that $x_{n+2} = \sqrt{\alpha x_{n+1} - x_n}$ for all $n \geq 1$

5. A strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that for ever positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1.

6. Consider the sequence $a_{n+1} = \frac{1}{1-a_n}$, $n \geq 1$ .Given that $a_1 = \frac{1}{2}$ find sum of first 100 terms of this sequence

7. Find the value of $\dfrac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots+\frac{1}{997^2}+\frac{1}{999^2}-\frac{1}{1002^2}-\frac{1}{1004^2}-\frac{1}{1006^2}-\ldots-\frac{1}{2000^2}}{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{999^2}+\frac{1}{1000^2}}$

8. A geometrical progression consists of 37 positive integers. The first and the last terms are relatively prime numbers. Prove that the 19^{th} terms of the progression in the 18^{th} power of some positive integer.

Exercise-2

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

1. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is the value of $\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}$?= **[PRMO 2013]**

2. Let m be the smallest odd positive integer for which 1 + 2 + ... + m is a square of an integer and let n be the smallest even positive integer for which 1 + 2 + ... + n is a square of an integer. What is the value of m + n ? **[PRMO 2013]**

3. What is the maximum possible value of k for which 2013 can be written as sum of k consecutive positive integers ? **[PRMO 2013]**

4. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014th term of the sequence ? **[PRMO 2014]**

5. A sequence of positive $(a_1, a_2, \ldots a_n)$ is called good if $a_i = a_1 + a_2 + \ldots + a_{i-1}$ for all $2 \le i \le n$. What is the maximum possible value of n for a good sequence such that $a_n = 9216$? **[PRMO 2015]**

6. Find the sum $S = \sum_{k=1}^{2015} (-1)^{\frac{k(k+1)}{2}} k$ **[PRMO 2015]**

7. A new sequence is obtained from the sequence of positive integers 1, 2, ..., by deleting all the perfect squares. What is the 2015-th term from the beginning of the new sequence ? **[PRMO 2015]**

8. Let E(n) denote the sum of the even digits of n. For example, E(1243) = 2 + 4 = 6. What is the value of E(1) + E(2) + E(3) + ... + E(100) ? **[PRMO 2015]**

9. Consider the sequence 1, 3, 3, 3, 5, 5, 5, 5, 5, 7, 7, 7, 7, 7, 7, 7,..... and evaluate its 2016th term. **[PRMO 2016Delhl]**

10. Find the value of the expression
$$\frac{(3^4+3^2+1).(5^4+5^2+1).(7^4+7^2+1).(9^4+9^2+1).(11^4+11^2+1).(13^4+13^2+1)}{(2^4+2^2+1).(4^4+4^2+1).(6^4.6^2+1).(8^4+8^2+1).(10^4+10^2+1).(12^4+12^2+1)}$$
When written in lowest form. **[PRMO 2016Delhi]**

11. Let $S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{99}} + \frac{1}{\sqrt{100}}$. Find [S] **[PRMO 2016WB]**
You may use the fact that $\left(\sqrt{n} < \frac{1}{2}\right)\left(\sqrt{n}+\sqrt{n+1}\right) < \sqrt{n+1}$ for all integers $n \ge 1$.

12. For positive real numbers x and y, define their special mean to be average of their arithmetic and geometric means. Find the total number of pairs of integers (x, y), with $x \le y$, from the set of numbers (1, 2, ...,2016), such that the special mean of x and y is a perfect square. **[PRMO 2016WB]**

13. Let u, v, w be real numbers in geometric progression such that u > v > w. Suppose $u^{40} = v^n = w^{60}$. Find the value of n. **[PRMO 2017]**

14. Let the sum $\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of q – p. **[PRMO 2017]**

15. Suppose x is a positive real number such that {x}, [x] and x are in the geometric progression. Find the least positive integer n such that $x^n > 100$. (Here [x] denotes the integer part of x and {x} = x – [x]) **[PRMO 2017]**

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term. **[PRMO 2017]**

17. Let N = 6 + 66 + 666 + ... + 666....66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N ? **[PRMO 2018]**

18. What is the value of $\sum_{\substack{1\le i<j\le 10\\ i+j=\text{odd}}}(i+j)-\sum_{\substack{1\le i<j\le 10\\ i+j=\text{even}}}(i+j)$? **[PRMO 2018]**

19. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n ? **[PRMO 2018]**

PART – II : PREVIOUSLY ASKED QUESTION OF RMO

1. Prove that there exist two infinite sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ of positive integers such that the following conditions hold simultaneously : **[RMO-2006]**
   (i) $1 < a_1 < a_2 < a_3 < ...;$
   (ii) $a_n < b_n < a_n^2$, for all $n \ge 1$;
   (iii) $a_n - 1$ divides $b_n - 1$, for all $n \ge 1$;
   (iv) $a_n^2 - 1$ divides $b_n^2 - 1$, for all $n \ge 1$.

2. Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$. Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which a = 20 and b divides c. **[RMO-2008]**

3. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off ? **[RMO-2009]**

4. For each integer $n \ge 1$, define $a_n = \left[\frac{n}{[\sqrt{n}]}\right]$, where [x] denotes the largest integer not exceeding x, for any real number x. Find the number of all n is the set {1, 2, 3,....,2010} for which $a_n > a_{n+1}$. **[RMO-2010]**

5. Let $a_1, a_2, ... a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let
   (i) $a_1^2 + a_3^2 + ... + a_{2n-1}^2 = x$, **[RMO-2014]**
   (ii) $a_2^2 + a_4^2 + ... + a_{2n}^2 = y$, and
   (iii) $a_n + a_{n+1} = z$.
   Express d in terms of x, y, z, n.

6. Let $\langle a_1, a_2, a_3, ...\rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression. **[RMO-2016]**

7. **(a)** Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.
   (b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression. **[RMO-2016]**

8. **(i)** Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.
   (ii) Find with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube. **[RMO-2016]**

Answers

Exercise-1

PART – I : PRE RMO

1.	17	2.	3	3.	$\frac{7}{81}\left(89+\frac{1}{10^{10}}\right)$			4.	$\frac{3n-2}{5^{n-1}}$
5.	27/8	6.	3080	7.	¾			8.	n
9.	$\frac{1}{201}$	10.	1002 × 1003 a's and $(1002)^2$ b's					11.	$4 \nmid a_{2004}$
12.	$\frac{2005\times2006}{2}$	13.	1003	14.	$\frac{4}{5}$			15.	495
16.	503	17.	9	18.	403			19.	5
20.	3	21.	4	22.	28				
23.	m = 7, n = 15.	24.	3059	25.	3050			26.	4, 9, 14
27.	12, 18, 27	28.	$\lvert d\rvert = 32$, $\lvert c\rvert = 320$			29.	0	31.	31
33.	$(4,-2)$, $\left(\frac{5}{2},-5\right)$			34.	$x = 2000, -\frac{1}{2000}$			35.	$\underbrace{333.......3}_{n\ times}$
36.	1/2			37.	$\frac{n\ (2n+1)\ (4n+1)}{3}-1$				
38.	$\frac{n(n+1)^2(n+2)}{12}$			39.	1	40.	40	41.	6
42.	0	43.	1	44.	50	45.	925		

PART – II : RMO

5.	504	6.	50	7.	$\frac{3}{4}$

Exercise-2

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

1.	9	2.	9	3.	61	4.	370	5.	12
6.	0	7.	2060	8.	400	9.	89	10.	61
11.	18	12.	506	13.	48	14.	83	15.	10
16.	36	17.	33	18.	55	19.	17		

PART – II : PREVIOUSLY ASKED QUESTION OF RMO

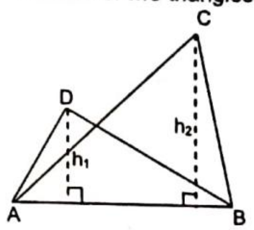
2. (20, 39, 780), (20, 38, 380), (20, 36, 180), (20, 135, 140), (20, 30, 60)

3. 3   4. $2^2-1, 3^2-1, ..., 44^2-1$ 5. $d = \frac{y-x}{zn}$

Geometry

Geometry

1. The area of two triangles having equal bases (heights) are in the ratio of their heights (bases).



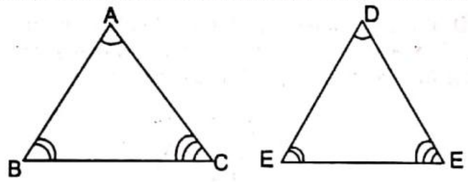
$$\frac{\text{Area of } \Delta\ ADB}{\text{Area of } \Delta\ ACB} = \frac{h_1}{h_2}$$

2. If ABC and DEF are two triangles, then the following statements are equivalent :

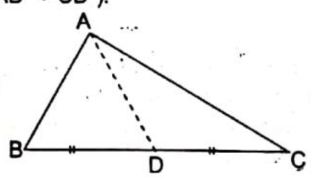
(A) $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ (B) $\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$.

(C) $\frac{AB}{AC} = \frac{DE}{DF}$ and $\angle A = \angle D$.

Two triangles satisfying any one of these conditions are said to be similar to each other.



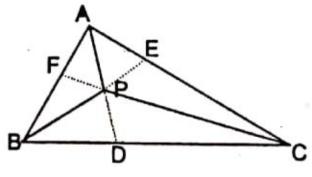
3. Appolonius Theorem : If D is the midpoint of the side BC in a triangle ABC then $AB^2 + AC^2 = 2(AD^2 + BD^2) = 2(AD^2 + CD^2)$.



4. Ceva's Theorem: If ABC is a triangle, P is a point in its plane and AP, BP, CP meet the sides BC, CA, AB in D, E, F respectively then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1$

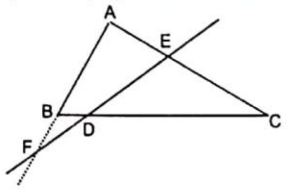
Conversely, if D, E, F are points on the (possibly extended) sides BC, CA, AB respectively and the above relation holds good, then AD, BE, CF concur at a point.

Lines such as AD, BE, CF are called Cevians.

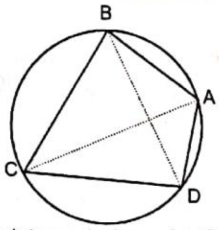


5. Menelaus's Theorem : If ABC is a triangle and a line meets the sides BC, CA, AB in D, E, F respectively then $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1$

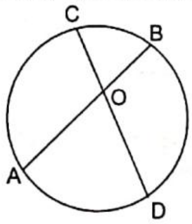
taking directions of the line segments into consideration i.e., for example, CD = –DC. Conversely if on the sides BC, CA, AB (possibly extended) of a triangle ABC, points D, E, F are taken respectively such that the above relation holds good, then D, E, F are collinear.



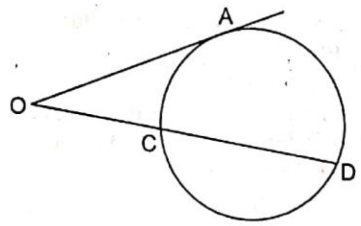
6. Ptolemy's theorem : If ABCD is a cyclic quadrilateral, then AB. CD + AD. BC = AC. BD
Conversely, if in a quadrilateral ABCD the above relation is true, then the quadrilateral is cyclic.



7. If two chords AB and CD of a circle intersect at a point O (which may lie inside or outside the circle), then AO.OB = CO. OD. Conversely, if AB and CD are two line segments intersecting at O such that AO.OB = CO.OD, then the four points A, B, C, D are concyclic.



8. If OA is tangent to a circle at A from a point O outside the circle and OCD is any secant of the circle (that is, a straight line passing through O and intersecting the circle at C and D), then OA^2 = OC.OD. Conversely, if OA and OCD are two distinct line segments such that OA^2 = OC .OD, then OA is a tangent at A to the circumcircle of triangle ACD.



9. If AB is a line segment in a plane, then set of points P in the plane such that $\frac{AP}{PB}$ is a fixed ratio $\lambda(\neq 0$ or $1)$ constitute a circle, called the Appolonius circle. If C and D are two points on AB dividing the line segment AB in the ratio λ : 1 internally and externally, then C and D themselves are two points on

the circle such that CD is a diameter. Also, for any point P on the circle, PC and PD are internal and external bisectors of $\angle APB$.

EXAMPLE-1.

Let ABC be an isosceles triangle with AB = AC and let Γ denote its circumcircle. A point D is on the arc AB of Γ not containing C and a point E is on the arc AC of Γ not containing B such that AD = C E. Prove that BE is parallel to AD.

Sol. We note that triangle AEC and triangle BDA are congruent. Therefore AE = BD and hence $\angle ABE = \angle DAB$. This proves that AD is parallel to BE.

EXAMPLE-2.

Let ABCD be a convex quadrilateral with AB = a, BC = b, CD = c and DA = d. Suppose $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$ and the area of ABCD is 60 square units. If the length of one of the diagonals is 30 units, determine the length of the other diagonal.

Solution.

Given AB = a, BC = b, CD = c and DA = d and $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$

$\Rightarrow \quad 2(a^2 + b^2 + c^2 + d^2 - ab - bc - cd - da) = 0$

$\Rightarrow \quad (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 = 0$

$\Rightarrow \quad$ So $a - b = b - c = c - d = d - a = 0$

$\Rightarrow \quad a = b = c = d$

Hence ABCD a Rhombus

Area of Rhombus = 60 sq. units

$\frac{1}{2} d_1 d_2 = 60$

$\frac{1}{2} \times 30 \times d_2 = 60$

$d_2 = 4$

So length of the other diagonal is 4 units

EXAMPLE-3.

In a triangle ABC, points D and E are on segments BC and AC such that BD = 3DC and AE = 4EC. Point P is on line ED such that D is the midpoint of segment EP. Lines AP and BC intersect at point S. Find the ratio BS/SD.

Solution.

Let F denote the midpoint of the segment AE. Then it follows that DF is parallel to AP. Therefore, in triangle ASC we have C D/SD = C F /F A = 3/2. But DC = BD/3 = (BS + SD)/3. Therefore BS/SD = 7/2.

Example 4.

Three straight lines are drawn through a point P lying inside a triangle ABC, parallel to its sides. The area of the resulting triangles are 1,4 and 9 sq.units. Find the area of ΔABC in square units.

Sol. Let the lines through P, parallel to BC meet AB and AC at L and M respectively. Similarly for lines parallel to AB and AC. Note that triangle PLQ, PNM and PRO are similar to the triangle ABC. Let Δ be the area of ΔABC.

then $\frac{1}{\Delta} = \frac{LP^2}{BC^2}, \frac{4}{\Delta} = \frac{PM^2}{BC^2}$ and $\frac{9}{\Delta} = \frac{OR^2}{BC^2}$

Now BC = BO + OR + RC

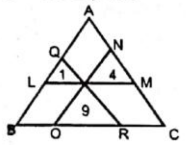
$\Rightarrow$ BC = LP + OR + PM

$\Rightarrow$ $BC = \frac{BC}{\sqrt{\Delta}} + \frac{3BC}{\sqrt{\Delta}} + \frac{2BC}{\sqrt{\Delta}}$

$\Rightarrow$ $BC = \frac{6}{\sqrt{\Delta}} BC$

$\Rightarrow$ $\sqrt{\Delta} = 6$

$\Rightarrow$ Δ = 36 sq.cm.



Example 5.

Inside a unit square, all isosceles triangles whose base is a side of the square and whose vertex is the midpoint of the opposite side are drawn. If the area of the octagon determined by the intersection of these four triangles is $\frac{1}{\lambda}$ (where $\lambda \in N$) then find the value of λ.

Solution.

The octagon is PQRSTUVW

Let O be the centre of PQRSTUVW

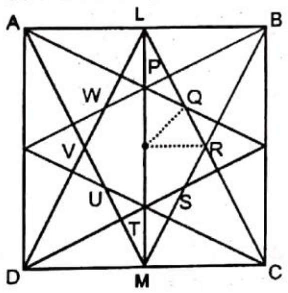
O(1/2, 1/2)

P(1/2, 3/4)

R(3/4, 1/2)

Q is point of intersection of lines $2x + y = 2$ and $x + 2y = 2$ $\Rightarrow$ Q(2/3, 2/3)

So area of PQRSTUVW = 4 × area of quad. ARQP = 1/24



Exercise-1

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

1. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?

2. In ΔABC, we have AC = BC = 7 and AB = 2. Suppose that D is a point on line AB such that B lies between A and D and CD = 8. What is the length of the segment BD?

3. In rectangle ABCD, AB = 5 and BC = 3. Points F and G are on line segment CD so that DF = 1 and GC = 2. Lines AF and BG intersect at E. What is the area of ΔAEB?

4. ABCD is a square and AB = 1. Equilateral triangle AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY?

5. O and I are the circumcentre and incentre of ΔABC respectively. Suppose O lies in the interior of ΔABC and I lies on the circle passing through B, O and C. What is the magnitude of $\angle BAC$ in degrees?

6. PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of region QXROQ bounded by the two semicircular arcs?

7. Let AD and BC be the parallel sides of a trapezium ABCD. Let P and Q be the midpoints of the diagonals AC and BD. If AD = 16 and BC = 20, what is the length of PQ?

8. In a triangle ABC, let H, I and O be the orthocenter, incentre and circumcentre, respectively. If the points B, H, I, C lie on a circle, what is the magnitude of $\underline{|BOC}$ in degrees?

9. Three points X, Y, Z are on a straight line such that XY = 10 and XZ = 3. What is the product of all possible values of YZ ?

10. Let ABC be an equilateral triangle. Let P and S be point on AB and AC respectively and let Q and R be point on BC such that PQRS is a rectangle. If PQ = $\sqrt{3}$ PS and the area of PQRS is $28\sqrt{3}$, what is the length of PC?

11. Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral ABCD and let A_2, B_2, C_2, D_2 be the midpoints of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of ABCD?

12. Let S be a circle with centre O. A chord AB, not a diameter, divides S into two regions R_1 and R_2 such that O belongs to R_2. Let S_1 be a circle with centre in R_1, touching AB at X and S internally. Let S_2 be circle with centre in R_2, touching AB at Y, the circle S internally and passing through the centre of S. The point X lies on the diameter passing through the centre of S_2 and $\underline{|YXO} = 30°$. If the radius of S_2 is 100 then what is the radius of S_1?

13. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. if the ratio of the area of quadrilateral BXYC be the area of triangle ABC is 13 : 18 and BC = 12 then which is the length of AC?

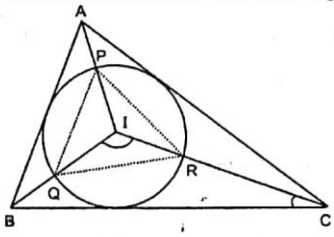
14. Let ABCD be a convex quadrilateral with perpendicular diagonals. If AB = 20, BC = 70 and CD = 90, then what is the value of DA?

15. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that AX:XB= 1 : 2 and AY : YC = 2 : 1. If the area of triangle AXY is 10 then what is the area of triangle ABC?

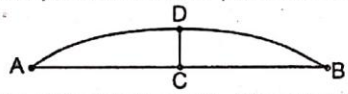
16 Let ABCD be a convex quadrilateral with $\angle DAB = \angle BDC = 90°$. Let the incircles of triangle ABD and BCD touch BD at P and Q, respectively with P lying in between B and Q. If AD = 999 and PQ = 200, then what is the sum of the radii of the incircles of triangles ABD and BDC?

17. Let XOY be a triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that XN = 19 and YM = 22. What is XY?

18. In a triangle ABC, let I denote the incentre. Let the lines AI, BI and CI intersect the incircle at P, Q and R, respectively. If $\angle BAC = 40°$, what is the value of $\angle QPR$ in degrees?



19. The figure below shows a broken piece of a circular plate made of glass.



C is midpoint of AB and D is the midpoint of arc AB. Given that AB = 24cm and CD = 6cm, what is the radius of the plate in centimetre? (The figure is not drawn to scale).

20. A 2 × 3 rectangle and a 3 × 4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square?

21. In rectangle ABCD, AB = 8 and BC = 20. Let P be a point on AD such that $\angle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?

22. In acute angled triangle ABC, let D be the foot of the altitude from A and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\angle BAE = 40°$. If $\angle DAE = \angle DEF$, what is the magnitude of $\angle ADF$ in degrees?

23. The circle ω touches the circle Ω internally at P. The centre O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume PY > PX. Let PY intersect ω at Z. If YZ = 2PZ, what is the magnitude of $\angle PYX$ in degrees?

24. Let ΔABC be an equilateral triangle with each side $2\sqrt{3}$. Let P be a point outside the triangle such that the points A and P lie in the opposite sides of the straight line BC. Let PD, PE, PF be the perpendiculars dropped on the sides BC, AC and AB respectively where D, foot of the perpendicular, lies inside the line segment BC. Let PD = 2. What is the value of PE + PF.

25. In trapezium PQRS, QR || PS. Let QR = 1001, PS = 2015. Also, let $\angle P = 37°$ and $\angle S = 53°$. Finally, let X and Y be the midpoints of QR and PS, respectively. Find the length of XY.

26. A square PQRS has length of its side equal to $3 + \sqrt{5}$. Let M be the mid-point of the side RS. Also, let C_1 be the in-circle of ΔPMS and C_2 be the circle that touches the sides PQ, QR and PM. Find the radius of the circle C_2.

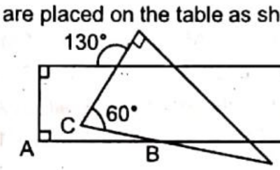
27. Let ΔABC be a triangle with base AB. Let D be the mid-point of AB and P be the mid-point of CD. Extend AB in both direction. Assuming A to be on the left of B, let X be a point on BA extended further left such that XA = AD. Similarly, let Y be a point on AB extended further right such that BY = BD. Let PX cut AC at Q and PY cut BC at R. Let the sides of ΔABC be AC = 13, BC = 14 and AB = 15. What is the area of the pentagon PQABR?

28. Suppose we wish to cut four equal circles from a circular piece of wood whose area is equal to 25π square ft. We want these circles (of wood) to be the largest in area that can possible be cut from the piece of wood. Let R ft. be the radius of each of the four new circles. Find the integer nearest to R.

29. In right-angled triangle ABC with hypotenuse AB, AC = 12, BC = 35. Let CD be the perpendicular from C to AB. Let Ω be the circle having CD as a diameter. Let I be a point outside ΔABC such that AI and BI are both tangents to the circle Ω. Let the ratio of the perimeter of ΔABI and the length of AI be m/n, where m, n are relatively prime positive integers. Find m + n.

30. Given a rectangle ABCD, determine two points K and L on the sides BC and CD such that the triangles ABK, AKL and ADL have same area.

31. Two of the Geometry box tools are placed on the table as shown. Determinant the angle $\angle ABC$.



32. Let AD be an altitude in a right triangle ABC with $\angle A = 90°$ and D on BC. Suppose that the radii of the incircles of the triangles ABD and ACD are 33 and 56 respectively. Let r be the radius of the incircle of triangle ABC. Find the value of 3(r + 7).

33. In triangle ABC right angled at vertex B, a point O is chosen on the side BC such that the circle γ centered at O of radius OB touches the side AC. Let AB = 63 and BC = 16, and the radius of γ be of the form $\frac{m}{n}$ where m, n are relatively prime positive integers. Find the value of m + n.

34. The hexagon OLYMPI has a reflex angle at O and convex at every other vertex. Suppose that $LP = 3\sqrt{2}$ units and the condition
$\angle O = 10\angle L = 2\angle Y = 5\angle M = 2\angle P = 10\angle I$
holds. Find the area (in sq units) of the hexagon.

35. Points G and O denote the centroid and the circumcenter of the triangle ABC. Suppose that $\angle AGO = 90°$ and AB = 17, AC = 19. Find the value of BC^2.

36. Consider a right-angled triangle ABC with $\angle C = 90°$. Suppose that the hypotenuse AB is divided into four equal parts by the points D, E, F, such that AD = DE = EF = FB. If $CD^2 + CE^2 + CF^2 = 350$, find the length of AB.

37. Consider a triangle ABC with AB = 13, BC = 14, CA = 15. A line perpendicular to BC divides the interior of ΔABC into two regions of equal area. Suppose that the aforesaid perpendicular cuts BC at D, and cuts ΔABC again at E. If L is the length of the line segment DE, find L^2.

38. Suppose a circle C of radius $\sqrt{2}$ touches the Y-axis at the origin (0, 0). A ray of light L, parallel to the X-axis, reflects on a point P on the circumference of C, and after reflection, the reflected ray L′ becomes parallel to the Y-axis. Find the distance between the ray L and the X-axis.

39. In a rectangle ABCD, E is the midpoint of AB , F is point on AC such that BF is perpendicular to AC and FE perpendicular to BD. Suppose BC = $8\sqrt{3}$. Find AB.

40. Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

41. Let ABCD be a rectangle and let E and F be points on CD and BC respectively such that area (ADE) = 16, area (CEF) = 9 and area (ABF) = 25. What is the area of triangle AEF ?

42. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose AB = 6, CD = 8. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n) / k$, where m, n, k are positive integers with gcd(m, n, k) = 1. What is the value of m + n + k ?

43. Let Ω_1 be a circle with centre O and let AB be diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5$, $AB_1 = 15$ and OP = 10, find the radius of Ω_1.

44. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium ABCD. The product of these areas, taken two at time, are computed. If among the six products so obtained, two product are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

45. In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r ?

46. Let ABCD be a trapezium in which AB || CD and AD $\perp$ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, Find PQ.

47. A point P in the interior of a regular hexagon is at distance 8,8,16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r ?

48. Let AB be a chord of circle with centre O. Let C be a point on the circle such that $\angle ABC = 30°$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20°$. Find the measure of $\angle CDO$ in degrees.

49. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

50. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.

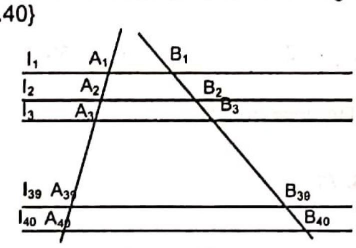
PART – II : PREVIOUSLY ASKED QUESTION OF RMO

1. Let ABC be a right angled triangle with $\angle B = 90°$ and let BD be the altitude from B on to AC. Draw DE $\perp$ AB and DF $\perp$ BC. Let P, Q, R and S be respectively the incenter of triangle DFC, DBF, DEB and DAE. Suppose S, R, Q are collinear. Prove that P, Q, R, D be on a circle. **[RMO-2015]**

2. Let ABC be a right-angled triangle with $\angle B$ = 90°. Let I be the incentre of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and prove that ID = $\sqrt{b(b-a)}$ where BC = a and CA = b **[RMO-2016]**

3. Let ABC be a triangle with centroid G. Let the circumcircle of triangle AGB intersect the line BC in X different from B; and the circumcircle of triangle AGC intersect the line BC in Y different from C. Prove that G is the centroid of triangle AXY. **[RMO-2016]**

4. Let $\ell_1, \ell_2, \ell_3, \ldots\ldots \ell_{40}$ be fourty parallel lines. As shown in the diagram, let m be another line that intersects the lines ℓ_1 to ℓ_{40} in the points $A_1, A_2, A_3, \ldots\ldots A_{40}$ respectively. Similarly, let n be another line that intersects the lines ℓ_1 to ℓ_{40} in the points $B_1, B_2, B_3, \ldots\ldots B_{40}$ respectively.
Given that $A_1B_1 = 1$, $A_{40}B_{40} = 14$, and the areas of the 39 trapeziums $A_1B_1B_2A_2$, $A_2B_2B_3A_3$,........$A_{39}B_{39}B_{40}A_{40}$ are all equal ; then count the number of segments A_iB_i whose length is a positive integer; where $i \in \{1, 2, \ldots\ldots 40\}$



[RMO-2017]

5. Let ΔABC be acute - angled; and let Γ be its circumcircle, let D be a point on minor arc BC of Γ . Let E and F be points on lines AD and AC respectively, such that BE $\perp$ AD and DF $\perp$ AC. Prove that EF|| BC if and only if D is the midpoint arc BC. **[RMO-2017]**

Answers

Exercise-1

PART – I

1.	$\frac{3\sqrt{7}}{4}$	**2.**	3 units	**3.**	12.5	**4.**	$\sqrt{3}-1$	**5.**	60°
6.	$\frac{3\pi-4}{2}$	**7.**	2	**8.**	120°	**9.**	91	**10.**	14
11.	208	**12.**	60	**13.**	36	**14.**	60	**15.**	45
16	799	**17.**	26	**18.**	55	**19.**	15	**20.**	25
21.	8	**22.**	40	**23.**	15	**24.**	5	**25.**	507
26.	2	**27.**	56	**28.**	2	**29.**	12753		

30. K is $\left(\frac{\sqrt{5}-1}{2}\right)$ BC units from B and L is $\left(\frac{\sqrt{5}-1}{2}\right)$ DA units from D

31. 10 **32.** 216 **33.** 71

34. 9 **35.** 325 **36.** 20 **37.** 112 **38.** 1

39. 24 **40.** Bonus **41.** 30 **42.** 75 **43.** 20

44. 13 **45.** (8) **46.** (84) **47.** (14) **48.** 80

49. 24 **50.** 24

Combinatorics

Combinatorics

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3\times2\times1 & ; \text{ if } n\in N \\ 1 & ; \text{ if } n=0 \end{cases}$$

Note : $n! = n\,.\,(n-1)!$; $n \in N$

Statement of binomial theorem :

$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$
where $n \in N$

or $\quad (a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$

General term :

$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + ... + {}^nC_r\, x^{n-r} y^r + ... + {}^nC_n\, x^0 y^n$
$(r+1)^{th}$ term is called general term and denoted by T_{r+1}.
$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Example # 1 : Expand the binomial $\left(\frac{2}{x}+x\right)^{10}$ up to four terms

Solution : $\left(\frac{2}{x}+x\right)^{10} = {}^{10}C_0\left(\frac{2}{x}\right)^{10} + {}^{10}C_1\left(\frac{2}{x}\right)^{9} x + {}^{10}C_2\left(\frac{2}{x}\right)^{8} x^2 + {}^{10}C_3\left(\frac{2}{x}\right)^{7} x^3 + ...$

Example # 2 : The number of dissimilar terms in the expansion of $(1 + x^4 - 2x^2)^{15}$ is
(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1-x^2)^{30}$
Therefore number of dissimilar terms = 31.

Example # 3 : Find (i) 15^{th} term of $(2x-3y)^{20}$ (ii) 4^{th} term of $\left(\frac{3x}{5}-y\right)^7$

Solution : (i) $T_{14+1} = {}^{20}C_{14}\,(2x)^6\,(-3y)^{14} = {}^{20}C_{14}\,2^6\,3^{14}\,x^6.y^{14}$

(ii) $T_{3+1} = {}^7C_3\left(\frac{3x}{5}\right)^4 (-y)^3 = -{}^7C_3\left(\frac{3}{5}\right)^4 x^4y^3$

Example # 4 : Find the number of rational terms in the expansion of $\left(2^{\frac{1}{3}}+3^{\frac{1}{5}}\right)^{600}$

Solution : The general term in the expansion of $\left(2^{\frac{1}{3}}+3^{\frac{1}{5}}\right)^{600}$ is

$$T_{r+1} = {}^{600}C_r\left(2^{\frac{1}{3}}\right)^{600-r}\left(3^{\frac{1}{5}}\right)^{r} = {}^{600}C_r\; 2^{\frac{600-r}{3}}\; 3^{\frac{r}{5}}$$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{600-r}{3}$ and $\frac{r}{5}$ must be integers.

The possible set of values of r is {0, 15, 30,45............,600}

Hence, number of rational terms is 41

Example # 5 : Find the middle term(s) in the expansion of

(i) $(1 + 2x)^{12}$ (ii) $\left(2y-\frac{y^2}{2}\right)^{11}$

Solution : (i) $(1 + 2x)^{12}$

Here, n is even, therefore middle term is $\left(\frac{12+2}{2}\right)^{th}$ term.

It means T_7 is middle term $T_7 = {}^{12}C_6\,(2x)^6$

(ii) $\left(2y-\frac{y^2}{2}\right)^{11}$

Here, n is odd therefore, middle terms are $\left(\frac{11+1}{2}\right)^{th}$ & $\left(\frac{11+1}{2}+1\right)^{th}$.

It means T_6 & T_7 are middle terms

$T_6 = {}^{11}C_5\,(2y)^6\left(-\frac{y^2}{2}\right)^5 = -2\ {}^{11}C_5\,y^{16} \Rightarrow T_7 = {}^{11}C_6\,(2y)^5\left(-\frac{y^2}{2}\right)^6 = \frac{{}^{11}C_6}{2}y^{17}$

Self practice problems

(1) Write the first three terms in the expansion of $\left(2-\frac{y}{3}\right)^6$.

(2) Expand the binomial $\left(\frac{x^2}{3}+\frac{3}{x}\right)^5$.

(3) Find term which is independent of x in $\left(x^2-\frac{1}{x^6}\right)^{16}$

Ans. (1) $64 - 64y + \frac{80}{3}y^2$ (2) $\frac{x^{10}}{243}+\frac{5}{27}x^7+\frac{10}{3}x^4+30x+\frac{135}{x^2}+\frac{243}{x^5}$.

(3) T_5 is independent of x.

Finding Remainder :

Example # 6 : Show that $7^n + 5$ is divisible by 6, where n is a positive integer.

Solution : $7^n + 5 = (1 + 6)^n + 5 = {}^nC_0 + {}^nC_1 . 6 + {}^nC_2 . 6^2 + ... + {}^nC_n\,6^n + 5$.

$= 6. C_1 + 6^2. C_2 + ... + C_n . 6^n + 6$.

$= 6\lambda$, where λ is a positive integer

Hence, $7^n + 5$ is divisible by 6.

Example # 7 : What is the remainder when 7^{81} is divided by 5.

Solution : $7^{81} = 7.7^{80} = 7. (49)^{40} = 7\,(50 - 1)^{40}$

$= 7\,[{}^{40}C_0\,(50)^{40} - {}^{40}C_1\,(50)^{39} + ... - {}^{40}C_{39}\,(50)^1 + {}^{40}C_{40}\,(50)^0]$

$= 5(k) + 7$(where k is a positive integer) $= 5\,(k + 1) + 2$

Hence, remainder is 2.

Example # 8 : Find the last digit of the number $(13)^{12}$.

Solution : $(13)^{12} = (169)^6 = (170 - 1)^6$

$= {}^6C_0\,(170)^6 - {}^6C_1\,(170)^5 + ... - {}^6C_5\,(170)^1 + {}^6C_6\,(170)^0$

Hence, last digit is 1

Example-9 : Which number is larger $(1.1)^{100000}$ or 10,000 ?

Solution : By Binomial Theorem

$(1.1)^{100000} = (1 + 0.1)^{100000} = 1 + {}^{100000}C_1\,(0.1)$ + other positive terms

$= 1 + 100000 \times 0.1$ + other positive terms

$= 1 + 10000$ + other positive terms

Hence $(1.1)^{100000} > 10,000$

Self practice problems :

(1) If n is a positive integer, then show that $6^n - 5n - 1$ is divisible by 25.

(2) What is the remainder when 3^{257} is divided by 80 .

(3) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(4) Which number is larger $(1.3)^{2000}$ or 600

Ans. **(2)** 3 **(3)** 1, 01, 001 **(4)** $(1.3)^{2000}$.

Binomial theorem for negative and fractional indices :

If $n \in R$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2\,!}x^2 + \dfrac{n(n-1)(n-2)}{3\,!}x^3 + \ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots + \dfrac{n(n-1)(n-2)\ldots\ldots.(n-r+1)}{r\,!}x^r + \ldots\ldots\ldots\ldots\ldots \infty.$

Remarks

(i) The above expansion is valid for any rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x+y)^n = \begin{cases} x^n\left(1+\dfrac{y}{x}\right)^n = x^n\left\{1+n\ .\ \dfrac{y}{x}+\dfrac{n\ (n-1)}{2\ !}\left(\dfrac{y}{x}\right)^2+\ldots..\right\} & \text{if } \left|\dfrac{y}{x}\right|<1 \\[2ex] y^n\left(1+\dfrac{x}{y}\right)^n = y^n\left\{1+n\ .\ \dfrac{x}{y}+\dfrac{n\ (n-1)}{2\ !}\left(\dfrac{x}{y}\right)^2+\ldots..\right\} & \text{if } \left|\dfrac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \dfrac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r\,!}x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots + (-1)^r x^r + \ldots\ldots\ldots\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots + x^r + \ldots\ldots\ldots\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots + (-1)^r (r + 1)\, x^r + \ldots\ldots\ldots\infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots + (r + 1)x^r + \ldots\ldots\ldots\ \infty$

Example # 10 : Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Solution: $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots\ldots(-n-r+1)}{r\,!}(-x)^r$$

$$= (-1)^r\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r\,!}(-x)^r = \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r\,!}x^r$$

$$= \frac{(n-1)!\ n(n+1)\ldots\ldots(n+r-1)}{(n-1)\,!\ r\,!}x^r$$ Hence, coefficient of x^r is $\dfrac{(n+r-1)!}{(n-1)!\ r!} = {}^{n+r-1}C_r$ Proved

Example-11 : If x is so small such that its square and higher powers may be neglected, then find the value of $\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}}$

Solution : $\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}} = \frac{1-\frac{2}{3}x+1-\frac{15x}{2}}{3\left(1+\frac{x}{9}\right)^{1/2}} = \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1+\frac{x}{9}\right)^{-1/2}$

$= \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1-\frac{x}{18}\right) = \frac{1}{2}\left(2-\frac{x}{9}-\frac{49}{6}x\right) = 1-\frac{x}{18}-\frac{49}{12}x = 1-\frac{149}{36}x$

Self practice problems :

(1) Find the possible set of values of x for which expansion of $(3-2x)^{1/2}$ is valid in ascending powers of x.

(2) Find the coefficient of x^{50} in $\frac{2-3x}{(1-x)^3}$

Ans. **(1)** $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ **(2)** -1173

Meaning of Permutation & Combination :

Fundamental counting principle :

Suppose that an operation O_1 can be done in m different ways and another operation O_2 can be done in n different ways.

(i) **Addition rule :** The number of ways in which we can do exactly one of the operations O_1, O_2 is m + n

(ii) **Multiplication rule :** The number of ways in which we can do both the operations O_1, O_2 is mn.

Note : The addition rule is true only when O_1 & O_2 are mutually exclusive and multiplication rule is true only when O_1 & O_2 are independent (The reader will understand the concepts of mutual exclusiveness and independence, in the due course)

Example # 12 : There are 8 buses running from Kota to Jaipur and 10 buses running from Jaipur to Delhi. In how many ways a person can travel from Kota to Delhi via Jaipur by bus?

Solution : Let E_1 be the event of travelling from Kota to Jaipur & E_2 be the event of travelling from Jaipur to
Delhi by the person.
E_1 can happen in 8 ways and E_2 can happen in 10 ways.
Since both the events E_1 and E_2 are to be happened in order, simultaneously,
the number of ways = 8 × 10 = 80.

Example # 13 : How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if
(i) No digit is repeated in any number. (ii) Digits can be repeated.

Solution : (i) Number of two digit numbers = 5 × 4 = 20
Number of three digit numbers = 5 × 4 × 3 = 60
Number of four digit numbers = 5 × 4 × 3 × 2 = 120
Total = 200

(ii) Number of two digit numbers = 5 × 5 = 25
Number of three digit numbers = 5 × 5 × 5 = 125
Number of four digit numbers = 5 × 5 × 5 × 5 = 625
Total = 775

Self Practice Problems :

(1) How many 4 digit numbers are there, without repetition of digits, if each number is divisible by 5 ?

(2) Using 6 different flags, how many different signals can be made by using atleast three flags, arranging one above the other?

Ans. (1) 952 (2) 1920

Arrangements :

If ${}^nP_r$ denotes the number of permutations (arrangements) of n different things, taking r at a time, then

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Example # 14 : How many three digit can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits? How many of these are even?

Solution : Three places are to be filled with 5 different objects.

$\therefore$ Number of ways = ${}^5P_3 = 5 \times 4 \times 3 = 60$

For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^4P_2$ ways.

$\therefore$ Number of even numbers = $2 \times {}^4P_2 = 24$.

Example # 15 : If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the given word.

Solution : Number of words beginning with E = ${}^4P_4 = 24$

Number of words beginning with QE = ${}^3P_3 = 6$

Number of words beginning with QS = 6

Number of words beginning withQT = 6.

Next word is 'QUEST'

$\therefore$ its rank is 24 + 6 + 6 + 6 + 1 = 43.

Self Practice Problems :

(1) Find the sum of all four digit numbers (without repetition of digits) formed using the digits 1, 2, 3, 4, 5.

(2) Six horses take part in a race. In how many ways can these horses come in the first, second and third place, if a particular horse is among the three winners (Assume No Ties)?

(3) Find the sum of all three digit numbers those can be formed by using the digits. 0, 1, 2, 3, 4.

Ans. (1) 399960 (2) 60 (3) 27200

Combination :

If ${}^nC_r$ denotes the number of combinations (selections) of n different things taken r at a time, then

$${}^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!} \text{ where } r \le n\,;\ n \in N \text{ and } r \in W.$$

NOTE : (i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_r = 0$ if $r \notin \{0, 1, 2, 3\ldots\ldots, n\}$

Example # 16 : There are fifteen players for a cricket match.

(i) In how many ways the 11 players can be selected?

(ii) In how many ways the 11 players can be selected including a particular player?

(iii) In how many ways the 11 players can be selected excluding two particular players?

Solution : (i) 11 players are to be selected from 15

Number of ways = ${}^{15}C_{11}$ = 1365.

(ii) Since one player is already included, we have to select 10 from the remaining 14

Number of ways = ${}^{14}C_{10}$ = 1001.

(iii) Since two players are to be excluded, we have to select 11 from the remaining 13.

Number of ways = ${}^{13}C_{11}$ = 78.

Example # 17 : A regular polygon has 20 sides. How many triangles can be drawn by using the vertices, but not using the sides?

Solution : The first vertex can be selected in 20 ways. The remaining two are to be selected from 17 vertices so that they are not consecutive. This can be done in ${}^{17}C_2 - 16$ ways.

∴ The total number of ways = 20 × (${}^{17}C_2$ – 16)

But in this method, each selection is repeated thrice.

∴ Number of triangles = $\frac{20 \times ({}^{17}C_2 - 16)}{3}$ = 800.

Example # 18 : 15 persons are sitting in a row. In how many ways we can select three of them if adjacent persons are not selected ?

Solution : Let $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}$ be the persons sitting in this order.

If three are selected (non consecutive) then 12 are left out.

Let P,P,P,P,P,P,P,P,P,P,P,P be the left out & q, q, q be the selected. The number of ways in which these 3 q's can be placed into the 13 positions between the P's (including extremes) is the number ways of required selection.

Thus number of ways = ${}^{13}C_3$ = 286.

Example # 19 : In how many ways we can select 4 letters from the letters of the word MISSISSIPPI?

Solution : M

IIII

SSSS

PP

Number of ways of selecting 4 alike letters = ${}^2C_1$ = 2.

Number of ways of selecting 3 alike and 1 different letters = ${}^2C_1 \times {}^3C_1$ = 6

Number of ways of selecting 2 alike and 2 alike letters = ${}^3C_2$ = 3

Number of ways of selecting 2 alike & 2 different = ${}^3C_1 \times {}^3C_2$ = 9

Number of ways of selecting 4 different = ${}^4C_4$ = 1

Total number of ways = 2 + 6 + 3 + 9 + 1 = 21

Self Practice Problems :

(1) In how many ways 7 persons can be selected from among 5 Indian, 4 British & 2 Chinese, if atleast two are to be selected from each country ?

(2) In how many ways 6 boys & 6 girls can sit at a round table so that girls & boys sit alternate?

(3) In how many ways 4 persons can occupy 10 chairs in a row, if no two sit on adjacent chairs?

(4) In how many ways we can select 3 letters of the word PROPORTION ?

Ans. (1) 100 (2) 86400 (3) 840 (4) 36

Arrangement of n things, those are not all different :

The number of permutations of 'n' things, taken all at a time, when 'p' of them are same & of one type, q of them are same & of second type, 'r' of them are same & of a third type & the remaining $n - (p + q + r)$ things are all different, is $\dfrac{n!}{p!\,q!\,r!}$.

Example # 20: In how many ways we can arrange 3 red flowers, 4 yellow flowers and 5 white flowers in a row? In how many ways this is possible if the white flowers are to be separated in any arrangement? (Flowers of same colour are identical).

Solution : Total we have 12 flowers 3 red, 4 yellow and 5 white.

Number of arrangements = $\dfrac{12!}{3!\,4!\,5!}$ = 27720.

For the second part, first arrange 3 red & 4 yellow

This can be done in $\dfrac{7!}{3!\,4!}$ = 35 ways

Now select 5 places from among 8 places (including extremes) & put the white flowers there. This can be done in ${}^8C_5$ = 56.

$\therefore$ The number of ways for the 2[nd] part = 35 × 56 = 1960.

Example # 21 : In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative positions of vowels & consonants?

Solution : The consonants in their positions can be arranged in $\dfrac{4!}{2!}$ = 12 ways.

The vowels in their positions can be arranged in $\dfrac{3!}{2!}$ = 3 ways

$\therefore$ Total number of arrangements = 12 × 3 = 36

Self Practice Problems :

(1) How many words can be formed using the letters of the word ASSESSMENT if each word begin with A and end with T?

(2) If all the letters of the word ARRANGE are arranged in all possible ways, in how many of words we will have the A's not together and also the R's not together?

Ans. (1) 840 (2) 660

Selection of one or more objects

(a) Number of ways in which atleast one object may be selected out of 'n' distinct objects, is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n - 1$$

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type, 'q' alike objects of second type and 'r' alike objects of third type, is

$$(p + 1)(q + 1)(r + 1) - 1$$

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type, 'q' alike of second type and 'r' alike of third type and rest $n - (p + q + r)$ are different, is $(p + 1)(q + 1)(r + 1)\,2^{n-(p+q+r)} - 1$

Example # 22 : There are 12 different books in a shelf. In how many ways we can select atleast one of them?

Solution : We may select 1 book, 2 books,........, 12 books.

$\therefore$ The number of ways = ${}^{12}C_1 + {}^{12}C_2 + \ldots\ldots + {}^{12}C_{12} = 2^{12} - 1.$ = 4095

Example # 23: There are 11 fruits in a basket of which 6 are apples, 3 mangoes and 2 bananas (fruits of same species are identical). How many ways are there to select atleast one fruit?

Solution : Let x be the number of apples being selected

y be the number of mangoes being selected and

z be the number of bananas being selected.
Then $x = 0, 1, 2, 3, 4, 5, 6$
$y = 0, 1, 2, 3$
$z = 0, 1, 2$
Total number of triplets (x, y, z) is $7 \times 4 \times 3 = 84$
Exclude (0, 0, 0)
$\therefore$ Number of combinations = 84 – 1 = 83.

Self Practice Problems

(1) In a shelf there are 6 physics, 4 chemistry and 3 mathematics books. How many combinations are there if (i) books of same subject are different? (ii) books of same subject are identical?

(2) From 5 apples, 4 mangoes & 3 bananas, in how many ways we can select atleast two fruits of each variety if fruits of same species are identical ?

Ans. (1) (i) 8191 (ii) 139 (2) 24

Circular Permutation :

The number of circular permutations of n different things taken all at a time is $(n - 1)!$.

If clockwise & anti–clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

Note : Number of circular permutations of n things when p are alike and the rest are different, taken all at a time, distinguishing clockwise and anticlockwise arrangement is $\frac{(n-1)!}{p!}$.

Example # 24. In how many ways can we arrange 6 different flowers in a circle? In how many ways we can form a garland using these flowers?

Solution : The number of circular arrangements of 6 different flowers = $(6 - 1)! = 120$
When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the number of ways of forming garland = $\frac{1}{2}(6 - 1)! = 60$.

Example # 25 In how many ways 6 persons can sit at a round table, if two of them prefer to sit together?

Solution : Let $P_1, P_2, P_3, P_4, P_5, P_6$ be the persons, where P_1, P_2 want to sit together.
Regard these person as 5 objects. They can be arranged in a circle in $(5 - 1)! = 24$ ways. Now P_1, P_2 can be arranged in 2! ways. Thus the total number of ways = $24 \times 2 = 48$.

Self Practice Problems :

(1) In how many ways letters of the word 'MONDAY' can be written around a circle, if vowels are to be separated in any arrangement ?

(2) In how many ways we can form a garland using 3 different red flowers,5 different yellow flowers and 4 different blue flowers, if flowers of same colour must be together?

Ans. (1) 72 (2) 17280

Method of fictious partition :

Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ${}^{n+p-1}C_n$.

Example # 26 : Find the number of solutions of the equation $x + y + z = 6$, where $x, y, z \in W$.

Solution : Number of solutions = coefficient of x^6 in $(1 + x + x^2 + ... x^6)^3$
$= \text{coefficient of } x^6 \text{ in } (1 - x^7)^3 (1 - x)^{-3}$
$= \text{coefficient of } x^6 \text{ in } (1 - x)^{-3}$
$= {}^{3+6-1}C_6 = {}^8C_2 = 28.$

Example # 27 : In a bakery four types of biscuits are available. In how many ways a person can buy 10 biscuits if he decide to take atleast one biscuit of each variety?

Solution : Let the person select x biscuits from first variety, y from the second, z from the third and w from the fourth variety. Then the number of ways = number of solutions of the equation

$x + y + z + w = 10.$

where $x = 1, 2, ...,7$
$y = 1, 2, ...,7$
$z = 1, 2, ...,7$
$w = 1, 2, ...,7$

So, number of ways = coefficient of x^{10} in $(x + x^2 + ... + x^7)^4$
$=$ coefficient of x^6 in $(1 + x + ... + x^6)^4$
$=$ coefficient of x^6 in $(1 - x^7)^4 (1 - x)^{-4}$
$=$ coefficient x^6 in $(1 - x)^{-4}$
$= {}^{4+6-1}C_6 = {}^9C_3 = 84$

Self Practice Problems:

(1) Three distinguishable dice are rolled. In how many ways we can get a total 15?

(2) In how many ways we can give 5 apples, 4 mangoes and 3 oranges (fruits of same species are similar) to three persons if each may receive none, one or more?

Ans. (1) 10 (2) 3150

Formation of Groups :

Number of ways in which (m + n + p) different things can be divided into three different groups containing m, n & p things respectively is $\frac{(m+n+p)\,!}{m\,!\,n\,!\,p\,!}$.

If m = n = p and the groups have identical qualitative characteristic then the number of groups $= \frac{(3n)!}{n!\; n!\; n!\; 3!}$.

Note : If 3n different things are to be distributed equally among three people then the number of ways $= \frac{(3n)!}{(n!)^3}$.

Example #28 : 12 different toys are to be distributed to three children equally. In how many ways this can be done ?

Solution : The problem is to divide 12 different things into three different groups.

Number of ways $= \frac{12!}{4!\;4!\;4!} = 34650.$

Example # 29 : In how many ways 10 persons can be divided into 5 pairs?

Solution : We have each group having 2 persons and the qualitative characteristic are same (Since there is no purpose mentioned or names for each pair).

Thus the number of ways $= \frac{10!}{(2!)^5 5!} = 945.$

Self Practice Problems :

(1) 9 persons enter a lift from ground floor of a building which stops in 10 floors (excluding ground floor), if it is known that persons will leave the lift in groups of 2, 3, & 4 in different floors. In how many ways this can happen?

(2) In how many ways one can make four equal heaps using a pack of 52 playing cards?

Ans. (1) 907200 (2) $\frac{52!}{(13!)^4\;4!}$

Inclusion Exclusion Principle:

If A, B, C are finite sets and U be the finite universal set then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

If $A_1, A_2, A_3 \ldots. A_n$ are finite sets then

$$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \Sigma\, n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) - \ldots.\ (-1)^{n-1}\, n\,(A_1 \cap A_2 \cap A_3 \ldots. \cap A_n)$$

Example # 30 : Amongst first hundred natural numbers how many are divisible by 2, 3 or 5

Solution : $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 50 + 33 + 20 - 16 - 6 - 10 + 3 = 74$

Self Practice Problems :

(1) A,A,B,B,C,C,D,E,F are arranged in a row so that no two alike alphabets are together. Find number of such arrangement

Ans. (1) 21960

Derrangements :

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is $n!\left(1-\frac{1}{1\,!}+\frac{1}{2\,!}-\frac{1}{3\,!}+\frac{1}{4\,!}-\ldots\ldots\ldots+(-1)^n\frac{1}{n\,!}\right)$

Example # 31 : In how many ways we can put 5 writings into 5 corresponding envelopes so that no writing go to the corresponding envelope?

Solution : The problem is the number of dearragements of 5 digits.

This is equal to $5!\left(\frac{1}{2\,!}-\frac{1}{3\,!}+\frac{1}{4\,!}-\frac{1}{5\,!}\right) = 44.$

Example # 32 : Four slip of papers with the numbers 1, 2, 3, 4 written on them are put in a box. They are drawn one by one (without replacement) at random. In how many ways it can happen that the ordinal number of atleast one slip coincide with its own number?

Solution : Total number of ways = 4 ! = 24.

The number of ways in which ordinal number of any slip does not coincide with its own number is the number of dearrangements of 4 objects $= 4\,!\left(\frac{1}{2\,!}-\frac{1}{3\,!}+\frac{1}{4\,!}\right) = 9$

Thus the required number of ways. = 24 – 9 = 15

Self Practice Problems:

(1) In a match the column question, Column I contain 10 questions and Column II contain 10 answers written in some arbitrary order. In how many ways a student can answer this question so that exactly 6 of his matching are correct ?

(2) In how many ways we can put 5 letters into 5 corresponding envelopes so that atleast one letter go to wrong envelope ?

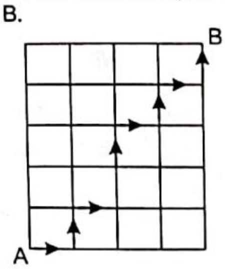
Ans. (1) 1890 (2) 119

Geometrical & Grid Problems :

Example # 33 : Let each side of smallest square of chess board is one unit in length. Find the sum of area of all possible squares whose side parallel to side of chess board.

Solution: $(8^2 \times 1^2) + (7^2 \times 2^2) + (6^2 \times 3^2) + \ldots\ldots\ldots\ldots + (1^2 \times 8^2) = 1968$

Example # 34 : A person is to walk from A to B. However, he is restricted to walk only to the right of A or upwards of A. but not necessarily in the order shown in the figure. Then find the number of paths from A to B.



Solution: No matter which path the person choses, he must walk 9 steps in total, 4 in the right direction and 5 in the upwards direction. So we have to arrange 9 steps (or which 4 are of one kind and 5 of the other), which can be done in $\frac{9!}{4!5!}$ ways.

Hence, the required number of ways $= \frac{9!}{4!5!} = \frac{9.8.7.6.5!}{4.3.2.1.5!} = 126$

Self Practice Problems :

(1) Let each side of smallest square of chess board is one unit in length. Find the total number of rectangles (including squares) whose side parallel to side of chess board.

(2) Find the number of ways of selecting pair of black squares in chessboard such that they have exactly one common corner

Ans. (1) 1296 (2) 49

Pigeon hole principle: If there are 'n' pigeons and r holes ($n \geq r$) then there is atleast one hole having more than one pigeon

Example # 35 Prove that given 13 points with integer coordinates, one can always find 4 of them such that their center of gravity has integer coordinates.

Sol. When we divide 13 integers by number 4, we get 4 possibilities of remainders (0, 1, 2,3). By PHP, atleast 4 numbers have same remainder. If we find centre of gravity of these numbers , it also an integer.

Self Practice Problems:

(1) 16 composite integers are chosen from 1 to 2800. Prove that the selection includes at least two integers which are not co-prime

Exercise-1

PART – I : PRE RMO

1. There are seven even greeting cards each of a different colour and seven envelopes of the same seven colours .Find the number of ways in which the cards can be put in the envelopes so that exactly four of the cards go into the envelopes of the right colour

2. In how many ways is it possible to separate the nine letters a, b, c, d, e, f, g, h, i into three non-empty batches ?

3. There an n persons sitting around a circular table. Each person shakes hands with everybody except the person sitting on both sides of him. The total number of hand shakes are 90. Then find n.

4. In a plane there are 37 straight lines of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no lines passes through both points A and B, and no two are parallel. Then find the number of intersection points the lines have.

5. Find the number of selections of six letters from the letters of the word 'K A R N A T A K A'

6. Find the number of ways by which 4 green, 3 red and 2 white balls can be arranged in a row such that no two balls of the same colour are together. All balls of the same colour are identical

7. There are 10 stations enroute. A train has to be stopped at 3 of them. Let N be the no of ways is which the train can be stopped if atleast two of the stopping stations are consecutive. Find N.

8. Number n is selected from the set {1, 2, ...200}. and the number $2^n + 3^n + 5^n$ is formed. Find the total number of ways of selecting n so that the formed number is divisible by 4

9. Find of the total number of words which are formed by using the all the letters of the word "SUCCESS" such the no two alike letters are together

10. Let N be the number of integer solutions of the equation $n_1 + n_2 + n_3 + n_4 = 17$ when $n \geq 1$, $n_2 \geq -1$, $n_3 \geq 3$, $n_4 \geq 0$. If $N = {}^{17}C_K$,then find K .

11. The number of different permutations of all the letters of the word PERMUTATION such that any two consecutive letters in the arrangement are neither both vowels nor both identical is K . 5! . 5! . Find K

12. There are ten boxes numbered from 1 to 10 and 10 balls numbered from 1 to 10. If the number of ways of putting all balls in given boxes such that no box remains empty, odd numbered balls go to odd numbered box and no ball goes to the box having same number which is written on the ball is K^2, then find K .

13. Let set A = {1, 2, 3 ....50}. Set B is a subset of A and B has exactly 20 elements. if the sum of elements of all possible subsets of B is ${}^{49}C_{19}$. × 25 × K

14. Mr. john has x children by his first wife and Ms. Bashu has x + 1 children by her first husband. They marry and have children of their own. The whole family has 10 children. Assuming that two children of the same parents do not fight, Find the maximum number of fight that can take place among children

15. What is the total number of ways of selecting atleast one item from each of the two sets containing 6 identical items each?

PART – II : RMO

1. All the 7-digit numbers containing each of the digits 1,2,3,4,5,6,7 exactly once, and not divisible by 5, are arranged in increasing order. Find the 3200th number in the list

2. Determine the number of subsets of {1,2,3 .... 70} whose sum is larger than 1243.

3. Find the number of 4×4 array where entries are from the set {0,1,2,3} and which are such that the sum of the numbers in each of the four rows and in each of the four columns is divisible by 4.

4. Find the number of 9 digit natural numbers in which each digit appears at least thrice.

5. Find all 6-digit natural numbers $a_1a_2a_3a_4a_5a_6$ formed by using the digits 1,2,3,4,5,6 once each such that number $a_1\ a_2a_3........a_k$ is divisible by k for $1 \le k \le 6$.

6. In a group of 10 people, the sum of the age of the members is 542 years. Prove that four members can be chosen so that their ages is not less than 216 years.

7. Find number of three digit numbers equal to the sum of the factorials of their digits.

8. Find the sum of all four digit numbers formed by the digits {1,2,3.......9} in which exactly two digits are prime and repetition of digits is not allowed

9. If 11 distinct integers are chosen from among 1,2,3 ......20 then show that selection includes at least one pair of integers which are relatively prime.

10. Let n be a positive integer. Find the maximum number of non congruent triangles whose side lengths are integers less than or equal to n.

11. Find all the three digit numbers for which one obtains when dividing the number by 11, the sum of the squares of the digits of the initials

12. Find all 8-digit numbers such that the sum of their digits is 14 and each of the digits 0,1,2,3,4 occurs at least once in them.

13. Find all 7 digit numbers formed by using only the digits 5 and 7 and divisible by both 5 and 7.

14. Two boxes contain 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any 5 balls of the same colour at least two of them will always be of the same size (radius) Prove that there are at least 3 balls which lie in the same box, have same colour and have same size (radius)

15. A set of 10 positive integers in given such that decimal expansion of each of them has two digits. Prove that there are two disjoint subsets of the set with equal sum of their elements.

16. Find the number of solutions of the equation $\sum_{i=1}^{501} x_i = 1501$, where x_i's are odd natural numbers:

17. Find the number of different seven digit numbers that can be written using only the three digits 1, 2, 3 with the condition that the digit 2 occurs atleast 2 times in each number.

18. Find the number of permutations of the letters of the word CONTRADICTORY such that neither the pattern "CON" nor "RAD" nor "ORY" appears

19. Find the number of quintuples (x, y, z, u, v) of positive integers satisfying both equations
$x + y + z + u = 100$ and $x + y + z + v = 70$

20. In how many ways can 10 persons take seats in a row of 24 seats so that no two persons take consecutive seats

21. In how many we can put 6 identical white balls and 5 identical black balls in 10 different boxes if each box must contain atleast one ball.

Exercise-2

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

PRMO (2012)

1. A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different ways could he do this so that exactly two of the five houses receive the correct letters ?

PRMO (2013)

2. There are n – 1 red balls, n green balls and n + 1 blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of n ?

3. To each element of the set S = {1, 2, ..., 1000} a colour is assigned. Suppose that for any two elements a, b of S, if 15 divides a + b then they are both assigned the same colour. What is the maximum possible number of distinct colours used ?

PRMO (2015) WB

4. Consider all the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once, and not divisible by 5. Arrange them in decreasing order. What is the 2015-th number (from the beginning) in this list?

5. Determine the largest 2-digit prime factor of the integer $\binom{200}{100}$, i.e., ${}^{200}C_{100}$.

6. Find the sum of all the distinct prime divisors of $\sum_{r=0}^{2015} r^2 \binom{2015}{r}$ i.e., $\sum_{r=0}^{2015} r^2 \cdot {}^{2015}C_r$

7. Let n!, the factorial of a positive integer n, be defined as the product of the integers 1, 2 , ... n. In other words, n! = 1 × 2 × .... × n. What is the number of zeros at the end of the integer
$10^2! + 11^2! + 12^2! + \_\_\_ + 99^2!$?

8. Find the number of ordered pairs of positive integers (a, b) such that a + b = 1000 and neither a nor b has a zero digit. Note that (2,998) and (998,2) should be counted as two distinct solutions.

9. Find the largest positive integer n such that 2^n divides $3^{4096} - 1$.

PRMO (2015)

10. How many line segments have both their endpoints located at the vertices of a given cube?

11. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party?

PRMO (2016) Chd

12. On a stormy night ten guests came to dinner party and left their shoes outside the room in order to keep the carpet clean. After the dinner there was a blackout, and the guests leaving one by one, put on at random, any pair of shoes big enough for their feet,(Each pair of shoes stays together). Any guest who could not find a pair big enough spent the night there. What is the largest number of guests who might have had to spend the night there ?

PRMO (2016) Delhi

13. There are three kinds of fruits in the market. How many ways are there to purchase 25 fruits from among them if each kind has at least 25 of its fruit available?

PRMO (2016) WB

14. Find the coefficient of $a^5b^5c^5d^6$ in the expansion of the following expression. $(bcd + acd + abd + abc)^7$

15. Find the total number of times the digit '2' appears in the set of integers {1, 2,.......2016} For example, the digit '2' appears twice in the integer 229.

16. Find the number of pairs of positive integers (m, n), with $m \le n$, such that the 'least common multiple' (LCM) of m and n equals 600.

PRMO (2017)

17. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3 ?

18. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once ? (The order in which he visits the cities also matters. e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)

19. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?

20. Find the number of ordered triples (a, b, c) of positive integers such that abc = 108.

21. Suppose in the plane 10 pair wise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?

PRMO (2018)

22. There are several tea cups in the kitchen, some with handle and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen ?

23. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, \ldots, \epsilon_8)$ such that $\epsilon_1, \epsilon_2, \ldots, \epsilon_8 \in \{1, -1\}$ and $\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + \ldots + 8\epsilon_8$ is a multiple of 3.

PART – II : PREVIOUSLY ASKED QUESTION OF RMO

1. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0, 1, 2, 3 occurs at least once in them. **[RMO-2008]**

2. Find the number of all integer-sided isosceles obtuse-angled triangles with perimeter 2008. **[RMO-2008]**

3. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit. **[RMO-2009]**

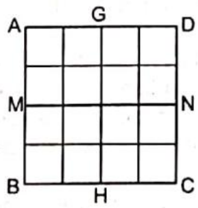
4. Let $(a_1, a_2, a_3 \ldots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers 1, 2, 3 . . . , 2011. Show that there exist two numbers j, k. such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$. **[RMO-2011]**

5. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not). **[RMO-2011]**

6. A finite non-empty set S of integers is called 3-good if the the sum of the elements of S is divisible by 3. Find the number of 3-good non-empty subsets of $\{0, 1, 2, \ldots, 9\}$. **[RMO-2013]**

7. Suppose for some positive integers r and s, the digits of 2^r is obtained by permuting the digits of 2^s in decimal expansion. Prove that r = s. **[RMO-2014]**

8. Is it possible to write the numbers 17, 18, 19, ... 32 in a 4 × f grid of unit squares, with one number in each square, such that the product of the numbers in each 2 × 2 sub-grids AMRG, GRND, MBHR and RHCN is divisible by 16? **[RMO-2014]**



9. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits. **[RMO-2016]**

10. Consider a chessboard of size 8 units × 8 units (i.e. each small square on the board has a side length of 1 unit). Let S be the set of all the 81 vertices of all the squares on the board. What is number of line segments whose vertices are in S_1 and whose length is a positive integer ? (The segments need not be parallel to the sides of the board.) **[RMO-2017]**

Answers

Exercise-1

PART – I : PRE RMO

1.	70	**2.**	3025	**3.**	n = 15	**4.**	535	**5.**	19
6.	79	**7.**	64	**8.**	99	**9.**	96	**10.**	3
11.	57	**12.**	44	**13.**	51	**14.**	33	**15.**	36

PART – II : RMO

1. 6172453 **2.** 2^{69} **3.** 4^{9} **4.** 1,98,207

5. 123654 , 321654 **7.** 1

8. $1111[2 + 3 + 5 + 7)^3C_1.\ ^5C_2.3! + (1 + 4 + 6 + 8 + 9)\ ^3C_2.^5C_1.3!]$

10. $\begin{cases} \dfrac{n(n+2)(2n+5)}{24}, & n \text{ is even} \\ \dfrac{(n+1)(n+3)(2n+1)}{24}, & n \text{ is odd} \end{cases}$ **11.** 550, 803

12. $^7C_2.\ \dfrac{6!}{3!.} + {}^7C_2.\ \dfrac{6!}{2!.2!.} + {}^7C_3\dfrac{5!}{2!.} + {}^7C_1.\ \dfrac{7!}{3!.2!}$

13. 7775775, 7757575, 5577775, 7575575, 5777755, 7755755
5755575, 5557755, 7555555

16. $^{1000}C_{500}$ **17.** $3^7 - 2^7 - 7.2^6$ **18.** $\dfrac{13!}{(2!)^4} - \dfrac{11!}{2!.2!} - \dfrac{11!}{(2!)^3} - \dfrac{11!}{2!.2!} + \dfrac{9!}{2!} + \dfrac{9!}{4!} + \dfrac{9!}{2!} - \dfrac{7!}{2!}$

19. $^{69}C_3$ **20.** $^{15}C_{10}$. 10! **21.** $\dfrac{10!}{6!.3!} + \dfrac{10!}{5!.4!} + \dfrac{10!}{5!.4!}$

Exercise-2

PART – I : PREVIOUS ASKED QUESTION FOR PRE RMO

1.	20	**2.**	10	**3.**	8	**4.**	4712536
5.	61	**6.**	61	**7.**	24	**8.**	738
9.	14	**10.**	28	**11.**	12	**12.**	5
13.	351						
14.	630	**15.**	619	**16.**	53	**17.**	28
18.	60	**19.**	48	**20.**	60	**21.**	36
22.	29	**23.**	88				

PART – II : PREVIOUSLY ASKED QUESTION OF RMO

1.	490	**2.**	86	**3.**	370775	**5.**	520
6.	351	**8.**	Not possible	*9.*	*281250*	**10.**	780