Random Numbers

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***Abstract*—This document contains the solution for Random Numbers**

Problem 1.1: Generate 10^6 samples of U using a C program and save into a file called uni.dat.
Problem 1.2: Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as
Solution:

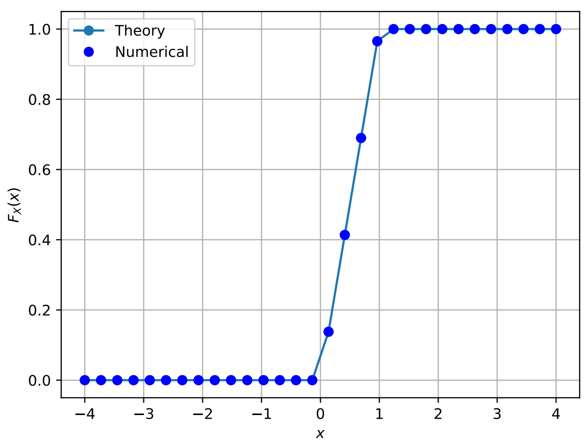

Fig. 1. The CDF of U

Problem 1.3: Find a theoretical solution for $F_U(x)$
Solution:

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1}$$

1) Case 1: $x < 0$: $p_X(x) = 0 \Rightarrow F_U(x) = 0$
2) Case 2: $0 \leq x < 1$: $F_U(x) = \int_0^x du = x$
3) Case 3: $x \geq 1$: Put x = 1 to get $F_U(x) = 1$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{2}$$

Problem 1.4: The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{3}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{4}$$

Write a C program to find the mean and variance of U.
Problem 1.5: Verify your result theoretically given that $E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x)$
Solution:

$$E[U] = \int_0^1 x dx = \frac{1}{2} \tag{5}$$

$$\tag{6}$$

$\Rightarrow$ theoretically empirical mean is 0.50 which closely agrees with numerical value of 0.500007

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{7}$$

$$\text{var}[U] = E[U - E[U]]^2 \tag{8}$$

$$= E[U^2 - 2UE[U] + E^2[U]] \tag{9}$$

$$= E[U^2] - E^2[U] = \frac{1}{12} \tag{10}$$

$\Rightarrow$ theoretically empirical variance is 0.083333 which closely agrees with numerical value of 0.083301
Problem 2.1: Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{11}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Problem 2.2: Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution:

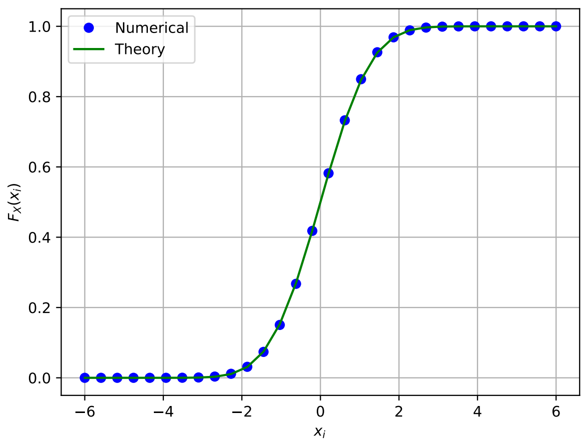

Fig. 2. The CDF of X

Problem 2.3: Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{12}$$

What properties does the PDF have?
Solution:

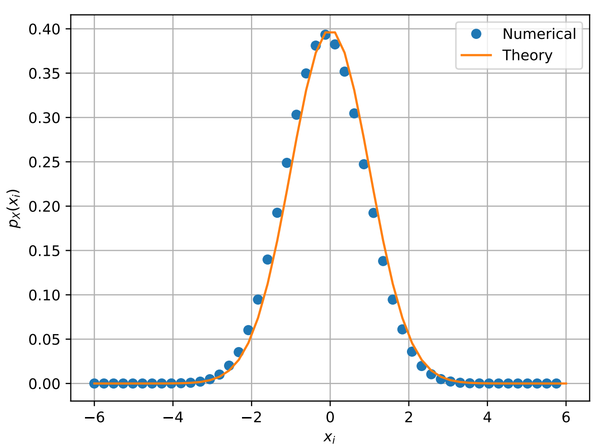

Fig. 3. The PDF of X

Problem 2.4: Find the mean and variance of X by writing a C program.
Problem 2.5: Given that $p_X(x) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}}$ & repeat the above exercise theoretically.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0 \tag{13}$$

$\Rightarrow$ theoretically empirical mean is 0 which closely agrees with numerical value of 0.000326

$$\begin{aligned} var[X] &= E[X^2] - E^2[X] && (14) \\ &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}} dx = 1 && (15) \\ & && (16) \end{aligned}$$

$\Rightarrow$ theoretically empirical variance is 1 which closely agrees with numerical value of 1.000906
Problem 3.1: Generate samples of

$$V = -2\ln(1-U) \tag{17}$$

and plot its CDF.
Solution:

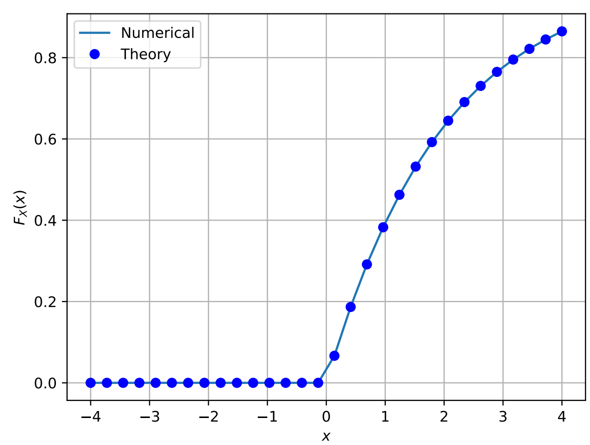

Fig. 4. The PDF of X

Problem 3.2 Find the theoretical expression for $F_V(x)$
Solution:

$$v = f(u) = -2\ln(1-U) \tag{18}$$

$$u = f^{-1}(v) = 1 - e^{-\frac{v}{2}} \tag{19}$$

$$\Rightarrow F_V(v) = F_U(1 - e^{-\frac{v}{2}}) \tag{20}$$

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - e^{-\frac{v}{2}} & v \geq 0 \end{cases} \tag{21}$$