CSC 418 Assignment 1 Chungchen Yang 1000285880 yangch25

1. Curves

(a) $X(t) = 2\sin t \qquad y(t) = 5\cos t \sin t$

$X^2 = 4\sin^2 t \Rightarrow 2 - X^2 = 2 - 4\sin^2 t \Rightarrow 2 - X^2 = 2\cos 2t$

$\Rightarrow \cos 2t = 1 - \frac{X^2}{2}$

$y = 5\cos t \sin t \Rightarrow y = 2.5\sin 2t \Rightarrow \sin 2t = \frac{y}{2.5}$

$\cos^2 2t + \sin^2 2t = 1 \Rightarrow (1 - \frac{X^2}{2})^2 + (\frac{y}{2.5})^2 = 1$

$\Rightarrow 1 + \frac{X^4}{4} - X^2 + \frac{y^2}{6.25} = 1$

$\Rightarrow \frac{X^4}{4} - X^2 + \frac{y^2}{6.25} = 0$

$\Rightarrow f(x,y) = \frac{X^4}{4} - X^2 + \frac{y^2}{6.25} = 0$ (implicit form)

(b) tangent vector is $\vec{T} = \vec{r}'(t)$

$\vec{r}(t) = 2\sin t\,\vec{i} + 5\cos t\sin t\,\vec{j} = 2\sin t\,\vec{i} + 2.5\sin 2t\,\vec{j}$

$\vec{T}(t) = \vec{r}'(t) = 2\cos t\,\vec{i} + 5\cos 2t\,\vec{j}$

normal vector $\vec{N}(t) = \vec{T}'(t)$

$\vec{N}(t) = \vec{T}'(t) = -2\sin t\,\vec{i} - 10\sin 2t\,\vec{j}$

(c) For symmetry of X-axis

we have $f(x,y) = \frac{X^4}{4} - X^2 + \frac{y^2}{6.25} = 0$

If $f(x,y) = f(X,-y)$

then we know the curve is symmetric around the X-axis

$f(X,-y) = \frac{X^4}{4} - X^2 + \frac{(-y)^2}{6.25} = \frac{X^4}{4} - X^2 + \frac{y^2}{6.25} = f(x,y)$

so the curve is symmetric around the X-axis

For symmetry of y-axis

we need to prove $f(X,y) = f(-X,y)$

$f(-X,y) = \frac{(-X)^4}{4} - (-X)^2 + \frac{y^2}{6.25} = \frac{X^4}{4} - X^2 + \frac{y^2}{6.25} = f(X,y)$

so the curve is symmetric around the y-axis as well

(d) $A = \int_0^{2\pi} y\, dX = \int_0^{2\pi} 5\sin t\cos t\; d\sin 2t$

$= \int_0^{2\pi} 5\sin t\cos t \cdot 2\sin t\; dt = 2\times\int_0^{\pi} 10\sin t\cos^2 t\; dt$

$= 2\times\left(-10\times\frac{\cos^3 t}{3}\right)\Big|_0^{\pi} = 2\times\frac{10}{3} - \left(-\frac{10}{3}\right) = \frac{40}{3}$

$= 13.33$

The enclosed area for bow tie is 13.33

(e) In order to calculate the preimeter of the bowtie
We could take the pieces of the lengths and add them together.
For example, take the $0-\frac{\pi}{16}$, $\frac{\pi}{16}\sim\frac{2\pi}{16}$ ~
and add all linear length to get the approximation.

2. Transformations

(a) Assume translation $T_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$

translation $T_2 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$

$$T_1 \cdot T_2 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 \cdot T_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$$

$T_1 \cdot T_2 = T_2 \cdot T_1$, so T_1 and T_2 commute

(b) Assume translation $T = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$

rotation $R = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$T \cdot R = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & a \\ \sin t & \cos t & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$R \cdot T = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & a\cos t - b\sin t \\ \sin t & \cos t & a\sin t + b\cos t \\ 0 & 0 & 1 \end{bmatrix}$$

$T \cdot R \neq R \cdot T$ so T and R do not commute

(c) Assume scaling $S = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$ about origin $(0,0)$

rotation $R = \begin{bmatrix} \cos t & -\sin t & \sin t - \cos t + 1 \\ \sin t & \cos t & 1 - \sin t - \cos t \\ 0 & 0 & 1 \end{bmatrix}$ about $(1,1)$

$$S \cdot R = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & \sin t - \cos t + 1 \\ \sin t & \cos t & 1 - \sin t - \cos t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a\cos t & -a\sin t & a(\sin t - \cos t + 1) \\ b\sin t & b\cos t & b(1 - \sin t - \cos t) \\ 0 & 0 & 1 \end{bmatrix}$$

$$R \cdot S = \begin{bmatrix} \cos t & -\sin t & \sin t - \cos t + 1 \\ \sin t & \cos t & 1 - \sin t - \cos t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a\cos t & -b\sin t & \sin t - \cos t + 1 \\ a\sin t & b\cos t & 1 - \sin t - \cos t \\ 0 & 0 & 1 \end{bmatrix}$$

$S \cdot R \neq R \cdot S$ So R and S do not commute

(d) Assume scaling $S_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$

scaling $S_2 = \begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$S_1 \cdot S_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ac & 0 & 0 \\ 0 & bd & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_2 \cdot S_1 = \begin{bmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ac & 0 & 0 \\ 0 & bd & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S_1 \cdot S_2 = S_2 \cdot S_1$ So S_1 and S_1 commute

3. Homography

(a) Assume the Affine transformation is $A = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$

we have A map the points (1,0)(0,1)(1,1)(0,0)
to points (6,2)(7,3)(6,3)(7,2)

$\vec{p'} = A \cdot \vec{p}$, so we have

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a+t_x \\ c+t_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} b+t_x \\ d+t_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+t_x \\ c+d+t_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

we have
$$\begin{cases} a+t_x = 6 \\ c+t_y = 2 \\ b+t_x = 7 \\ d+t_y = 3 \\ a+b+t_x = 6 \\ c+d+t_y = 3 \\ t_x = 7 \\ t_y = 2 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 0 \\ c = 0 \\ d = 1 \\ t_x = 7 \\ t_y = 2 \end{cases}$$

$$\Rightarrow A = \begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) point $(2,5)$ the vector for it is $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ after applying Affine transformation A

$$\begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$$

we get the point $(2,5)$ mapped to $(5,7)$

4. Polygons

Given three vertices $A(X_1, Y_1)$ $B(X_2, Y_2)$ $C(X_3, Y_3)$, point q (X_4, Y_4)

(a) The area of a triangle is defined as

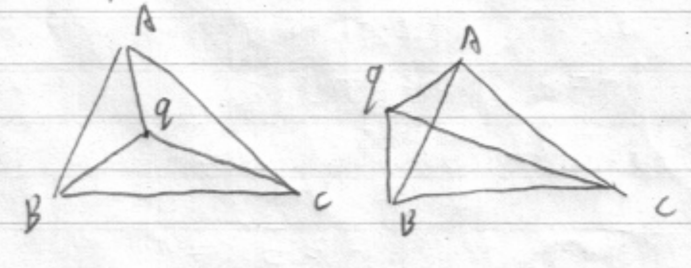
$$A = \frac{X_1(Y_2 - Y_3) + X_2(Y_3 - Y_1) + X_3(Y_1 - Y_2)}{2}$$

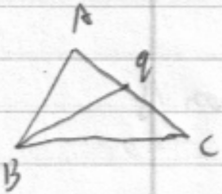
so we can calculate the areas of ABq, BCq, ACq

if $A_{ABq} + A_{BCq} + A_{ACq} = A_{ABC}$

then we know the point q is inside the triangle

if not, then it is outside the triangle.





(b) Use the same method to calculate areas in (a)
If one of the A_{ABq}, A_{ACq}, A_{BCq} is 0
then the point q is on the edge.

(c) The area of the triangle can be calculated
with formula $A = \frac{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)}{2}$

The centroid of the triangle can be calculated
with $x = \frac{x_1+x_2+x_3}{3}$

$y = \frac{y_1+y_2+y_3}{3}$

$\Rightarrow$ centroid $= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$