CSC 418 A2 Part A Chengchen Yang 1002285880 yangch25

1\.

(a) Assume a 2D affine transformation matrix M (3×3 matrix)

assume $a = \begin{bmatrix} a_{1x} & a_{2x} & a_{3x} \\ a_{1y} & a_{2y} & a_{3y} \\ 1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ 1 & 1 & 1 \end{bmatrix}$

we have $M \cdot a = b \Rightarrow M = b \cdot a^{-1}$

by doing so, we could find the affine transformation M

$$M = \begin{bmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{1x} & a_{2x} & a_{3x} \\ a_{1y} & a_{2y} & a_{3y} \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

In order to fully determine this mapping, we need know the values of all 6 points. And the a_1, a_2, a_3 should be non-colinear as stated.

(b) A general 2D homography transformation has 9 unknown variables, which means it needs 3 unmapped points and 3 correspondingly mapped points, a total of 6 points.

A 2D similarity transformation has 6 unknown variables. so it needs 2 pairs of points to determine

(C) Assume the three vertices of a triangle is A, B, C

$$A = \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} X_3 \\ Y_3 \\ 1 \end{bmatrix}$$

the affine transformation matrix $M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Apply affine transformation to three vertices

$$A' = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} aX_1 + bY_1 + t_x \\ cX_1 + dY_1 + t_y \\ 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} aX_2 + bY_2 + t_x \\ cX_2 + dY_2 + t_y \\ 1 \end{bmatrix}$$

$$C' = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} aX_3 + bY_3 + t_x \\ cX_3 + dY_3 + t_y \\ 1 \end{bmatrix}$$

new centroid is $\begin{bmatrix} (aX_1 + bY_1 + t_x + aX_2 + bY_2 + t_x + aX_3 + bY_3 + t_x)/3 \\ (cX_1 + dY_1 + t_y + cX_2 + dY_2 + t_y + cX_3 + dY_3 + t_y)/3 \\ 1 \end{bmatrix}$

The original centroid is $\begin{bmatrix} (X_1 + X_2 + X_3)/3 \\ (Y_1 + Y_2 + Y_3)/3 \\ 1 \end{bmatrix}$

Apply affine transformation to original centroid

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (X_1+X_2+X_3)/3 \\ (Y_1+Y_2+Y_3)/3 \\ 1 \end{bmatrix} = \begin{bmatrix} a(X_1+X_2+X_3)/3 + b(Y_1+Y_2+Y_3)/3 + t_x \\ c(X_1+X_2+X_3)/3 + d(Y_1+Y_2+Y_3)/3 + t_y \\ 1 \end{bmatrix}$$

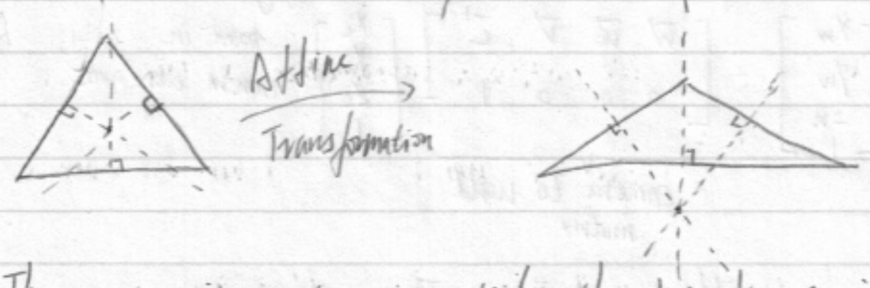
It is the same as the new centroid

$$(aX_1+bY_1+t_x+aX_2+bY_2+t_x+aX_3+bY_3+t_x)/3 = a(X_1+X_2+X_3)/3 + b(Y_1+Y_2+Y_3)/3 + t_x$$

$$(cX_1+dY_1+t_y+cX_2+dY_2+t_y+cX_3+dY_3+t_y)/3 = c(X_1+X_2+X_3)/3 + d(Y_1+Y_2+Y_3)/3 + t_y$$

So centroid is affine invariant.

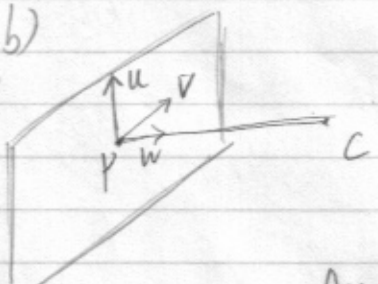
As for circumcenter. It is not affine invariant.
Here is one counter-example.



The new circumcenter is outside the triangle, so it is not an affine transformation. In this case, we know that the circumcenter is not affine invariant.

2.
(a) The light travels through air in a perfect straight line. So that light travels from the top of an object, straight through the pinhole. To the bottom of the image. This results in an inverted image in pinhole camera.

(b)



Let $\vec{w}$ be the unit vector follow the direction of $P \to C$, perpendicular to plane

$$\vec{w} = \frac{\vec{C} - \vec{P}}{\|\vec{C} - \vec{P}\|}$$

Assume $\vec{v}$ be the vector parallel to the horizontal axis of the screen.

$$\vec{V} = \frac{\vec{u} \times \vec{w}}{\|\vec{u} \times \vec{w}\|}$$

now, we have 3 different unit direction vectors. $\vec{u}, \vec{v}, \vec{w}$.
A world point can be determined by:

$$\underset{\text{point in world coordinate}}{\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}} = \underbrace{\begin{bmatrix} \vec{w} & \vec{u} & \vec{v} & \vec{c} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{camera to world matrix}} \underset{\text{point in camera coordinate}}{\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}}$$

So the world to camera matrix is

$$M = \begin{bmatrix} \vec{w} & \vec{u} & \vec{v} & \vec{c} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

(c) When the vector $\vec{v}(V_x, V_y, V_z)$ is parallel to the screen plane, a family of lines remain parallel to the vector $\vec{v}$

(d) All lines in the family converge at a single 2D point, which is called vanishing point.

3.

(a) $f(x,y,z) = (R - \sqrt{x^2+y^2})^2 + z^2 - r^2 = 0$, $R > r$

surface normal $\vec{n} = \nabla f = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right)$

$$\frac{df}{dx} = 2x - \frac{2Rx}{\sqrt{x^2+y^2}} \qquad \vec{n} = \left(2x - \frac{2Rx}{\sqrt{x^2+y^2}},\ 2y - \frac{2Ry}{\sqrt{x^2+y^2}},\ 2z\right)$$

$$\frac{df}{dy} = 2y - \frac{2Ry}{\sqrt{x^2+y^2}}$$

$$\frac{df}{dz} = 2z$$

(b) The tangent plane is perpendicular to $\nabla f(\vec{p})$

so we have $\nabla f(\vec{p}) = 0$

$$\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \left(2p_x - \frac{2Rp_x}{\sqrt{p_x^2+p_y^2}}\right)(x-p_x) + \left(2p_y - \frac{2Rp_y}{\sqrt{p_x^2+p_y^2}}\right)(y-p_y) + 2p_z(z-p_z) = 0$$

(c) $q(\lambda) = (R\cos\lambda, R\sin\lambda, r)$

$f(R\cos\lambda, R\sin\lambda, r) = (R - \sqrt{R^2\cos^2\lambda + R^2\sin^2\lambda})^2 + r^2 - r^2 = (R-R)^2 + r^2 - r^2 = 0$

$\Rightarrow q(\lambda)$ curve lie on the surface

(d) $q(\lambda) = (R\cos\lambda, R\sin\lambda, r)$

$$\vec{T} = \frac{dq}{d\lambda} = (-R\sin\lambda, R\cos\lambda, 0)$$

(e) $\vec{T} = (-R\sin\lambda,\ R\cos\lambda\ , 0)$

$q(\lambda) = (R\cos\lambda\ , R\sin\lambda\ , r)$

$x - P_x = -R\sin\lambda, \quad y - P_y = R\cos\lambda\ , \quad z - P_z = 0$

$P_x = R\cos\lambda, \quad P_y = R\sin\lambda\ , \quad P_z = r$

$$\left(2R\cos\lambda - \frac{2R^2\cos\lambda}{R}\right)(-R\sin\lambda) + \left(2R\sin\lambda - \frac{2R^2\sin\lambda}{R}\right)(R\cos\lambda) + 2r\cdot 0 =$$

$$= (2R\cos\lambda - 2R\cos\lambda)(-R\sin\lambda) + (2R\sin\lambda - 2R\sin\lambda)(R\cos\lambda)$$

$$= 0(-R\sin\lambda) + 0(R\cos\lambda) = 0 + 0$$

$$= 0$$

$\Rightarrow$ This tangent vector lie on the implicit equation of the tangent plane

4.

(a) use $p(t) = f_1(t)P_1 + f_2(t)P_2 + f_3(t)P_3 + f_4(t)P_4$

$$f_i(t) = \binom{n}{i}(1-t)^{(n-i)}t^i$$

$$B_1(t) = (1-t)^3P_1 + 3(1-t)^2tP_2 + 3(1-t)t^2P_3 + t^3P_4$$

$$B_1'(t) = -3(1-t)^2P_1 + 3(1-t)^2P_2 - 6(1-t)tP_2 + 6(1-t)tP_3 - 3t^2P_3 + 3t^2P_4$$

$$= 3(1-t)^2(P_2-P_1) + 6(1-t)t(P_3-P_2) + 3t^2(P_4-P_3)$$

B_1' at P_4, $t=1 \Rightarrow B_1'(1) = 3(P_4-P_3)$

$$B_2(s) = (1-s)^3P_4 + 3(1-s)^2sP_5 + 3(1-s)s^2P_6 + s^3P_7$$

$s=t-1$ $\quad B_2'(s) = 3(1-s)^2(P_5-P_4) + 6(1-s)s(P_6-P_5) + 3s^2(P_7-P_6)$

B_2' at P_4, $t=1$ $\Rightarrow B_2'(0) = 3(P_5-P_4)$

$s=0$

(b) $B_1''(t) = -6(1-t)(P_2-P_1) + 6(1-t)(P_3-P_2) - 6t(P_3-P_2) + 6t(P_4-P_3)$

$$= 6(1-t)(P_3-P_2-P_2+P_1) + 6t(P_4-P_3-P_3+P_2)$$

$$= 6(1-t)(P_3-2P_2+P_1) + 6t(P_4-2P_3+P_2)$$

B_1'' at P_4, $t=1 \Rightarrow B_1''(1) = 6(P_4-2P_3+P_2)$

$$B_2''(s) = 6(1-s)(P_6-2P_5+P_4) + 6s(P_7-2P_6+P_5)$$

B_2'' at P_4 $\quad t=1 \Rightarrow B_2''(0) = 6(P_6-2P_5+P_4)$

$s=0$

(c) If combined curve is C^2 continuous, then the second derivative is equal to first derivative.

so that we can have $\begin{cases} B_1' = B_1'' \\ B_2' = B_2'' \\ B_1' = B_2'' \\ B_2' = B_1'' \end{cases}$

since P_1 P_2 P_3 P_4 are known
then we have

$$\begin{cases} 3(P_5 - P_4) = 6(P_6 - 2P_5 + P_4) \\ 3(P_4 - P_3) = 6(P_6 - 2P_5 + P_4) \\ 3(P_5 - P_4) = 6(P_4 - 2P_3 + P_2) \end{cases} = \begin{cases} P_5 - P_4 = 2P_6 - 4P_5 + 2P_4 \\ P_4 - P_3 = 2P_6 - 4P_5 + 2P_4 \\ P_5 - P_4 = 2P_4 - 4P_3 + 2P_2 \end{cases}$$

Above equations are the constraints to P_5 P_6 P_7

(d)

1. Bezier curves' mathematical description are compact, intutive and elegant

2. Bezier curves are easy to compute and use in higher dimensions

3. Bezier curves can be stiched together to represent any shape

4. Bezier curves allow the specification of a path as a piece-wise continuous polynomial