Dynamic Asset Allocation with Options

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Abstract

We derive replicating portfolios for some commonly used dynamic trading rules. In particular, we show that two commonly used dynamic rules can be replicated with static option portfolios. These replicating portfolios are more precise than the corresponding dynamic rules in the sense that the exposure is always correct at each price. For a rule that changes notional exposure linearly with price the error is linear in variance.

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1 Introduction

A Dynamic Asset Allocation (DAA) rule is a function from price of an asset $S(t)$ and its valuation S^* to a target exposure:

$$DAA(S(t), S^*) \rightarrow \omega(t)$$

The rule typically specifies a positive position if the price is below the valuation S^* and a negative exposure if it is above. There are two ways to implement such a rule:

1. Adjust a futures exposure to the asset as $S(t)$ or S^* moves
2. Hold a portfolio of options on the asset that generates a delta (option sensitivity the asset) equal to the target DAA weight for any value of $S(t)$

The approaches are identical in the generation of the target DAA exposure at each rebalancing point. Between these points the option portfolio is more correct since the delta matches the rule at each point, even if the price jumps, whereas the futures portfolio is always "late".

A typical DAA rule might linearly increase the exposure if the price is below the valuation and decrease if the price is above valuation between some bounds. A replicating option portfolio sells a constant number of options at evenly spaced strikes between a lower bound S_{lower} and upper bound S_{upper}. Each put option below the value adds a fixed amount of delta to the total profile so the combination creates a step function (figure 1). As the interval approaches zero the option delta profile approaches the target.

This portfolio is linear in delta, but not in notional delta (delta * underlying price). A linear delta notional profile can be generated with a $-dK/K^2$ weights to option strikes K.

This case provides an interesting alternative replication. Option weights of $-dK/K^2$ is part of the replicating portfolio for a log contact, which in turn is part of the replicating

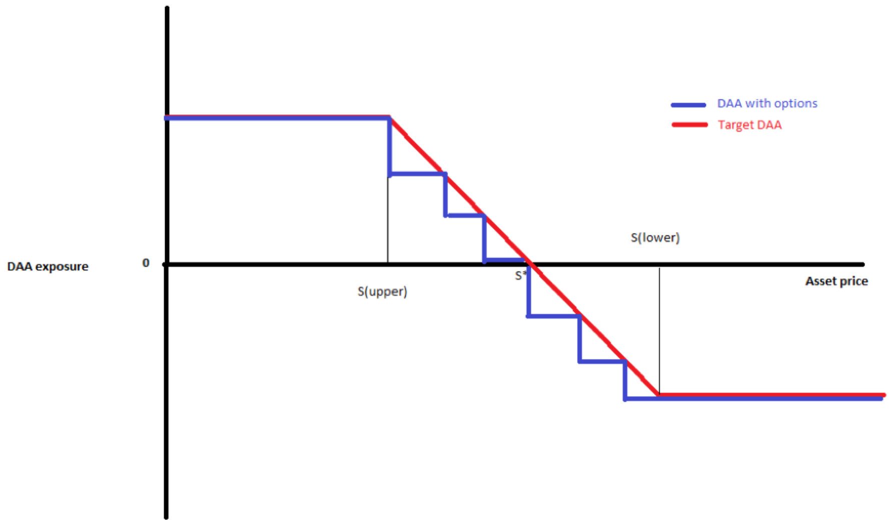

Figure 1: A DAA rule implemented with options

portfolio for a variance swap. Certain DAA rules can therefore be seen explicitly as a variance swap plus a rebalanced future.

A more general technique to find the option portfolio to match an arbitrary DAA rule is developed in the appendix. The remainder of the paper demonstrates the machinery more concretely.

2 Preliminaries

We are interested in portfolio sensitivity (delta) generated by asset allocation rules $\omega(t)$.

$$D(\omega(t), S(t)) = \frac{\partial V(\omega(t), S(t))}{\partial S(t)} \tag{1}$$

Where V is the value of a portfolio given a function $\omega(t)$ specifying weights to replicating assets (for our purposes: futures, options, bonds, log contracts, and variance swaps).

Delta replicating portfolios are a class of replicating portfolios that produce the same delta:

$$R^D(\omega(t)) = \{\omega'(t) : D(\omega'(t), S(t)) = D(\omega(t), S(t))\}$$

In particular we show how a trading rule that varies with price can be generated with either:

- A rebalanced future
- A static log contract and future
- A static strip of options
- A rebalanced future and a static variance swap

3 The relationship between delta and option weights

The appendix shows that for a weighting function on strikes $\omega(K)$ the delta profile given an expiry price $S(T)$ is

$$D(\omega, S(T)) = \Omega(S(T)) - \Omega(S^*)$$

where $\Omega(K) = \int \omega(K)dK$.

So for constant weights $\bar{\omega}$ the delta is linear in the final price:

$$D(\omega, S(T)) = \Omega(S(T)) - \Omega(S^*) = \bar{\omega}\left(S(T) - S^*\right)$$

For weights $\bar{\omega}/K^2$ we have

$$D(\omega, S(T)) = \Omega(S(T)) - \Omega(S^*) = \bar{\omega}\left(\frac{1}{S(T)} - \frac{1}{S^*}\right)$$

For these weights the notional delta (delta * price) is linear in the final price

$$D(\omega, S(T))S(T) = \bar{\omega}\left(1 - \frac{S(T)}{S^*}\right) \tag{2}$$

4 Spanning with the log contract

The appendix shows that the log contract paying $\log \frac{S(T)}{S^*}$ has

$$d\log S(t) = \frac{1}{S^*}dS(t) + \int_0^{S^*} dP(K)\frac{-1}{K^2}dK + \int_{S^*}^{\infty} dC(K)\frac{-1}{K^2}dK$$

The replicating portfolio is:

- $1/S^*$ futures
- $-dK/K^2$ puts from 0 to S^* and calls from S^* onward

5 Spanning with a variance swap

The linear notional delta option weights ($-dK/K^2$) are the same as for a log contract. The intuition is that a log contract pays the growth rate of the asset, which is the change of value on a constant notional. This means that a DAA rule with a linear notional delta rule can be partly replicated by a variance swap.

The appendix shows that the value of a linear notional delta portfolio can represented as the weighted option payoff or as a function of variance and rebalanced futures

$$\begin{aligned} V(S(T), S^*) &= \int_0^{S^*} P(K)\frac{-1}{K^2}dK + \int_{S^*}^{\infty} C(K)\frac{-1}{K^2}dK \\ &= \int_0^T \frac{dS(t)}{S(t)} - \frac{T}{2}(\sigma^2(t, ...) - K^{var}) - \frac{1}{S^*}(S(T) - S^*) \end{aligned}$$

Where K^{var} is a variance swap strike and $\sigma^2(t, ...) = \frac{1}{T}\int_0^T d\log(S(t))^2$. Differentiating gives the payoff in each period

$$dV(S(T), S^*) = dS(t)\left(\frac{1}{S(t)} - \frac{1}{S^*}\right) - 0.5((d\log(S(t)))^2 - K^{var}) \tag{3}$$

This holds approximately in the discrete case

$$\Delta V(S(T), S^*) \approx \Delta S(t)\left(\frac{1}{S(t)} - \frac{1}{S^*}\right) - 0.5((\Delta \log(S(t)))^2 - K^{var})$$

6 Example replications

Suppose we have a DAA rule that targets \$5 notional delta exposure for each 5% below the valuation and -\$5 for each 5% below. If the asset is initially at target price $S(0) = S^* = 100$ then from (2) we have $\bar{\omega} = 100$ and there are four possible portfolios that produce the target notional delta:

Portfolio 0: $100(1/S(t) - 1/S^*)$ futures (rebalanced)

Portfolio 1: $-100/K^2$ (taking $\Delta K = 1$) put options for $K \leq S^* = 100$ and call options for $K > S^* = 100$ (static)

Portfolio 2: 100 log contracts contract and 0.01 futures (static)

Portfolio 3: $100(1/S(t) - 1/S^*)$ futures (rebalanced) and -50 variance swap notional (static)

The portfolio is illustrated for each portfolio across prices between 95 and 105 in tables 1-4

Table 1: Portfolio 0: DAA rebalancing

# Contracts across $S(t)$	95	...	100	...	105
$S(t)$	0.0526	...	0	...	-0.0476
$\log S(t)$	0	...	0	...	0
$\omega(K)$	0	...	0	...	0
Variance	0	...	0	...	0

Table 2: Portfolio 1: Static option replication

# Contracts across $S(t)$	95	...	100	...	105
$S(t)$	0	...	0	...	0
$\log S(t)$	0	...	0	...	0
$\omega(K)$	$-100/K^2$	...	$-100/K^2$	...	$-100/K^2$
Variance	0	...	0	...	0

Table 3: Portfolio 2: Log contract replication

# Contracts across $S(t)$	95	...	100	...	105
$S(t)$	-0.01	...	-0.01	...	-0.01
$\log S(t)$	100	...	100	...	100
$\omega(K)$	0	...	0	...	0
Variance	0	...	0	...	0

Table 4: Portfolio 3: Variance swap replication

# Contracts across $S(t)$	95	...	100	...	105
$S(t)$	0.0526	...	0	...	-0.0476
$\log S(t)$	0	...	0	...	0
$\omega(K)$	0	...	0	...	0
Variance	50	...	50	...	50

7 Discussion

Asset allocation rules that vary exposure with price are common in the wild. By construction these rules are "late" in the sense that the exposure after rebalancing is relevant to the previous price move. Using a portfolio of options solves this problem since by construction the delta adjusts correctly as the price moves.

This construction is short volatility: intuitively since it sells as the price increases and buys as the price decreases, and explicitly in the replicating portfolio which is short a strip of options. In the special case of a rule targeting linear notional delta changes the portfolio can be made precise by adding a short variance swap.

Appendix: Spanning results

Spanning delta with options

The delta of a portfolio of puts up to K^* and calls beyond is

$$D(\omega, S(t)) = \int_0^{K^*} \omega^{put}(K) D(P(K)) dK + \int_{K^*}^{\infty} \omega^{call}(K) D(C(K)) dK$$

Where $D(P)$, $D(C)$ are the delta of the puts and calls. This simplifies to

$$D(\omega, S(t)) = \int_0^{\infty} \omega(K) D(P(K)) dK + \int_{K^*}^{\infty} \omega(K) dK$$

At expiry this is

$$D(\omega, S(T)) = \int_S^{\infty} -\omega(K) dK \int_{K^*}^{\infty} \omega(K) dK$$

Writing $\Omega(K) = \int \omega(K) dK$

$$D(\omega, S(T)) = \Omega(S(T)) - \Omega(K^*)$$

Spanning the log contract

Any function $W(S(T))$ of the final payoff can be spanned with (see (4))

$$W(S(T)) = W(A)e^{-rt} + W'(A)(S(T) - Ae^{-rt}) + \int_0^A P(K) W''(K) dK + \int_A^{\infty} C(K) W''(K) dK$$

For $W(S(T)) = \log(S(T))$ and $A = K^*$ and assuming rates are zero this is

$$\log(S(T)) = \log(K^*) + \frac{1}{K^*}(S(T) - K^*) + \int_0^{K^*} P(K) \frac{-1}{K^2} dK + \int_{K^*}^{\infty} C(K) \frac{-1}{K^2} dK$$

Differentiating gives

$$d \log S(t) = \frac{1}{K^*} dS(t) + \int_0^{K^*} dP(K) \frac{-1}{K^2} dK + \int_{K^*}^{\infty} dC(K) \frac{-1}{K^2} dK$$

This means the log contract is replicated with

- $1/K^*$ futures

- $-dK/K^2$ puts from 0 to K^* and calls from K^* onward

Spanning the variance swap

We use the standard construction for a variance swap (see (3)) with a general price process

$$\frac{dS(t)}{S(t} = \mu(t, ...)dt + \sigma(t, \ldots)dZ(t)$$

It's lemma on $\log S(t)$ gives

$$d \log S(t) = (\mu - 0.5\sigma^2)dt + \sigma dZ(t)$$

Taking the difference between the orginal process and this is

$$\frac{dS(t)}{S(t)} - d \log S(t) = 0.5\sigma^2 dt$$

Integrating between 0 and T gives the variance

$$V \equiv \frac{1}{2} \int \sigma^2(t, \ldots) = \frac{2}{T} \left(\int_0^T \frac{dS(t)}{S(t)} - \log \frac{S(T)}{S(0)} \right)$$

This means that a variance contract can be replicated with a rebalanced $\frac{1}{S(t)}$ futures positon and a short log contract.

The discrete version is

$$\frac{\Delta S(t)}{S(t)} - \Delta \log S(t) \approx 0.5(\Delta \log S(t))^2 dt = 0.5r(t)^2$$

Where $r(t)$ is the log return.

References

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