Principle of data science coursework

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1 Preliminary information

I run the entire code on a Ubuntu-22.04 WSL on my Huawei MateBook 16 laptop with Windows 11. To run the code I used Anaconda 23.9.0. I have allocated 8 GBs of RAM to the WSL, half of the total laptop's available RAM. The laptop has an AMD Ryzen 5 5600 Hz CPU with Radeon Graphics.

- Part c took 0.2 seconds to run, provided the plots aren't displayed. If the `--plots` is applied, the time rises up to 4.9 seconds.
- Part d took 0.48 seconds to run without the `--plots` flag, whereas it takes 2.5 seconds to run with the flag.
- Part e takes 0.8 seconds and 2.8 seconds without and with the flag, respectively.
- Part f takes 4.5 minutes if the `--fit` option is selected, whereas if the data is already saved and just needs to be loaded it just takes 0.35 seconds to generate the plots.
- Similarly, part g takes roughly 26 minutes to run with the `--fit` flag on, whereas it takes only 1.7 seconds to generate the plots.

Therefore, running all the steps require less than an hour.

2 Section a

In this section I will be presenting the work done on parts a-e.

The statistical model is presented in Equation (1), where the background follows an exponentially decaying distribution $b(M;\lambda) = \Theta(M)^{1}\lambda e^{-\lambda M}$ and the signal is a Gaussian distribution with mean μ and variance σ^2.

$$p(M; f, \lambda, \mu, \sigma) = f \cdot s(M; \mu, \sigma) + (1 - f) \cdot b(M; \lambda) \tag{1}$$

a) Prove that the probability distribution is normalised in the range $M \in [-\infty, \infty]$.

[1] $\Theta(x)$ is the Heaviside step function, which returns 0 for values smaller than 0, and 1 for values greater

Number of models	α	β	Mean	Variance
10000	5	5.6	0.9999	6.1×10^{-17}
10000	0	10	1	3.3×10^{-14}
10000	-5	5	0.9999	3.2×10^{-14}
10000	-5	8	1	5.6×10^{-9}

Table 1: Some example results obtained when running the code, which display that the pdf defined in Equation (4) is normalised.

Proof. $\int_{\mathbb{R}} p(M; f, \lambda, \mu, \sigma) dM = (1-f)\lambda \int_{\mathbb{R}^+} e^{-\lambda M} dM + \frac{f}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left(\frac{-(M-\lambda)^2}{2\sigma^2}\right)$. The first integral is $1 - f$, because the integral of the exponential from 0 to ∞ is $-1/\lambda$, whereas the integral of the second function is f, because the Gaussian distribution is normalised in $\mathbb{R}$. Therefore, the sum of the two integrals is $f + 1 - f = 1$. □

b) Because $M \in [5, 5.6]$, we need to change the normalisation factor so that the total probability equals 1. In order to do so, recall the cumulative density function (CDF) $F(X) = \int_{\infty}^{X} f(X')dX'$. Given that for the exponential $F(X) = 1 - e^{-\lambda X}$ and for the Gaussian $F(X) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{X-\mu}{\sqrt{2}\sigma}\right)\right)$, we can normalise the two different distributions separately, as in Equations (2) and (3), respectively for signal and background.

$$N_S^{-1}(\lambda, \mu, \sigma; \alpha, \beta) = \frac{1}{2}\left(\text{erf}\left(\frac{\beta-\mu}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{\alpha-\mu}{\sqrt{2}\sigma}\right)\right) \tag{2}$$

$$N_B^{-1}(\lambda, \mu, \sigma; \alpha, \beta) = e^{-\lambda\alpha} - e^{-\lambda\beta} \tag{3}$$

Therefore the resulting pdf, displayed in Equation (4), is normalised in the range $M \in [\alpha, \beta]$

$$\text{pdf}(M; \boldsymbol{\theta}, \alpha, \beta) = N_S(\lambda, \mu, \sigma; \alpha, \beta) \cdot f \cdot s(M; \mu, \sigma) + N_B(\lambda, \mu, \sigma; \alpha, \beta) \cdot (1-f) \cdot b(M; \lambda) \tag{4}$$

c) In order to prove that the pdf is correctly normalised, I used the code `src/solve_part_c.py`. The details to run it are in the README file. I generated 10000 random values of $\boldsymbol{\theta} = (f, \lambda, \mu, \sigma)$ uniformly distributed in a specified range of values, and then I numerically evaluated the integral of the pdf in $[\alpha, \beta]$ with the `np.trapz` function. I tested different values of α and β, and for all values the average of the 10000 integrals was 1, with a very low variance, as displayed in Table 1.

d) The true values for the parameters $\boldsymbol{\theta}$ are $f = 0.1$, $\lambda = 0.5$, $\mu = 5.28$ and $\sigma = 0.018$. Therefore, the true PDF (Equation (4)) has the shape displayed in green in Figure 1. From the plot we can clearly see the exponential decay of the background distribution, as well as the Gaussian representing the signal distribution, with the two components being represented separately in blue and red, respectively.

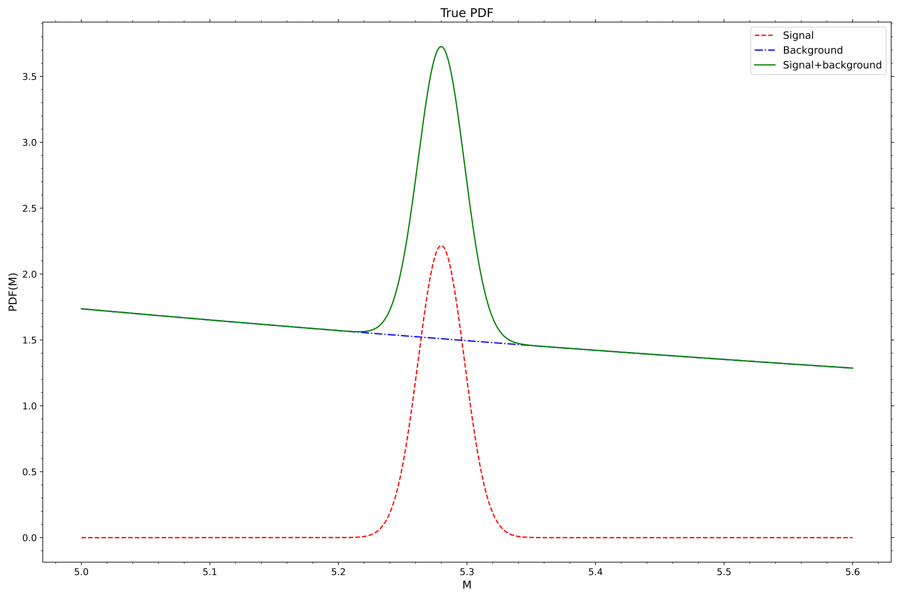

Figure 1: Plot displaying the true PDF alongside the distributions for signal and background, each rescaled by their relative fraction (10% signal, 90% background).

e) In order to generate data according to the PDF, the accept/reject (AR) method was used. This method is the more general, in that it doesn't require any knowledge of the CDF, since only the PDF is needed. In this case the CDF was analytically known, so the CDF method could have been used. Nonetheless, I decided to use the AR method in order to give more generality to the problem, so that my class could be used also in problems where the analytical expression of the CDF is not known. The price to pay for an increased in generality is a consequential increase in the computational cost of the project, since the generation of random values requires more time than the evaluation of the CDF and its inverse function.

The AR method works as follow:

$i)$ Generate $\vec{x} = \{x_1, \ldots, x_N\}$ random uniformly distributed points in the range $[\alpha, \beta]$

$ii)$ Generate $\vec{y} = \{y_1, \ldots, y_N\}$ random uniformly distributed points in the range $[0, Y_{max}]$, where $Y_{max} \doteq \max_{x\in[\alpha,\beta]} [\text{pdf}(x)]$

$iii)$ Evaluate $\vec{pdf}(\vec{x})$ and accept the points $\vec{x}$ for which $\vec{pdf}(\vec{x}) > \vec{y}$

My method is a variant of the usual AR method, because instead of evaluating the PDF for each point separately inside a for loop, I evaluate the PDF for all the points at the same time. The issue with this method is that I have no control on the number of generated points, as I found out after discussing with the other CDT students. This is due to the fact that of the N points generated for $\vec{x}$ and $\vec{y}$, only $L < N$ points are generated as an output, which are the points for which the PDF is greater than y, with L being usually $N/2$. To solve this, I generated $M = 5 \times N$ points, and then selected only the first N

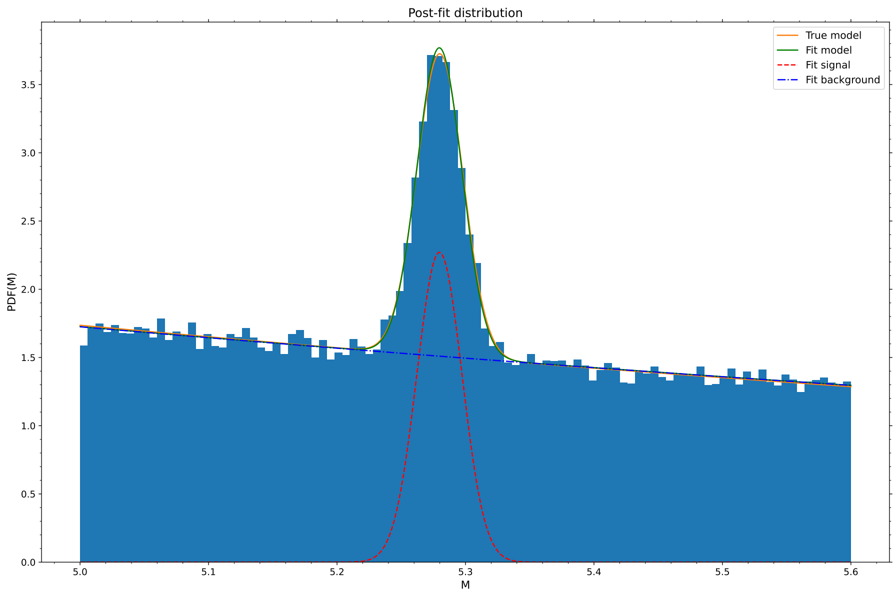

Figure 2: Plot displaying the generated data alongside the true distribution (orange) and the distribution of the best-fit model for signal (red), background (blue) and combined (green).

Parameter	MLE	Uncertainty	True value
f	0.0965	0.0024	0.1
μ	5.2799	0.0005	5.28
σ	0.0180	0.0005	0.018
λ	0.495	0.029	0.5

Table 2: MLE for the four parameters. We can see that for all parameters there is a good agreement between the MLE and the true value.

points of the output array containing values of x for which $y < pdf(x)$. This is not the most efficient way, since it requires the generation of 5 times more points than needed, but it is still significantly faster than the usual AR method. Furthermore, the time difference between generating N events of $5N$ events is almost negligible, being of the order of few milliseconds. Therefore, I decided to use this method for the rest of the coursework.

In this way, we are able to generate data according to the PDF.

Using this function, I generated 100k events following the true PDF (Equation (4)), and then I performed an unbinned ML fit using the `Minuit.migrad` algorithm. The algorithm returns an estimate of the free parameters $\vec{\theta}$ that minimise the negative log-Likelihood. The values obtained are displayed in Table 2, and we can see how all the estimates are in agreement with the true values. The ML estimate for the PDF can be seen in Figure 2. As expected we can see that the orange and green distributions are very similar, once again ensuring the fairness of the fit.

3 Section b

In this section I will be presenting the work done on parts f and g

f) The goal of this part is to estimate how much data is needed to discover the signal 90% of the times. For "discovery" we consider the HEP discovery limit, meaning 5σ of a normal distribution away from the null hypothesis, which requires a p-value smaller than 2.9×10^{-7}.

For null hypothesis (H_0) we consider the background-only hypothesis, meaning we set $f = 0$, having only an exponentially decaying distribution with 1 free-floating parameter, λ. The alternate hypothesis (H_1) is the hypothesis that both signal and background are present in our data. For this hypothesis we have 4 free-floating parameters, which are f, μ, σ and λ, respectively the fraction of signal, the central value of the signal Gaussian distribution with its variance, and the decay constant λ for the background distribution.

As a test statistics I used the likelihood ratio, following Neyman-Pearson lemma (Equation (5))

$$T = -2 \ln \left(\frac{L(x|H_0)}{L(x|H_1)} \right) \tag{5}$$

This test statistics is more powerful than the χ^2, and it holds the property, thanks to Wilks' theorem, of being asymptotically distributed as a χ^2 distribution with m degrees of freedom, where m is the difference between the number of free parameters between denominator and numerator. Therefore, it is trivial to evaluate the p-value for this hypothesis, which is given by Equation (6), where F is the CDF for the χ^2 distribution[2].

$$p = 1 - F_{\chi^2}(T; dof) \tag{6}$$

The significance Z of the test, meaning how many Gaussian standard deviations away we are from the hypothesis H_0, can be obtained using the percent point function (PPF, F^{-1}), which is the inverse of the CDF function, as per Equation (7).

$$Z = F^{-1}_{\chi^2}(1 - p) \tag{7}$$

Therefore, the goal of this exercise is to evaluate the size of the generated data for which the discovery rate is above 90%. In order to do, I have performed the following steps

i) Select number of toy experiments. I have chosen 1000, meaning we want at least 900[3] experiments that give a significance greater than 5σ

ii) Select dataset size. The dataset size tells how many points are generated with the

[2] $F(x; k) = \dfrac{\gamma\left(\frac{k}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$, where k is the number of degrees of freedom (dof)

[3] This is not technically correct, since sometimes the fit fails, and therefore the failed values need to be removed from the counting. The actual number of toy experiments that give us a discovery rate above 90% is then $0.9 \times (1000 - N_{\text{failed}})$

AR method for each toy experiment. I have tested different values, ranging from 50 to 2000, and then zooming in the values that gave a discovery rate close to 90%. In the final version of the code not all these values are displayed, and only the significant ones are kept in order to speed up the code, which already takes ~ 4.5 minutes to run.

iii) For each dataset size:

- Evaluate the test statistics T (Equation (5)). In order to do so, the negative-log likelihoods for both the null hypothesis and the alternate hypothesis are minimised, and T is evaluated as the difference between the two minima. If either one or the two fits fails, meaning it is not converging, the T value is not saved. In order to minimise the number of fails, the `iterate` option of `Minuit.migrad` was set to 100. This significantly slows the code down, but it reduces the number of failed fits, thus giving us more statistics.
- Evaluate p-value. Using Equation (6), the p-value is evaluated, using the χ^2 approximation with $m = 3$ degrees of freedom, where $m = N_{free}^{H1} - N_{free}^{H0} = 4 - 1 = 3$, where N_{free}^{H} refers to the number of free-floating parameters of the hypothesis H.
- Evaluate the significance, using Equation (7).
- Evaluate the discovery rate, as the ratio between number of converging fits with $Z \geq 5\sigma$ and the number of converging fits.

The plot displaying the discovery rate for each dataset size can be found in Figure 3. As discussed among the other CDT students, because this experiment is a Bernoulli-like experiment, given we have two possible outcomes, the uncertainty on each rate r is $r(1-r)/N$, where N is the number of toys, set to 1000. We can see how there are some fluctuations, due to some fits failing in reaching a convergence. Nonetheless, it is clear that the greater the size of the dataset, the higher the discovery rate will be, because with more data points it will be easier to fit the correct function. The fits that fail are always for the alternate hypothesis, and this number ranges from 100 with 400 data points to 23 for 1000.

Overall, a 90% discovery rate is obtained when the size is of the order or 800 points, with the discovery rate for 750 points being 0.89.

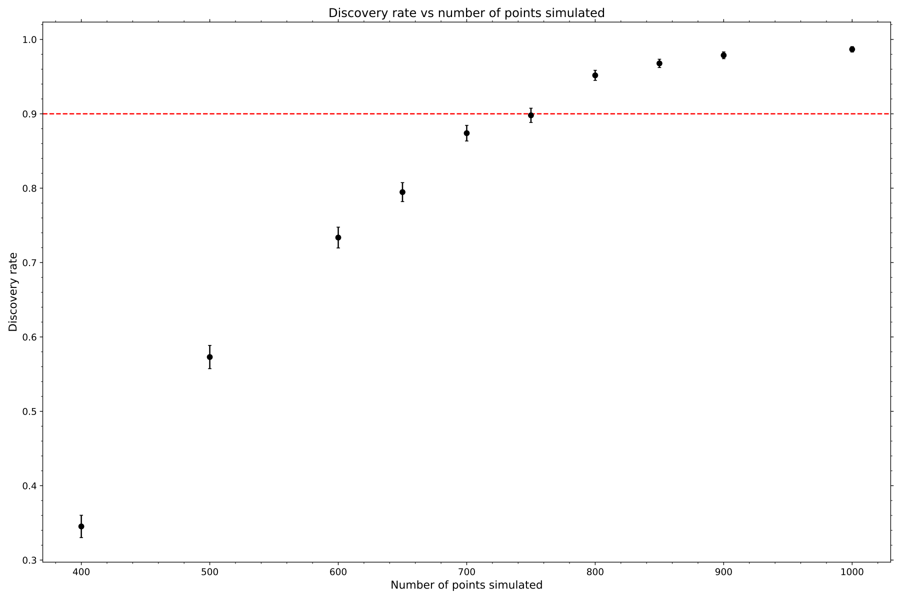

Figure 3: Discovery rate for each dataset size, in the 400-1000 range.

g) Finally, for this part of the coursework the PDF in Equation (4) was modified, with a second Gaussian signal being added. The PDF then takes the form in Equation (8), where $s_i(M;\mu_i,\sigma)$ are two Gaussian centred in μ_i with same variance σ^2 representing the two signal distributions, each carrying a fraction f_i of the total PDF.

$$\mathrm{pdf}(M;\boldsymbol{\theta},\alpha,\beta) = f_1 \cdot s_1(M;\mu_1,\sigma) + f_2 \cdot s_2(M;\mu_2,\sigma) + (1 - f_1 - f_2) \cdot b(M;\lambda) \quad (8)$$

The same procedure applied to the PDF in Equation (4) was carried out here, with each component being normalised separately by dividing by the CDF in the range $[\alpha,\beta]$, so that the integral of the PDF is $f_1 + f_2 + (1 - f_1 - f_2) = 1$.

Setting as true values for the PDF $f_1 = 0.1$, $f_2 = 0.05$, $\mu_1 = 5.28$, $\mu_2 = 5.35$, $\sigma = 0.018$ and $\lambda = 0.5$, the true PDF of Equation (8) has the shape represented in Figure 4, where we can clearly see two separate peaks, with a very small overlap only at the tails. Before testing the discovery rate, I checked a single fit for this model to assure the fairness of the AR method, and I obtained the MLE displayed in Table 3. Once again, we have a good agreement for all the six MLE, as we can also see from the plot in Figure 5. Finally, I moved on to the evaluation of the discovery rate. This time, the null-hypothesis H_0 is that we have only one signal, therefore either f_1 or f_2 must be 0, whereas the alternate hypothesis is that f_1, $f_2 \neq 0$, meaning we have two different signals. The fit for the null-hypothesis was then made to the PDF in Equation (4), because it is a PDF with only one signal, whereas for H_1 the PDF in Equation (8) was used.

The same procedure of exercise f) was used, except this time the dataset size range was increased in order to reach a discovery rate of 90%, because this time it was harder to discard H_0 due to the two signals being close. The plot displaying the different values

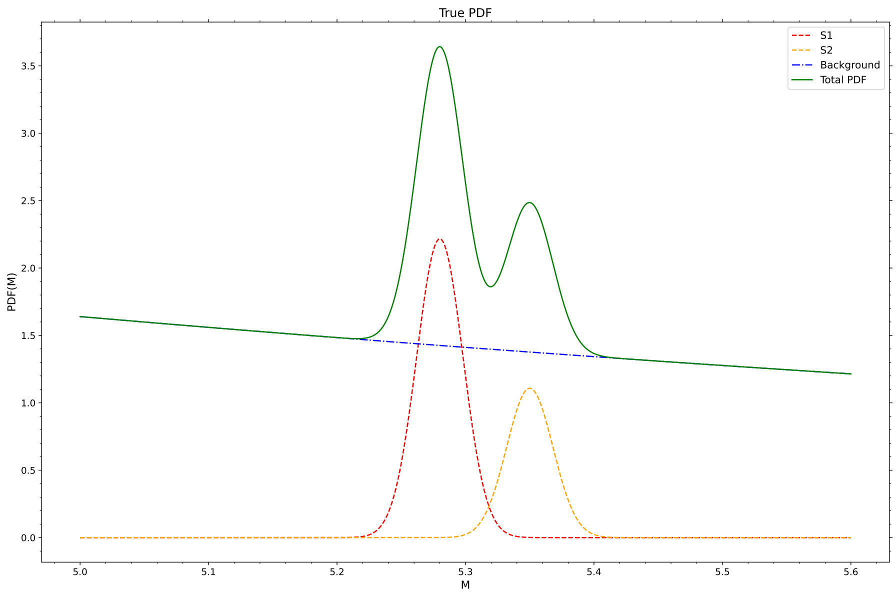

Figure 4: Plot displaying the true PDF alongside the distributions of the two signals and the background, each rescaled by their relative fraction (10%, 5%) and 85%.

Parameter	MLE	Uncertainty	True value
f_1	0.1017	0.0023	0.1
f_2	0.0499	0.0019	0.05
μ_1	5.2796	0.0005	5.28
μ_2	5.3495	0.0009	5.35
σ	0.0186	0.0004	0.018
λ	0.487	0.030	0.5

Table 3: MLE for the siz free-floating parameters. We can see that for all parameters there is a good agreement between the MLE and the true value.

of discovery rate can be found in Figure 6. We observe a similar pattern to what was observed in part f) (Figure 3), with the only difference being in the numbers being much greater. In detail, for 2900 generated points the discovery rate is 0.91 ± 0.01, ensuring us that slightly less than 3000 points are needed in order to have a 90% discovery rate. Once again, this number takes into account the failed fits, which ranged from 89 for 1000 data points to 35 for 4000 points, with the majority of fails being from H_0 fits.

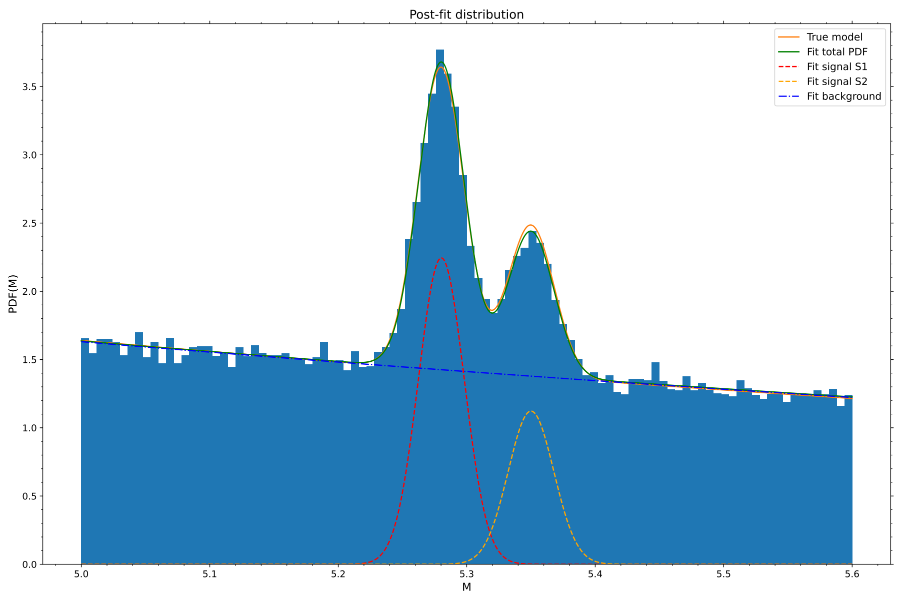

Figure 5: Plot displaying the generated data alongside the true distribution (orange) and the distribution of the best-fit model for the two signals (red and yellow), background (blue) and combined (green)

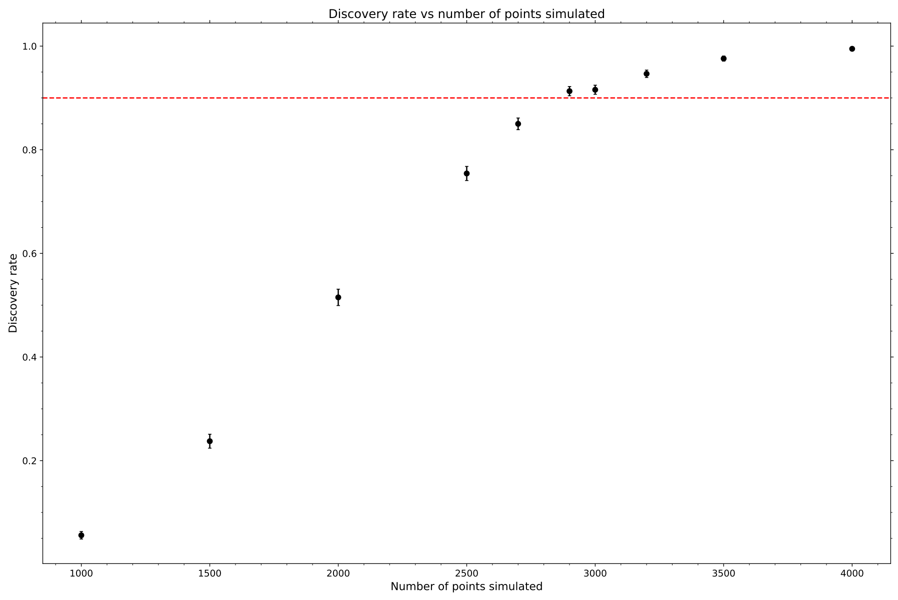

Figure 6: Discovery rate for each dataset size, in the 1000-4000 range.

A Appendix: README

Instructions on how to run the code for the coursework.

The repository can be cloned with `git clone` git@gitlab.developers.cam.ac.uk:phy/data-intensive-science-mphil/s1_assessment/ap2387.git

Anaconda

The conda environment can be created using the conda_env.yml file, which contains all the packages needed to run the code.

```
conda env create -n mphil --file conda_env.yml
```

Report

The final report is presented in ap2387.pdf. The file is generated using LaTeX, but all LaTeX-related files are not being committed as per the instructions in the .gitignore file.

Code Structure

The codes to run the exercises can be found in the `src` folder, whereas the file HelperFunctions.py contains the definition for the `Model()` classes, as well as additional functions used in the code.

Exercise c

The code for this exercise can be found in src/solve_part_c.py. The code can be run using `parser` options, which can be accessed with the following command

```
python src/solve_part_c.py -h
```

The available options are `-a, --alpha`, `-b, --beta`, and `-n, --nentries`, which allow changing the interval range $[\alpha, \beta]$ and the number of models to be tested. The default options are $\alpha = 5$, $\beta = 5.6$, and $n\_entries = 1000$. Additionally, a flag `--plots` can be used if one wants the plots to be displayed and not only saved.

The code will then generate $N = n\_entries$ models, selecting $n\_entries$ random values for the parameters θ, uniformly distributed in a selected range. If the `--plots` flag is selected, the code will also produce a plot displaying the pdfs of all the models. This plot is then saved in the `plots/` folder, under the name `Part_c/a_alpha_beta_n_entries.pdf`. It is recommended to not use the flag if the number of entries is greater than 1000 because otherwise, the plot is too messy and takes a lot of time to be opened (roughly 85 seconds for 10000).

Afterwards, the code will numerically evaluate the integral of the pdf in the $[\alpha, \beta]$ range, and then evaluate the mean and the standard deviation of the $n\_entries$ integrals of the pdfs. These two values are then printed out, and then the code is exited.

Exercise d

The code for this exercise can be found in src/solve_part_d.py. This code simply plots the true distributions of signal, background, and of the combined pdf on the same canvas, fixing the values of θ. The plot generated can be found in plots/Part_d/true_pdf.pdf. Similarly to before, the `--plots` flag can be selected if you want to see the plot while running the code.

Exercise e

The code for this exercise can be found in src/solve_part_e.py. For this part, an additional option can be passed using `-p, --points`, which specifies how many points the user wants to generate, with this number set by default at 100000.

This code generates p points according to the pdf of the Model, using the accept/reject method, and then performs an ML fit in order to estimate the parameters θ. The code uses the `minuit` package, and after the minimization of the NLL, it prints out the best values of the parameters with their uncertainties, as well as the Hessian matrix for the parameters. As an output, this code plots the generated sample alongside the estimates of signal, background, and total probability all overlaid. This plot can be found in plots/Part_e/fit.pdf.

Exercise f

The code for this exercise can be found in src/solve_part_f.py. This code tests the discovery rate of our model, evaluating how much data must be collected to discover at 5σ the signal at least 90% of the time. Multiple different values of dataset size were tested, ranging from 10 to 2000, and for each a fixed number of 1000 datasets is generated. For each dataset, the "discovery" hypothesis is tested against the "null" hypothesis (background-only), and then the p-value is evaluated using the Neyman-Pearson lemma ($T = -2\Delta \ln \mathcal{L}$). In the final version of the code only 10 values are evaluated, chosen such that it's possible to see the value for which the 90% rate is met. The code prints out the number of failed fits for each number of models, and as expected this number decreases significantly as the dataset size increases because with more points it's less likely that the fit fails. The code then prints out the discovery rate for each dataset size and generates a plot of these values (plots/Part_f/discovery_rates.pdf). Furthermore, the discovery rate values are stored in a numpy array in data/discovery_rates.npy, so that the plot can be generated easily without having to re-run the entire code, which takes some time (roughly 4.5 minutes). For this purpose, a `--fit` flag was implemented, which forces the code to re-do the fit step for all the different dataset sizes. This flag is turned off by default, so that the plot can be generated more quickly.

Exercise g

The code for this exercise can be found in src/solve_part_g.py. The structure of this code is basically the same structure of parts e and f, except this time a different model is used. Therefore, there are three options that can be passed via the command line, `-p`, `--points`, which specifies the number of points to be generated for a single fit to the generated data (plot visible in plots/Part_g/true_pdf.pdf), as well as the usual `--plots` and `--fit`, respectively to show the plots and to do the fit step to evaluate the discovery rate for each dataset size. Similarly to part e), the plot displaying the generated sample alongside the estimates of signal, background, and total probability all overlaid is shown in plots/Part_g/fit.pdf, whereas the true pdf distribution, like the one generated in part d, can be found in plots/Part_g/true_pdf.pdf. Finally, the discovery rate for different dataset sizes is plotted in plots/Part_g/discovery_rates.pdf, and similarly to part f) the numpy array containing the discovery rate values is saved in data/discovery_rates_g.npy, due to the code taking roughly 26 minutes to run.