Assignment 2: Orbital Mechanics

Abstract

Through this assignment the student will learn how to implement orbital mechanics to model satellites flying in elliptical orbits around the Earth.

1 Objective

The objective with this assignment is for the student to learn

- How to simulate a satellite in an elliptic orbit
- The definition of orbital parameters
- The concept of the angular velocity of the orbit frame relative to the inertial frame
- How to implement Runge Kutta 45 solver in Python with logging of variables.

2 Introduction

2.1 Notation

Vectors are small bold letters, $\mathbf{v}$, while matrices are capital bold leters, $\mathbf{M}$, and scalars are written with small letters. Superscripts on vectors denote their frame of reference, where i denotes the inertial frame, b denotes the body frame, o denotes the orbit frame, e denotes the Earth Centered Earth Fixed (ECEF), pqr denotes the pqr-frame, while d denotes the desired frame. Depending on the assignment, other reference frames might be used.

2.2 Reference Frames

The Earth Centered Inertial (ECI) frame has its origin in the center of the Earth, with the $\mathbf{x}^i$ axis pointing towards the Vernal equinox, $\mathbf{z}^i$ axis going through the North pole, while $\mathbf{y}^i$ completes the right-handed orthonormal frame (Sidi, 1997, p23).

The orbit frame has its origin in the center of the spacecraft, and follows the spacecraft as it propagates in its orbit. The $\mathbf{x}^o$ axis is parallel with the radius vector ($\mathbf{r}$), the $\mathbf{z}^o$ axis is parallel with the angular momentum vector ($\mathbf{h}$), and $\mathbf{y}^o$ completes the right-handed orthonormal reference frame,

$$\mathbf{z}^o = \frac{\mathbf{h}}{||\mathbf{h}||} \qquad \mathbf{x}^o = \frac{\mathbf{r}}{||\mathbf{r}||} \qquad \mathbf{y}^o = \mathbf{z}^o \times \mathbf{x}^o. \tag{1}$$

Note that $||\cdot||$ denotes the Euclidean norm, meaning that each of the vectors here are of unit length.

The orbit is defined by the classical orbital parameters and has similar definitions as the orbit frame, but with origin in the center of the Earth. In (Sidi, 1997), this frame is known as pqw, and is used to describe the orbits orientation and map the radius, velocity and acceleration vectors to the inertial frame.

3 Rotation Matrices

Rotation matrices are needed to rotate a vector from one reference frame to another and describe the orientation (or attitude) of one reference frame relative to another. A rotation matrix can be given as $\mathbf{R}^{to}_{from}$, where the subscript denote which reference frame to rotate from, and the superscript denotes which reference frame that shall be rotated to. As such, a rotation matrix that rotates a vector from the orbit frame to the body frame can be denoted as $\mathbf{R}^b_o$. Now assume a vector is defined in the orbit frame $\mathbf{v}^o$, then it can be rotated to the body frame by premultiplication as $\mathbf{v}^b = \mathbf{R}^b_o \mathbf{v}^o$.

The rotation matrix is orthonormal and satifies the property that

$$\mathbf{R}^b_o = (\mathbf{R}^o_b)^{-1} = (\mathbf{R}^o_b)^\top. \tag{2}$$

This means that if the rotation matrix between the orbit and body frame ($\mathbf{R}^b_o$) is known, the inverse rotation from body to orbit frame ($\mathbf{R}^o_b$) can be found simply by transposing the matrix.

A rotation from inertial to body frame $\mathbf{R}^b_i$ can be described as a composite rotation made by a rotation from inertial to orbit frame, $\mathbf{R}^o_i$, and then by a rotation from the orbit to body frame $\mathbf{R}^b_o$. This gives the rotation matrix from the inertial frame to the body frame as

$$\mathbf{R}^b_i = \mathbf{R}^b_o \mathbf{R}^o_i. \tag{3}$$

This rotation matrix gives us the attitude (or orientation) of the spacecraft relative to the inertial frame, where the first matrix $\mathbf{R}^b_o$ commonly is described using attitude dynamics, while the second rotation matrix $\mathbf{R}^o_i$ is found from the orbital mechanics.

4 Orbital Mechanics

This assignment will simulate how a spacecraft propagates in an elliptical orbit. This means that the objective is to describe the radius, velocity and acceleration vectors in the inertial frame, as well as finding the angular velocity of the orbit. From (Sidi, 1997, p24), the orbit of a spacecraft can be defined by the six orbital parameters:

- a - the semimajor axis
- e - the eccentricity
- i - the inclination
- Ω - the right ascension of the ascending node
- ω - the argument of perigee
- $M = n(t - t_0)$ the mean anomaly (where n is the mean motion and t is the time).

Figure 1 shows the parameters needed to define the location of the spacecraft in orbit, and allows the complete orbit to be defined through a series of

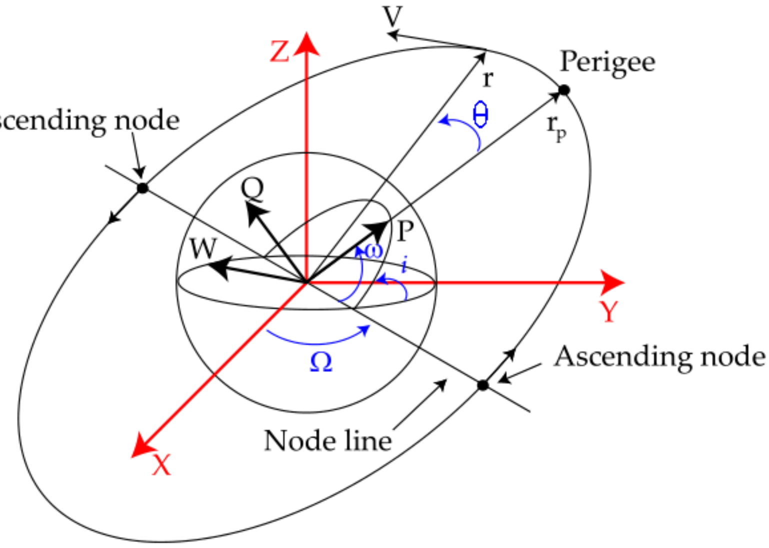

Figure 1: Orbital Parameters

angles. This then allows transformation back to cartesian frame where e.g. the position/velocity vectors can be defined in the ECI frame.

The eccentricity is defined as (Sidi, 1997, p15) as

$$e = \frac{r_a - r_p}{r_a + r_p} \tag{4}$$

where r_a denotes the distance from the center of Earth to the apogee, while r_p denotes the distance from the center of Earth to the perigee.

Another parameter that is of interest is the semi-major axis, which can be found as (Sidi, 1997, p15)

$$a = \frac{r_a + r_p}{2} \tag{5}$$

and allows the mean motion n to be found as

$$n = \sqrt{\frac{\mu}{a^3}} \tag{6}$$

where $\mu = GM_{Earth}$ with the gravitational constant and mass of Earth defined as

$$\begin{aligned} G &= 6.669 \cdot 10^{-11} \\ M_{Earth} &= 5.9742 \cdot 10^{24}. \end{aligned} \tag{7}$$

Finally, the orbital period can be found as (Sidi, 1997, p20) $T = \frac{2\pi}{n}$.

For circular orbits, it is straight forward to find the location of the spacecraft in the orbit. For elliptical orbits it is a little more involved. To that end, the

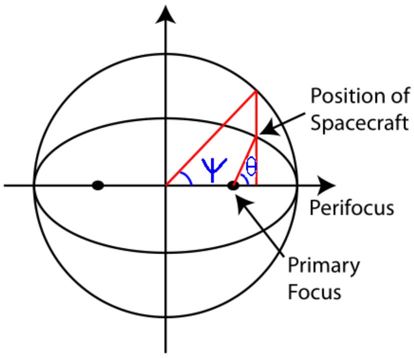

Figure 2: Relationship between eccentric and true anomaly. Adapted from (Sidi, 1997).

orbit is described using what is known as true (θ) and eccentric anomalies (ψ). The true anomaly θ describes the angle between the major axis and the position of the spacecraft and is given as (Sidi, 1997, p26)

$$\cos(\theta) = \frac{\cos(\psi) - e}{1 - e\cos(\psi)} \tag{8}$$

where ψ is the eccentric anomaly. The relationship between the true and eccentric anomaly can be found by considering Figure 2. While the true anomaly considers the true location using the elliptic orbit, the eccentric anomaly uses a circle with radius equal to half the semi-major axis.

The eccentric anomaly cannot be found analytically, but through successive iterations it is possible to obtain an approximation for elliptical orbits. First the eccentric anomaly is set equal to the mean anomaly:

$$\psi_0 = M = n(t - t_0) \tag{9}$$

where t_0 is the initial time, and is set to zero in this assignment. The next iterations are then found as

$$\begin{aligned}
\psi_1 &= M + e\sin(\psi_0) \\
\psi_2 &= M + e\sin(\psi_1) \\
&\vdots \\
\psi_n &= M + e\sin(\psi_{n-1})
\end{aligned}$$

Hence, it is possible to use an iterative method for finding the eccentric anomaly that can run until it has an error of new value relative to old value below a threshold such as $1 \cdot 10^{-6}$.

With the eccentric anomaly available, the true anomaly can be calculated using Equation 8. However, this leads to some problems as the definition of the true anomaly is bounded between $0, 2\pi$, while the definition of the eccentric anomaly is unbounded. This creates a mismatch between the true and eccentric anomaly. To that end, consider the derivative of the true anomaly, which can be found as (Kristiansen, 2008)

$$\dot{\theta} = \frac{n(1 + e\cos(\theta))^2}{(1 - e^2)^{\frac{3}{2}}}, \tag{10}$$

which can be integrated over time to provide a continuous true anomaly to be used for the simulations.

The rotation matrix from the inertial frame to the *pqw* frame can now be found as (Sidi, 1997, p25)

$$\mathbf{R}_i^{pqw} = \begin{bmatrix} \cos(\omega)\cos(\Omega) - \cos(i)\sin(\omega)\sin(\Omega) & \cos(\omega)\sin(\Omega) + \sin(\omega)\cos(i)\cos(\Omega) & \sin(\omega)\sin(i) \\ -\sin(\omega)\cos(\Omega) - \cos(i)\sin(\Omega)\cos(\omega) & -\sin(\omega)\sin(\Omega) + \cos(\omega)\cos(i)\cos(\Omega) & \cos(\omega)\sin(i) \\ \sin(i)\sin(\Omega) & -\sin(i)\cos(\Omega) & \cos(i) \end{bmatrix}.$$

The radius, velocity and acceleration vector can be defined in the *pqw* frame as (Sidi, 1997, pp. 26-27)

$$\mathbf{r}^{pqw} = \begin{bmatrix} a\cos(\psi) - ae & a\sin(\psi)\sqrt{1 - e^2} & 0 \end{bmatrix}^\top \tag{11}$$

$$\mathbf{v}^{pqw} = \begin{bmatrix} -\frac{a^2 n}{r}\sin(\psi) & \frac{a^2 n}{r}\sqrt{1 - e^2}\cos(\psi) & 0 \end{bmatrix}^\top \tag{12}$$

$$\mathbf{a}^{pqw} = \begin{bmatrix} -\frac{a^3 n^2}{r^2}\cos(\psi) & -\frac{a^3 n^2}{r^2}\sqrt{1 - e^2}\sin(\psi) & 0 \end{bmatrix}^\top, \tag{13}$$

where $r = ||\mathbf{r}^i||$ is the length of the radius vector. Each of these vectors can be rotated to the inertial frame using the rotation matrix $\mathbf{R}_{pqw}^i = (\mathbf{R}_i^{pqw})^\top$, such that $\mathbf{r}^i = \mathbf{R}_{pqw}^i \mathbf{r}^{pqw}$, $\mathbf{v}^i = \mathbf{R}_{pqw}^i \mathbf{v}^{pqw}$ and $\mathbf{a}^i = \mathbf{R}_{pqw}^i \mathbf{a}^{pqw}$.

The orbit frame has its origin of the spacecraft, which can be accounted for by including the true anomaly in the rotation matrix as (Sidi, 1997, p26):

$$\mathbf{R}_i^{o} = \begin{bmatrix} \cos(\omega+\theta)\cos(\Omega) - \cos(i)\sin(\omega+\theta)\sin(\Omega) & \cos(\omega+\theta)\sin(\Omega) + \sin(\omega+\theta)\cos(i)\cos(\Omega) & \sin(\omega+\theta)\sin(i) \\ -\sin(\omega+\theta)\cos(\Omega) - \cos(i)\sin(\Omega)\cos(\omega+\theta) & -\sin(\omega+\theta)\sin(\Omega) + \cos(\omega+\theta)\cos(i)\cos(\Omega) & \cos(\omega+\theta)\sin(i) \\ \sin(i)\sin(\Omega) & -\sin(i)\cos(\Omega) & \cos(i) \end{bmatrix}.$$

The angular velocity of the orbit frame relative to the inertial frame can be found using the radius and velocity vectors, enabling its calculations through

$$\boldsymbol{\omega}_{i,o}^i = \frac{\mathbf{r}^i \times \mathbf{v}^i}{(\mathbf{r}^i)^\top \mathbf{r}^i},$$

which can be differentiated to find the angular acceleration as

$$\dot{\boldsymbol{\omega}}_{i,o}^i = \frac{(\mathbf{r}^i \times \mathbf{a}^i)(\mathbf{r}^i)^\top \mathbf{r}^i - 2(\mathbf{r}^i \times \mathbf{v}^i)(\mathbf{v}^i)^\top \mathbf{r}^i}{((\mathbf{r}^i)^\top \mathbf{r}^i)^2} \tag{14}$$

where $\mathbf{a}^i$ is the linear acceleration of the satellite.