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On maximizing the utility of uplink transmissions in sensor networks under explicit fairness constraints*

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***Abstract*— We consider wireless sensor networks with multiple gateways, multiple classes of traffic, and no restrictions on routing and transmission scheduling other than those imposed by the wireless medium. The objective is to schedule uplink transmissions in order to maximize the overall system utility under explicit fairness constraints. We propose a decomposition algorithm drawing upon large-scale decomposition ideas in mathematical programming. We show that in the region of small powers, in which most sensor networks operate, this algorithm terminates with an optimal solution in a finite number of iterations. Moreover, we show that an associated subproblem can be transformed to a maximum weighted matching problem and is therefore solvable in polynomial time. We also consider how to optimize sensor power levels in order to save energy while achieving a certain utility goal. Our approach can efficiently determine the optimal transmission policy for dramatically larger problem instances than an alternative enumeration approach.**

***Index Terms*— Wireless sensor networks, transmission scheduling, utility maximization, fairness, mathematical programming.**

I. INTRODUCTION

Wireless Sensor NETworks (WSNETs) consist of a potentially large number of typically small devices – the sensor nodes or sensors – used to monitor some physical process or system. Sensors are often powered by batteries, have limited computational capabilities, communicate wirelessly, and often operate in noisy and potentially adverse environments. As a result, efficient resource allocation and aggressive optimization of network operations is not merely a desirable luxury but rather an indispensable necessity.

In this paper we view the WSNET as a network that collects data to relay them to some other processing or communication infrastructure. To that end, it utilizes a host of *gateways* whose role is information collection (and fusion) from the sensor nodes. A plethora of applications fit this paradigm, including industrial automation, condition monitoring in manufacturing systems, surveillance, environmental monitoring, military, and homeland defense.

There is, by now, a fair amount of work in WSNETs. [1] considers throughput maximization and proposes an approach that essentially enumerates all possible transmission strategies. As will become apparent later on, such an approach becomes impractical as the size of the network increases. [2] proposes an algorithm to determine the achievable rates in multi-hop wireless networks with fixed link capacities.

In the quest to optimize network performance, *quality-of-service (QoS)* and *fairness* have been recognized as important considerations. Although there has been work along these lines in wireline networks (e.g., [3]) its application to wireless networks is limited. In WSNETs collecting data for instance, a "geographic" bias might be introduced into the data collection process. That is, sensors close to the gateway (i.e., with high channel gains) have a significant advantage over sensors that happen to be further away. This is a rather unfair operation of the WSNET and is due to the wireless medium rather than the actual needs. One way to mitigate it is to explicitly introduce fairness constraints into the problem formulation. The resulting transmission strategy could use multi-hop routing to achieve a more balanced operation.

In this paper we seek to optimize how sensors transmit information to other sensors or to the gateways (uplink transmissions) by carefully scheduling the use of the wireless medium. Carrier sensing used in the IEEE 802.11 protocol and other random *medium access control (MAC)* strategies are inefficient and energy wasteful in most sensor network applications. We formulate the problem as a *utility maximization* problem over the convex hull of feasible transmission schemes. As a result, the optimal policy involves time-sharing among several such schemes. Our approach borrows from ideas in large-scale decomposition methods in mathematical programming. The algorithm we develop consists of a *master problem* and a *subproblem*. An instance of the subproblem is solved for each master problem iteration. Apart from establishing convergence of the proposed algorithm, a key contribution is the efficient solution of the subproblem. The subproblem is an *integer linear programming*; however, we discover enough special structure to be able to solve it in polynomial-time by relating it to a maximum weighted matching problem.

Our methodology dramatically improves the size of problems that can be solved. Compared to very small instances (with 5-6 nodes) solvable by enumeration [1], we are able to solve instances with 50 or so nodes in less than a minute. By maximizing system utility under fairness constraints, we cover a variety of interesting cases that appear in practice, including throughput optimization [1, 2]. We offer a unified framework of addressing these problems, including

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the optimization of sensor power levels. This is critical as energy preservation is essential in sensor networks. Utility maximization is also considered in [4], but in the absence of fairness constraints. In that work, utility is separable (i.e., a sum of individual sensor utilities) which allows for a distributed approach. Our work can handle more general utilities that can accommodate potential interdependencies among sensor objectives; the price to pay is the need for a centralized approach.

This rest of the paper is organized as follows. In Section II we present the system model and formulate the utility maximization problem. Section III presents our decomposition algorithm and establishes its convergence. In Section IV we discuss how to solve the subproblem efficiently (in polynomial-time). Optimization over power limits is considered in Section V. Some illustrating numerical results are presented in Section VI. Conclusions are in Section VII.

Notational Conventions: Throughout the paper all vectors are assumed to be column vectors. We use lower case boldface letters to denote vectors and for economy of space we write $\mathbf{x} = (x_1, \dots, x_R)$ for the column vector $\mathbf{x}$. $\mathbf{x}'$ denotes the transpose of $\mathbf{x}$ and $\mathbf{0}$ the vector of all zeroes. We use upper case boldface letters to denote matrices. We use script letters to define sets and denote by $\text{Conv}(\mathscr{A})$ the convex hull of a set $\mathscr{A}$, and by $|\mathscr{A}|$ its cardinality. We denote by $1_{\mathscr{A}}(\mathbf{x})$ the indicator function of $\mathbf{x} \in \mathscr{A}$. When $\mathscr{A}$ is described by a simple condition, say $\mathbf{x} \geq \mathbf{0}$, we simply write $1(\mathbf{x} \geq \mathbf{0})$.

II. Network Model and Problem Formulation

We consider a WSNET with N sensor nodes each of which can receive, transmit and relay information with a single port/antenna that it carries. We assume that sensor nodes do not multicast information, so each transmission is from one node to another. Since they carry a single antenna, nodes cannot receive and transmit simultaneously. Furthermore, receiving nodes cannot receive information from multiple nodes simultaneously. In addition to the sensor nodes, the network uses M gateways which receive information from the sensors and relay it to some other networking or processing infrastructure. Gateways can only receive information but they are more capable than the sensor nodes, thus, they can be receiving from multiple nodes simultaneously. Henceforth, we will refer to all $M + N$ sensor and gateway nodes alike as *nodes of the WSNET*. Nodes $1, \dots, N$ will correspond to sensor nodes and nodes $N + 1, \dots, N + M$ to gateways.

Sensors in the WSNET collect different types of data depending on the physical system or process they monitor (e.g., temperature, pressure, levels of harmful agents, etc.) and want to relay them to other (sensor or gateway) nodes. As a result, the WSNET carries multiple types of traffic, differing in information content and utility associated with their successful transmission. We use the term *traffic class* to refer to types of traffic with a particular origin and destination. Let K be the total number of traffic classes. We denote by $s(k)$ and $d(k)$ the source and destination of class k, for $k = 1, \dots, K$.

We model the background noise in the WSNET as a single source of additive, white and Gaussian noise, with power spectral density η and bandwidth W. Let p_{ijk} denote the power used by node i to transmit class k traffic to node j, for $i = 1, \dots, N + M$, $j = 1, \dots, N + M$, $k = 1, \dots, K$. We will refer to such a transmission as the (i, j, k) transmission. Let G_{ij} be the channel gain between nodes i and j when i is transmitting. When node i transmits class k traffic the received power at node j is $p_{ijk}G_{ij}$. Sensor nodes have limited power resources; we let $\bar{p}_i$ denote the maximum power available at node i for $i = 1, \dots, N$. Thus, for any $i, j = 1, \dots, M+N$, $k = 1, \dots, K$, it follows that p_{ijk} is upper bounded by

$$\bar{p}_{ijk} \triangleq \begin{cases} 0, & \text{if } i = N+1, \dots, N+M, \\ & \text{or } i = d(k), \text{ or } i = j, \\ \bar{p}_i, & \text{otherwise,} \end{cases} \tag{1}$$

where $\bar{p}_{ijk}$ denotes the maximum power available for the (i, j, k) transmission. The first branch of the above is an immediate consequence of the assumptions made heretofore.

Consider next an (i, j, k) transmission. The SINR, γ_{ijk}, is given by

$$\gamma_{ijk} = \frac{p_{ijk}G_{ij}}{\eta W + \sum_{v=1}^{K} \sum_{l=1, l\neq i}^{N+M} \sum_{u=1}^{N+M} p_{luv}G_{lj}}. \tag{2}$$

We use the Shannon capacity to determine the maximum rate for an (i, j, k) transmission and assume that the sending node i transmits with the maximum possible rate. Let r_{ijk} denote the net flow rate for an (i, j, k) or a (j, i, k) transmission, i.e.,

$$r_{ijk} = W \log_2(1 + \gamma_{ijk}) - W \log_2(1 + \gamma_{jik}). \tag{3}$$

When an (i, j, k) transmission is in progress, and under the transmission restrictions adopted, it follows that $\gamma_{ijk} \geq 0$, $\gamma_{uiv} = 0$ for all u, v, and $r_{ijk} \geq 0$. Otherwise, when a (j, i, k) transmission is in progress, $\gamma_{jik} \geq 0$, $\gamma_{ujv} = 0$ for all u, v, and $r_{ijk} \leq 0$. Clearly, $r_{ijk} = -r_{jik}$. We write $\mathbf{r}$ for the $(N+M)^2K$-dimensional vector of r_{ijk}'s and denote by r_{ijk} its component that corresponds to the net flow rate for an (i, j, k) or a (j, i, k) transmission. Similarly, we write $\mathbf{p}$ for the $(N + M)^2K$-dimensional vector of powers and denote by p_{ijk} its component corresponding to the (i, j, k) transmission.

In this work, we concentrate on sensor networks in which power levels are on the order of mWatt or even lower and the objective function can be well approximated by a linear function. Taking the Taylor series expansion of (3) around $\mathbf{p} = \mathbf{0}$ and maintaining up to first order terms we obtain

$$r_{ijk} = \frac{p_{ijk}G_{ij}}{\eta \ln 2} - \frac{p_{jik}G_{ji}}{\eta \ln 2}, \quad \forall i, j, k. \tag{4}$$

In matrix notation we have $\mathbf{r} = \mathbf{H}\mathbf{p}$, where the matrix $\mathbf{H}$ is appropriately defined.

The transmission restrictions introduced thus far translate into the following set of conditions

$$\begin{array}{ll} p_{ijk}p_{uiv} = 0, & \forall i,j,k,u,v, \\ p_{ijk}p_{iuv} = 0, & \forall (j,k) \neq (u,v), \\ p_{ijk}p_{ujv} = 0, & \forall (i,k) \neq (u,v),\ j \leq N, \\ 0 \leq p_{ijk} \leq \bar{p}_{ijk}, & \forall i,j,k. \end{array}$$

These conditions respectively state that at any point in time *(i)* nodes cannot transmit and receive simultaneously, *(ii)* can only transmit traffic of a single class to a single node and, *(iii)* except for gateways, nodes can receive only a single traffic class from a single node. We denote by $\mathscr{P}$ the set of all $\mathbf{p} \in \mathbb{R}^{(N+M)^2 K}$ that satisfy the conditions above. We call *valid* a transmission scheme with $\mathbf{p} \in \mathscr{P}$. Clearly $\mathscr{P}$ is bounded. We also denote $\mathscr{R} = \{\mathbf{r} \mid \mathbf{r} = \mathbf{H}\mathbf{p},\ \mathbf{p} \in \mathscr{P}\}$. The next lemma establishes some useful properties of $\mathscr{P}$ and $\mathscr{R}$.

Lemma II.1 *(i) Conv($\mathscr{P}$) and Conv($\mathscr{R}$) are polytopes (i.e., bounded polyhedra). (ii) Conv($\mathscr{R}$) $= \{\mathbf{r} \mid \mathbf{r} = \mathbf{H}\mathbf{p},\ \mathbf{p} \in$ Conv($\mathscr{P}$)$\}$. (iii) For any extreme point $\mathbf{r} \in$ Conv($\mathscr{R}$), there exists an extreme point $\mathbf{p} \in$ Conv($\mathscr{P}$) such that $\mathbf{r} = \mathbf{H}\mathbf{p}$.*

Suppose next we choose L valid transmission schemes. To every valid transmission scheme n corresponds a rate vector in $\mathscr{R}$, say $\mathbf{r}^n$. Let us consider the information flow in the network in a potentially large but finite time interval. Normalize the length of this interval to 1. At different times, the network may employ different transmission schemes, e.g., in order to implement multi-hop routing. Suppose that during this time interval, the network uses the L selected schemes only and spends a fraction of time α_n transmitting according to scheme $n = 1, \ldots, L$. Then the total amount of information delivered during this interval is characterized by $\mathbf{r} = \sum_{n=1}^{L} \alpha_n \mathbf{r}^n$. This is also the long-term average transmission rate vector.

Over the long run, the WSNET obeys flow conservation laws, i.e., the traffic of each class should not accumulate in any node other than its intended destination. This implies that the average transmission rate vector satisfies

$$\sum_{j=1}^{N+M} r_{ijk} = 0, \qquad \forall i \neq s(k), d(k),\ \forall k,$$

that is class k traffic flow into i equals class k traffic outflow from node i.

We seek to maximize the overall utility of transmissions in the WSNET, expressed as a function $F(\mathbf{r})$ of the long-term average transmission rate vector $\mathbf{r}$. We assume that $F(\mathbf{r})$ is continuous, concave, and bounded in Conv($\mathscr{R}$). Note that by considering system utility, we cover a large variety of objectives studied in the literature, including weighted throughput which is a linear function of $\mathbf{r}$. Moreover, $F(\mathbf{r})$ needs not be a sum of individual utilities associated with each traffic class. Rather, it can represent quite general performance metrics that model interdependent behavior of the various sensors, e.g., when, for instance, clusters of sensors collaborate towards a common goal.

We are interested in utility maximization subject to explicit fairness constraints. We model fairness considerations as a set of R linear inequalities $\mathbf{A}\mathbf{r} \leq \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{R \times (N+M)^2 K}$ and $\mathbf{b} \in \mathbb{R}^{(N+M)^2 K}$ are given. For example, these constraints can impose equality among all average transmission rates. Let $\mathscr{S}$ be the set of rates that satisfy fairness constraints and flow conservation, i.e.,

$$\mathscr{S} \triangleq \left\{\mathbf{r} \mid \mathbf{A}\mathbf{r} \leq \mathbf{b}, \sum_{j=1}^{N+M} r_{ijk} = 0, \forall i \neq s(k), d(k), \forall k\right\}$$

and to exclude trivial cases assume Conv($\mathscr{R}$) $\cap\, \mathscr{S} \neq \emptyset$.

We can formulate the utility optimization problem as

$$\begin{array}{ll} \max & F(\mathbf{r}) \\ \text{s.t.} & \mathbf{r} \in \text{Conv}(\mathscr{R}) \cap \mathscr{S}. \end{array}$$

An important observation is that we seek to maximize utility over the convex hull of $\mathscr{R}$ rather than $\mathscr{R}$ itself. This is bound to yield higher system utility and as we have seen the WSNET operates by time-sharing among different transmission schemes.

Let $\mathbf{r}^1, \ldots, \mathbf{r}^L$ denote the extreme points of Conv($\mathscr{R}$). Any $\mathbf{r} \in$ Conv($\mathscr{R}$) can be expressed as a convex combination of those. Incorporating the definition of $\mathscr{S}$ and writing it as a minimization problem the problem above is equivalent to

$$\begin{array}{ll} \min & -F(\mathbf{r}) \\ \text{s.t.} & \mathbf{r} - \sum_{n=1}^{L} \alpha_n \mathbf{r}^n = 0, \\ & \sum_{n=1}^{L} \alpha_n = 1, \\ & \sum_{j=1}^{N+M} r_{ijk} = 0,\ \forall i \neq s(k), d(k), \forall k, \\ & \mathbf{A}\mathbf{r} \leq \mathbf{b}, \\ & \alpha_n \geq 0,\ n = 1, \ldots, L. \end{array} \tag{5}$$

Note that $\mathbf{r}^1, \ldots, \mathbf{r}^L$ are also points of $\mathscr{R}$, thus, there exist corresponding valid transmission schemes (i.e., points in $\mathscr{P}$) $\mathbf{p}^1, \ldots, \mathbf{p}^L$ with $\mathbf{r}^n = \mathbf{H}\mathbf{p}^n$ for all $n = 1, \ldots, L$. The problem above maximizes a concave function over a polyhedron. It can be solved using, for example, the *conditional gradient method* [5]. If $F(\mathbf{r})$ is linear, then it is a linear programming problem for which very efficient algorithms exist.

Of course, Conv($\mathscr{R}$) can have a humongous number of extreme points and this is the key challenge in solving (5). A simpler version of (5), maximizing throughput and with no fairness constraints, was considered in [1] and proposed to be solved by simply enumerating all extreme points and including them in the formulation (5). As indicated in [1] and clearly illustrated in Section VI this approach can quickly run out of steam (that is, memory) in very small networks. As we will see in Section III, there are more efficient ways to solve (5). The decomposition algorithm we propose does not need to know $\mathbf{r}^1, \ldots, \mathbf{r}^L$ (or equivalently, the corresponding transmission schemes) in advance. It generates them as needed and identifies the ones that should be used to achieve optimality.

III. A DECOMPOSITION METHOD

In this section we propose a decomposition method for solving (5). For linear utilities the method is a *column generation* method for solving large-scale linear programming problems.

To develop the decomposition approach consider the problem (5), to which we will be referring as the *master problem*. The dual function is

$$\begin{aligned}
G(\boldsymbol{\lambda},\mu,\boldsymbol{\nu},\boldsymbol{\sigma}) &= \inf_{\boldsymbol{\alpha}\geq\mathbf{0},\mathbf{r}} \Big\{ -F(\mathbf{r}) + \boldsymbol{\lambda}'(\mathbf{r} - \textstyle\sum_n \alpha_n \mathbf{r}^n) \\
&\quad + \mu(\textstyle\sum_n \alpha_n - 1) + \boldsymbol{\sigma}'(\mathbf{A}\mathbf{r} - \mathbf{b}) \\
&\quad + \textstyle\sum_k \sum_{i\neq s(k),d(k)} \nu_{ik} \sum_j r_{ijk} \Big\} \\
&= G_1(\boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\sigma}) + G_2(\boldsymbol{\lambda},\mu) - \mu - \boldsymbol{\sigma}'\mathbf{b},
\end{aligned}$$

where

$$\begin{aligned}
G_1(\boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\sigma}) &= \inf_{\mathbf{r}} \Big\{ -F(\mathbf{r}) + (\boldsymbol{\lambda}' + \boldsymbol{\sigma}'\mathbf{A})\mathbf{r} \\
&\qquad + \textstyle\sum_k \sum_{i\neq s(k),d(k)} \nu_{ik} \sum_j r_{ijk} \Big\}, \\
G_2(\boldsymbol{\lambda},\mu) &= \inf_{\boldsymbol{\alpha}\geq\mathbf{0}} \textstyle\sum_n (\mu - \boldsymbol{\lambda}'\mathbf{r}^n)\alpha_n.
\end{aligned}$$

Let

$$\begin{aligned}
\mathscr{D}_1 &= \{(\boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\sigma}) \mid G_1(\boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\sigma}) > -\infty\} \\
\mathscr{D}_2 &= \{(\boldsymbol{\lambda},\mu) \mid G_2(\boldsymbol{\lambda},\mu) > -\infty\}
\end{aligned}$$

and note that

$$\mathscr{D}_2 = \{(\boldsymbol{\lambda},\mu) \mid \mu - \boldsymbol{\lambda}'\mathbf{r}^n \geq 0,\ n = 1,\ldots,L\},$$

$$G_2(\boldsymbol{\lambda},\mu) = \begin{cases} 0, & \text{if } (\boldsymbol{\lambda},\mu) \in \mathscr{D}_2, \\ -\infty, & \text{otherwise,} \end{cases}$$

and $\mathscr{D}_1$ is independent of $\mathbf{r}^1,\ldots,\mathbf{r}^L$. Then the dual of the master problem (5) is

$$\begin{array}{ll}
\max & G_1(\boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\sigma}) - \mu - \boldsymbol{\sigma}'\mathbf{b} \\
\text{s.t.} & (\boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\sigma}) \in \mathscr{D}_1, \\
& \mu - \boldsymbol{\lambda}'\mathbf{r}^n \geq 0, \quad n = 1,\ldots,L, \\
& \boldsymbol{\sigma} \geq \mathbf{0}.
\end{array} \tag{6}$$

Since the master problem is a convex optimization problem there is no duality gap [5].

Suppose now we have an extreme point of Conv($\mathscr{R}$), say $\mathbf{r}^1$, which belongs to $\mathscr{S}$. Let $m \in \{1,\ldots,L\}$, and consider

$$\begin{array}{ll}
\min & -F(\mathbf{r}) \\
\text{s.t.} & \mathbf{r} - \sum_{n=1}^m \alpha_n \mathbf{r}^n = 0, \\
& \sum_{n=1}^m \alpha_n = 1, \\
& \sum_{j=1}^{N+M} r_{ijk} = 0,\ \forall i \neq s(k), d(k), \forall k, \\
& \mathbf{A}\mathbf{r} \leq \mathbf{b}, \\
& \alpha_n \geq 0,\ n = 1,\ldots,m,
\end{array} \tag{7}$$

which we call the *restricted master problem* at the mth iteration. Suppose we solve this problem to optimality. The dual of this problem is identical to (6) with the exception that only constraints $\mu - \boldsymbol{\lambda}'\mathbf{r}^n \geq 0$, for $n = 1,\ldots,m$, appear. We refer to this latter problem as the *restricted dual problem* at the mth iteration. Let $(\mathbf{r}^{(m)}, \boldsymbol{\alpha}^{(m)}; \boldsymbol{\lambda}^{(m)}, \mu^{(m)}, \boldsymbol{\nu}^{(m)}, \boldsymbol{\sigma}^{(m)})$ be an optimal primal-dual pair for the restricted master problem. The dual variables are dual feasible and satisfy $(\boldsymbol{\lambda}^{(m)}, \boldsymbol{\nu}^{(m)}, \boldsymbol{\sigma}^{(m)}) \in \mathscr{D}_1$, $\boldsymbol{\sigma}^{(m)} \geq \mathbf{0}$, and $\mu^{(m)} - \boldsymbol{\lambda}^{(m)'}\mathbf{r}^n \geq 0$, for all $n = 1,\ldots,m$. If it happens that $\mu^{(m)} - \boldsymbol{\lambda}^{(m)'}\mathbf{r}^n \geq 0$ for all $n = 1,\ldots,L$ then we have a primal-dual pair for (5) and we are done. Otherwise, we need to generate an extreme point, say $\mathbf{r}^{m+1}$, of Conv($\mathscr{R}$) that violates dual feasibility, solve the $m+1$st restricted master problem, and continue iterating in this fashion. We next examine how to produce "cuts" in the dual.

A. The subproblem

At the mth iteration we seek an extreme point $\mathbf{r}^{m+1}$ of Conv($\mathscr{R}$) satisfying $\mu^{(m)} - \boldsymbol{\lambda}^{(m)'}\mathbf{r}^{m+1} < 0$. As we argued earlier, the extreme points of Conv($\mathscr{R}$) are also in in $\mathscr{R}$. So we might as well generate a point $\mathbf{r}$ that minimizes $\mu^{(m)} - \boldsymbol{\lambda}^{(m)'}\mathbf{r}$ over $\mathscr{R}$. This suggests the subproblem

$$\begin{array}{ll}
\max & \boldsymbol{\lambda}'\mathbf{r} \\
\text{s.t.} & \mathbf{r} = \mathbf{H}\mathbf{p}, \\
& \mathbf{p} \in \mathscr{P},
\end{array} \tag{8}$$

with cost vector $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(m)}$.

Before we proceed showing that the proposed decomposition algorithm converges we establish some properties of (8). $\boldsymbol{\lambda} \in \mathbb{R}^{(M+N)^2K}$ is the dual vector corresponding to the first constraint of (5). Denote by λ_{ijk} the element of $\boldsymbol{\lambda}$ corresponding to r_{ijk} and let $\pi_{ijk} = \lambda_{ijk} - \lambda_{jik}$. Then

$$\begin{aligned}
\boldsymbol{\lambda}'\mathbf{H}\mathbf{p} &= \textstyle\sum_{k=1}^K \sum_{i=1}^{N+M} \sum_{j=1}^{N+M} \lambda_{ijk}\big(\frac{p_{ijk}G_{ij}}{\eta \ln 2} - \frac{p_{jik}G_{ji}}{\eta \ln 2}\big) \\
&= \textstyle\sum_{k=1}^K \sum_{i=1}^{N+M} \sum_{j=1}^{N+M} \frac{\pi_{ijk}G_{ij}}{\eta \ln 2} p_{ijk},
\end{aligned}$$

hence the subproblem is equivalent to

$$\begin{array}{ll}
\max & \sum_{k=1}^K \sum_{i=1}^{N+M} \sum_{j=1}^{N+M} \frac{\pi_{ijk}G_{ij}}{\eta \ln 2} p_{ijk} \\
\text{s.t.} & \mathbf{p} \in \mathscr{P}.
\end{array} \tag{9}$$

Next we reduce it to an *integer linear programming problem.*

Proposition III.1 *Problem (9) is equivalent to the ILP:*

$$\begin{array}{ll}
\textit{max} & \sum_{(i,j,k)|\psi_{ijk}>0} \psi_{ijk} s_{ijk} \\
\textit{s.t.} & \displaystyle\sum_{j=1}^{N+M} \sum_{k=1}^{K} s_{ijk} + \sum_{j=1}^{N+M} \sum_{k=1}^{K} s_{jik} \leq 1,\ \forall i \leq N, \\
& 0 \leq s_{ijk} \leq I_{ijk}, \\
& s_{ijk} \in \{0,1\},
\end{array} \tag{10}$$

where $\psi_{ijk} = \frac{\pi_{ijk}\bar{p}_{ijk}G_{ij}}{\eta \ln 2}$ *and* $I_{ijk} = 1(\psi_{ijk} > 0)$.

We summarize the discussion on the subproblem as follows: to compute an optimal solution $\mathbf{r}^*$ of (8) we first solve (10) to obtain an optimal solution $\mathbf{s}^*$, then compute $\mathbf{p}^*$ as

$$p^*_{ijk} = \begin{cases} \bar{p}_{ijk}, & \text{if } s_{ijk} = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

and finally compute $\mathbf{r}^* = \mathbf{H}\mathbf{p}^*$. It is evident that $\mathbf{s}^*$ prescribes how to operate the network under the transmission scheme $\mathbf{p}^*$: (i,j,k) transmissions occur only if $s_{ijk} = 1$ and if so at maximum power.

B. The decomposition algorithm

We now have all the ingredients to present the decomposition algorithm and show its convergence. The algorithm is in Fig. 1 and the next theorem establishes its convergence.

Theorem III.2 *Assume that (5) is feasible. Then the decomposition algorithm of Fig. 1 terminates with an optimal solution of (5) in a finite number of iterations.*

1) **Initialization**: Let $\mathbf{r}^1 \in \text{Conv}(\mathscr{R}) \cap \mathscr{S}$ and set $m = 1$.
2) **m-th iteration**:
   a) Solve the restricted master problem (7) with $\mathbf{r}^1, \ldots, \mathbf{r}^m$ to obtain an optimal primal-dual pair $(\mathbf{r}^{(m)}, \boldsymbol{\alpha}^{(m)}; \boldsymbol{\lambda}^{(m)}, \mu^{(m)}, \boldsymbol{\nu}^{(m)}, \boldsymbol{\sigma}^{(m)})$.
   b) Solve the subproblem (8) with cost vector $\boldsymbol{\lambda}^{(m)}$ as outlined in Section III-A. Let $\mathbf{r}^{m+1}$ be the optimal solution obtained.
   c) If $\mu^{(m)} - \boldsymbol{\lambda}^{(m)'} \mathbf{r}^{m+1} \geq 0$ stop; $(\mathbf{r}^{(m)}, \boldsymbol{\alpha}^{(m)})$ is an optimal solution of (5). Otherwise, set $m := m + 1$ and go to step 2a.

Fig. 1. The decomposition algorithm.

We conclude this section by noting that the feasibility assumption can be relaxed. Using a slightly modified version of the decomposition algorithm above, we can determine whether problem (5) is feasible and if so identify the initial transmission vector $\mathbf{r}^1$.

IV. Solving the Subproblem

The efficiency of the algorithm of Fig. 1 critically depends on how efficiently we can solve the subproblem. As outlined in Section III-A, solving the subproblem amounts to solving an ILP. ILPs are hard to solve (they are NP-complete); solvers invariably use branch-and-bound methods which, depending on the problem and its size, can take a long time. Fortunately, our subproblem has enough of special structure that makes it polynomially solvable. In this section, we establish that (10) is equivalent to a *maximum weighted matching* problem.

Let us define the following sets: $\mathscr{A} = \{1, \ldots, N\}$ and $\mathscr{B}_l = \{Nl + 1, \ldots, Nl + N\}$ for $l = 1, \ldots, M$. Each element of $\mathscr{A}$ corresponds to a sensor node of the WSNET and set $\mathscr{B}_l$ corresponds to the gateway l of the WSNET. Let $\mathscr{V} = \mathscr{A} \cup (\cup_{l=1}^{M} \mathscr{B}_l)$ and consider the undirected graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{E}$ is the complete set of edges between nodes in $\mathscr{V}$. With each edge $(i, j) \in \mathscr{E}$ we associate a weight w_{ij} such that

$$w_{ij} = \begin{cases} \max\limits_{k=1,\ldots,K} \max\{\psi_{ijk}, \psi_{jik}\}, & \forall i, j \in \mathscr{A}, \\ \max\limits_{k=1,\ldots,K} \max\{\psi_{i,N+l,k}, 0\}, & \forall i \in \mathscr{A}, j \in \mathscr{B}_l, \\ \max\limits_{k=1,\ldots,K} \max\{\psi_{j,N+l,k}, 0\}, & \forall i \in \mathscr{B}_l, j \in \mathscr{A}, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Note that $w_{ij} = w_{ji} \geq 0$, $\forall i, j$. Also for any i, u, v, if $u, v \in \mathscr{B}_l$ for some l, then $w_{iu} = w_{iv}$, that is, the weight of the link between i and any node in $\mathscr{B}_l$ is the same. Let us also construct a set $\mathscr{K}$ as follows: for each $1 \leq i \leq N, 1 \leq j \leq N + M$, we select only one, if any, k satisfying the conditions

$$k = \begin{cases} \text{argmax}_{t=1,\ldots,K} \max\{\psi_{ijt}, \psi_{jit}\}, & \text{if } j \leq N, \\ \text{argmax}_{t=1,\ldots,K} \max\{\psi_{ijt}, 0\}, & \text{otherwise,} \end{cases}$$

and $\psi_{ijk} > 0$, and let (i, j, k) be an element of $\mathscr{K}$. The next theorem is the main result of this section.

Theorem IV.1 *Suppose* $\mathbf{x}^*$ *is an optimal solution to the maximum weighted matching problem*

$$\begin{array}{ll} \textit{max} & \sum_{(i,j)\in\mathscr{E}} w_{ij} x_{ij} \\ \textit{s.t.} & \sum_{j|(i,j)\in\mathscr{E}} x_{ij} \leq 1, \quad \forall i \\ & x_{ij} = x_{ji}, \quad \forall i, j, \\ & x_{ii} = 0, \quad \forall i, \\ & x_{ij} \in \{0, 1\}, \quad \forall i, j. \end{array} \tag{13}$$

Then, an optimal solution $\mathbf{s}^*$ *to the subproblem (10) satisfies*

$$s^*_{ijk} = \begin{cases} 1_{\mathscr{K}}(i, j, k) x^*_{ij}, & \textit{if } 1 \leq i, j \leq N, \\ 1_{\mathscr{K}}(i, j, k) \sum\limits_{v \in \mathscr{B}_{j-N}} x^*_{iv}, & \textit{if } 1 \leq i \leq N, j \geq N + 1, \\ 0, & \textit{otherwise.} \end{cases}$$

The maximum weighted matching problem is a well studied problem in graph theory and can be solved in polynomial time. Due to the connection asserted in the above theorem, the subproblem is solvable in $O(KN(N + M) + (M + 1)^3 N^3)$ time.

V. Optimization over Power Limits

So far we have assumed that the power limits $\bar{p}_i$ of all sensor nodes $i = 1, \ldots, N$ are fixed. As we will see higher power limits lead to higher utility, but, of course, higher energy consumption. As energy preservation is critical in WSNETs, it becomes of interest to optimize the power limits used by the sensor nodes to achieve a certain utility target. In this section we discuss how this can be done.

Let us view the utility maximization problem formulated in Section II as parametrized by the vector of power limits, denoted by $\bar{\mathbf{p}} = (\bar{p}_1, \ldots, \bar{p}_N)$. In particular, consider

$$\begin{array}{rll} \bar{F}(\bar{\mathbf{p}}) \stackrel{\triangle}{=} \max & F(\mathbf{r}) \\ \text{s.t.} & \mathbf{r} \in \text{Conv}(\mathscr{R}(\bar{\mathbf{p}})) \cap \mathscr{S}, \end{array} \tag{14}$$

where we write $\mathscr{R}(\bar{\mathbf{p}})$ to explicitly denote the fact that the set of transmission rate vectors depends on $\bar{\mathbf{p}}$. The first, and rather intuitive, property we show is monotonicity.

Theorem V.1 *(Monotonicity) Suppose* $\bar{\mathbf{p}}^1$ *and* $\bar{\mathbf{p}}^2$ *are two vectors of power limits. If* $\bar{\mathbf{p}}^1 \geq \bar{\mathbf{p}}^2$*, then* $\bar{F}(\bar{\mathbf{p}}^1) \geq \bar{F}(\bar{\mathbf{p}}^2)$.

Next we show that the optimal utility is concave in $\bar{\mathbf{p}}$ if we scale the power limits in the whole network uniformly.

Theorem V.2 *(Concavity) Suppose the power limit vector* $\bar{\mathbf{p}}$ *belongs to the set* $\mathscr{W} = \{\bar{\mathbf{p}} \mid \bar{\mathbf{p}} = \phi \bar{\mathbf{p}}^0, \phi > 0\}$ *where* $\bar{\mathbf{p}}^0 > \mathbf{0}$ *is a constant vector. Then* $\bar{F}(\bar{\mathbf{p}})$ *is concave in* $\bar{\mathbf{p}}$ *over* $\mathscr{W}$.

The above theorem is critical in trading-off energy consumption with achieved utility. Suppose we are interested in minimizing energy consumption subject to achieving a utility level equal to some given value, say $F_{\min}$. Assuming that power limits are scaled uniformly for the whole network by a factor ϕ, we can formulate the problem as

$$\begin{array}{ll} \min & \phi \\ \text{s.t.} & \bar{F}(\phi\bar{\mathbf{p}}_0) \geq F_{\min}, \end{array} \tag{15}$$

where $\bar{F}(\phi\bar{\mathbf{p}}_0)$ is defined in (14). Theorem V.2 asserts that the above is a convex optimization problem, thus, a global minimum can be obtained using standard gradient-based algorithms [5].

It is also interesting to consider what are the implications of power optimization to the lifetime of the network. Actually, scaling the power limits enables us to cast and resolve the tradeoff between system utility and the lifetime of the network. We omit the details due to space limitations.

VI. Numerical Results

In this section we present some illustrative numerical results to assess the efficiency of the proposed approach.

The example we consider is a WSNET with sensor nodes uniformly distributed in the box $[-10m, 10m] \times [-10m, 10m]$. The network has a single gateway at the origin. We use identical parameters with [1]. In particular, $G_{ij} = K(d_0/d_{ij})^\alpha$, where $K = 10^{-6}$, $d_0 = 10$, d_{ij} is the distance between nodes i and j, $\alpha = 4$, and $\bar{p}_i = 0.1$ Watts for all nodes i. The noise is characterized by $\eta = 10^{-10}$ and $W = 10^6$. We compare our algorithm with what we call the *enumeration* approach proposed in [1]. This latter approach does not make the linear approximation we made in (4); it instead uses the exact expression for transmission rates given in (3). It solves (5) by enumerating all feasible transmission rate vectors in Conv($\mathscr{R}$). To that end, it discretizes the possible values $\mathbf{p} \in \mathscr{P}$ can take, generates all possible transmission schemes, and from those it derives the corresponding rate vectors $\mathbf{r}$.

TABLE I

Computational efficiency comparison (single gateway).

N	Enumeration	Time	Decomposition	Time	Single-hop
2	284.19	0.01	285.49	0.01	132.7
3	593.09	0.02	594.37	0.01	309.8
4	844.77	0.12	846.29	0.02	495.5
5	1668.75	68.4	1668.78	0.03	866.9
6	out of memory	-	1768.33	0.04	966.5

Table I contains the results. In all cases, the objective is to maximize total throughput (reported in bps) and the fairness constraints have the form $\rho_{i+1} \leq 2\rho_i$, $i = 1, \ldots, N$, where ρ_i denotes the throughput of node i. The 1st column of Table I lists the number of nodes in the network. The 2nd and 3rd columns list the throughput achieved by the enumeration approach and the corresponding CPU time in seconds. The 4th and 5th columns list the throughput achieved by our algorithm and the corresponding CPU time in seconds. Finally, the last column reports the throughput achieved by the single-hop strategy, i.e., when each node sends directly to the gateway.

A couple of remarks are in order. First, comparing columns 2 and 4 of Table I suggests that even at power levels of 0.1 Watts our (linear) approximation (cf. (4)) is very accurate. Typical sensor networks will operate at lower power levels which is bound to improve accuracy. Second, the inherent combinatorial explosion of possible transmission schemes limits the use of the enumeration method to very small instances (in the 6-node case we run out of memory). In comparison, computational requirements in our method scale rather nicely. Without particular effort at optimizing the code we can currently solve problem instances with 50 nodes in less than 1 minute ! Third, it is interesting to note that time-sharing (multi-hop strategy) can dramatically improve performance over the naive single-hop strategy. For the cases reported in Table I the improvement is on the order of 50%. For a test network of 40 nodes, the improvements reaches 854% ! Finally, to demonstrate the effects of power optimization we considered the 3-node case in Table I. Setting $F_{\min} = 450$, the approach of Section V yields $\phi = 0.759$. That is, sensors can scale down their power by ϕ and this is sufficient to achieve a throughput equal to $F_{\min}$. Of course, this results in significant energy savings and extended lifetime of the WSNET.

VII. Conclusion

We considered the problem of scheduling transmissions in WSNETs to maximize the total system utility subject to explicit fairness constraints. We proposed a decomposition algorithm and established its convergence in a finite number of iterations. To the best of our knowledge, there is no alternative in the existing literature other than enumerating all feasible transmission schemes for solving the utility maximization problem in the general setting we consider.

We adopt a linear approximation of achievable rates which is asymptotically exact in the regime of low power levels. This regime is appropriate for WSNETs with rather dispersed nodes or operating in noisy environments. Still, the subproblem is an integer linear programming problem. Nevertheless, we show that it is polynomially solvable.

Finally, our framework allows us to optimize sensor power levels to achieve a given utility. This translates into significant energy savings and increased WSNET lifetime.

The numerical results we presented suggest that the linear approximation we adopted is very accurate even for power levels that are higher than typical WSNET applications. They also convincingly demonstrate that our approach can handle sizable instances of the problem.

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