Eina B. Jørgensen, Anna Lina P. Sjur and Jan-Adrian H. Kallmyr

December 7, 2019

Abstract

1 Introduction

2 Theory

2.1 The general Problem

The equation to be solved with different methods in this project is a simple diffusion equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad t > 0 \quad x \in [0, L] \quad (1)$$

Or $u_{xx} = u_t$. Using $L = 1$, with initial condition at t=0:

$$u(x, 0) = \sin(\pi x) \quad (2)$$

and Dirichlet boundry conditions

$$u(0, t) = u(L, t) = 0 \quad t \geq 0$$

This problem can for instance model the temperature of rod that has been heated in the middle, and as time progresses the temperature is transported through the rod and falls.

2.2 Discretization

For time discretization, as time is only used in first order derivative, we will use the explicit Forward Euler Scheme, which gives an error proportional to Δt (SOURCE).

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x, t + \Delta t) - u(x,t)}{\Delta t} \quad (3)$$

For the spatial discretization we use centered difference, which has an error proportional to Δx^2 (SOURCE).

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u(x + \Delta x, t) - 2u(x,t) + u(x - \Delta x, t)}{\Delta x^2}$$
$$(4)$$

On a discrete time and space grid, $u(x,t) = u(x_i, t_n)$, $t + \Delta t = t_{n+1}$ and so on. For simplicity we use the notation $u_i^n = u(x_i, t_n)$. The equation in it's discrete form is then

$$u_{xx} = u_t$$
$$[u_{xx}]_i^n = [u_t]_i^n$$
$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$
$$(5)$$

Solving this for u_i^{n+1} we can caluclate the next time step for each spatial point i:

$$u_i^{n+1} = \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + u_i^n \quad (6)$$

Which has a stability level for the grid resolution

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

2.3 The Exact Solution

As we will be comparing the precision of the different ways of solving the partial differential equation, we need to calculate the the exact solution in order to calculate the error. Through sepeation of variables, the equation can be expressed as

$$u(x,t) = X(x)T(t) \quad (7)$$

Differentiating this according to (1) and moving some terms, we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

As the to sides of this equation are not dependant on the same variables, they must both be equal to a consant. (We can choose this constant to be $-\lambda^2$). This gives the tho equations.

$$X''(x) = -\lambda^2 X(x)$$
$$T'(t) = -\lambda^2 T(t)$$

For X can have three possible forms given by the characteristic equation. In order to satisfy the inital condition (2), $X(x)$ must be on the form

$$X(x) = B\sin(\lambda x) + C\cos(\lambda x)$$

The inital condition then rules $C = 0, \lambda = \pi$. For $T(t)$ the solution is on the form

$$T(t) = Ae^{-\lambda^2 t}$$

As we know $\lambda = \pi$ the solution is then:

$$u(x,t) = X(x)T(t) = Ae^{-\pi^2 t}B\sin(\pi x)$$

And finally from the initial condition, we know that $A \cdot B = 1$, and the exact solution is

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x) \qquad (8)$$

3 Results

4 Discussion

5 Conclusion

References