

# Solving differential equations using neural networks

Eina B. Jørgensen, Anna Lina P. Sjur and Jan-Adrian H. Kallmyr

December 18, 2019

## Abstract

In this paper we use a neural network approach for solving the diffusion equation and compare this with a traditional finite difference approach using Forward Euler (FE) and Centered Difference (CD) methods. We also find eigenpairs of a real symmetric matrix using a similar neural network approach. Our findings show that neural networks can be used to solve partial differential equations. We find that the neural network compares to the finite difference approach, but starts out 3 orders of magnitude slower, reaching the same run time for the same MSE after  $10^5$  time steps and  $10^4$  iterations, respectively. We also looked at the effect of learning rate on MSE, and detected a possible minimum at 0.006. While temporal resolution had an effect on the neural network, we found that the overall trend was dominated by increasing iterations. When computing eigenvalues, we found that the neural network gave results that differed from `numpy.linalg` first at the 5th decimal place, and the eigenvectors differed first at the 3th decimal place, when the threshold of the cost function was set to  $10^{-4}$ . We also find that the network was considerably slower than the numpy approach, with the convergence time and needed iterations seemingly dependent on the matrix.

## 1 Introduction

In physics, as well as many other fields, differential equations (DEs) play a central role in analysis of a wide variety of problems. As such, it is important to use an efficient and accurate algorithm to produce reliable results. There exists many such algorithms of different orders, and each have their uses for different DEs. More complex DEs usually require higher order algorithms such as fourth order Runge-Kutta, while simple DEs can do with a first or second order algorithm such as Forward Euler (FE), or Centered Difference (CD), respectively. Higher order methods usually perform slower, and for complex coupled models, it is important to choose just the right algorithm for solving each sub-problem, as a slow method might significantly slow down model runs. It is therefore of interest if a general method for

solving DEs exist which is efficient. A possible candidate are Neural Networks (NNs), which can approximate any function. In general, NNs for solving DEs could be useful if they offer a good accuracy/efficiency trade-off for many different types of equations, or if they outperform traditional methods. We will focus on the latter, and compare a traditional CD/FE-approach with a NN-approach. Research provided by [Lagaris et al. \(1998\)](#) and [Chiaramonte and Kiener \(2013\)](#) suggest that one can indeed solve PDEs and ODEs with Neural Networks with high precision and several advantages, but that it comes at a computational cost. We will see if we when solving the Diffusion Equation, come to the same conclusion. Furthermore, we will also look at an immediate application of the NN-approach in finding extrema eigenpairs ([Yi et al., 2004](#)) and compare this with the standard approach.

The Partial Differential Equation (PDE) we will solve (Eq. 1), as well as all background theory and methods, can be found in Section 2. Our most important results are showcased in Section 3, and mainly consists of comparisons between different methods. In Section 4 we discuss the pros and cons of each method, as well as taking a deeper look at interesting results. Finally, in section 5 we summarise the article, presenting the most important takeaways, as well as possible future uses.

## 2 Theory and implementation

### 2.1 The general problem

The first problem to be solved in this project is a simple diffusion equation

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t} \quad t > 0 \quad x \in [0, L]. \quad (1)$$

Alternatively, this can be written as  $u_{xx} = u_t$ . Using  $L = 1$ , the initial condition at  $t = 0$  is given by

$$u(x, 0) = \sin(\pi x). \quad (2)$$

Dirichlet boundary conditions are used, i.e.

$$u(0, t) = u(L, t) = 0 \quad t \geq 0.$$

This problem can for instance model the temperature of a rod that has been heated in the middle, with constant temperature at the ends. As time progresses the heat is transported through the rod and the temperature falls. We will first look at how an explicit scheme can be used, followed by a method using neural networks.

We will look at how we can compute the eigenvalues and eigenvectors of a real, symmetric  $6 \times 6$  matrix  $A$ . In particular, we will see how methods used to solve ((1)) can be applied to this problem.

To construct such a matrix, we let

$$A = \frac{Q^T + Q}{2}$$

where  $Q$  is a random, real matrix.

### 2.2 Exact solution of the diffusion equation

As we will be comparing the precision of the different ways of solving the diffusion equation, we need to calculate the exact solution in order to calculate the error. Through separation of variables, the equation can be expressed as

$$u(x, t) = X(x)T(t) \quad (3)$$

Differentiating this according to (1) and moving some terms, we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

As the two sides of this equation are not dependent on the same variables, they must both be equal to a constant. (We can choose this constant to be  $-\lambda^2$ ). This gives the two equations.

$$\begin{aligned} X''(x) &= -\lambda^2 X(x) \\ T'(t) &= -\lambda^2 T(t) \end{aligned}$$

$X$  can have three possible forms given by the characteristic equation. In order to satisfy the initial condition (2),  $X(x)$  must be on the form

$$X(x) = B \sin(\lambda x) + C \cos(\lambda x)$$

The initial condition then rules  $C = 0, \lambda = \pi$ . For  $T(t)$  the solution is on the form

$$T(t) = A e^{-\lambda^2 t}$$

As we know  $\lambda = \pi$  the solution is then:

$$u(x, t) = X(x)T(t) = A e^{-\pi^2 t} B \sin(\pi x)$$

And finally from the initial condition, we know that  $A \cdot B = 1$ , and the exact solution is

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x) \quad (4)$$

### 2.3 Discretization of the diffusion equation

For time discretization, we have a first order derivative, and so we will use the explicit Forward Euler Scheme, which gives an error proportional to  $\Delta t$ . This is given as

$$\frac{\partial u(x, t)}{\partial t} \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \quad (5)$$

For the spatial discretization we use centered difference, which has an error proportional to  $\Delta x^2$ , given by

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2} \quad (6)$$

On a discrete time and space grid,  $u(x, t) = u(x_i, t_n)$ ,  $t + \Delta t_n = t_{n+1}$  and so on. For simplicity we use the notation  $u_i^n = u(x_i, t_n)$ . The equation in it's discrete form is then

$$\begin{aligned} u_{xx} &= u_t \\ [u_{xx}]_i^n &= [u_t]_i^n \\ \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} &= \frac{u_i^{n+1} - u_i^n}{\Delta t} \end{aligned} \quad (7)$$

Solving this for  $u_i^{n+1}$  we can calculate the next time step for each spatial point  $i$ :

$$u_i^{n+1} = \frac{\Delta t}{\Delta x^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + u_i^n \quad (8)$$

Which has a stability level for the grid resolution given by

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

### 2.4 Solving PDEs with Neural Networks

A different approach for solving PDEs is with the aid of a neural network. For an more in-depth description of how a Neural Network functions, see Jørgensen et al. (2019). Much

of the same logic will here be applied to solve a partial differential equation.

In order to solve PDEs with a Neural Network, a trial function  $\Psi(x, t)$  must be approximated, and our aim is to get this  $\Psi$  as close to the true function  $u$  as possible (Lagaris et al., 1998). When aiming to solve (1), the corresponding equation, substituting with the trial function, is

$$\frac{\partial^2 \Psi(x, t)}{\partial x^2} = \frac{\partial \Psi(x, t)}{\partial t} \quad t > 0 \quad x \in [0, L]$$

The error (or residual) in the approximation is then

$$E = \frac{\partial^2 \Psi(x, t)}{\partial x^2} - \frac{\partial \Psi(x, t)}{\partial t} \quad (9)$$

The cost function to be minimized by the Neural Network is the sum of this  $E$ , evaluated at each point in the space and time grid.

For each iteration in the calculations, we will update our trial function based on the Neural Network's previous calculations. Therefore we must choose a fitting form for our trial function. This is based on the order of the PDE, and its initial condition. To satisfy the initial condition and Dirichlet conditions of ((1)), we choose:

$$\Psi(x, t) = (1-t)I(x) + x(1-x)tN(x, t, p), \quad (10)$$

where  $I(x)$  is the initial condition,  $N(x, t, p)$  is the output from the neural network, and  $p$  is the weights and biases.

Then, for each iteration in the network, the partial derivatives of  $\Psi$  are calculated according to the new state of the network  $N(x, t, p)$ , updating the cost. As the cost is minimized, the error term  $E$  gets closer to zero, and our trial function  $\Psi(x, t)$  will approach the solution of the PDE.

How small we are able to get the cost is dependent on the maximum number of iterations we allow the Network to execute, the learning rate, and the structure of the Neural Network in terms of the number of hidden layers, and the number of nodes in each layer.

## 2.5 Computing eigenpairs with Neural Networks

Following the discussion by Yi et al. (2004), it is possible to compute the eigenvector  $v_{max}$  corresponding to the largest eigenvalue  $w_{max}$  of the  $n \times n$  matrix  $A$  by solving the ordinary differential equation

$$\frac{dv(t)}{dt} = -v(t) + f(v(t)), \quad t \geq 0 \quad (11)$$

where  $v = [v_1, v_2, \dots, v_n]^T$  and  $f(v)$  is given as

$$f(v) = [v^T v A + (1 - v^T A v) I] v \quad (12)$$

Here,  $I$  is the  $n \times n$  identity matrix.

As shown by Yi et al. (2004), when  $t \rightarrow \infty$ , any random non-zero initial  $v$ , as long as it's not orthogonal to  $v_{max}$ , will approach  $v_{max}$ . To compute the corresponding eigenvalue  $w_{max}$ , the following equation can be used.

$$w = \frac{v^T A v}{v^T v}$$

To compute the eigenvector  $v_{min}$  corresponding to the smallest eigenvalue  $w_{min}$  of  $A$ , one can simply substitute  $A$  with  $-A$  in (11).

As described earlier, a trial function is needed to solve (11) with a Neural Network. Since  $v \in \mathbb{R}^n$ , we choose a trial function  $\Psi(x, t)$  dependent both on position  $x$  and time  $t$ , so that for each time step, the approximated eigenvector is given as  $[\Psi(1, t), \Psi(2, t), \dots, \Psi(n, t)]^T$ . (11) can then be rewritten as

$$\frac{\partial \Psi(x, t)}{\partial t} = -\Psi(x, t) + f(\Psi(x, t)) \quad (13)$$

with  $t \geq 0$  and  $x = 1, 2, \dots, n$ . We defined the trial function as

$$\Psi(x, t) = v_0 + tN(x, t, p)$$

where  $v_0$  is the initial  $v$ , chosen at random. The error is then the difference between the two hand sides of (13). For this problem, the mean squared error (MSE) was used as cost function.

## 2.6 Implementing the Neural Network

There are many ways of implementing the Neural Network method for solving differential equations. In this project, we have chosen to use *TensorFlow* for python3, as it is fast, stable and relatively simple to use. For implementation, see the git repository of [janadr](#).

## 3 Results

### 3.1 Diffusion equation

In Figure 1 two different instants in the time evolution of the diffusion equation (1), as obtained from finite difference methods, are shown. For both time stamps, a comparison is made with the exact solution and a run with higher spatial resolution ( $\Delta x = 1/100$ ) and lower spatial resolution ( $\Delta x = 1/10$ ). In both cases, the graphs of the exact solution and the one with higher spatial resolution are difficult to distinguish.

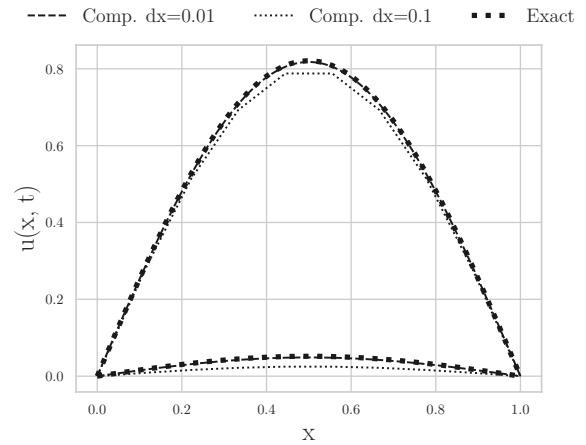


Figure 1: Time evolution of (1) solved using finite difference methods: CD and FE for time's  $t = 0.02$  s and  $t = 0.3$  s. For each point in time a visual comparison is made with the exact solution.

Likewise, in Figure 2 we show a parallel to Figure 1 with solutions from a neural network. In this case, we also see agreement between the computed and exact solutions for both points in time.

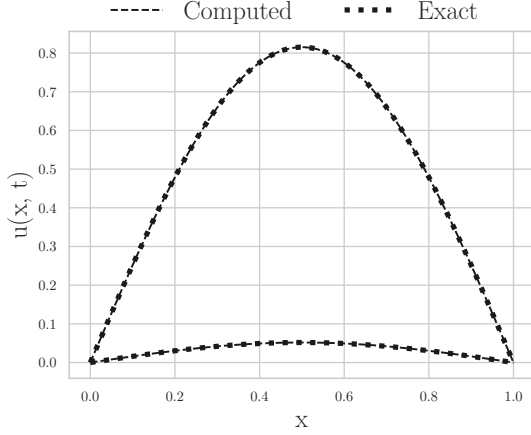


Figure 2: Time evolution of (1) solved using a neural network. For both instants in time,  $t = 0.02$  s and  $t = 0.3$  s, a visual comparison is made with the exact solution, represented by dotted lines. The computations are done with  $N_t = 30$  and  $N_x = 100$ .

Shown in Figure 3 are the MSE's for two instants in time for both finite difference methods and a neural network. For the finite difference methods we observe a decrease in MSE for increasing temporal resolution. For  $t = 0.02$  (a.), the MSE is minimal at around  $10^{-8.5}$  for a little under  $10^5$  time points, while for  $t = 0.3$  (c.) it only becomes around  $10^{-8}$  for  $10^5$  time points.

For the neural network, we observe a decrease in MSE for an increasing number of iterations. In the case of  $t = 0.02$  s (b.), we see that the MSE decreases to almost  $10^{-9}$  after less than  $10^4$  iterations. As for  $t = 0.3$  s (d.), the MSE is stays at  $10^{-8}$  after around 4000 iterations and is mostly unchanged afterwards.

Accompanying Figure 3, we show the CPU

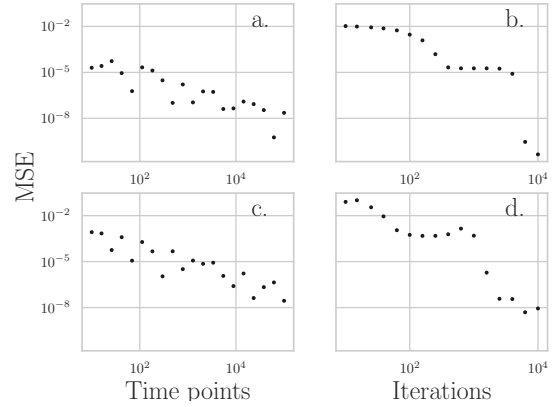


Figure 3: Shown on the left-hand side, top-to-bottom (a., c.), are the MSE's as a function of temporal resolution, obtained from using finite difference methods for  $t = 0.02$  and  $t = 0.3$ , respectively. Likewise, on the right-hand side (b., d.) the MSE's yielded from the neural network are shown as a function of iterations with network parameters  $N_t = 10$ ,  $N_x = 100$ , and  $\gamma = 0.006$ .

times as a function of temporal resolution and number of iterations, respectively, for the different scenarios in Figure 4. We see that the general curve is very similar for all scenarios. For the CD/FE method, we see that the CPU time becomes slightly longer for shorter  $t$  at  $10^5$  time points. The neural network looks to have the same curve in both b. and d. All four graphs peaks at around 100 seconds.

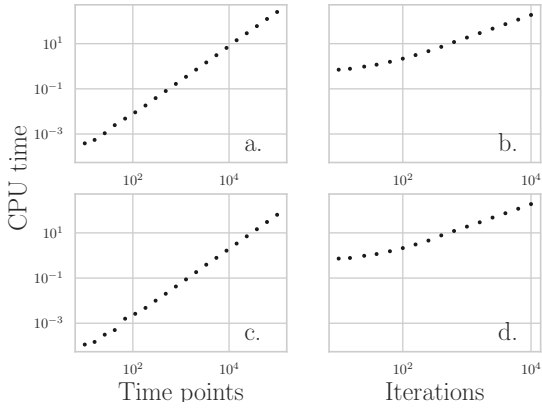


Figure 4: Shown on the left-hand side, top-to-bottom (a., c.), are the CPU time's as a function of temporal resolution, obtained from using finite difference methods for  $t = 0.02$  and  $t = 0.3$ , respectively. Likewise, on the right-hand side (b., d.) the CPU time's yielded from the neural network are shown as a function of iterations with network parameters  $N_t = 10$ ,  $N_x = 100$ , and  $\gamma = 0.006$ .

Looking at Figures 5 and 6 we again observe that the MSE decreases for an increasing number of iterations of the neural network. We also see that, in general, this MSE decrease happens after fewer iterations for higher learning rates. The minima in both figures can be found in the upper right corner, i.e. the area of high learning rate and many iterations. In Figure 5 the minimum MSE is on the order of  $10^{-10} - 10^{-11}$ , while for Figure 6 it is on the order of  $10^{-8} - 10^{-9}$ .

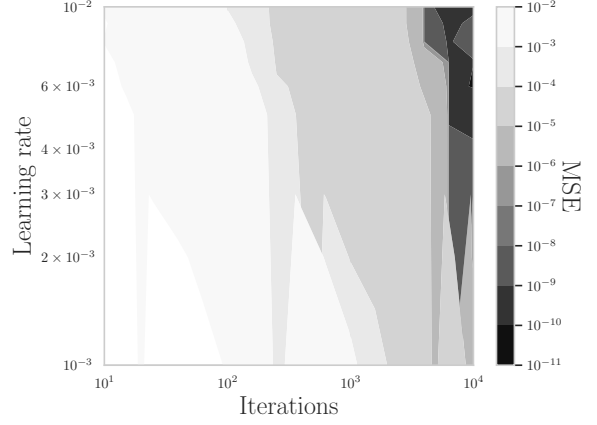


Figure 5: Contour plot showing MSE (grayscale) as a function of number of iterations and learning rate for  $t = 0.02$  s,  $N_t = 10$ , and  $N_x = 100$ .

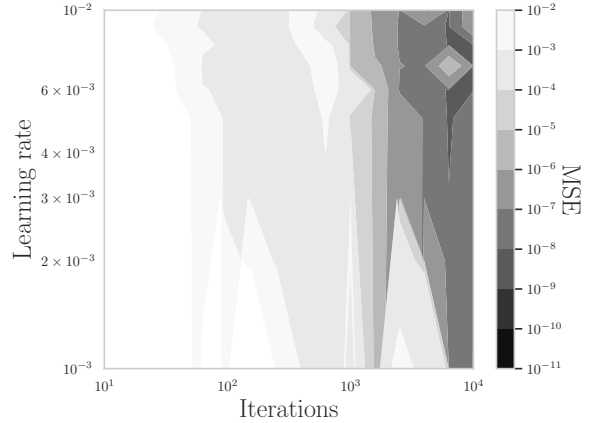


Figure 6: Contour plot showing MSE (grayscale) as a function of number of iterations and learning rate for  $t = 0.3$  s,  $N_t = 10$ , and  $N_x = 100$ .

We also tested the temporal resolution dependency of the neural network. Our findings are shown in Figure 7, where we see that the maximum MSE is found for  $N_t = 10$ , while the minimum MSE is found for  $N_t = 50$ . It is also noteworthy that the three graphs do not diverge before approaching  $10^4$  iterations.

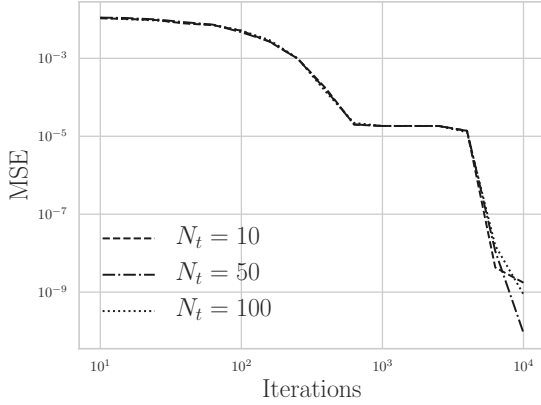


Figure 7: Comparison of the MSE for  $N_t = 10$ ,  $N_t = 50$ , and  $N_t = 100$  respectively. Here the learning rate is 0.004 at  $t = 0.02$  s.

### 3.2 Eigenpairs

Looking at the eigenpairs of matrix  $A$  (A.1), the maximum eigenvalue computed with `numpy.linalg` is given under as  $w_{max}^{np}$ , with corresponding eigenvector  $v_{max}^{np}$ . The same eigenpair, computed with a neural network, is given as  $w_{max}^{nn}$  and  $v_{max}^{nn}$ . Note that the eigenvectors have been normalized, to allow for comparison. We see that the two eigenvalues only differ at the 5th decimal place, while the eigenvector elements first differ at the 3th decimal place. Comparing the CPU times of the two methods for one run, the neural network is slower by a factor of about  $10^3$ .

$$v_{max}^{np} = \begin{bmatrix} 0.4025950 \\ 0.4790814 \\ 0.3357358 \\ 0.3835011 \\ 0.3542414 \\ 0.4723555 \end{bmatrix} \quad w_{max}^{np} = 2.83515$$

$$v_{max}^{nn} = \begin{bmatrix} 0.40165637 \\ 0.47992723 \\ 0.33576365 \\ 0.38384145 \\ 0.35298366 \\ 0.47294087 \end{bmatrix} \quad w_{max}^{nn} = 2.83514$$

How  $v_{max}^{nn}$  evolves over time is shown in Figure 8. The network used for this computation had three hidden layers, all with ten neurons. The learning rate was 0.001, while the time step was 0.1. The precision, which was the MSE at which the training was stopped, was set to  $10^{-4}$ , resulting in 10820 iterations.

Analogously to what was shown above for the maximum eigenvalue, the minimum eigenvalue computed with `numpy.linalg` is shown below as  $w_{min}^{np}$  together with the corresponding eigenvector  $v_{min}^{np}$ . Similarly, the same pair computed with neural network is denoted as  $w_{min}^{nn}$  and  $v_{min}^{nn}$ . Again, the eigenvalues differ first at the 5th decimal place, and the eigenvectors differ at the 3th decimal place.

$$v_{min}^{np} = \begin{bmatrix} 0.6928048 \\ -0.0074343 \\ -0.7048516 \\ -0.1487365 \\ 0.0122096 \\ 0.0296413 \end{bmatrix} \quad w_{min}^{np} = -0.515594$$



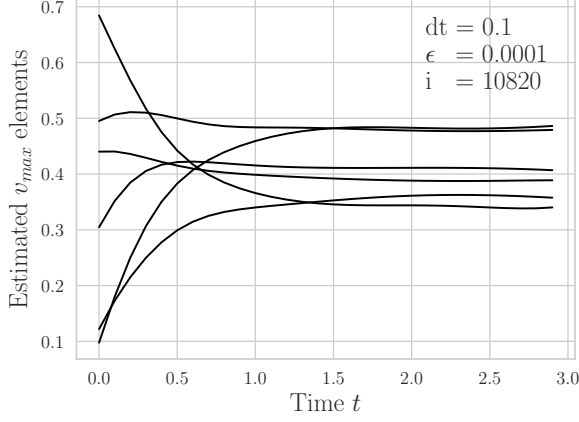


Figure 8: Figure showing how the value of the elements of the estimated eigenvector  $v_{max}$  evolves over time, when computed by a neural network. Each line represents one of the elements of the vector. Time step  $dt$ , precision  $\epsilon$  and iterations  $i$  needed to achieve said precision is also shown.

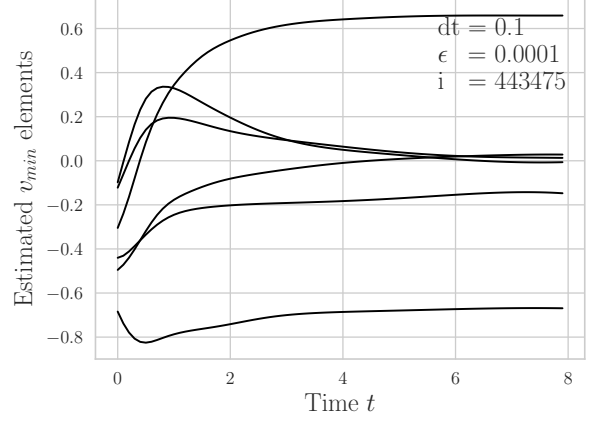


Figure 9: Figure showing how the value of the elements of the estimated eigenvector  $v_{min}$  evolves over time, when computed by a neural network. Each line represents one of the elements of the vector. Time step  $dt$ , precision  $\epsilon$  and iterations  $i$  needed to achieve said precision is also shown.

$$v_{min}^{nn} = \begin{bmatrix} 0.69273272 \\ -0.00676815 \\ -0.70356177 \\ -0.15496268 \\ 0.0137166 \\ 0.02957751 \end{bmatrix} \quad w_{min}^{nn} = -0.515579$$

Similarly to Figure 8, the graph in Figure 9 shows the evolution of the eigenvector  $v_{min}^{nn}$ . Note the different time values in the two figures. The same network, with the same configurations as described above, is used, except that  $A$  was replaced with  $-A$  in the cost function. To reach a precision of  $10^{-4}$ , 443475 iterations was needed.

## 4 Discussion

Our work shows that PDEs can be solved using neural networks (Figure 2), yielding similar results as a finite difference approach, i.e. Figure

1. Looking back at Figure 3 we see that the two methods yields approximately the same MSE after  $10^5$  time steps and  $10^4$  iterations respectively. However, Figure 3a. and 3d. shows a more consistent decrease compared with Figure 3b. and 3d. Furthermore, the shape of the curve in a. and c. looks similar, while there is a difference between b. and d. Figure 5 and 6 also indicates that this difference is dependent on the chosen learning rate. This might be expected due to the cost function of the neural network being dependent on the shape of the curve, which is different in time.

We also studied the effect of varying time interval for the neural network and found that the overall trend was dominated by the number of iterations (see Figure 7). One interesting point, however, is that  $N_t = 50$  performed better than  $N_t = 100$ , and so there isn't a linear relationship between increasing  $N_t$  and decreasing MSE. In the FE/CD scheme there is a stability criterion between  $\Delta t$  and  $\Delta x$  which is



taken into account. There is no such criterion for the neural network, but our results in Figure 5 might indicate that the neural network considers this relationship.

Comparing the two methods in CPU time, we see that the neural network starts out at being around 3 orders of magnitude, if not more, slower than the finite difference methods. After  $10^5$  time points and  $10^4$  iterations, however, they are around the same. This was also intentional as we wanted to present a fair comparison in MSE as a function of CPU time. Looking at the inclines, it looks as if the increase in CPU time for the neural network is decreasing, while for the finite difference method approach, it is increasing at a constant rate. With more computing power, the neural network might win out in an MSE per CPU time perspective.

While we have shown that neural networks can solve PDEs at a level of accuracy close to a finite difference method, the results in this study are not enough to conclude which approach yields the best results. Future studies can choose more sophisticated finite difference methods such as higher order Runge-Kutta, which could yield a smaller MSE. Furthermore, varying the size of the time step could speed up the algorithm where possible and slowing it down where needed. Likewise, for the neural network, an adaptive learning rate can be implemented, which might speed up convergence and mitigate difference in MSE between target times. More of the hyperparameters can also be studied, such as the amount of hidden layers and neurons.

When it comes to calculating eigenpairs with the neural network, it seems to yield good results that agrees with the `numpy.linalg` method, when it comes to precision. In this report, the network was set to run until the cost function went below a threshold of  $10^{-4}$ . When experimenting with the network, it was clear that there was a direct link between the threshold of the cost function, and the level

of agreement between the network results and the numpy results. In other words, it seems to be possible to tune the network corresponding to the level of precision needed. However, the higher the precision, the more iterations is needed.

The maybe biggest drawback of the neural network is its time use, with a CPU time about  $10^3$  times that of `numpy.linalg` for a precision of  $10^{-4}$  for the largest eigenvalue. The smallest eigenvalue demanded even more iterations to reach that precision. This leads us to another feature of the neural network. As can be seen in Figure 8 and 9, the time  $t$  at which the estimated eigenvectors stabilises varies from matrix to matrix (here from  $A$  to  $-A$ ), and the number of iterations needed varies as well. So although it is possible to compute eigenpairs with a high level of precision, the computation can possibly become very time consuming.

Lastly, this method only gives us the largest and the smallest eigenvalues, together with the corresponding eigenvectors. In other words, this method can not be used to find all the eigenpairs of a real symmetric matrix.

## 5 Conclusion

We used a neural network to solve the diffusion equation, and compared this with a finite difference solution. Our findings show that a neural network can be used to solve a partial differential equation. The neural network is for the most part slower than the forward euler and centered difference schemes for a given MSE. However, for an MSE of around  $10^{-8}$  they are around even. The neural network might be faster for higher precision, but this would require more computing power to confirm. Further tuning of the neural network could also be done to decrease the MSE further.

We then used a neural network to find the

extreme eigenpairs of a real, symmetric matrix by solving an ordinary differential equation. Comparing with `numpy.linalg`, the difference in eigenvalues was in the order of  $10^{-5}$ , when the network was set to train until the cost function went below  $10^{-4}$ . But the high precision made the network considerably slower than `numpy.linalg`, indicating that the latter method is favourable. In addition, the converging time and iterations seemed to be dependent on the matrix. Possibly, a better implementation of the network could deal with some of these factors.

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## Appendix

$$A = \begin{bmatrix} 0.37454012 & 0.50439896 & 0.78221829 & 0.51530175 & 0.30604431 & 0.38176969 \\ 0.50439896 & 0.86617615 & 0.40672706 & 0.49965086 & 0.40288023 & 0.57021699 \\ 0.78221829 & 0.40672706 & 0.18182497 & 0.3976287 & 0.25195801 & 0.29490401 \\ 0.51530175 & 0.49965086 & 0.3976287 & 0.13949386 & 0.40318954 & 0.65762369 \\ 0.30604431 & 0.40288023 & 0.25195801 & 0.40318954 & 0.59241457 & 0.50604122 \\ 0.38176969 & 0.57021699 & 0.29490401 & 0.65762369 & 0.50604122 & 0.80839735 \end{bmatrix} \quad (\text{A.1})$$