Planetary waves: a numerical study of Rossby waves

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Abstract

In this article, we study Rossby waves using different both implicit and explicit schemes for solving the wave equation numerically. We consider both the advantages and disadvantages of both methods, as well as their efficiency and accuracy. From this analysis, we find that

1 Introduction

We encounter wave phenomena everywhere in the natural sciences. From quantum mechanics to oceanography, we find that be it the motion of a particle or the ocean, we require knowledge of wave-like behaviour to solve the problem. In quantum mechanics, a particle's wave function is described by a complex-valued diffusion equation, the Schrödinger equation, while in oceanography, we can describe ocean waves using the wave equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where x and t denote the spatial and temporal coordinates, respectively. The wave equation will be the topic of this paper, in particular, we will model Rossby waves, first identified by Rossby (1939). These are inertial, planetary waves in the Earth's atmosphere and ocean which motions contribute to extreme weather (Mann et al., 2017), might drive the El-Ninõ southern oscillation (ENSO) (Bosc and Delcroix, 2008), and is also produced by ENSO, see Battisti (1989). While we hope the reader appreciate the wide range of phenomena related to these waves, our article presents a numerical study of the waves isolated from other processes. We therefore begin by describing fundamental theory of waves and partial differential equations in the Theory section, present our algorithm and the technicalities relating to its implementation. In the Results section, we present our data as figures, before discussing their implications in the Discussion section. Concluding our paper, we present our final thoughts on the topic of simulating Rossby waves.

2 Theory

2.1 Properties of waves

Waves are solutions of the wave equation (see eq. 1), and have certain properties including

2.2 Rossby wave equation

Rossby waves are low frequency waves induced by the meridional variation of the Coriolis parameter f. This parameter depends on the rotation of the Earth Ω and the latitude φ, and is given by

$$f = \Omega \sin \varphi. \tag{2}$$

An approximation where f is set to vary linear in space is called the β-plane approximation, and can be written as

$$f = f_0 + \beta y, \tag{3}$$

where $\beta = \frac{df}{dy}\Big|_{\varphi_0} = \frac{2\Omega}{a}\cos\varphi_0$, a being the radius of the Earth. Combining the β-plane approximation with the shallow water vorticity equation, you get the quasi-geostrophic vorticity equation. This can be linearised, and by assuming a constant mean flow without bottom topography, you get the barotropic Rossby wave equation:

$$(\partial_t + U\partial_x)\nabla_H\psi + \beta\partial_x\psi = 0. \quad (4)$$

Here, ψ is the stream function describing the velocity perturbation, ∂_x denotes $\frac{\partial}{\partial x}$, ∇_H is the horizontal divergence $\partial x + \partial y$ and U is the mean velocity. In this report, we will assume no mean velocity, i.e. $U = 0$, in which case equation (4) simplifies to

$$\partial_t\nabla_H\psi + \beta\partial_x\psi = 0. \quad (5)$$

Two forms of boundaries will be examined in this report, that is periodic and constant boundaries. The first case can be used to describe an atmosphere that wraps around the earth, where the stream function is equal at the end-points. The latter case, where the stream function has a constant value at the boundaries, can be used to describe an ocean basin.

A possible solution to (5) in one dimension with periodic boundaries, where $x \in [0, L]$, is given by

$$\psi = A\cos(kx - \omega t), \quad (6)$$

where $k = \frac{2n\pi}{L}$ and $\omega = -\frac{\beta L}{2n\pi}$. The phase speed c can be calculated through the dispersion relation, given by

$$c = \frac{\omega L}{2n\pi} = -\beta\left(\frac{L}{2n\pi}\right)^2. \quad (7)$$

Since β is positive for all latitudes, the phase speed will be negative, implying that Rossby waves travels from east to west in a bounded domain.

The same problem, but with constant boundaries equal to zero, has the possible solution

$$\psi = A\sin\left(\frac{\pi n}{L}x\right)\cos(kx - \omega t), \quad (8)$$

with $k = \frac{L}{\pi n}$ and $\omega = -\frac{\beta}{2k}$. Here, the phase speed is given by

$$c = \frac{\omega L}{2n\pi} = -\frac{\beta}{4} \quad (9)$$

Again, the phase speed is negative. Equation (8) describes a cosine wave where the amplitude is dependent on the position, following a sine curve with zeros at the boundaries.

2.2.1 Differential equations

The forward difference is of first order, meaning that the error is proportional to Δt. The centred difference, on the other hand, is of second order, with an error proportional to Δt^2 for the time derivative and Δx^2 for the spatial derivative.

2.3 Algorithms

2.3.1 1-dimensional

Scaling eq. 5, we essentially wanted to solve two equations

$$\partial_t\zeta + \partial_x\psi = 0 \quad (10)$$

$$\partial_{xx}\psi = \zeta, \quad (11)$$

where the latter is Poisson's equation. To discretise, we use the following schemes:

$$\partial_q f \approx \frac{f_{q+1} - f_q}{\Delta q}, \quad (12)$$

$$\partial_q f \approx \frac{f_{q+1} - f_{q-1}}{2\Delta q}, \quad (13)$$

$$\partial_{qq} f \approx \frac{f_{q+1} - 2f_q + f_{q-1}}{(\Delta q)^2}, \quad (14)$$

where f is arbitrary and q a general coordinate. Here eq. 12 is the explicit forward scheme, and eq. 13 the implicit centered scheme. Letting $t^n = n\Delta t$ and $x_j = j\Delta x$, eq. 10 becomes

$$\zeta_j^{n+1} = \zeta_j^n - \frac{\Delta t}{2\Delta x}\left(\psi_{j+1}^n - \psi_{j-1}^n\right) \tag{15}$$

in the explicit scheme, and

$$\zeta_j^{n+1} = \zeta_j^{n-1} - \frac{\Delta t}{\Delta x}\left(\psi_{j+1}^n - \psi_{j-1}^n\right) \tag{16}$$

in the implicit scheme. For Poisson's equation, we simply have

$$\frac{\psi_{j+1}^{n+1} - 2\psi_j^{n+1} + \psi_{j-1}^{n+1}}{(\Delta x)^2} = \zeta_j^{n+1}, \tag{17}$$

which could be solved as outlined in one of our earlier papers (Sjur and Kallmyr, 2018). Our general algorithm is then Algorithm **??**. What remains now is to determine how to solve

```
initialise wave;
for n = 0, 1, ..., T do
    for j = 0, 1, ..., X do
        solve for ζ_j^{n+1};
        solve for ψ_j^{n+1};
```

Algorithm 1: Algorithm for solving the 1+1 dimensional Rossby wave equation. Here T is a final time, and X a upper spatial bound.

the equations for closed and periodic boundary conditions. we looked at two type of boundary conditions, closed and periodic. We also tested different schemes, the explicit forward scheme and the implicit centered scheme. This essentially resulted in two different algorithms.

2.4 Implementation

We implemented our algorithms in C++ using the armadillo and LAPACK libraries to handle matrix operations. To analyse data and produce figures, we used python 3.6 with a standard set of modules: matplotlib, numpy and seaborn.

We used methods as outlined by Sjur and Kallmyr (2018). For closed boundaries we had the Dirichlet boundary conditions and could use gaussian elimination to solve eq. 11. In the case of periodic boundaries our matrix became non-tridiagonal, and we had to use LU decomposition to solve the differential equation.

3 Results

Looking at Figure 1 and **??**, we see the time evolution of a sine wave in a bounded and periodic domain respectively. In both domains, the wave propagates towards the east (left). Comparing the two, we see that in the bounded domain there is a clear distinction between extremas, while in the periodic domain

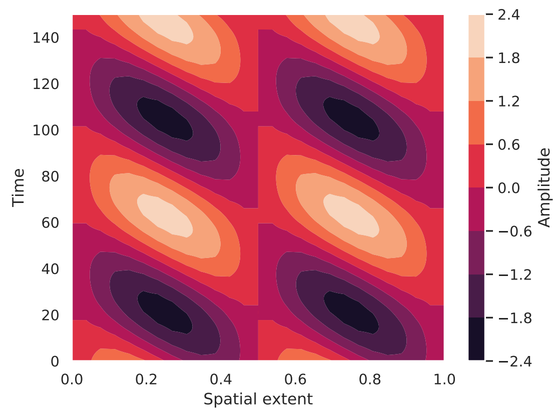

Figure 1: Hov Muller diagram of a bounded Rossby wave with a initial sine wave using a explicit scheme.

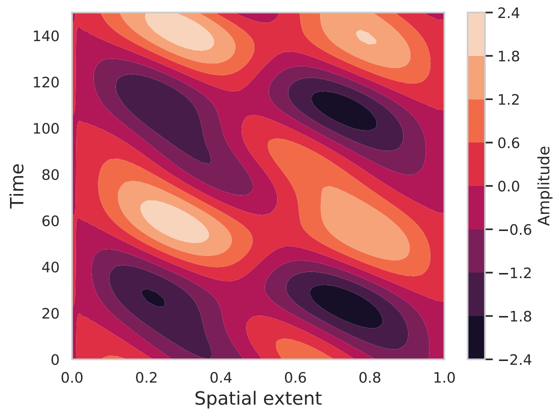

Figure 2: Hov Muller diagram of a Rossby wave with periodic boundary conditions, initially a sine wave using a explicit scheme.

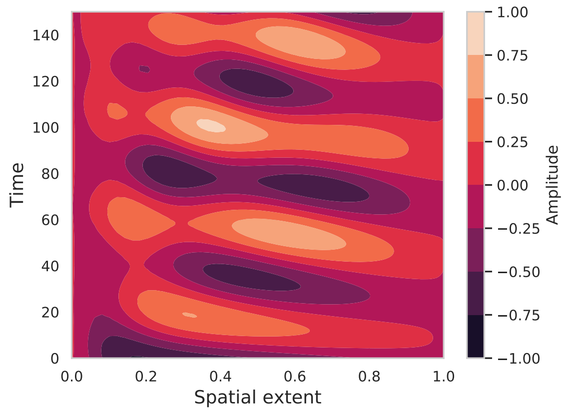

Figure 4: Hov Muller diagram of a Rossby wave with periodic boundary conditions, initially a gaussian using a explicit scheme.

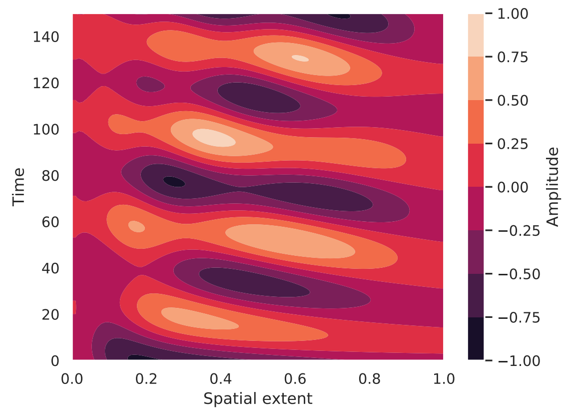

Figure 3: Hov Muller diagram of a bounded Rossby wave initially a gaussian using a explicit scheme.

$\mathbf{n}$	$\mathbf{t_g/t_s}$	$\mathbf{t_{LU}/t_s}$
10	2.08	3.70
10^2	1.89	$1.00 \cdot 10^2$
10^3	1.48	$1.05 \cdot 10^4$
10^4	1.43	$1.18 \cdot 10^6$
10^5	1.39	-
10^6	1.41	-
10^7	1.39	-

Table 1: Ratio between CPU time for the general algorithm ($\mathbf{t_g}$), the special algorithm ($\mathbf{t_g}$) and the LU decomposition algorithm ($\mathbf{t_{LU}}$) for different matrix sizes ($\mathbf{n}$). The LU decomposition crashed for $\mathbf{n}$ greater than 10^4.

4 Discussion

5 Conclusion

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A

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \dots & \dots & 0 \\ a_1 & b_2 & c_2 & 0 & \dots & 0 \\ 0 & a_2 & b_3 & c_3 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \ddots & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & \dots & \dots & 0 & a_{n-1} & b_n \end{bmatrix},$$