Two methods of solving Poisson's equation

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Abstract

State problem. Briefly describe method and data. Summarize main results.

1 Introduction

Physics is a field concerned with the behaviour of nature, and nature is everchanging. It is therefore no surprise that differential equations appear everywhere in physics. From global climate dynamics to statistical mechanics, what we find is that differential equations, often many and coupled, are required to explain or model the phenomena. For such large models, efficiency is important, as we would, for example, like to have timely weather forecasts. One way to make a model more efficient is by using an efficient algorithm for solving differential equations.

In this report, we compare two different numerical methods of solving linear second-order differential equations with the Dirichlet boundary conditions. To do this, we will solve the one-dimensional Poisson's equation:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = -4\pi\rho(r). \tag{1}$$

In the methods section, we develop an approximation for the 2nd derivative to the 2nd order. We will then solve eq. 1 numerically, using gaussian elimination and lower-upper decomposition. As for the former, we will further specialise it to solve eq. 1 more efficiently. Next, in the results section we present the comparison between our numerical solutions and the analytical solution, as well as the error. We then compare the efficiency of all three algorithms, and finally, in the discussion section we will consider the advantages and disadvantages of each algorithm, and discuss their uses.

2 Methods

We would like to solve eq. 1 numerically. Generalising the equation, we get

$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x), \tag{2}$$

where we have assumed that $\rho \propto \frac{1}{r}e^{-r}$ and let $r \to x$, $\phi \to u$. Summing the backward and forward Taylor expansions of $u(x)$ and discretising the equation for n integration points, we get:

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + \mathcal{O}\big(h^2\big), \tag{3}$$

where $h = \frac{1}{n+1}$. Using the Dirichlet boundary conditions $v_0 = v_{n+1} = 0$, we can rewrite the equation in matrix form:

$$A\mathbf{v} = \mathbf{d}, \tag{4}$$

where

$$\begin{bmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \ddots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{bmatrix}$$

3 Results

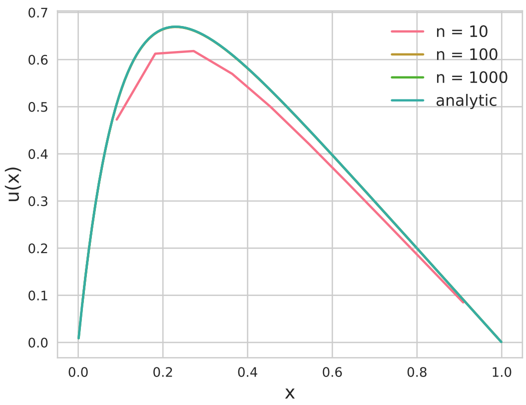

Figure 1: text

$\mathbf{n}$	$\mathbf{t_g/t_s}$	$\mathbf{t_{LU}/t_s}$
10	2.08	3.70
10^2	1.89	$1.00 \cdot 10^2$
10^3	1.48	$1.05 \cdot 10^4$
10^4	1.43	$1.18 \cdot 10^6$
10^5	1.39	-
10^6	1.41	-
10^7	1.39	-

Table 1:

4 Discussion

References

[Górski et al.(1994)] Górski, K. M., Hinshaw, G., Banday, A. J., Bennett, C. L., Wright, E. L., Kogut, A., Smoot, G. F., and Lubin, P. 1994, ApJL, 430, 89