

Analysis 1 - Zsolt Patavflaki

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1 Proofs and the reals

1.1 Some general proofs

A valid proof is set of lines where each line logically follows from the next. A most famous proof is that $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$
 Now we have that $\sqrt{2}b = a$ which means that $2b^2 = a^2$. As result, $2|a^2$ hence also $2|a$. Thus we get that $a = 2k$ which also means that $b^2 = 2k^2$ hence $2|b$. As result, $\gcd(a, b) = 2$ which is a contradiction. Therefore, $\sqrt{2}$ must be irrational. \square

Quite interestingly, we can also construct a 'wrong' proof just through one fallacious assumption and a set of correct steps.

Claim: 1 is the largest integer.
Proof:
 Let n be the largest integer. Then we have $n \geq n^2$. Which also means $0 \geq n^2 - n = n(n - 1)$.
 Now we have that either $n < 0$ or $n - 1 < 0$. But we know that $n \not< 0$ as n is at least 1. Hence, $n - 1 < 0$ giving us the result $n < 1$ proving our theorem. Note that the mistake here is solely the assumption we made at the start that there was a largest integer.

1.2 Proofs relating to infinite processes

Consider the claim that $0.999\dots = 1$. One way to prove this claim, rather naively is this.

$$\begin{aligned} & 9 \times 0.999\dots \\ &= (10 - 1) \times 0.999\dots \\ &= 9.999\dots - 0.999\dots = 1 \end{aligned}$$

Now a more formal proof is to use an infinite sum and limits. Here it is.

Analysis proof of $0.999\dots = 1$

$$\begin{aligned} 0.999\dots &= 9 \lim_{k \rightarrow \infty} \sum_{i=1}^k (10^{-i}) \\ \lim_{k \rightarrow \infty} \sum_{i=1}^k (10^{-i}) &= \frac{10^{-1} - 10^{-(k+1)}}{1 - 10^{-1}} \\ &= 9 \times \frac{1}{10} \times \frac{10}{9} \\ &= 1 \end{aligned}$$

1.3 Basic notions of sets

The breakdown of sets used in 'standard' analysis are $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. There are also some common set related notation that must be known.

- a **subset** $a \subseteq b$ is defined as $\{x \in b \mid \text{"condition"}\}$
- a **open interval** is defined as $]a, b[; r \in A, a < r < b$
- an **open ball** $B(a, \lambda) =]a - \lambda, a + \lambda[$

1.4 The Reals

The reals, denoted \mathbb{R} are an ordered field. Here is a more precise definition.

The reals are a set that have the 3 following axioms:

- \mathbb{R} is an abelian group under $(+)$ and \mathbb{R}^* is an abelian group under (\times) . In addition to this, multiplication distributes over addition.
- The order relation \leq holds $\forall x \in \mathbb{R}$ That is:

$$\begin{aligned} x &\leq y \otimes y \leq x \\ x &\leq y, y \leq x \implies x = y \\ x &\leq y \implies \forall a \in \mathbb{R}, x + a \leq y + a \\ 0 &\leq x, 0 \leq y \implies 0 \leq xy \end{aligned}$$

- The inf and sup axioms hold

We shall now come the **inf** and **sup** axioms. It should be intuitively clear that any subset of \mathbb{R}

1.5 Bounds

Take some subset S in \mathbb{R} . An element B is called an upper bound of S if $\forall x \in S, B \geq x$. Similarly, it is a lower bound of S if $\forall x \in S, B \leq x$.

The maximum B of a set S denoted $\max(S)$ is such that $B \in S, \forall x \in S, B \geq x$.

The supremum of a set S (if it exists) is the lowest upper bound. That is $\sup(S) = b$ is such that,

$$\forall x \in S, b \geq x \tag{1.5.1}$$

$$\forall \epsilon > 0, \exists x_\epsilon, b - x_\epsilon \leq \epsilon \tag{1.5.2}$$

Remark 1.1. In our above definition, b does not have to be in S .

Remark 1.2. Condition 1 states that b is an upper bound of S .

Remark 1.3. Given condition 1, b is the minimum of the upper bounds of S .

Some examples of \sup and \inf

Example 1.1.

$$\begin{aligned} \sup]a, b[&= b \\ \inf]a, b[&= a \\ \sup\{x \in \mathbb{R} | x = 2k\} &\implies \text{Sup doesn't exist.} \end{aligned}$$

We now establish the infimum axiom.

Axiom 1. All non-empty subsets of \mathbb{R}_+^* have a highest lower boundary(aka.infimum)

1.6 \mathbb{Q} is dense in \mathbb{R}

We claim that between every real number, one can find a rational number. Here's the proof,

Proof. Let $x < y \in \mathbb{R}$ Suppose now that $\exists a \in \mathbb{Q}$ such that $x < a < y$. By the Archimedean principle(there is always a greater natural number, $n > \frac{1}{y-x}$ which implies $ny > nx + 1$ Now since $ny > nx + 1$, there is guaranteed to be some integer in the open bound $]nx, ny[$ which we denote P . Dividing by n , we get that $\frac{P}{n} \in]x, y[$ which proves the theorem. \square

1.7 Integer and fractional part

Any number $x \in \mathbb{R}$ has a integer and fractional part(at least intuitively). Let's formally define these. For some $x \in \mathbb{R}$, let $S := \{n \in \mathbb{N} | n > x\}$. Now since S is bounded from below, letting N be the minimum of this set, we obtain that $N \notin S$. $N-1$ is thus called the integer part of x denoted $[x]$. ie. $[6.4] = 6$. Similarly, the fractional part of x denoted x is simply $x = x - [x]$

1.8 Pinning it down:Sup/Inf, bounds, max/min

Definition 1. for a given set $S \subseteq \mathbb{R}$, we have the following:

Sup $s = b \iff \forall \epsilon > 0, \exists x_\epsilon \in S, s.t. b - x_\epsilon < \epsilon$ (resp. Inf s has the flipped argument)

Upper bound $= b \iff \forall x \in S, b \geq x$ (resp. lower bound)

Max $s = b \iff b$ is an upper bound and $b \in S$

And for the sake of repeating the early axiom(but very important) the infimum axiom is:

Axiom 2. For all non-empty subsets of \mathbb{R} , $\inf S$ exists.

Now we make the first claim in this course that uses an epsilon proof.

[Proposition] 1. Whenever $S \subseteq \mathbb{N}$, then $\inf S = \min S$

Proof. Now, by our axiom, we have that $\mathbb{N} \subseteq \mathbb{R}$ hence we know that $\inf S$ exists. We now have to show that $\inf S = \min S$.

Suppose that $\inf S \neq \min S$ and let $\inf S = b$. Now clearly, $b + \epsilon$ is not a lower bound of S . Now, let $\epsilon = \frac{1}{2}$. Because, $b + \epsilon$ is not a lower bound, we know that $\exists s_\epsilon < b + \epsilon$.

Now s_ϵ is also not a lower bound, so let's pick $\epsilon'' = s_\epsilon - d$. Now again, $s_{\epsilon''}$ must exist. We obtain yet the following:

$$d < s_{\epsilon''} < s_\epsilon < d + 1/2$$

Now, two natural numbers clearly can not be in an interval which is only $\frac{1}{2}$ units long. Hence, contradiction which means that $d \in S$ □

Let's now prove that $\sqrt{2}$ belongs to the reals. For this, we need the following corollary and axiom.

Corollary 1. Every non-empty subset of \mathbb{R} with an upper boundary admits a supremum.

Axiom 3. An ordered field F , which \mathbb{R} is, has the Archimedean property if given any positive x and y in F , $\exists n \in \mathbb{Z}$ s.t. $nx > y$

[Proposition] 2. $\sqrt{2} \in \mathbb{R}$

Proof. Suppose we define a set $S = \{r \in \mathbb{R} | r \geq 0, r^2 < 2\}$. Now, as S is a non-empty subset of \mathbb{R} bounded from above (i.e. 2 is an upper bound) we know that $\sup S = x$ exists. Our goal is to show that both $x^2 < 2$ and $x^2 > 2$ lead to a contradiction.

Case 1: Suppose $x^2 < 2$. We want to find $x + \frac{1}{n} \in S$ which implies that x is not an upper bound as $x < x + \frac{1}{n}$

$$(x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2}{n} + \frac{1}{n} = x^2 + \frac{1}{n}(2x + 1)$$

Now, we want to show that we can pick an n s.t. $x^2 + \frac{1}{n}(2x + 1) < 2$. If we can pick such an n , then we know by transitivity of $<$ that $(x + \frac{1}{n})^2 < 2$ as $x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2}{n} + \frac{1}{n}$

Reordering the terms, we get $\frac{1}{n} < \frac{2-x^2}{2x+1}$ and clearly, $\frac{2-x^2}{2x+1}$ is positive as $x^2 < 2$ and $x \geq 0$. This way, we apply the archimedean property to know that n exists s.t. $\frac{1}{n} < \frac{2-x^2}{2x+1}$. Given this, we now know that $x^2 + \frac{1}{n}(2x+1) < 2$ which in turn implies $x + \frac{1}{n} \in S$. This contradicts that $x = \sup S$ hence $x^2 \not< 2$

Case 2: In turn, we consider the case where $x^2 > 2$ and try to derive a contradiction. We want to show that $\exists m \in \mathbb{N}$ s.t. $x - \frac{1}{m}$ is also an upper bound of S which would mean that $x \neq \sup S$. Now:

$$(x - \frac{1}{m})^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

We want to choose m s.t. $x^2 - \frac{2x}{m} > 2$. This way, if $x^2 - \frac{2x}{m} > 2$ holds, then since $(x - \frac{1}{m})^2 > x^2 - \frac{2x}{m}$, we will have that $x - \frac{1}{m}$ is an upper bound. We obtain:

$$\frac{x^2 - 2}{2x} > \frac{1}{m}$$

Now as $\frac{x^2-2}{2x}$ is positive, $1/m$ does exist. Hence, $x - \frac{1}{m}$ is also an upper bound that implies $x \neq \sup S$ if $x^2 > 2$.

Therefore, $\sup S = x = \sqrt{2}$ and since every $\sup S \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$

□

1.9 More theorems about \mathbb{R}

[Proposition] 3. If $a < b$ are real numbers, then $\exists c \in \mathbb{Q}, a < c < b$

Proof. Now our goal is to show that for any real number a, b we can always find such a c . Now take some arbitrary n and set it to $n = \lfloor \frac{1}{b-a} \rfloor + 1$. Now clearly, $n > \frac{1}{b-a}$ and hence $\frac{1}{n} < b-a$. We will now use this result. Realize that $a = \frac{an}{n} < \frac{[an]+1}{n} \leq \frac{an+1}{n} = a + \frac{1}{n} < a + b - a = b$. Therefore, we have found that $a < \frac{[an]+1}{n} < b$ where $c = \frac{[an]+1}{n}$

□

[Proposition] 4. *If $a < b$ are real numbers, then $\exists c \in \mathbb{R} \setminus \mathbb{Q}, a < c < b$*

Proof. Using the above proposition, we know that $\exists c$ for any $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ and now we get that $a < \sqrt{2}c < b$ where $\sqrt{2}c$ is irrational as whenever one term in $x \times y = z$ is rational and the other irrational, we have (supposing x is irrational) $x = \frac{z}{y}$ and if also z were rational, it would make x rational which is a contradiction. \square

Definition 2. *The absolute value function x is a function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ such that:*

$$\begin{cases} f(x) = x, x \geq 0 \\ f(x) = -x, x < 0 \end{cases}$$

Absolute value respects multiplication and division, that is:

$$\begin{aligned} |a||b| &= |ab| \\ \frac{|a|}{|b|} &= \left| \frac{a}{b} \right| \end{aligned}$$

But this doesn't hold for addition. For addition we have the triangle inequality:

$$|x + y| \leq |x| + |y|$$

To prove the above:

Proof. Take $|x + y| < 0$. Then $|x + y| = -(x + y) = -x - y \leq |x| + |y|$ Take $|x + y| \geq 0$. Then $|x + y| = x + y \leq |x| + |y|$ \square

2 Sequences

2.1 Basics

Let's begin by formally defining sequences. A sequence is a function $f : \mathbb{N} \rightarrow \mathbb{R}$ generally denoted $(x_n)_{n \geq 0}$. And here are some more definitions on sequences:

A sequence is:

- constant if $\exists C \in \mathbb{R}; x_n = C \forall n \in \mathbb{N}$
- bounded from below (resp. above) if $\exists m \in \mathbb{R}; m \leq x_n \forall n \in \mathbb{N}$
- bounded if bounded from both directions
- increasing (resp. decreasing) if $x_{n+1} \geq x_n \forall n \in \mathbb{N}$
- strictly increasing (resp. decreasing) if $x_{n+1} > x_n \forall n \in \mathbb{N}$
- monotonous if it is increasing or decreasing (resp. strictly)

Let's now consider a proof on the following proposition:

Define $x_n = \sqrt{4 + x_{n-1}}$, $x_0 = 1$. We claim that x_n is bounded and more precisely that $1 \leq x_n \leq 3$.

Proof. Base case: $x_0 = 1$ hence holds.

Now supposing proposition is true for all $n - 1$ we get

$$\begin{aligned} 1 &\leq x_{n-1} \leq 3 \\ 5 &\leq x_{n-1} + 4 \leq 7 \end{aligned}$$

Now we get:

$$\begin{aligned} \sqrt{5} &\leq \sqrt{x_{n-1} + 4} \leq \sqrt{7} \\ 1 &\leq \sqrt{5} \leq \sqrt{x_{n-1} + 4} \leq \sqrt{7} \leq 3 \end{aligned}$$

□

Definition 3. A sequence x_n converges to x if $\forall \epsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $n \geq n_0 \rightarrow |x_n - x| \leq \epsilon$

Now in an intuitive sense, suppose the sequence converges to x from both the right and the left. x being our central point, we move a distance ϵ away from this x . Now, if we can pick some n_0 such that for another $n \geq n_0$ we have that $|x_n - x| \leq \epsilon$ this means that no matter how small we make epsilon we are able to find some x_n in this region.

Having established convergence, any sequence that is not convergent is said to be **divergent**.

Let's now prove that a sequence is divergent.

Proof. Take $x_n = (-1)^n$. Now suppose that x_n converges to x . Then for $\epsilon = \frac{1}{2}$, $\exists n_{\frac{1}{2}} \in \mathbb{N}$ s.t. $\forall n \geq n_{\frac{1}{2}}$ we would have $|x_n - x| \leq \frac{1}{2}$. In particular if n' is any other integer $n' > n_{\frac{1}{2}}$ then we would have $|x_n - x_{n'}| \leq |x_n - x| + |x - x_{n'}| \leq \frac{1}{2} + \frac{1}{2}$ which implies a contradiction as $|x_n - x_{n+1}| = 2 > 1$. □

2.2 Limits and their algebra

Definition 4. If a sequence x_n converges to some x , we say that x is the **limit** of the sequence and is denoted $\lim_{n \rightarrow \infty} x_n = x$

Here are some properties of limits:

For sequences x_n and y_n with limits x, y we have:

•

$$\lim_{n \rightarrow \infty} x_n \cdot y_n = x \cdot y$$

•

$$\lim_{n \rightarrow \infty} x_n + y_n = x + y$$

•

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}; y \neq 0$$

• if $\exists n_0 \in \mathbb{N}, x_n \leq y_n \forall n \geq n_0$ then $x \leq y$

We now introduce the famous **squeeze theorem** and prove it.

Theorem 2.1. *Squeeze theorem*

Let a_n and b_n be both sequences that converge to a . In addition, let c_n be such that $\exists n_0 \in \mathbb{N}, \forall n \geq n_0, a_n \leq c_n \leq b_n$. Then we clearly have the following:

Proof.

$$\forall \epsilon > 0, \exists N_1, n \geq N_1 \rightarrow |a_n - a| < \epsilon \equiv a - \epsilon < a_n < a + \epsilon \quad (2.2.1)$$

Similarly, for b_n we have that:

$$\forall \epsilon > 0, \exists N_2, n \geq N_2 \rightarrow |b_n - a| < \epsilon \equiv b - \epsilon < b_n < a + \epsilon \quad (2.2.2)$$

Now set $N = \max\{N_1, N_2, n_0\}$. Now since $N \geq N_1, N_2, n_0$ we have that both 2.2.1 and 2.2.2 hold $\forall n > N$. This further gives us the result that:

$$\begin{aligned} a - \epsilon < a_n \leq c_n \leq b_n < a + \epsilon \\ |c_n - a| < \epsilon \end{aligned}$$

□

We now list some useful inequalities that may be used along with the squeeze theorem and also a sample limit problem and a solution to it:

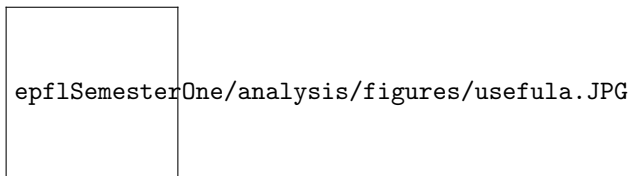


Figure 1: Useful inequalities

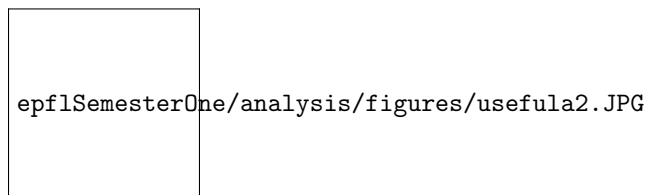


Figure 2: Solution to hard limit problem

2.3 More on sequences

Suppose we want to show that some geometric sequence does not have a limit, simply that it is not converging.

We first establish the *Bernoulli inequality* which we shall prove later. We will use this result immediately.

$$q^n \geq 1 + n(q - 1) \tag{2.3.1}$$

Take $x_n = 4^n$. Now we claim that x_n is not bounded.

Proof. It is not bounded if we can show that it increasing in increments that do not decrease. Suppose now that b is some upper bound to x_n . Now by Bernoulli, we have that $4^n \geq 1 + n \cdot 3$. If we can show that $4^n \geq 1 + n \cdot 3 > b$ we indeed get that there can be no upper bound b . We have that $n > \frac{b-1}{3}$ and such an $n \in \mathbb{N}$ exists if we set it to $n := \lceil \frac{b-1}{3} \rceil + 1$. Now since $4^n \geq 1 + n \cdot 3$ we have that x_n is not bounded. □

Theorem 2.2. *Every converging sequence is bounded (a lower or upper bound exists obviously and if convergent the latter also exists.)*

Let's now get on to proving the rules we established for limit arithmetic.

Proof. (Proof to sum rule) Now we are given that x_n converges to x and that y_n converges to y . We want to show that $x_n + y_n$ converges to $x + y$. By definition, this is true if we can show that

$$|(x_n + y_n) - (x + y)| \leq \epsilon, \forall \epsilon \in \mathbb{R}$$

Now luckily we have that $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$. And by the triangle inequality we know that $|(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$. Thus if we can show that $|x_n - x| + |y_n - y| < \epsilon$ we are guaranteed that $|(x_n - x) + (y_n - y)| < \epsilon$. We will succeed with the latter part if we can show that both parts $(x_n - x)$ and $(y_n - y)$ are smaller than $\frac{\epsilon}{2}$. Now we have by definition of convergence that:

$$\exists n \geq n_{\frac{\epsilon}{2}}^x \rightarrow |x_n - x| \leq \frac{\epsilon}{2}$$

Similarly:

$$\exists n \geq n_{\frac{\epsilon}{2}}^y \rightarrow |y_n - y| \leq \frac{\epsilon}{2}$$

Now fixing $n_\epsilon := \max n_{\frac{\epsilon}{2}}^x, n_{\frac{\epsilon}{2}}^y$ (we do this since it assures both conditions to hold we have that because each of $(x_n - x)$ and $(y_n - y)$ are smaller than $\frac{\epsilon}{2}$, so must $|x_n - x| + |y_n - y|$ by the triangle inequality. \square)

Let's now do more applications of the squeeze theorem to show limits.

Example 2.1. We want to show that aq^n converges to 0 for $a \neq 0$ and $|q| < 1$. Now as a property we use that $\lim_{n \rightarrow \infty} x_n = 0 \rightarrow \lim_{n \rightarrow \infty} |x_n| = 0$. It is clear that we have an inequality of the form $0 \leq |aq^n| \leq ?$. Doing more algebra (our goal is to find another sequence of form $\frac{1}{x}$ converging to 0 to get:

$$\frac{1}{?} \leq \frac{1}{|a \cdot q^n|} = \frac{1}{|a|} \cdot \left(\frac{1}{|q|}\right)^n$$

Now using Bernoulli we have:

$$\left(\frac{1}{|q|}\right)^n \geq 1 + n\left(\frac{1}{|q|} - 1\right)$$

which happily means:

$$\left(\frac{1}{a} \cdot \frac{1}{|q|}\right)^n \geq \left(\frac{1}{a} \cdot \left(1 + n\left(\frac{1}{|q|} - 1\right)\right)\right)$$

And finally taking the reciprocal all to get back to aq^n we are left with

$$0 \leq |aq^n| \leq \frac{1}{\frac{1}{a} \cdot \left(1 + n\left(\frac{1}{|q|} - 1\right)\right)}$$

and clearly we see that the RHS is also a sequence that converges to 0 since all terms in the denominator but n are constants. Hence, we have **squeezed** our sequence. The key here was that we found a RHS sequence which we wanted to be of form $\frac{1}{x}$. And in addition, we took the reciprocal of the inequality at the start simply to be able to use the Bernoulli inequality.

Let's now consider a harder example.

Example 2.2. Consider the sequence $x_n = \sqrt[n]{n}$. Now is this sequence converging? Well we know that $1 \leq \sqrt[n]{n}$ and now another sequence we know which approaches to 1 is $1 + \frac{1}{\sqrt[n]{n}}$. Now we only need to show that:

$$\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt[n]{n}}$$

holds and if we can show this, we'll have that our sequence converges to 1.

Now we get:

$$n \leq \left(1 + \frac{1}{\sqrt{n}}\right)^n$$

And notice how $\left(1 + \frac{1}{\sqrt{n}}\right)^n$ is simply a binomial hence if any one of the terms in the binomial expansion is greater than n our inequality will hold (this works as all n are positive). Well for $\sum_{i=0}^n \binom{n}{i} \left(\frac{1}{\sqrt{n}}\right)^i$ when observe that for $i = 4$ we get:

$$\frac{(n-1)(n-2)(n-3)}{24n} \geq n$$

further reducing to:

$$\frac{24n^2}{(n-1)(n-2)(n-3)} \leq 1$$

and we know this is valid as

$$\lim_{n \rightarrow \infty} \frac{24n^2}{(n-1)(n-2)(n-3)} = 0$$

Hence we get that

$$\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}}$$

meaning that $\sqrt[n]{n}$ converges to 1.

Let's do one last example:

Example 2.3. Consider $x_n = \frac{2^n}{n!}$ Now clearly $0 \leq \frac{2^n}{n!}$ and observing that $\frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \dots$ it is obvious that $\frac{2^n}{n!} \leq 2 \cdot \left(\frac{2}{3}\right)^{n-2}$ But we know that $\lim_{n \rightarrow \infty} 2 \cdot \left(\frac{2}{3}\right)^{n-2} = 0$ hence we have again squeezed our sequence!

Theorem 2.3. If $\lim_{n \rightarrow \infty} x_n = 0$ and y_n is bounded, then $\lim_{n \rightarrow \infty} x_n \cdot y_n = 0$

Theorem 2.4. If a sequence is bounded and increasing (monotone) it converges to the supremum (resp. infimum).

Proof. Now because y_n is bounded, we have that $\exists M$ s.t. $M \geq y_n \forall n$. Hence we obtain that $x_n y_n \leq M x_n$ and clearly since $M x_n$ converges to 0 we have squeezed $x_n y_n$ and have that it also converges to 0. \square

We consider the famous sequence of **Fibonacci quotients**

Now the fibonacci sequence is defined as $x_0 = x_1 = 1$, $x_{n+1} = x_n + x_{n-1}$ and we define the sequence of fibonacci quotients as $y_n = \frac{x_{n+1}}{x_n}$. Our first theorem is that fibonacci quotient sequence is bounded between 1 and 2. Let's now find its limit. Notice firstly that $y_{n+1} = 1 + \frac{1}{y_n}$

$$y = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} y_{n+1} = 1 + \frac{1}{\lim_{n \rightarrow \infty} y_n}$$

Hence we get:

$$y = 1 + \frac{1}{y}$$

which gives $\frac{1+\sqrt{5}}{2}$ as the only valid solution. But we still have to show that our sequence converges. A smart way to do so is to show that $z_n := |y_n - \frac{1+\sqrt{5}}{2}|$ converges which by limit arithmetic would imply $\lim_{n \rightarrow \infty} y_n = \frac{1+\sqrt{5}}{2}$. Our goal is to squeeze z_n in doing so. Notice that $z_{n+1} = |y_{n+1} - \frac{1+\sqrt{5}}{2}| =$

$$1 + \frac{1}{y_n} - \left(1 + \frac{2}{1+\sqrt{5}}\right) \text{ This yields the inequality } z_{n+1} = \frac{|y_n - \frac{1+\sqrt{5}}{2}|}{y_n \frac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} \frac{|y_n - \frac{1+\sqrt{5}}{2}|}{y_n} \text{ to give us that}$$

using definition
 $z_{n+1} \leq \frac{2}{1+\sqrt{5}} |y_n - \frac{1+\sqrt{5}}{2}|$ because we know that $y_n \geq 1$. Now finally this simplifies to $z_{n+1} \leq \frac{2}{1+\sqrt{5}} z_n$ which applying the definition of z_n gives $z_n \leq \frac{2}{1+\sqrt{5}}^2 z_{n-1}$ to result in $z_n \leq \frac{2}{1+\sqrt{5}}^n z_0$ which converges to 0. Hence we have squeezed z_n

In general this is the scheme one should use for finding limits of recursive sequences:

(Limit of recursive sequence)

1. Assuming there is a limit, compute it using limit algebra.
2. Showing some upper and lower bound(using induction) exclude any extra answers.
3. Finally show that the sequence converges by showing that $\lim_{n \rightarrow \infty} x_n - x = 0$

We now come to define **approaching infinities**.

Definition 5. The definition of approaching ∞ is as intuitive as saying for any real number I pick, the sequence has a term larger than it. Hence we define $\lim_{n \rightarrow \infty} x_n = \infty$ as $\forall A \in \mathbb{R} \exists n_A \in \mathbb{N}$ such that $n \geq n_A$ and $x_n \geq A$. The similar definition applies for approaching $-\infty$

Example 2.4. Notice that for a geometric sequence $x_n = aq^n$ if $a > 0$ and $q > 1$ it approaches ∞ and if $a < 0$ and $q > 1$ it approaches $-\infty$

A set of useful theorems on approaching infinities is the following:

Theorem 2.5. (Theorems on approaching infinity)

If $\lim_{n \rightarrow \infty} x_n = \infty$ and y_n is bounded from below, then $\lim_{n \rightarrow \infty} x_n + y_n = \infty$

If x_n and y_n both approach infinity, so does their product.

If y_n is bounded and x_n approaches infinity, then $\lim_{n \rightarrow \infty} \frac{y_n}{x_n} = 0$

We note that when it is the case that $\lim_{n \rightarrow \infty} x_n = \infty$ and $\lim_{n \rightarrow \infty} x_n = -\infty$ we may have different cases such as:

$$\lim_{n \rightarrow \infty} \underbrace{n}_{x_n} + \underbrace{(-n)}_{y_n} = 0$$

$$\lim_{n \rightarrow \infty} \underbrace{2n}_{x_n} + \underbrace{(-n)}_{y_n} = \infty$$

$$\lim_{n \rightarrow \infty} \underbrace{2n + (-1)^n n}_{x_n} + \underbrace{(-2n)}_{y_n} = (-1)^n n \text{ which is unbounded hence does not approach anything}$$

Theorem 2.6. (Squeeze theorem for approaching infinities) For sequences x_n and y_n if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, x_n \leq y_n$ we have:

(1) If $\lim_{n \rightarrow \infty} x_n = \infty$ then $\lim_{n \rightarrow \infty} y_n = \infty$

(2) If $\lim_{n \rightarrow \infty} y_n = -\infty$ then $\lim_{n \rightarrow \infty} x_n = -\infty$

Theorem 2.7. (Quotient criterion)

$\forall x_n \neq 0$ and $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \infty$ then we have that x_n diverges.

And now we provide an example of sandwich for infinities.

Example 2.5. Define $x_n := \frac{x_1}{2^n}$. We have that $\frac{n}{2} \cdot \frac{n-1}{2} \dots \frac{3}{2} \dots \geq \frac{n}{2} \cdot \frac{3^{q-1}}{2}$ for some n . And because the latter is a geometric sequence which noticeably approaches infinity we have that x_n approaches infinity.

Proof. (We present a proof to theorem 2.4)

Now set $S := \sup\{x_n | n \in \mathbb{N}\}$ and let $0 < \epsilon \in \mathbb{R}$. By definition S is the smallest bound hence $S - \epsilon$ is not the smallest bound. By def. again, $\exists n_\epsilon$ such that $S - \epsilon < x_{n_\epsilon}$ and we now get (for $n \geq n_\epsilon$)

$$S - \epsilon < x_{n_\epsilon} < x_n < S < S + \epsilon$$

and this is exactly the definition of convergence hence S is the limit. \square

(Exploration of e)

We define $x_n = (1 + \frac{1}{n})^n$. We first ask is x_n increasing. Consider the claim $(1 + \frac{1}{n})^n < (1 + \frac{1}{n+1})^{n+1}$. Now let's consider both the expansions of $(1 + \frac{1}{n})^n$ and of $(1 + \frac{1}{n+1})^{n+1}$.

$$(1 + \frac{1}{n})^n = \sum_{i=0}^n \binom{n}{i} (\frac{1}{n})^i = \sum_{i=0}^n \frac{1}{i!} \frac{n \cdot (n-1) \dots (n-i+1)}{n^i} = \sum_{i=0}^n \frac{1}{i!} (1 - \frac{1}{n}) \dots (1 - \frac{i-1}{n})$$

Similarly we have:

$$(1 + \frac{1}{n+1})^{n+1} = \sum_{i=0}^{n+1} \frac{1}{i!} (1 - \frac{1}{n+1}) \dots (1 - \frac{i-1}{n+1})$$

and because generally $\frac{q}{j} > \frac{q}{j+1}$ we have that terms on the RHS of the first expression are larger hence $(1 + \frac{1}{n})^n < (1 + \frac{1}{n+1})^{n+1}$. And we also have that our sequence is less than 3 (which we do not show).

Thus we define

$$\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$$

And now we present an example of a bounded sequence that is decreasing which converges to its infimum. Consider $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_n})$ and $x_0 = 2$.

We claim 1 is a lower bound which is true by induction (easy to show). Similarly, the sequence is decreasing as $x_n - x_{n+1} = \frac{1}{2}(x_n - \frac{1}{x_n}) \geq 0$ since $x_n \geq 1$. And applying the recursive formula as usual we find that the limit is either -1 or 1 and ruling out -1 we get 1.

2.4 More definitions and theorems

We come to an interesting definition. That of \limsup \liminf .

Definition 6. (\limsup and \liminf) Let x_n be a bounded sequence. Then:

$$y_n := \sup \{x_k | n \leq k \in \mathbb{N}\} \text{ (resp. inf)}$$

Now the sequence y_n is clearly decreasing as we are looking over a smaller set for each $n+1$. Similarly considering the respective inf definition, we have that it is increasing as each time we are removing elements from the largest set for $n=0$ meaning that our inf is at least as large as y_0 or bigger. Now since y_n is valid sequence definition, it naturally has a limit as well defined as:

$$y_n := \lim_{n \rightarrow \infty} \sup \{x_k | n \leq k \in \mathbb{N}\} \text{ (resp. inf)}$$

Remark 2.1. The purpose in defining \limsup and \liminf is that when a limit on itself doesn't exist, by taking \limsup , we limit the values of our sequence and if for instance our sequence consists of purely -1 and 1 \limsup tells us that the largest occurring value is 1. In the case where our sequence approached ∞ so does \limsup and whenever a limit exists, \limsup is the limit.

Here's an example:

Example 2.6. Consider $x_n = (-1)^n$. Defining $y_n = \sup \{x_k | n \leq k \in \mathbb{N}\}$ we get that the limit of y_n as n goes to infinity is 1.

And now we define what it means to be a subsequence.

Definition 7. Let x_n be a sequence. Then x_{n_k} is a subsequence of x_n where each k is mapped to some n_k by some rule $f: \mathbb{N} \rightarrow \mathbb{N}$.

Example 2.7. Suppose we define $x_n = \frac{1}{n} \sin n$. And define the subsequence $x_{2\pi k}$ we notice that this subsequence is constant with all values mapping to 0.

And yet a meatier example is:

Example 2.8. Define $x_n = (1 + \frac{2}{n})^n$ and a subsequence x_{2k} . We notice that

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{k})^{2k} = \lim_{k \rightarrow \infty} (1 + \frac{1}{k})^k)^2 = e^2$$

Theorem 2.8. If a sequence x_n converges to a , then so do all subsequences of x_n

Proof. (Simply using invoking definition of convergence) Now accepting the if true, we have that:

$$\forall \epsilon > 0, \exists n_\epsilon \text{ s.t. } n \geq n_\epsilon \rightarrow |x_n - a| \leq \epsilon$$

We now fix $n_\epsilon = n_k$ hence proof. (this proof is to be revisited, it may be false. □)

Example 2.9. Consider a sequence that jumps between e and $-e$ as $n \rightarrow \infty$ defined by $x_n = (-1)^n(1 + \frac{1}{n})^n$. For all subsequences with an even domain, the limit is e and for subsequences with an odd domain, limit is $-e$

Theorem 2.9. (Bolzano Weierstrass) Every bounded sequence contains a convergent subsequence.

Proof. Let's define $y_n = \sup x_k : k \geq n$. Now given that y_n is bounded from below (because x_n is bounded) and that it is decreasing, we have that y_n converges to some y meaning:

$$\forall \epsilon > 0, \forall N, \exists n_\epsilon \geq N, \quad \underbrace{|y_n - y| \leq \frac{1}{2}\epsilon}_{\text{definition of convergence}}$$

And by definition of sup and how we defined y_n we have that

$$\forall \epsilon > 0, \exists n_1 \geq n \rightarrow |x_{n_1} - y_n| \leq \frac{1}{2}\epsilon$$

Considering that we want an expression like $|x_{n_1} - y| \leq \epsilon$ And we get this by:

$$|x_{n_1} - y| = |x_{n_1} - y_n + y_n - y| \leq \underbrace{|x_{n_1} - y_n| + |y_n - y|}_{\text{using triangle ineq.}} \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

Now the last part is enough for the proof because we are assured that for any subsequence choice we make, it holds that there is an N such that $n_k \geq N$ □

Definition 8. (Cauchy convergence) A sequence is Cauchy convergent if $\forall \epsilon > 0, \exists n_\epsilon$ such that $\forall n, m \geq n_\epsilon$ we have that $|x_n - x_m| \leq \epsilon$

A very important theorem concerning Cauchy convergence is:

3 Series

Definition 9. A series S_n is defined in terms of the summation of some sequence x_n .

$$S_n = \sum_{k=0}^n x_k$$

Theorem 3.1. (Bernoulli Inequality (Known as the negative case of it))

Whenever $-1 < x < 0$ we have that $(1 + x)^n \geq 1 + nx$

Definition 10. (Cauchy criterion for series) A series S_n is convergent iff:

$$\forall \epsilon > 0, \exists n_\epsilon \forall n, m > n_{\epsilon} \quad \sum_{k=n+1}^m |x_k| \leq \epsilon$$

Theorem 3.2. *As follows from the Cauchy criterion, we have that $\forall \epsilon > 0, \exists n_\epsilon \forall n, m > n_{\epsilon} \sum_{k=n+1}^m |x_k| \leq \epsilon$ which means that if we let $m = n + 1$ we obtain $|x_{n+1}| \leq \epsilon$ which by definition of the limit says that x_{n+1} goes to 0.*

(Yet another way to obtain e)

Consider the sequence $S_n = \sum_{k=0}^{\infty} \frac{1}{k!}$. Now clearly S_n is increasing and is bounded. It is bounded because:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \leq 1 + \sum_{k=1}^n \frac{1}{2^{k-1}} \leq 1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 3$$

Thus we know that a limit must exist. Now let's find that limit. Observe that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \geq \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)}_{\text{each term smaller than 1}} \geq 2 + \sum_{k=2}^{\infty} \frac{1}{k!} \underbrace{\left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)}_{\text{first 2 terms are 1}}$$

And further realizing that:

$$2 + \sum_{k=2}^{\infty} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \geq 2 + \sum_{k=2}^{\infty} \frac{1}{k!} \left(1 - \frac{k-1}{n}\right)^{k-1} \geq \sum_{k=0}^n \frac{1}{k!} \underbrace{\left(1 - \frac{k-1}{n}\right)^{k-1}}_{\text{Using negative Bernoulli}}$$

Now the very last term on the RHS is equal to:

$$\sum_{k=0}^n \frac{1}{k!} - \frac{1}{n} \sum_{k=2}^n \frac{(k-1)^2}{k!}$$

Now we are in a very good position because we have that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \geq \sum_{k=0}^{\infty} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \geq \sum_{k=0}^n \frac{1}{k!} - \frac{1}{n} \sum_{k=2}^n \frac{(k-1)^2}{k!}$$

Which means that if we can show that $\frac{1}{n} \sum_{k=2}^n \frac{(k-1)^2}{k!}$ tends to 0 and given that $\sum_{k=0}^{\infty} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \rightarrow e$ we will have squeezed e between $\sum_{k=0}^n \frac{1}{k!}$

Now notice that:

$$\frac{1}{n} \rightarrow 0, \frac{1}{n} \sum_{k=2}^n \frac{(k-1)^2}{k!} \leq \frac{1}{n} \sum_{k=2}^n \frac{1}{(k-2)!} = \frac{1}{n} \sum_{k=0}^{n-2} \frac{1}{(k)!} \leq 3$$

Thus we gloriously obtain that :

$$\sum_{k=0}^n \frac{1}{k!} \geq e \geq \sum_{k=0}^n \frac{1}{k!}$$

Definition 11. A series is convergent if simply S_n is convergent and it is absolute convergent if $S_n = \sum_{k=0}^n |x_k|$ is convergent.

Now a very important theorem is the following:

Theorem 3.3. If $\sum_{k=0}^{n=\infty} x_k$ is convergent, then $\lim_{n \rightarrow \infty} x_n = 0$

And squeeze theorem for series is;

Theorem 3.4. assume $\exists n_0$ s.t. $0 \leq x_n \leq y_n \forall n \geq n_0$ Then:

1. if $\sum_{k=0}^{\infty} y_k$ is convergent, so is $\sum_{k=0}^{\infty} x_k$
2. if $\sum_{k=0}^{\infty} x_k$ is divergent, so is $\sum_{k=0}^{\infty} y_k$

And more theorems:

Theorem 3.5. Whenever $x_k \geq 0$ a series is:

$$\sum_{k=0}^{\infty} x_k \begin{cases} \text{Convergent if } S_n \text{ is bounded.} \\ \text{Divergent else.} \end{cases}$$

And now we define a **Leibniz series** as being of form $S_n = \sum_{k=0}^n (-1)^k (x_k)$ And the important Leibniz criterion is:

Theorem 3.6. For a Leibniz S_n , S_n diverges if x_k is decreasing and $\lim_{n \rightarrow \infty} (x_n) = 0$

[Proposition] 5. If S_n is absolute convergent, then it is convergent.

The above is obvious to see using Cauchy criterion. Now absolute convergence implies:

$$\sum_{k=n+1}^m |x_k| \leq \epsilon$$

and the triangle inequality gives us that

$$\left| \sum_{k=n+1}^m x_k \right| \leq \sum_{k=n+1}^m |x_k|$$

hence we know that

$$\left| \sum_{k=n+1}^m x_k \right| \leq \epsilon$$

which is exactly what we want. And the last thing we do on series is to present the Cauchy and Alembert convergence criteria:

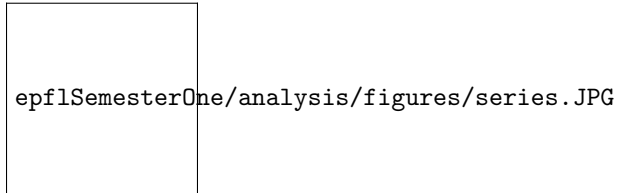


Figure 3: Yes I was lazy to write this down

Theorem 3.7. Yet another important fact is that the harmonic series defined in terms of summing $\frac{1}{n}$ diverge (one would have expected) it to converge.

And another important and rather obvious theorem is:

Theorem 3.8. Whenever a sequence maps $\mathbb{N} \rightarrow \mathbb{R}^+$ or the opposite and all of its partial sums are bounded, then the series converges.

4 Real-valued functions of 1-variable

Definition 12. (Defining what it means to be a function of one variable)

Let $E \subseteq \mathbb{R}$. Then f is subset of $E \times \mathbb{R}$ where each element of f is of form (a, b) and each a occurs only once.

Something taught at high-school is the local maximum. We now formally define it:

Definition 13. (Local maximum) Some $f : E \rightarrow \mathbb{R}$ has a local maximum at x_0 if $\forall \sigma > 0, \exists x$ such that $|x_0 - x| < \sigma \rightarrow f(x_0) > f(x)$

We say that some $f : E \rightarrow \mathbb{R}$ is defined a pointed neighbourhood of $x_0 \in \mathbb{R}$ if \exists an interval $]x_0 - a, x_0 + a[$ contained in $E \setminus \{x_0\}$

We now define what it means to be the limit of a function.

Definition 14. Suppose $f : E \rightarrow \mathbb{R}$ is defined on a pointed neighbourhood of $x_0 \in \mathbb{R}$. We say that $\lim_{x \rightarrow x_0} f(x) = L$ if:

1. $\forall 0 < \epsilon \in \mathbb{R}, \exists 0 < \sigma \in \mathbb{R}$ such that $|x - x_0| < \sigma \rightarrow |f(x) - L| < \epsilon$
2. for every sequence $x_n \subseteq E - \{x_0\}$ we have that $\lim_{n \rightarrow \infty} x_n = x_0 \rightarrow \lim_{n \rightarrow \infty} f(x_n) = L$

Example 4.1. Let's show that $\lim_{x \rightarrow 2} (x^2) = 4$ using the first definition. We need:

$$|x - 2| < \sigma \rightarrow |x^2 - 4| < \epsilon$$

Now pick suppose that $|x - 2| \leq 1$ which yields that $1 \leq x \leq 3 \equiv 3 \leq x + 2 \leq 5$ Particularly because $|x + 2| \leq 5$ we get that:

$$|x - 2||x + 2| = |x^2 - 4| \leq 5|x - 2|$$

Now finally set $\sigma := \min 1, \frac{\epsilon}{5}$ because then if:

$$|x - 2| \leq 1 \text{ and } |x - 2| \leq \frac{\epsilon}{5}$$

we get that:

$$|x^2 - 4| \leq \underbrace{5|x - 2|}_{\text{because } |x + 2| \leq 5} \leq 5 \frac{\epsilon}{5}$$

Let's now go through an example demonstration using the sequence definition of the limit.

Example 4.2. Suppose we want to show that for $f(x) = x^2 + 1$ $\lim_{x \rightarrow 0} x^2 + 1 = 1$. Now define $x_n = n$ which gives us a sequence such that $\lim_{n \rightarrow \infty} x_n = 0$. Now we want $\lim_{n \rightarrow \infty} x_n^2 + 1 = 1$. By limit algebra we get:

$$= (\lim_{n \rightarrow \infty} x_n)^2 + 1 = 0^2 + 1 = 1$$

Definition 15. some f is continuous in x_0 if $\lim_{x \rightarrow x_0} f(x)$ exists and is equal to $f(x_0)$ which extrapolates to, $f : E \rightarrow \mathbb{R}$ is continuous if it is continuous $\forall x \in E$

And now let's take the example of a function for which a limit does not exists.

Example 4.3. Suppose f is given by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Now notice that we can pick $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\sqrt{2}}{n} = 0$ but we have that the latter has limit 1 and the former limit 0.

And now using our earlier construct of the limit of a function, we define what it means for f to be continuous at x_0

$f(x)$ is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ And in general, f is **continuous** if it is continuous $\forall x \in \mathbb{R}$

And now let's show that $\sin(1/x)$ is not continuous at 0.

Example 4.4. define $x_n := \frac{1}{\pi(2n+\frac{1}{2})}$ and $y_n := \frac{1}{\pi(2n+\frac{3}{2})}$ Now both sequences converge to 0. Yet we have that $f(x_n)$ converges to 1 whereas $f(y_n)$ converges to -1.

And now we highlight limit algebra for functions. It is all very similar to limit algebra of sequences since the sequence definition tells us that $\lim_{x \rightarrow \infty} f(x_n) = L$ and supposing also that $\lim_{x \rightarrow \infty} g(x_n) = k$ we have:

$$\begin{aligned}\lim_{x \rightarrow \infty} (f + g)(x_n) &= L + K \\ \lim_{x \rightarrow \infty} (f \cdot g)(x_n) &= L \cdot K \\ \lim_{x \rightarrow \infty} \left(\frac{f}{g}\right)(x_n) &= \frac{L}{K} \text{ whenever } x \neq 0\end{aligned}$$

And now squeeze for limits of functions:

Theorem 4.1. Suppose that $\lim_{x \rightarrow \infty} g(x_n) = k$ and that $\lim_{x \rightarrow \infty} f(x_n) = L$ such that a pointed neighbourhood of some x_0 is also a subset of the domain of $h(x)$. Now on some neighbourhood of x_0 :

$$f(x) \leq h(x) \leq g(x)$$

$$l = k$$

$$\text{then } \lim_{x \rightarrow \infty} h(x) = l$$

Example 4.5. Consider $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$. We have that $\frac{\sin(x)}{x} \leq 1$ and also that $\frac{\sin(x)}{x} \geq \cos(x)$ As both approach 1 so does our function.