Analysis 2 - Thomas Mountford

Alp Ozen

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1 Reviewing $\mathbf{R}^n$

Let's recall that $\mathbb{R}^n$ is a Euclidean vector space. We define a scalar product on $\mathbb{R}^n$ as follows:

Definition 1.

$$< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

1. $< x, x > \geq 0$
2. $< x, y > = < y, x >$
3. $< ax + by, z > = a < x, z > +b < y, z >$

1.1 Introducing topological properties on $\mathbf{R}^n$

A *norm* is defined as function that maps some real vector space E to $\mathbb{R}$ and satisfies:

$$\begin{aligned} 1)\ |x| \geq 0\ \forall x \in E,\ |x| = 0 \iff x = 0 \\ 2)\ |\lambda \cdot x| = |\lambda| \cdot |x| \\ 3)\ |x + y| \leq |x| + |y| \end{aligned}$$

In our intuitive understanding of $\mathbb{R}^n$ we are actually thinking about the Euclidian space $\mathbb{R}^n$ equipped with the Euclidian norm.

Definition 2. ***Euclidian norm***

$$|x|_2 = \sqrt{< x, x >} = (\sum_i^n x_k^2)^{\frac{1}{2}}$$

And from this naturally follows the definition of Euclidian distance:

Definition 3.

$$d(x, y) = |x - y|$$

We note that d satisfies the same 3 properties as the norm. Thus, the couple (E, d) is called a metric space.

And now more definitions:

Definition 4. ***Open sets***

1. ***Open ball*** $B(a, r) := \{x \in \mathbb{R}^n : d(x, a) < r\}$
2. ***Open subset*** *Some subset* $S \subset \mathbb{R}^n$ *is open if* $\forall x \in \mathbb{R}^n,\ \exists \epsilon > 0\ B(x, \epsilon) \subset S$
3. ***Closed subset*** *Some* S *is closed if* $\mathbb{R}^n - S$ *is open, note that the empty set and* $\mathbb{R}^n$ *are both open and closed.*
4. ***The interior and boundary of a set*** a *is in the interior of* S *if* $\exists \epsilon > 0\ B(a, \epsilon) \subset S$ *and* b *is in the boundary of a set* S *if any* $B(a, \epsilon)$ *contains points from both* S *and* $\mathbb{R}^n - S$. *The set of all interior points is denoted* (^{o}S) *and set of all boundary points is denoted* ∂S
5. ***Closure of a set*** a *is a closure of* S *if for any* $B(a, \epsilon)$ *we have* $B(a, \epsilon) \cap S \neq \emptyset$