Analysis

SemesterOne analysis at EPFL

1 Proofs and the reals

1.1 Some general proofs

A valid proof is set of lines where each line logically follows from the next. A most famous proof is that $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $gcd(a, b) = 1$
Now we have that $\sqrt{2}b = a$ which means that $2b^2 = a^2$. As result, $2|a^2$ hence also $2|a$. Thus we get that $a = 2k$ which also means that $b^2 = 2k^2$ hence $2|b$. As result, $gcd(a, b) = 2$ which is a contradiction. Therefore, $\sqrt{2}$ must be irrational. □

Quite interestingly, we can also construct a 'wrong' proof just through one fallacious assumption and a set of correct steps.

Claim: 1 is the largest integer.
Proof:
Let n be the largest integer. Then we have $n \geq n^2$. Which also means $0 \geq n^2 - n = n(n-1)$. Now we have that either $n < 0$ or $n - 1 < 0$. But we know that $n \not< 0$ as n is at least 1. Hence, $n - 1 < 0$ giving us the result $n < 1$ proving our theorem. Note that the mistake here is solely the assumption we made at the start that there was a largest integer.

1.2 Proofs relating to infinite processes

Consider the claim that $0.999\ldots = 1$. One way to prove this claim, rather naively is this.

$$\begin{aligned} &9 \times 0.999\ldots \\ &= (10 - 1) \times 0.999\ldots \\ &= 9.999\ldots - 0.999\ldots = 1 \end{aligned}$$

Now a more formal proof is to use an infinite sum and limits. Here it is.

Analysis proof of $\mathbf{0.999\ldots = 1}$

$$\begin{aligned} 0.999\ldots &= 9 \lim_{k\to\infty} \sum_{i=1}^{k} (10^{-k}) \\ \lim_{k\to\infty} \sum_{i=1}^{k} (10^{-k}) &= \frac{10^{-1} - 10^{-(k+1)}}{1 - 10^{-1}} \\ &= 9 \times \frac{1}{10} \times \frac{10}{9} \\ &= 1 \end{aligned}$$

1.3 Basic notions of sets

The breakdown of sets used in 'standard' analysis are $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.
There are also some common set related notation that must be known.

- a **subset** $a \subseteq b$ is defined as $\{x \in b|\text{"condition"}\}$
- a **open interval** is defined as $]a, b[; r \in A, a < r < b$
- an **open ball** $B(a, \lambda) =]a - \lambda, a + \lambda[$

1.4 The Reals

The reals, denoted R are an ordered field. Here is a more precise definition.

The reals are a set that have the 3 following axioms:

- R is an abelian group under $(+)$ and R* is an abelian group under $(\times)$. In addition to this, multiplication distributes over addition.
- The order relation $\leq$ holds $\forall x \in \mathbb{R}$ That is:

$$\begin{aligned} x \leq y \otimes y \leq x \\ x \leq y, y \leq x \implies x = y \\ x \leq y \implies \forall a \in \mathbb{R},\ x + a \leq y + a \\ 0 \leq x, 0 \leq y \implies 0 \leq xy \end{aligned}$$

- The inf and sup axioms hold

We shall now come the **inf** and **sup** axioms. It should be intuitively clear that any subset of R

1.5 Bounds

Take some subset S in R. An element B is called an upper bound of S if $\forall x \in S, B \geq x$. Similarly, it is a lower bound of S if $\forall x \in S, B \leq x$.
The maximum B of a set S denoted $max(S)$ is such that $B \in S, \forall x \in SB \geq x$.
The supremum of a set S(if it exists) is the lowest upper bound. That is $sup(S) = b$ is such that,

$$\forall x \in S, b \geq x \tag{1}$$
$$\forall \epsilon > 0, \exists x_\epsilon, b - x_\epsilon \leq \epsilon \tag{2}$$

Remark 1. *In our above definition, b does not have to be in S.*

Remark 2. *Condition 1 states that b is an upper bound of S.*

Remark 3. *Given condition 1, b is the minimum of the upper bounds of S.*

Some examples of sup and inf

Example 1.1.

$$\begin{aligned} Sup]a, b[= b \\ Inf]a, b[= a \\ Sup\{x \in \mathbb{R}|x = 2k\} \implies Sup\ doesn't\ exist. \end{aligned}$$

We now establish the infinimum axiom.

Axiom 1. *All non-empty subsets of $\mathbb{R}^*_+$ have a highest lower boundary(aka.infinimum)*

1.6 Q is dense in R

We claim that between every real number, one can find a rational number. Here's the proof,

Proof. Let $x < y \in \mathbb{R}$ Suppose now that $\exists a \in \mathbb{Q}$ such that $x < a < y$. By the Archimedean principle(there is always a greater natural number, $n > \frac{1}{y-x}$ which implies $ny > nx + 1$ Now since $ny > nx + 1$, there is guaranteed to be some integer in the open bound $]nx, ny[$ which we denote P. Dividing by n, we get that $\frac{P}{n} \in]x, y[$ which proves the theorem. □

1.7 Integer and fractional part

Any number $\in \mathbb{R}$ has a integer and fractional part(at least intuitively). Let's formally define these. For some $x \in \mathbb{R}$, let $S := \{n \in \mathbb{N} | n > x\}$ Now since S is bounded from below, letting N be the minimum of this set, we obtain that $N \notin S$. N-1 is thus called the integer part of x denoted [x]. ie. [6.4] = 6
Similarly, the fractional part of x denoted x is simply $x = x - [x]$

1.8 Pinning it down:Sup/Inf, bounds, max/min

Definition 1. *for a given set $S \subseteq \mathbb{R}$, we have the following:*
Sup s = b $\iff \forall \epsilon > 0, \exists x_\epsilon \in S, s.t. b - x_\epsilon < \epsilon$ *(resp. Inf s has the flipped argument)*
Upper bound = b $\iff \forall x \in S, b \geq x$ *(resp. lower bound)*
Max s = b $\iff$ *b is an upper bound and $b \in S$*

And for the sake of repeating the early axiom(but very important) the infimum axiom is:

Axiom 2. *For all non-empty subsets of $\mathbb{R}$, $infS$ exists.*

Now we make the first claim in this course that uses an epsilon proof.

[Proposition] 1. *Whenever $S \subseteq \mathbb{N}$, then $infS = minS$*

Proof. Now, by our axiom, we have that $\mathbb{N} \subseteq \mathbb{R}$ hence we know that $infS$ exists. We now have to show that $infS = minS$.

Suppose that $infS \neq minS$ and let $infS = b$. Now clearly, $b + \epsilon$ is not a lower bound of S. Now, let $\epsilon = \frac{1}{2}$. Because, $b + \epsilon$ is not a lower bound, we know that $\exists s_e < b + \epsilon$.

Now s_e is also not a lower bound, so let's pick $\epsilon'' = s_\epsilon - d$. Now again, $s_{\epsilon''}$ must exist. We obtain yet the following:

$$d < s_{\epsilon''} < s_\epsilon < d + 1/2$$

Now, two natural numbers clearly can not be in an interval which is only $\frac{1}{2}$ units long. Hence, contradiction which means that $d \in S$ □

Let's now prove that $\sqrt{2}$ belongs to the reals. For this, we need the following corollary and axiom.

Corollary 1. *Every non-empty subset of R with an upper boundary admits a supremum.*

Axiom 3. *An ordered field F, which R is, has the Archimedean property if given any positive x and y in F, $\exists n \in \mathbb{Z}$ s.t. $nx > y$*

[Proposition] 2. $\sqrt{2} \in \mathbb{R}$

Proof. Suppose we define a set $S = \{r \in \mathbb{R} | r \geq 0, r^2 < 2\}$. Now, as S is a non-empty subset of $\mathbb{R}$ bounded from above(i.e. 2 is an upper bound) we know that $supS = x$ exists. Our goal is to show that both $x^2 < 2$ and $x^2 > 2$ lead to a contradiction.

Case 1: Suppose $x^2 < 2$. We want to find $x + \frac{1}{2} \in S$ which implies that x is not an upper bound as $x < x + \frac{1}{2}$

$$(x + \frac{1}{2})^2 = x^2 \frac{2}{x} + \frac{1}{n^2} \leq x^2 + \frac{2}{x} + \frac{1}{n} = x^2 + \frac{1}{n}(2x + 1)$$

Now, we want to show that we can pick an n s.t. $x^2 + \frac{1}{n}(2x + 1) < 2$. If we can pick such an n, then we know by transitivity of $<$ that $(x + \frac{1}{n})^2 < 2$ as $x^2 \frac{2}{x} + \frac{1}{n^2} \leq x^2 + \frac{2}{x} + \frac{1}{n}$
Reordering the terms, we get $\frac{1}{n} < \frac{2-x^2}{2x+1}$ and clearly, $\frac{2-x^2}{2x+1}$ is positive as $x^2 < 2$ and $x \geq 0$. This way, we apply the archimedean property to know that n exists s.t. $\frac{1}{n} < \frac{2-x^2}{2x+1}$. Given this, we now know that $x^2 + \frac{1}{n}(2x+1) < 2$ which in turn implies $x + \frac{1}{n} \in S$ This contradicts that $x = supS$ hence $x^2 \nless 2$

Case 2: In turn, we consider the case where $x^2 > 2$ and try to derive a contradiction. We want to show that $\exists m \in \mathbb{N}$ s.t. $x - \frac{1}{m}$ is also an upper bound of S which would mean that $x \neq infS$. Now:

$$(x - \frac{1}{m})^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

We want to choose m s.t. $x^2 - \frac{2x}{m} > 2$. This way, if $x^2 - \frac{2x}{m} > 2$ holds, then since $(x - \frac{1}{m})^2 > x^2 - \frac{2x}{m}$, we will have that $x - \frac{1}{m}$ is an upper bound. We obtain:

$$\frac{x^2 - 2}{2x} > \frac{1}{m}$$

Now as $\frac{x^2-2}{2x}$ is positive, $1m$ dpes exist. Hence, $x - \frac{1}{m}$ is also an upper bound that implies $x \neq supS$ if $x^2 > 2$.
Therefore, $supS = x = \sqrt{2}$ and since every $supS \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$

□

1.9 More theorems about R

[Proposition] 3. *If $a < b$ are real numbers, then $\exists c \in \mathbb{Q}, a < c < b$*

Proof. Now our goal is to show that for any real number a, b we can always find such a c. Now take some arbitrary n and set it to $n = [\frac{1}{b-a}] + 1$ Now clearly, $n > \frac{1}{b-a}$ and hence $\frac{1}{n} < b - a$. We will now use this result. Realize that $a = \frac{an}{n} < \frac{[an+1]}{n} \leq \frac{an+1}{n} = a + \frac{1}{n} < a + b - a = b$ Therefore, we have found that $a < \frac{[an]+1}{n} < b$ where $c = \frac{[an]+1}{}$

□

[Proposition] 4. *If $a < b$ are real numbers, then $\exists c \in \mathbb{R} \backslash \mathbb{Q}, a < c < b$*

Proof. Using the above proposition, we know that $\exists c$ for any $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ and now we get that $a < \sqrt{2}c < b$ where $\sqrt{2}c$ is irrational as whenever one term in $x \times y = z$ is rational and the other irrational, we have (supposing x is irrational) $x = \frac{z}{y}$ and if also z were rational, it would make x rational which is a contradiction. □

Definition 2. *The absolute value function x is a function $f : \mathbb{R} \to \mathbb{R}^+$ such that:*

$$\begin{cases} f(x) = x, x \geq 0 \\ f(x) = -x, x < 0 \end{cases}$$

Absolute value respects multiplication and division, that is:

$$|a||b| = |ab|$$
$$\frac{|a|}{|b|} = |\frac{a}{b}|$$

But this doesn't hold for addition. For addition we have the triangle inequality:

$$|x + y| \leq |x| + |y|$$

To prove the above:

Proof. Take $|x + y| < 0$. Then $|x + y| = -(x + y) = -x - y \leq |x| + |y|$ Take $|x + y| \geq 0$. Then $|x + y| = x + y \leq |x| + |y|$ □