Analysis 2 - Thomas Mountford

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1 Notions on $\mathbf{R}^n$

Let's recall that $\mathbb{R}^n$ is a Euclidean vector space. We define a scalar product on $\mathbb{R}^n$ as follows:

Definition 1.

$$< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

1. $< x, x > \geq 0$

2. $< x, y > = < y, x >$

3. $< ax + by, z > = a < x, z > +b < y, z >$

1.1 Introducing topological properties on $\mathbf{R}^n$

A *norm* is defined as function that maps some real vector space E to $\mathbb{R}$ and satisfies:

$$1)\ |x| \geq 0\ \forall x \in E,\ \ |x| = 0 \iff x = 0$$
$$2)\ |\lambda \cdot x| = |\lambda| \cdot |x|$$
$$3)\ |x + y| \leq |x| + |y|$$

In our intuitive understanding of $\mathbb{R}^n$ we are actually thinking about the Euclidian space $\mathbb{R}^n$ equipped with the Euclidian norm.

Definition 2. ***Euclidian norm***

$$|x|_2 = \sqrt{< x, x >} = (\sum_{i}^{n} x_k^2)^{\frac{1}{2}}$$

And from this naturally follows the definition of Euclidian distance:

Definition 3.

$$d(x, y) = |x - y|$$

We note that d satisfies the same 3 properties as the norm. Thus, the couple (E, d) is called a metric space.

And now more definitions:

Definition 4. ***Open sets***

1. ***Open ball*** $B(a, r) := \{x \in \mathbb{R}^n : d(x, a) < r\}$

2. ***Open subset*** *Some subset* $S \subset \mathbb{R}^n$ *is open if* $\forall x \in \mathbb{R}^n,\ \exists \epsilon > 0\ B(x, \epsilon) \subset S$

3. ***Closed subset*** *Some* S *is closed if* $\mathbb{R}^n - S$ *is open, note that the empty set and* $\mathbb{R}^n$ *are both open and closed.*

4. ***The interior and boundary of a set*** *a is in the interior of* S *if* $\exists \epsilon > 0\ B(a, \epsilon) \subset S$ *and b is in the boundary of a set* S *if any* $B(a, \epsilon)$ *contains points from both* S *and* $\mathbb{R}^n - S$*. The set of all interior points is denoted* $(^{\circ}S)$ *and set of all boundary points is denoted* ∂S

5. ***Closure of a set*** *a is a closure of* S *if for any* $B(a, \epsilon)$ *we have* $B(a, \epsilon) \cap S \neq \emptyset$

Definition 5. ***Topology***

A Topology exists whenever the following are satisfied:

$$\textit{For a given } M \subset \mathbb{R}^n \textit{we define } O \subset P(M)$$

1. $\emptyset \in O$, $M \in O$
2. $U \in O, V \in O \to U \cap V \in O$
3. $U_\alpha \in O \to \bigcup_\alpha U_\alpha \in O$

Definition 6. ***Closure of a set*** *A point $a \in \mathbb{R}^n$ is a closure point of S if for any $B(a,\epsilon)$ we have:*

$$B(a,\epsilon) \cap S \neq \emptyset$$

The set of all closure points called the closure of S is denoted $\bar{S}$

Theorem 1.1. *Important results on closures and boundaries*

$$\begin{aligned} S^{\mathrm{o}} \subset S \subset \bar{S} \\ \bar{S} = S^{\mathrm{o}} \cup \partial S \\ S \text{ is open iff } S = S^{\mathrm{o}} \\ S \text{ is closed iff } S = \bar{S} \end{aligned}$$