Analysis 3

1. Week 1: Differential operators

Gradient: For Ω open, $f : \Omega \to \mathbb{R}$ is defined as:

$$\nabla f(x) = (\frac{\partial f}{\partial x_1}, ...)$$

Divergence: For Ω open, $f : \Omega \to \mathbb{R}^n$ is defined as:

$$div F(x) = (\nabla \cdot F)(x) = \frac{\partial F_1}{\partial x_1} + ... + \frac{\partial F_n}{\partial x_n}$$

Rotational: Let $F : \Omega \to \mathbb{R}^n$

if $n = 2$ then:

$$rot F(x, y) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

if $n = 3$ then:

$$rot F(x, y) = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$$

The best way to remember the formula for the case $n = 3$ is by using the determinant formula for the following matrix:

$$\begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Laplacian: Let $F : \Omega \to \mathbb{R}$, then:

$$lap(f) = \Delta f = div(grad(f)) = \frac{\partial F_1^2}{\partial x_1^2} + ... + \frac{\partial F_n^2}{\partial x_n^2}$$

If $\Delta f = 0$ then f is harmonic.

Important result: Let $\Omega \subset \mathbb{R}^n$ and f a scalar map with $f \in C^2$ and $F : \Omega \to \mathbb{R}^n$ with $F \in C^2$ then:

1. $div\ grad f = \Delta f$
2. for $n = 2$, $rot\ grad f = 0$
3. for $n = 3$, $rot\ grad f = \vec{0}$

2. Week 2/3: Line integrals, Greens theorem

$R \subset \mathbb{R}^n$ is a simple regular curve if there exists an interval $[a, b] \subset \mathbb{R}$ and a function $f : [a, b] \to \mathbb{R}^n$ such that:

1. $R = f([a, b])$
2. r is injective on $[a, b[$
3. $r \in C^1$
4. $||r'(t)|| \neq 0$

If the first two properties hold, the curve is said to be *simple*. If the two start and endpoints meet, the curve is said to be *closed*. If all the above hold, then the curve is said to be a *regular* curve. A *regular* curve is one where the velocity is never zero.
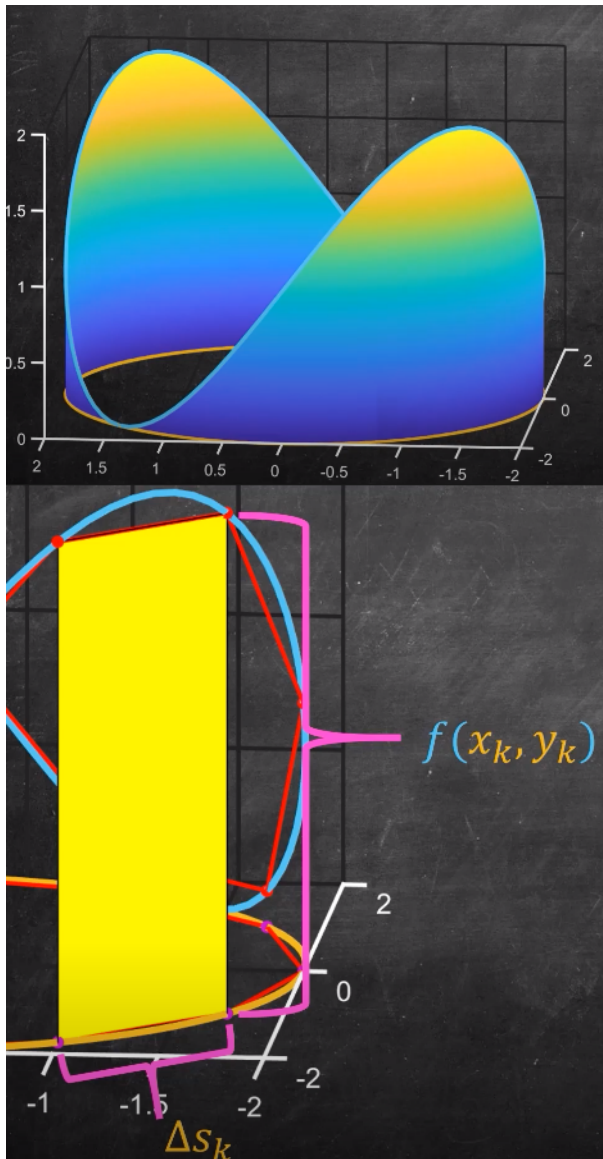
Some visuals below:
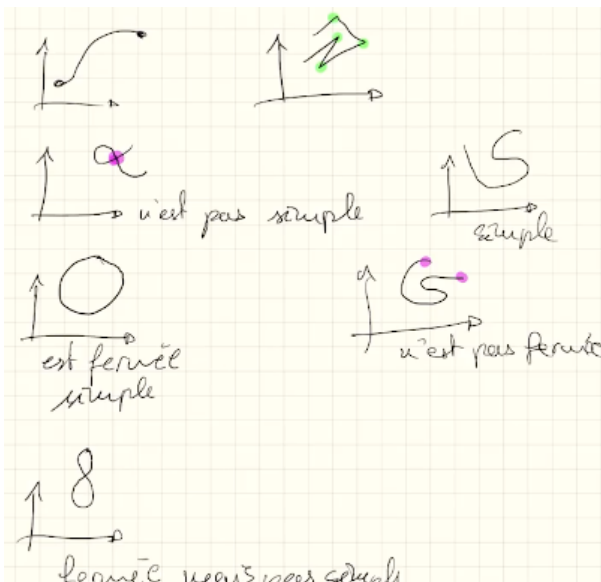
Line integral over a scalar field

Let $r : [a, b] \to \mathbb{R}^n$ and $f : R \to \mathbb{R}$. Then the integral of f along R is defined as:

$$\int_{} f = \int_{a}^{b} f(r(t)) ||r'(t)|| dt$$

Here is some intuition for the line integral:

The best way to see a line integral is thus to realize that it is merely the area encapsulated by some curve which is a function of t with the height given by some other function f.

We find the length of a curve as:

$$longL = \int 1dl$$

that is we let the height function be 1.

Potential field

F is said to be derived from a potential if there exists f such that $F = \nabla f$. Then we also have that $rotF = 0$ that is:

$$\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} = 0$$

But the above condition doesn't guarantee the existance of a potential f. We must have that the domain of f which is Ω must be convex or a simply connected domain.
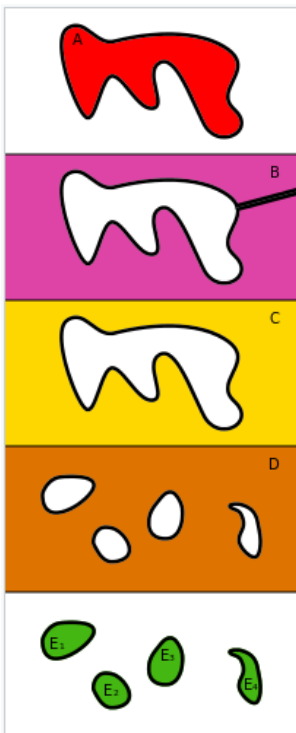
Convex sets

A set $\Omega \in \mathbb{R}^n$ is convex if $\forall A \in \Omega$ and $\forall B \in \Omega$, a straight line joining A and B is also in Ω.
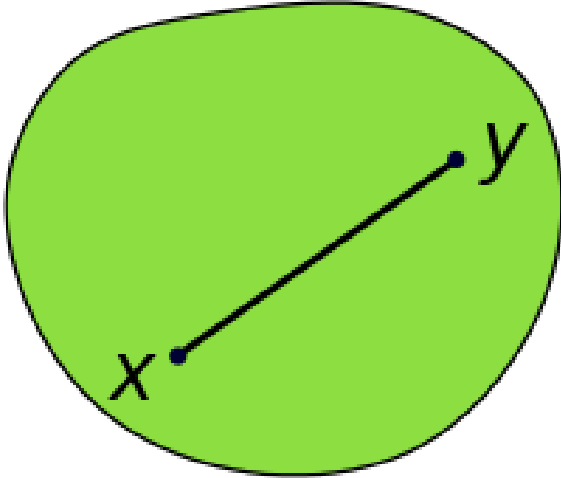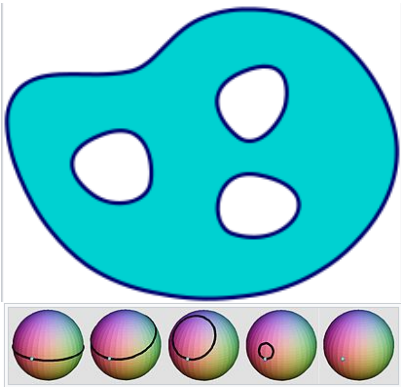

Connected set

A set $\Omega \in \mathbb{R}^n$ is connected if the set cannot be represented as the union of two or more disjoint non-empty open subsets.

All the above sets except for the one in green are connected. The last one is clearly not connected because there is no way of us representing it as the union of two or more disjoint non empty open subsets since we would not get the entire set.

Simply connected sets

We provide a rather informal definition for a simply connected set. A set is said to be simply connected if any simple closed curve can continuously be shrunk to a point.

So for instance a sphere would be a simply connected space whereas the blue segment in $\mathbb{R}^2$ would not be simply connected but in $\mathbb{R}^3$ it would be simply connected.

Simply connected sets in $\mathbb{R}^2$

A simply connected set in $\mathbb{R}^2$ is one that does not have any holes.

Neccesary condition

The neccesary condition for the potential field theorem to hold is that the domain set Ω is connected, and the optional condition is that it is simply connected.

We also have the following:

$$\Omega \text{ is not connected} \implies \Omega \text{ is not simply connected} \implies \Omega \text{ is not convex}$$

Important equivalence

The following two statements are equivalent:

1. F derives from a potential on Ω
2. $\forall A, B \in \Omega$ and curves $R_1, R_2 \in \Omega$ joining A, B we have $\int_{R_1} F dl = \int_{R_2} F dl$
3. $\int_R F dl = 0$ for all simple regular closed curves in Ω.

Determining if F derives from a potential

These are the steps we will follow:

1. Verify that $rot F = 0$ if yes then move to step 2
2. Verify that Ω is convex if yes then we are done, otherwise step 3
3.