Analysis 2 - Thomas Mountford

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Contents

1 Notions on $\mathbf{R}^n$

1.1 Introducing topological properties on $\mathbf{R}^n$

Let's recall that $\mathbb{R}^n$ is a Euclidean vector space. We define a scalar product on $\mathbb{R}^n$ as follows:

Definition 1.

$$< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

1. $< x, x > \geq 0$
2. $< x, y > = < y, x >$
3. $< ax + by, z > = a < x, z > + b < y, z >$

A *norm* is defined as function that maps some real vector space E to $\mathbb{R}$ and satisfies:

$$1)\ |x| \geq 0\ \forall x \in E,\ |x| = 0 \iff x = 0$$
$$2)\ |\lambda \cdot x| = |\lambda| \cdot |x|$$
$$3)\ |x + y| \leq |x| + |y|$$

In our intuitive understanding of $\mathbb{R}^n$ we are actually thinking about the Euclidian space $\mathbb{R}^n$ equipped with the Euclidian norm.

Definition 2. ***Euclidian norm***

$$|x|_2 = \sqrt{< x, x >} = (\sum_i^n x_k^2)^{\frac{1}{2}}$$

And from this naturally follows the definition of Euclidian distance:

Definition 3.

$$d(x, y) = |x - y|$$

We note that d satisfies the same 3 properties as the norm. Thus, the couple (E, d) is called a metric space.

And now more definitions:

Definition 4. ***Open sets***

1. ***Open ball*** $B(a, r) := \{x \in \mathbb{R}^n : d(x, a) < r\}$
2. ***Open subset*** *Some subset* $S \subset \mathbb{R}^n$ *is open if* $\forall x \in \mathbb{R}^n,\ \exists \epsilon > 0\ B(x, \epsilon) \subset S$
3. ***Closed subset*** *Some* S *is closed if* $\mathbb{R}^n - S$ *is open, note that the empty set and* $\mathbb{R}^n$ *are both open and closed.*
4. ***The interior and boundary of a set*** a *is in the interior of* S *if* $\exists \epsilon > 0\ B(a, \epsilon) \subset S$ *and* b *is in the boundary of a set* S *if any* $B(a, \epsilon)$ *contains points from both* S *and* $\mathbb{R}^n - S$. *The set of all interior points is denoted* $(^{\circ}S)$ *and set of all boundary points is denoted* ∂S
5. ***Closure of a set*** a *is a closure of* S *if for any* $B(a, \epsilon)$ *we have* $B(a, \epsilon) \cap S \neq \emptyset$

Definition 5. ***Topology***
A Topology exists whenever the following are satisfied:

$$\text{For a given } M \subset \mathbb{R}^n \text{we define } O \subset P(M)$$

1. $\emptyset \in O$, $M \in O$

2. $U \in O, V \in O \to U \cap V \in O$

3. $U_\alpha \in O \to \bigcup_\alpha U_\alpha \in O$

Definition 6. ***Closure of a set*** *A point $a \in \mathbb{R}^n$ is a closure point of S if for any $B(a, \epsilon)$ we have:*

$$B(a, \epsilon) \cap S \neq \emptyset$$

The set of all closure points called the closure of S is denoted $\bar{S}$

$$\bar{S} = S \cup \partial S$$

Theorem 1.1. *Important results on closures and boundaries*

$$\begin{aligned} S^o \subset S \subset \bar{S} \\ \bar{S} = S^o \cup \partial S \\ S \text{ is open iff } S = S^o \\ S \text{ is closed iff } S = \bar{S} \end{aligned}$$

Definition 7. ***Sequence in R^n***

$f : \mathbb{N} \to \mathbb{R}^n$ such that:

$$x_k = f(k) \in \mathbb{R}^n \forall k \in \mathbb{N}$$

Definition 8. ***Convergence***

The definition of convergence in $\mathbb{R}^n$ is very similar to its counterpart in $\mathbb{R}$. We say that x_k converges to x iff:

$$\forall \epsilon > 0 \; \exists N_\epsilon \; \forall k \geq N_\epsilon \; d_2(x_k, x) < \epsilon$$

Definition 9. ***Complete spaces aka. Banach spaces***
A space is called complete if every Cauchy sequence in this space converges to a limit. Some example of complete spaces are $\mathbb{R}, \mathbb{R}^n$

We restate the axioms of a **Norm** and of a **Metric**. The subtle difference between the two is that a norm can only be applied to some vector space whereas a metric is applicable to other spaces.

Remark 1.1. ***Axioms of norm and metric***
Norm:

1. $||x|| \geq 0$

2. $||\lambda x|| = |\lambda| ||x||$

3. $||x + y|| \leq ||x|| + ||y||$

Metric:

1. $d(x, y) \geq 0$

2. $d(x, y) = d(y, x)$

3. $d(x, y) \leq d(x, z) + d(z, y)$

1.2 Difficult problems and notes from week 2

We note that any open set is equal to its interior.

Example 1.1. *Consider the set defined as* $T = \{(x,y) \in \mathbb{R}^2 | 1 < x^2 + 4y^2 < 4\}$ *We observe the following about this set:*

$$T^{\circ} = T$$
$$\partial T = \{(x,y) \in \mathbb{R}^2 | 1 = x^2 + 4y^2 \, or \, x^2 + 4y^2 = 4\}$$

Remark 1.2. *Rather interestingly, the rational numbers are neither closed nor open because both the irrationals are dense as well as the rationals being dense which was proved in analysis 1. We have the following properties for* $\mathbb{Q}$

$$\partial \mathbb{Q} = \bar{\mathbb{Q}} = \mathbb{R}$$

Definition 10. *If* x_k *is in a bounded closed set* D *then* $\exists$ *a convergent subsequence with limit in* D*.*

Definition 11. ***Continuity open ball definition***
$\forall a \in \mathbb{R}^n$ *we have* $\forall \epsilon > 0 \; \exists \delta > 0$ *such that* $f(B(a,\delta)) \subset B(f(a),\epsilon)$

Definition 12. ***Continuity open set definition***
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ *Then* f *is continuous on* $\mathbb{R}^n$ *iff. for all* $O \subset \mathbb{R}^m$ *where* O *is open, we have* $f^{-1}(O)$ *open in* $\mathbb{R}^n$

Proof. **(only if part)** Let f be continuous, we want to show that whenever O is open, $f^{-1}(O)$ is also open. Now take $f^{-1}(O)$. Let $p \in f^{-1}(O)$. Then $f(p) \in O$ Now we have that O is open hence $\exists \epsilon > 0$ s.t. $B(f(p),\epsilon) \subset O$ Now because f is continuous at p we have that $\exists \delta > 0$ s.t. $\underbrace{f(B(p,\delta)) \subseteq B(f(p),\epsilon)}_{\text{applying open ball cont.}} \subseteq O$
This further shows that $B(p,\delta) \subseteq f^{-1}(O)$ which shows that f is open in S.

(if part) Now suppose that for all open sets $O \in \mathbb{R}^m$ we have $f^{-1}(V)$ are open in S. To show that f is continuous, we must show f is continuous on all $p \in S$. Let $p \in S$ s.t. $y = f(p)$. Then $\forall \epsilon > 0$ $B(y,\epsilon)$ is open. By assumption then, $f^{-1}(B(f(p),\epsilon))$ is open in S. Now, $f(p) \in B(f(p),\epsilon)$ we have $p \in f^{-1}(B(f(p),\epsilon))$ is open in S, $\exists \delta > 0$ s.t. $B(p,\delta) \subseteq f^{-1}(B(f(p),\epsilon))$ and applying f to both sides we get our result. ∎