Analysis

SemesterOne analysis at EPFL

1 Proofs and the reals

1.1 Some general proofs

A valid proof is set of lines where each line logically follows from the next. A most famous proof is that $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $gcd(a, b) = 1$
Now we have that $\sqrt{2}b = a$ which means that $2b^2 = a^2$. As result, $2|a^2$ hence also $2|a$. Thus we get that $a = 2k$ which also means that $b^2 = 2k^2$ hence $2|b$. As result, $gcd(a, b) = 2$ which is a contradiction. Therefore, $\sqrt{2}$ must be irrational. □

Quite interestingly, we can also construct a 'wrong' proof just through one fallacious assumption and a set of correct steps.

Claim: 1 is the largest integer.
Proof:
Let n be the largest integer. Then we have $n \geq n^2$. Which also means $0 \geq n^2 - n = n(n-1)$. Now we have that either $n < 0$ or $n - 1 < 0$. But we know that $n \nless 0$ as n is at least 1. Hence, $n - 1 < 0$ giving us the result $n < 1$ proving our theorem. Note that the mistake here is solely the assumption we made at the start that there was a largest integer.

1.2 Proofs relating to infinite processes

Consider the claim that $0.999\ldots = 1$. One way to prove this claim, rather naively is this.

$$\begin{aligned}
&9 \times 0.999\ldots \\
&= (10 - 1) \times 0.999\ldots \\
&= 9.999\ldots - 0.999\ldots = 1
\end{aligned}$$

Now a more formal proof is to use an infinite sum and limits. Here it is.

Analysis proof of $0.999\ldots = 1$

$$0.999\ldots = 9 \lim_{k\to\infty} \sum_{i=1}^{k} (10^{-k})$$

$$\begin{aligned}
\lim_{k\to\infty} \sum_{i=1}^{k} (10^{-k}) &= \frac{10^{-1} - 10^{-(k+1)}}{1 - 10^{-1}} \\
&= 9 \times \frac{1}{10} \times \frac{10}{9} \\
&= 1
\end{aligned}$$

1.3 Basic notions of sets

The breakdown of sets used in 'standard' analysis are $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.
There are also some common set related notation that must be known.

- a **subset** $a \subseteq b$ is defined as $\{x \in b | \text{"condition"}\}$
- a **open interval** is defined as $]a, b[; r \in A, a < r < b$
- an **open ball** $B(a, \lambda) =]a - \lambda, a + \lambda[$

1.4 The Reals

The reals, denoted R are an ordered field. Here is a more precise definition.

The reals are a set that have the 3 following axioms:

- R is an abelian group under (+) and R* is an abelian group under (×). In addition to this, multiplication distributes over addition.
- The order relation $\leq$ holds $\forall x \in \mathbb{R}$ That is:

$$\begin{aligned} x \leq y \otimes y \leq x \\ x \leq y, y \leq x \implies x = y \\ x \leq y \implies \forall a \in \mathbb{R},\ x + a \leq y + a \\ 0 \leq x, 0 \leq y \implies 0 \leq xy \end{aligned}$$

- The inf and sup axioms hold

We shall now come the **inf** and **sup** axioms. It should be intuitively clear that any subset of R

1.5 Bounds

Take some subset S in R. An element B is called an upper bound of S if $\forall x \in S, B \geq x$. Similarly, it is a lower bound of S if $\forall x \in S, B \leq x$.
The maximum B of a set S denoted $max(S)$ is such that $B \in S, \forall x \in SB \geq x$.
The supremum of a set S(if it exists) is the lowest upper bound. That is $sup(S) = b$ is such that,

$$\forall x \in S, b \geq x \tag{1.5.1}$$
$$\forall \epsilon > 0, \exists x_\epsilon, b - x_\epsilon \leq \epsilon \tag{1.5.2}$$

Remark 1.1. *In our above definition, b does not have to be in S.*

Remark 1.2. *Condition 1 states that b is an upper bound of S.*

Remark 1.3. *Given condition 1, b is the minimum of the upper bounds of S.*

Some examples of sup and inf

Example 1.1.

$$\begin{aligned} Sup]a, b[= b \\ Inf]a, b[= a \\ Sup\{x \in \mathbb{R} | x = 2k\} \implies Sup\ doesn't\ exist. \end{aligned}$$

We now establish the infinimum axiom.

Axiom 1. *All non-empty subsets of $\mathbb{R}^*_+$ have a highest lower boundary(aka.infinimum)*

1.6 Q is dense in R

We claim that between every real number, one can find a rational number. Here's the proof,

Proof. Let $x < y \in \mathbb{R}$ Suppose now that $\exists a \in \mathbb{Q}$ such that $x < a < y$. By the Archimedean principle(there is always a greater natural number, $n > \frac{1}{y-x}$ which implies $ny > nx + 1$ Now since $ny > nx + 1$, there is guaranteed to be some integer in the open bound $]nx, ny[$ which we denote P. Dividing by n, we get that $\frac{P}{n} \in]x, y[$ which proves the theorem. □

1.7 Integer and fractional part

Any number $\in \mathbb{R}$ has a integer and fractional part(at least intuitively). Let's formally define these. For some $x \in \mathbb{R}$, let $S := \{n \in \mathbb{N} | n > x\}$ Now since S is bounded from below, letting N be the minimum of this set, we obtain that $N \notin S$. N-1 is thus called the integer part of x denoted [x]. ie. [6.4] = 6
Similarly, the fractional part of x denoted x is simply $x = x - [x]$

1.8 Pinning it down:Sup/Inf, bounds, max/min

Definition 1. *for a given set $S \subseteq \mathbb{R}$, we have the following:*
Sup s = b $\iff \forall \epsilon > 0, \exists x_\epsilon \in S, s.t. b - x_\epsilon < \epsilon$ *(resp. Inf s has the flipped argument)*
Upper bound = b $\iff \forall x \in S, b \geq x$*(resp. lower bound)*
Max s = b $\iff$ *b is an upper bound and* $b \in S$

And for the sake of repeating the early axiom(but very important) the infimum axiom is:

Axiom 2. *For all non-empty subsets of $\mathbb{R}$, $infS$ exists.*

Now we make the first claim in this course that uses an epsilon proof.

[Proposition] 1. *Whenever $S \subseteq \mathbb{N}$, then $infS = minS$*

Proof. Now, by our axiom, we have that $\mathbb{N} \subseteq \mathbb{R}$ hence we know that $infS$ exists. We now have to show that $infS = minS$.

Suppose that $infS \neq minS$ and let $infS = b$. Now clearly, $b + \epsilon$ is not a lower bound of S. Now, let $\epsilon = \frac{1}{2}$. Because, $b + \epsilon$ is not a lower bound, we know that $\exists s_e < b + \epsilon$.

Now s_e is also not a lower bound, so let's pick $\epsilon'' = s_\epsilon - d$. Now again, $s_{\epsilon''}$ must exist. We obtain yet the following:

$$d < s_{\epsilon''} < s_\epsilon < d + 1/2$$

Now, two natural numbers clearly can not be in an interval which is only $\frac{1}{2}$ units long. Hence, contradiction which means that $d \in S$ □

Let's now prove that $\sqrt{2}$ belongs to the reals. For this, we need the following corollary and axiom.

Corollary 1. *Every non-empty subset of R with an upper boundary admits a supremum.*

Axiom 3. *An ordered field F, which R is, has the Archimedean property if given any positive x and y in F, $\exists n \in \mathbb{Z}$ s.t. $nx > y$*

[Proposition] 2. $\sqrt{2} \in \mathbb{R}$

Proof. Suppose we define a set $S = \{r \in \mathbb{R} | r \geq 0, r^2 < 2\}$. Now, as S is a non-empty subset of $\mathbb{R}$ bounded from above(i.e. 2 is an upper bound) we know that $supS = x$ exists. Our goal is to show that both $x^2 < 2$ and $x^2 > 2$ lead to a contradiction.

Case 1: Suppose $x^2 < 2$. We want to find $x + \frac{1}{2} \in S$ which implies that x is not an upper bound as $x < x + \frac{1}{2}$

$$(x + \frac{1}{2})^2 = x^2 \frac{2}{x} + \frac{1}{n^2} \leq x^2 + \frac{2}{x} + \frac{1}{n} = x^2 + \frac{1}{n}(2x + 1)$$

Now, we want to show that we can pick an n s.t. $x^2 + \frac{1}{n}(2x + 1) < 2$. If we can pick such an n, then we know by transitivity of $<$ that $(x + \frac{1}{n})^2 < 2$ as $x^2 \frac{2}{x} + \frac{1}{n^2} \leq x^2 + \frac{2}{x} + \frac{1}{n}$
Reordering the terms, we get $\frac{1}{n} < \frac{2-x^2}{2x+1}$ and clearly, $\frac{2-x^2}{2x+1}$ is positive as $x^2 < 2$ and $x \geq 0$. This way, we apply the archimedean property to know that n exists s.t. $\frac{1}{n} < \frac{2-x^2}{2x+1}$. Given this, we now know that $x^2 + \frac{1}{n}(2x+1) < 2$ which in turn implies $x + \frac{1}{n} \in S$ This contradicts that $x = supS$ hence $x^2 \not< 2$

Case 2: In turn, we consider the case where $x^2 > 2$ and try to derive a contradiction. We want to show that $\exists m \in \mathbb{N}$ s.t. $x - \frac{1}{m}$ is also an upper bound of S which would mean that $x \neq infS$. Now:

$$(x - \frac{1}{m})^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}$$

We want to choose m s.t. $x^2 - \frac{2x}{m} > 2$. This way, if $x^2 - \frac{2x}{m} > 2$ holds, then since $(x - \frac{1}{m})^2 > x^2 - \frac{2x}{m}$, we will have that $x - \frac{1}{m}$ is an upper bound. We obtain:

$$\frac{x^2 - 2}{2x} > \frac{1}{m}$$

Now as $\frac{x^2-2}{2x}$ is positive, $1m$ dpes exist. Hence, $x - \frac{1}{m}$ is also an upper bound that implies $x \neq supS$ if $x^2 > 2$.
Therefore, $supS = x = \sqrt{2}$ and since every $supS \in \mathbb{R}$, $\sqrt{2} \in \mathbb{R}$

□

1.9 More theorems about R

[Proposition] 3. *If $a < b$ are real numbers, then $\exists c \in \mathbb{Q}, a < c < b$*

Proof. Now our goal is to show that for any real number a, b we can always find such a c. Now take some arbitrary n and set it to $n = [\frac{1}{b-a}] + 1$ Now clearly, $n > \frac{1}{b-a}$ and hence $\frac{1}{n} < b - a$. We will now use this result. Realize that $a = \frac{an}{n} < \frac{[an+1]}{n} \leq \frac{an+1}{n} = a + \frac{1}{n} < a + b - a = b$ Therefore, we have found that $a < \frac{[an]+1}{n} < b$ where $c = \frac{[an]+1}{}$

□

[Proposition] 4. *If $a < b$ are real numbers, then $\exists c \in \mathbb{R} \backslash \mathbb{Q}, a < c < b$*

Proof. Using the above proposition, we know that $\exists c$ for any $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ and now we get that $a < \sqrt{2}c < b$ where $\sqrt{2}c$ is irrational as whenever one term in $x \times y = z$ is rational and the other irrational, we have (supposing x is irrational) $x = \frac{z}{y}$ and if also z were rational, it would make x rational which is a contradiction. □

Definition 2. *The absolute value function x is a function $f : \mathbb{R} \to \mathbb{R}^+$ such that:*

$$\begin{cases} f(x) = x, x \geq 0 \\ f(x) = -x, x < 0 \end{cases}$$

Absolute value respects multiplication and division, that is:

$$|a||b| = |ab|$$
$$\frac{|a|}{|b|} = |\frac{a}{b}|$$

But this doesn't hold for addition. For addition we have the triangle inequality:

$$|x + y| \leq |x| + |y|$$

To prove the above:

Proof. Take $|x + y| < 0$. Then $|x + y| = -(x + y) = -x - y \leq |x| + |y|$ Take $|x + y| \geq 0$. Then $|x + y| = x + y \leq |x| + |y|$ □

2 Sequences

2.1 Basics

Let's begin by formally defining sequences. A sequence is a function $f : \mathbb{N} \to \mathbb{R}$ generally denoted $(x_n)_{n\geq 0}$ And here are some more definitions on sequences:
A sequence is:

- constant if $\exists\ C \in \mathbb{R};\ x_n = C\ \forall\ n \in \mathbb{N}$
- bounded from below(resp. above) if $\exists\ m \in \mathbb{R}\ ; m \leq x_n \forall\ n \in \mathbb{N}$
- bounded if bounded from both directions
- increasing(resp. decreasing) if $x_{n+1} \geq x_n \forall\ n \in \mathbb{N}$
- strictly increasing(resp.decreasing) if $x_{n+1} \leq x_n \forall\ n \in \mathbb{N}$
- monotonous if it is increasing or decreasing(resp. strictly)

Let's now consider a proof on the following proposition:

Define $x_n = \sqrt{4 + x_{n-1}},\ x_0 = 1$. We claim that x_n is bounded and more precisely that $1 \leq x \leq 3$.

Proof. Base case: $x_0 = 1$ hence holds.
Now supposing proposition is true for all $n - 1$ we get

$$1 \leq x_{n-1} \leq 3$$

$$5 \leq x_{n-1} + 4 \leq 7$$

Now we get:

$$\sqrt{5} \leq \sqrt{x_{n-1} + 4} \leq \sqrt{7}$$

$$1 \leq \sqrt{5} \leq \sqrt{x_{n-1} + 4} \leq \sqrt{7} \leq 3$$

□

Definition 3. *A sequence x_n conveges to x if $\forall \epsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $n \geq n_0 \to |x_n - x| \leq \epsilon$*

Now in an intuitive sense, suppose the sequence converges to x from both the right and the left. x being our central point, we move a distance ϵ away from this x. Now, if we can pick some n_0 such that for another $n \geq n_0$ we have that $|x_n - x| \leq \epsilon$ this means that no matter how small we make epsilon we are able to find some x_n in this region.
Having established convergence, any sequence that is not convergent is said to be **divergent**.
Let's now prove that a sequence is divergent.

Proof. Take $x_n = (-1)^n$ Now suppose that x_n converges to x. Then for $\epsilon = \frac{1}{2}$, $\exists n_{\frac{1}{2}} \in \mathbb{N}$ s.t. $\forall n \geq n_{\frac{1}{2}}$ we would have $|x_n - x| \leq \frac{1}{2}$. In particular if n' is any other integer $n' > n_{\frac{1}{2}}$ then we would have $x_n - x_{n'} \leq |x_n - x| + |x - x_{n'}| \leq \frac{1}{2} + \frac{1}{2}$ which implies a contradiction as $|x_n - x_{n+1}| = 2 > 1$ □

2.2 Limits and their algebra

Definition 4. *If a sequence x_n converges to some x, we say that x is the **limit** of the sequence and is denoted* $\lim_{n\to\infty} x_n = x$

Here are some properties of limits:

For sequences x_n and y_n with limits x, y we have:

- $$\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$$
- $$\lim_{n\to\infty} x_n + y_n = x + y$$
- $$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}; y \neq 0$$
- if $\exists\ n_0 \in \mathbb{N},\ x_n \leq y_n\ \forall n \geq n_0$ then $x \leq y$

We now introduce the famous **squeeze theorem** and prove it.

Theorem 2.1. *Squeeze theorem*
Let a_n and b_n be both sequences that converge to a. In addition, let c_n be such that $\exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ a_n \leq c_n \leq b_n$ Then we clearly have the following:

Proof.

$$\forall \epsilon > 0,\ \exists N_1,\ n \geq N_1 \to |a_n - a| < \epsilon \equiv a - \epsilon < a_n < a + \epsilon \tag{2.2.1}$$

Similarly, for b_n we have that:

$$\forall \epsilon > 0,\ \exists N_2,\ n \geq N_2 \to |b_n - a| < \epsilon \equiv b - \epsilon < b_n < a + \epsilon \tag{2.2.2}$$

Now set $N = \max\{N_1, N_2, n_0\}$. Now since $N \geq N_1, N_2, n_0$ we have that both 2.2.1 and 2.2.2 hold $\forall n > N$ This further gives us the result that:

$$a - \epsilon < a_n \leq c_n \leq b_N < a + \epsilon$$
$$|c_n - a| < \epsilon$$

□

We now list some useful inequalities that may be used along with the squeeze theorem and also a sample limit problem and a solution to it:

$$0 \leq \sin(x) \leq x \leq \mathrm{tg}(x) \quad \Rightarrow \quad 1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)} \quad \Rightarrow \quad \cos(x) \leq \frac{\sin(x)}{x} \leq 1$$
$$\Rightarrow \quad \cos(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \quad \Rightarrow \quad 1 - \sin(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1$$
$$\Rightarrow \quad 1 - x^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \quad \Rightarrow \quad \sqrt{1 - x^2} \leq \frac{\sin(x)}{x} \leq 1\ .$$

Figure 1: Useful inequalities

(c) We have

$$\sqrt{1-\left(\frac{2n+3}{n^3}\right)^2} \leq \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}} \leq 1 \ .$$

Like part $ii)$ we show that

$$\lim_{n\to\infty} \sqrt{1-\left(\frac{2n+3}{n^3}\right)^2} = 1 \ ,$$

According to the two policemen theorem

$$\lim_{n\to\infty} \frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}} = 1 \ .$$

So

$$\begin{aligned}\lim_{n\to\infty}\left(n \sin\left(\frac{2n+3}{n^3}\right)\right) &= \lim_{n\to\infty}\left(\frac{2n+3}{n^2}\cdot\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right) \\ &= \left(\lim_{n\to\infty}\frac{2n+3}{n^2}\right)\cdot\left(\lim_{n\to\infty}\frac{\sin\left(\frac{2n+3}{n^3}\right)}{\frac{2n+3}{n^3}}\right) = 0\cdot 1 = 0 \ .\end{aligned}$$

Note that we can split the limits because one is bounded and the other one converges to zero.

Figure 2: Solution to hard limit problem

2.3 More on sequences

Suppose we want to show that some geometric sequence does not have a limit, simply that it is not converging.

We first establish the *Bernoulli inequality* which we shall prove later. We will use this result immediately.

$$q^n \geq 1 + n(q-1) \tag{2.3.1}$$

Take $x_n = 4^n$. Now we claim that x_n is not bounded.

Proof. It is not bounded if we can show that it increasing in increments that do not decrease. Suppose now that b is some upper bound to x_n. Now by Bernoulli, we have that $4^n \geq 1 + n\cdot 3$. If we can show that $4^n \geq 1 + n \cdot 3 > b$ we indeed get that there can be no upper bound b. We have that $n > \frac{b-1}{3}$ and such an $\in \mathbb{N}$ exists if we set it to $n := [\frac{b-1}{3} + 1]$ Now since $4^n \geq 1 + n \cdot 3$ we have that x_n is not bounded. $\square$

Theorem 2.2. *Every converging sequence is bounded(a lower or upper bound exists obviously and if convergent the latter also exists.)*

Let's now get on to proving the rules we established for limit arithmetic.

Proof. (Proof to sum rule) Now we are given that x_n converges to x and that y_n converges to y. We want to show that $x_n + y_n$ converges to $x + y$. By definition, this is true if we can show that

$$|(x_n + y_n) - (x + y)| \leq \epsilon,\ \forall \epsilon \in \mathbb{R}$$

Now luckily we have that $|(x_y + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$ And by the triangle inequality we know that $|(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$ Thus if we can show that $|x_n - x| + |y_n - y| < \epsilon$ we are guaranteed that $|(x_n - x) + (y_n - y)| < \epsilon$ We will succeed with the latter part if we can show that both parts $(x_n - x)$ and $(y_n - y)$ are smaller than $\frac{\epsilon}{2}$. Now we have by definition of convergence that:

$$\exists n \geq n^x_{\frac{\epsilon}{2}} \to |x_n - x| \leq \frac{\epsilon}{2}$$

Similarly:

$$\exists n \geq n^y_{\frac{\epsilon}{2}} \to |y_n - y| \leq \frac{\epsilon}{2}$$

Now fixing $n_\epsilon := \max n^x_{\frac{\epsilon}{2}}, n^y_{\frac{\epsilon}{2}}$(we do this since it assures both conditions to hold we have that because each of $(x_n - x)$ and $(y_n - x)$ are smaller than ϵ, so must $|x_n - x| + |y_n - y|$ by the triangle inequality. □

Let's now do more applications of the squeeze theorem to show limits.

Example 2.1. *We want to show that aq^n converges to 0 for $a \neq 0$ and $|q| < 1$. Now as a property we use that $\lim_{n\to\infty} x_n = 0 \to \lim_{n\to\infty} |x_n| = 0$ It is clear that we have an inequality of the form $0 \leq |aq^n| \leq ?$ Doing more algebra(our goal is to find another sequence of form $\frac{1}{x}$ converging to 0 to get:*

$$\frac{1}{?} \leq \frac{1}{|a \cdot q^n|} = \frac{1}{|a|} \cdot (\frac{1}{|q|})^n$$

Now using Bernoulli we have:

$$(\frac{1}{|q|})^n \geq 1 + n(\frac{1}{|q|} - 1)$$

which happily means:

$$(\frac{1}{a} \cdot \frac{1}{|q|})^n \geq (\frac{1}{a} \cdot (1 + n(\frac{1}{|q|} - 1))$$

And finally taking the reciprocal all to get back to aq^n we are left with

$$0 \leq |aq^n| \leq \frac{1}{\frac{1}{a} \cdot (1 + n(\frac{1}{|q|} - 1)}$$

and clearly we see that the RHS is also a sequence that converges to 0 since all terms in the denominator but n are constants. Hence, we have ***squeezed*** *our sequence. The key here was that we found a RHS sequence which we wanted to be of form $\frac{1}{x}$ And in addition, we took the reciprocal of the inequality at the start simply to be able to use the bernoulli inequality.*

Let's now consider a harder example.

Example 2.2. *Consider the sequence $x_n = \sqrt[n]{n}$ Now is this sequence converging? Well we know that $1 \leq \sqrt[n]{n}$ and now another sequence we know which approaches to 1 is $1 + \frac{1}{\sqrt{n}}$. Now we only need to show that:*

$$\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}}$$

holds and if we can show this, we'll have that our sequence converges to 1.

Now we get:

$$n \leq (1 + \frac{1}{\sqrt{n}})^n$$

And notice how $(1 + \frac{1}{\sqrt{n}})^n$ *is simply a binomial hence if any one of the terms in the binomial expansion is greater than n our inequality will hold(this works as all n are positive). Well for* $\sum_{i=0}^{n} \binom{n}{i} (\frac{1}{\sqrt{n})^i}$ *when observe that for* $i = 4$ *we get:*

$$\frac{(n-1)(n-2)(n-3)}{24n} \geq n$$

further reducing to:

$$\frac{24n^2}{(n-1)(n-2)(n-3)} \leq 1$$

and we know this is valid as

$$\lim_{n\to\infty} \frac{24n^2}{(n-1)(n-2)(n-3)} = 0$$

Hence we get that

$$\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}}$$

meaning that $\sqrt[n]{n}$ *converges to 1.*

Let's do one last example:

Example 2.3. *Consider* $x_n = \frac{2^n}{x!}$ *Now clearly* $0 \leq \frac{2^n}{x!}$ *and observing that* $\frac{2^n}{x!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \ldots$ *it is obvious that* $\frac{2^n}{x!} \leq 2 \cdot (\frac{2}{3})^{n-2}$ *But we know that* $\lim_{n\to\infty} 2 \cdot (\frac{2}{3})^{n-2} = 0$ *hence we have again squeezed our sequence!*

Theorem 2.3. *If* $\lim_{n\to\infty} x_n = 0$ *and* y_n *is bounded, then* $\lim_{n\to\infty} x_n \cdot y_n = 0$

Theorem 2.4. *If a sequence is bounded and increasing(monotone) it converges to the supremum (resp. infimum).*

Proof. Now because y_n is bounded, we have that $\exists M$ s.t. $M \geq y_n \forall n$. Hence we obtain that $x_n y_n \leq M x_n$ and clearly since $M x_n$ converges to 0 we have squeezed $x_n y_n$ and have that it also converges to 0. □

We consider the famous sequence of **Fibonacci quotients**
Now the fibonacci sequence is defined as $x_0 = x_1 = 1$, $x_{n+1} = x_n + x_{n-1}$ and we define the sequence of fibonacci quotients as $y_n = \frac{x_{n+1}}{x_n}$ Our first theorem is that fibonacci quotient sequence is bounded between 1 and 2. Let's now fnd it's limit. Notice firstly that $y_{n+1} = 1 + \frac{1}{y_n}$

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = 1 + \frac{1}{\lim_{n\to\infty} y_n}$$

Hence we get:

$$y = 1 + \frac{1}{y}$$

which gives $\frac{1+\sqrt{5}}{2}$ as the only valid solution. But we still have to show that our sequence converges. A smart way to do so is to show that $z_n := |y_n - \frac{1+\sqrt{5}}{2}|$ converges which by limit arithmetic would imply $\lim_{n\to\infty} y_n = \frac{1+\sqrt{5}}{2}$ Our goal is to squeeze z_n in doing so. Notice that $z_{n+1} = |y_{n+1} - \frac{1+\sqrt{5}}{2}| = \underbrace{1 + \frac{1}{y_n} - (1 + \frac{2}{1+\sqrt{5}})}_{\text{using definition}}$ This yields the inequality $z_{n+1} = \frac{|y_n - \frac{1+\sqrt{5}}{2}|}{y_n \frac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} \frac{|y_{n-\frac{1+\sqrt{5}}{2}}|}{y_n}$ to give us that $z_{n+1} \leq \frac{2}{1+\sqrt{5}} |y_{n-\frac{1+\sqrt{5}}{2}}|$ because we know that $y_n \geq 1$. Now finally this simplifies to $z_{n+1} \leq \frac{2}{1+\sqrt{5}} z_n$ which applying the definition of z_n gives $z_n \leq \frac{2}{1+\sqrt{5}}^2 z_{n-1}$ to result in $z_n \leq \frac{2}{1+\sqrt{5}}^n z_0$ which converges to 0. Hence we have squeezed z_n

In general this is the scheme one should use for finding limits of recursive sequences:

(Limit of recursive sequence)

1. Assuming there is a limit, compute it using limit algebra.
2. Showing some upper and lower bound(using induction) exclude any extra answers.
3. Finally show that the sequence converges by showing that $\lim_{n\to\infty} x_n - x = 0$

We now come to define **approaching infinities**.

Definition 5. *The definition of approaching ∞ is as intuitive as saying for any real number I pick, the sequence has a term larger than it. Hence we define $\lim_{n\to\infty} x_n = \infty$ as $\forall A \in \mathbb{R}\ \exists n_A \in \mathbb{N}$ such that $n \geq n_A$ and $x_n \geq A$. The similar definition applies for approaching $-\infty$*

Example 2.4. *Notice that for a geometric sequence $x_n = aq^n$ if $a > 0$ and $q > 1$ it approaches ∞ and if $a < 0$ and $q > 1$ it approaches $-\infty$*

A set of useful theorems on approaching infinities is the following:

Theorem 2.5. *(Theorems on approaching infinity)*
If $\lim_{n\to\infty} x_n = \infty$ and y_n is bounded from below, then $\lim_{n\to\infty} x_n + y_n = \infty$
If x_n and y_n both approach infinity, so does their product.
If y_n is bounded and x_n approaches infinity, then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$

We note that when it is the case that $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} x_n = -\infty$ we may have different cases such as:

$$\lim_{n\to\infty} \underbrace{n}_{x_n} + \underbrace{(-n)}_{y_n} = 0$$

$$\lim_{n\to\infty} \underbrace{2n}_{x_n} + \underbrace{(-n)}_{y_n} = \infty$$

$$\lim_{n\to\infty} \underbrace{2n + (-1)^n n}_{x_n} + \underbrace{(-2n)}_{y_n} = (-1)^n n \text{ which is unbounded hence does not approach anything}$$

Theorem 2.6. *(Squeeze theorem for approaching infinities) For sequences x_n and y_n if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $x_n \leq y_n$ we have:*
(1) If $\lim_{n\to\infty} x_n = \infty$ then $\lim_{n\to\infty} y_n = \infty$
(2) If $\lim_{n\to\infty} y_n = -\infty$ then $\lim_{n\to\infty} x_n = -\infty$

Theorem 2.7. *(D'Alembert Theorem)*
$\forall x_n \neq 0$ and $\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \infty$ then we have that x_n diverges.

And now we provide an example of sandwich for infinities.

Example 2.5. *Define $x_n := \frac{x!}{2^n}$ We have that $\frac{n}{2} \cdot \frac{n-1}{2} \ldots \cdot \frac{3}{2} \ldots \geq \frac{n}{2} \cdot \frac{3}{2}^{q-1}$ for some n. And because the latter is a geometric sequence which noticeably approaches infinity we have that x_n approaches infinity.*

Proof. (We present a proof to theorem 2.4)
Now set $S := sup\{x_n | n \in \mathbb{N}\}$ and let $0 < \epsilon \in \mathbb{R}$ By definition S is the smallest bound hence $S - \epsilon$ is not the smallest bound. By def. again, $\exists n_\epsilon$ such that $S - \epsilon < x_{n_\epsilon}$ and we now get (for $n \geq n_\epsilon$

$$S - \epsilon < x_{n_\epsilon} < x_n < S < S + \epsilon$$

and this is exactly the definition of convergence hence S is the limit. □

(Exploration of e)

We define $x_n = (1+\frac{1}{n})^n$ We first ask is x_n increasing. Consider the claim $(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}$
Now let's consider both the expansions of $(1+\frac{1}{n})^n$ and of $(1+\frac{1}{n+1})^{n+1}$

$$(1+\frac{1}{n})^n = \sum_{i=0}^{n}\binom{n}{i}(\frac{1}{n^n}) = \sum_{i=0}^{n}\frac{1}{i!}\frac{n\cdot(n-1)\dots(n-i+1)}{n^i} = \sum_{i=0}^{n}\frac{1}{i!}(1-\frac{1}{n})\dots(1-\frac{i-1}{n})$$

Similarly we have:

$$(1+\frac{1}{n+1})^{n+1} = \sum_{i=0}^{n}\frac{1}{i!}(1-\frac{1}{n+1})\dots(1-\frac{i-1}{n})$$

and because generally $\frac{q}{j} > \frac{q}{j+1}$ we have that terms on the RHS of the first expression are larger hence $(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}$ And we also have that our sequence is less than 3(which we do not show).
Thus we define

$$\lim_{x\to\infty}(1+\frac{1}{n})^n = e$$

And now we present an example of a bounded sequence that is decreasing which converges to its infimum. Consider $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_2})s$ and $x_0 = 2$.
We claim 1 is a lower bound which is true by induction(easy to show). Similarly, the sequence is decreasing as $x_n - x_{n+1} = \frac{1}{2}(x_n - \frac{1}{x_2} \geq 0$ since $x_n \geq 1$ And applying the recursive formula as usual we find that the limit is either -1 or 1 and ruling out -1 we get 1.