NUMBER THEORETIC STUDY OF THE FIRST NON-ZERO DIGIT OF N!

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Contents

1. Introduction

In this study our goal is examine the factorial function and a property it takes. That is, after inputs of $n \geq 5$ our number ends with zeros. Our goal is to remove these zeros and determine the first non-zero digit. This is an interesting question mainly because it can be solved using number theory. From this, we pose the following research question; **Does there exist a number theoretic algorithm to obtain the first non-zero digit of n!**

Overall, the reason why we require an algorithm to answer this question follows from the fact that $n!$ becomes laboriously difficult to compute for large numbers. Thus, we follow a method where we first try to come up with our own algorithm. This way, we ensure some proximity to the question and have a more established background when beginning a research of what others have done. After having found our own algorithm, we make use of computer assisted findings as well to further guide our intuition. This idea was proposed to us by professor Oguzhan Kulekci who is a proffesor of computer science. Indeed his proposal was useful because the computer assisted results signal to a pattern in the first non-zero digit of $n!$ which otherwise could not have been observed. Finally, we reach the peak of this study where we explain an algorithm found by S.Kakutani.

The significance of this question is twofold. Initially, I began investigating the question just out of pure interest and the fact that the question was easy to understand aided the process. However, later on, upon understanding the research that had been done on the question, my appreciation of it skyrocketed. That is, this question is a perfect exemplar of the beauty that can arise from mathematics. Furthermore, having studied it, I have witnessed that mathematical beauty or perfection exists in many more things than we normally assume. What I mean by this is seen in the last section of this study where we explain an algorithm found by the mathematician S.Kakutani. On a more academic context, this question has been of interest to number theory and statistical mechanics. In the mathematical world, many authors have rigorously studied this question which is also the reason the second part of this essay exists. We mainly use findings of S.Kakutani and G.Dreseden which we found from the online encyclopedia of integer sequences [1]. We also add the subtle note that S.Kakutani studied this problem for his particular interest in Ergodic theory which is a branch of statistical mechanics where he describes an algorithm he found in his paper called ‘Ergodic theory of Shift transformations’ [2].

2. Notation

To simplify our use of language in this study we ~~arbitrarily~~ will use the following definitions.

Definition 2.1. $n!$ is recursively defined as $n! = n(n-1)!$ where $0! = 1$

Definition 2.2. $\lambda(n!) =$ first non-zero digit of $n!$

Definition 2.3. $p(n) =$ number of zeros at the end of n!

Definition 2.4. Floor division: $\left\lfloor \frac{a}{b} \right\rfloor = c$ where c is the number of times that b goes into a ignoring a remainder

Definition 2.5. ex(p) = the exponent that some p a prime factor of n! has in the prime factorization of n!

Definition 2.6. $[\log_p n]$ is equal to the integer part of some real number $\log_p n$.

Definition 2.7. $l(a^x) =$ returns the last digit of a^x where a and x are integers.

Definition 2.8. $c(a)$ determines the period that some exponential a^x has. For example $c(2) = 4$

* A note on notation of n and p = Whenever we have used n, we refer to the input that the n! function takes. And whenever we use p, we refer to some prime integer.

3. A FIRST APPROACH

3.1. General observations.

3.1.1. *Core idea.* The main problem with determining $\lambda(n!)$ is the fact that n! as a positive integer grows rapidly and to calculate n! becomes a laborious job. It is for this reason that we are in search of shortcuts that we can synthesize into forming an algorithm. Primarily, we realize that n! outputs a positive integer. Thus, if we apply the fundamental theorem of arithmetic, we can express n! as the unique product of primes.

$$\text{Fundamental theorem of arithmetic:} \forall n \in \mathbb{N}, n = 2^a \times 3^b \times 5^c ..... = \prod_{n=1}^{k} p^i \tag{3.1}$$

We now consider how eq. (3.1) is of use to us. Let's consider remark 3.1

Remark 3.1. We notice that for any integer $n!$ may take where $n \geq 5$, $n!$ ends with at least 1 zero. This is rather obvious. For 5! we obtain a multiplication of 5×2 and similarly for 6! we also obtain a multiplication of 5×2 and so on. From this, we observe that in the prime factorization of any $n \geq 5$, we are guaranteed to find 5 and 2. This way, we may easily remove the 0's at then end of $n!$

Thus we are able to establish eq. (3.2).

$$n! = 2^a \times 3^b \times 5^c ..... \implies \frac{n!}{10^{p(n)}} = 2^{a-c} \times 3^b \times 7^d ..... \tag{3.2}$$

Let's consider example 3.2 to see what we mean in eq. (3.2).

Example 3.2.

$$5! = 2^3 \times 3^1 \times 5^1$$
$$c = 1, p(5) = 1$$
$$\therefore \frac{5!}{10^1} = 2^{3-1} \times 3^1$$

We further observe that if we evaluate $\frac{n!}{10^{p(n)}}$ in mod10 we obtain $\lambda(n!)$. This follows from the fact that the $\frac{n!}{10^{p(n)}}$ operation removes the zero's from the end of n! Then all that the modulo operation does is to find how close our new number is to the largest multiple of 10 less than it. Clearly, this is determined by whatever the last digit of $\frac{n!}{10^{p(n)}}$ is. Hence, we establish eq. (3.3).

$$\lambda(n!) \equiv \frac{n!}{10^{p(n)}} \bmod 10 \tag{3.3}$$

To demonstrate our point in eq. (3.3), let's consider example 3.3.

Example 3.3. Suppose we want to find $\lambda(6!)$ Now,

$$6! = 720$$
$$p(6) = 1$$
$$\therefore \frac{720}{10^1} = 72, 72 \equiv 2 \bmod 10$$

Now, all of eq. (3.1),eq. (3.2) and eq. (3.3) are useful to us only if we actually have the prime factorization of n!. Clearly, our next step is to find a method with which we obtain the prime factorization of n! Considering how we compose $n!$, that is $1 \times 2 \times 3.....(n-1) \times (n)$, we know from eq. (3.1) that each one of these terms can be expressed as the product of primes. From this, we may deduce that the largest prime factor of $n!$ is less than or equal to n. We also deduce quite clearly that all primes in the range $[2, n]$ are prime factors of $n!$ because every prime less than n is a factor of n! Thus, we arrive at eq. (3.4).

3.1.2. *Prime factorization.*

$$\forall p \in [2, n], \frac{n!}{p} \equiv 0 \bmod p \tag{3.4}$$

From eq. (3.4), we are left to determine the exponent that each prime will have in the factorization. Let's take the example of 36. To see how we may do this, we now consider example 3.4.

Example 3.4. Let's take the prime factorization of 36! To implement what we found in example 3.2, we want to determine the exponent of 2 in the prime factorization of 36! We ask, how many of the numbers in the range $[2, 36]$ are divisible by 2. Clearly all multiples of 2 less than 36, that is $2, 4, 6...$ are divisible by 2. We may express this as $\left\lfloor \frac{36}{2} \right\rfloor = 13$ We use floor division instead of division because if say we had n=37 instead of 36, then still 13 numbers in the range $[2, 37]$ would have been divisible by 2. Thus we use floor division because it is only the quotient that matters to us. We know that 2 goes into 36! at least 13 times. But this isn't enough. There exists 4 that is divisible by 2 twice, but we have only counted the 2 in 4 once. We must now ask, how many multiples of 4 lie in the range $[2, 36]$. That will be all the numbers [4,8,12...]. We now perform $\left\lfloor \frac{36}{4} \right\rfloor = 9$. We again realize that we have counted the 2 in 8 twice, but there is actually one more 2. It thus becomes clear that we must repeat the floor operation once again to account for all multiples of 8 in our range. In fact, we must repeat this process for all powers of 2 less than or equal to n and finally sum all the results of our floor division. This sum gives us the exact number of times that 2 goes into 36!

What we have shown in example 3.4 is generalized now in eq. (3.5). The formula presented in eq. (3.5) is known as [3] Legendre's formula and is presented in the [4] famous proofs from the book. Despite the formula not being our work, example 3.4 presents our understanding of it.

$$ex(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \tag{3.5}$$

We do notice that the upper boundary of the summation in eq. (3.5) ∞ sign may be replaced. If we refer to *Definition 1.5.*, we may replace ∞ with $[\log_p n]$.

$$ex(p) = \sum_{k=1}^{[\log_p n]} \left\lfloor \frac{n}{p^k} \right\rfloor \tag{3.6}$$

From eq. (3.6), it is more obvious that to find the exponent of some prime p, we ought to iterate the floor division until we reach the largest power of p less than n. But also from eq. (3.5) this is seen because after $p^k > n$, $\left\lfloor \frac{n}{p^k} \right\rfloor = 0$.

It seems now that we are much closer to a fully functioning algorithm. The only remaining question is, how do we reach $\lambda(n!)$ after having the prime factorization of $n!$

3.1.3. *Cyclical values of exponentials.* One idea suggests that it would be very useful if every exponential integer had a last digit on a cyclical basis. This is very useful because knowing the cyclical value of the last digit for some p^x is equivalent to performing $p^x \mod 10$. If we observe Table 1. we notice that for powers in the range $[2, 9]$ a pattern appears for the last digit.

digit	a^1	a^2	a^3	a^4	a^5
2	2	4	8	6	2
3	3	9	7	1	3
4	4	6	4	6	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	9	3	1	7
8	8	4	2	6	8
9	9	1	9	1	9

TABLE 1. pattern in the last digit of powers in range $[2, 9]$

How may we however conjecture that the last digit of an exponential is in fact periodic, and may we do this for all integers? Clearly, from table 1 we verify that the last digit of exponentials with a base in the range $[2, 9]$ are periodic because some k steps later, the same last digit value is reached. This also means that these k steps is the period for the last digit of the exponential.

Let's now create an algorithm that will serve us in finding the last digit of exponentials with a base in the range $[2, 9]$. Our algorithm uses the value of $c(a)$ or in other words, the period of the last digit of the exponential. To create our algorithm we use the idea of equivalence classes. For example, from table 1, it is clear that the period for the last digit of 2^x is 4. This means that $l(2^x) = l(2^{x+4}) = l(2^{x+4+4})$ This is because since $c(2) = 4$, all of $x, x+4, x+8$ fall under the same equivalence class. From this, we establish eq. (3.7)

$$c(a) = z \implies l(a^z) = l(a^{xz}) | x \in \mathbb{Z} \tag{3.7}$$

Our point in eq. (3.7) is applied to create remark 3.5.

Remark 3.5. The following remarks list all we need to know in order to determine the last digit of exponentials with base in range $[2, 9]$. *We note that bases $1, 5, 6$ all have a period of 0 as seen in table 1 thus we do not list them.

$$l(2^x) = 2 \iff x \equiv 1 \bmod 4$$

$$l(2^x) = 4 \iff x \equiv 2 \bmod 4$$

$$l(2^x) = 8 \iff x \equiv 3 \bmod 4$$

$$l(2^x) = 6 \iff x \equiv 0 \bmod 4$$

$$l(3^x) = 3 \iff x \equiv 1 \bmod 4$$

$$l(3^x) = 9 \iff x \equiv 2 \bmod 4$$

$$l(3^x) = 7 \iff x \equiv 3 \bmod 4$$

$$l(3^x) = 1 \iff x \equiv 0 \bmod 4$$

$$l(4^x) = 4 \iff x \equiv 1 \bmod 2$$

$$l(4^x) = 6 \iff x \equiv 0 \bmod 2$$

$$l(7^x) = 7 \iff x \equiv 1 \bmod 4$$

$$l(7^x) = 9 \iff x \equiv 2 \bmod 4$$

$$l(7^x) = 3 \iff x \equiv 3 \bmod 4$$

$$l(7^x) = 1 \iff x \equiv 0 \bmod 4$$

$$l(8^x) = 8 \iff x \equiv 1 \bmod 4$$

$$l(8^x) = 4 \iff x \equiv 2 \bmod 4$$

$$l(8^x) = 2 \iff x \equiv 3 \bmod 4$$

$$l(8^x) = 6 \iff x \equiv 0 \bmod 4$$

$$l(9^x) = 9 \iff x \equiv 1 \bmod 2$$

$$l(9^x) = 1 \iff x \equiv 0 \bmod 2$$

Finally, we will consider a way to evaluate $l(a^x)$ for a base a larger than 9. [5] proposes that we may find $l(a^x)$ using the binomial expansion combined with modular arithmetic. Using an example of our own, this may be done as follows;

Example 3.6. Suppose we want to evaluate $l(63^{23})$ We may write 63^{23} as $(60+3)^{23}$ Now recalling the binomial theorem;

$$(a+b)^N = \sum_{r=0}^{N} \binom{N}{r} \times a^r b^{N-r}$$

We notice that 63^{23} may be written as a binomial expansion.

$$(3+60)^{23} = \binom{23}{0} 3^{23} 60^0 + \binom{23}{1} 3^{22} 60^1 + \binom{23}{2} 3^{21} 60^2 + ...$$

Further on, we notice that, from the second term onwards, all terms equal 0 mod 10.

$$\binom{23}{1} 3^{22} 60^1 \mod 10 = 0$$

$$\binom{23}{2} 3^{21} 60^2 \mod 10 = 0$$

$$\vdots$$

It is clear that any term containing 60^1 or greater powers of 60, will end with a zero. From this, it appears that the last digit of our first term in the expansion will reveal the value of $l(63^{23})$ simply because the last digit of all terms is 0.

Thus, we ought to evaluate $\binom{23}{0} 3^{23} 60^0 \mod 10$. Since we can only choose nothing from 23 things only once and since $60^0 = 1$, we simplify our expression to $3^{23} \mod 10$ and we further notice that $3^{23} \mod 10 = l(3^{23})$ Using our algorithm in remark 3.5 we find;

$$23 \mod 4 = 3 \rightarrow l(3^{23}) = 7$$

Thus,

$$l(63^{23}) = 7$$

From this example, we curiously observe that $l(63^{23}) = l(3^{23})$. Thus we conjecture the following;

Conjecture 3.7. For some integer a^x expressed in base 10 as $((a_n, a_{(n-1)} .....a_1)^x)$, $l((a_n, a_{(n-1)} .....a_1)^x) = l(a_1^x)$

Now, if we are able to show the truth of this conjecture, we will have found a way of finding $l(a^x)$ for any positive integer. And ofcourse, this conjecture is particularly useful because just by using our findings in remark 3.5, we may evaluate $l(a^x)$ and this is obvious from the fact that a_1 is in the range $[1, 9]$.

Let's now consider an informal proof to conjecture 3.7

Proof. We notice that every $((a_n, a_{(n-1)}...a_2, a_1)^x)$ may be written in the form $((a_n, a_{(n-1)}...a_2, 0+a_1)^x)$. Similar to *Example 2.5.*, we take the binomial expansion of this. We obtain ;

$$\binom{x}{0}(a_n, a_{(n-1)}...a_2, 0)^0 a_1^x + \binom{x}{1}(a_n, a_{(n-1)}...a_2, 0)^1 a_1^{x-1} + \binom{x}{2}(a_n, a_{(n-1)}...a_2, 0)^2 a_1^{x-2}$$

Just as we had observed in example 3.6, we notice that from the second term onwards, every termmod10 is congruent to 0. This is seen from the fact that the term $(a_n, a_{(n-1)}...a_2, 0)$ taken to some power in the range $[1, x]$ will always be a multiple of 10. Thus, it is then obvious that we only need to look at the first term $\binom{x}{0}(a_n, a_{(n-1)}...a_2, 0)^0 a_1^x$ to obtain $l((a_n, a_{(n-1)}.....a_1)^x)$. Ofcourse, our first term simplifies to a_1^x and now we notice that it suffices to know $l(a_1^x$ to know the last digit of $((a_n, a_{(n-1)}...a_2, 0 + a_1)^x)$.

□

3.2. **Bringing it together.** Considering all we have found thus far, it seems that we may now propose a somewhat primitive algorithm to obtain $\lambda(n!)$. That is, we know how to obtain a prime factorization for $n!$ and we know how to find the last digit for all factors and of course it is clear that to reach $\lambda(n!)$, we multiply these last digits and evalute the result $\bmod 10$.

Theorem 3.8. *To obtain $\lambda(n!)$ for any $n!$, we follow the following steps.*
1. Obtain prime factorization using the formula in (6);

$$ex(p) = \sum_{k=1}^{[\log_p n]} \left\lfloor \frac{n}{p^k} \right\rfloor$$

2. For every prime factor, we then evaluate the factors $\bmod 10$ *using our findings about the cyclical last digit of exponentials.*

3. After having found all factors $\bmod 10$ *we multiply all the results and operate the final result* $\bmod 10$ *to obtain $\lambda(n!)$.*

The algorithm that we have just established is clearly quicker than calculating n! by hand and observing its first non-zero digit. However, it still presents some challenges given that for some very large n it will be a very long process to obtain the prime factorization of $n!$. In addition, we also realize that for some very large n, we will almost never directly obtain $\frac{n!}{10^{\mu(n)}} \bmod 10$. That is, we will always take the prime factorization of $\frac{n!}{10^{\mu(n)}} \bmod 10$, determine every factor modulo 10(in other words find the last digit of every factor) multiply the results and finally evaluate it modulo 10 again to obtain $\lambda(n!)$. This will clearly be a laborius process as even after having zero's removed from some very large n!, all the prime factors in the range $[2, n]$ excluding the prime 5 will still be factors. Thus, we will first demonstrate some sample calculations to examplify these challenges and then look for more efficient algorithms.

3.3. **Applying theorem 2.7. through sample calculation.** Suppose we want to determine $\lambda(45!)$ We know that all primes in range $[2, 45]$ will be in the prime factorization. We now list them.

$$[2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43]$$

Using (6), we now obtain the exponent each prime will have in the factorization.

$$ex(2) = \sum_{k=1}^{[\log_2 45]} \left\lfloor \frac{45}{2^k} \right\rfloor = \left\lfloor \frac{45}{2} \right\rfloor + \left\lfloor \frac{45}{4} \right\rfloor + \left\lfloor \frac{45}{8} \right\rfloor + \left\lfloor \frac{45}{16} \right\rfloor + \left\lfloor \frac{45}{32} \right\rfloor = 22+11+6+2+1 = 42$$

$$ex(3) = \sum_{k=1}^{[\log_3 45]} \left\lfloor \frac{45}{3^k} \right\rfloor = \left\lfloor \frac{45}{3} \right\rfloor + \left\lfloor \frac{45}{9} \right\rfloor + \left\lfloor \frac{45}{27} \right\rfloor = 15 + 5 + 1 = 21$$

$$ex(5) = \sum_{k=1}^{[\log_5 45]} \left\lfloor \frac{45}{5^k} \right\rfloor = \left\lfloor \frac{45}{5} \right\rfloor + \left\lfloor \frac{45}{25} \right\rfloor = 9 + 1 = 10$$

$$ex(7) = \sum_{k=1}^{[\log_7 45]} \left\lfloor \frac{45}{7^k} \right\rfloor = \left\lfloor \frac{45}{7} \right\rfloor = 6$$

$$ex(11) = \sum_{k=1}^{[\log_{11} 45]} \left\lfloor \frac{45}{11^k} \right\rfloor = \left\lfloor \frac{45}{11} \right\rfloor = 4$$

$$ex(13) = \sum_{k=1}^{[\log_{13} 45]} \left\lfloor \frac{45}{13^k} \right\rfloor = \left\lfloor \frac{45}{13} \right\rfloor = 3$$

$$ex(17) = \sum_{k=1}^{[\log_{17} 45]} \left\lfloor \frac{45}{17^k} \right\rfloor = \left\lfloor \frac{45}{17} \right\rfloor = 2$$

$$ex(19) = \sum_{k=1}^{[\log_{19} 45]} \left\lfloor \frac{45}{19^k} \right\rfloor = \left\lfloor \frac{45}{19} \right\rfloor = 2$$

$$ex(23) = \sum_{k=1}^{[\log_{23} 45]} \left\lfloor \frac{45}{23^k} \right\rfloor = \left\lfloor \frac{45}{23} \right\rfloor = 1$$

$$\vdots$$

$$ex(p \geq 23) = 1$$

It is clear that now, the rest of the prime factors will all have $ex(p) = 1$. This follows from the fact that $\forall p \in [23, 45], [\log_p 45] = 1 \wedge \left\lfloor \frac{45}{p} \right\rfloor = 1$

We may now write the prime factorization.

$$45! = 2^{42} \times 3^{21} \times 5^{10} \times 7^6 \times 11^4 \times 13^3 \times 17^2 \times 19^2 \times 23 \times 29 \times 31 \times 41 \times 43$$

Now, following step 2 of theorem 3.8, we remove matching powers of 2 and 5. We perform $ex(2) - ex(5)$ to find the new exponent of 2.

$$\frac{45!}{10^{\mu(45)}} = 2^{32} \times 3^{21} \times 7^6 \times 11^4 \times 13^3 \times 17^2 \times 19^2 \times 23 \times 29 \times 31 \times 41 \times 43$$

We now take every factor and find it's last digit.

$$\begin{aligned} 32 \mod 4 = 0 &\to l(2^{32}) = 6 \\ 21 \mod 4 = 1 &\to l(3^{21}) = 3 \\ 6 \mod 4 = 2 &\to l(7^6) = 9 \end{aligned}$$

$$l(11^4) = l(1^4) = 1$$
$$l(13^3) = l(3^3) \wedge 3 \mod 4 = 3 \rightarrow l(13^3) = 7$$
$$l(17^2) = l(7^2) \wedge 2 \mod 4 = 2 \rightarrow l(17^3) = 9$$
$$l(19^2) = l(9^2) \wedge 2 \mod 2 = 0 \rightarrow l(19^2) = 1$$
$$l(23) = 3$$
$$l(29) = 9$$
$$l(31) = 1$$
$$l(41) = 1$$
$$l(43) = 3$$

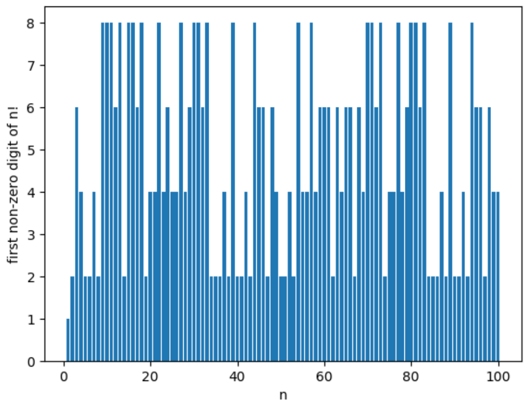
Finaly we multiply our results and operate $\mod 10$ to obtain $\lambda(45!)$.

$$\lambda(45) = 6 * 3 * 9 * 1 * 7 * 9 * 1 * 3 * 9 * 1 * 1 * 3 \mod 10 = 826686 \mod 10 = 6$$

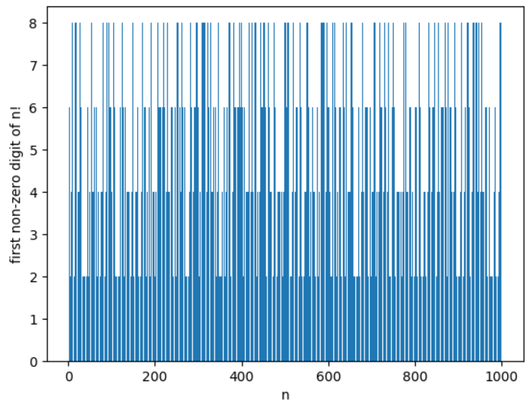
We verify this result using Wolfram Alpha[6] with the entry ('last non-zero digit of n!') and it is indeed correct.

4. Long range data analysis

In this section, our goal is to plot values of $\lambda(n!)$ and to observe whether we can discern any useful patterns that either verify our findings or reveal something new. To obtain output values of $\lambda(n!)$, we use code that we have written in python which is included in appendix A. The one problem with the code is that it slows down significantly at finding $\lambda(n!)$ at values larger than 3000 which means our graphs can only have a limited domain. We will thus create bar graphs with input values $100, 1000, 3000$ respectively.

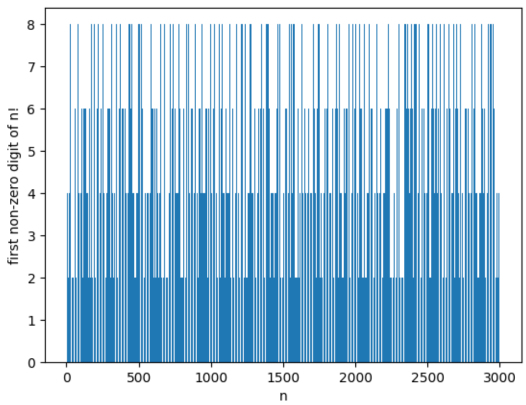


density for first 100 inputs



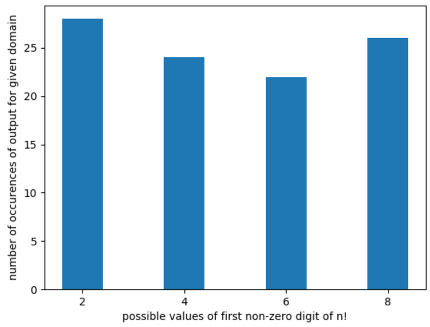
density for first 1000 inputs

Comparison of density distribution for 3 domains

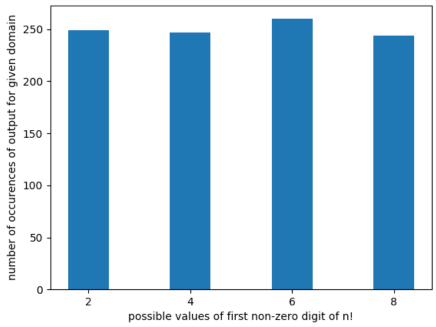


density for first 3000 inputs

What we confirm from our 3 graphs is that as we had already seen, all outputs of $\lambda(n!)$ excluding the case of $n = 1$ map to the codomain $[2, 4, 6, 8]$. Let's now plot the density of $2, 4, 6, 8$ for the each domain to make further observations. The code for this is also included in appendix A.

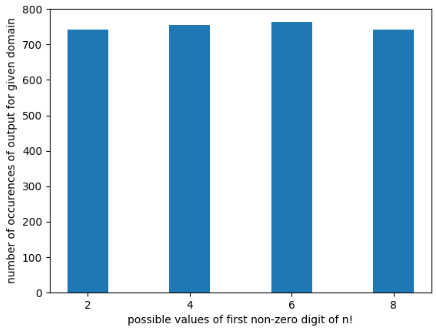


density for first 100 inputs



density for first 1000 inputs

Comparison of density distribution for 3 domains



density for first 3000 inputs

From our graphs, it seems possible that a pattern could indeed exist. In order to verify this, we use a method called data compression. That is, we take all of our $\lambda(n!)$ outputs for the range $[1, 3000]$, save our data in an excel sheet, observe the file size of this file and then observe it's file size after having compressed it into zip format. This method was proposed to me after I spoke to an interview with Assoc. Prof. Kulekci at the Istanbul Technical University. Kulekci told me that during file compression, the computer removes any redundancies that include recurring patterns for the case of an integer sequence. Below in fig. 3, what we have done is to take a file containing a list of 3000 random numbers in the range [1,9] and compressed it. This is the file named (fakenumbers3000). We also have the first 3000 values of $\lambda(n!)$ saved in the file named range3000. Now, after compressing both files, we observe that despite both files having the same initial size range3000 has a much smaller new size. This proves that a pattern exists within the first 3000 values of $\lambda(n!)$ as it has a smaller size after compression compared to a file that we know has no pattern.

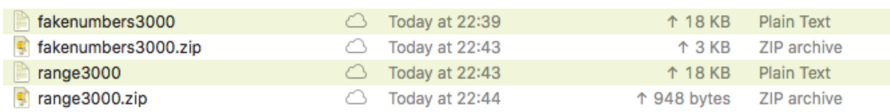


FIGURE 3. The proof that indeed a pattern exists in the range $[2, 3000]$

5. Literature result

Based on our computer aided finding, we did a literature survey in track of any patterns. In this section all the theory belongs to [7] that we found as a link on OEIS A008904 [1]. Despite this source however, the original theory which [7] describes first appeared in the work of Japanese mathematician S.Kakutani who wrote it in his paper called 'Ergodic theory of shift transformations' [2]. Yet, our explanation is based on [7] as the source provides a simplified read of Kakutani's algorithm. In addition, at certain points [7] does not provide all the reasoning and explanations to certain claims and we have made an effort to add in our own reasoning to some of these claims.

We realize that first of all $\lambda(n!) = \lambda(\lambda(n)\lambda((n-1)!))$ unless n is a multiple of 5.

Remark 5.1. $\lambda(n!) = \lambda(\lambda(n)\lambda((n-1)!)) \wedge n \mod 5 \neq 0$

The truth of remark 5.1 follows from the fact that the value of $\lambda(n!)$ is determined solely by multiplying $\lambda((n-1)!)$ with whatever the last digit of n is. This follows from the fact that $n!$ is recursively defined as $n \times (n-1)!$ To demonstrate this, we consider example 5.2.

Example 5.2. $\lambda(13!) = 12! \times 13 = \lambda(479001600) \times \lambda(13) = 8$

Using this example, we may also see why a number that is a multiple of 5 can not be found using remark 5.1. That is, a multiple of 5 has last digit 5 or 0 and we know that the last digit of $(n-1)$ is even, thus when we multiply 5 or 0 with $\lambda((n-1)!)$ an extra 0 is added. Let's consider example 5.3 to see this.

Example 5.3.

$$\lambda(15!) = 14! \times 15 = \lambda(87178291200) \times \lambda(5) = 0$$

Having seen this, we recognize that all we need to be determine is $\lambda(5n!)$ because then we may know the values of all $\lambda((5^x n + j)!)$ where $j = 0, 1, 2, 3, 4$. This way, if we find a way to figure $\lambda(5n!)$, then we can compute $\lambda(n!)$ for any integer n.

We now create a list of all the values of $\lambda((5^x n)!)$ up to $n = 49$ and $x = 4$. From this list, we obtain the following.

					n					
digit	1	5	10	15	20	25	30	35	40	45
$\lambda(n!)$	1264	22428	88682	88682	44846	44846	88682	22428	22428	66264
$\lambda(5n!)$	2884	48226	24668	48226	48226	86442	24668	62884	24668	24668
$\lambda(25n!)$	4244	82622	82622	28488	46866	64244	82622	82622	28488	46866
$\lambda(125n!)$	8824	68824	26648	68824	42286	26648	26648	42286	26648	84462
$\lambda(625n!)$	6264	22428	88682	88682	44846	44846	88682	22428	22428	66264

Table 2. Values of $\lambda(n!)$ for factors of five

It is clear that all of our values come in strings of 5 where the first term determines the rest. Curiously, our main source [7] does not explain why we only create a table upto values of $n = 49$. We have however found an explanation to this. We explain it in 5.4

Remark 5.4. We notice that after obtaining the value of $\lambda(5^x)$, we multiply this number with the least significant digit of the next 4 numbers. For $n!$(column 1) in table 2, these numbers are $\{1, 2, 3, 4\}$ and $\{6, 7, 8, 9\}$. We notice that $\{1, 2, 3, 4\}$ and $\{6, 7, 8, 9\}$ act the same on the set $\{2, 4, 6, 8\}$. Since we observe that the 10 multiples of 5 in the range $[1, 49]$ have taken all the values of $\{2, 4, 6, 8\}$, we have obtained all possbile strings of 5. That is, it is unnecesary to continue our list any further because we will not obtain a new string that has not appeared before. The same explanation also applies to the 4 columns below $n!$ in table 2.

Now from table 2 we observe that the values of $\lambda(5n!)$ equal that of $\lambda(625n!)$. Using this observation, we come up with lemma 5.5.

Lemma 5.5. $\lambda(5^{j+4}n!) = \lambda(5^j n!)$

Despite [7] not providing a proof to lemma 5.5 we prefer to prove it as it will form the skeleton of our algorithm. We use a proof provided by [8].

Proof. If we express n in base 5 notation as $\sum_{i=0}^{N} a_i 5^i$ we may find $\lambda(n!)$ using

$$\lambda(n!) = 6 \prod_{i=0}^{N} (a_i!) 2^{ia_i} \mod 10$$

We will not prove this formula. We will use it to show that $\lambda(5^{j+4}n!) = \lambda(5^j n!)$. We notice that if we write $5^j n!$ in base 5 as $\sum_{i=0}^{N} a_i 5^i$, then $5^{j+4}n! = (\sum_{i=0}^{N} a_i 5^i) \times 5^4$. By the formula for $\lambda(n!)$ we have $\lambda(5^{j+4}n!) = \lambda(5^j n!) \times (1!)2^{4\times 1} \mod 10$. We notice that $2^4 \mod 10 = 6$ and 6 acts as an identity on all of the numbers $\{2, 4, 6, 8\}$. That is when we multiply 6 with any number from this set, we still obtain the number as the least significant digit. Thus, we now have completed our proof. ∎

Having proven that $\lambda(5^{j+4}n!) = \lambda(5^j n!)$, we now create a table containing all possible strings of 5 that may occur for some value of j in $\lambda(5^{j+4}n!)$. For convience, we also let $\lambda(0) = 0$ and $\lambda(1) = 1$

$j \mod 4$	Look up table for $\lambda(5^j n!)$				
0	06264	22428	44846	66264	88682
1	02884	24668	48226	62884	86442
2	04244	28488	46866	64244	82622
3	08824	26648	42286	68824	84462

TABLE 3. The table containing all possible strings of 5 needed to know $\lambda(n!)$

One assumption table 3 makes is that for every string starting with an integer $\{2, 4, 6, 8\}$, there exists one unique string starting with $\{2, 4, 6, 8\}$ in every column. This follows from the fact that our operands in each column act the same on the integers $\{2, 4, 6, 8\}$.

We are now ready to establish our algorithm. We will use the key idea that $\lambda(n!)$ can be calculated recursively by $\lambda(\lambda(n)(\lambda(n-1)!)$. From this, we recall that we may find all of $\lambda((5^x n + j)!)$ where $j = 0, 1, 2, 3, 4$. Now, our only challenge is to know the value of some $\lambda(5^x)$ in order to compute any $\lambda((5^x n + j)!)$. To demonstrate how we may do this, we will use an example.

Example 5.6. Suppose we want to evaluate $\lambda(343!)$. We have to find a way such that just by using table 3 we will obtain $\lambda(343!)$. To do this, we convert 343 to base 5 to obtain $(0, 2, 3, 3, 3)$ or in other words $((0\times 625+2\times 125+3\times 25+3\times 5+3\times 1)!)$. Our strategy is first find $\lambda((2 \times 125)!)$, then using the last digit of this, find $\lambda((2 \times 125+3\times 25)!)$ and move our way up to $((0\times 625+2\times 125+3\times 25+3\times 5+3\times 1)!)$. And to do this, we follow the steps we outlined. Now, let's go step by step. All of our references are to table 3.

To find $\lambda((2\times 125)!)$, we look at $j \mod 4 = 3$ at the string starting with 0 because $\lambda((0 \times 625)!) = 0$. We take the 2nd term which is 8.

To find $\lambda((2 \times 125 + 3 \times 25)!)$, we look at $j \mod 4 = 2$ at the string starting with 8 because $\lambda((2 \times 125)!) = 8$. We take the 3rd term which is 6.

To find $\lambda((2\times 125+3\times 25+3\times 5)!)$, we look at $j \mod 4 = 1$ at the string starting with 6 because $\lambda((2 \times 125 + 3 \times 25)!) = 6$. We take the 3rd term which is 8.

To find $\lambda((2 \times 125 + 3 \times 25 + 3 \times 5+)!)$, we look at $j \mod 4 = 0$ at the string starting with 8 because $\lambda((2\times 125+3\times 25+3\times 5)!) = 8$. We take the 3rd term which is 6. Thus $\lambda(343!) = 6$. We verify this answer using Wolfram Alpha and confirm that it is correct.

As we see from example 5.6 our method is to simply separate our original number into multiples of 5^x and by respectively finding their last digits, work our way up to our number n. One key advantage of this algorithm is that for any given n, we know with how many steps we will reach $\lambda(n)$. That is, the number of steps we follow is equal to how many digits our number n comprises in base 5.

6. Synthesis

Overall, in this study we have surfaced two possible algorithms that can output the first non-zero digit of $n!$. It is clear that the latter of these two is far more ingenious and as a result more efficient than the former which is our own finding. So we ask, how does the second algorithm differ in essence? Clearly, it has a different approach. When creating this algorithm, Kakutani focuses on the idea that knowing some $\lambda((n-1)!)$ will yield knowledge of $\lambda(n!)$. Indeed, Kakutani successfully finds such a relationship which results in the beautiful look-up table that we show in table 3. It is this whole notion of cyclicity that yields this result. Quite similarly however, cyclicity also appears in our own algorithm as we build our algorithm on the basis of the cyclic last digit of exponentials. On a more metamathematical note, we might observe that cyclicity neccesarily appears because the essence of an algorithm is to find a way of obtaining information without necessarily needing more detail. However, Kakutani's algorithm is way more successful at achieving this goal as in our own algorithm, the information needed to reach $\lambda(n!)$ grows linearly with the size of the input. In contrast, the detail we need in Kakutani's algorithm grows only on a logarithmic scale. To see this, consider the example below.

Example 6.1. Using Kakutani's algorithm, for some input n, the number of steps we need to reach $\lambda(n!)$ is exactly equal to the number of terms in the base 5 expansion of n. This means that in general we would have $[\log_5 n]$ steps. In contrast, our own algorithm has most nearly a linear growth in the number of steps needed to reach $\lambda(n!)$ as for every next input of n it is likely that a new term in the prime factorization will appear.

7. Conclusion

In conclusion, we note that this study has only scratched the surface of the research question **Does there exist a number theoretic algorithm to obtain the first non-zero digit of n!**. It is very likely that many more algorithms exist and very possible that there exist numerous unrecognized patterns within the first nonzero digit of $n!$. In addition to this, a limitation of this study is that our lack of knowledge on ergodic theory has possibly prevented us from reaching profound intuitions about the significance of this question on a larger scale. For instance, S.Kakutani studies this problem not for the sake of number theory but because it is relevant to his paper on 'Ergodic theory of shift transformations'. In addition, we also acknowledge that our use of the computer aided result in part 4 was rather limited. All in all however, this study has been fruitful in that both us and the reader hopefully gains a further insight into how interesting a simple question of number theory may be.

Appendix A.

Code that evaluates $\lambda(n)$ for a domain (2,1001). These listed values go on the y-axis.

```
import math as mt

x = 2
f = []
while x < 1001:
    z = mt.factorial(x)
    b = [int(z) for z in str(z)]
    c = str(z).rstrip("0")
    d = len(c)
    e = c[(d - 1):]
    f.append(int(e))
    x = x + 1
print(f)
```

Code that lists all integers for a range we specify. This way, we have an input that goes on the x-axis.

```
import math as mt

x = 1
f = []
while x < 3000:
    e = x + 1
    f.append(int(e))
    x = x+1

print(f)
```

Code that uses the matplotlib library in python to generate the graphs we used.

```
from matplotlib import pyplot as plt

x = [2, 3, 4, 5, 6, 7, 8, 9, 10]
y = [2, 6, 4, 2, 2, 4, 2, 8, 8]
plt.bar(x,y)
plt.show()
```

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