HI MY FELLOW VIEWERS,

I WRITE x AS A LOWER-CASE X. DON'T BE CONFUSED

THE REGULATION (OR SET-POINT STABILIZATION) PROBLEM

Check Khalil's book

Consider the system (all functions are assumed sufficiently smooth)

$$\begin{cases} \dot{x} = f(x, u) \\ y_r = h_r(x) \\ y_m = h_m(x) \end{cases}$$

state — input

$x(t) \in \mathbb{R}^{n_x}$ (STATE) (space of state), $u(t) \in \mathbb{R}^{n_u}$ (CONTROL) (space of input)

$y_r(t) \in \mathbb{R}^{n_r}$ (REGULATED OUTPUT) (output to drive the system somewhere), $y_m(t) \in \mathbb{R}^{n_m}$ (MEASURED OUTPUT)

variable on which we have performance specifications

WE ONLY MEASURE $y_m(t)$

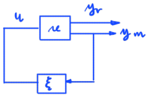
desired value for y_r

Let $y_r^* \in \mathbb{R}^{n_r}$ be a desired value (SET-POINT) for $y_r(t)$

GOAL: $y_r(t) \xrightarrow{\text{control to}} y_r^*$

PROBLEM: design a controller of the form (still autonomous)

$$\begin{cases} \dot{\xi} = g(\xi, y_m) \\ u = \gamma(\xi, y_m) \end{cases} \qquad \xi(t) \in \mathbb{R}^{n_\xi} \qquad \boxed{(\xi = \text{CONTROLLER'S STATE})}$$



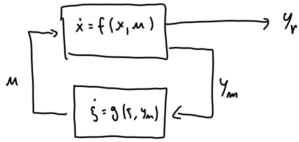
A SUBCASE IS THE STATIC CONTROLLER:

$$u = \gamma(y_m)$$

such that the closed-loop system

$$\begin{cases} \dot{x} = f(x, \gamma(\xi, h_m(x))) \\ \dot{\xi} = g(\xi, h_m(x)) \\ y_r = h_r(x) \end{cases} \qquad (*)$$

we want to have this to be steady-state



Satisfies the following:

1. There exists an equilibrium point $(x^*, \xi^*) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$ of $(*)$ such that:

solutions must converge

$$y_r^* = h_r(x^*)$$

<- The regulated output y_r equals y_r^* at the equilibrium

2. The equilibrium (x^*, ξ^*) is (LOCALLY / GLOBALLY / ...) asymptotically stable for $(*)$

stability is a local property

local / non-local / global

Global stability is only feasible on linear-system.

robustness: a small push will not explode to infinity

If this problem is solved, then all solutions of the closed-loop system converge to a steady state (x^*, ξ^*) at which y_r attains its desired value y_r^*

This problem will be our main focus throughout the module

BIBLIOGRAPHY: H. KHALIL, Nonlinear systems (Chap. 12.2)

Solving steps:

1. Using Local Linear Control (locality in Lyapunov).
2. Using Robust Linear Control via integral action.
3. Using Non-Local which is Gain Scheduling
4. Using Non-Local / Global (non-linear control).
   Sometimes it is non-robust or fragile
5. Using Robust Control via "high-gain." (we will not reach this in this course)

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we want to see the equilibrium conditions in (x^*, ξ^*)

$$\begin{cases} 0 = f(x^*, \overbrace{\gamma(\xi^*, h_m(x^*))}^{u^*}) \\ 0 = g(\xi^*, h_m(x^*)) \\ y_r^* = h_r(x^*) \end{cases}$$

Call a dummy variable $u^* = \gamma(\xi^*, h_m(x^*))$ and split to two sets of equations.

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

$\exists x^*, u^*$ solvable equations

$$\begin{cases} 0 = g(\xi^*, h_m(x^*)) \\ u^* = \gamma(\xi^*, h_m(x^*)) \end{cases}$$

• **NECESSARY CONDITION:** If points 1 and 2 are satisfied, then there exist $(x^*, u^*) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_m}$ such that:

CAN BE FOUND IN CHAPTER 12 OF KHALIL

$$\text{Plant}\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

→ **Starting point**

(SOLVABILITY Eq.)

↑ Property of the plant that tells us when a problem

what to define the system

and the point ξ^* satisfies

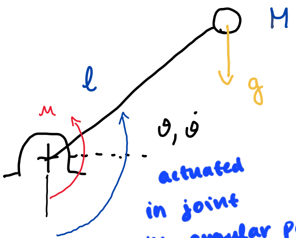
$$\text{controller}\begin{cases} 0 = g(\xi^*, h_m(x^*)) \\ u^* = \gamma(\xi^*, h_m(x^*)) \end{cases}$$

(REGULATOR Eq.)

↑ Prop. of the controller that tells us what must be the steady-state behavior of the controller

what we should have to control the system

EXAMPLE: ACTUATED PENDULUM



actuated in joint with angular param θ and θ̇

GOAL: stabilize the pendulum to a given position θ^*

ASSUMPTION: we measure the full state X

pendulum equations: $(x_1 = \theta, \quad x_2 = \dot{\theta})$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{\beta}{M\ell^2}x_2 + \frac{1}{M\ell^2}u \\ y_m = X \\ y_r = x_1 \end{cases}$$

β as the friction

u as the torque input

regulation goal: drive the pendulum to $\theta = 45°$

SET-POINT: $y_r^* \doteq \theta^*$

There must exist (x^*, u^*)

Let us solve the SOLVABILITY EQUATIONS : we look for (x^*, u^*) such that

what we found on the previous page

$$\begin{cases} 0 = x_2^* \quad = 0 \\ 0 = -\frac{g}{\ell}\sin x_1^* - \frac{\beta}{M\ell^2} x_2^* + \frac{1}{M\ell^2} u^* \\ y_r^* = x_1^* \qquad \theta^* = x_1^* \end{cases} \rightarrow \begin{cases} \overbrace{x_1^* = \theta^*, \quad x_2^* = 0}^{x^* = [\theta^*, 0]} \\ u^* = Mg\ell \sin\theta^* \end{cases}$$

$u^* = Mg\ell \sin\theta$

REGULATOR must give us u^*

the controller does not switch off at steady-state except when $\theta^* = k\pi \quad (k \in \mathbb{N})$

the REGULATOR Eqs. tell us that only controller solving the problem must provide this control action at steady-state

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LOCAL STATE-FEEDBACK SOLUTION TO THE REGULATION PROBLEM

- STATE-FEEDBACK : $\boxed{y_m(x) = x}$ (we measure the full state)

- SOLUTION APPROACH BY LINEAR FEEDBACK + LYAPUNOV INDIRECT METHOD

STEP 1) given y_r^*, we solve the SOLVABILITY Eqs :

Linearized

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

what is this ?

STEP 2) we LINEARIZE the system $\dot{x} = f(x, u)$ around (x^*, u^*) by defining the matrices

$$A := \frac{\partial f}{\partial x}(x^*, u^*) \qquad B := \frac{\partial f}{\partial u}(x^*, u^*)$$

We're now dealing with LOCAL-LINEAR CONTROL. It's also STATE-FEEDBACK.

Local

$$\begin{cases} \dot{x} = f(x, u) \\ y_m = h_m(x) = x \\ y_r = h_r(x) \end{cases}$$

y_r^* is given

STEP 1: Call the solvability equation

reference eq. point

$\exists (x^*, u^*)$:

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

STEP 2:

We know $f(x, u) = A(x - x^*) + B(u - u^*) + f_{HOT}(x, u)$ and now we <u>linearize</u> it by (approximation)

$$A = \frac{\partial f}{\partial x}(x, u)\Big|_{\substack{x = x^* \\ u = u^*}} = \frac{\partial f}{\partial x}(x^*, u^*)$$

$$B = \frac{\partial f}{\partial u}(x^*, u^*)$$

If it's stabilizable, proceed to STEP 3

STEP 3:

$K \in \mathbb{R}^{n_u \times n_x}$

Choose $u(t) = u^* + K(x(t) - x^*)$. So $A + BK$ is Hurwitz. When we apply all the $u(t)$, it because

$u(t) = u^* + K(x(t) - x^*)$

$$f(x, u) = (A + BK)(x - x^*) + f_{HOT}(x, u^* + K(x - x^*))$$

Touching the HOT is a problem!

STEP 3) We choose the controller $K \in \mathbb{R}^{n_u \times n_x}$

If A is Hurwitz already then $u(t) = u^*$ is enough

$$\boxed{u(t) = u^* + K\,(x(t) - x^*)} \qquad (*)$$

(u^* = FEEDFORWARD ACTION)

where K is chosen so that $\underbrace{A+BK \text{ is Hurwitz}}$

we need (A,B) stabilizable !

RESULT. With the controller $(*)$, the equilibrium x^* is LAS

Hence, there exists an open set $\mathcal{D} \subset \mathbb{R}^n$ such that $x^* \in \mathcal{D}$, and

$$\forall\, x(0) \in \mathcal{D}, \qquad \lim_{t\to\infty} y_r(t) = y_r^*$$

Remember that : $\cancel{\dot{\xi} = g(\xi, y_m)}$

$u = \gamma(\xi, y_m)$, $y_m = x$

$\gamma(x) = u^* + K(x - x^*)$

PROOF.

We can write the plant as

$\dot{x} = f(x, u^* + K(x - x^*))$

$$\dot{x} = A\tilde{x} + B\tilde{u} + f_{HOT}(x,u) \qquad (\blacksquare) \qquad (\tilde{x} = x - x^*, \ \tilde{u} = u - u^*)$$

$\dot{x} = f(x, u^* + K(x-x^*))$

$= F(x)$ where $x = x^*$

So, $F(x^*) = f(x^*, u^*) = 0$

where

$$f_{HOT}(x,u) = f(x,u) - A\tilde{x} - B\tilde{u}$$

satisfies

$$\frac{\partial f_{HOT}}{\partial(x,u)}(x^*,u^*) = \left[\frac{\partial f_{HOT}}{\partial x}(x^*,u^*) - A \;\vdots\; \frac{\partial f_{HOT}}{\partial u}(x^*,u^*) - B \right] = 0$$

we have to proof that $\bar{A} = \dfrac{\partial F}{\partial x}(x^*)$ is Hurwitz

We can rewrite $(*)$ as

$$\tilde{u} = K\tilde{x}$$

Plugging this into $(\blacksquare)$ leads to the closed-loop system:

$$\dot{x} = (A+BK)\tilde{x} + f_{HOT}(x, u^* + K\tilde{x}) \ \dot{=}\ F(x)$$

$$\frac{dF}{dx}(x) = \frac{d}{dx} f(x, u^* + K(x - x^*))$$

$$= \frac{\partial}{\partial x} f(x, u^* + K(x-x^*)) +$$

$$\frac{\partial}{\partial u} f(x, u^* + K(x-x^*)) \cdot \frac{\partial}{\partial x}(u^* + K(x-x^*))$$

$$\bar{A} = \frac{\partial F}{\partial x}(x^*) = \overbrace{\frac{\partial}{\partial x} f(x^*, u^*)}^{A} + \overbrace{\frac{\partial}{\partial u} f(x^*, u^*) K}^{BK}$$

It is proven that $\bar{A} = A + BK$ is Hurwitz and implies $x = x^*$ is <u>linear asymptotically stable</u>. for the closed-loop system with K.

Addendum : $u = u^* + K(x - x^*)$.

What if $f(x, u) = A(x - x^*) + B(u - u^*) + f_{HOT}(x, u)$ already Hurwitz?

Hence, $u(t) = u^*$, which is an open-loop system.

The linearization of $\dot{x} = F(x)$ around x^* is

$$\frac{\partial F}{\partial x}(x^*) = \frac{\partial}{\partial x}\Big[(A+BK)(x-x^*)\Big]\Big|_{x=x^*} + \frac{\partial}{\partial x} f_{HOT}(x, u^* + K\hat{x})\Big|_{x=x^*}$$

$$= A + BK + \underbrace{\cancel{\frac{\partial}{\partial x} f_{HOT}(x^*, u^*)}}_{=0}$$

(see above)

Hence:

$$\frac{\partial F}{\partial x}(x^*) = A + BK$$

Since $A+BK$ is Hurwitz by design, the result follows from Lyapunov's indirect theorem. ☒

EXAMPLE:

Case 1: Regulate $y_r^* = 1$

Case 2: Regulate $y_r^* = 0$

Case 3: Regulate $y_r^* = -1$

consider the system

$$\begin{cases} \dot{x} = x^3 + u^2 \\ y_r = y_m = x \end{cases}$$

we want to regulate x

CASE 1: regulation to $\boxed{y_r^* = 1}$

The SOLVABILITY CONDITIONS read

$$\begin{cases} 0 = x^{*3} + u^{*2} \\ 1 = x^* \end{cases} \Rightarrow u^{*2} = -1 \qquad \text{IMPOSSIBLE}$$

we cannot solve case 1.

CASE 2. Regulation to $\boxed{y^* = 0}$

. the SOLVABILITY CONDITIONS are:

$$\begin{cases} 0 = x^{*3} + u^{*2} \\ 0 = x^* \end{cases} \Rightarrow \begin{cases} x^* = 0 \\ u^* = 0 \end{cases}$$

CASE 1

⊙ Step 1 : Solvability eq.

$$\begin{cases} 0 = (x^*)^3 + (u^*)^2 \\ x^* = 1 \end{cases}$$

No way to solve this in real number because : $(u^*)^2 = -1$

CASE 2

⊙ Step 1 : Solvability eq.

$$\begin{cases} 0 = (x^*)^3 + (u^*)^2 \\ x^* = 0 \end{cases}$$

We found $u^* = 0$.

⊙ Step 2 :

$$A = \frac{\partial}{\partial x}(x^3 + u^2)\Big|_{\substack{x = x^* \\ u = u^*}} = 3x^2 \Big|_{x = x^*} = 0$$

$$B = \frac{\partial}{\partial u}(x^3 + u^2)\Big|_{u=0} = 0$$

A and B are not stabilizable because it is impossible to put eigval on 0.

CASE 3

- Step 1 : Solvability

$$\begin{cases} 0 = (x^*)^3 + (u^*)^2 \\ x^* = -1 \end{cases}$$

$u^* = \pm 1$ (pick $u^* = 1$)

- Step 2 :

$$A = \frac{\partial}{\partial x}(x^3 + u^2)\Big|_{x = x^* = -1} = 3$$

$$B = \frac{\partial}{\partial u}(x^3 + u^2)\Big|_{u = u^* = 1} = 2$$

$A + BK = 3 + 2K < 0 \iff \forall K < -\frac{3}{2}$

How does it affect our control eq.?

$$\begin{cases} \dot{x} = x^3 + u^2 \\ u = 1 + K(x + 1) \end{cases}$$

Substitute u, we get $\dot{x} = x^3 + (1 + K(x+1))^2 = F(x)$.

Call $\tilde{x} = x - x^* = x + 1$, so we get $x = \tilde{x} - 1$

$$\dot{\tilde{x}} = \dot{x} = (\tilde{x} - 1)^3 + (1 + K\tilde{x})^2$$

$$= \tilde{x}^3 - 2\tilde{x}^2 + \tilde{x} - \tilde{x}^2 + 2\tilde{x} - 1 + 1 + 2K\tilde{x} + K^2\tilde{x}^2$$

$$\dot{\tilde{x}} = \tilde{x}^3 + (K^2 - 3)\tilde{x}^2 + (3 + 2K)\tilde{x}$$

$$= [\tilde{x}^2 + (K^2 - 3)\tilde{x} + (3 + 2K)]\tilde{x}$$

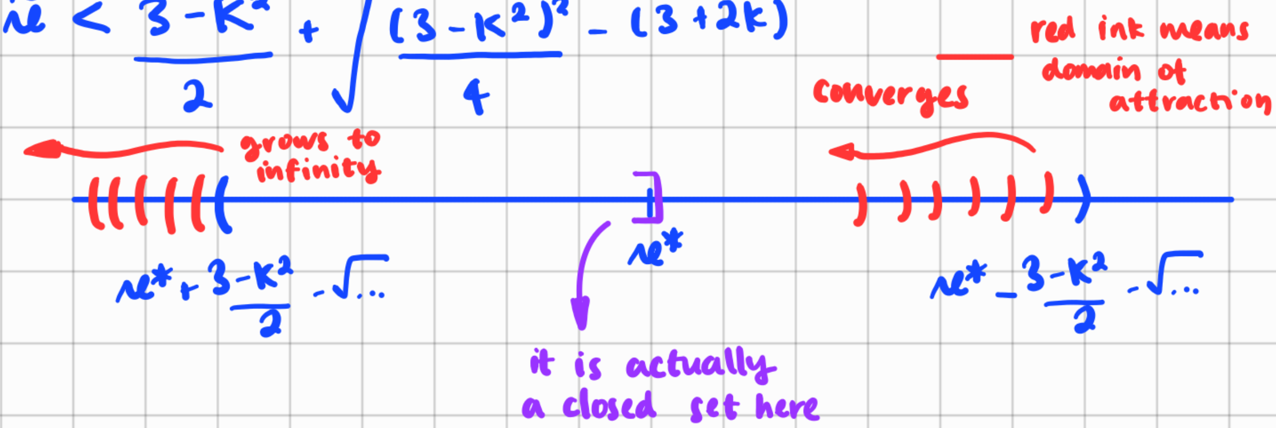
$$V(x) = (x - x^*)^T P (x - x^*) = \tilde{x}^T P \tilde{x}$$
$$= \frac{1}{2}\tilde{x}^2$$

$$V(x) = \frac{\partial V}{\partial x}(x)\, F(x) = [\tilde{x}^2 + (u^2 - 3)\tilde{x} + (3 + 2K)]\,\tilde{x}^2$$

Take the Lyapunov eq. $\tilde{x}^2 + (K^2 - 3)\tilde{x} + (3 + 2K) < 0$
and the solutions are :

$$\tilde{x} > \frac{3 - K^2}{2} - \sqrt{\frac{(3 - K^2)^2}{4} - (3 + 2K)}$$

$$\tilde{x} < \frac{3 - K^2}{2} + \sqrt{\frac{(3 - K^2)^2}{4} - (3 + 2K)}$$



Other example:

$$\begin{cases} x = xu + u \\ y_r = x \end{cases} \longrightarrow y_r^* = 0$$

- Step 1

$$\begin{cases} 0 = x^* u^* + u^* \longrightarrow u^* = 0 \\ 0 = x^* \end{cases}$$

- Step 2

$$A = \frac{\partial}{\partial x}(xu + u)\Big|_{\substack{x=0 \\ u=0}} = 0$$

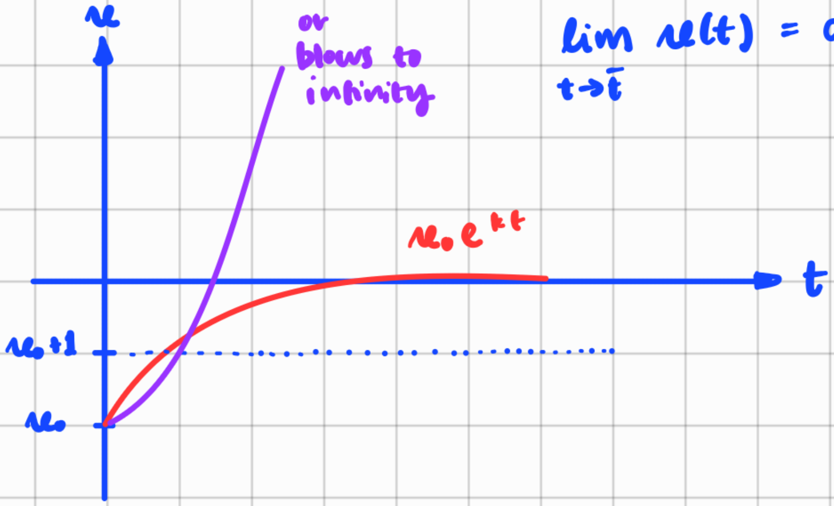
$$B = \frac{\partial}{\partial u}(xu + u)\Big|_{\substack{x=0 \\ u=0}} = x + 1\Big|_{x=0} = 1$$

From this, we find $A + BK = K$. Thus $u(t) = Kx$

$$\dot{x} = Kx^2 + Kx = Kx(x+1) \Rightarrow x(t) = \frac{x_0 e^{kt}}{x_0 + 1 - x_0 e^{kt}}$$

$\boxed{x_0 < -1}$

What's weird from this solution?



$$\lim_{t \to \bar{t}} x(t) = \infty$$

$$x_0 e^{kt} = x_0 + 1$$

$$Kt = \log\left(\frac{x_0 + 1}{x_0}\right)$$

$$t = \frac{1}{K} \log\left(\frac{x_0 + 1}{x_0}\right)$$

$$= \frac{1}{|K|} \log\left(\frac{|x_0|}{|x_0 + 1|}\right) = \bar{t}$$

LOCALITY:

$$f(x, u) = A(x - x^*) + B(u - u^*) + f_{HOT}(x, u)$$

$$= (A + B)(x - x^*) + f_{HOT}(x, u^* + K(x - x^*))$$

• To define the control law we first need to linearize the system around (x^*, u^*)

$$A = \frac{\partial f}{\partial x}(x^*, u^*) = \frac{\partial}{\partial x}\left(x^3 + u^2\right)\Big|_{\substack{x=0\\u=0}} = 3x^2\Big|_{x=0} = 0$$

$$B = \frac{\partial f}{\partial u}(x^*, u^*) = \frac{\partial}{\partial u}\left(x^2 + u^2\right)\Big|_{\substack{x=0\\u=0}} = 2u\Big|_{u=0} = 0$$

$\rightsquigarrow$ $A = 0$, $B = 0$ $\Rightarrow$ (A, B) IS NOT STABILIZABLE

$\Rightarrow$ we cannot solve case 2 with a linear controller

CASE 3. Regulation to $\boxed{y_r^* = -1}$

the SOLVABILITY CONDITIONS are:

$$\begin{cases} 0 = x^{*3} + u^{*2} \\ -1 = x^* \end{cases} \Rightarrow \begin{cases} x^* = -1 \\ u^* = 1 \end{cases}$$

The linearization matrices are

$$A = 3x^2\Big|_{x=-1} = 3$$

$$B = 2u\Big|_{u=1} = 2$$

$\Rightarrow$ (A, B) is controllable

For every $K < -\frac{3}{2}$, the matrix $A + BK$ is Hurwitz since:

$$A + BK = 3 + 2K < 0$$

Then, for every $K < -\frac{3}{2}$, the controller

$$u(t) = \overset{u^*}{\overset{\shortparallel}{1}} + K\left(x(t) + \overset{-x^*}{\overset{\shortparallel}{1}}\right) \qquad (\bullet)$$

locally stabilizes the eq. point where $y^* = -1$.

What about the domain of attraction?

Plugging $(\bullet)$ into the system's equations leads to

$$\dot{x} = x^3 + \left(1 + K(x+1)\right)^2 = x^3 + 1 + K^2(x+1)^2 + 2K(x+1)$$

changing coordinates from x to $\tilde{x} := x - x^* = x+1$ leads to

$$\dot{\tilde{x}} = \dot{x} + \dot{1} = \dot{x} = x^3 + 1 + k^2(x+1)^2 + 2k(x+1)$$

$$= (\tilde{x}-1)^3 + 1 + k^2\tilde{x}^2 + 2k\tilde{x}$$

$$= \tilde{x}^3 - 3\tilde{x}^2 + 3\tilde{x} + k^2\tilde{x}^2 + 2k\tilde{x}$$

$$= \tilde{x}\left(3 + 2k + (k^2-3)\tilde{x} + \tilde{x}^2\right)$$

this is suggested by Lyapunov's indirect method

Considering the Lyapunov candidate $V(x) = (x+1)^2 = \tilde{x}^2$, we get

$$\frac{\partial V}{\partial x}(x)\cdot f(x) = 2\tilde{x}\dot{\tilde{x}} = 2\tilde{x}^2\left(3+2k + (k^2-3)\tilde{x} + \tilde{x}^2\right)$$

$$= 2V(x)\cdot\underbrace{\left[3+2k + (k^2-3)\tilde{x} + \tilde{x}^2\right]}$$

we want this to be negative

$$\Rightarrow \frac{\partial V}{\partial x}(x) f(x) < 0 \quad \text{as long as} \quad 3+2k + (k^2-3)\tilde{x} + \tilde{x}^2 < 0$$

so, we need to solve $\tilde{x}^2 + (k^2-3)\tilde{x} + 3+2k < 0$, which gives:

$$\tilde{x} < \frac{3-k^2}{2} + \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)} \quad \vee \quad \tilde{x} > \frac{3-k^2}{2} - \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)}$$

Thus, the domain of attraction is

$$D = \left(\underset{-1}{\underset{\shortparallel}{x^*}} + \frac{3-k^2}{2} - \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)}\ ,\ \underset{-1}{\underset{\shortparallel}{x^*}} + \frac{3-k^2}{2} + \sqrt{\frac{(3-k^2)^2}{4} - (3+2k)}\right) \qquad \square$$

notice that as $|k|$ grows to infinity D converges to

$$\lim_{\substack{|k|\to\infty \\ k<0}} D = (-\infty, x^*]$$

Thus, by "highering" the control gain ($|k| \nearrow$) enlarges the domain of attraction on the left, but reduces it on the right

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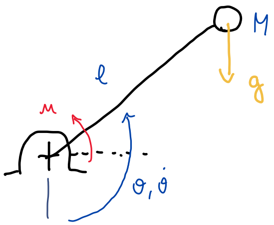
EXAMPLE: ACTUATED PENDULUM



GOAL: stabilize the pendulum to a given position θ^*

pendulum equations: $(x_1 = \theta \,,\; x_2 = \dot{\theta})$

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\dfrac{g}{\ell}\sin x_1 - \dfrac{\beta}{M\ell^2}x_2 + \dfrac{1}{M\ell^2}u \\
y_m = x \\
y_r = x_1 \\
y_r^* = \theta^*
\end{cases}
$$

we saw before that the SOLVABILITY Eqs. have the solution

$$
\begin{cases}
x_1^* = \theta^* \,, \quad x_2^* = 0 \\
u^* = Mg\ell \sin\theta^*
\end{cases}
\qquad \left[\text{in short: } x^* = (\theta^*, 0)\right]
$$

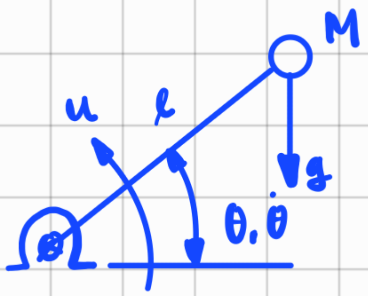
Next, we linearize the system around (x^*, u^*). We have

$$
\frac{\partial f}{\partial x} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} \quad \text{where} \quad
\begin{cases}
f_1(x,u) = x_2 \\
f_2(x,u) = -\dfrac{g}{\ell}\sin x_1 - \dfrac{\beta}{M\ell^2}x_2 + \dfrac{1}{M\ell^2}u
\end{cases}
$$

we have:

$$
\frac{\partial f_1}{\partial x_1}(x,u) = 0 \,, \quad \frac{\partial f_1}{\partial x_2}(x,u) = 1 \,, \quad \frac{\partial f_2}{\partial x_1}(x,u) = -\frac{g}{\ell}\cos x_1 \,, \quad \frac{\partial f_2}{\partial x_2}(x,u) = -\frac{\beta}{M\ell^2}
$$

Example:



$x_1 = \theta, \quad x_2 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{\beta}{Ml^2} x_2 + \dfrac{1}{Ml^2} u \\ y_m = x \\ y_r = \theta \quad y_r = \theta^* \end{cases}$$

- Step 1:

$$\begin{cases} x^* = (\theta^*, 0) \\ u^* = Mgl \sin\theta^* \end{cases} \longrightarrow f(x,u) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \end{pmatrix}$$

- Step 2:

$$A = \frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} 0 & 1 \\ -g/l \cos\theta^* & -\beta/Ml^2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{Ml^2} \end{bmatrix}$$

Based on controllability canonical form, AB is controllable.

We then have to find K so that $K = [K_1 \quad K_2]$

$$A + BK = \begin{bmatrix} 0 & 1 \\ -g/l \cos\theta^* & -\beta/Ml^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{K_1}{Ml^2} & \frac{K_2}{Ml^2} \end{bmatrix}$$

cont...

$$A + BK = \begin{bmatrix} 0 & 1 \\ \underbrace{-\frac{g}{l}\cos\theta^* + \frac{K_1}{Ml^2}}_{-a_0} & \underbrace{\frac{K_2 - \beta}{Ml^2}}_{a_1} \end{bmatrix}$$

The eigval charact. polynom. is:

$\lambda^2 + \alpha_1 \lambda + a_0$

$\alpha_1 > 0$

$\alpha_2 > 0$

$$\frac{K_2 - \beta}{Ml^2} < 0, \quad \cos\theta^* > 0, \quad \frac{K_1}{Ml^2} - \frac{g}{l}\cos\theta^* < 0,$$

$$K_1 < Mgl \overbrace{\cos\theta^*}^{-1 \leftarrow \text{worst case}}$$

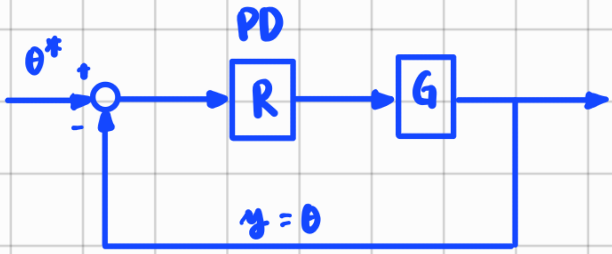
$\forall K_2 < 0$

⊙ Step 3:

Pick $u = Mgl \sin\theta^* + [K_1 \quad K_2] \begin{bmatrix} \theta - \theta^* \\ \dot\theta - 0 \end{bmatrix} \begin{matrix} \rightarrow e \\ \rightarrow \dot e \end{matrix}$

$$u = Mgl \sin\theta^* + K_1 e + K_2 \dot e$$

it's a PD controller



Thus, we obtain:

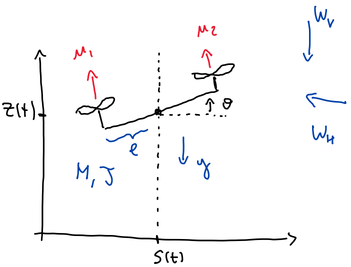
$$A = \frac{\partial f}{\partial x}(x^*, u^*) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos\theta^* & -\frac{\beta}{M\ell^2} \end{pmatrix}$$

$\leftarrow$ If $\theta^* > \frac{\pi}{2}$, the forced equilibrium x^* is unstable

$$B = \frac{\partial f}{\partial u}(x^*, u^*) = \begin{pmatrix} \frac{\partial f_1}{\partial u}(x^*, u^*) \\ \frac{\partial f_2}{\partial u}(x^*, u^*) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{M\ell^2} \end{pmatrix}$$

- Since (A, B) is controllable, we can always design K so that $A + BK$ is Hurwitz, for instance: $K = \left[M\ell g(\cos\theta^* - \varepsilon) \quad -\mu \right]$ for any $\varepsilon > 0$ and $\mu \geq 0$ (PD controller)

EXAMPLE: PLANAR DRONE



$$\begin{cases} M\ddot{s} = -(u_1 + u_2)\sin\theta - W_H - \delta\dot{s} \\ M\ddot{z} = (u_1 + u_2)\cos\theta - Mg - W_V - \delta\dot{z} \\ J\ddot{\theta} = -\beta\dot{\theta} + \ell(u_2 - u_1) \end{cases}$$

β, δ = friction coefficients

$W_H \geq 0$ and $W_V \geq 0$ = (constant) wind

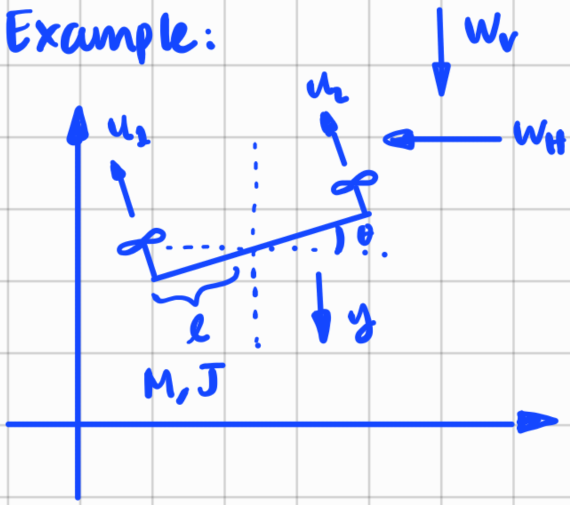
GOAL: regulate both velocities $\dot{s}$ and $\dot{z}$ to zero

$\hookrightarrow \quad y_r = \begin{pmatrix} \dot{s} \\ \dot{z} \end{pmatrix}$ and $y_r^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We neglect the vertical position and call

$$x_1 = \dot{s}, \quad x_2 = \dot{z}, \quad x_3 = \theta, \quad x_4 = \dot{\theta}$$

Example:



Model Eq:

$$\begin{cases} M\ddot{s} = -(u_1+u_2)\sin\theta - \delta\dot{s} - W_H \\ M\ddot{z} = (u_1+u_2)\cos\theta - \delta\dot{z} - W_V - Mg \\ J\ddot{\theta} = -\beta\dot{\theta} + \ell(u_2-u_1) \end{cases}$$

neglect the position dynamics

$x_1 = \dot{s}, \quad x_2 = \dot{z}, \quad x_3 = \theta \quad x_4 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 = -\frac{1}{M}(u_1+u_2)\sin x_3 - \frac{\delta}{M}x_1 - \frac{W_H}{M} \\ \dot{x}_2 = \frac{1}{M}(u_1+u_2)\cos x_3 - \frac{\delta}{M}x_2 - \frac{W_V}{M} - g \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\frac{\beta}{J}x_4 + \frac{\ell}{J}(u_2-u_1) \\ y_r = (x_1, x_2) \quad y_r^* = 0 \end{cases}$$

① Step 1:

$$\begin{cases} 0 = -(u_1^*+u_2^*)\sin x_3^* - \delta x_1^* - W_H \\ 0 = (u_1^*+u_2^*)\cos x_3^* - \delta x_2^* - W_V - Mg \\ 0 = x_4^* \\ 0 = \frac{\ell}{J}(u_2^*-u_1^*) \qquad u_1^* = u_2^* = U^* \ \# \\ x_1^* = x_2^* = 0 \end{cases}$$

#

$\#$ $2U^* \sin x_3^* = -W_H$

$2U^* \cos x_3^* = W_V + Mg$

$\tan x_3^* = -\dfrac{W_H}{W_V + Mg} \iff x_3^* = -\tan^{-1}\left(\dfrac{W_H}{W_V+Mg}\right)$

$\Downarrow$

$$(2U^*)^2 \underbrace{(\sin^2 x_3^* + \cos^2 x_3^*)}_{=1} = W_H^2 + (W_V + Mg)^2$$

$$U^* = \frac{1}{2}\sqrt{W_H^2 + (W_V + Mg)^2}$$

$$x^* = \begin{bmatrix} 0 \\ 0 \\ -\tan^{-1}\left(\dfrac{W_H}{W_V+Mg}\right) \\ 0 \end{bmatrix}$$

⊙ Step 2:

$$A: \left[\begin{array}{cc|cc} -\dfrac{\gamma}{M} & 0 & -\dfrac{2U^*}{M}\cos x_3^* & 0 \\ 0 & -\dfrac{\delta}{M} & -\dfrac{2U^*}{M}\sin x_3^* & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\dfrac{\beta}{J} \end{array}\right]$$

$$B = \begin{bmatrix} \dfrac{-\sin x_3^*}{M} & -\dfrac{\sin x_3^*}{M} \\ \dfrac{\cos x_3^*}{M} & \dfrac{\cos x_3^*}{M} \\ 0 & 0 \\ -\dfrac{l}{J} & \dfrac{l}{J} \end{bmatrix}$$

$$K = \left[\begin{array}{cc|cc} 0 & 0 & \alpha & a \\ 0 & 0 & \alpha & -\alpha \end{array}\right]$$

⊙ Step 3:

$$u(t) = \begin{pmatrix} u^* \\ u^* \end{pmatrix} + K\left(x - \overbrace{\begin{bmatrix} 0 \\ 0 \\ -\tan\left(\frac{W_H}{W_V + u_0}\right) \\ 0 \end{bmatrix}}^{x^*} \right)$$

$$U^* = \sqrt{W_H^{\,2} + (W_V + Mg)^2}$$

The model in the state space reads

$$\begin{cases} \dot{X}_1 = -\frac{1}{M}(u_1+u_2)\cdot\sin X_3 - \frac{W_H}{M} - \frac{\delta}{M}X_1 \\ \dot{X}_2 = \frac{1}{M}(u_1+u_2)\cdot\cos X_3 - g - \frac{W_V}{M} - \frac{\delta}{M}X_2 \\ \dot{X}_3 = X_4 \\ \dot{X}_4 = -\frac{\beta}{J}X_4 + \frac{\ell}{J}(u_2-u_1) \\ y_m = X \\ y_r = (X_1, X_2) \end{cases}$$

GOAL: $y_r^* = 0$

SOLVABILITY EQUATIONS:

$$\begin{cases} 0 = -(u_1^*+u_2^*)\sin X_3^* - W_H - \delta \overset{0}{\overset{\shortparallel}{X_1^*}} \\ 0 = (u_1^*+u_2^*)\cos X_3^* - Mg - W_V - \delta \overset{0}{\overset{\shortparallel}{X_2^*}} \\ 0 = X_4^* \\ 0 = -\beta \underset{0}{\underset{\shortparallel}{X_4^*}} + \ell(u_2^*-u_1^*) \\ 0 = X_1^* = X_2^* \end{cases}$$

$$\rightarrow \begin{cases} \tan X_3^* = -\frac{W_H}{Mg+W_V} \quad \rightarrow \quad X_3^* = -\operatorname{atan}\left(\frac{W_H}{Mg+W_V}\right) \\ u_1^* = u_2^* \doteq U^*, \quad U^* \doteq \frac{1}{2}\sqrt{W_H^2+(Mg+W_V)^2} \\ X_1^* = X_2^* = X_4^* = 0 \end{cases}$$

Let us linearize the system about (x^*, u^*). We have

$$\dot{x} = f(x,u) \qquad f(x,u) = \begin{pmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \\ f_4(x,u) \end{pmatrix}$$

with

$$f_1(x,u) = -\frac{1}{M}(u_1+u_2)\cdot \sin x_3 - \frac{W_H}{M} - \frac{\delta}{M} x_1$$

$$f_2(x,u) = \frac{1}{M}(u_1+u_2)\cdot \cos x_3 - g - \frac{W_V}{M} - \frac{\delta}{M} x_2$$

$$f_3(x,u) = x_4$$

$$f_4(x,u) = -\frac{\beta}{J} x_4 + \frac{\ell}{J}(u_2 - u_1)$$

Then:

$$\frac{\partial f_1}{\partial x}(x^*,u^*) = \begin{bmatrix} -\frac{\delta}{M} & 0 & -\frac{1}{M} 2U^* \cdot \cos x_3^* & 0 \end{bmatrix}$$

$$\frac{\partial f_2}{\partial x}(x^*,u^*) = \begin{bmatrix} 0 & -\frac{\delta}{M} & -\frac{1}{M} 2U^* \sin x_3^* & 0 \end{bmatrix}$$

$$\frac{\partial f_3}{\partial x}(x^*,u^*) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial f_4}{\partial x}(x^*,u^*) = \begin{bmatrix} 0 & 0 & 0 & -\frac{\beta}{J} \end{bmatrix}$$

$$\frac{\partial f}{\partial u}(x^*, u^*) = \begin{bmatrix} -\frac{1}{M}\sin x_3^* & -\frac{1}{M}\sin x_3^* \\ \frac{1}{M}\cos x_3^* & \frac{1}{M}\cos x_3^* \\ 0 & 0 \\ -\frac{l}{J} & \frac{l}{J} \end{bmatrix}$$

Thus, the linearization is given by the matrices

$$A = \begin{bmatrix} -\frac{\delta}{M} & 0 & -\frac{1}{M}2U^* \cdot \cos x_3^* & 0 \\ 0 & -\frac{\delta}{M} & -\frac{1}{M}2U^* \sin x_3^* & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{\beta}{J} \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{M}\sin x_3^* & -\frac{1}{M}\sin x_3^* \\ \frac{1}{M}\cos x_3^* & \frac{1}{M}\cos x_3^* \\ 0 & 0 \\ -\frac{l}{J} & \frac{l}{J} \end{bmatrix}$$

where $x_3^* = -\operatorname{atan}\left(\frac{W_H}{Mg + W_V}\right)$ and $U^* = \frac{1}{2}\sqrt{W_H^2 + (Mg + W_V)^2}$

The pair (A,B) is always stabilizable. Indeed, the matrix

$$K = \begin{bmatrix} 0 & 0 & \alpha & \alpha \\ 0 & 0 & -\alpha & -\alpha \end{bmatrix} \qquad \alpha > 0$$

is such that $A+BK$ is Hurwitz. It can be also shown that (A,B) is controllable whenever $x_3^* \neq k\frac{\pi}{2}$, $k \in \mathbb{N}$

The control design is then concluded by taking K such that $A+BK$ is Hurwitz and choosing

$$u(t) = u^* + K\left(x(t) - x^*\right)$$

that in this case reads: (K_{ij} is the (i,j)-th entry of K)

$$u_1 = \frac{1}{2}\sqrt{w_H^2 + (mg + w_v)^2} + K_{11}x_1 + K_{12}x_2 + K_{13}\left(x_3 + \operatorname{atan}\left(\frac{w_H}{mg + w_v}\right)\right) + K_{14}x_4$$

$$u_2 = \underbrace{\frac{1}{2}\sqrt{w_H^2 + (mg + w_v)^2}}_{\text{FEED FORWARD ACTION}} + \underbrace{K_{21}x_1 + K_{22}x_2 + K_{23}\left(x_3 + \operatorname{atan}\left(\frac{w_H}{mg + w_v}\right)\right) + K_{24}x_4}_{\text{ERROR - FEEDBACK}}$$

EXERCISE:
- study the controllability properties of the system
- design K to place the eigenvalues where desired (whenever possible)

DESIGNING AN OBSERVER

Let a model equation:

$$\begin{cases} \dot{x} = f(x,u) \\ y_m = h_m(x) \neq x \\ y_r = h_r(x) \end{cases}$$

$$u = u^* + K\overbrace{(x - x^*)}^{\tilde{x}}$$

Consider the state equation where:

$$\dot{x} = A(x - x^*) + B\underbrace{(u - u^*)}_{= \tilde{u}} + f_{HOT}(x,u)$$

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + f_{HOT}(x,u)$$

To create a Luenberger Observer, we just copy the dynamic. Hence,

$$\begin{aligned} u &= u^* + K\hat{\tilde{x}} \\ \dot{\hat{\tilde{x}}} &= A\hat{\tilde{x}} + B\tilde{u} + \underbrace{L}_{\text{observer gain}}\big[\overbrace{C_m\hat{\tilde{x}}}^{\hat{\tilde{y}}_m} - \tilde{y}_m\big] \end{aligned}$$

So $A + LC_m$ is Hurwitz

We can linearize the output:

$$\begin{aligned} \tilde{y}_m = y_m - h_m(x^*) &= h_m(x) - h_m(x^*) \\ &= C_m(x - x^*) + h_{m,HOT}(x) \end{aligned}$$

where $C_m = \dfrac{\partial h_m}{\partial x}(x^*)$

After we linearize the estimation:

$$\begin{cases} \dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + BK\hat{\tilde{x}} + L\left[C_m\hat{\tilde{x}} - \tilde{y}_m\right] \\ u = u^* + K\hat{\tilde{x}} \end{cases}$$

$$\tilde{y}_m = y_m - u_m(x^*)$$

Remember we introduced:

$\dot{\xi} = g(\xi, y_m)$, $\xi = \hat{\tilde{x}}$

$u = \gamma(\xi, y_m)$

What is $\hat{\tilde{x}}$ considering (x^*, ξ^*) where $x \longrightarrow x^*$?

It is 0 because $\hat{\tilde{x}} = x - x^*$.

NB: we don't change the coordinate in this case.

Let the controller dynamic as:

$$\begin{cases} \dot{x} = A(x - x^*) + BK\hat{\tilde{x}} + f_{HOT}(x, u) \\ \dot{\hat{\tilde{x}}} = (A + BK + LC_m)\hat{\tilde{x}} - LC_m(x - x^*) - L h_{m,HOT}(x) \end{cases}$$

$$\boxed{\text{Remember: } -L\tilde{y}_m = -L\left(C_m\tilde{x} + h_{m,HOT}(x)\right)}$$

$$\begin{bmatrix} f_{HOT}(x, u^* + K\hat{\tilde{x}}) \\ -L h_m(x) \end{bmatrix}$$

It's a tradeoff between speed of conv. and stability

In matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{\tilde{x}}} \end{bmatrix} = \underbrace{\begin{bmatrix} A & BK \\ -LC_m & A + BK + LC_m \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x - x^* \\ \hat{\tilde{x}} \end{bmatrix} + \begin{bmatrix} f_{HOT}(x, u) \\ -L h_{m,HOT}(x) \end{bmatrix}$$

LOCALITY PROBLEM

LOCAL OUTPUT - FEEDBACK SOLUTION

In this case $\boxed{y_m(x) \neq x}$, typically we only measure a part of the state

We can resort to the linear output - feedback theory, by which we implement

$$u(t) = u^* + K\hat{\tilde{x}}(t) \qquad \left(\text{instead of } u = u^* + K(x - x^*)\right)$$

in which $\hat{\tilde{x}}$ is an estimate of $\tilde{x} \doteq x - x^*$.

We can write the plant as before as

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + f_{HOT}(x, u)$$

where

$$\tilde{u} \doteq u - u^* \quad , \quad A = \frac{\partial f}{\partial x}(x^*, u^*) \quad , \quad B = \frac{\partial f}{\partial u}(x^*, u^*)$$

$$f_{HOT}(x, u) = f(x, u) - A\tilde{x} - B\tilde{u} \quad \Big) \longrightarrow \text{HIGHER - ORDER TERMS}$$

We add the output equation

$$\tilde{y}_m = h_m(x) - h_m(x^*) = C\tilde{x} + h_{m,HOT}(x)$$

where

$$C \doteq \frac{\partial h_m}{\partial x}(x^*) \qquad \text{and} \qquad \underbrace{h_{m,HOT}(x) = h_m(x) - h_m(x^*) - C\tilde{x}}_{= \text{ HIGHER ORDER TERM}}$$

If (C, A) is detectable, we can find L such that $A + LC$ is Hurwitz, and we can define the Luemberger observer

$$\boxed{\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + B\tilde{u} + L(C\hat{\tilde{x}} - \tilde{y}_m)}$$

$$= A\hat{\tilde{x}} + BK\hat{\tilde{x}} + L\left(C\hat{\tilde{x}} - y_m + h_m(x^*)\right)$$

The overall controller is then

$$(\blacktriangle) \quad \begin{cases} \dot{\hat{\tilde{x}}} = (A + BK + LC)\hat{\tilde{x}} - L\tilde{y}_m \\ u = u^* + K\hat{\tilde{x}} \end{cases} \quad \longrightarrow$$

of the form (see page 1)

$$\begin{cases} \dot{\xi} = g(\xi, y_m) \\ u = \gamma(\xi, y_m) \end{cases}$$

with

$$\xi = \hat{\tilde{x}}$$

RESULT. Suppose that (A,B) is stabilizable and (C,A) detectable. The dynamic controller $(\blacktriangle)$ locally stabilizes the equilibrium $(x, \hat{\tilde{x}}) = (x^*, 0)$. Therefore, there exists an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ containing $(x^*, 0)$ such that

$$\forall\, (x(0), \hat{\tilde{x}}(0)) \in \Omega, \qquad \lim_{t \to \infty} y_r(t) = y_r^*$$

PROOF.

Let us analyze the closed-loop system, which has state $(\tilde{x}, \hat{\tilde{x}})$. Changing variables from $\hat{\tilde{x}}$ to

$$e := \hat{\tilde{x}} - \tilde{x}$$

leads to the closed-loop system

$$\dot{\tilde{x}} = A\tilde{x} + B\underbrace{\tilde{u}}_{\tilde{u} = K\hat{\tilde{x}} = K(e + \tilde{x})} + f_{HOT}(x,u) = (A+BK)\tilde{x} + BKe + f_{HOT}(x,u)$$

$$\dot{e} = \dot{\hat{\tilde{x}}} - \dot{\tilde{x}} = (A + BK + LC)\hat{\tilde{x}} - L\tilde{y}_m - (A+BK)\tilde{x} - BKe - f_{HOT}(x,u)$$

$$= (A+BK+LC)(e+\tilde{x}) - (A+BK)\tilde{x} - BKe - L\left(C\tilde{x} + h_{m,HOT}(x)\right) - f_{HOT}(x,u)$$

$$= (A+LC)e - \left(f_{HOT}(x,u) + L\,h_{m,HOT}(x)\right)$$

Proof (in more direct way)

$$T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \Leftrightarrow T^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

Locality problem

So,

$$T \begin{bmatrix} x \\ \hat{\tilde{x}} \end{bmatrix} = \begin{bmatrix} x \\ \hat{\tilde{x}} - x \end{bmatrix}$$

$$\boxed{\bar{A} = \begin{bmatrix} A & BK \\ -LC_m & A+BK+LC_m \end{bmatrix}}$$

What is $T\bar{A}T^{-1}$?

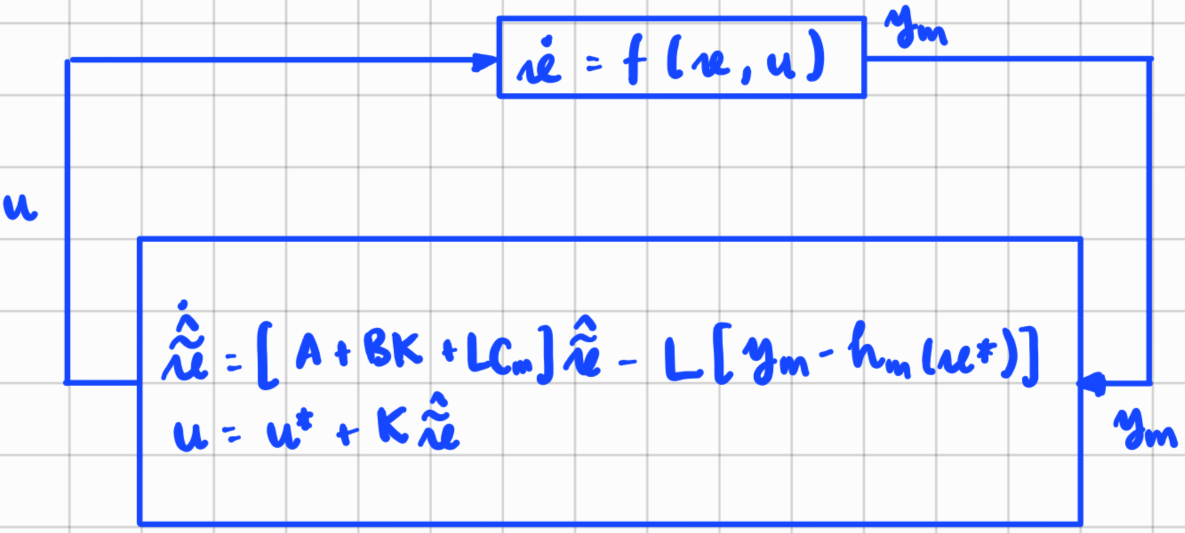
$$T\bar{A} = \begin{bmatrix} A & BK \\ -A-LC_m & A+LC_m \end{bmatrix}$$

$$\boxed{T\bar{A}T^{-1} = \left[\begin{array}{c|c} A+BK & BK \\ \hline 0 & ALC_m \end{array}\right]}$$

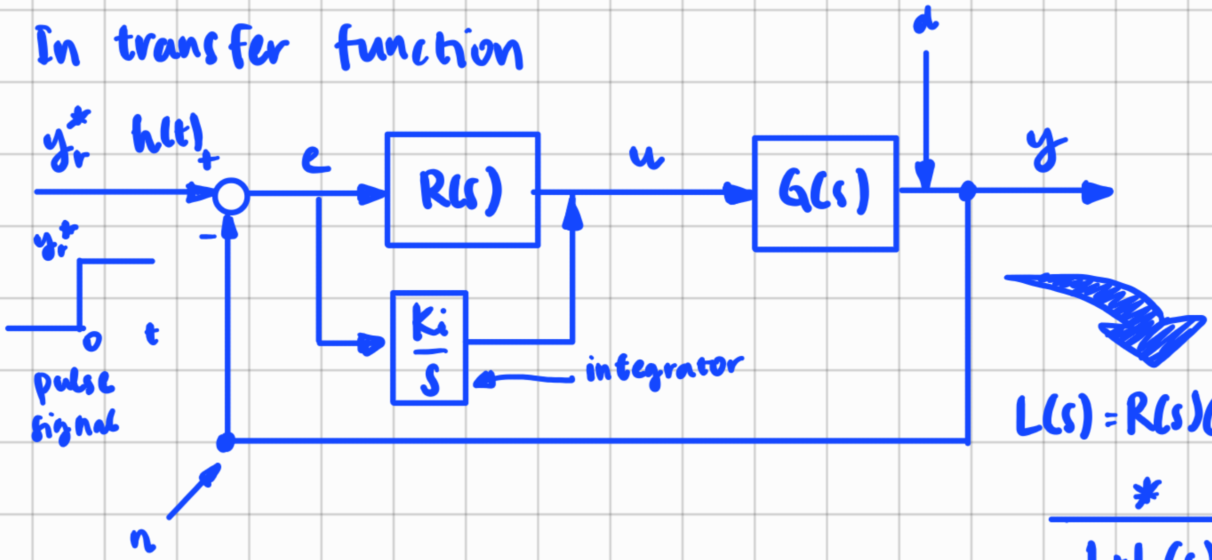
The robustness problem

$u^*,\ h_m(x^*)$
$\quad\quad\quad\ \ \| $
$\quad\quad\quad\ y_m^*$

Assume:
$y_m = y_r$



In transfer function



$L(s) = R(s)G(s)$

$$\frac{*}{1 + L(s)}$$

$$L(s) = \frac{1}{s} L'(s) \quad \text{where } L'(0) \neq 0$$

$$E(j\omega) = \frac{1}{1 + L(j\omega)} \cdot Y_R^*(j\omega)$$

The error function :

$$e(t) \approx \left| \frac{1}{1+L(j\bar{\omega})} \right| \cdot Y_r^* = 0 \Rightarrow L(j\omega) \xrightarrow{\omega \to 0} \infty$$

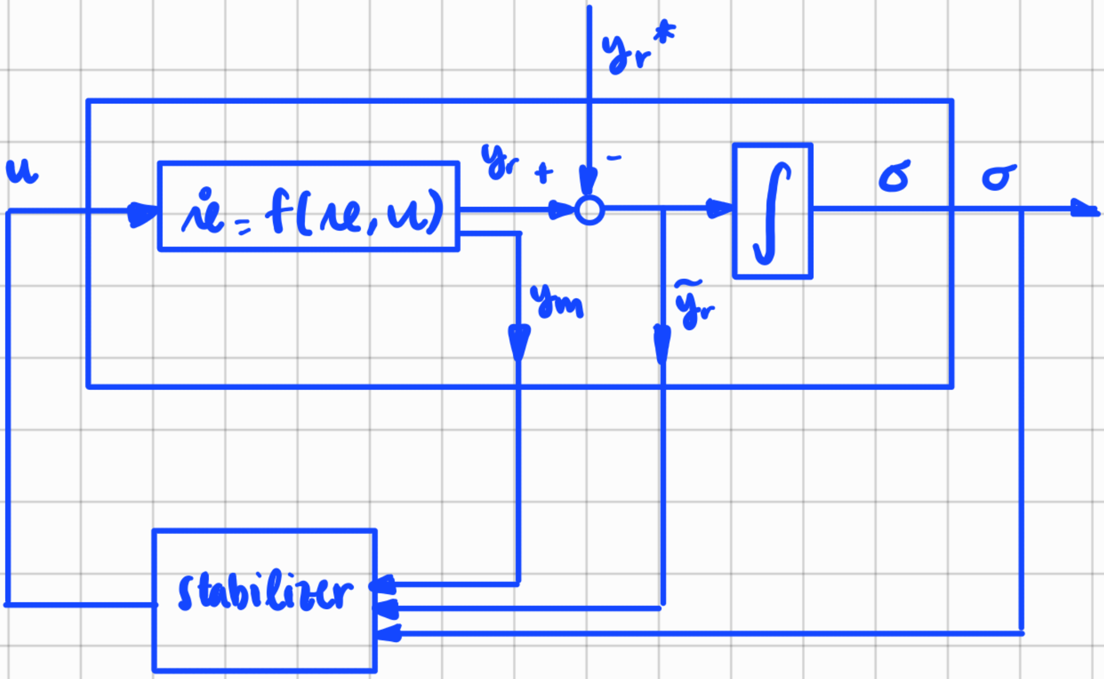
$$y_r^*(t) = Y_r^*$$

Say we have $\frac{1}{s}E(s) \longrightarrow \int_0^t e(\tau)\,d\tau = \sigma(t)$

$$\Sigma(s) = \frac{1}{s}E(s) \longrightarrow \int_0^t e(\tau)\,d\tau = \sigma(t)$$

$$\dot{\sigma} = 0$$

In non-linear system:



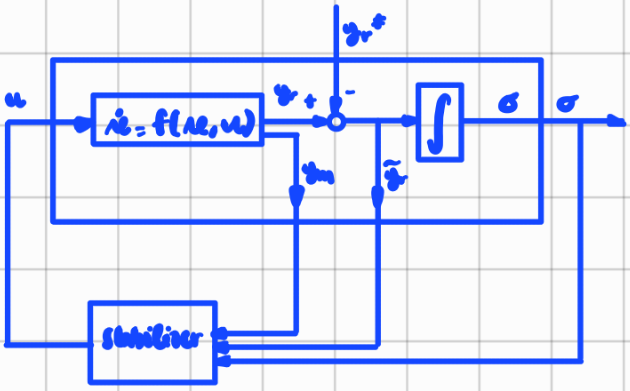
* $y_m = \begin{bmatrix} x \\ \tilde{y}_r \end{bmatrix}$

* $n_u = n_r \quad \begin{cases} n_u = \dim(u) \\ n_r = \dim(y_r) \end{cases}$

$\dot{x} = f(x, u)$

$y_r = h_r(x), \quad \tilde{y}_r = y_r - y_r^*$

$\begin{cases} \dot{\sigma} = \tilde{y}_r \\ u = \hat{u}^* + K_1(x - \hat{x}^*) + K_2\sigma \end{cases}$



$\dot{\sigma}_i = \tilde{y}_{r,i} \quad \forall i = 1, \dots, n_r$

K_1: modifiable $\quad$ K_2: modifiable

Why with wrong estimate $\hat{x}^*$ can still lead to stability?

$\dot{x} = A(x - x^*) + B(u - u^*) + f_{HOT}(x, u)$

$\dot{\sigma} = h_r(x) - y_r^* = h_r(x) - h_r(x^*)$

$= C_r(x - x^*) + h_{r,HOT}(x)$

$u = \hat{u}^* + K_1(x - x^* + x^* - \hat{x}^*) + K_2\sigma$

$\dot{x} = A(x - x^*) + BK_1(x - x^*) + BK_2\sigma$

$+ B(\hat{u}^* - u^* + K_1(x^* - \hat{x}^*)) + f_{HOT}(x, u)$

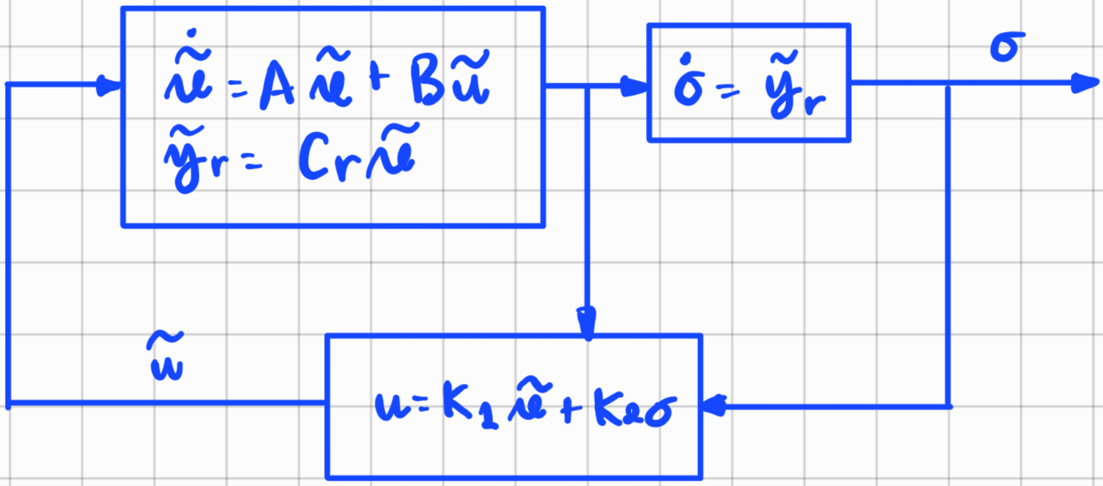
$\dot{\sigma} = C_r(x - x^*) + h_{r,HOT}(x)$

The matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix} \begin{bmatrix} x - x^* \\ \sigma \end{bmatrix} + \begin{bmatrix} f_{NOT}(x,u) \\ h_r(x) \end{bmatrix}$$

$$+ \begin{bmatrix} B \\ 0 \end{bmatrix} \left[\hat{u}^* - u^* + K_1 (x^* - \hat{x}^*) \right]$$

consider a linear system



This cascade is stabilizable if:

$$\bar{A} = \begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}$$

$\exists K_1, K_2 : \bar{A}$ is Hurwitz

1) A, B are stabilizable

2) $\text{rank} \begin{bmatrix} A & B \\ C_r & 0 \end{bmatrix} = n_x + n_r$
(non-resonance cond.)

Proof using PBH-test

$$\begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ C_r & 0 \end{bmatrix}}_{A_o} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_o} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

this is stabilizable if A_o and B_o stabilizable

$$\boxed{\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y}_r &= C_r\tilde{x} \end{aligned}} \longrightarrow \boxed{\dot{\sigma} = \tilde{y}_r}$$

We found that

$$\begin{cases} \dot{x} = f(x, u) \\ y_r = h_r(x), \quad y_r^* \\ y_m = (x, \underbrace{y_r - y_r^*}_{\tilde{y}_r}) \end{cases} \longrightarrow \begin{cases} \dot{\sigma} = \tilde{y}_r \\ u = \hat{u}^* + K_1(x - \hat{x}^*) \\ \qquad + K_2 \sigma \end{cases}$$

where $n_u = n_r$

From those expressions above, we can form:

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x - x^* \\ \sigma \end{bmatrix} + \begin{bmatrix} f_{HOT}(x, u) \\ h_{r,HOT}(x) \end{bmatrix}$$

$$+ \begin{bmatrix} B \\ 0 \end{bmatrix} \left[\hat{u}^* - u^* + K_1(x^* - \hat{x}^*) \right]$$

where $A = \frac{\partial f}{\partial x}(x^*, u^*)$, $B = \frac{\partial f}{\partial u}(x^*, u^*)$, $C_r = \frac{\partial h_r}{\partial x}(x^*)$

The result : $\exists K_1, K_2$ such that $\bar{A}$ is Hurwitz if:

1) (A, B) is stabilizable
2) $\text{Rank} \begin{bmatrix} A & B \\ C_r & 0 \end{bmatrix} = \text{n. of rows} = n_x + n_r$

NON-RESONANCE CONDITION

Proof: $\bar{A} = \underbrace{\begin{bmatrix} A & 0 \\ C_r & 0 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_0} \underbrace{\begin{bmatrix} K_1 & K_2 \end{bmatrix}}_{K}$

$\exists K_1, K_2$ such that $\bar{A}$ is Hurwitz iff (A_0, B_0) stabilizable

PBH — Controllability → $\operatorname{rank}\begin{bmatrix}\lambda I - A_0 & B_0\end{bmatrix}$ $= \text{n. of rows} \ \forall \lambda$ (full)

— Stabilizability → $\operatorname{rank}\begin{bmatrix}\lambda I - A_0 & B_0\end{bmatrix}$ $= \text{n. of rows} \ \forall \lambda : \text{Real}[\lambda] \geq 0$ (full)

recommended to study the geometry

The rank of $\begin{bmatrix}\lambda I - A & 0 & B \\ -C_r & \lambda I & 0\end{bmatrix}$ is:

* CASE 1 : $\lambda \neq 0$

$$\begin{bmatrix}\lambda I - A & B & 0 \\ -C_r & 0 & \lambda I\end{bmatrix}$$

$$\operatorname{rank}\left[\begin{array}{c|cc}\lambda I - A & 0 & B \\ -C_r & \lambda I & 0\end{array}\right] = \text{n. of rows (full rank)}$$

switch the col. → to see the matrix is full rank

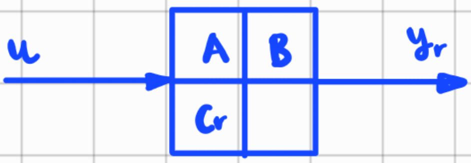
* CASE 2 : $\lambda = 0$

n. of rows (full rank)

$$\operatorname{rank}\begin{bmatrix}-A & 0 & B \\ -C_r & 0 & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix}A & B & 0 \\ C_r & 0 & 0\end{bmatrix}$$

switch the col. $= \text{n. of rows (full rank)}$

It indicates:



Back to the closed-loop system:

$$\begin{bmatrix}\dot{x} \\ \dot{\sigma}\end{bmatrix} = \underbrace{\begin{bmatrix}A + BK_1 & BK_2 \\ C_r & 0\end{bmatrix}}_{}\begin{bmatrix}x - x^* \\ \sigma\end{bmatrix} + \begin{bmatrix}f_{HOT}(x, u) \\ h_{r\,HOT}(x)\end{bmatrix}$$

$$= \bar{A} + \begin{bmatrix}B \\ 0\end{bmatrix}\left[\hat{u}^* - u^* + K_1(x^* - \hat{x}^*) \ \circledast - K_2\sigma^*\right)$$

Fix K_1, K_2 such that A is Hurwitz:

Set $x = x^*$, $\sigma = 0$, we got

$u = \hat{u}^* + K_1(x - \hat{x}^*) + K_2\sigma = \hat{u}^* + K_1(x^* - \hat{x}^*) \neq u^*$.

We find that $\sigma \neq 0$!

Remember regulator $\begin{cases} \dot{\xi} = g(\xi, y_m) \\ u = \gamma(\xi, y_m) \end{cases}$ ⊛

(x^*, ξ^*) asymptot. stable equilibrium,

$h_r(x^*) = y_r^*$

Solv. eq:

$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$

Regulator eq.

$\begin{cases} 0 = g(\xi^*, y_m^*) \\ u^* = \gamma(\xi^*, y_m^*) \end{cases}$ ⊛

Make $\xi = \sigma$, $g(\xi, y_m) = \tilde{y}_r$, $\gamma(\xi, y_m) = \hat{u}^* + K_1(x - \hat{x}^*) + K_2\sigma$

⊛ $\begin{cases} \dot{\sigma} = \tilde{y}_r \\ u = \hat{u}^* + K_1(x - \hat{x}^*) + K_2\sigma \end{cases}$

⊛ $\begin{cases} 0 = 0 \\ u^* = \hat{u}^* + K_1(x^* - \hat{x}^*) + \underbrace{K_2\sigma^*}_{} \end{cases}$

$K_2\sigma^* = u^* - \hat{u}^* + K_1(\hat{x}^* - x^*)$ ⊛

$n_r = n_u$

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A+BK_1 & BK_2 \\ C_r & 0 \end{bmatrix} \begin{bmatrix} x - x^* \\ \sigma \end{bmatrix} - \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix} K_2 \sigma^*}_{\begin{bmatrix} 0 & -BK_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x - x^* \\ \sigma^* \end{bmatrix}} + \begin{bmatrix} f_{HOT}(x, u) \\ h_{r,HOT}(x, u) \end{bmatrix}$$

$$= \begin{bmatrix} A+BK_1 & BK_2 \\ C_r & 0 \end{bmatrix} \begin{bmatrix} x - x^* \\ \sigma - \sigma^* \end{bmatrix} - \begin{bmatrix} f_{HOT}(x, u) \\ h_{r,HOT}(x, u) \end{bmatrix}$$

$$u \Big|_{\substack{x = x^* \\ \sigma = \sigma^*}} = \hat{u}^* + K_1(x - \hat{x}^*) + K_2\sigma \underset{-K_2\sigma^*}{} \Big|_{\substack{x = x^* \\ \sigma = \sigma^*}}$$

$$= \hat{u}^* + K_1(x^* - \hat{x}^*) + K_2\sigma^*$$

$$= u^*$$

The good news is: result if $\begin{bmatrix} A+BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}$ Hurwitz,

then K_2 is invertible!

Proofing by contradiction :

Assume $\begin{bmatrix} A+BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}$ Hurwitz and K_2 is not invertible.

K_2 is not invertible $\Rightarrow \exists\, w \neq 0$ such that $\underline{K_2 w = 0}$

$\mathbb{R}^{n_r}$

$$\begin{bmatrix} A+BK_1 & BK_2 \\ C_r & 0 \end{bmatrix} \begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} BKw \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ w \end{bmatrix} \longrightarrow \text{eigval eq!}$$

$$\begin{bmatrix} A+BK_1 & BK_2 \\ C_r & 0 \end{bmatrix} v = \lambda v \iff \lambda = 0,\ v = \begin{bmatrix} 0 \\ w \end{bmatrix} \neq 0$$

$\bar{A}$ cannot be Hurwitz if K_2 is not invertible

All in all, it results $\begin{cases} \dot{\sigma} = \tilde{y}_r \\ u = \hat{u}^* + K_1(x - \hat{x}^*) + K_2\sigma \end{cases}$

where K_1, K_2, such that $\begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}$ is Hurwitz,

where $A = \frac{\partial f}{\partial x}(x^*, u^*)$, $B = \frac{\partial f}{\partial u}(x^*, u^*)$, $C_r = \frac{\partial h_r}{\partial x}(x^*)$

is such that $\exists \sigma^*$ such that (x^*, σ^*) is a Lyapunov Asymp. Stable equilibrium for the closed loop system.

Hence $\exists \mathcal{D} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ open continuity (x^*, σ^*) such that $\forall (x(0), \sigma(0)) \in \mathcal{D}$, $\lim_{t \to \infty} y_r(t) = y_r^*$

No matter how badly we estimate $\hat{u}^*$ and $\hat{x}^*$, can it affect at all? Not quite really because K_1 and K_2 are based on K_1 and K_2.

$$\begin{cases} \dot{\sigma} = \tilde{y}_r, \quad \sigma(0) = 0 \\ u = \underset{u^*}{\hat{u}^*} + K_1(x - \underset{x^*}{\hat{x}^*}) + K_2\sigma \end{cases}$$

Because we need $\underbrace{\sigma(0) - \sigma^*}_{= -\sigma^*}$ small enough

Then, what we can do if we already knew u^* and x^*?

$u = u^* + K(x - x^*)$ from $K_2\sigma^* = u^* - \hat{u}^* + K_1(\hat{x}^* - x^*)$

will result:

$$u = u^* + K_1(x - x^*) + K_2(\sigma - \sigma^*)$$

$$= u^* - K_2\sigma^* + K_1(\cancel{x^*} - x^*) + K_1(x - \cancel{x^*}) + K_2\sigma$$

$$= u^* + K_1(x - x^*) + K_2(\sigma - \sigma^*)$$

Because of the integration until these parameters

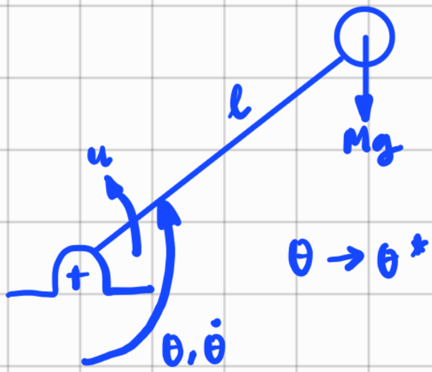
1) σ approaches σ^*
2) x approaches x^*
3) u approaches u^*

$$f_{HOT}(x, u^* + K_1(x - x^*) + K_2(\sigma - \sigma^*))$$

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} & * & \\ & & \end{bmatrix} \begin{bmatrix} x - x^* \\ \sigma - \sigma^* \end{bmatrix} + \begin{bmatrix} f_{HOT}(x, u) \\ h_{r,HOT}(x) \end{bmatrix}$$

if we estimate u^* and x^* wrong enough, it messes the HOT!

PENDULUM EXAMPLE



$\theta \to \theta^*$

$$u = Mgl\cos\theta^* + K\left(x - \begin{bmatrix} \theta^* \\ 0 \end{bmatrix}\right)$$

$$x^* = \begin{bmatrix} \theta^* \\ 0 \end{bmatrix} \qquad u^* = Mgl\cos\theta^*$$

$$A = \frac{\partial f}{\partial x}(x^*, u^*) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta^* & -\frac{\beta}{Ml^2} \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(x^*, u^*) = \begin{bmatrix} 0 \\ \frac{1}{Ml^2} \end{bmatrix}$$

The error equation would be:

$$\begin{cases} \dot{\sigma} = x_1 - \theta^* \\ u = K_1\left(x - \begin{bmatrix} \theta^* \\ 0 \end{bmatrix}\right) + K_2\sigma \end{cases}$$

$$\hat{u}^* = 0, \quad \hat{x}^* = x^* = \begin{bmatrix} \theta^* \\ 0 \end{bmatrix}$$

Design K_1 and K_2 so the system is stabilizable.

Check rank $\begin{bmatrix} A & B \\ C_r & 0 \end{bmatrix} = 3$ where $C_r = [1 \quad 0]$.

full rank.

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{g}{l}\cos\theta^* & -\frac{\beta}{Ml^2} & \frac{1}{Ml} \\ \hline 1 & 0 & 0 \end{array}\right] \longrightarrow \text{proofed it is full rank}$$

$$K_1 = [K_{11} \; K_{12}]$$

Form a matrix $\begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{g}{l}\cos\theta^* + \frac{K_{11}}{Ml^2} & \frac{K_{12}-\beta}{Ml^2} & \frac{K_2}{Ml^2} \\ 1 & 0 & 0 \end{bmatrix}$$

Find the determinant of eigval. matrix

$$\det \begin{bmatrix} \lambda & -1 & 0 \\ \frac{g}{l}\cos\theta^* - \frac{K_{11}}{Ml^2} & \lambda - \frac{K_{12}-\beta}{Ml^2} & -\frac{K_{2}}{Ml^2} \\ -1 & 0 & \lambda \end{bmatrix}$$

$$= \lambda^2\left(\lambda + \frac{\beta - K_{12}}{Ml^2}\right) + \lambda\left(\frac{g}{l}\cos\theta^* - \frac{K_{11}}{Ml^2}\right) - \frac{K_2}{Ml^2}$$

$$= \lambda^3 + \underbrace{\frac{\beta - K_{12}}{Ml^2}}_{\alpha_2}\lambda^2 + \underbrace{\left(\frac{g}{l}\cos\theta^* - \frac{K_{11}}{Ml^2}\right)}_{\alpha_1}\lambda \underbrace{- \frac{K_2}{Ml^2}}_{\alpha_0}$$

$$\begin{cases} \alpha_2 > 0, \; \alpha_1 > 0, \; \alpha_0 > 0 \\ \alpha_1 \cdot \alpha_2 > \alpha_0 \end{cases} \Rightarrow \text{Real}[\lambda] < 0$$

$\hookrightarrow$ it's ROUTH-HURWITZ

Because *) $\alpha_0 > 0$, $-\frac{K_2}{Ml^2} > 0 \iff K_2 < 0$

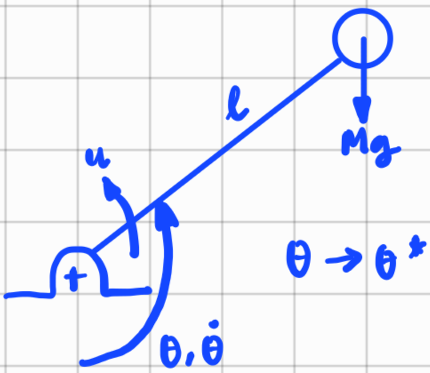
*) $\frac{g}{l}\cos\theta^* - \frac{K_{11}}{Ml^2} > 0 \iff K_{11} < Mgl\cos\theta^*$

*) $\frac{\beta - K_{12}}{Ml^2} > 0 \iff K_{12} < \beta$ //

Then, $\left(\frac{g}{l}\cos\theta^* - \frac{K_{11}}{Ml^2}\right) \cdot \underbrace{\frac{\beta - K_{12}}{Ml^2}}_{>0} > \underbrace{\frac{K_2}{Ml^2}}_{>0}$

This implies K_{11} with any negative number

So, we can fix $K_2 < 0$, $K_{12} < 0$, $K_{11} < 0$ with $|K_{11}|$ is VERY LARGE!



Last time we arrived at

$$\begin{cases} \dot{\sigma} = x_1 - \theta^* \\ u = K_{11}(x_1 - \theta^*) + K_{12}x_2 + K_2\sigma \end{cases}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$

We can set $K_2 < 0$, $K_{12} < 0$, $K_{11} < 0$ and $|K_{11}|$ could be LARGE.

$$\begin{bmatrix} A + BK_1 & BK_2 \\ C_r & 0 \end{bmatrix}$$

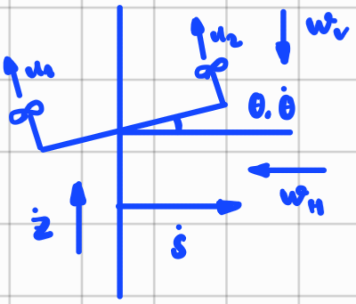
So, $\sigma^* = \frac{1}{K_2}u^* = \frac{1}{K_2}Mgl\sin\theta^*$

$$e(t) = \tilde{y}_r(t) = x_3(t) - \theta^*$$
$$\dot{e}(t) = x_2(t)$$

$$\sigma(t) = \int_0^t e(\tau)\,d\tau$$

$$u = \underbrace{K_{11}\, e}_{P} + \underbrace{K_{12}\, \dot{e}}_{D} + \underbrace{K_2 \int_0^t e(\tau)\,d\tau}_{I}$$

DRONE EXAMPLE



$$y_r = (\dot{s}, \dot{z}) \longrightarrow y_r^* = 0$$

$x_1 = \dot{s}$ $\quad x_3 = \theta$

$x_2 = \dot{z}$ $\quad x_4 = \dot{\theta}$

The model equation is:

$$u(t) = \begin{bmatrix} \frac{1}{2}\sqrt{w_H^2 + (Mg + w_v)^2} \\ \\ \frac{1}{2}\sqrt{w_H^2 + (Mg + w_v)^2} \end{bmatrix} + K \begin{bmatrix} x_1 \\ x_2 \\ x_3 + \tan^{-1}\left(\frac{w_H}{Mg + w_v}\right) \\ x_4 \end{bmatrix}$$

$$\dot{\sigma} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \sigma \in \mathbb{R}^2$$

$\longrightarrow$ estimate no wind

$$\Longrightarrow \quad u = \underbrace{\begin{bmatrix} \frac{1}{2}\hat{M}\hat{g} \\ \\ \frac{1}{2}\hat{M}\hat{g} \end{bmatrix}}_{\hat{u}^*} + \underset{\mathbb{R}^{2\times 4}}{\underset{\Uparrow}{K_1}}\, x + \underset{\mathbb{R}^{2\times 2}}{\underset{\Uparrow}{K_2}}\, \sigma$$

$$\hat{x}^* = 0$$

hence :

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} A+BK & BK \\ 0 & A+LC \end{pmatrix}}_{\text{LINEAR PART}} \begin{pmatrix} \hat{x} \\ e \end{pmatrix} + \underbrace{\begin{pmatrix} f_{HOT}(x,u) \\ -f_{HOT}(x,u) - L h_{m,HOT}(x) \end{pmatrix}}_{\text{HIGHER ORDER TERMS}}$$

The linear part is Hurwitz (recall the SEPARATION PRINCIPLE) $\rightarrow$ By Lyapunov indirect theorem we conclude that $(\hat{x}, e) = 0$ is LAS

Thus we conclude that the equilibrium point $(x^*, 0)$ is LAS for $(x, \hat{x})$

PROBLEMS OF CONTROL VIA LINEARIZATION :

P1) LOCALITY the guarentees are given by Lyapunov's indirect Theorem that, However, only guarentees the existence of an open non-empty domain of attraction, but it does not say anything about its size

sometimes taking large gains K enlarges the domain of attraction. But this is not always the case.

$\downarrow$

this can be seen in the drone example and in the examples below.

EXAMPLE consider the system

$$\dot{x} = xu + u$$

and let us try to stabilize $x^* = 0$ (requiring $u^* = 0$).
The linear part of the dynamics is defined as

$$A = u^* = 0$$
$$B = x^* + 1 = 1$$

The state-feedback controller is: $u(t) = u^* + K\,(x(t) - x^*) = K\,x(t)$ for some $K<0$

This leads to

$$\dot{x} = (x+1)\,Kx = K\,(x^2 + x) \qquad (+)$$

No matters how large is $|K|$, for all $x(0) < -1$ the closed-loop trajectories are divergent.

$\Rightarrow$ we cannot enlarge the domain of attraction

In particular the solutions of $(+)$ are (recall, $K<0$)

$$x(t) = x_0 \frac{e^{Kt}}{x_0 + 1 - x_0 e^{Kt}} \rightarrow \text{ If } x_0 < -1, \ \exists\, \bar{t} = \bar{t}(x_0) > 0 \text{ s.t. } \lim_{t \to \bar{t}} x(t) = \infty$$

P2. ROBUSTNESS ISSUE The controller depends on the quantities x^* and u^* that are highly sensitive to model uncertainties

To find x^* and u^* we need to solve the SOLVABILITY Eqs.

$$\begin{cases} 0 = f(x^*, u^*) \\ y_r^* = h_r(x^*) \end{cases}$$

Any slight uncertainty in the knowledge of f, h_m, and h_r may produce some wrong values of x^* and u^*

$\downarrow$

A control law that depends so critically from x^* and u^* is FRAGILE and its implementation in practice may lead to problems

(TYPICAL PROBLEM OF FEEDFORWARD CONTROL)

EXAMPLE

- In the previous example regarding the actuated pendulum we had:

$$\begin{cases} x^* = (\theta^*, 0) \\ u^* = Mg\ell \sin\theta^* \end{cases}$$

$\rightarrow$ u^* depends on $\begin{cases} \bullet \text{ mass } M \\ \bullet \text{ length } \ell \\ \bullet \text{ gravity } g \end{cases}$ ALL UNCERTAIN QUANTITIES !

- In the previous example of the planar drone, we had:

$$\begin{cases} x_3^* = -\operatorname{atan}\left(\dfrac{W_H}{Mg + W_V}\right) \\ u_1^* = u_2^* = \dfrac{1}{2}\sqrt{W_H^2 + (Mg + W_V)^2} \end{cases}$$

$\rightarrow$ both x^* and u^* depend on the wind (W_H and W_V), the gravity (g) and the mass (M), all uncertain quantities!

Some of the effects of using wrong x^* and u^* are:

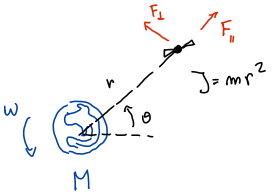
- wrong set-point: the state may not converge to x^*

  $\rightarrow$ $\lim_{t \to \infty} y_p(t) \neq y_p^*$

- destabilization: $x(t)$ may exit the stability domain

EXAMPLE: GEOSTATIONARY SATELLITE STABILIZATION

MODEL (relative to a ref. frame rotating with the Earth)



gravity   centrifugal force

$$
\begin{cases}
m\ddot{r} = -\dfrac{GmM}{r^2} + mr(\dot{\theta}+w)^2 + F_\parallel \\[2ex]
J\ddot{\theta} = -2mr\dot{r}(\dot{\theta}+w) + rF_\perp
\end{cases}
$$

thrust

torque due to the thrust

where:

G = Gravitational constant $\simeq 6.675 \cdot 10^{-11} \dfrac{m^3}{kg \cdot s^2}$

M = Earth's mass $\approx 5{,}972 \cdot 10^{24}$ Kg

w = Earth's rotational speed $\simeq 2\pi \dfrac{rad}{24H} \simeq 7.3 \cdot 10^{-5} \dfrac{rad}{s}$

GOAL: stabilize the satellite to a fixed angle θ^* relative to the Earth

Let $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, $u_1 = F_\parallel$, $u_2 = F_\perp$

Then, we obtain the following model in the state space

$$
\begin{cases}
\dot{x}_1 = x_2 \\[1ex]
\dot{x}_2 = -\dfrac{GM}{x_1^2} + x_1(x_4+w)^2 + \dfrac{1}{m}u_1 \\[1ex]
\dot{x}_3 = x_4 \\[1ex]
\dot{x}_4 = -2\dfrac{x_2}{x_1}(x_4+w) + \dfrac{1}{m x_1}u_2 \\[1ex]
y_r = x_3
\end{cases}
$$

with objective

$$y_r^* = \theta^*$$

Let us solve the SOLVABILITY EQUATIONS :

$$\begin{cases} 0 = x_2^* \\ 0 = -\dfrac{GM}{x_1^*} + x_1^* \left(x_4^* + w \right)^2 + \dfrac{1}{m} u_1^* \\ 0 = x_4^* \\ 0 = -2 \dfrac{x_2^*}{x_1^*} \left(x_4^* + w \right) + \dfrac{1}{m x_1^*} u_2^* \\ \theta^* = x_3^* \end{cases}$$

These give :

$$\begin{cases} x_2^* = 0 \\ x_3^* = \theta^* \\ x_4^* = 0 \\ u_2^* = 0 \\ 0 = -\dfrac{GM}{x_1^*} + x_1^* w^2 + \dfrac{1}{m} u_1^* \end{cases}$$

Note: for every value x_1^* for the altitude of the satellite we can find u_1^* for which the problem is solvable

Conversely, given a value u_1^* for the steady-state control action, we can find an altitude x_1^* for which the problem is solvable

To save as much energy as possible let us take $\boxed{u_1^* = 0}$

The solvability conditions give:

$$x_1^* = \sqrt[3]{\frac{GM}{\omega^2}} \approx 42\,241 \text{ km}$$

At this hight, the centrifugal effect balances gravity, and we can keep a satellite in a geostationary orbit "for free"

$\downarrow$

It is important though to stabilize such an orbit as otherwise any small perturbation would result in a trajectory that either falls toward the Earth, or diverges into deep space.

Let us compute the linearization around $x^* = (x_1^*, 0, 0, 0)$, $u^* = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\frac{GM}{(x_1^*)^3} + \omega^2 & 0 & 0 & 2x_1^*\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\frac{\omega}{x_1^*} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2R\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\frac{\omega}{R} & 0 & 0 \end{bmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & \frac{1}{m x_1^*} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & \frac{1}{mR} \end{pmatrix}$$

The reachability matrix reads

$$R = \Bigg[\underbrace{\begin{matrix} 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & \frac{1}{mR} \end{matrix}}_{B} \quad \underbrace{\begin{matrix} 1/m & 0 \\ 0 & \frac{2w}{m} \\ 0 & \frac{1}{mR} \\ -\frac{2W}{mR} & 0 \end{matrix}}_{AB} \quad \underbrace{\cdots}_{\cdots} \Bigg]$$

we can stop here as this is full rank

$\operatorname{rank} R = 4 \implies$ the system is controllable

Hence, we can find K so as $A+BK$ is Hurwitz, an a state-feedback control law is

$$u(t) = K\left(x(t) - x^*\right) \qquad \text{with} \quad x^* = \begin{pmatrix} \sqrt[3]{\frac{GM}{w^2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Can we stabilize the geostationary orbit via OUTPUT-FEEDBACK from the output $y_m = \vartheta$?

In this case we have

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

and the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2w}{\bar{r}} & 0 & 0 \\ -\frac{6w^3}{\bar{r}} & 0 & 0 & -4w^2 \end{bmatrix}$$

$\mathcal{O}$ is full-rank! so we can build a local output-feedback control system as detailed by the theory:

$$\begin{cases} \dot{\hat{\tilde{x}}} = (A + BK + LC)\,\hat{\tilde{x}} - L\,y_m \\ u = K\,\hat{\tilde{x}} \end{cases} \qquad (y_m^* = x_3^* = 0)$$

where L is such that $A + LC$ is Hurwitz

EXERCISE: find a possible value for K and L