Stability

A behavior displayed in systems where a perturbation / disturbance is applied.

Epsilon - Delta Stability Notion

Norm
$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots}$
(Distance to origin)

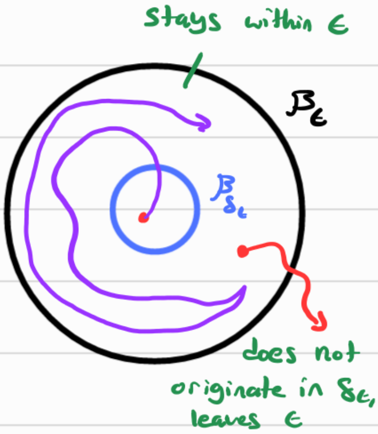
- Three ways to describe an equilibrium point's (ex: $x=0$) stability:

given limit of perturbed trajectory

1) **Stable**: for all ϵ, there exists a δ_ϵ, dependent on ϵ, where all initial conditions within $\|x(t=0)\| \leq \delta_\epsilon$ will not exceed ϵ

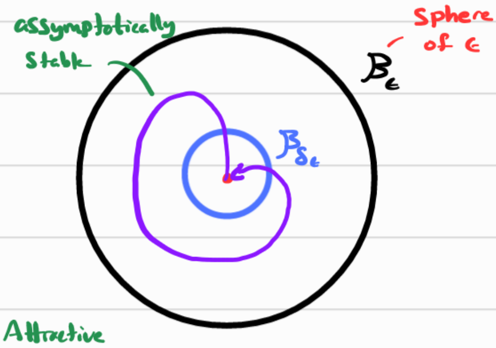
$[\|x(t)\| \leq \epsilon;\ \forall\, t>0]$.

$\forall \epsilon > 0, \exists \delta_\epsilon > 0 : \|x(t)\| \leq \epsilon \ \forall\, t>0 \ \text{ if } \ \|x(0)\| \leq \delta_\epsilon$

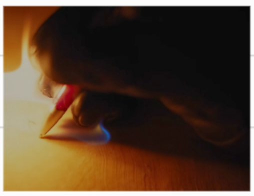


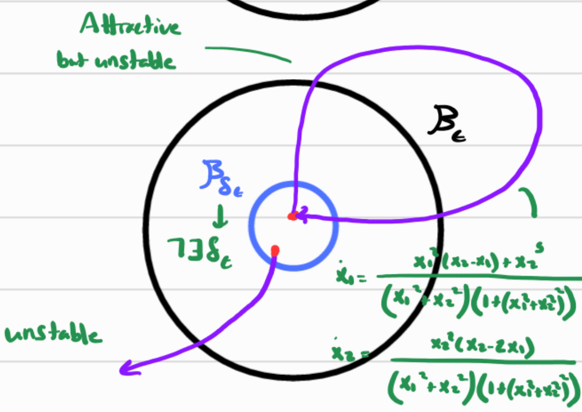
2) **Assymptotically Stable**: Stable with the addition of a attraction domain ($\mathcal{A}$) where $\lim_{t\to\infty}(x(t)) = 0$. $\mathcal{A} \supset \{0\}$.

Attractivity does not imply stability.



3) **Unstable**: the equilibrium point is unstable if it is not stable.





* Linear Systems:-

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) = A\,\phi(t)\,x_0 = A\,e^{At}x_0 \\ x(t+1) &= A\,x(t) = A\,\phi(t)\,x_0 = A\,A^t x_0 \end{aligned}\right]\ x_0 = x(0)$$

- So since $\|x(t)\| = \|\phi(t)x_0\| \leq \epsilon \ \ \forall t>0$, $\|\phi(t)x_0\| \leq \|\phi(t)\|\ \underbrace{\|x_0\|}_{\leq \delta_\epsilon} \leq \|\phi(t)\|\,\delta_\epsilon$, thus:

given $\|\phi(t)\| \leq M \ \ \forall t \geq 0$,

then if $\exists M>0$ then the system is stable. $\left(\delta_\epsilon = \dfrac{\epsilon}{M}\right)$

if $\exists M>0$ AND $\lim_{t\to\infty}(\phi(t)) = 0$ then the system is A.S.

- A C-T linear system is: stable if eigenvalues of A have non-positive real values, or potentially real parts equal to zero, and $g = a$.

: A.S. if eigenvalues of A have non-positive real values, without real parts equal to zero, and $g = a$.

} A Hurwitz

- A D-T linear system is
  - Stable if amplitude ≤ 1 (w/ amplitude $= 1$)
  - A.S. if amplitude < 1 (w/o amplitude $= 1$)

  ] A Schur

- Trajectories are generic, with A Hurwitz trajectory being bounded (BIBS) to any bounded input $u(t)$.
  - Proof:

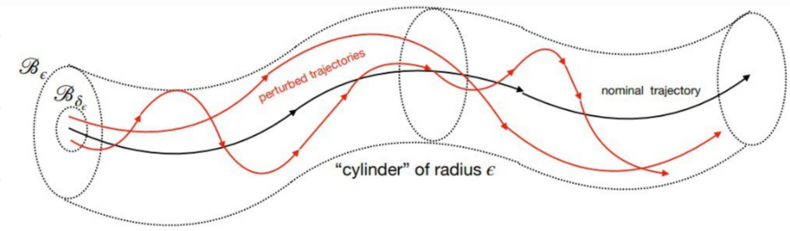
$$x(t) = \phi(t)\, x_0 + \cancel{\Psi(t)\, u([0,t))}$$

$$- \quad x(t) = \phi(t)(x_0 + \delta_\epsilon) + \cancel{\Psi(t)\, u([0,t))}$$

$$[x(t) - x(t)] = \phi(t)[-\delta_\epsilon]$$

$$\|x(t) - x(t)\| = \|\phi(t)\|\, \|\delta_\epsilon\|$$

If $\phi(t)$ is bounded, $\frac{\epsilon}{\delta_\epsilon}$ condition is fulfilled. (Stable, $\exists M > \phi(t)$)



Lyapunov

A tool that checks if an equilibrium point is Stable, A.S, or Unstable.

- Is geometric instead of time varying.
- Energy interpretation: a stable e.q is related to an "energy" function not increasing for stable systems, and is decreasing for A.S. systems.

* Definitions for Lyapunov Functions (V):-

- Positive Definite: V is +ve definite wrt point $\bar{x} \in$ domain D if $V(\bar{x}) = 0$ and $V(x) > 0 \quad \forall x \in D \setminus \{\bar{x}\}$. (set minus/excluding)
- Positive Semi-Definite: V is +ve semi-definite wrt point $\bar{x} \in D$ if $V(\bar{x}) = 0$ and $V(x) \geq 0$.
- Negative Definite: $V(\bar{x}) = 0$ and
  - → $V(x) < 0$; Negative Definite
  - → $V(x) \leq 0$; Negative Semi-Definite

Ex 1: What types from above are the following Lyapunov functions categorised as?

$$\left.\begin{array}{l} 1)\ V(x) = x_1^2 + x_2^2 \\ 2)\ V(x) = x_1^2 + \int_0^{x_2} g(s)\,ds \quad \text{where } g(s) \text{ is symmetric } (sg(s) > 0) \\ 3)\ V(x) = x_1^2 + (1-\cos(x_2)) \quad [D : |x_2| < \pi/2] \\ 4)\ V(x) = x_1^4 \end{array}\right] \bar{x} = (0,0)$$

1) Since x_1^2 and x_2^2 cannot be -ve: Pos definite

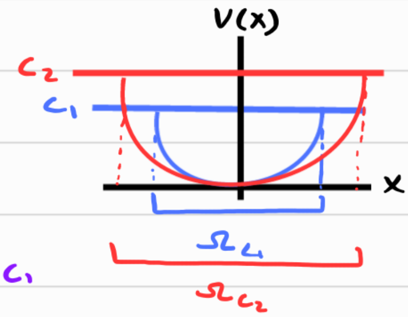
2) Since $g(s)$ is symmetric and positive: Pos definite

3) Since x_2 cannot be equal to $\pm\pi$ in D: Pos definite

4) Since $(x_1, x_2) = (0, S) \neq \bar{x} = (0,0)$ and results in $V(x) = 0$: Pos semi-definite

- Level Sets (Ω_c): A level set is a set such that for $x \in \Omega_c$,

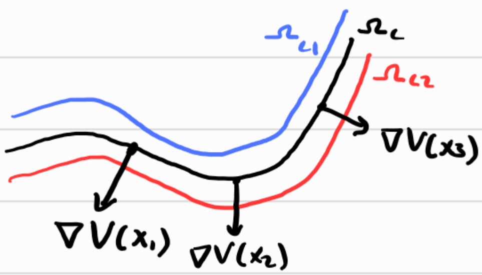
$$V(x) \leq c$$



- Level sets can be nested geometrically such that where $c_1 < c_2$,

then $\Omega_{c_2} \subset \Omega_{c_1}$ and $V_2(x) - V_1(x) = \leq c_2 - \leq c_1$

- The divergence of the lyapunov (∇V) points towards the level set of the greater c (c_2)



for $x \in \Omega_c$

$c_2 > c > c_1$

- The Quadratic form $(V(x))$:

$$V(x) = x^T \underset{n \times n}{P} x \qquad \nabla V(x) = 2x^T P$$

- $P \geq 0$ and $x^T P x \geq 0 \quad x \in \mathbb{R}^n$ (Pos semi-def)
- $P > 0$ and $x^T P x > 0 \quad x \in \mathbb{R}^n \setminus \{0\}$ (Pos def)
- $P \geq 0$ and $x^T P x \leq 0 \quad x \in \mathbb{R}^n$ (negative semi-def)
- $P < 0$ and $x^T P x > 0 \quad x \in \mathbb{R}^n \setminus \{0\}$ (negative def)

- P can be taken as symmetric $(P = P^T)$ w/o losing generality. (See Proof 1)

- Take D_k as the determinant after removing the last $n-k$ rows and columns:

$n=3$, $\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$ $\quad D_2 = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$

$\quad D_1 = \det(x_1)$

Matrix P is pos def if $D_k > 0 \quad \forall k = 1, 2, \dots, n$

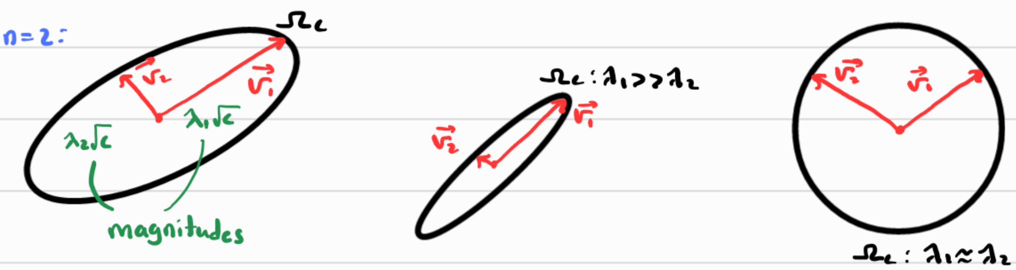
- Since P is symmetric, then if it is pos. definite:

1) P is diagonalisable $(a_i = g_i)$
2) real eigenvalues and eigenvectors $(\lambda_i \in \mathbb{R}^n)$ $(\vec{v_i} \in \mathbb{R}^n)$ $i = 1, 2, \dots, n$
3) Positive eigenvalues w/ orthogonal eigenvectors

- Geometry of level sets:

Take a level set $\Omega_c = \{x \in \mathbb{R}^n : x^T P x \le c\}$, the geometry is ellipsoid w/ principal axes direct as the eigenvalues of P, and amplitude $\propto \lambda$.

Examples w/ $n=2$:



Ex2: $P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Is P pos. definite?

$n=2$, $D_k > 0$ for $k = 1, 2$?

$D_1 = \det(1) = 1 > 0$ ✓

$D_2 = \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (1)(1) - (-1)(0) = 1 > 0$ ✓

$\therefore$ P is positive definite.

Proof 1: let's say $P \neq P^T$ (Asymmetric):

Decompose P into symmetric and asymmetric components:

$$\left.\begin{aligned} P_S &= \frac{P + P^T}{2} = P_S^T \\ P_A &= \frac{P - P^T}{2} = -P_A^T \end{aligned}\right] \quad V = x^T P x = x^T (P_S + P_A) x = \underbrace{x^T P_S x}_{\text{Scaler}} + \underbrace{x^T P_A x}_{\text{Scaler}=0}$$

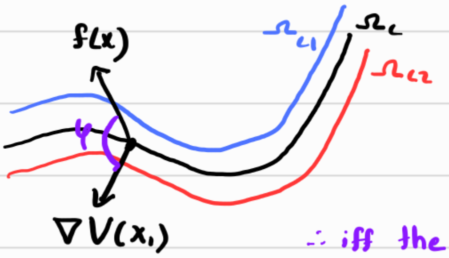
$(WMV)^T = V^T M^T W^T$

$-(x^T P_A x) = x^T P_A^T x$

*** Direct Lyapunov Theorem:-**

- Given an equilibrium point $\bar{x} \in \mathbb{R}^n$ ($\text{for } \dot{x} = f(x)$), let $V(x)$ be a pos def function wrt $\bar{x} \in D$ ($D \subseteq \mathbb{R}^n$) and $\dot{V}(x) = \nabla V(x)\, f(x)$ ($\dot{V}(x): D \to \mathbb{R}$):
  1) $\bar{x}$ is a stable eq if $\dot{V}(x) \leq 0 \ \forall x \in D$
  2) $\bar{x}$ is an A.S. eq if $\dot{V}(x) < 0 \ \forall x \in D \setminus \{0\}$ w/ a domain of attraction $\mathcal{A}$ (no magnitude of attractivity)

- $\dot{V}(x)$ is the scalar product between the divergence ($\nabla V(x)$) and the vector field ($f(x) = \dot{x}$)



for $x \in \Omega_c$ — $|\varphi| > \frac{\pi}{2}$ iff $\dot{V}(x) < 0$

$c_2 > c > c_1$ — $|\varphi| = \pm\frac{\pi}{2}$ iff $\dot{V}(x) = 0$

$\therefore$ iff the vector field, $f(x)$, points towards c_1, $\dot{V}(x) < 0$

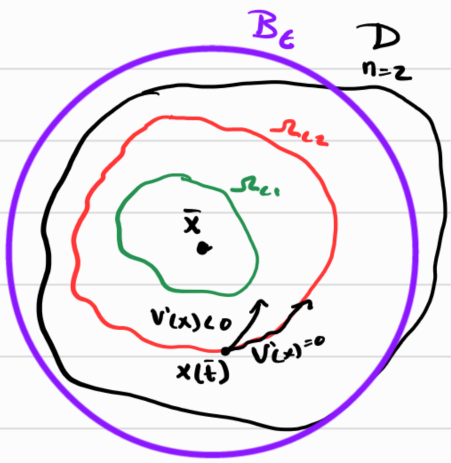
↳ inner level set

- $V(x)$ and $\dot{V}(x)$ are geometric functions, not temporal (independent of time, only x)

- For D-T: $\dot{V}(x) = V(f(x)) - V(x)$

{B is boundary}

- Graphically:



- For $\dot{V}(x(t)) \leq 0$, level sets c_1 and c_2 are invariant (trajectories starting in a level set do not escape it).

Using math: $\dot{V}(x) \leq 0 \to \forall c > 0: \ x(\bar{t}) \in \Omega_c$ then $x(t) \in \Omega_c \ \forall t > \bar{t}$

- Thus, 1) choose a $c > 0: \Omega_{c_2} \subseteq D$, $\Omega_{c_1} \subseteq \Omega_{c_2}$

  2) choose $\delta_\epsilon > 0 : \Omega_{c_2} \subseteq B_\epsilon$

This gives us a qualitative answer that $\delta_\epsilon > 0$, not trying to find the largest δ_ϵ.

- To find the domain of attractivity, use the maximum level set Ω_{c_2} within domain D.

(This is a conservative estimation)

- Local Asymptotic Stability (LAS): $\mathcal{A} \subset \mathbb{R}^n$
- Global Asymptotic Stability (GAS): $\mathcal{A} = \mathbb{R}^n$

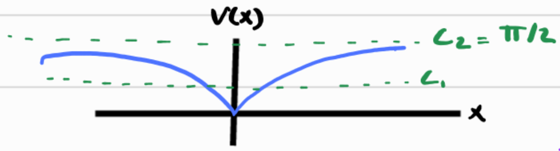
* Converse Lyapunov Theorem:- if $\bar{x}$ is a stable eq of $\dot{x}=f(x)$, then $\exists V(x)$ pos def w/ $\dot{V}(x) \leq 0$ for all $x \in D$.

- If $V(x)>0$ and $\dot{V}(x) \nleq 0$: $V(x)$ is not Lyapunov or eq point is unstable.

- Global Asymptotic Stability: $\bar{x}$ is GAS if it meets three conditions:
    1) $V(x)$ is Positive definite $\left(V(x)>0 : \forall x \in D \setminus \{0\}\right)$.] Not enough
    2) $\dot{V}(x)<0$ everywhere
    3) $\bar{x}$ is radially unbounded $\left(\|x\| \to \infty \Rightarrow V(x) \to \infty\right)$

- Invariance of level sets do not guarantee stability.

Example: $V(x) = \tan^{-1}(x)$



$\Omega_{c_2} = \mathbb{R}$

$\therefore$ not radially unbounded.

- Linear Systems: Let $\left.\begin{matrix}\dot{x}(t)\\ x(t+1)\end{matrix}\right\} = Ax(t)$, $V(x) = x^T P x$ (quadratic form), then the system is Hurwitz or Schur iff $P=P^T>0$ and $\underset{\sim}{Q}=Q^T>0$

↳ Solution of the Lyapunov matrix equation

$$PA + A^T P = Q \quad \text{(Proof 2)} \qquad A^T P A - P = Q \quad \text{(Proof 3)}$$

- Note that Q is arbitrary: choose any symmetric pos. def matrix and solve the LME for P.

Congrats, Stable system. (If $Q \geq 0$ then only simply stable)

Proof 2:

C-T case: $\dot{x} = Ax$ $\quad V(x) = x^T P x$, $\quad \dot{V}(x) = \nabla V(x)\, f(x) = (2x^T P)(Ax) = x^T \underset{\sim}{(2PA)}\, x$

Not necessarily symmetric $\therefore$ Decompose

$$(2PA)_s = \frac{(2PA)+(2PA)^T}{2} = \frac{2PA + 2A^T P^T}{2} \overset{P=P^T}{=} PA + A^T P, \quad \therefore \dot{V}(x) = x^T(PA + A^T P)x$$

Let $\exists Q \geq 0$: $\underset{\substack{\text{Pos}\\ \text{def}}}{\underset{\sim}{PA}} + \underset{\substack{\text{Pos}\\ \text{def}}}{\underset{\sim}{A^T P}} = -\underset{\sim}{Q}$

$Q \geq 0$: $\dot{V}(x) = x^T Q x \leq 0 \;\; \forall x$ (stable)

$Q > 0$: $\dot{V}(x) = x^T Q x < 0 \;\; \forall x \in \mathbb{R}^n \setminus \{0\}$ (A.S)

Proof 3:

D-T Case: $x(t+1) = Ax(t)$, $V(x) = x^T P x$,

Already Symmetric

$$V'(x) = V(\underbrace{f(x)}_{Ax}) - V(x) = (Ax)^T P (Ax) - x^T P x = x^T \underbrace{(A^T P A - P)}x$$

Let $\exists Q \geq 0$: $A^T P A - P = -Q = -I$ (Q can be whatever we like)

Special Case: iff $\exists P > 0$; $\exists Q = Q^T > 0$; $PA + A^T P = Q \xleftrightarrow{\text{By Lyapunov}}$ A is hurwitz (All $\lambda_i \in \mathbb{R}$ and $\lambda_i < 0$)

well defined as A is hurwitz

Choose $P = \int_0^\infty e^{A^T s} Q e^{As} ds$ (w/ any Q)

$$LME = \int_0^\infty (A^T e^{A^T s} Q e^{As} A) ds = \int_0^\infty \left(\frac{d}{\cancel{ds}} (e^{A^T s} Q e^{As}) \right) \cancel{ds} = e^{A^T s} Q e^{As} \Big]_0^\infty = (0) - (Q) = -Q$$

$\therefore$ P solves LME $\forall Q$ and if $Q = Q^T > 0$, $\exists P = P^T > 0$

$$P^T = \int_0^\infty (e^{A^T s} Q^T e^{As}) ds = \int_0^\infty (e^{A^T s} Q e^{As}) ds = P; \quad x^T P x = \int_0^\infty (\underbrace{x^T e^{A^T s} Q e^{As} x}_{> 0 \, \forall s}) ds, \therefore x^T P x > 0 \quad \forall x$$

*** Indirect Lyapunov Theorem:-**

- Using Taylor Series Expansion: $\dot{x} = f(x) = \overbrace{\cancel{f(\bar{x})}}^{\triangleq 0} + A(x - \bar{x}) + \cdots \overbrace{\cancel{g(x)}}^{HOT \approx 0}$

negligible: higher order terms $\to 0$ especially near $\bar{x}$.

$$A = \frac{d}{dx} f(x) \Big|_{x=\bar{x}} = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1}\Big|_{x=\bar{x}} & \cdots & \frac{\partial f_1(x)}{\partial x_n}\Big|_{x=\bar{x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1}\Big|_{x=\bar{x}} & \cdots & \frac{\partial f_n(x)}{\partial x_n}\Big|_{x=\bar{x}} \end{pmatrix}$$

If A is Hurwitz, $\bar{x}$ is LAS

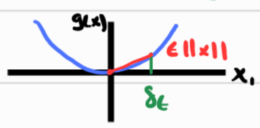
(Proof 4)

Proof 4: Suppose that $\exists P = P^T > 0$ and LME Solution $Q = Q^T > 0$, then $\bar{x}$ is LAS in A w/ $V(x) = (x - \bar{x})^T P (x - \bar{x})$ as a possible Lyapunov function. If A has at least one positive, real eigenvalue, then $\bar{x}$ is unstable for $\dot{x} = f(x)$.

$$V'(x) = \nabla V(x) f(x) = (2x^T P)(Ax + g(x)) = \underbrace{2x^T P A x}_{-x^T Q x < 0 \; \forall x \in \mathbb{R}^n \setminus \{0\}} + 2x^T P \underbrace{g(x)}_{HOT}$$

HOT $\hookrightarrow \forall \epsilon > 0, \exists \delta_\epsilon > 0 : (\|g(x)\|) \leq (\epsilon \|x\|)$

$\forall x . \|x\| \leq \delta_\epsilon$



$\epsilon\|x\|$ slope becomes horizontal approaching origin

$\therefore V'(x) \leq -x^T Q x + 2 \|x\| \|P\| \|g(x)\|$ Since eigenvalues are real and +ve

$\underline{\lambda} \|x\|^2 \leq x^T Q x \leq \bar{\lambda} \|x\|^2$

$\llcorner$ min λ $\quad$ $\llcorner$ max λ

So if $\|x\| \le \delta_\epsilon \le -(1-2\|P\|\epsilon)\|x\|^2$, let $\epsilon = \frac{1}{2\|P\|}$ and δ_ϵ according to $g(x)$

$V'(x) \le \frac{-1}{2}\|x\|^2 < 0$ (negative definite everywhere except $x=0$)

$\Rightarrow$ $\bar{x}=0$ is LAS for $\dot{x}=f(x)$

- Exception: When A has eigenvalues on the $\mathbb{I}$ axis, the non-linear system is non-hyperbolic.

eq is either stable, A.S, or unstable

- HOT plays a role now in determining stability properties.

Ex3: $\dot{x} = \alpha x^3$ (random constant) $\quad 0 = \alpha\bar{x}^3 \therefore \bar{x}=0$, $n=1$; $A = \left.\frac{df}{dx}\right|_{x=\bar{x}} = \left.\frac{\partial}{\partial x}(\alpha x^3)\right|_{x=0} = \left.3\alpha x^2\right|_{x=0} = 0$, $A=[0]$ $\quad \sigma(A)=\{0\}$

$\bar{x}=0$

Let $V(x) = x^2$

$V'(x) = \nabla V(x)\cdot f(x) = (2x)(\alpha x^3) = 2\alpha x^4$ $\qquad x(t) = e^{\alpha t}x_0$

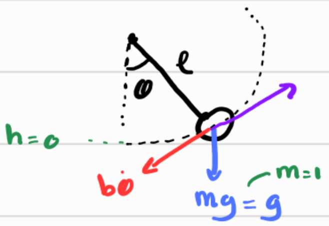
$\hookrightarrow$ $\alpha = +ve$, $V'(x) > 0$ $\therefore$ either wrong $V(x)$ or unstable.

$\hookrightarrow$ $\alpha = -ve$, $V'(x) < 0$ $\therefore$ Asymptotically Stable (GAS)

Krasovski La-Salle Criterion

$J\ddot{\theta} = \cancel{m}\ell^2\ddot{\theta} = -\cancel{m}g\ell\sin(\theta) - \underset{b\dot{\theta}\ell}{b v}$

- Motivating Example: Pendulum



$x_1 = \theta$, $\dot{x}_1 = \dot{\theta} = x_2$

$x_2 = \dot{\theta}$, $\dot{x}_2 = \ddot{\theta} = \frac{-g}{\ell}\sin(\theta) - b\ell\dot{\theta}$

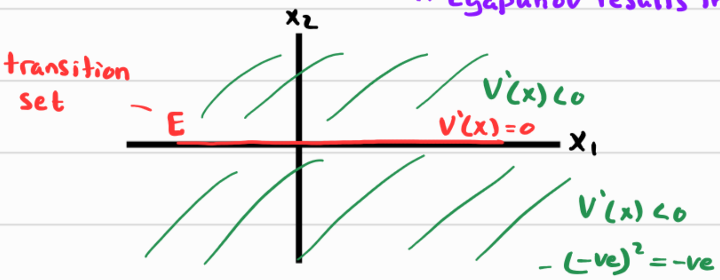
Use an energy based Lyapunov Function: $V(x) = \frac{1}{2}\cancel{m}v^2 + \cancel{m}gh = \frac{1}{2}\dot{\theta}^2 + g(\ell - \ell\cos(\theta))$

$= \left(\frac{\ell^2}{2}x_2^2 + g\ell(1-\cos(x_1))\right)\frac{1}{\ell^2}$

$V(x) = \frac{g}{\ell}(1-\cos(x_1)) + \frac{1}{2}x_2^2$ (pos def wrt $x_1=0, x_2=0$)

$$V'(x) = \begin{pmatrix} \underbrace{\frac{g}{\ell}\sin(x_1)}_{\frac{\partial V}{\partial x_1}} & \underbrace{x_2}_{\frac{\partial V}{\partial x_2}} \end{pmatrix} \underbrace{\begin{pmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - h\ell x_2 \end{pmatrix}}_{f(x)=\dot{x}} = \cancel{\frac{g}{\ell}\sin(x_1)x_2} - \cancel{\frac{g}{\ell}\sin(x_1)x_2} - h\ell x_2^2 = -h\ell x_2^2 \le 0$$

$\therefore$ Lyapunov results in origin being simply stable instead of A.S.



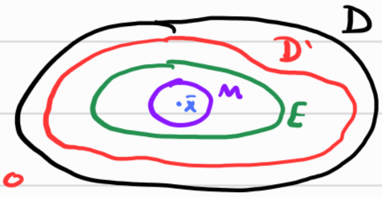
(Energy not enough)

(decrease of system energy)

$-(-ve)^2 = -ve$

* **Criterion:-** Let $V(x)$ $(D \subseteq \mathbb{R}^n \rightarrow \mathbb{R})$ be positive definite to $\bar{x}$, $\dot{x}=f(x)$, and suppose $\dot{V}(x) \leq 0$ $(D' \subseteq D)$

Let $E \subseteq D'$ be the set where $\dot{V}(x)=0$, and let M be the largest set in E invariant to the system trajectories. We can conclude that:



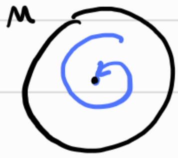
1) $\bar{x}$ is stable from direct Lyapunov

2) M is attractive for the system trajectories $\lim_{t \to \infty} (dist(x(t), M)) = 0$

- Invariance of M: M is invariant to the trajectories $(\dot{x}=f(x))$ if $x_0 \in M \Rightarrow x(t, x_0) \in M$ $\forall t \in \mathbb{R}$.

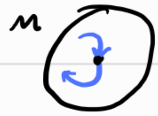
A special case occurs if $M=\bar{x}$, then $\bar{x}$ is A.S.

Forward Invariance



Backward invariance

Invariance



- Ex4A: Pendulum from the previous example w/ La Salle

In the previous example: we need $\dot{x}_2(t) \equiv 0$ for $x(t, x_0) \in M$, thus $x_2(t) \equiv 0$.

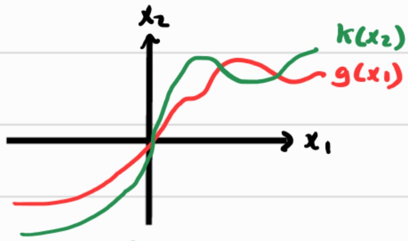
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \cancel{b x_2}^{\,x_2 \equiv 0} \equiv 0 \quad \therefore \quad -\frac{g}{l}\sin(x_1) \equiv 0, \quad x_1(t) = 0.$$

$$\therefore M = \{0\}$$

- Ex4B: Generalised Pendulum

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g(x_1) - h(x_2)$$



Generalisation of gravity's/friction's contribution (generic non-linear functions, arbitrary shapes)

To conclude if GAS or LAS: Use Lyapunov La-Salle TEAM-UP!!

$$\text{Lyapunov: } V(x) = \underbrace{\frac{1}{2}x_2^2}_{} ^{\text{K.E}} + \underbrace{\int_0^{x_1} g(s)\,ds}_{} ^{\text{P.E}}$$

(When x_1, x_2 go to ∞, so does $V(x)$ since it's radially unbounded)

$$\dot{V}(x) = \nabla V(x)\, f(x) = \begin{pmatrix} g(x_1) & x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -g(x_1) - h(x_2) \end{pmatrix} = \cancel{g(x_1)x_2} - \cancel{g(x_1)x_2} - h(x_2)x_2 = -h(x_2)\,x_2$$

So $\dot{V}(x) = -h(x_2)\,x_2 \leq 0$ ($V(x)$ is pos. semi-def) implying simply stable.

La Salle: $E = \{x \in \mathbb{R}^n : x_2 = 0\} \Rightarrow M = \{0\}$ $\therefore$ $\bar{x}$ is GAS

Continue Generalising: Starting w/ energy, find a $V(x)$ w/ $\dot{V}(x) < 0$,

$$V(x) = \underbrace{\tfrac{1}{2}x^T P x}_{\text{H.E Quadratic form}} + \underbrace{\tfrac{g}{\ell}(1-\cos(x_1))}_{\text{gets cancelled anyway so unchanged}}$$

We need $\dot{V}(x)$ to be negative definite, so add extra terms to H.E:

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}, \quad D_1 = \det(P_{11}) = P_{11} > 0$$

$$D_2 = \det\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} = P_{11}P_{22} - P_{12}^2 > 0 \;\Downarrow\; P_{22} > 0$$

w/ $D_1, D_2 > 0$: P is pos. def, $V(x) = \frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{g}{\ell}(1-\cos(x_1))$

$$= \tfrac{1}{2}\begin{pmatrix} P_{11}x_1 + P_{12}x_2 & P_{12}x_1 + P_{22}x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \tfrac{g}{\ell}(1-\cos(x_1))$$

$$= \tfrac{1}{2}(P_{11}x_1^2 + 2P_{12}x_1x_2 + P_{22}x_2^2) + \tfrac{g}{\ell}(1-\cos(x_1))$$

$$\dot{V}(x) = \nabla V(x)\, f(x) = \begin{pmatrix} P_{11}x_1 + P_{12}x_2 + \frac{g}{\ell}\sin(x_1) & P_{12}x_1 + P_{22}x_2 \end{pmatrix}\begin{pmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - h(x_2) \end{pmatrix}$$

$$= \left(P_{11}x_1x_2 + P_{12}x_2^2 + \tfrac{g}{\ell}\sin(x_1)x_2\right) + (P_{12}x_1 + P_{22}x_2)\left(-\tfrac{g}{\ell}\sin(x_1) - h(x_2)\right)$$

$$= P_{11}x_1x_2 + P_{12}x_2^2 + \tfrac{g}{\ell}\sin(x_1)x_2 - P_{12}x_1\tfrac{g}{\ell}\sin(x_1) - P_{22}x_2\tfrac{g}{\ell}\sin(x_1) - P_{12}x_1 h x_2 - P_{22}x_2^2 h$$

$$= \underbrace{(P_{12} - P_{22}h)x_2^2}_{(4)} \underbrace{- \tfrac{g}{\ell}P_{12}\sin(x_1)x_1}_{(3)} + \underbrace{x_1x_2(P_{11} - P_{12}h)}_{(2)} + \underbrace{x_2\left(\tfrac{g}{\ell}\sin(x_1) - P_{22}\tfrac{g}{\ell}\sin(x_1)\right)}_{(1)}$$

Good term: x_2^2 always +ve so easy to make $P_{12} - P_{22}h < 0$

Good term: always < 0

Bad terms: cross terms should be cancelled out.

$\therefore$ Select P_{11}, P_{12}, and P_{22} to eliminate bad terms while respecting constraints.

Ex 4 c: Variable Gradient Method

Constraints: a) $P_{11} > 0$ b) $P_{22} > 0$ c) $P_{11}P_{22} - P_{12}^2 > 0$

Cancellations:
1) $P_{22} = 1$ — $P_{22} > 0$ ✓
2) $P_{11} = P_{12}k$ — $P_{11} > 0$ ✓
3) $P_{12} > 0$
4) $P_{12} < P_{22}k$ — $P_{11}P_{12} - P_{12}^2 = P_{12}kP_{12} - P_{12}^2 = P_{12}^2(k-1) > 0$ ✓

$\therefore \dot{V}(x) < 0$

Ex 4D: Adaptive Control Systems

$\dot{y} = ay + u$; $a \in \mathbb{R}$ — we know nothing of a

Goal: Design u : $\lim_{t\to\infty} (y) = 0$

1) Trivial Solution:

if we know that $|a| \le \bar{a}$ (a has an upper bound $\bar{a}$), $u = -ky$: $k > \bar{a}$

$\dot{y} = -(k-a)y$ (easy)

Problem is, we don't

2) Harder Solution:

$\dot{y} = ay + u$ — tune control input

let $u = -ky$, $\dot{k} = \gamma y^2$

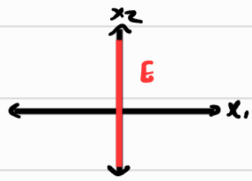
$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ w/ $\begin{matrix} x_1 = y, & \dot{x}_1 = \dot{y} = ay + ky = ax_1 - x_2 x_1 \\ x_2 = k, & \dot{x}_2 = \dot{k} = \gamma y^2 = \gamma x_1^2 \end{matrix}$

unknown constant $\quad \dot{x} = f(x) = \begin{pmatrix} (a-x_2)x_1 \\ \gamma x_1^2 \end{pmatrix}$

Let $V(x) = \frac{x_1^2}{2} + \frac{1}{2\gamma}(x_2 - b)^2$; $V(x) > 0$ wrt $x_1 = 0$ and $x_2 = b$ $\quad \bar{x} = (0, b)$

$$V'(x) = \nabla V f(x) = \begin{pmatrix} x_1 & \frac{1}{\gamma}(x_2-b)(1) \end{pmatrix} \begin{pmatrix} (a-x_2)x_1 \\ \gamma x_1^2 \end{pmatrix} = (a-x_2)x_1^2 + (x_2-b)x_1^2 = x_1^2(a-b\cancel{+x_2-x_2}) \le 0$$

So since $V'(x) \le 0$, use La Salle w/ E on the line $x_1 = 0$:



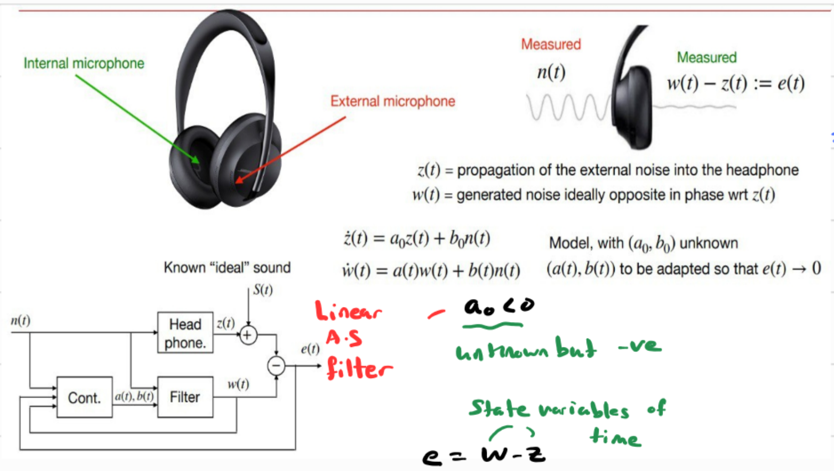
$\lim_{t\to\infty} (x(t)) \big|_M = 0$,

for $\dot{x} = \begin{pmatrix} (a-x_2)x_1 \\ \gamma x_1^2 \end{pmatrix} \in \underset{(0,x_2)}{E}$, then substituting $x_1 = 0$ gets us $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

$\therefore M \equiv E$

Since we aren't looking for a GAS eq, $x_1 = y = 0$ for $t \to \infty$, thus robust A.S reached in original system, showing how good $u = -ky$ and $\dot{k} = \gamma y^2$ is.

Exs: Microphone



Linear A.S filter

$a_0 < 0$ — unknown but -ve

State variables of time

$e = w - z$

$\dot{e} = aw + bn - a_0 z - b_0 n + a_0 w - a_0 w$

$= (a - a_0)w + a_0(w - z) + n(b - b_0)$

$\dot{a} = V_a$

$\dot{b} = V_b$

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} e \\ a-a_0 \\ b-b_0 \end{pmatrix}$ (eq Points at $e=0$, $a=a_0$, $b=b_0$)

$x_1 = e$, $\dot{x}_1 = \dot{e} = (a-a_0)w + a_0(\overset{e}{\cancel{w-z}}) + n(b-b_0) = x_2 w + a_0 x_1 + n x_3$

$x_2 = a-a_0$, $\dot{x}_2 = \dot{a} = v_a$

Cancels w/ 2 from $\nabla V(x)$

$x_3 = b-b_0$, $\dot{x}_3 = \dot{b} = v_b$

Let $V(x) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma_a}x_2^2 + \frac{1}{\gamma_b}x_3^2\right)$,

$$V'(x) = \nabla V f(x) = \begin{pmatrix} x_1 & \frac{1}{\gamma_a}x_2 & \frac{1}{\gamma_b}x_3 \end{pmatrix} \begin{pmatrix} a_0 x_1 + w x_2 + n x_3 \\ v_a \\ v_b \end{pmatrix} = \underbrace{a_0 x_1^2}_{\substack{\text{Good} \\ \text{term} \\ a_0 < 0}} + w x_1 x_2 + n x_1 x_3 + \frac{v_a}{\gamma_a} x_2 + \frac{v_b}{\gamma_b} x_3$$

Bad terms

use v_a and v_b to cancel them

$v_a = -\gamma_a w x_1$, $v_b = -\gamma_b n x_1 \rightarrow V'(x) = a_0 x_1^2 + \cancel{w x_1 x_2} + \cancel{n x_1 x_3} - \cancel{w x_1 x_2} - \cancel{n x_1 x_3} = a_0 x_1^2 < 0$

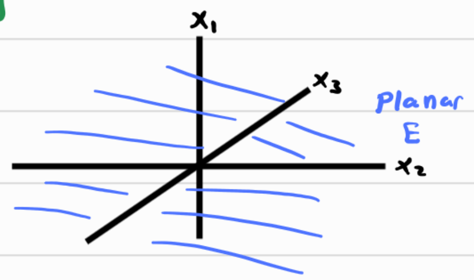
La Salle: $x_1 = 0$, $\dot{x}_1 = 0$

$\dot{x}_2 = v_a$

$\dot{x}_3 = v_b$

$E = \{ x \in \mathbb{R}^3 \mid x_1 = e = 0 \}$

$\lim_{t \to \infty} (\text{Dist}(x(t), M)) = 0$, $M \subseteq 0$ $\Big]$ $\lim_{t \to \infty} (e(t)) = 0$



We can stop here since M is attractive and is contained in E ($M \equiv E$ in this case). $x_1 = e = 0$ and that's our only goal