GAIN SCHEDULING : INTRO AND "MOTIVATIONAL APPROACH"

- The integral controller seen in the previous part is ROBUST, but it is still LOCAL
- The remainder of the course will be concerned with NON-LOCAL controllers (possibly GLOBAL)
- The first non-local solution we investigate is obtained by "combining" different local controllers

INTUITIVE IDEA : suppose our goal is to drive $y_r(t)$ to y_r^* such that the corresponding x^* is "far" from $x(0)$.
We can think to define a sequence $\alpha_1, \alpha_2, \dots, \alpha_N$ of set points for $y_r(t)$ that "gradually" bring $y_r(t)$ to y_r^*.

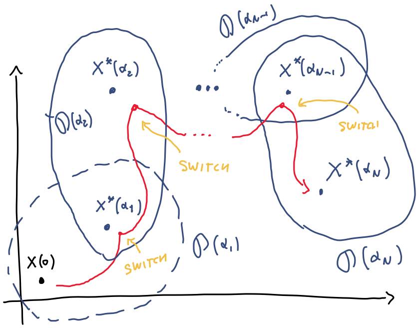
$\downarrow$

Namely: $\forall i = 1, \dots, N$ let $(x^*(\alpha_i), u^*(\alpha_i))$ be the solution to the SOLVABILITY EQS.

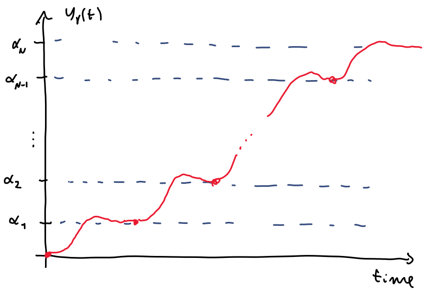
$$\begin{cases} 0 = f\left(x^*(\alpha_i),\ u^*(\alpha_i)\right) \\ \alpha_i = h_r\left(x^*(\alpha_i)\right) \end{cases}$$

Then :

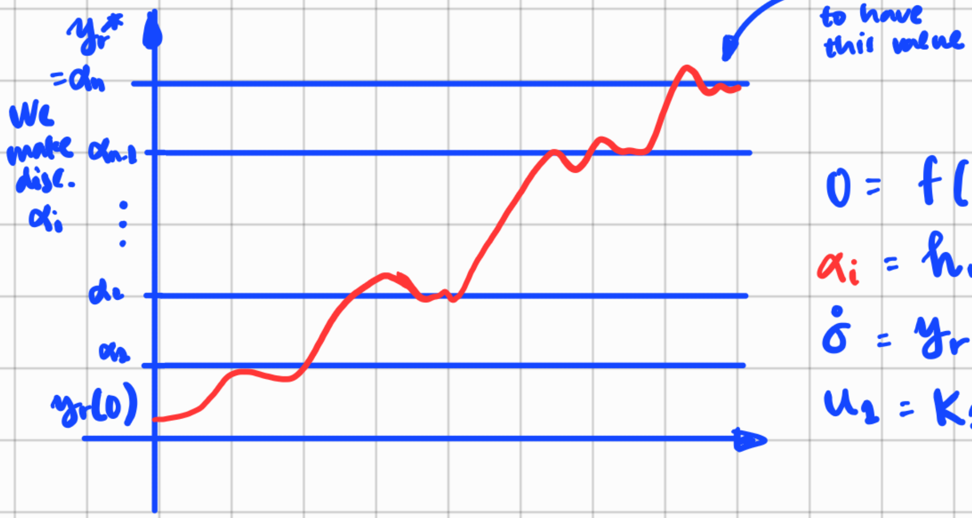
- we choose $\alpha_N = y_r^*$
- we take α_1 so that $x^*(\alpha_1)$ is "sufficiently close" to $x(0)$ so that we can design a local controller stabilizing $x^*(\alpha_1)$ with a domain of attraction $\mathcal{D}(\alpha_1)$ including $x(0)$



- We take α_2 so that we can find a local controller stabilizing $x^*(\alpha_2)$ with a domain of attraction $\mathcal{D}(\alpha_2)$ containing $x^*(\alpha_1)$
- We do similarly for $\alpha_3, \alpha_4, \dots, \alpha_N$
- We SWITCH from one controller to the subsequent when we reach the corresponding domain of attraction



GAIN SCHEDULING



$$0 = f(x^*(\alpha_1), u^*(\alpha_1))$$

$$\alpha_i = h_r(x^*(\alpha_i)) \quad \forall i = 1, \dots, n$$

$$\dot{\sigma} = y_r - \alpha_1$$

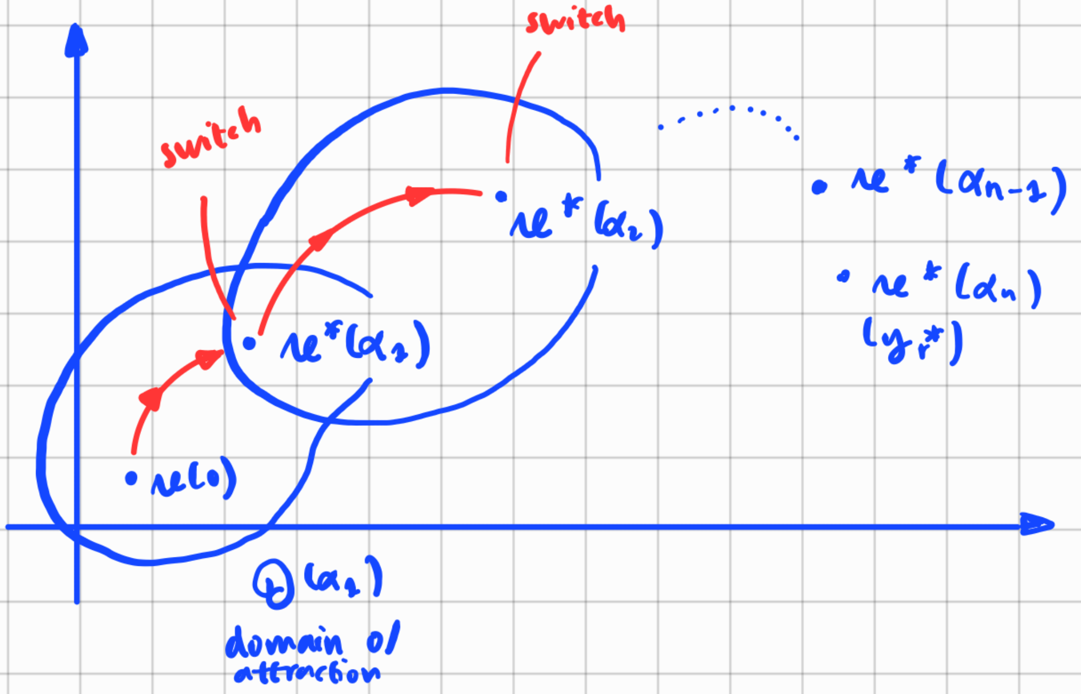
$$u_1 = K_1(\alpha_1)x + K_2(\alpha_1)\sigma$$

We can form a matrix

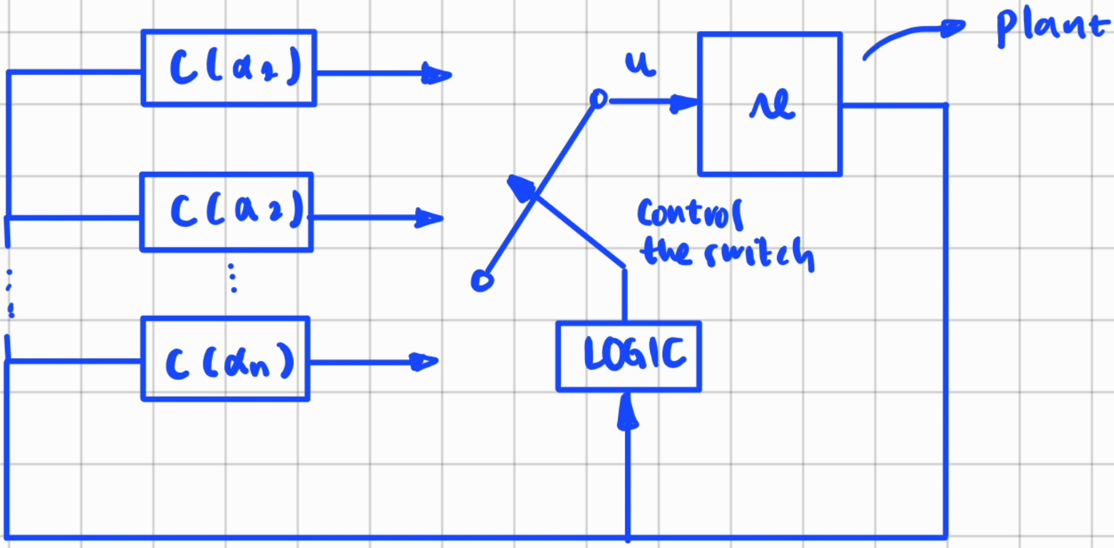
$$\begin{bmatrix} A(\alpha_1) + B(\alpha_1)K_1(\alpha_1) & B(\alpha_1)K_2(\alpha_1) \\ C_r(\alpha_1) & 0 \end{bmatrix}$$

which has to be Hurwitz.

where $A(\alpha_1)$, $B(\alpha_1)$, $C_r(\alpha_1)$ are linearizable around $x^*(\alpha_1)$, $u^*(\alpha_1)$



No local controller since we can go far away from the initial state. But by intuition...



We're designing a new function

$\alpha_1, \cdots, \alpha_n \Rightarrow \alpha(t)$

where $\alpha : \{1, \dots, n\} \longrightarrow \mathbb{R}^{n_r} \Rightarrow \alpha : [0, \infty) \to \mathbb{R}^{n_r}$

Now we have a new equation

$$\begin{cases} \dot{\sigma} = y_r - \alpha \\ u = K_1(\alpha) x + K_2(\alpha)\sigma \end{cases} \Rightarrow \begin{bmatrix} A(\alpha)+B(\alpha)K_1(\alpha) & BK_2(\alpha) \\ C_r(\alpha) & 0 \end{bmatrix}$$

is Hurwitz

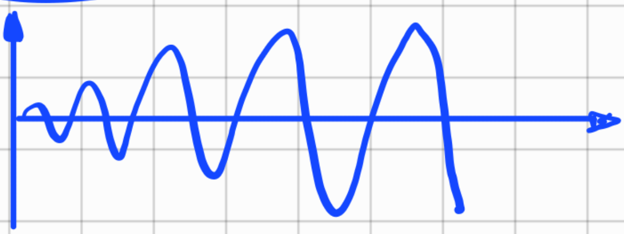
The result $\rightarrow$ assume α is bounded and C^1. can be der. continuously

Then $\exists$ $\varepsilon_1, \varepsilon_2$, such that $\alpha(t) = t \cdot \cos(t)$

if : • $\forall t \geq 0$, $\|\dot{\alpha}(t)\| \leq \varepsilon_1$

• $\|x(0) - x^*(\alpha(0))\| \leq \varepsilon_2$

$\|\sigma(0) - \sigma^*(\alpha(0))\| \leq \varepsilon_2$



We can follow this property:

1) The closed-loop solutions are BOUNDED

$$\begin{cases} \dot{x} = f(x, K_1(\alpha)x + K_2(\alpha)\sigma) \\ \dot{\sigma} = h_r(x) - \alpha \end{cases}$$

α acts as an input

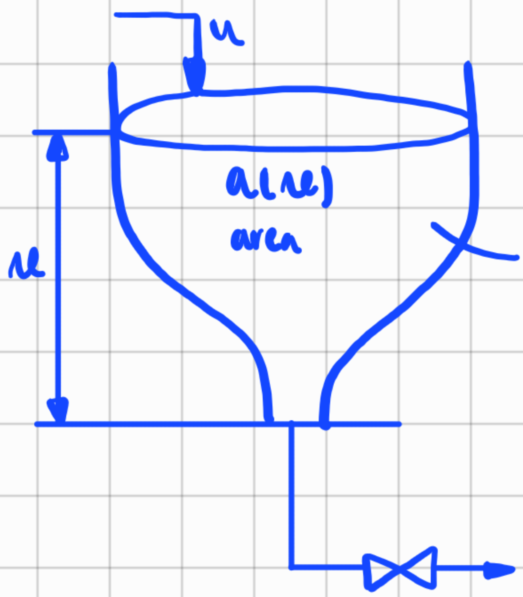
2) $\exists \underbrace{C, T}_{\text{constant}} > 0,\ \forall t \geq T,\ \| \underbrace{y_r(t) - \alpha(t)}_{\text{transitory}} \| \leq C \cdot \varepsilon_1$

It's a guarantee we don't go far from $\alpha(t)$

3) If $\lim_{t \to \infty} \alpha(t) = y_r^*$ and $\lim_{t \to \infty} \dot{\alpha}(t) = 0$, then

$\lim \lim_{t \to \infty} y_r(t) = y_r^*$

TANK EXAMPLE



$P(x) = \rho g x$

ρ density

$P_{env} = 0$

$w_0 = K\sqrt{\rho g x} = C\sqrt{x}$, with $C = K\sqrt{\rho g}$

The volume : $V(x) = \int_0^x a(s)\, ds$

Model eq: $\dot{V}(x(t)) = u(t) - W_0(t) = u(t) - C\sqrt{x(t)}$

We also have $\dot{V} = a(x) \cdot \dot{x}$

$$\boxed{\dot{x} = \frac{u - c\sqrt{x}}{a(x)}}$$

Our goal is $y_r = x$ from $x(0)$ to $y_r^*(0)$.

We design a signal $\alpha(t)$ where $\alpha(0) = x(0)$

$\alpha(t) \rightarrow y_r^*$

and $\alpha \in C^1$ bounded

$\dot{\alpha}$ bounded.

Solvable equation (α)

$$\begin{cases} 0 = \dfrac{u^*(\alpha) - C\sqrt{x^*(\alpha)}}{a(x^*(\alpha))} \\ \alpha = x^*(\alpha) \end{cases} \longrightarrow \boxed{u^*(\alpha) = c\sqrt{\alpha}}$$

$$A(\alpha) = \left. \frac{-\frac{c}{2}\frac{1}{\sqrt{x}} a(x) - a'(x)\cdot(u - c\sqrt{x})}{a(x^2)} \right|_{\substack{x = \alpha \\ u = c\sqrt{\alpha}}}$$

$$= -\frac{c}{2} \frac{1}{a(\alpha)\sqrt{\alpha}}$$

$$B(\alpha) = \frac{1}{a(\alpha)} \qquad C_r(\alpha) = 1$$

$\dot{\sigma} = y_r - \alpha$

$u = K_1(\alpha)x + K_2(\alpha)\sigma$

Make it Hurwitz:

$$\begin{bmatrix} A(\alpha) + B(\alpha)K_1(\alpha) & BK_2(\alpha) \\ C_r(\alpha) & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{-c}{2\sqrt{\alpha}} + K_1(\alpha)}{a(\alpha)} & \dfrac{K_2(\alpha)}{a(\alpha)} \\ 1 & 0 \end{bmatrix}$$

Analyzing the characteristic polynomial :

$$\det[\lambda I - (*)] = \det \begin{vmatrix} \dfrac{\lambda + \frac{c}{2\sqrt{a}} - K_1(\alpha)}{a(\alpha)} & -\dfrac{K_2(\alpha)}{a(\alpha)} \\ -1 & \lambda \end{vmatrix}$$

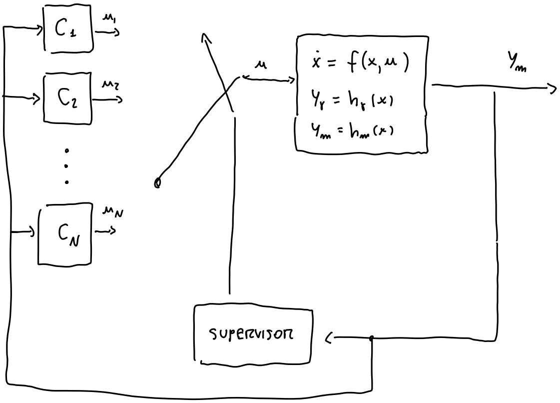
$$= \lambda^2 + \underbrace{\left(\frac{\frac{c}{2\sqrt{\alpha}} - K_1(\alpha)}{a(\alpha)} \right)}_{>0} \lambda - \underbrace{\frac{K_2(\alpha)}{a(\alpha)}}_{>0}$$

We can get the boundary of K, such that

$$K_2(\alpha) < 0$$

$$K_1(\alpha) < \frac{c}{2\sqrt{\alpha}}$$

intuitive control scheme:



This scheme can be implemented but has many problems:

- we need a way to tell when we enter the domain of attraction of the next set-point
- switching must be typically <u>slow</u> as it introduces discontinuities and transient effects

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CONTINUOUS GAIN SCHEDULING

We now pass from a sequence $\alpha_1, \dots, \alpha_N$ to a continuous trajectory $\alpha = \alpha(t)$

As before, for each α, let $x^*(\alpha)$ and $u^*(\alpha)$ be the solutions of the SOLVABILITY eqs.

$$\begin{cases} 0 = f(x^*(\alpha), u^*(\alpha)) \\ \alpha = h_r(x^*(\alpha)) \end{cases}$$

and let

$$A(\alpha) = \frac{\partial f}{\partial x}(x^*(\alpha), u^*(\alpha)) \quad , \quad B(\alpha) = \frac{\partial f}{\partial u}(x^*(\alpha), u^*(\alpha)) \quad , \quad C_r(\alpha) = \frac{\partial h_r}{\partial x}(x^*, u^*)$$

and let us design, for each α, an integral controller of the form

$$\begin{cases} \dot{\sigma} = y_r - \alpha \\ u = K_1(\alpha)\, x + K_2(\alpha)\, \sigma \end{cases}$$

(we put $\hat{u}^*(\alpha) = 0$ and $\hat{x}^*(\alpha) = 0$)

with $K_1(\alpha)$ and $K_2(\alpha)$ are such that

$$\begin{pmatrix} A(\alpha) + B(\alpha) K_1(\alpha) & B K_2(\alpha) \\ C_r(\alpha) & 0 \end{pmatrix}$$

is Hurwitz.

Then the following result holds:

RESULT. Let $\alpha : \mathbb{R} \to \mathbb{R}^{n_r}$ be bounded and continuously differentiable.

Then, there exist* $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that if

1) $\forall t \geq 0 \quad \| \dot{\alpha}(t) \| \leq \varepsilon_1$ → (we need to go slow enough)

2) $\| x(0) - x^*(\alpha(0)) \| \leq \varepsilon_2$ and $\| \sigma(0) - \sigma^*(\alpha(0)) \| \leq \varepsilon_2$

↳ (we need to start close to the path $(x^*(\alpha(t)), u^*(\alpha(t)))$)

* the value of ε_1 and ε_2 depends on the specific system under concern

then the following hold:

a) The trajectories of the closed-loop system

$$\begin{cases} \dot{x} = f(x, u) \\ y_r = h_r(x) \\ \dot{\sigma} = y_r - \alpha \\ u = K_1(\alpha)\, x + K_2(\alpha)\, \sigma \end{cases}$$

are bounded;

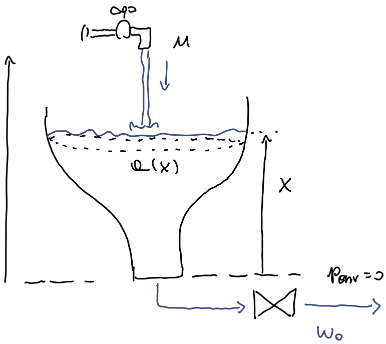
b) There exist $c > 0$ and $T > 0$ such that

$$\forall t \geq T, \qquad \| y_r(t) - \alpha(t) \| \leq c\, \varepsilon_1$$

c) If $\lim_{t \to \infty} \alpha(t) = y_r^*$ and $\lim_{t \to \infty} \dot{\alpha}(t) = 0$, then

$$\lim_{t \to \infty} y_r(t) = y^*$$

EXAMPLE : TANK



x = height

u = incoming flow rate

$a(x)$ = cross-sectional area

$$V(x) = \int_0^x a(s)\, ds \quad = \text{ volume of water in the tank}$$

$p_{env} = 0$ = environmental pressure

$\Delta p = p(x) - p_{env}$ = pressure difference

$p(x) = \rho g x$ (ρ = liquid density, g = gravity)

$w_0 = K\sqrt{\Delta p}$ = outgoing flow rate

$\stackrel{!}{=} c\sqrt{x}$ with $c = K\sqrt{\rho g}$

MODEL:

$$\frac{d}{dt} V(x(t)) = u(t) - w_0(t)$$

$$\downarrow$$

we have:

$$\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x}(x(t)) \cdot \dot{x}(t) = a(x)\,\dot{x}$$

$$w_0(t) = c\sqrt{x(t)}$$

so we obtain the state equation

$$\dot{x} = \frac{u - c\sqrt{x}}{a(x)}$$

GOAL: drive $y_r = x$ to a desired level y_r^*

Let $\alpha(t)$ be the scheduling variable such that $\alpha(0) = x(0)$, $\lim_{t\to\infty} \alpha(t) = y^*$,

and $\lim_{t\to\infty} \dot\alpha(t) = 0$. The SOLVABILITY EQS read

$$\begin{cases} 0 = \dfrac{u^*(\alpha) - c\sqrt{x^*(\alpha)}}{a(\alpha)} \\ \alpha = x^*(\alpha) \end{cases} \Rightarrow \begin{cases} x^*(\alpha) = \alpha \\ u^*(\alpha) = c\sqrt{x^*(\alpha)} \end{cases}$$

The linearization matrices are:

$$A(\alpha) = \frac{\partial}{\partial x}\left(\frac{u - c x^{\frac{1}{2}}}{a(x)}\right)\Bigg|_{\substack{x=\alpha \\ u=\sqrt{\alpha}}} = \frac{1}{a(x)^2}\cdot\left(-\frac{c}{2}x^{-\frac{1}{2}}a(x) - (u - c x^{\frac{1}{2}})\,a'(x)\right)$$

$$= -\frac{c}{2\sqrt{\alpha}\cdot a(\alpha)}$$

$$B(\alpha) = \frac{1}{a(\alpha)}, \qquad C_r(\alpha) = 1$$

Using the integral controller

$$\begin{cases} \dot\sigma = x - \alpha \\ u = K_1(\alpha)\,x + K_2(\alpha)\,\sigma \end{cases}$$

the closed-loop matrix reads as

$$A_{cl}(\alpha) = \begin{pmatrix} A(\alpha) + B(\alpha)K_1(\alpha) & B(\alpha)K_2(\alpha) \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a(\alpha)}\left(K_1(\alpha) - \frac{c}{2\sqrt{\alpha}}\right) & \frac{1}{a(\alpha)} K_2(\alpha) \\ 1 & 0 \end{pmatrix}$$

Let s_1 and s_2 be complex numbers with negative real part.

Then we choose

$$K_1(\alpha) = \frac{c}{2\sqrt{\alpha}} + a(\alpha)(s_1 + s_2)$$

$$K_2(\alpha) = -a(\alpha)\, s_1 s_2$$

The resulting closed-loop matrix is

$$A_{cl}(\alpha) = A_{cl} = \begin{bmatrix} s_1 + s_2 & -s_1 s_2 \\ 1 & 0 \end{bmatrix}$$

and its characteristic polynomial is

$$\varphi(A_{cl}) = \det(\lambda I - A_{cl}) = \det \begin{pmatrix} \lambda - (s_1 + s_2) & s_1 s_2 \\ -1 & \lambda \end{pmatrix}$$

$$= \lambda^2 - (s_1 + s_2)\lambda + s_1 s_2$$

$$= (\lambda - s_1)(\lambda - s_2)$$

$\Rightarrow \sigma(A_{cl}) = \{s_1, s_2\}$ $\Rightarrow$ we can apply the previous result!

The resulting controller is

$$\begin{cases} \dot{\sigma} = x - \alpha \\ u = \left(\frac{c}{2\sqrt{\alpha}} + a(\alpha)(s_1 + s_2) \right) x - a(\alpha) s_1 s_2 \sigma \end{cases}$$

BIBLIOGRAPHY. H. KHALIL, Nonlinear Systems (Chap. 12.5)

REMARK. According to the result shown above, gain scheduling suits regulation problems where the dynamics is "slow"

$\downarrow$

To control "fast" systems like drones or inverted pendulums we need a different theory that will be the subject of the next parts

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