Controllability and Reachability

Master degree in Automation Engineering

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Controllability/Reachability

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t)+Bu(t) \qquad x(0)=x_0$$

$$\boxed{x(t) = \overbrace{\phi(t)\,x(t_0)}^{\text{free evolution}} + \overbrace{\Psi(t)\,u([0,t))}^{\text{convolution sum}}}$$

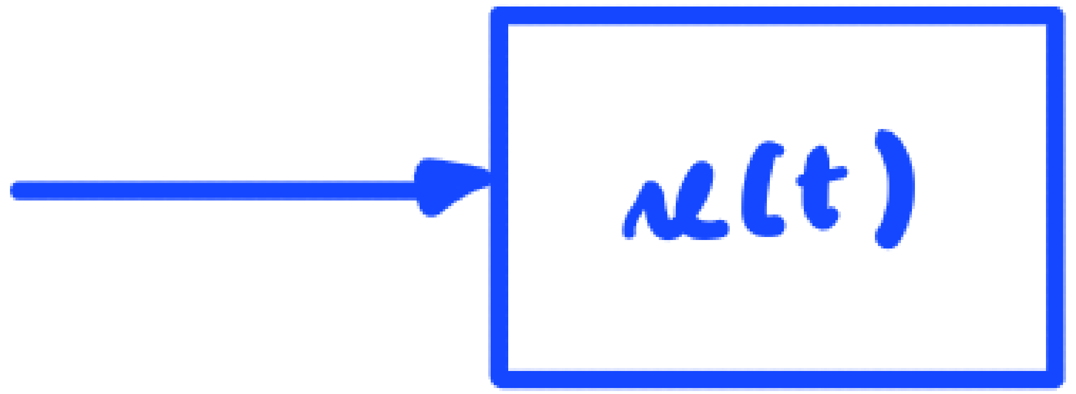
exp. with power of A

No outputs involved in the following analysis

Main question to be answered: given the system with a given initial condition (maybe the origin) can we reach in an appropriate time interval any "target state" in $\mathbb{R}^n$? Are there some "regions" of $\mathbb{R}^n$ that cannot be reached or regions where we remain "trapped"?

Reachability: Which state can be reached starting from the origin by playing with $u(\cdot)$? —> $\mathscr{R}^+$

Controllabilty: Which initial states can be steered to the origin by playing with $u(\cdot)$? —> $\mathscr{R}^-$



Reachability

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = \quad Ax(t) + Bu(t) \qquad x(0) = x_0$$

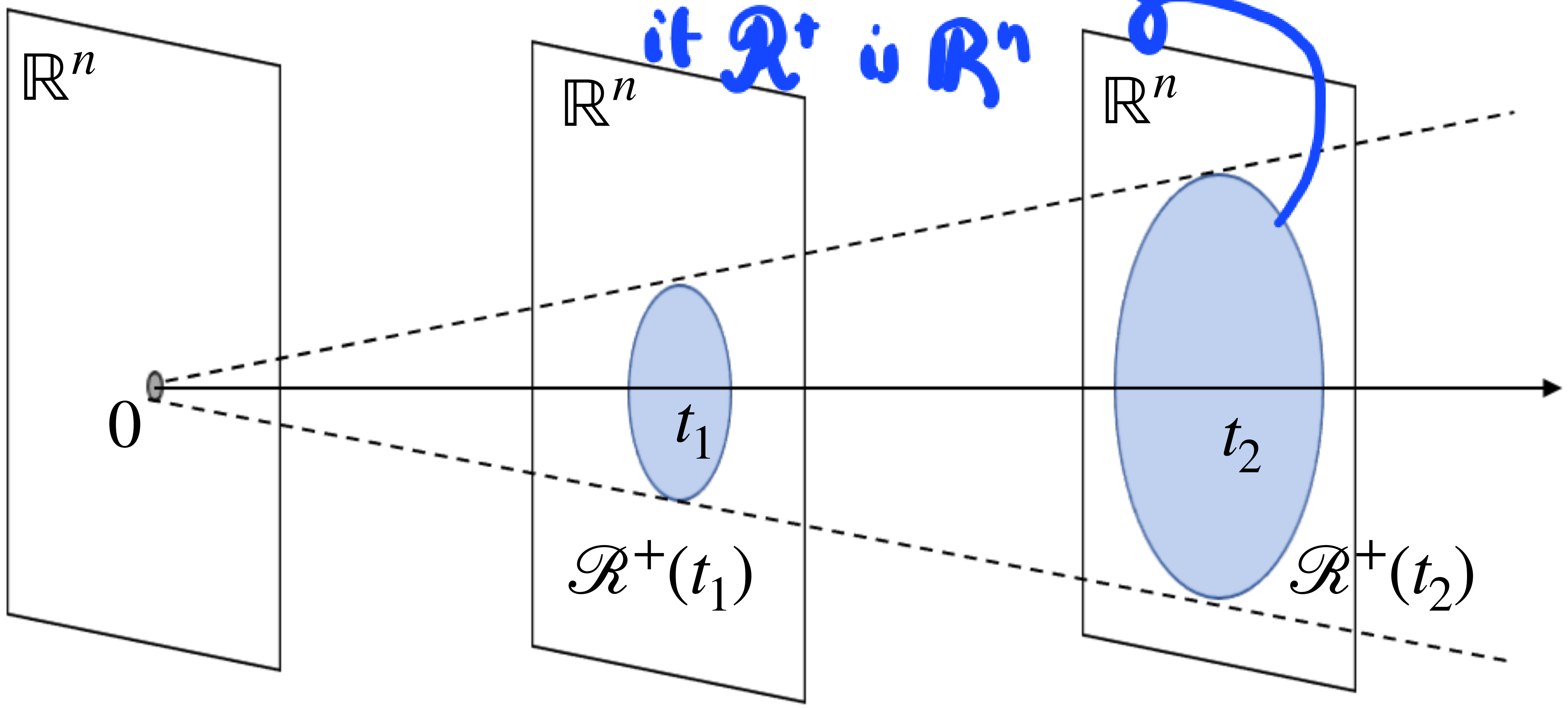
$$\boxed{x(t) = \phi(t)\, x(t_0) \; + \; \Psi(t)\, u([0,t))}$$

No outputs involved in the following analysis

Set of reachable states at time t_1: $\mathscr{R}^+(t_1) = \{x \in \mathbb{R}^n \;\;:\;\; x = \psi(t_1)\, u([0,t_1)) \;$ for some $\; u([0,t_1)) \in \mathbb{R}^m\}$

completely reachable if $\mathscr{R}^+$ is $\mathbb{R}^n$

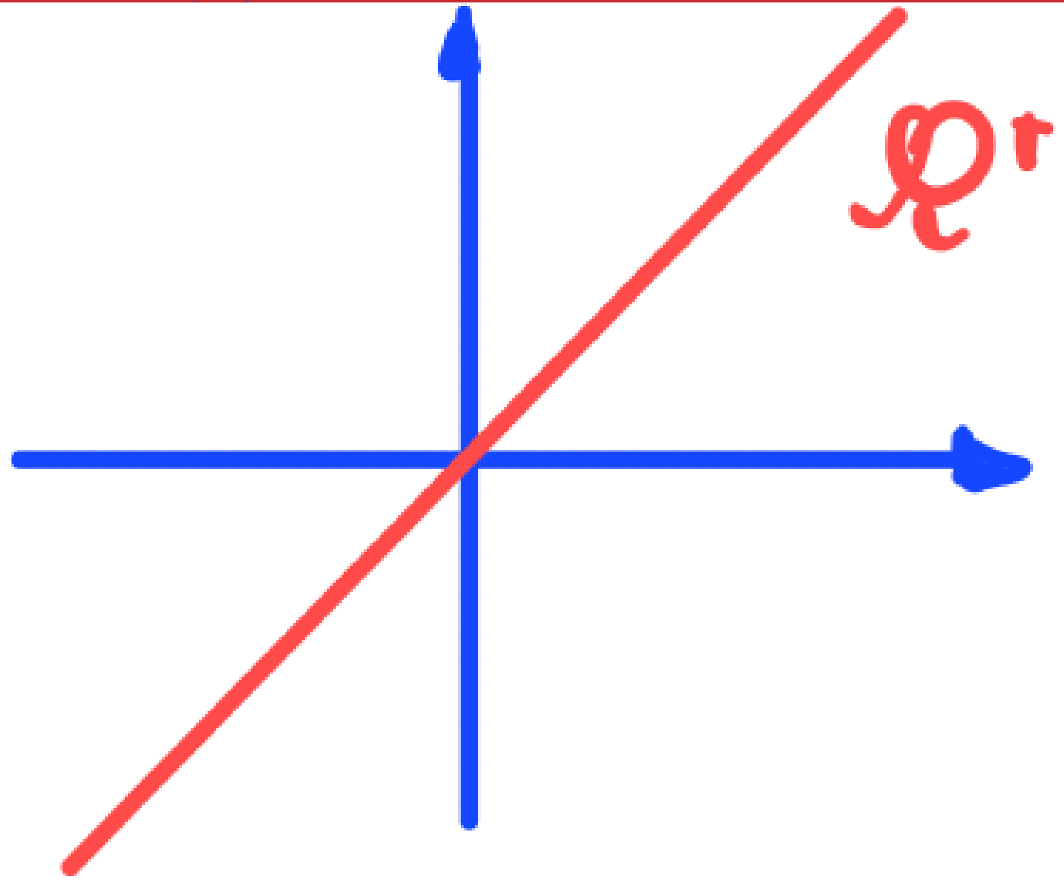
If $t_2 > t_1$ then $\mathscr{R}^+(t_1) \subseteq \mathscr{R}^+(t_2)$



Reachable set: $\mathscr{R}^+ := \mathscr{R}^+(\infty)$

Definition: The system is said to be completely reachable if $\mathscr{R}^+ := \mathbb{R}^n$

For $n = 2$



Result: the reachable set $\mathscr{R}^+$ is a subspace of $\mathbb{R}^n$ (much more than a set!)

Theorem : $\mathscr{R}^+ = \operatorname{Im} R$ $\qquad$ $R = \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}}_{mn} \Big\} n$

Reachability matrix $\qquad$ Dimension = $n \times mn$: $\begin{cases} \text{“Fat” matrix if } m > 1, \\ \text{“Square” matrix if } m = 1 \end{cases}$

Proof … (for D-T systems and intuition for C-T systems)

Image $M = \{ v \in \mathbb{R}^{\#\text{row } M} :$
$v = M w, \; w \in \mathbb{R}^{\#\text{row } M} \}$

Remarks:

- If $\operatorname{rank}(R) = n$ (full row rank) then $\mathscr{R}^+ = \mathbb{R}^n$ (all the states in $\mathbb{R}^n$ can be reached from the origin by applying a certain control input)
- For C-T (D-T) systems if a state can be reached from the origin (namely if it is in the span of R), then it can be reached in an arbitrarily small amount of time (in at most n steps)

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = 0$$

$$x(t) = \sum_{i=0}^{t-1} A^{t-i-1} B u(i) \quad t \geq 1$$

$x(1) = Bu(0)$ $\quad \mathcal{R}^+(1) = \text{Image } B$

$x(2) = ABu(0) + Bu(1)$ $\quad \mathcal{R}^+(2) = \text{Image } [B \ AB]$

$x(3) = A^2Bu(0) + ABu(1) + Bu(2)$

$x(t) = A^{t-1}Bu(0) + A^{t-2}Bu(1) + \dots + Bu(t-1)$

$$R_t \triangleq [B \mid AB \mid A^{t-1}B]$$

$$U_0^{t-1} \triangleq \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

collecting the input from $t-1$ up to 0

discrete-time

$$x(t) = R_t U_0^{t-1}$$

$$\begin{array}{ccccccc} \text{Image } R_1 & \subseteq & \text{Image } R_2 & \subseteq \quad \subseteq & \text{Image } R_t & \subseteq & \text{Image } R_n \\ \| & & \| & & \| & & \| \\ \text{Image } B & & \text{Image } [B \ AB] & & \text{Image } [B \ A^{t-1}B] & & \text{Image } R \end{array}$$

Goal to be proved set of states reachable in n steps.

$$\boxed{\mathcal{R}^+ = \text{Image } R}$$

$\Uparrow$

$A^nB \leftarrow$ linearly dependent on $[B \ AB \ A^{n-1}B]$

$\text{Image } R_n \ \& \ \text{Image } R_{n+1}$

$=$

$= \text{Im } R_{n+2}$

Cayley - Hamilton:

$$\varphi_A(\lambda) = \det(\lambda I - A) = A$$
$$= \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n$$

$$\varphi_A(A) = [A^n + \alpha_1 A^{n-1} + \dots + \alpha_n I]B$$
$$= 0 \cdot B$$

$$A^n B = -\alpha_1 A^{n-1} B - \alpha_n B$$

$$A^n B = -\alpha_1 A^n B - \dots - \alpha_n AB$$

Image R_n & Image R_{n+1}
$=$
$$= \text{Image } R_{n+2} = [R_{n+1} \,\vdots\, A^{n+1} B]$$

Homework:

For cont. time system,
$$x^{[k]}(0) = R_k U_0^{k-1} = \begin{bmatrix} u^{[k-1]}(0) \\ \dots \\ u^{[1]}(0) \\ u(0) \end{bmatrix}$$

For discrete time system:
$$x(t) = R_t U_0^{t-1}$$

For continuous - time
$$x^{[k]}(0) = R_k U_0^{k-1} \quad \text{where } U_0^{k-1} = \begin{bmatrix} u^{[k-1]}(0) \\ \vdots \\ u^{[0]}(0) \end{bmatrix}$$

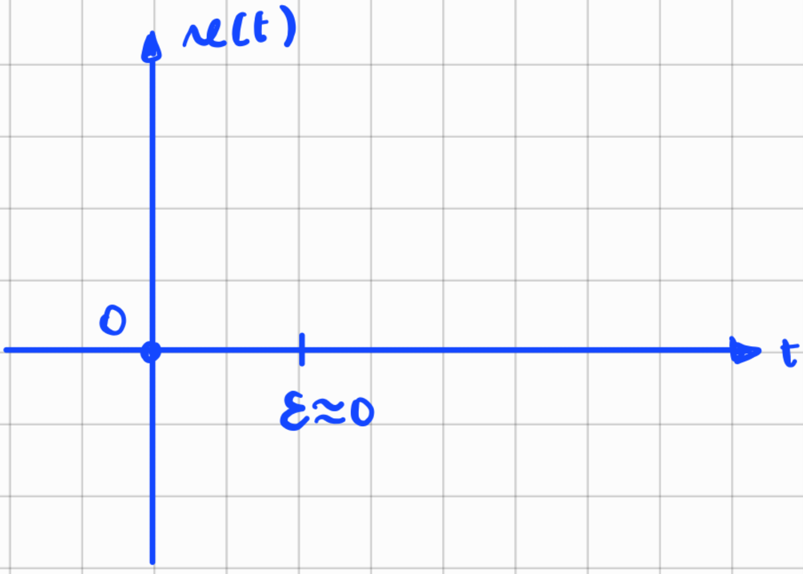
R_k → reachability matrix

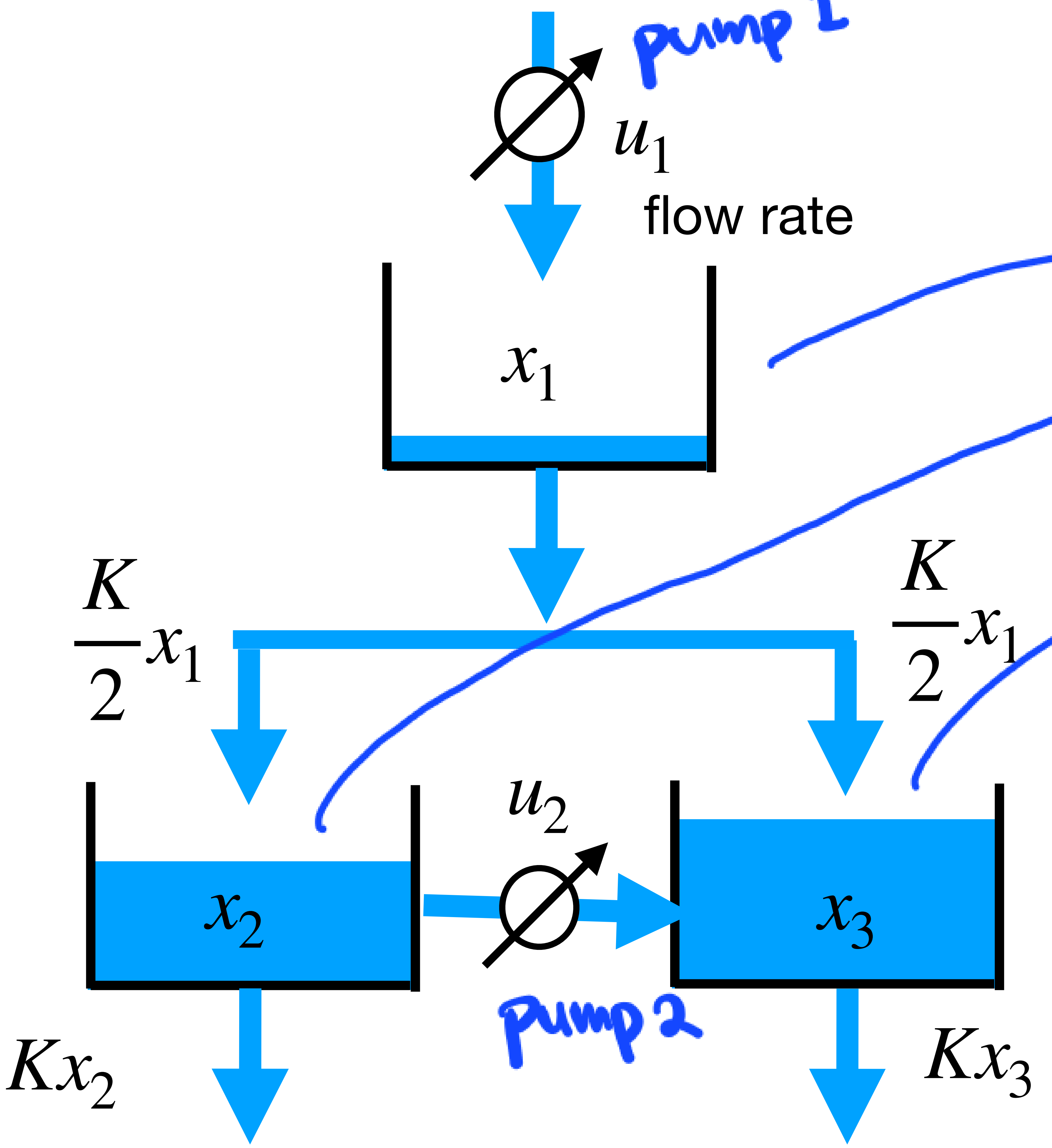
Suppose that $\mathcal{R}^+ = \mathbb{R}^n$, the image of $R = \mathbb{R}^n$, by Taylor,

$t_0 = 0$

$$x(t) = x(t_0) + \frac{1}{1!} x^{[1]}(t_0)(t - t_0) + \frac{1}{2!} x^{[2]}(t_0)(t - t_0)^2 + \dots + \frac{1}{n!} x^{[n]}(t_0)(t - t_0)^n$$

Indeed, $x(t) \equiv$ linear combination of $x^{[k]}(0)$, where $k = 1, 2, \ldots, n, n+1, \ldots, \infty$





Is this system completely reachable?

1. Modeling the system

$$\dot{x}_1 = u_1 - \frac{K}{2}x_1 - \frac{K}{2}x_1 = u_1 - x_1$$

$$\dot{x}_2 = \frac{K}{2}x_1 - Kx_2 - u_2$$

$$\dot{x}_3 = \frac{K}{2}x_1 - Kx_3 + u_2$$

2. A, B matrix

$$A = \begin{bmatrix} -K & 0 & 0 \\ K/2 & -K & 0 \\ K/2 & 0 & -K \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

3. Reachability matrix

this col. is not needed because we already know the rank of R = 3

$$R = \left[\begin{array}{cc|cc|c} 1 & 0 & -K & * & \\ 0 & -1 & K/2 & * & \\ 0 & 1 & K/2 & * & \end{array}\right]$$

$$\underbrace{\quad}_{B} \quad \underbrace{\quad}_{AB} \quad \underbrace{\quad}_{A^2B}$$

According to the theory, the system is reachable.

Now, how if pump 2 is faulty?

1. New B matrix:

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

NOTE :
this analysis only applies on linear sys.

2. Reachability matrix:

$$R = \begin{bmatrix} 1 & -K & K^2 \\ 0 & K/2 & -K^2 \\ 0 & K/2 & -K^2 \end{bmatrix}$$

$$\begin{array}{ccc} B & AB & A^2B \end{array}$$

3rd col

$$\begin{bmatrix} K^2 \\ -K^2 \\ -K^2 \end{bmatrix} = -2K^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2nd col

$$-2K \begin{bmatrix} -K \\ K/2 \\ K/2 \end{bmatrix}$$

rank R = 2

$\mathcal{R}^+$ = image R

PBH Test

Popov - Belevitch- Hautus test: it presents a sufficient and necessary condition for complete reachability not requiring the computation of the reachability matrix

pick a generic A, eigval λ, and multiply with I matrix.

The pair (A, B) is completely reachable (rank$R = n, ; \mathscr{R}^{+} = \mathbb{R}^{n}$) **iff**

$$\textbf{rank}\,[\lambda I - A \quad B] = n \qquad \forall\, \lambda\ (\in \sigma(A))$$

full row-rank (if λ is not in $\sigma(A)$ the condition is always fulfilled)

Proof (rank$R = n \quad \Rightarrow$ rank$[\lambda I - A \quad B] = n$ **)**

The proof:

$$\text{rank}\left(\left[\underbrace{(\lambda I - A)}_{\text{the determinant should be } \neq 0}{}_{n\times n} \;\vdots\; \underbrace{B}_{m}\right]\right) = n, \ \forall \lambda$$

Iff $\lambda \notin \sigma(A)$, $\det(\lambda I - A) \neq 0 \Rightarrow (\lambda I - A)$ is not singular. As long as the previous condition holds, the test is always fulfilled.

Suppose that the system is reachable, $\mathcal{R}^+ = \mathbb{R}^n$, rank $R = n$, and there exists $\exists \lambda^*$: rank $(\lambda^* I - A \mid B) < n$, by contradiction.

$\Rightarrow \exists\, w^* \in \mathbb{R}^n$, $w^{*T}(\lambda^* I - A \mid B) = 0$

non-zero

$$w^{*T}(\lambda^* I - A)^B = 0^B \qquad \boxed{w^{*T} B = 0}$$

$$\underbrace{w^{*T}\lambda^* B}_{=0} = \boxed{\underbrace{w^{*T} AB}_{=0}}$$

$$\boxed{w^{*T} A^2 B = 0}$$

By these relations, we can conclude $w^{*T}\overbrace{[B \mid AB \mid \dots \mid A^{n-1}B]}^{\text{reachability matrix } R} = 0$

Computing the input - The D-T case

Suppose (A,B) is reachable

How to practically compute the input steering the state of a system from the origin to a target state in $\mathscr{R}^+$? $= \mathbb{R}^n$

We know that $x(t) = R_t\, u_0^{t-1}$ with $\underbrace{R_t}_{n \times t \cdot m} = [B\ AB\ \cdots\ A^{t-1}B]$ and $u_0^{t-1} = [u(t-1)\ u(t-2)\ \cdots\ u(0)]^T$

- Case $\mathscr{R}^+ = \mathbb{R}^n$ (all states of $\mathbb{R}^n$ are reachable in at most n steps). Let $\bar{x}$ a target state.

$t \geq n,\ m \geq 1$

$$\bar{x} = R_t\, u_0^{t-1} \qquad u_0^{t-1} = \boxed{R_t^T\,(R_t\,R_t^T)^{-1}}\,\bar{x}$$

generic

"Fat" matrix (full row rank)

Right inverse of R_t

Result: Given any full row rank matrix M, the square matrix $M\,M^T$ is nonsingular

Special case $t = n, m = 1$

$$\bar{x} = R_n\, u_0^{n-1} = R\, u_0^{n-1} \qquad u_0^{n-1} = R^{-1}\,\bar{x}$$

generic

square (not singular)

$$\underbrace{\bar{x}}_{\text{fixed}} = \underbrace{x(t)}_{\text{fixed}} = \underbrace{R_t}_{\text{fixed}} \; \underbrace{u_0^{t-1}}_{\text{to be computed}}$$

Discrete-time :

$t \geq n$

$$\bar{x} = \overbrace{\left(\quad R_t \quad \right)}^{nmt} n \; nmt \overbrace{\left(\quad R_t^T \quad \right)}^{n} \left(\; u^{t-1} \; \right)$$

how to invert R_t?

$$\left[R_t R_t^T \right]^{-1}$$

NB : $[R_t R_t^T][R_t R_t^T]^{-1} = I$

In case of $t \geq n$

$$R_t = [\, \underbrace{B \quad AB \quad \dots \quad A^{n-1}B}_{n \times nm} \quad A^n B \quad A^{t-1} B \,]$$

Computing the input - The D-T case

no assumption on t

- In all the other cases (namely $t < n$ or $\mathscr{R}^+ \subset \mathbb{R}^n$) the target state $\bar{x}$ is not generic but, to be reached, must fulfil $\bar{x} \in \operatorname{Im} R_t$. In these cases the equation $\bar{x} = R_t\, u_0^{t-1}$ can be solved for u_0^{t-1}:

must be the image of R_t

$\bar{x} \in R_1 = [B]$ single col. tall matrix

$$u_0^{t-1} = R_t^{\dagger}\, \bar{x}$$

Generalized (Moore Penrose) inverse of R_t

The Moore-Penrose pseudoinverse is defined for any matrix and is unique. If the matrix is (right) invertible it boils down to the canonical (right) inverse. Otherwise, it provides the unique solution to $\bar{x} = R_t\, u_0^{t-1}$ if the latter has a solution (namely $\bar{x} \in \operatorname{Im} R_t$). If the equation in question does not admit a solution (namely $\bar{x} \notin \operatorname{Im} R_t$) the Moore-Penrose inverse provides the "closest" solution to $\bar{x}$ in the Euclidean sense

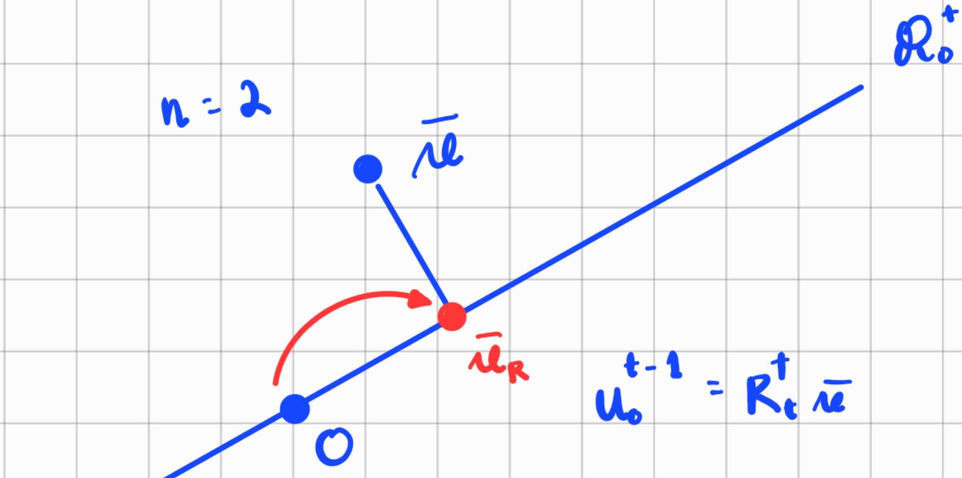
$\bar{x}_t = R_t\, u_0^{t-1}$

$u_0^{t-1} = R_t^{\dagger}\, \bar{x}$

$$\|\bar{x} - R_t\, R_t^{\dagger}\, \bar{x}\| \leq \|\bar{x} - R_t\, u_0^{t}\|$$

generic input sequence

Suppose that the system is not completely reachable



$$u_0^{t-1} = R_t^\dagger \bar{x}$$

If we start from the origin, we will reach the state close to the $\bar{x}$, which is projected by Moore-Penrose matrix $R_t^\dagger$

Example: Taxi company

$u(t)$: # new cars bought at year t

$x_1(t)$: # of 1-year old cars at year t

$x_2(t)$: # of cars older than 1 year at year t

P: Probability that in a generic year a car has not an irreparable accident

Problem: Compute the number of car to be bought so that after two years the park has $x_1^\star$ 1-Y old cars, and $x_2^\star$ older cars (suppose the park is empty at $t = 0$). What's the input profile if the park at $t = 0$ is not empty?

$$x_1(t+1) = Pu(t)$$
$$x_2(t+1) = Px_1(t) + Px_2(t)$$

1. Modelling the eq. $\rightarrow$ new cars bought

$$x_1(t+1) = P u(t)$$
$$x_2(t+1) = P x_1(t) + P x_2(t)$$

2. The matrix

$$A = \begin{bmatrix} 0 & 0 \\ P & P \end{bmatrix} \qquad B = \begin{bmatrix} P \\ 0 \end{bmatrix}$$

3. Reachability matrix

$$R = \begin{bmatrix} \underbrace{\begin{matrix} P \\ 0 \end{matrix}}_{B} & \vdots & \underbrace{\begin{matrix} 0 \\ P^2 \end{matrix}}_{AB} \end{bmatrix} \rightarrow \text{rank} = 2$$

The system is reachable, so it is possible to reach any number of cars.

$$R^{-1} = \begin{bmatrix} 1/P & 0 \\ 0 & 1/P^2 \end{bmatrix}$$

4. To reach the x_1^* and x_2^* :

$$\underbrace{\begin{bmatrix} u(1) \\ u(0) \end{bmatrix}}_{u_0^1} = \begin{bmatrix} 1/P & 0 \\ 0 & 1/P^2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$$

any

5. In case of three years

$$\begin{bmatrix} u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix} = R_3^T \left[R_3 R_3^T \right] \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$$

Computing the input - The C-T case

initial state

$$\dot{x} = Ax + Bu, \quad x(0) = 0 \qquad x(t) = \int_0^t e^{A(t-s)}\, B\, u(s)\, ds$$

How to compute the input $u(s)$, $s \in [0,t)$ steering the state of a system from the origin to a target state in $x(t) = \bar{x} \in \mathscr{R}^+$?

suppose the system is completely reachable

Reachability/Controllability $\boxed{\text{Gramian}}$ at time t

$$\boxed{W(t) = \int_0^t e^{A(t-s)}\, B\, B^T\, e^{A^T(t-s)}\, ds}$$

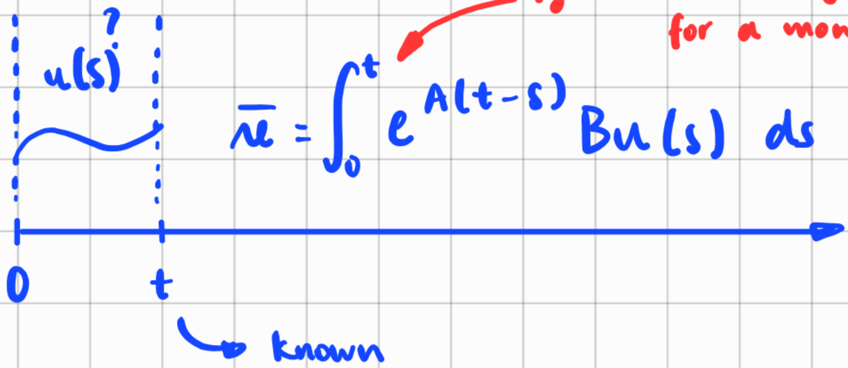
Theorem: The Gramian is not singular for each $t > 0$ **iff** $\mathscr{R}^+ = \mathbb{R}^n$ always non-singular

Proof…

Result: Assume that the system is completely reachable. Let $\bar{x} \in \mathbb{R}^n$ a generic target state. Let $t > 0$ be an arbitrary time. then the control input able to steer the state of (A, B) from the origin to $\bar{x}$ in the interval $[0,t)$ is

target state

$$u(s) = B^T\, e^{A^T(t-s)}\, W^{-1}(t)\, \bar{x} \qquad s \in [0,t)$$



ignore the integral for a moment,

$$\bar{x} = \int_0^t e^{A(t-s)} B u(s)\, ds$$

$s \in [0, t)$

$m = 1$

$$\underbrace{\left[e^{A(t-s)}B\right]\left[B^T e^{A^T(t-s)}\right]}_{n\times n}\left[e^{A(t-s)}BB^T e^{A^T(-s)}\right]^{-1\,?}\bar{x}$$

$$u(s) = \left[B^T e^{A^T(t-s)}\right]\left[e^{A(t-s)}BB^T e^{A^T(-s)}\right]^{-1}\bar{x}$$

rank[$v\ v^T$] = rank v

is not an invertible matrix

$$W(t) = \int_0^t \boxed{e^{A(t-s)}BB^T e^{A^T(t-s)}}\, ds$$

$n \times n$ singular

Controllability/reachability Gramian

$$\bar{x} = \underbrace{\int_0^t e^{A(t-s)}B\left[B^T e^{A^T(t-s)}\right] ds}_{W(t)}\; W^{-1}(t)\,\bar{x}$$

$u(s) = B^T e^{A^T(t-s)} W^{-1}(t)\,\bar{x}$

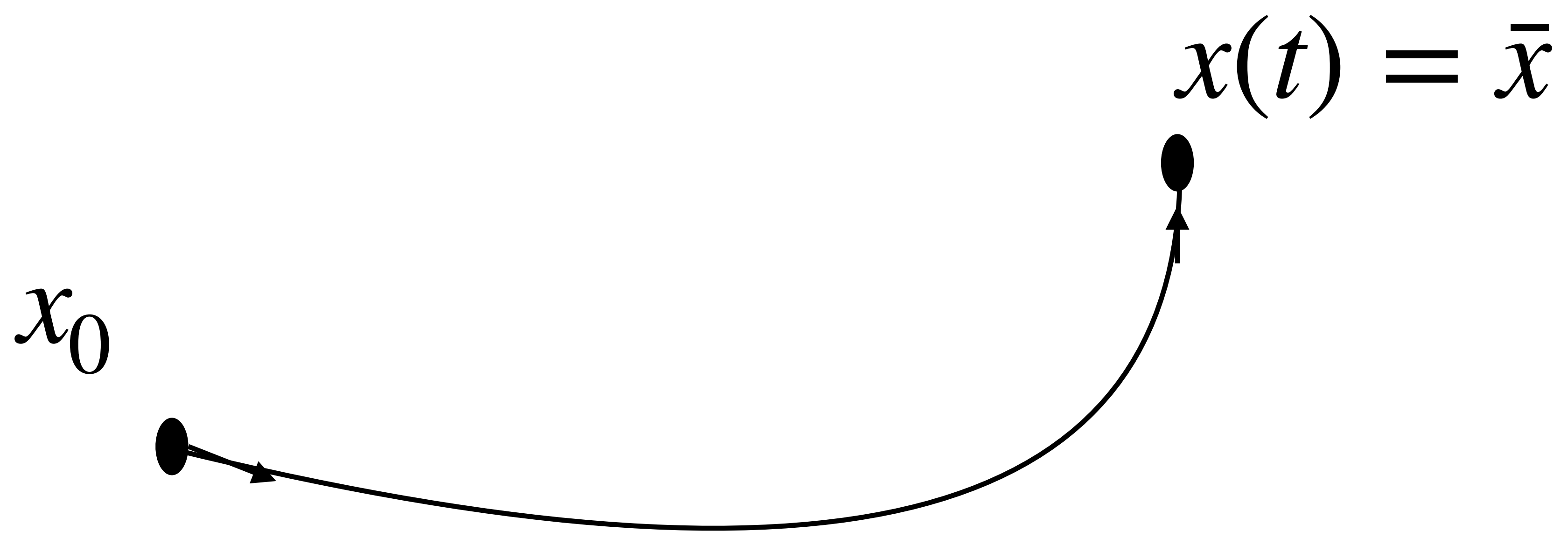
for $s \in [0, t)$

$$x(t) = \int_0^t e^{A(t-s)} B\, u(s)\, ds$$

$$\begin{bmatrix} A & B \\ & \end{bmatrix} \xrightarrow{?} x(t)$$

Computing the input - The C-T case

General case: How to compute the control input steering the state of a linear system from a generic initial condition (not necessarily the origin) to a generic final state in a predetermined time interval?



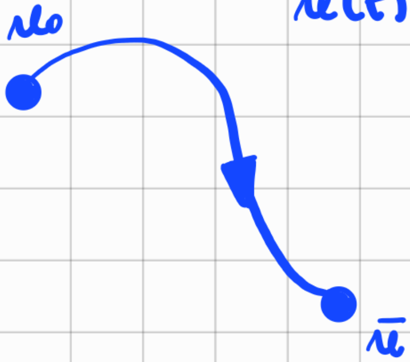
$$x(t) - e^{At}x_0 = \int_0^t e^{A(t-s)}\, B\, u(s)\, ds$$

Because of linearity the problem can be cast as the problem of staring the state of the system from the origin to the final target $\bar{x}' = \bar{x} - e^{At}$ in a predetermined interval.

$$\boxed{u(s) = B^T\, e^{A^T(t-s)}\, W^{-1}(t)\, (\bar{x} - e^{At}x_0) \qquad s \in [0,t)}$$

For the Discrete-Time case:

Lagrange formula: $$x(t) = A^t x_0 + \underbrace{\sum_{i=0}^{t-1} A^{t-i} B u(i)}_{R_t u_0^{t-1}}$$ ✓



$$x(t) - A^t x_0 = R_t u_0^{t-1} \quad R_t u_0^{t-1}$$

$$u_0^{t-1} = R_t^{\dagger}(\bar{x} - A^t x_0)$$

$t \geq n \quad \mathcal{R}^+ = \mathbb{R}^n$

For the Continuous-Time case:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} B u(s)\, ds$$ ✓

$$x(t) - e^{At} x_0 = \int_0^t e^{A(t-s)} B u(s)\, ds$$

Controllability

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t) \qquad x(0) = x_0$$

$$x(t) = \phi(t)\, x(t_0) \ + \ \Psi(t)\, u([0,t))$$

No outputs involved in the following analysis

controllable to the origin

Set of controllable states at time t_1:

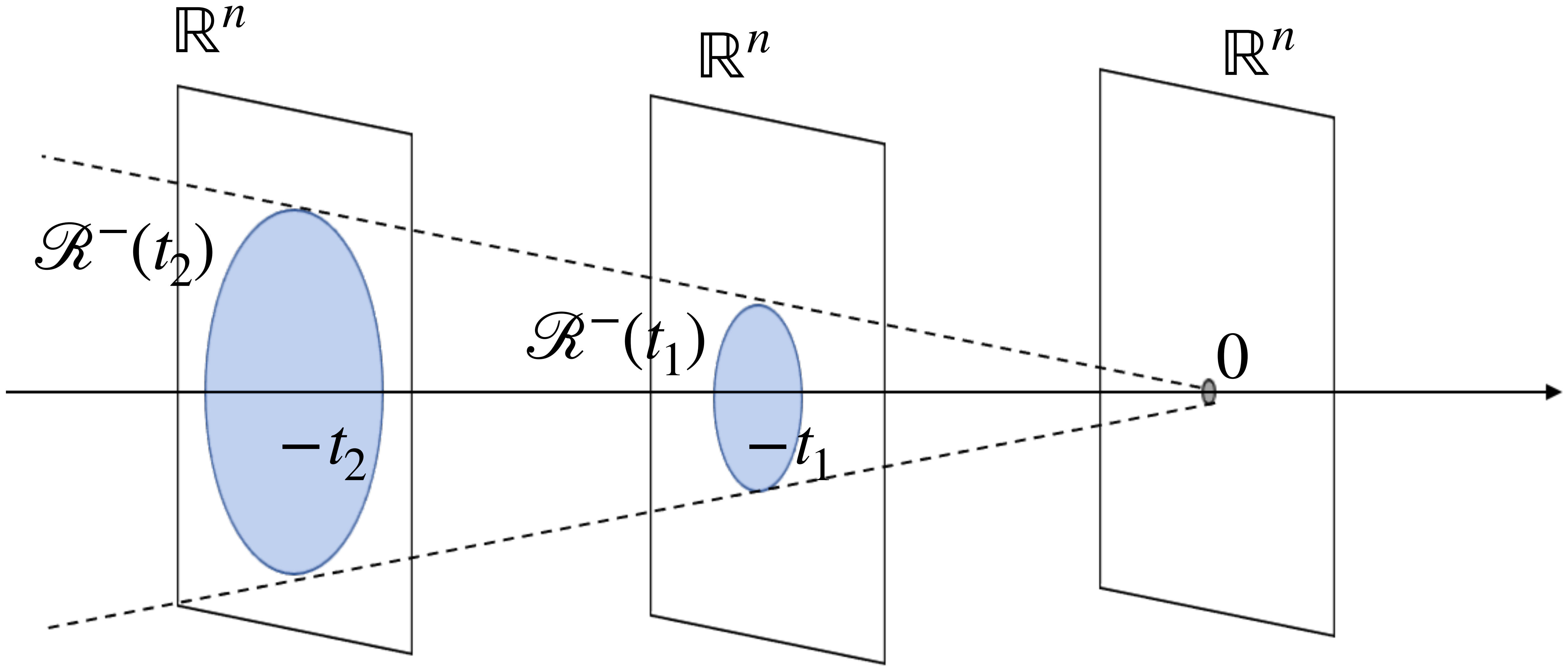
$$\mathscr{R}^{-}(t_1) = \{x \in \mathbb{R}^n \quad : \quad 0 = \phi(t_1)x + \psi(t_1)\, u([0,t_1)) \ \text{ for some } \ u([0,t_1)) \in \mathbb{R}^m\}$$

If $t_2 > t_1$ then $\mathscr{R}^{-}(t_1) \subseteq \mathscr{R}^{-}(t_2)$

Controllable set:

$\mathscr{R}^{-} := \mathscr{R}^{-}(\infty)$ (Reverse from $\mathscr{R}^{-}$)

Definition: The system is said to be completely controllable if $\boxed{\mathscr{R}^{-} := \mathbb{R}^n}$



Controllability

not only a set, but a subspace

Result: the reachable set $\mathscr{R}^-$ is a subspace of $\mathbb{R}^n$ (much more than a set!)

Theorem : $\mathscr{R}^+ \subseteq \mathscr{R}^-$ in general. $\mathscr{R}^+ = \mathscr{R}^-$ if the system is reversible

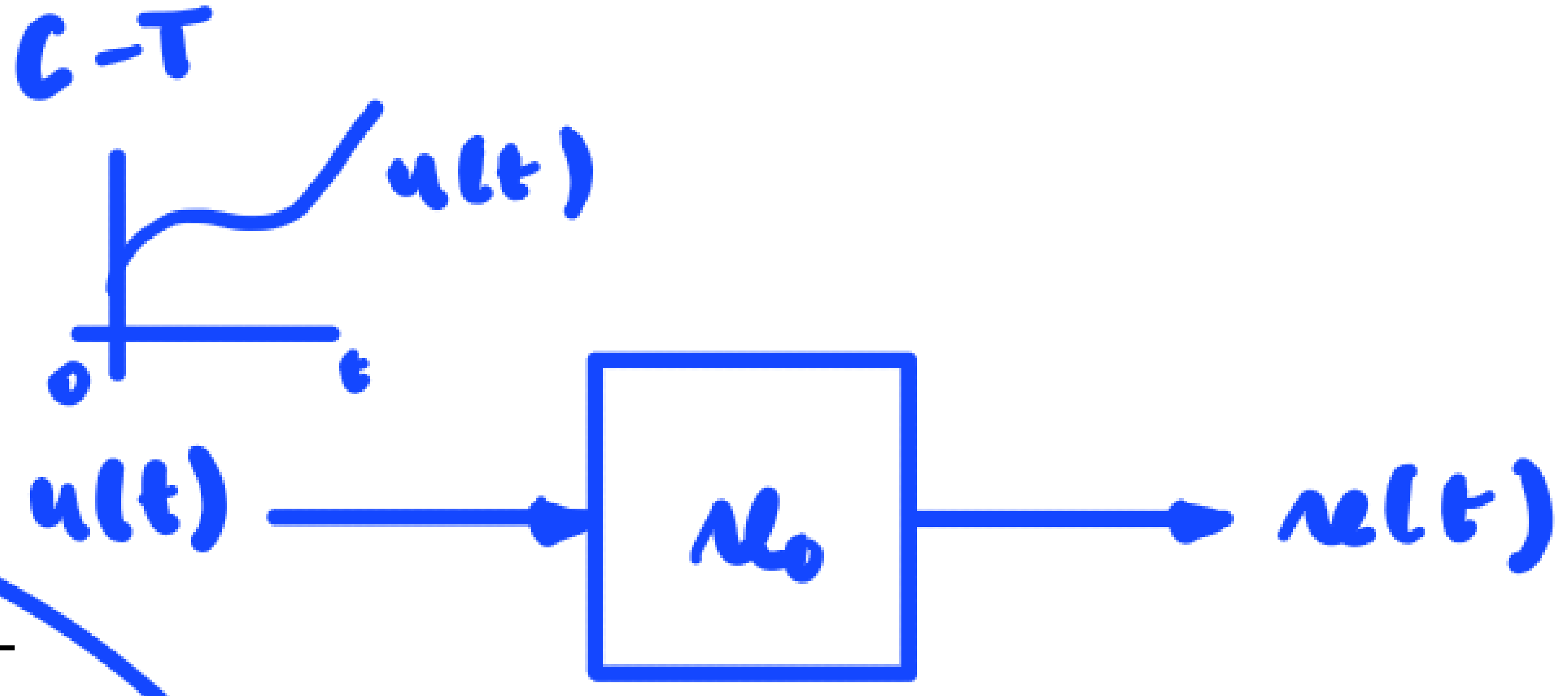
Remark: There could be systems with state that are controllable to the origin but that cannot be reached from the origin. **Complete reachabilty ⇒ complete controllability**

Remark: All continuous-time systems are reversible, and thus $\mathscr{R}^+ = \mathscr{R}^-$. However there could be discrete time systems for which the inclusion holds true

Example: trivially $\mathscr{R}^+$ is the origin for this case

$$x(t+1) = Ax(t) + Bu(t) \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

C-T



$x_1(t+1) = x_2(t)$

$x_2(t+1) = 0$

$x(0) = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} \in \mathbb{R}^2$

Remark: D-T sampled-data systems are reversible and thus $\mathscr{R}^+ = \mathscr{R}^-$

Remark: From now on controllability and reachability will be confused

For the Discrete-Time case:

$$x_1(t+1) = x_2(t)$$

$$x_2(t+1) = 0 \implies \begin{aligned} x_2(1) &= 0 \\ x_1(1) &= x_2^* \\ x_1(2) &= 0 \end{aligned}$$

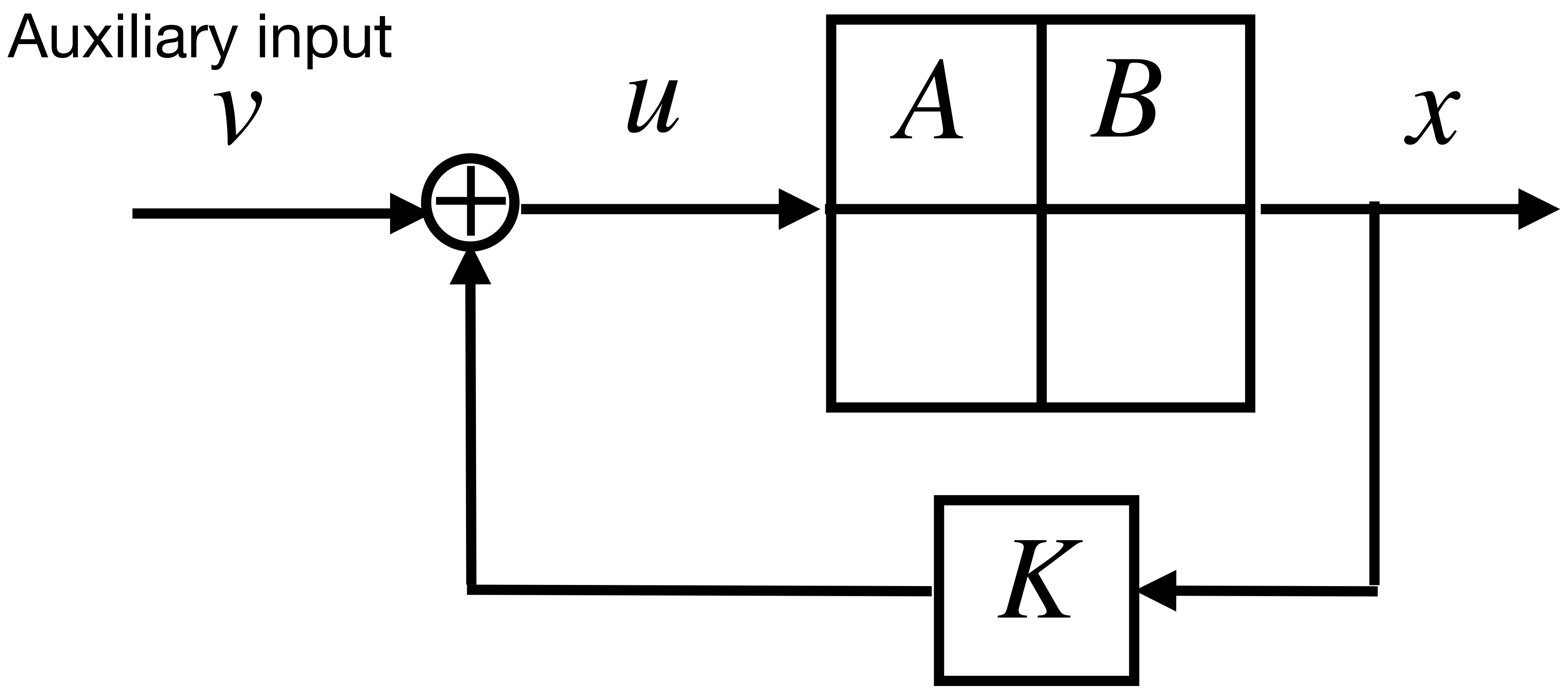
$$x_2(0) = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \in \mathbb{R}^2$$

Controllability and state feedback

we close the system loop

Up to now we focused on computing a control input steering the state between two desired states in an “open loop” way. If the system is completely controllable/reachable we have full authority to steer the state of the systems between two arbitrary states (also in an arbitrary time for C-T systems). Now we are interested to link the controllability property of a system to the ability of designing a state feedback “improving” in some way the resulting closed- loop

Not imposing trajectories but imposing dynamics



Static (matrix) gain

$$\dot{x} = Ax + Bu$$

gain matrix with dim. mxn

$$\Downarrow \quad u = Kx + v$$

$$\dot{x} = (A + BK)x + Bv$$

could be D.O.F.

Can the static gain K be chosen to change the system dynamics (eigenvalues) ?

Controllability and state feedback

Question: Are controllability properties affected by change of coordinates? For instance, if (A, B) is completely controllable, is a $\boxed{\text{“similar pair” } (\tilde{A}, \tilde{B})} = (TAT^{-1}, TB)$ also completely controllable?

$$\tilde{R} = T\,R$$

$$\tilde{R} = (\tilde{B}\ \tilde{A}\tilde{B}\ \cdots\ \tilde{A}^{n-1}\tilde{B}) \qquad R = (B\ AB\ \cdots\ A^{n-1}B)$$

$\Rightarrow\ \text{rank}R = \text{rank}\tilde{R}$

Controllability/reachability is a “structural property”, not affected by the coordinate framework used to describe the system

Controllability/reachability of the system

Question: Are controllability properties affected by static state feedback?

Result

(A, B) completely controllable $\Rightarrow$ $(A + BK, B)$ completely controllable $\forall\ K$

Homework: prove it

Proving with any K, a fully controllable system is controllable.

(A, B) $\quad R = [B \quad AB \quad A^2B \quad A^{n-1}B]$

$\downarrow T$ $\quad Q^1 = \text{image } R$

$(\tilde{A}, \tilde{B}) = (TAT^{-1}, TB)$ $\quad \tilde{R} = [TB \quad \underbrace{TAT^{-1}}_{\tilde{A}} \underbrace{TB}_{\tilde{B}} \quad \underbrace{TAT^{-1}TAT^{-1}}_{\tilde{A}^2}\underbrace{TB}_{\tilde{B}}]$

Proven $\tilde{R} = TR$ $\quad$ rank $\tilde{R}$ = rank $\tilde{R}$

$R = T^{-1}\tilde{R}$

So, with any K, it does not change the dynamic of the system.

Can we then identify a coordinate framework where the design of K is easier ?!

Controllability canonical form $m = 1$)

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix} \quad B_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \varphi_A(\lambda) &= \det(\lambda I - A) \\ &= \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n \end{aligned}$$

always 1 on the last row

non-zero on the last row

single column

Theorem $(m = 1)$

A pair (A, B) is similar to (A_c, B_c) **iff** the system is completely controllable

change of coord. in controllability canonical form

$$T = T_c := R_c\, R^{-1}$$

$R_c = T_c R$

$B_c = T_c B$

$$R_c = (B_c \ A_c B_c \ \cdots \ A_c^{n-1} B_c) \qquad R = (B \ AB \ \cdots \ A^{n-1}B)$$

$A_c = T_c A T_c^{-1}$

Controllability and state feedback

eigenvalues assignment theorem

Theorem $(m \geq 1)$

A pair (A, B) is completely controllable **iff** for all $\{\lambda_1^\star, \ldots, \lambda_n^\star\}$ (set of desired eigenvalues) there exists a K such that $\sigma(A + BK) = \{\lambda_1^\star, \ldots, \lambda_n^\star\}$ desired eigvals

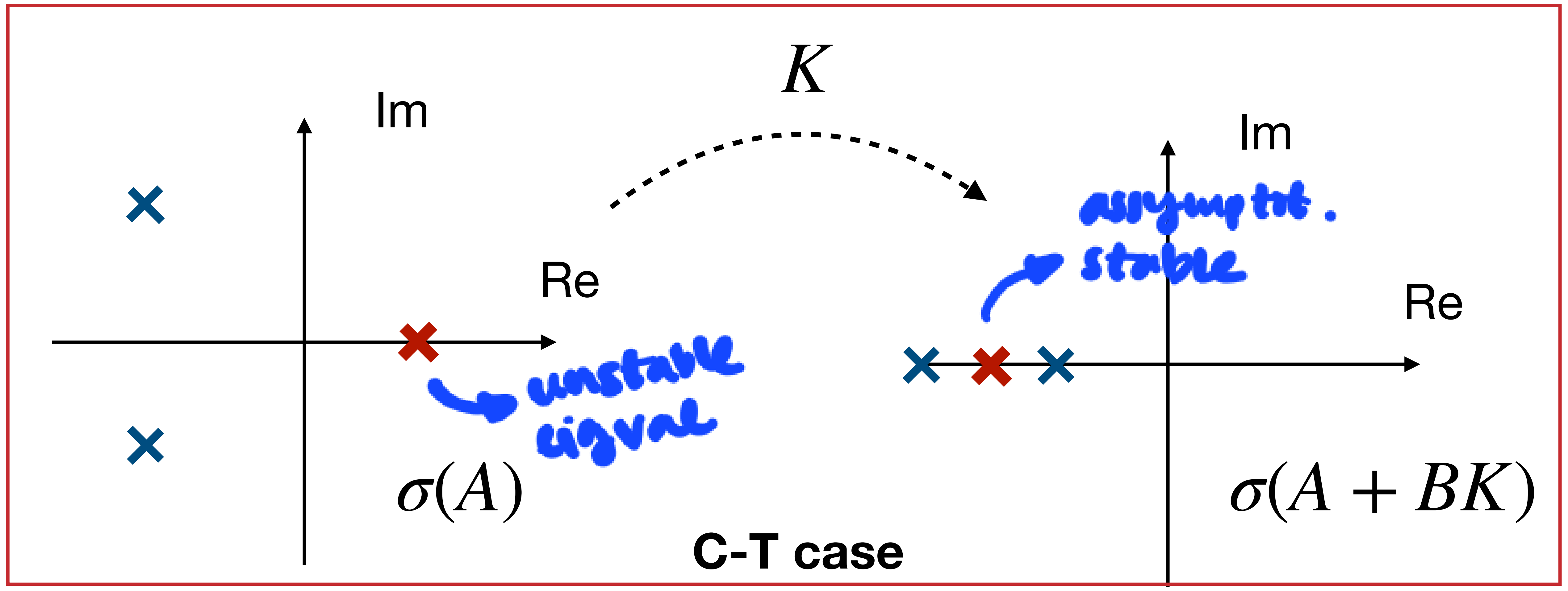
Constructive proof of the if part $(m = 1)$:

- Let $(\alpha_1^\star, \ldots, \alpha_n^\star)$ be such that $\{\lambda_1^\star, \ldots, \lambda_n^\star\}$ are roots of $\lambda^n + \alpha_1^\star \lambda^{n-1} + \ldots + \alpha_{n-1}^\star \lambda + \alpha_n^\star = 0$
- Let $K_c = (\alpha_n - \alpha_n^\star \;\; \cdots \;\; \alpha_1 - \alpha_1^\star)$
- Pick $K = K_c\, T_c$

Corollary

A completely controllable system can be always stabilised by static state feedback (…and much more!)

Full authority in eigenvalues assignment



Suppose a system that (A, B) has a set of eigvals. $\{\lambda_1^* \dots \lambda_n\} = \sigma(A + BK)$

$$T_c(A + BK)T_c^{-1} = A_c + B_c \overbrace{K T_c^{-1}}^{K_c}$$

Recall:

$K_c = K T_c^{-1}$

$K = K_c T_c$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & \cdots & -\alpha_1 \end{bmatrix}}_{A_c} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_{c1} & \cdots & K_{cn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \underbrace{-\alpha_n + K_{c1}}_{-\alpha_n^*} & \cdots & \cdots & \cdots & \underbrace{-\alpha_1 + K_{cn}}_{-\alpha_1^*} \end{bmatrix} \Rightarrow$$

The eigenvalues are: $\lambda^n + \alpha_1^* \lambda^{n-1} + \dots + \alpha_n^* = 0$ have roots: $\{\lambda_1^*, \dots \lambda_n^*\}$

Kalman decomposition

Suppose now the (A, B) is not completely controllable, namely $\operatorname{rank} R = n_r < n$ ($\dim \mathscr{R}^+ = n_r < n$)

Let $\mathscr{R}^+_\perp$ be the orthogonal complement of $\mathscr{R}^+$. It turns out that $\dim \mathscr{R}^+_\perp = n - n_r$

Let $\{v_1, \ldots, v_{n_r}\}$ be a base of $\mathscr{R}^+$ and let $\{v_{n_r+1}, \ldots, v_n\}$ be a base of $\mathscr{R}^+_\perp$. The two sets of vectors are all linearly independent

base of $\mathscr{R}^+$

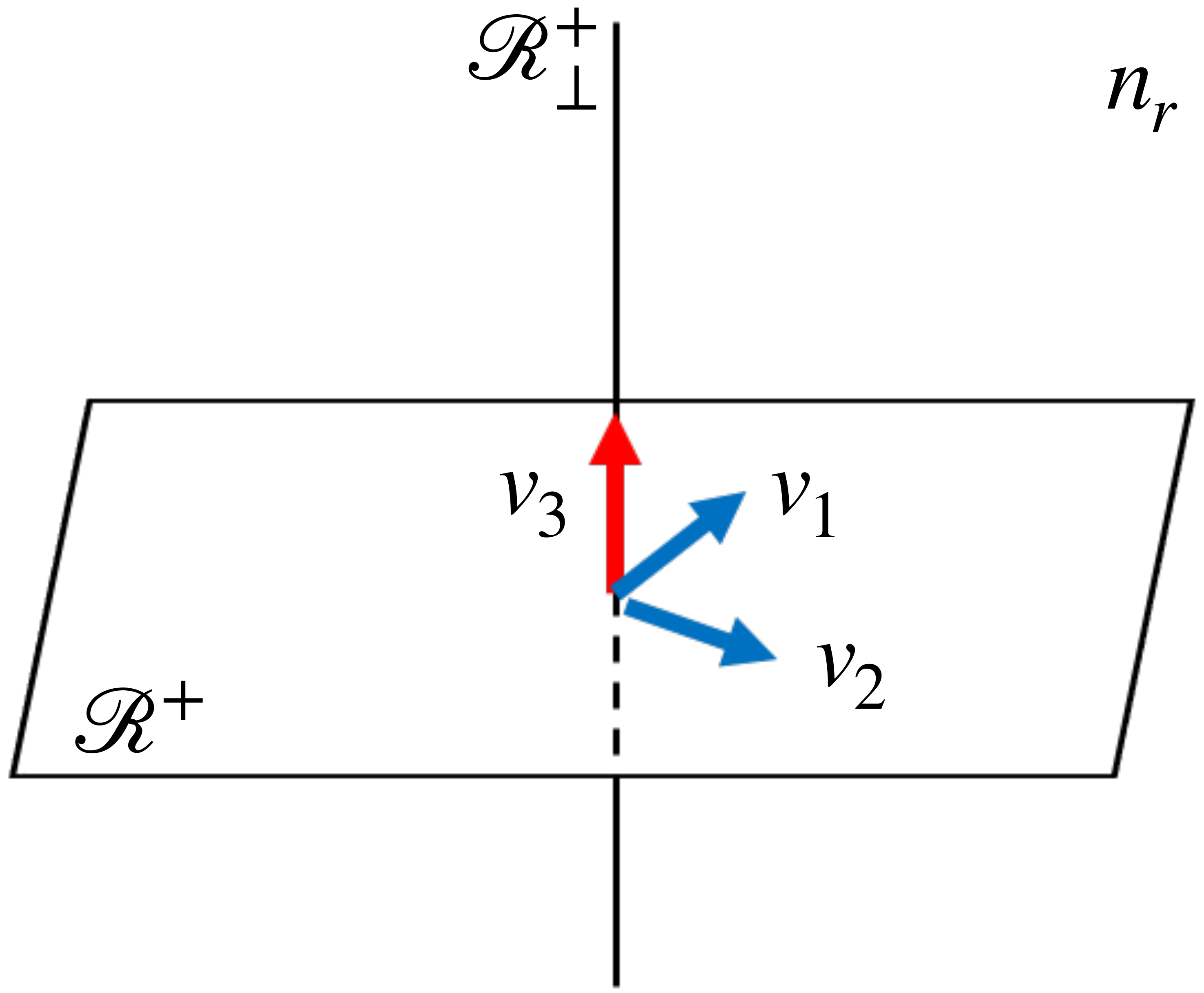
Consider the change of variables $T_K^{-1} = \begin{bmatrix} v_1 & \ldots & v_{n_r} & v_{n_r+1} & \ldots & v_n \end{bmatrix}$

$$z = T_K x = \begin{pmatrix} z_r \\ z_{nr} \end{pmatrix} \qquad x = T_K^{-1} z$$

It turns out that $x \in \mathscr{R}^+ \;\Rightarrow\; z = \begin{pmatrix} \star \\ 0 \end{pmatrix} \qquad x \in \mathscr{R}^+_\perp \;\Rightarrow\; z = \begin{pmatrix} 0 \\ \star \end{pmatrix}$

$n = 3$

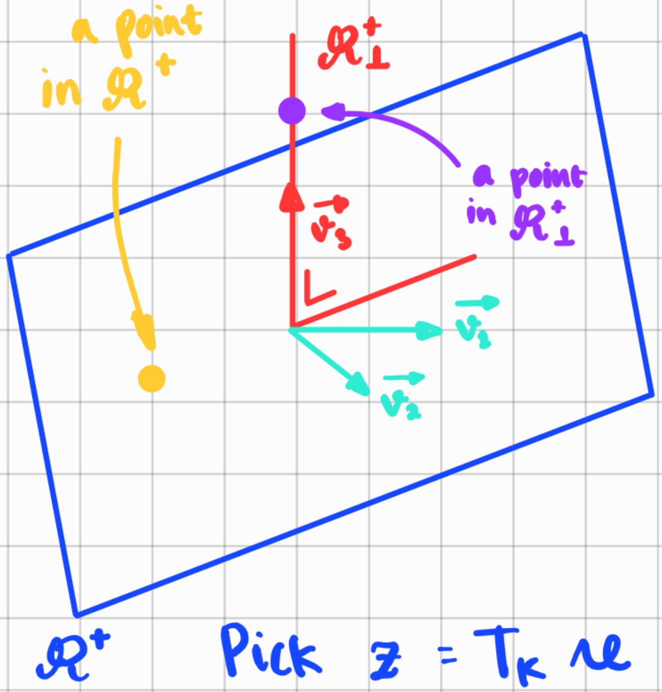
$n_r = 2$



$\mathcal{R}^+ = \mathcal{R}^- \subset \mathbb{R}^n$ rank $R = n_R < n$

dim $\mathcal{R}^+ = n_R$

Suppose $n = 3$, $n_R = 2$



$\{\vec{v}_1, \vec{v}_2\}$ base for $\mathcal{R}^+$

$\{\vec{v}_3\}$ base for $\mathcal{R}^+_\perp$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ base for $\mathbb{R}^3$ → not necessarily orthogonal but linearly indep.

$$T^{-1} = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$$

Pick $z = T_K x$, $x = T_K^{-1} z$, for

an orange point, $z = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$

a violet point, $z = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$, $z = \begin{bmatrix} z_R \\ z_{NR} \end{bmatrix} \begin{array}{l} \} n_R \\ \} n - n_R \end{array}$

$$z \in \mathcal{R}^+ \begin{cases} z_{NR} = 0 \\ z_R \neq 0 \end{cases}$$

$$z \in \mathcal{R}^+_\perp \begin{cases} z_{NR} \neq 0 \\ z_R = 0 \end{cases}$$

Kalman decomposition

Result (by Kalman)

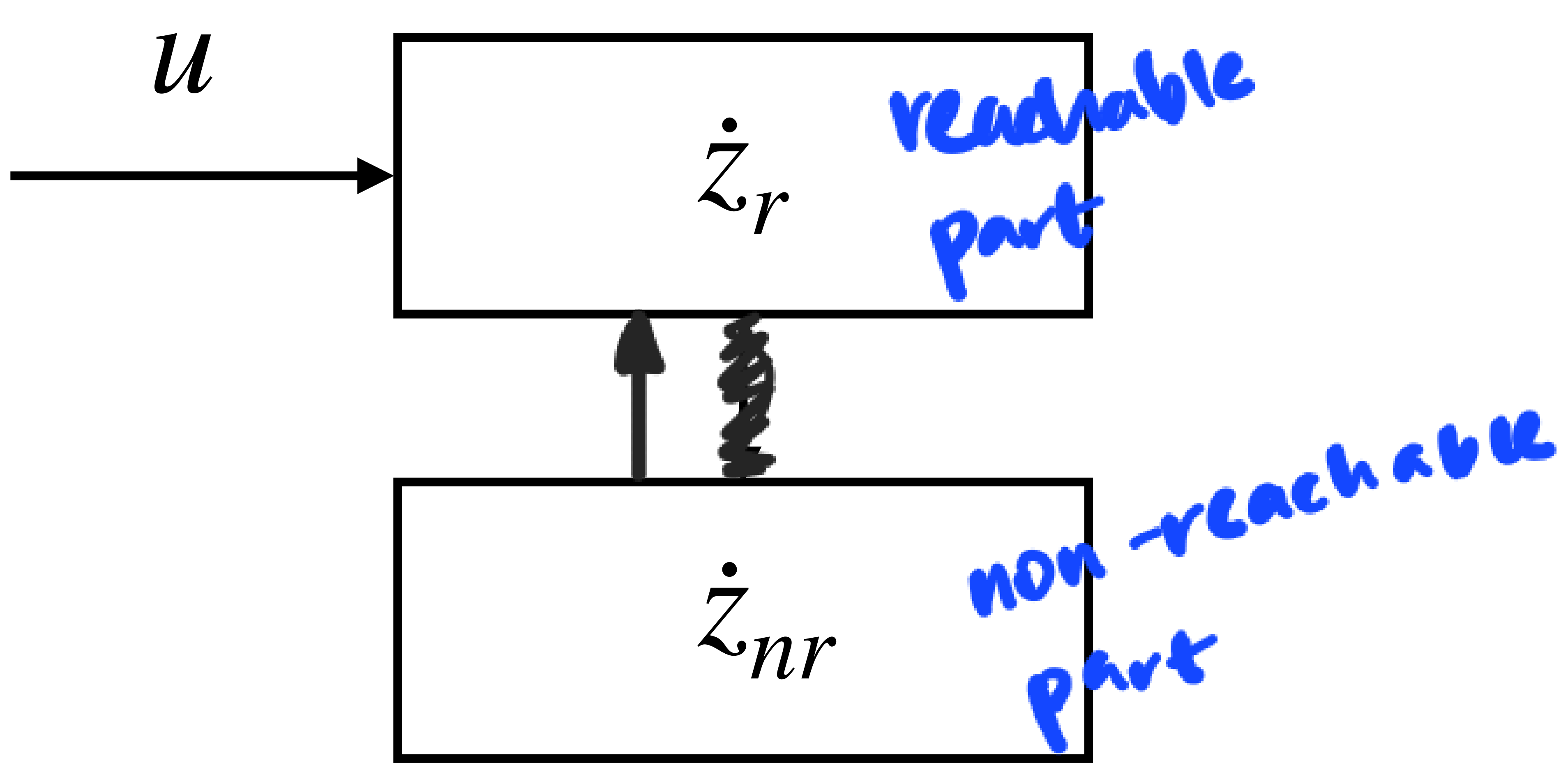
$$\tilde{A}_K = T_K A T_K^{-1} = \begin{array}{c} n_r \times n_r \\ \begin{pmatrix} \tilde{A}_R & \tilde{A}_J \\ 0 & \tilde{A}_{NR} \end{pmatrix} \\ n - n_r \times n - n_r \end{array} \qquad \tilde{B}_K = T_K B = \begin{pmatrix} \tilde{B}_R \\ 0 \end{pmatrix} \qquad (\tilde{A}_R, \tilde{B}_R) \quad \text{Completely reachable}$$

autonomous adj.

$$\dot{x} = Ax + Bu \quad \Rightarrow \quad \begin{cases} \dot{z}_r = \tilde{A}_R z_r + \tilde{A}_J z_{nr} + \tilde{B}_R u \\ \dot{z}_{nr} = \tilde{A}_{nr} z_{nr} \end{cases}$$

reachable

non-reachable



- The set $\mathscr{R}^+$ is forward invariant for the system dynamics. The "internal dynamics" are completely reachable/controllable

- If $\tilde{A}_{nr}$ is Hurwitz then trajectories starting outside $\mathscr{R}^+$ asymptotically converge to it

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\Big\downarrow\, T_K$$

$$\dot{z} = \tilde{A}_K z + \tilde{B}_K u \qquad z \in \mathbb{R}^n$$

$$\begin{cases} \dot{z}_R = \tilde{A}_R z_R + \tilde{B}_R u + \tilde{A}_J z_{NR} & z_R \in \mathbb{R}^{n_R} \\ \dot{z}_{NR} = \tilde{A}_{NR} z_{NR} & z_{NR} \in \mathbb{R}^{n-n_R} \end{cases}$$

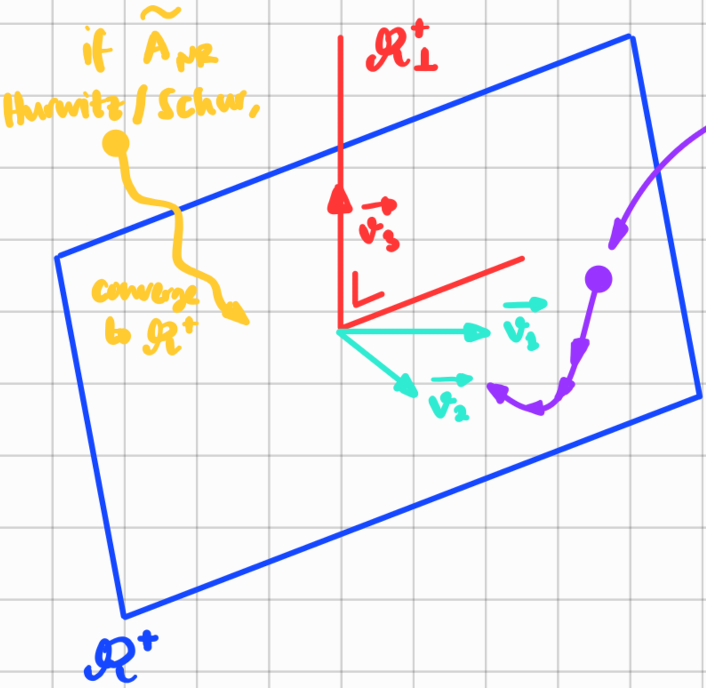
completely controllable/reachable

$z_{NR}(0) = 0$

$z_{NR}(t) = 0$

For $R_R = [\,\tilde{B}_R \;\; \tilde{A}_R \tilde{B}_R \;\; \cdots \;\; \tilde{A}_R^{\,n_R-1} \tilde{B}_R\,]$,

rank $R_R = n_R$



For an initial condition in $\mathcal{R}^+$:

$$\begin{array}{c} \begin{bmatrix} * \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \} \, n_R \\ \} \, n-n_R \end{array}$$

$$\dot{z}_R = \tilde{A}_R z_R + \tilde{B}_R u$$

Pick $x(0) \in \mathfrak{R}^+$

$x(t) \in \mathfrak{R}^+ \quad \forall t, \forall u$

Pick $x(0) \in \mathfrak{R}^+_\perp$

$z_R(0) = 0, \quad z_{NR}(0) \neq 0,$

$z_R(t) \neq 0, \quad z_{NR}(t) \neq 0, \quad x \notin \mathfrak{R}^+_\perp$

Stabilisability

Definition

A system (A, B) is said to be **stabilizable** if there exists a K such that $A + BK$ is Hurwitz (Schur)

Remark

(A, B) completely reachable $\overset{\Rightarrow}{\cancel{\Leftarrow}}$ (A, B) stabilizable

Theorem

(A, B) is stabilizable **iff** $\tilde{A}_{nr}$ is Hurwitz (Schur)

Constructive proof ($\tilde{A}_{nr}$ Hurwitz/Schur $\Rightarrow \exists\, K \;:\; (A + BK)$ is Hurwitz/Schur

The selection of K is: $\quad K = K_K\, T_K \qquad K_K = [K_{Kr} \;\; \star]$ with K_{Kr} so that $\tilde{A}_r + \tilde{B}_r K_{Kr}$ is Hurwitz

doesn't play any role

Suppose that (A, B) is not controllable but stabilizable. The goal is design K that makes $A + BK$ is Hurwitz/Schur.

coord. change on Kalman

$$T_K (A + BK) T_K^{-1} = \tilde{A}_K + \tilde{B}_K \underbrace{K_K}_{KT_K^{-1}} \Longrightarrow \begin{aligned} K_K &= K T_K^{-1} \\ K &= K_K T_K \end{aligned}$$

$$\underbrace{\left[\begin{array}{c|c} \tilde{A}_R & \tilde{A}_J \\ \hline 0 & \tilde{A}_{NR} \end{array}\right]}_{\tilde{A}_K} + \underbrace{\left[\begin{array}{c} \tilde{B}_R \\ 0 \end{array}\right]}_{\tilde{B}_K} \underbrace{\left[\begin{array}{c|c} K_{KR} & K_{KNR} \end{array}\right]}_{K_K}$$

assignable

completely controllable

$$\sigma \left[\begin{array}{c|c} \tilde{A}_R + \tilde{B}_R K_{KR} & \tilde{A}_J + \tilde{B}_R K_{KNR} \\ \hline 0 & \tilde{A}_{NR} \end{array}\right] = \begin{array}{c} \sigma(\tilde{A}_R + \tilde{B}_R K_{KR}) \\ \cup \\ \sigma(\tilde{A}_{NR}) \end{array}$$

Controllability of a cascade

$$u \in \mathbb{R} \longrightarrow \boxed{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx\end{array}}_{\,x \in \mathbb{R}^n} \xrightarrow{\;y \in \mathbb{R}\;} \boxed{\dot{z} = Fz + Gy}_{\,z \in \mathbb{R}^\ell}$$

(A, B), (F, G) controllable

$\Downarrow$?? No?

$\begin{pmatrix} A & 0 \\ GC & F \end{pmatrix}$ controllable

Result

The cascade is controllable iff the pairs (A, B), (F, G) are controllable and

$$\text{rank} \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} = n + 1 \qquad \forall\, \lambda \in \sigma(F)$$

Non resonance condition