

# Mathematical Methods for Automation Engineering M

## – *Combinatorial Analysis* –

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# Combinatorial Analysis

*we need to compute the cardinality using combinatorics*

**Problem.** In many contexts, it is necessary to compute **the number of possible outcomes** of an experiment

Some examples

- How many triples are possible in the lottery?
- In how many different ways can 10 persons sit at a round table?
- From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed, knowing that 2 of the men are feuding and refuse to serve on the committee together?

# Combinatorial Analysis

## Definition (Combinatorial analysis)

*Combinatorial analysis* is the branch of mathematics dealing with computing the cardinality (i.e. the number of elements) of sets made from available objects *following some prescribed rules*

## Theorem (The basic principle of counting)

If *an experiment  $E$*  can be ideally decomposed into  *$M$  sub-experiments*, the  $k^{\text{th}}$  of which has  *$n_k$  possible outcomes*, and different sequences of the  $M$  sub-experiments result in differentiated final outcomes, then *the total number  $n$  of outcomes* of the experiment  $E$  is

$$n = n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_M$$

$$n_1 \times n_2 \times n_3 \times \dots \times n_M$$

# Combinatorial Analysis

## Example

How many passwords are available on a login system allowing for sequences of 2 letters and 3 digits, like “AB 123”?

$$n = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot n_5 = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676\,000$$

This “experiment” is divided into 5 “sub-experiments”. The first two have 26 possible outcomes (each); the others have 10 possible outcomes (each)

## Example

In how many ways can a candidate answer the admission test to the School of Engineering? (80 quizzes, each with 5 available answers)

$$n = n_1 \cdot n_2 \cdot \dots \cdot n_{80} = 5 \cdot 5 \cdot \dots \cdot 5 = 5^{80} \approx 8.27 \times 10^{55}$$

(80 “sub-experiments”, each of which has 5 possible outcomes)

## Example

In how many ways can the podium of a competition to which 10 athletes take part be composed?

$$n = n_1 \cdot n_2 \cdot n_3 = 10 \cdot 9 \cdot 8 = 720$$

The reasoning goes as follows: The first position can be taken by any of the 10 athletes; the second position by any of the remaining 9 athletes; the third position by any of the remaining 8 athletes

# Combinatorial Analysis

## Example

Given a classroom of 20 students, in how many ways a pair of student delegates can be chosen?

$$n = n_1 \cdot n_2 = 20 \cdot 19 = 380? \quad \text{WRONG!}$$

*We only count A-B and B-A once!*

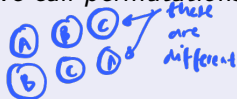
If we merely apply this formula, we end up with the result  $n = 20 \cdot 19 = 380$ , which is wrong. Here, the way in which the two students are selected does not matter (this is not a “ordered pair”). The 380 outcomes are composed by pairs  $(A, B)$  and  $(B, A)$  which have to be counted only once

The key observation is that the condition “**different sequences of the  $M$  sub-experiments result in differentiated final outcomes**” does not hold in this case

# Combinatorial Analysis

## Definition (Permutations) *↪ arrangement is important*

We call *permutations of  $n$  elements*



$P_n$  *↪ the number of possible permutation of  $n$  objects*

*the ordered groups of the  $n$  elements*, which differ one from the other *only* on the basis of the order of the elements

## Theorem

*The number of permutations of  $n$  elements is*

$$P_n = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 1 = n!$$

By definition:  $0! = 1$  *it's just a convention*

# Combinatorial Analysis

without repetition

## Definition ( $k$ -permutations)

We call  **$k$ -permutations** of  $n$  elements ( $k \leq n$ )

$$D_{n,k}$$

**the ordered groups of  $k$  of the  $n$  given elements**, which differ one from the other on the basis of the identity of the elements and their order

## Theorem

The number of  $k$ -permutations of  $n$  elements ( $k \leq n$ ) is

$$D_{n,k} = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

*↳  $k$  is number of required pair/group*



## Example

How many ordered pairs (tuples)  $N$  can be built from a set of 100 elements?

$$N = D_{100,2} = \frac{100!}{(100-2)!} = \frac{100!}{98!} = 100 \cdot 99 = 9\,900$$

For computational reasons, it is convenient first to simplify the ratio, rather than evaluating the factorials and subsequently compute the ratio. In fact

$$100! \simeq 0.933 \cdot 10^{158} \qquad 98! \simeq 0.943 \cdot 10^{154}$$

**Remark:** Working in *double precision* floating-point arithmetics (8 bytes), the largest factorial that can be stored in 64 bits is  
 $170! \simeq 7.257415615308 \times 10^{306}$

## Example

In a lottery, 100 000 tickets are sold. How many different permutations of tickets winning the first three prizes are possible?

$$\begin{aligned} N = D_{100\,000,3} &= \frac{100\,000!}{(100\,000 - 3)!} = \frac{100\,000!}{99\,997!} \\ &= 100\,000 \cdot 99\,999 \cdot 99\,998 \simeq 100\,000^3 = 10^{15} \end{aligned}$$

We don't even attempt a direct evaluation of the two factorials. Also, here we have done a very convenient approximation

# Combinatorial Analysis

## Definition (Permutations with repetition)

We call *permutations with repetition of  $n$  elements*

$$P'_n$$

the groups of all the  $n$  elements,  $n_1$  of which are indistinguishable one from the other,  $n_2$  of which are indistinguishable one from the other  $\dots n_r$  of which are indistinguishable one from the other ( $n = n_1 + n_2 + \dots + n_r$ )

## Theorem

*The number of permutations with repetition of  $n$  elements ( $n = n_1 + n_2 + \dots + n_r$ ) is*

$$P'_n = \binom{n}{n_1, n_2, \dots, n_r} \equiv \frac{n!}{n_1! n_2! \dots n_r!}$$

# Combinatorial Analysis

## Example

How many anagrams (non necessarily meaningful) can be extracted from the Italian name "Marinella"?

*A appears twice  
L appears twice*

*swappable letters*

$$N = P'_9 = \frac{9!}{1! 1! 1! 1! 1! 2! 2!} = \frac{362\,880}{2 \cdot 2} = 90\,720$$

*n = 9*

*M,  
A,  
R,  
I,  
N,  
E,  
L*

$$(n = 9, \quad n_M = n_R = n_I = n_N = n_E = 1, \quad n_A = n_L = 2)$$

# Combinatorial Analysis

## Definition (*k*-permutations with repetition)

now it is allowed

We call *k*-permutations with repetition of  $n$  elements taken  $k$ -by- $k$

$k$  times

$$D'_{n,k}$$

the groups made up with  $n$  elements taken  $k$ -by- $k$

## Theorem

The number of *k*-permutations with repetition of  $n$  elements taken  $k$ -by- $k$  is

$$D'_{n,k} = n^k$$

# Combinatorial Analysis

## Example

The passwords of length 8 that can be made with the extended-ASCII code are a neat example of 8-permutations with repetitions of  $n$  elements ( $n = 256$ , in this case). How many of these passwords are there?

$$N = D'_{256,8} = 256^8 = 2^{64} \simeq 1.84 \cdot 10^{19}$$

*Handwritten notes:*  $D'_{n,k}$  (under  $D'_{256,8}$ ),  $n$  (above  $256$ ),  $k$  (above  $8$ ), and a row of eight boxes labeled 256, 256, 256, 256, 256, 256, 256, 256.

**Remark:** The ASCII code (American Standard Code for Information Interchange) is a 7-bit standard code proposed by ANSI in 1963 (then officially recognized in 1968). ASCII is the standard code for microcomputers and, being a 7-bit code, it covers  $2^7 = 128$  symbols (integers from 0 to 127). Later, the code was extended to a 8-bit code ( $2^8 = 256$  symbols). In the extended set, the symbols from 128 to 255 represent special characters, mathematical symbols. . .

# Combinatorial Analysis

## Example

How many Internet addresses can be assigned using the IPv4 and the IPv6 protocols?

Since IPv4 addresses are 32 bits, we have

$192 \cdot 0 \cdot 0 \cdot 1$   
 $\underbrace{8+8+8+8}_{= 32 \text{ bits}} = 32 \text{ bits}$   
 $\rightarrow 0-255 = 2^8$

$$N_{v4} = D'_{2,32} = 2^{32} = 256^4 = 4\,294\,967\,296 \simeq 4.3 \cdot 10^9$$

IPv6 addresses are, instead, 128 bit addresses

$\rightarrow$  a lot but  
not enough

$$\begin{aligned} N_{v6} &= D'_{2,128} = 2^{128} \\ &= 340\,282\,366\,920\,938\,463\,463\,374\,607\,431\,768\,211\,456 \\ &\simeq 3.4 \cdot 10^{38} \end{aligned}$$

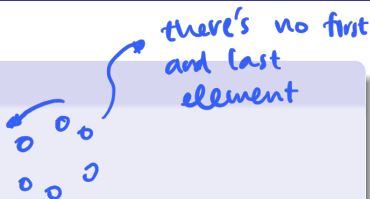
# Combinatorial Analysis

orders matter

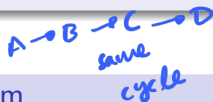
## Definition (Cyclic permutations)

We call *cyclic permutations* of  $n$  elements

$$P_n^c$$



the permutations of  $n$  elements along a closed path (a circle, for instance)

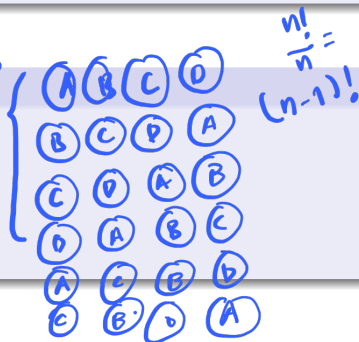


## Theorem

The number of cyclic permutations of  $n$  is



$$P_n^c = (n-1)!$$





## Example

In how many different ways can 10 persons take a seat at a round table?

$$N = P_{10}^c = 9! = 362\,880$$

# Combinatorial Analysis

Given a box filled with  $n$  balls we extract  $k$  balls (sample of size  $k$ )

- **Sampling with repetitions (Bernoullian sampling)**. Before extracting a ball, the previously extracted ball is put back in the box. In this case, how many possible **ordered** samples of size  $k$  are there?

$$N = D'_{n,k} = n^k$$

- **Sampling without repetitions (cluster sampling)**. After extraction, balls are not put back in the box. In this case, how many possible **ordered** samples of size  $k$  are there?

$$N = D_{n,k} = \frac{n!}{(n-k)!}$$

# Combinatorial Analysis

## Definition (Binomial coefficient)

natural number includes zero

Given  $n, k \in \mathbb{N}_0$  with  $n \geq k$ , we define the *binomial coefficient* as follows

$$\binom{n}{k} = \frac{n!}{k! (n-k)!} = \frac{(n-k+1) \cdot (n-k+2) \cdots n}{1 \cdot 2 \cdots k}$$

(to be read: " $n$  choose  $k$ ")

$$0! = 1 \text{ (by definition)} \Rightarrow \binom{0}{0} = 1 \quad \binom{n}{0} = 1 \quad \binom{n}{n} = 1$$

## Theorem (Binomial theorem)

The  $n^{\text{th}}$  power of a binomial can be written as follows

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

# Combinatorial Analysis

## Definition (Combinations)

We call *combinations of size  $k$  of  $n$  elements*

$$C_{n,k}$$

the groups of size  $k$  of the  $n$  elements, **no matter the order of appearance**

## Theorem

The number of combinations of size  $k$  of  $n$  elements is

where ( $k \leq n$ )

permutation

$$C_{n,k} = \frac{D_{n,k}}{k!} = \frac{n!}{k! (n-k)!} = \binom{n}{k}$$

Combinations of size  $k$  are nothing but  $k$ -permutations for which the order of appearance does not matter

1 2 3 4 5

## Definition (Combinations with repetition)

We call *combinations with repetition of size  $k$*  of  $n$  elements

$$C'_{n,k}$$

the groups of  $n$  elements, taken  *$k$ -by- $k$* , with possible repetition of each element (up to  $k$  times, obviously), in which the order does not matter

*less common*

## Theorem

The number of combinations with repetition of size  $k$  of  $n$  elements is

$$C'_{n,k} = C_{n+k-1,k} = \binom{n+k-1}{k}$$

# Combinatorial Analysis

## Example

How many groups of 4 elements can be built with the symbols X, Y and Z? how many groups of 2 elements?

In the first case, we have

$$n=3$$

$$k=4$$

$$N = C'_{3,4} = \binom{6}{4} = \frac{6!}{4!2!} = 15$$

$$N = \binom{n+k-1}{k} = \binom{3+4-1}{4} = \binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!}$$

$$n=3$$

$$k=4$$

In the second case

$$n=3$$

$$k=2$$

$$N = C'_{3,2} = \binom{4}{2} = \frac{4!}{2!2!} = 6$$


$$N = \binom{n+k-1}{k} = \binom{3+2-1}{2} = \binom{4}{2}$$

Notice that  $k$  can be either greater, smaller, or equal to  $n$

# Combinatorial Analysis

## Summary

$n!$   
 $n_1! \cdot n_2! \cdot \dots \cdot n_r!$



kind	differentiation	type	formulae
permutations	order	without repetitions	$P_n = n!$
		with repetitions	$P'_n = \binom{n}{n_1, n_2, \dots, n_r}$
		cyclic	$P_n^c = (n-1)!$
$k$ -permutations	type and order	without repetitions	$D_{n,k} = \frac{n!}{(n-k)!}$
		with repetitions	$D'_{n,k} = n^k$
combinations	type	without repetitions	$C_{n,k} = \binom{n}{k}$
		with repetitions	$C'_{n,k} = \binom{n+k-1}{k}$