NONLINEAR CONTROL

We now leverage the normal form to develop NON-LOCAL control strategies

Let us go back to our set-point control problem where the system under consideration has the form (we focus on a SISO setting)

$(n_u = n_r = 1 \quad , \quad n \doteq n_x)$

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y_r = h_r(x) \\ y_m = h_m(x) \end{cases}$$

and the objective is to drive y_r to y_r^*. We define the "error output"

$$\tilde{y} = h(x) \doteq h_r(x) - y_r^* \quad \rightarrow \quad \tilde{y} = y_r - y_r^* \qquad (= \text{REGULATION ERROR})$$

STANDING ASSUMPTION. The system with input u and output $\tilde{y}$ has global relative degree $r \leq n$ and it has a globally-defined normal form

We can change coordinates as $x \mapsto z = (\xi, \eta) = \Phi(x)$ obtaining the normal form:

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} \qquad i = 1, \dots, r-1 \\ \dot{\xi}_r = q(\xi,\eta) + b(\xi,\eta)u \\ \dot{\eta} = \psi(\xi,\eta) \\ \tilde{y} = \xi_1 \end{cases}$$

Notice that ξ_i are "ERROR VARIABLES", namely

$$\xi_1 = \tilde{y} = y_r - y^*$$

$$\xi_2 = \dot{\tilde{y}}$$

$$\xi_3 = \ddot{\tilde{y}}$$

$$\vdots$$

We have a system to drive from $y_r(t) \rightarrow y_r^*$

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y_m = h_m(x) = (x, \tilde{y}_r) \\ y_r = h_r(x) \end{cases}$$

Standing assumptions:

1) Affined system
2) SISO $n_m = n_r = 1$ $(n = n_x)$
3) State feedback
4) System from u to the output $\tilde{y}_r = h_r(x) - y_r^*$ has GLOBAL REL. DEGREE $r \leq n$ and GLOBALLY DEFINED normal form.

Now, we're working with error coordinate:

$y = \tilde{y}_r \; (h(x) = h_r(x) - y_r^*)$

Remember:

$$\phi(x) = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \begin{matrix} \xi \in \mathbb{R}^n \\ \eta \in \mathbb{R}^{n-r} \end{matrix}$$

Remember the normal form:

$$\begin{aligned}
\tilde{y}_r &= \xi_1 \\
\dot{\xi}_1 &= \xi_2 = \dot{\tilde{y}}_r \\
\dot{\xi}_2 &= \xi_3 = \ddot{\tilde{y}}_r \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r = \tilde{y}_r^{[r-1]} \\
\dot{\xi}_r &= q(\xi, \eta) + b(\xi, \eta)u \\
\dot{\eta} &= \psi(\xi, \eta)
\end{aligned}$$

We have three possible cases:

1) $r = n$
2) $r = 1$
3) $1 < r < n$

If $\xi(t) \to 0$, the $\tilde{y}_r \to 0$
$\eta(t) \to \eta^*$

CASE 1 $r = n$

The normal form would be:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = q(\xi) + b(\xi)u \end{cases}$$

$$b(\xi) = L_g L_f^{r-1} h(x) \Big|_{x = \phi^{-1}(\xi)}$$

$$u = \frac{1}{b(\xi)}(-q(\xi) + v)$$

$v = K\xi$ (where $A+BK$ is Hurwitz)

$$\downarrow$$

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_n = v \\ \tilde{y}_r = \xi_1 \end{cases} \longrightarrow \begin{aligned} \dot{\xi} &= A\xi + Bv \\ \tilde{y}_r &= C\xi \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 \cdots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [1 \; 0 \; \cdots \; 0]$$

$\dot{\sigma} = \xi_1 = \tilde{y}_r$

$v = K_1\xi + K_2\sigma$

$\dot{\sigma} = \tilde{y}_r$

$$u = \frac{1}{b(\xi)}(-q(\xi) + K_1\xi + K_2\sigma)$$

$$\xi = \begin{bmatrix} \tilde{y}_r \\ \dot{\tilde{y}}_r \\ \vdots \end{bmatrix}$$

See: Fredouin (2006/2008)
Khalil

It produces:

$$\begin{cases} \dot{\sigma} = \tilde{y}_r \\ u = \dfrac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + K_1 \phi(x) + K_2 \sigma\right) \end{cases}$$

We see that u highly depends on $f(x)$, $g(x)$, and $h(x)$ and we need to know them. It's <u>global</u> but <u>not</u> <u>robust</u>.

CASE 2 & 3 $r < n$

The normal form would be:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_{r-1} = \xi_r \\ \dot{\xi}_r = q(\xi, \eta) + b(\xi, \eta) u \\ \dot{\eta} = \psi(\xi, \eta) \\ \xi_1 = \tilde{y}_r \end{cases}$$

Solvability equations: find (ξ^*, η^*, u^*)

$$\begin{cases} \xi_1^* = 0 \\ 0 = \xi_2^* \\ \quad \vdots \\ 0 = \xi_r^* \end{cases} \qquad \begin{array}{l} 0 = q(\xi^*, \eta^*) + b(\xi^*, \eta^*) u^* \\ 0 = \psi(\xi^*, \eta^*) \end{array}$$

$$\downarrow$$

$$\begin{cases} \xi^* = 0 \\ 0 = \psi(0, \eta^*) \\ u^* = -\dfrac{q(0, \eta^*)}{b(0, \eta^*)} \end{cases}$$

system $\dot{\eta} = \psi(0, \eta)$ must have an equilibrium y^*

ZERO DYNAMICS OF THE SYSTEM

Assume $\eta^* = 0$

$$\tilde{\eta} = \tilde{\eta} - \eta^*$$
$$\dot{\tilde{\eta}} = \psi(\xi, \tilde{\eta} + \eta^*)$$
$$= \tilde{\psi}(\xi, \tilde{\eta})$$

Can we apply the control law on u?

$$u = \frac{1}{b(\xi,\eta)}\left(-a(\xi,\eta) + K\xi\right)$$

$$\dot{\xi} = (A + BK)\xi \longrightarrow \xi(t) \to 0$$
$$\dot{\eta} = \psi(\xi, \eta)$$

but we don't know $\eta(t)$

η may diverge:

1) $x = \phi^{-1}(\xi, \eta)$

2) Implementation problems since u depends on η

$$\dot{\eta} = \psi(\xi, \eta) \longrightarrow \dot{\eta} = \psi(0, \eta)$$

we can see zero dynamics plays key role

CONTROL VIA FEEDBACK LINEARIZATION

- If $\boxed{r = n}$, we have seen that

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v \right)$$

Produces

$$\begin{cases} \dot{\xi} = A\xi + Bv \\ \tilde{y} = C\xi \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

Since (A, B) is controllable we can design a linear controller of the kind

$$\begin{cases} \dot{\sigma} = \tilde{y} \\ v = K_1 \xi + K_2 \sigma \end{cases} \Rightarrow \quad u = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + K_1 \Phi(x) + K_2 \sigma \right)$$

non linear control law

- REMARKS:

1) The controller provides a GLOBAL stability guarantee

2) The controller is NOT ROBUST as it needs full knowledge of f, g, and h

3) There are "ROBUSTIFICATION APPROACHES" providing a robust variation of this control law (NOT TREATED IN THIS COURSE)

THE CASE $r<n$

If $r<n$, the η dynamics shows up, and things get more complex

The normal form is:

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} & i=1,\dots,r-1 \\ \dot{\xi}_r = q(\xi,\eta) + b(\xi,\eta)u \\ \dot{\eta} = \psi(\xi,\eta) \\ \hat{y} = \xi_1 \end{cases}$$

The SOLVABILITY EQS.: $\exists\,(\xi^*, \eta^*, u^*)$ so that

$$\begin{cases} 0 = \xi_1^* \\ 0 = \xi_{i+1}^* & i=1,\dots,r-1 \\ 0 = q(\xi^*,\eta^*) + b(\xi^*,\eta^*)u^* \\ 0 = \psi(\xi^*,\eta^*) \end{cases}$$

$\Rightarrow$ $\xi_i^* = 0 \quad \forall i=1,\dots,r$

$\Rightarrow$ $u^* = -\dfrac{q(0,\eta^*)}{b(0,\eta^*)}$ $\leftarrow$ $b(0,\eta^*) \neq 0$ since the system has global relative degree

$\Rightarrow$ $\psi(0,\eta^*) = 0$

$\Rightarrow$ If $\boxed{\dot{\eta} = \psi(0,\eta)}$ has an equilibrium point η^* the SOLVABILITY Eqs are always solvable and

$$u^* = -\frac{q(0,\eta^*)}{b(0,\eta^*)}$$

The dynamics

$$\dot{\eta} = \psi(0,\eta)$$

is called $\boxed{\text{ZERO DYNAMICS}}$ of the system

$\rightarrow$ In the linear case $\psi(0,\eta) = F\eta$ and the zeros of the fdt from u to $\tilde{y}$ are eigenvalues of the matrix F

By possibly considering the change of variables

$$\eta \mapsto \eta - \eta^*$$

we can assume without loss of generality that $\boxed{\eta^* = 0}$

It turns out that the stability properties of the zero dynamics are key to the existence of a controller solving the problem

$\downarrow$

we may try to proceed as in the previous feedback linearization case and try out a controller of the form (assuming $y_m = x$)

$$u = \frac{1}{b(\xi,\eta)} \left(-q(\xi,\eta) + K\xi \right)$$

This gives

$$\begin{cases} \dot{\xi} = (A+BK)\xi \\ \dot{\eta} = \psi(\xi,\eta) \end{cases}$$

As $A+BK$ is Hurwitz, then $\xi \to 0$, BUT:

1) $\dot{\eta} = \psi(\xi,\eta)$ may not be stable $\rightarrow$ we can have $\eta \to \infty$

$\Rightarrow$ the system's solutions may be unbounded

2) $u(t)$ depends on η $\Rightarrow$ we may not be able to implement such a controller if $\eta(t)$ blows up

Therefore it is clear that we cannot ignore the dynamics of η

(NON) MINIMUM PHASE

Consider the ZERO DYNAMICS of the system

$$\dot{\eta} = \psi(0, \eta)$$

we shall not consider this case

Then the system is called

- WEAKLY MINIMUM PHASE. If $\eta^* = 0$ is LAS for the ZERO DYNAMICS
- MINIMUM PHASE. If $\eta^* = 0$ is GAS for the ZERO DYNAMICS
- STRONGLY MINIMUM PHASE. If the ZERO DYNAMICS is $\boxed{\text{ISS}}$ (see part 4, module 1) with respect to the input ξ
- NON-MINIMUM PHASE. otherwise

For linear systems we have seen that the ZERO DYN. reads as

$$\dot{\eta} = F\eta$$

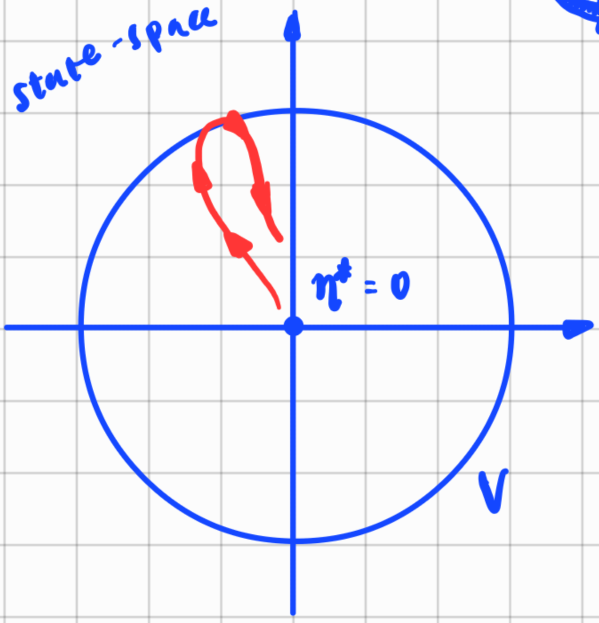
with $\sigma(F)$ = zeros of the FdT $G(s)$

$\Rightarrow$ For linear systems all the notions of MINIMUM PHASE are equivalent and are equivalent to ask that the zeros of $G(s)$ have strictly negative real part

The SYSTEM is:

1) **MINIMUM PHASE** if $\eta^* = 0$ is GLOBALLY ASYMP. STABLE for the ZERO DYNAMICS $\dot{\eta} = \psi(0, \eta)$

2) **STRONGLY MINIMUM PHASE** if $\dot{\eta} = \psi(\xi, \eta)$ is input-to-state stable (ISS)



By Lyapunov, if there exists $V : \mathbb{R}^{n-r} \to \mathbb{R}$, function on real $\alpha : \mathbb{R} \to \mathbb{R}$ such that

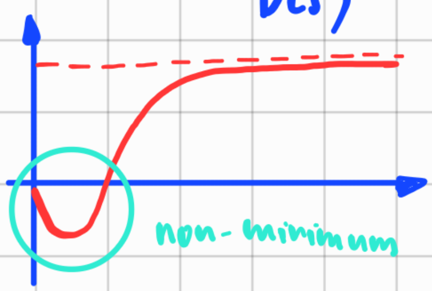
1) $V(\eta) > 0 \quad \forall \eta \neq 0$, wrt $V(0) = 0$
2) α is continuous, increasing, $\alpha(0) = 0$
3) if $\|\eta\| \geq \alpha(\|\xi\|)$, then $\underbrace{\frac{\partial V}{\partial \eta}(\eta)\, \psi(\xi, \eta) < 0}_{= \dot{V}}$

$$\boxed{u = \frac{1}{b(\xi, \eta)}\left(-a(\xi, \eta) + K\xi\right)}$$

It implies: min. phase + $\underbrace{\text{bounded-input-bounded}}_{\text{completeness property}}$ state properties.

3) **NON-MINIMUM PHASE**, otherwise.

MINIMUM PHASE in transfer function: $G(s) = \frac{N(s)}{D(s)}$, if it's negative, then it's minimum.



Normal form in linear case:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_r = H\xi + \Gamma\eta + Bu \\ \dot{\eta} = \underbrace{F}_{\text{if it's Hurwitz}}\eta + G\underbrace{\xi}_{\text{it's bounded}} \\ \tilde{y}_r = C\xi \end{cases}$$

EXTRAS

$\xi_r = \tilde{y}_r^{(r-1)}$ $\qquad \xi_r(s) = s^{r-1} y_r(s)$

$$\begin{cases} s\xi_r(s) = H\xi(s) + \Gamma\eta(s) + Bu(s) \\ \eta(s) = (sI - F)^{-1} G\xi(s) \end{cases}$$

$$\dot{\xi} = A\xi + \bar{B}(H\xi + \Gamma\eta + Bu)$$

$$\xi(s) = \left[A + \bar{B}H + \Gamma(sI - F)^{-1}G\right]\xi(s) + \bar{B}Bu(s)$$

$$\underbrace{(sI - * - \Gamma(sI - F)^{-1}G)}\xi(s) = \bar{B}B\,u(s)$$

$$M^{-1} = \frac{\operatorname{adj} M}{\det M} \Rightarrow (sI - F)^{-1} = \frac{*}{\det(sI - F)}$$

$$\Rightarrow \frac{*}{\det(sI - F)}\xi(s) = \bar{B}Bu(s)$$

$$\tilde{y}_r(s) = C\xi(s) = \underbrace{\det(sI - F) *}_{G(s)} u(s)$$

CASE 2 $(r = 1)$

$$\begin{cases} \dot{\xi} = q(\xi, \eta) + b(\xi, \eta) u \\ \dot{\eta} = \psi(\xi, \eta) \\ \tilde{y}_r = \xi \end{cases}$$

Can we use $u = \dfrac{1}{b(\xi,\eta)} \left(-q(\xi,\eta) + K\xi\right)$?

$\dot{\eta} = -\eta + \xi\eta^2 = -\eta + e^{Kt}\xi(0)\eta^2$

$\dot{\xi} = K\xi$, if $K<0$ $\xi(t) \rightarrow 0$

$\xi(t) = e^{Kt}\xi(0)$

$\psi(\xi,\eta) - \eta + \xi\eta^2$

$\psi(0,\eta) = -\eta$

$\dot{\eta} = -\eta$ $\eta^* = 0$, globally as. stable

$\xi(0) = 1$, $\eta(0) = 2 - k$

$$\eta(t) = \frac{(2-K)(1-K)}{2-K-e^{(1-K)t}} e^{Kt}$$

η could go to infinity

RESULT $\rightarrow$ If the system is MINIMUM PHASE, and it has $\dot{\eta} = \psi(\xi, \eta)$ the COMPLETENESS PROPERTIES, then $u = \dfrac{1}{b(\xi,\eta)}\left(-q(\xi,\eta) + K\xi\right)$ $\forall K < 0$, such that

$\xi(t) \rightarrow 0$ (converge to 0)

$\eta(t)$ is bounded

$\forall \xi(t)$ bounded,

1) $\eta(t)$ is defined $\forall t > 0$

2) $\eta(t)$ is bounded

ANOTHER RESULT $\longrightarrow$ Strong minimum phase implies minimum phase + completeness.

If the system is STRONGLY MINIMUM PHASE we can use linearization $u = \frac{1}{b(\xi, \eta)}(-q(\xi,\eta) + K\xi)$ such that: $\xi(t) \to 0$, $\eta(t) \to 0$

ESAME: no exercise, either Prof. Marconi, Prof Bin, or Alessandro

MATLAB basic command (diagonalizable, or Jordan form),

see the step to compute something,

two questions on paper, given few minutes to write brief explanations and explain the details to the examinator.

(+) Don't remember proof is ok but know the concept behind is a must.

(−) Even though writing something right but not understand it is a NO.

STATE FEEDBACK CONTROL - CASE $r=1$

We consider the normal form

$$\begin{cases} \dot{\eta} = \psi(\xi,\eta) \\ \dot{\xi} = q(\xi,\eta) + b(\xi,\eta)\,u \end{cases} \qquad \xi = \xi_1 \in \mathbb{R} \quad (r=1) \qquad \text{and} \qquad y_m(x) = \begin{pmatrix} \eta \\ \xi \end{pmatrix} \text{ (state feedback)}$$

As we commented before, the "feedback-linearization controller"

$$u = \frac{1}{b(\xi,\eta)}\left(-q(\xi,\eta) - k\,\xi\right) \qquad (k>0)$$

may lead to problems. This is clear if the system in NON-MINIMUM PHASE as in this case the η-dynamics is unstable.

BUT, what if we assume that $\eta^*=0$ is GAS for $\dot{\eta} = \psi(0,\eta)$ (MINIMUM-PHASE)?

(NEGATIVE) RESULT. a stable transitory of $\xi(t)$ can destabilize $\eta(t)$ even if $\dot{\eta} = \psi(0,\eta)$ is GAS.

↳ In general, if $\dot{\eta} = \psi(0,\eta)$ is GAS (i.e. system is minimum phase) the feedback-lineariz. controller can only guarantee LOCAL AS

Indeed, consider

$$\begin{cases} \dot{\eta} = -\eta + \xi\eta^2 \\ \dot{\xi} = -k\,\xi \end{cases} \qquad k>0$$

for every choice of $k>0$ there exists at least one initial condition such that $\eta(t)$ blows up to ∞ in FINITE TIME

Proof.

Take $\xi(0)=1$ and $\eta(0) = k+2$

Then

$$\xi(t) = e^{-kt} \quad \rightarrow \quad \dot{\eta}(t) = -\eta(t) + \eta(t)^2 e^{-kt}$$

The solution of the latter from $\eta(0)=k+2$ is *

$$\eta(t) = \frac{(k+2)(k+1)\,e^{kt}}{k+2 - e^{(k+1)t}}$$

Then as $t \to \frac{1}{k+1}\log(k+2)$, we have $\eta(t) \to \infty$

* Indeed, we have

$$\eta(0) = \frac{(k+2)(k+1)\cdot 1}{k+2 - 1} = k+2$$

$$\dot\eta = (k+2)(k+1)\,\frac{k e^{kt}\cdot\left(k+2-e^{(k+1)t}\right) + e^{kt}\cdot(k+1)\,e^{(k+1)t}}{\left(k+2-e^{(k+1)t}\right)^2}$$

$$= \frac{(k+2)(k+1)}{\left(k+2-e^{(k+1)t}\right)^2}\left((k+2)k + e^{(k+1)t}\right)e^{kt}$$

$$\dot\eta + \eta = \frac{(k+1)(k+2)}{\left(k+2-e^{(k+1)t}\right)^2}\,e^{kt}\left(k^2+2k+\cancel{e^{(k+1)t}} + k+2 - \cancel{e^{(k+1)t}}\right)$$

$$= \frac{(k+1)^2(k+2)^2}{\left(k+2-e^{(k+1)t}\right)^2}\,e^{kt} = \eta^2\,e^{-kt}$$

$$\Rightarrow \dot\eta = -\eta + \eta^2 e^{-kt}$$ ☒

So, when can we use the feedback lineariz. controller?

The dynamics η is said to have the <u>COMPLETENESS PROPERTY</u> if, for every bounded and continuous $\xi(t)$, the trajectories of

$$\dot\eta = \psi(\xi,\eta)$$

are defined for all times $t \geq 0$ and are bounded.
(we cannot have solutions that blow up in finite time)

<u>RESULT</u>. If the system is MINIMUM PHASE and η has the COMPLETENES PROPERTY, then for each $k>0$

$$u = \frac{1}{b(\xi,\eta)}\left(-q(\xi,\eta) - k\,\xi\right)\Bigg|_{(\xi,\eta)=\phi(x)}$$

<u>GLOBALLY STABILIZES</u> the origin of the system (η,ξ)

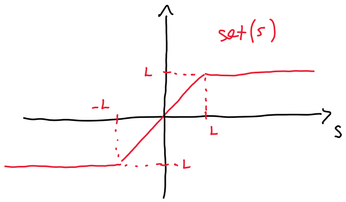
EXAMPLE

$$\begin{cases} \dot{\eta} = -\eta + \text{sat}_L(\xi\eta^2) \\ \dot{\xi} = q(\xi,\eta) + b(\xi,\eta)u \end{cases} \qquad L>0 \text{ Any}$$

$\Leftarrow$ NOT ISS

where

$$\text{sat}_L(s) = \begin{cases} L & \text{if } s \geq L \\ -L & \text{if } s \leq -L \\ s & \text{otherwise} \end{cases}$$



We have

$$\eta(t) = \underbrace{e^{-t}}_{\leq 1}\eta(0) + e^{-t}\int_0^t e^s \underbrace{\text{sat}_L(\xi(s)\eta(s)^2)}_{\leq L}\, ds$$

$$\leq \eta(0) + L \cdot \underbrace{e^{-t}\int_0^t e^s\, ds}_{= e^{-t}(e^t - 1) \leq 1} \leq \eta(0) + L$$

BOUNDED

$\Rightarrow$ η has the COMPLETENESS PROPERTY

RESULT. If the system is STRONGLY MINIMUM PHASE ($\dot{\eta} = \psi(0,\eta)$ is ISS)

then

$$u = \frac{1}{b(\xi,\eta)}\left(-q(\xi,\eta) - k\xi\right)\Bigg|_{(\xi,\eta)=\phi(x)}, \qquad k>0$$

GLOBALLY STABILIZES the system

(Indeed ISS $\Rightarrow$ GAS + COMPLETENESS PROP.)

What can we do when we only have MINIMUM PHASE ($\dot{\eta} = \psi(0,\eta)$ GAS) without COMPLETENESS ?

As $\eta^* = 0$ is GAS for $\dot{\eta} = \psi(0,\eta)$, there exists a Lyapunov function $W : \mathbb{R}^{n-r} \to \mathbb{R}$ satisfying

- $W(\eta) = 0$; if $\eta = \eta^* = 0$
- $W(\eta) > 0$, $\forall \eta \in \mathbb{R}^{n-1} / \{0\}$
- $\frac{\partial W}{\partial \eta}(\eta) \cdot \psi(0,\eta) < 0$, $\forall \eta \in \mathbb{R}^{n-r} / \{0\}$

good

f $\xi \neq 0$, we have:

bad

$$\frac{\partial W}{\partial \eta}(\eta) \cdot \psi(\xi,\eta) = \frac{\partial W}{\partial \eta}(\eta)\, \psi(0,\eta) + \frac{\partial W}{\partial \eta}(\eta) \cdot \big(\psi(\xi,\eta) - \psi(0,\eta)\big) \qquad (\mathrm{I})$$

Let $\gamma : [0,1] \to \mathbb{R}^{n-1}$ be given by

$$\gamma(s) = \psi(s\xi, \eta)$$

Then by definition

$$\psi(\xi,\eta) - \psi(0,\eta) = \gamma(1) - \gamma(0) = \int_0^1 \frac{d\gamma(s)}{ds}\, ds = \int_0^1 \left(\frac{\partial \psi}{\partial \xi}(s\xi,\eta) \cdot \xi \right) ds$$

$$= \underbrace{\int_0^1 \frac{\partial \psi}{\partial \xi}(s\xi,\eta)\, ds}_{= \rho(\xi,\eta)} \cdot \xi = \rho(\xi,\eta) \cdot \xi$$

Therefore, from (I) we get

$$\frac{\partial W}{\partial \eta} \psi(\xi,\eta) = \frac{\partial W}{\partial \eta}(\eta) \cdot \psi(0,\eta) + \frac{\partial W}{\partial \eta}(\eta) \cdot \rho(\xi,\eta) \cdot \xi$$

Consider the function

$$V(\xi,\eta) = W(\eta) + \frac{1}{2}\xi^2$$

Then:

- $V(\xi,\eta) = 0$ if $(\xi,\eta) = 0$
- $V(\xi,\eta) > 0, \quad \forall (\xi,\eta) \in \mathbb{R}^n / \{0\}$

Moreover

$$\dot{V}(\xi,\eta) := \frac{\partial V(\xi,\eta)}{\partial(\xi,\eta)} \cdot \begin{pmatrix} \psi(\xi,\eta) \\ q(\xi,\eta) + b(\xi,\eta)u \end{pmatrix} = \frac{\partial W}{\partial \eta}(\eta) \cdot \psi(\xi,\eta) + \xi \cdot \left(q(\xi,\eta) + b(\xi,\eta)u \right)$$

$$= \frac{\partial W}{\partial \eta}(\eta)\, \psi(0,\eta) + \left[\frac{\partial W}{\partial \eta}(\eta) \cdot e(\xi,\eta) + q(\xi,\eta) + b(\xi,\eta)u \right] \cdot \xi$$

Choosing the controller

$$u = \frac{1}{b(\xi,\eta)} \left(-q(\xi,\eta) - \frac{\partial W}{\partial \eta}(\eta) \cdot e(\xi,\eta) - k\xi \right) \qquad k>0 \qquad \text{(II)}$$

gives

$$\dot{V}(\xi,\eta) = \frac{\partial W}{\partial \eta}(\eta)\, \psi(0,\eta) - k\xi^2 \quad \underline{\underline{<0}} \quad \forall (\eta,\xi) \neq 0$$

<u>RESULT</u>. If the system is MINIMUM PHASE, the control (II) globally stabilizes the origin.

- GLOBAL SOLUTION ✓
- STATE FEEDBACK ✗
- NOT ROBUST ✗

compared to the feedback linearization controller, the law (II) includes the additional term

$$-\frac{1}{b(\xi,\eta)} \cdot \frac{\partial W}{\partial \eta}(\eta) \cdot e(\xi,\eta)$$

that handles the zero dynamics

What can we do if the system is <u>NON-MINIMUM PHASE</u> or η is not COMPLETE ?

$\downarrow$

Suppose we know a smooth control law $\alpha(\eta)$ with $\alpha(0)=0$ such that

1) $\dot{\eta} = \psi(\alpha(\eta), \eta)$ is GAS (namely, with $\xi = \alpha(\eta)$)

2) $\dot{\eta} = \psi(\alpha(\eta) + d, \eta)$ has the COMPLETENESS PROPERTY (with respect to input $d(t)$)

$d(t) =$ auxiliary input

only necessary if we want to use the feedback-linearizing controller

$\downarrow$

IDEA. use ξ as a "virtual controller"

change variables as

$$\xi \mapsto \tilde{\xi} = \xi - \alpha(\eta)$$

this gives

$$\dot{\eta} = \psi(\xi, \eta) = \psi(\alpha(\eta) + \underbrace{\tilde{\xi}}_{= d}, \eta) = \tilde{\psi}(\tilde{\xi}, \eta)$$

$$\dot{\tilde{\xi}} = \dot{\xi} - \frac{\partial \alpha}{\partial \eta}(\eta) \cdot \psi(\alpha(\eta) + \tilde{\xi}, \eta)$$

$$= \underbrace{q(\underbrace{\xi}_{= \tilde{\xi} + \alpha(\eta)}, \eta) - \frac{\partial \alpha}{\partial \eta}(\eta) \cdot \psi(\alpha + \tilde{\xi}, \eta)}_{= \tilde{q}(\tilde{\xi}, \eta)} + \underbrace{b(\xi, \eta)}_{= \tilde{b}(\tilde{\xi}, \eta) = b(\xi, \eta)\big|_{\xi = \tilde{\xi} + \alpha(\eta)}} u$$

Hence:

$$\dot{\eta} = \tilde{\psi}(\hat{\xi}, \eta)$$

$$\dot{\tilde{\xi}} = \tilde{q}(\tilde{\xi}, \eta) + \tilde{b}(\tilde{\xi}, \eta)\, u$$

This is a normal form in which the new ZERO DYNAMICS

$$\dot{\eta} = \tilde{\psi}(0, \eta)$$

is GAS and (POSSIBLY if point 2 above is guaranteed) η has the COMPLETENESS PROPERTY

$\Rightarrow$ This fits the previous cases so we can use the previous tools to drive $(\eta, \hat{\xi})$ to zero

Notice that

$$\hat{\xi}(t) \to 0 \quad \Rightarrow \quad \eta(t) \to 0 \quad \Rightarrow \quad \alpha(\eta(t)) \to 0$$

$$\Rightarrow \quad \xi(t) = \hat{\xi}(t) + \alpha(\eta(t)) \to 0$$

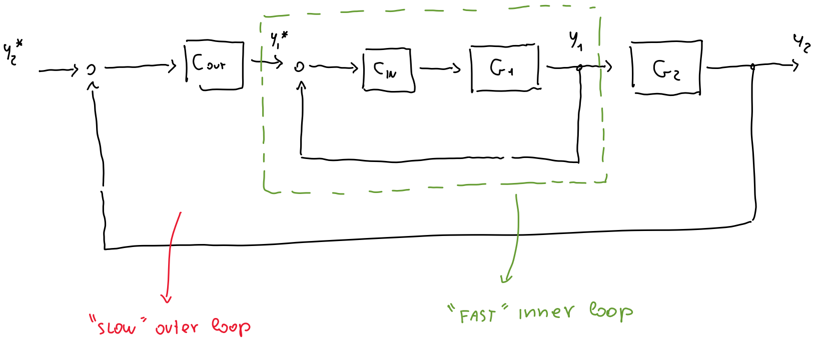
Hence our original goal is met

REMARK. If $\hat{u} = \gamma(\tilde{\xi}, \eta)$ is the control law driving $\tilde{\xi}$ to zero then

$$u = \gamma(\xi - \alpha(\eta), \eta)$$

is the control law to be implemented in the coordinates (η, ξ)

This control strategy is called BACKSTEPPING and is reminescent of "CASCADE CONTROL"



$y_2 \sim \eta$, $y_2^* \sim 0$, $y_1 \sim \xi$, $y_1^* \sim \alpha(\eta)$

EXAMPLE

$$\dot{\eta} = -\eta + \xi\eta^2 = \psi(\xi,\eta) \rightarrow \text{This is MINIMUM PHASE but NOT COMPLETE (see above)}$$

$$\dot{\xi} = u$$

Consider

$$\alpha(\eta) = -\eta^3$$

then

$$\psi(\alpha(\eta)+d,\ \eta) = -\eta + (-\eta^3 + d)\eta^2 = -\eta - \eta^5 + d\eta^2$$

take the ISS-Lyapunov function candidate (see part 4)

$$W(\eta) = \frac{1}{2}\eta^2$$

then

$$\frac{\partial W}{\partial \eta}(\eta) \cdot \psi(\alpha(\eta)+d,\ \eta) = \eta \cdot \psi(\alpha(\eta)+d,\ \eta) = -\eta^2 - \eta^6 + d\eta^3$$

Notice that for every two $a,b \geq 0$

$$ab = \frac{1}{2}\cdot 2ab = \frac{1}{2}\left(a^2+b^2-(a-b)^2\right) \leq \frac{1}{2}\left(a^2+b^2\right)$$

Therefore

$$d\eta^3 = \underbrace{\left(\sqrt{2}\,\eta^3\right)}_{a}\cdot\underbrace{\left(\frac{d}{\sqrt{2}}\right)}_{b} \leq \frac{1}{2}\left(\sqrt{2}\,\eta^3\right)^2 + \frac{1}{2}\left(\frac{d}{\sqrt{2}}\right)^2 = \eta^6 + \frac{1}{4}d^2$$

Then, we obtain

$$\frac{\partial W}{\partial \eta}(\eta)\,\psi(\alpha(\eta),\eta) \leq -\eta^2 + \frac{1}{4}d^2$$

$\Rightarrow$ If $|\eta| > \frac{1}{2}|d|$, then $\frac{\partial W}{\partial \eta}(\eta)\,\psi(\alpha(\eta),\eta) < 0$

$\Rightarrow$ W is an ISS-Lyapunov function

$\Rightarrow$ $\dot{\eta} = \psi(\alpha(\eta)+d,\eta)$ has the COMPLETENESS PROP.

$\Rightarrow$ $\dot{\eta} = \psi(\alpha(\eta),\eta)$ is GAS

Changing variables to

$$\tilde{\xi} = \xi - \alpha(\eta) = \xi + \eta^3$$

We have

$$\dot{\tilde{\xi}} = 3\eta^2\cdot\psi(\xi,\eta) + u = 3\eta^2\left(\alpha(\eta)+\tilde{\xi},\eta\right) + u$$

$$= \underbrace{3\eta^2\left(-\eta-\eta^5+\tilde{\xi}\eta^2\right)}_{\tilde{q}(\tilde{\xi},\eta)} + u$$

A stabilizing controller is

$$u = -K\tilde{\xi} - 3\eta^2\left(-\eta-\eta^5+\tilde{\xi}\eta^2\right) = -K\left(\xi+\eta^3\right) - 3\eta^2\left(-\eta-\eta^5+\left(\xi+\eta^3\right)\eta^2\right)$$

EXAMPLE

$$\begin{cases} \dot{\eta} = \eta^2 + \xi \\ \dot{\xi} = \xi\eta + u \end{cases} \qquad \leftarrow \text{ unstable if } \xi = 0$$

Consider

$$\alpha(\eta) = -\eta^2 - \eta$$

then $\psi(\alpha(\eta) + d, \eta) = -\eta + d$ $\Rightarrow$ MIN. PHASE + COMPLETENESS

In the variables

$$\tilde{\xi} = \xi - \alpha(\eta) = \xi + \eta^2 + \eta$$

we have

$$\dot{\tilde{\xi}} = \xi\eta + u - (2\eta + 1)\cdot(\eta^2 + \xi)$$

$$\Rightarrow u = -k\underbrace{(\xi + \eta^2 + \eta)}_{= \tilde{\xi}} - \xi\eta + (2\eta + 1)(\eta^2 + \xi)$$

CASE x

$$\begin{cases} \dot{\eta} = \psi(\xi, \eta) \\ \dot{\xi} = q(\xi, \eta) + b(\xi, \eta) u \end{cases}$$

ASSUME the system is MINIMUM PHASE

$$\Downarrow$$

$\dot{\eta} = \psi(0, \eta)$ globally as. stable

$\Rightarrow \exists W : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

1) $W(\eta) > 0, \ \forall \eta \neq 0$
2) $W(0) = 0$
3) $\frac{\partial W}{\partial \eta}(\eta) \cdot \psi(0, \eta) < 0 \quad \forall \eta$
4) W is radially unbounded

Consider $\dot{\eta} = \psi(\xi, \eta)$

$$\dot{W}(\eta) = \frac{\partial W}{\partial \eta}(\eta) \cdot \psi(\xi, \eta)$$

$$= \underbrace{\frac{\partial W}{\partial \eta}(\eta) \cdot \psi(0, \eta)}_{<0} + \frac{\partial W}{\partial \eta}(\eta)\left(\psi(\xi, \eta) - \overset{?}{\psi}(0, \eta)\right)$$

RESULT: $\exists e : \mathbb{R}^n \to \mathbb{R}$, $\frac{\partial W}{\partial \eta}(\eta)\left(\psi(\xi, \eta) - \psi(0, \eta)\right)$

$$= \frac{\partial W}{\partial \eta}(\eta) \cdot \rho(\xi, \eta) \cdot \xi$$

Let a function $\gamma : [0,1] \to \mathbb{R}$ be defined as

$$\gamma(s) = \psi(s\xi, \eta)$$

$\rightarrow$ fix ξ, η

Then,

$$\psi(\xi,\eta) - \psi(0,\eta) = \gamma(1) - \gamma(0) = \int_0^1 \dot{\gamma}(s)\,ds$$

$$= \int_0^1 \frac{\partial}{\partial \xi}\psi(s\xi,\eta)\,\xi\,ds = \underbrace{\boxed{\int_0^1 \frac{\partial}{\partial \xi}\psi(s\xi,\eta)\,ds}}_{p(\xi,\eta)} \cdot \xi$$

Minimum phase implies

$$\dot{W}(\eta) = \underbrace{\frac{\partial W}{\partial \eta}(\eta)\,\psi(0,\xi)}_{<0} + \frac{\partial W}{\partial \eta}(\eta)\,p(\xi,\eta)\cdot\xi$$

$$\dot{\eta} = \psi(\xi,\eta)$$

$$\dot{\xi} = q(\xi,\eta) + b(\xi,\eta)\,u$$

$\rightarrow$ choose u that drives Lyapunov neg.

$V(\xi,\eta) = W(\eta) + \frac{1}{2}\xi^2$ so that

1) $V(0,0) = 0$
2) $V(\xi,\eta) > 0 \quad \forall (\xi,\eta) \neq 0$
3) V is radially unbounded
4) $\dot{V}(\xi,\eta) = \frac{\partial W}{\partial \eta}(\eta)\,\psi(0,\eta) + K\xi^2 < 0 \quad \forall (\xi,\eta) \neq 0$

$$\dot{V}(\xi,\eta) = \dot{W}(\eta) + \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\xi\cdot\dot{\xi}$$

$$= \frac{\partial w}{\partial \eta}(\eta)\,\psi(0,\eta) + \frac{\partial w}{\partial \eta}(\eta)\,p(\xi,\eta)\cdot\xi$$

$$+ \xi\left[q(\xi,\eta) + b(\xi,\eta)u\right]$$

$$\dot V(\xi,\eta) = \underbrace{\frac{\partial w}{\partial \eta}(\eta)\,\psi(0,\eta)}_{\substack{=0 \text{ if } \eta=0,\\ \xi\neq 0}} + \left[\frac{\partial w}{\partial \eta}(\eta)\cdot p(\xi,\eta) + q(\xi,\eta) + b(\xi,\eta)u\right]\xi$$

$$u = \frac{1}{b(\xi,\eta)}\left(-q(\xi,\eta)\boxed{-\frac{\partial w}{\partial \eta}(\eta)\,p(\xi,\eta)} + K\xi\right) \Rightarrow (\xi,\eta)=0$$

$K<0$ is global as. stable

(COUNTER) EXAMPLE:

$$\begin{cases} \dot\eta = -\eta + \xi\eta^2 \\ \dot\xi = q(\xi,\eta) + b(\xi,\eta)u \end{cases}$$

$$\dot\eta = \psi(\xi,\eta) = -\eta + \xi\eta^2$$

$$\psi(0,\eta) = -\eta$$

Zero dynamic $\dot\eta = -\eta$

$$w(\eta) = \frac{1}{2}\eta^2$$

$$\dot w(\eta) = -\eta^2$$

$$\psi(\xi,\eta) - \psi(0,\eta) = \rho(\xi,\eta)\,\xi$$

$$-\eta + \xi\eta^2 - (-\eta) = \underbrace{\eta^2}_{\rho(\xi,\eta)=\eta^2} \cdot \xi$$

$$-\frac{\partial W}{\partial \eta}(\eta)\,\rho(\xi,\eta) = -\eta\cdot\eta^2 = -\eta^3$$

$$u = \frac{1}{b(\xi,\eta)}\left(-q(\xi,\eta) + k\xi - \eta^3\right)$$

For $r = 1$, how to stabilize ξ?

$$\begin{cases} \dot{\eta} = \psi(\xi,\eta) \\ \dot{\xi} = q(\xi,\eta) + b(\xi,\eta)\,u \end{cases}$$

Assume $\exists\, \alpha : \mathbb{R}^{n-1} \to \mathbb{R}$, $\alpha(0) = 0$ such that $\dot{\eta} = \psi(\alpha(\eta) + \tilde{\xi}, \eta)$ is ISS with respect to $\tilde{\xi}$

$\hookrightarrow$ transitory term on ξ / perturbation

$$\xi \to \alpha(\eta) \Rightarrow \eta(t) \to 0$$

$$\Rightarrow \alpha(\eta(t)) \to 0$$

$$\Rightarrow \xi(t) \to 0$$

$$\tilde{\xi} = \xi - \alpha(\eta) \qquad \xi = \tilde{\xi} + \alpha(\eta)$$

change variables $(\xi, \eta) \to (\tilde{\xi}, \eta)$

$$\begin{cases} \dot{\eta} = \psi(\alpha(\eta) + \tilde{\xi}, \eta) \\ \dot{\tilde{\xi}} = \dot{\xi} - \dot{\alpha}(\eta) \end{cases}$$

$$= \underbrace{q(\tilde{\xi} + \alpha(\eta), \eta)}_{\tilde{q}(\tilde{\xi}, \eta)} + \underbrace{b(\tilde{\xi} + \alpha(\eta), \eta)}_{\tilde{b}(\tilde{\xi}, \eta)} u - \underbrace{\frac{\partial \alpha}{\partial \eta}(\eta) \cdot \psi(\alpha(\eta) + \tilde{\xi}, \eta)}_{\tilde{q}(\tilde{\xi}, \eta)}$$

Now: $\dot{\eta} = \tilde{\psi}(\tilde{\xi}, \eta)$

$\dot{\tilde{\xi}} = \tilde{q}(\tilde{\xi}, \eta) + \tilde{b}(\tilde{\xi}, \eta) u$

is a system with input u and output $\tilde{\xi}$ it is STRONGLY MIN. PHASE

$$\Rightarrow \begin{cases} \eta(t) \to 0 \\ \tilde{\xi}(t) \to 0 \end{cases} \quad \xi(t) = \tilde{\xi}(t) + \alpha(\eta(t)) \to 0$$

$$u = \frac{1}{\tilde{b}(\tilde{\xi}, \eta)} \left(-q(\tilde{\xi}, \eta) + K \tilde{\xi} \right)$$

$$= \frac{1}{b(\xi, \eta)} \left(-q(\xi, \eta) + \frac{\partial \alpha}{\partial \eta}(\eta) \psi(\xi, \eta) + K(\xi - \alpha(\eta)) \right)$$

EXAMPLE:

$$\begin{cases} \dot{\eta} = \eta^2 + \xi \\ \dot{\xi} = \xi\eta + u \end{cases}$$

if $\xi = 0$, $\dot{\eta} = \eta^2$: $\xi = -\eta^2 - \eta = \alpha(\eta)$

$\dot{\eta} = \eta^2 + \alpha(\eta) = -\eta$

$\xi = \alpha(\eta) + \tilde{\xi} \rightarrow \dot{\eta} = -\eta + \tilde{\xi}$ (ISS)

$W(\eta) = \frac{1}{2}\eta^2$ satisfies ISS? Yes

$\xi \longrightarrow \tilde{\xi} = \xi - \alpha(\eta)$

$\dot{\eta} = -\eta + \tilde{\xi}$

$\dot{\tilde{\xi}} = (\tilde{\xi} + \alpha(\eta))\eta + u - \frac{\partial \alpha}{\partial \eta}(\eta)(-\eta + \tilde{\xi})$

$= (\tilde{\xi} - \eta^2 - \eta)\eta - (-2\eta - 1)(-\eta + \tilde{\xi}) + u = K\tilde{\xi}$

$u = -(\tilde{\xi} - \eta^2 - \eta)\eta - (2\eta + 1)(-\eta + \tilde{\xi}) + K\tilde{\xi}$ $(K < 0)$

$\dot{\eta} = -\eta + \tilde{\xi}$

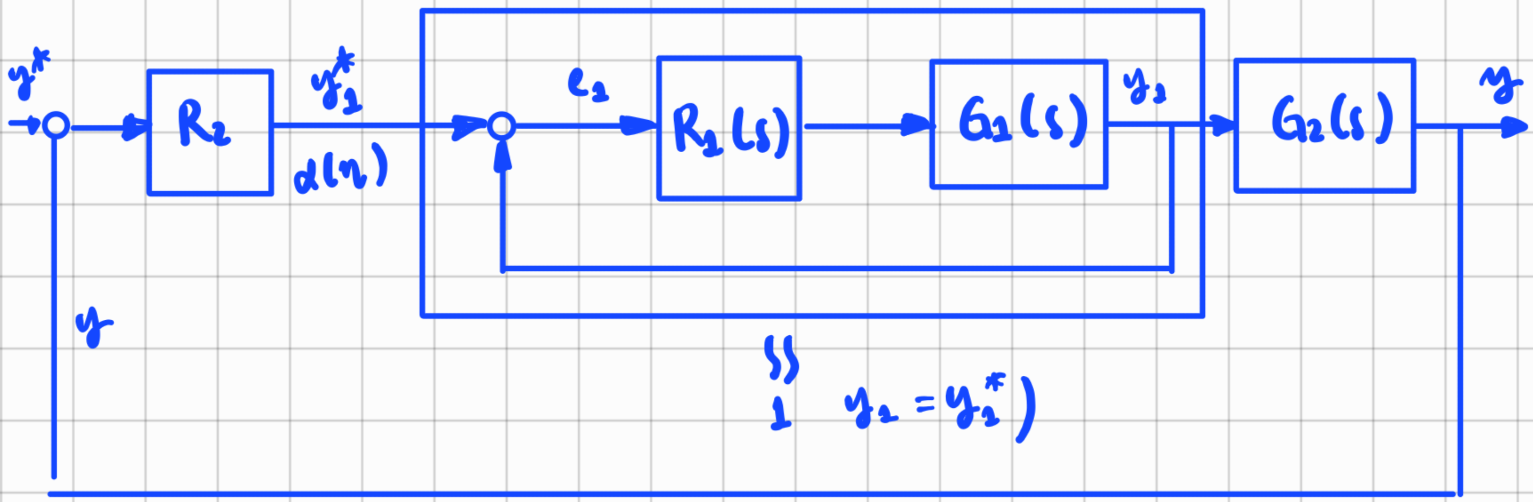
$\dot{\tilde{\xi}} = K\tilde{\xi}$

$$\begin{bmatrix} \dot{\eta} \\ \dot{\tilde{\xi}} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & K \end{bmatrix} \begin{bmatrix} \eta \\ \tilde{\xi} \end{bmatrix}$$

$\eta(t) \rightarrow 0$, $\tilde{\xi} \rightarrow 0$

$\xi = \tilde{\xi} - \eta^2 - \eta \rightarrow 0$

BACKSTEPPING



CASE $1 < r < n$

$$\begin{aligned}
\dot{\eta} &= \psi(\xi, \eta) \\
\dot{\xi}_1 &= \xi_2 \\
&\;\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= q(\xi, \eta) + b(\xi, \eta)u
\end{aligned}
\qquad \chi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{r-1} \end{bmatrix}$$

$$\dot{\eta} = \psi(\chi, \xi_r, \eta)$$

$$\dot{\chi} = A\chi + B\xi_r$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \bar{\eta} = \begin{bmatrix} \eta \\ \chi \end{bmatrix}$$

$$\dot{\xi}_r = q(\chi, \xi_r, \eta) + b(\chi, \xi_r, \eta)u$$

$$\dot{\bar{\eta}} = \bar{\psi}(\xi_r, \bar{\eta}) = \begin{bmatrix} \psi(x, \xi_r, \eta) \\ Ax + B\xi_r \end{bmatrix}$$

$$\dot{\xi}_r = \bar{q}(\xi_r, \bar{\eta}) + \bar{b}(\xi_r, \bar{\eta})u$$

$$= q(x, \xi_r, \eta) \qquad = b(x, \xi_r, \eta)$$

Assume $\dot{\eta} = \psi(\xi, \eta)$ ISS.

$\Rightarrow \dot{\eta} = \psi(x, \xi_r, \eta)$ ISS with respect to (x, ξ_r)

$\xi_r = \alpha(\bar{\eta}) = Kx$, $A + BK$ Hurwitz

$$\tilde{\xi}_r = \xi_r - Kx$$

$$\dot{\bar{\eta}} = \begin{bmatrix} \psi(x, \tilde{\xi}_r + Kx, \eta) \\ (A+BK)x + B\tilde{\xi}_r \end{bmatrix}$$

$$\tilde{\xi}_r = q(\ldots) + b(\ldots)u - K[(A+BK)x - B\tilde{\xi}_r]$$

$$= -H\tilde{\xi}_r$$

$$u = \frac{1}{b(\cdot\cdot)}\left(-q(\cdots) + K[(A+BK)x + B\tilde{\xi}_r] - H\tilde{\xi}_r\right)$$

STATE-FEEDBACK CONTROL - CASE $r > 1$

We now consider systems of the form

$$\begin{cases} \dot{\eta} = \psi(\xi, \eta) \\ \dot{\xi}_i = \xi_{i+1} \qquad i = 1, \dots, r-1 \\ \dot{\xi}_r = q(\xi, \eta) + b(\xi, \eta) u \end{cases}$$

STATE FEEDBACK $y_m = x \sim \begin{pmatrix} \eta \\ \xi \end{pmatrix}$

ASSUMPTION. The system is STRONGLY MINIMUM PHASE

The idea is to proceed as we did before via BACKSTEPPING

Call

$$\chi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{r-1} \end{pmatrix}$$

Then

$$\dot{\chi} = A\chi + B\xi_r \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ 0 & 0 & \cdots & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\xi_1 = C\chi \qquad\qquad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

So we can write

$$\dot{\eta} = \psi(\chi, \xi_r, \eta)$$

$$\dot{\chi} = A\chi + B\xi_r$$

$$\dot{\xi}_r = q(\chi, \xi_r, \eta) + b(\chi, \xi_r, \eta) u$$

Now, let $K = (k_1, \dots, k_{r-1})$ be such that $A + BK$ is Hurwitz

and let

$$\boxed{\vartheta \doteq \xi_r - K\chi}$$

Then, we obtain

$$\begin{cases} \dot{\eta} = \psi(\chi, \xi_r, \eta) \\ \dot{\chi} = A\chi + B(\theta + K\chi) = (A+BK)\chi + B\theta \\ \dot{\theta} = q(\chi, \theta+K\chi, \eta) + b(\chi, \theta+K\chi, \eta)\, u \quad - K(A+BK)\chi - KB\theta \end{cases}$$

Since $A+BK$ is Hurwitz, then χ is ISS with respect to θ

RESULT. The system $\bar{\eta} = \begin{pmatrix} \eta \\ \chi \end{pmatrix}$ is ISS with respect to θ

we can write

$$\begin{cases} \dot{\bar{\eta}} = \bar{\psi}(\theta, \bar{\eta}) \\ \dot{\theta} = \bar{q}(\theta, \bar{\eta}) + \bar{b}(\theta, \bar{\eta})\, u \end{cases}$$

$$\bar{\psi}(\theta, \bar{\eta}) = \begin{pmatrix} \psi(\chi, \theta+K\chi, \eta) \\ (A+BK)\chi + B\theta \end{pmatrix}$$

$$\bar{q}(\theta, \bar{\eta}) = q(\chi, \theta+K\chi, \eta) - K(A+BK)\chi - KB\theta$$

$$\bar{b}(\theta, \bar{\eta}) = b(\quad '' \quad)$$

$\downarrow$

This is a STRONGLY MINIMUM-PHASE normal form

$\Rightarrow$ the control

$$u = \frac{1}{\bar{b}(\theta, \bar{\eta})} \left(-\bar{q}(\theta, \bar{\eta}) - \bar{k}\theta \right) \qquad \bar{k} > 0$$

GLOBALLY stabilizes $(\eta, \xi) = 0$

Remark. We obtain u in the original coordinates by making explicit

$$\bar{\eta} = \begin{pmatrix} \eta \\ \xi_1 \\ \vdots \\ \xi_{r-1} \end{pmatrix} = \begin{pmatrix} \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \\ \phi_1(x) \\ \vdots \\ \phi_{r-1}(x) \end{pmatrix} \qquad \theta = \xi_r - K\chi = \phi_r(x) - k_1\phi_1(x) - \dots - k_{r-1}\phi_{r-1}(x)$$

EXTRA

NEW ASSUMPTION. $\psi(\xi,\eta)$ only depends on ξ_1 (that is, $\psi(\xi,\eta)=\psi(\xi_1,\eta)$)

and $\exists\, \alpha(\eta)$ such that $\dot\eta = \psi(\alpha(\eta)+d, \eta)$ is ISS with respect to d

$\downarrow$

We now consider systems of the form

$$\begin{cases} \dot\eta = \psi(\underline{\xi_1}, \eta) \\ \dot\xi_i = \xi_{i+1} \qquad i = 1, \dots, r-1 \\ \dot\xi_r = q(\xi,\eta) + b(\xi,\eta)u \end{cases}$$

By proceeding as before we change variables as

$$\xi_1 \mapsto \tilde\xi_1 \doteq \xi_1 - \underbrace{\alpha(\eta)}_{\text{call this } \alpha_1(\eta)}$$

$$\xi_2 \mapsto \tilde\xi_2 \doteq \xi_1 - \underbrace{\frac{\partial\alpha_1}{\partial\eta}(\eta)\cdot\psi(\xi_1,\eta)}_{\doteq\, \alpha_2(\eta)}$$

$$\xi_3 \mapsto \tilde\xi_3 \doteq \xi_3 - \underbrace{\frac{\partial\alpha_2}{\partial\eta}(\eta)\cdot\psi(\xi_1,\eta)}_{\doteq\, \alpha_3(\eta)}$$

$$\vdots$$

$$\xi_r \mapsto \tilde\xi_r \doteq \xi_r - \underbrace{\frac{\partial\alpha_{r-1}}{\partial\eta}(\eta)\cdot\psi(\xi_1,\eta)}_{\alpha_r(\eta)}$$

In the new variables we have

$$\dot\eta = \psi(\alpha(\eta)+\tilde\xi_1, \eta)$$ (STRONGLY MINIMUM PHASE)

$$\dot{\tilde\xi}_1 = \dot\xi_1 - \dot\alpha_1(\eta) = \xi_2 - \alpha_2(\eta) = \tilde\xi_2 \qquad (\alpha_2(\eta) = \dot\alpha_1(\alpha))$$

$$\dot{\tilde\xi}_2 = \dot\xi_2 - \dot\alpha_2(\eta) = \xi_3 - \alpha_3(\eta) = \tilde\xi_3 \qquad (\alpha_3(\eta) = \dot\alpha_2(\eta))$$

$$\vdots$$

$$\dot{\tilde\xi}_r = \dot\xi_r - \dot\alpha_r(\eta) = \underbrace{q(\xi,\eta)} + b(\xi,\eta)u \underbrace{- \frac{\partial\alpha_{r-1}}{\partial\eta}(\eta)\cdot\psi(\xi_1,\eta)}$$

$$= \tilde q(\tilde\xi,\eta) + \tilde b(\xi,\eta)u$$

EXTRA

where

$$\tilde{q}(\tilde{\xi},\tilde{\eta}) = \left[q(\xi,\eta) - \frac{\partial \alpha_{r-1}}{\partial \eta}(\eta) \cdot \psi(\xi_1,\eta) \right]_{\xi_i = \tilde{\xi}_i + \alpha_i(\eta),\ \forall i=1,\dots,r}$$

$$\tilde{b}(\xi,\eta) = b(\xi,\eta)\Big|_{\xi_i = \tilde{\xi}_i + \alpha_i(\eta),\ \forall i=1,\dots,r}$$

By letting

$$\tilde{\psi}(\tilde{\xi},\eta) \doteq \psi(\alpha_1(\eta) + \tilde{\xi}_1, \eta)$$

we finally obtain

$$\begin{cases} \dot{\eta} = \tilde{\psi}(\tilde{\xi},\eta) & (\text{STRONGLY MINIMUM PHASE}) \\ \dot{\tilde{\xi}}_i = \tilde{\xi}_{i+1} & i=1,\dots,r-1 \\ \dot{\tilde{\xi}}_r = \tilde{q}(\tilde{\xi},\tilde{\eta}) + \tilde{b}(\tilde{\xi},\tilde{\eta})\, u \end{cases}$$

We can use the previous result to obtain u

EXAMPLE

EXTRA

$$
\begin{cases}
\dot{\eta} = \xi_1 + \eta^2 \\
\dot{\xi}_1 = \xi_2 \\
\dot{\xi}_2 = \xi_3 \\
\dot{\xi}_3 = \eta\, \xi_1 + u
\end{cases}
\qquad \text{GOAL: } \xi_1 \to 0
$$

We first notice that

$$\alpha(\eta) \doteq -\eta - \eta^2$$

is such that $\xi_1 = \alpha(\eta) + d$ yields

$$\dot{\eta} = -\eta + d \qquad \rightsquigarrow \quad \text{ISS}$$

Proceeding as indicated above we et

$$\dot{\eta} = -\eta + \tilde{\xi}_1$$

g

$$\tilde{\xi}_1 \doteq \xi_1 - \overset{\alpha(\eta)}{\overset{\shortparallel}{\alpha_1(\eta)}} = \xi_1 + \eta + \eta^3$$

$$\tilde{\xi}_2 \doteq \xi_2 - \alpha_2(\eta) = \xi_2 - \left(\underbrace{\frac{\partial \alpha_1}{\partial \eta}(\eta)}_{1+3\eta^2} \cdot \underbrace{\psi(\alpha(\eta) + \tilde{\xi}_1, \eta)}_{-\eta + \tilde{\xi}_1} \right) = \xi_2 - (1+3\eta^2)(-\eta + \tilde{\xi}_1)$$

$$\tilde{\xi}_3 \doteq \xi_3 - \alpha_3(\eta) = \xi_3 - \left(\underbrace{\frac{\partial \alpha_2}{\partial \eta}(\eta)}_{= 6\eta(-\eta + \tilde{\xi}_1) - (1+3\eta^2)} \overbrace{\psi(\alpha(\eta) + \tilde{\xi}_1, \eta)} \right) = \xi_3 - 6\eta(-\eta + \tilde{\xi}_1)^2 + (1+3\eta^2)(-\eta + \tilde{\xi}_1)$$

In the new coordinates:

$$\dot{\eta} = -\eta + \tilde{\xi}_1$$

$$\dot{\tilde{\xi}}_1 = \tilde{\xi}_2$$

$$\dot{\tilde{\xi}}_2 = \tilde{\xi}_3$$

$$\dot{\tilde{\xi}}_3 = \dot{\xi}_3 - 6\dot{\eta}(-\eta + \tilde{\xi}_1)^2 - 12\eta(-\eta + \tilde{\xi}_1)(-\dot{\eta} + \dot{\tilde{\xi}}_1) + 6\eta\dot{\eta}(-\eta + \tilde{\xi}_1) + (1+3\eta^2)(-\dot{\eta} + \dot{\tilde{\xi}}_1)$$

$$= u + \underbrace{\eta(\xi_1 - \eta - \eta^3) - 6(-\eta + \tilde{\xi}_1)^2 - 12\eta(-\eta + \tilde{\xi}_1)(\eta - \tilde{\xi}_1 + \tilde{\xi}_2) + 6\eta(-\eta + \tilde{\xi}_1)^2 + (1+3\eta^2)(\eta - \tilde{\xi}_1 + \tilde{\xi}_2)}_{}$$

$$= \boxed{q(\eta, \tilde{\xi})} + u$$

EXTRA

Now, define

$$x = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix}, \qquad \vartheta = \tilde{\xi}_3 + \hat{\xi}_1 + 2\tilde{\xi}_2$$

Then we obtain

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{\xi}_3 = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}}_{\text{Hurwitz}} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \vartheta$$

$$\dot{\vartheta} = q(\eta, \tilde{\xi}) + u + \underbrace{\dot{\hat{\xi}}_1 + 2\dot{\tilde{\xi}}_2}$$

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \dot{x} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \vartheta$$

$$= -2\tilde{\xi}_1 - 3\tilde{\xi}_2 + 2\vartheta$$

$$= \underbrace{q(\eta, \hat{\xi}) - 2\tilde{\xi}_1 - 3\tilde{\xi}_2 + 2\vartheta}_{= \bar{q}(\eta, x)} + u$$

and where

$$\dot{\eta} = -\eta + \hat{\xi}_1$$

$\Rightarrow$ we can use

$$u = -\bar{q}(\eta, x) - k\vartheta \qquad \text{with } k > 0$$