Geometry of Trajectories

Master degree in Automation Engineering

Prof. Lorenzo Marconi

Dipartimento di Ingegneria dell’Energia Elettrica e dell’Informazione (DEI)
Università di Bologna

lorenzo.marconi@unibo.it
Tel. +39 051 2093788

LINEAR TIME-INVARIANT SYSTEM

Continuous Time : $x(t) = e^{At} x_0$

Discrete : $x(t) = A^t x_0$

$\boxed{\text{Remember: } \lambda \in (A) \quad Av = \lambda v}$

Result: natural number spectrum

$\forall k \in \mathbb{N},\ A^k v = \lambda^k v \quad (\lambda^k \in \sigma(\lambda^k))$

$\forall k \in \mathbb{R},\ e^{\lambda t} \in e^{\lambda t} v \quad (e^{\lambda t} \in \sigma(e^{At}))$

Assume that $A^{k-1} v = \lambda^{k-1} v$

$$\Longrightarrow \quad A^k v = A \cdot A^{k-1} v = A \lambda^{k-1} v$$
$$= \lambda^{k-1} \underbrace{Av}_{\lambda v} = \lambda^k v$$

so, $Av = \lambda v$ for $k = 2$

Proving $e^{At} v = e^{\lambda t} v$:

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{(At)^k}_{A^k t^k}$$

$$e^{At} v = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{A^k v}_{\lambda^k v} \cdot t^k = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k v t^k$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!} (\lambda t)^k}_{e^{\lambda t}} v$$

Assumptions (*):

1.) $\forall \lambda \in \sigma(A), \quad g_\lambda = e_\lambda$

2.) $\forall \lambda \in \sigma(A), \quad \lambda \in \mathbb{R}$

As the consequences from the assumptions above, v_i : eigenvalue related to λ_i, $\{v_1, \dots, v_n\}$ is e basis of $\mathbb{R}^n$:

$$\boxed{\begin{array}{l} x_0 \in \mathbb{R}^n, \quad x_0 = \sum_{i=1}^{n} \alpha_i^0 v_i \\ x(t) = \sum_{i=1}^{n} \alpha_i(t)\, v_i \end{array}}$$

α_i^0 is scalar

$$x(t) = e^{At} x_0 = e^{At} \sum_{i=1}^{n} \alpha_i^0 \cdot v_i$$

$$= \sum_{i=1}^{n} \alpha_i^0 \underbrace{e^{At} v_i}_{e^{\lambda_i t} \cdot v_i} = \sum_{i=1}^{n} \underbrace{\alpha_i^0 e^{\lambda_i t}}_{\alpha_i(t)} \cdot v_i$$

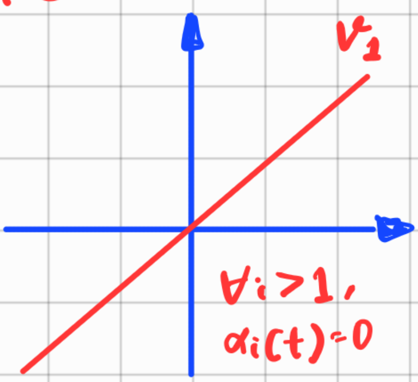
$\alpha_i(t) = \alpha_i^0 e^{\lambda_i t}$

$$x(t) = \alpha_1(t)\, v_1 + \dots + \alpha_n(t)\, v_n$$

the eigenspace is invariant

$$\alpha_2^0 = \alpha_3^0 = \dots = \alpha_n^0 = 0$$

$$x_0 = \alpha_1^0 v_1$$



$\forall i > 1$, $\alpha_i(t) = 0$ $\forall t \geq 0$

$$\sum_{i=1}^{n} \underbrace{\alpha_i^0 e^{\lambda_i t}}_{\alpha_i(t)} \cdot v_i = \alpha_1(t) v_1$$

If $\alpha_i^0 = 0$ for some i
implies $\alpha_i(t) = 0 \quad \forall t \geq 0$

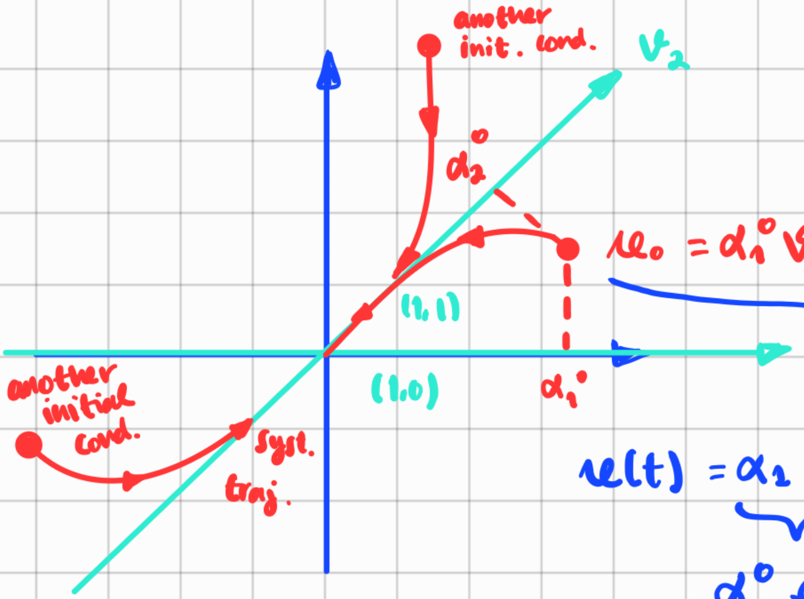
Example:

$$A = \begin{bmatrix} a & b-e \\ 0 & b \end{bmatrix} \qquad a \neq b$$

The eigenvalues are:
$\lambda_1 = a \qquad \lambda_2 = b$

and the eig.vectors are:
$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Plotting the eigenvectors and eigenspace

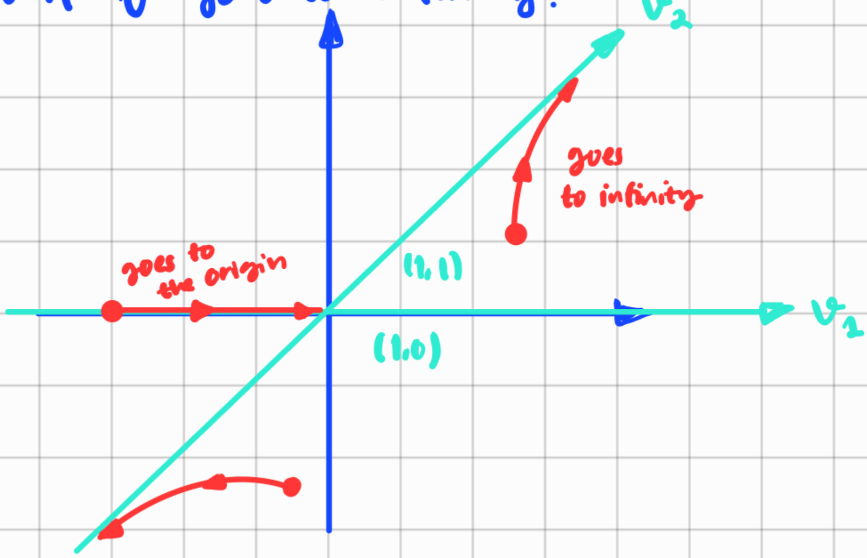


$$\boxed{\text{For } a<0,\ b<0}$$

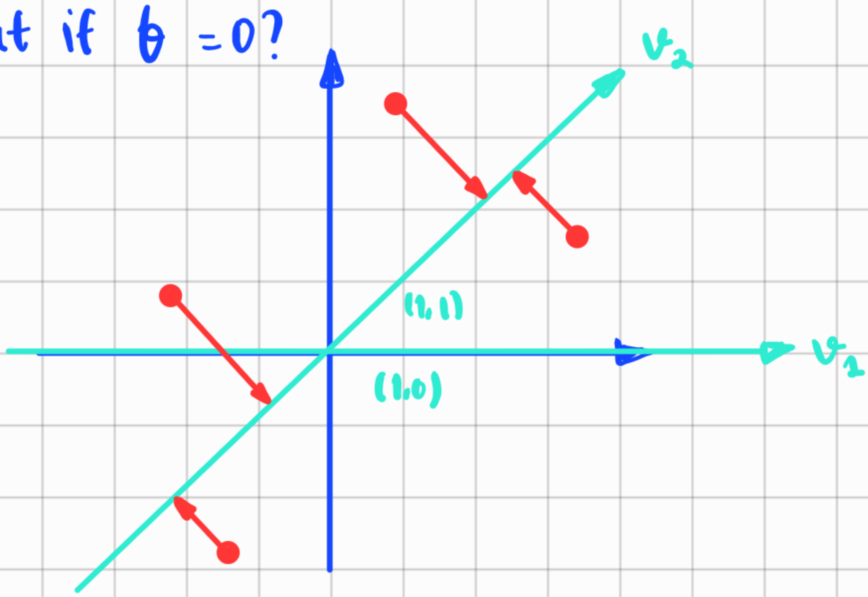
$$\underbrace{x_0 = \alpha_1^0 v_1 + \alpha_2^0 v_2}$$

$$x(t) = \underbrace{\alpha_1(t)}_{\alpha_1^0 e^{at}} v_1 + \underbrace{\alpha_2(t)}_{\alpha_2^0 e^{bt}} v_2$$

What if b goes to infinity?



What if $b = 0$?



Assumptions (*):

$$g_\lambda = e_\lambda \quad \forall \lambda_t \in \sigma(A)$$

$\lambda, \bar{\lambda} \in \sigma(A)$, $v, \bar{v}$ $\Longrightarrow$ $\lambda_1, \lambda_2 \in \sigma(A)$

$\lambda_1 = \bar{\lambda}_2 \quad v_1 = \bar{v}_2$

$\lambda = \sigma + j\omega$, $\sigma, \omega \in \mathbb{R}$

$v = q + jz$, $q, z \in \mathbb{R}^n$

$x_0 = \alpha_1^0 v_1 + \alpha_2^0 v_2 + \dots$

$\alpha_2^0 = \bar{\alpha}_1^0$

$$\alpha_1^0 v_1 + \alpha_2^0 v_2 = \alpha_1^0 v_1 + \overline{\alpha_1^0 v_1} = 2\,\text{Real}\,[\alpha_1^0 v_1]$$
$$= 2\,\text{Real}\,[(\alpha_{1R}^0 + j\alpha_{1I}^0)(q + jz)]$$
$$= 2\,\text{Real}\,[\alpha_{1R}^0 \cdot q - \alpha_{1I}^0 z + j(*)]$$
$$= \underline{\underline{2\alpha_{1R}^0 q - 2\alpha_{1I}^0 z}} \in \text{span}\{q, z\}$$

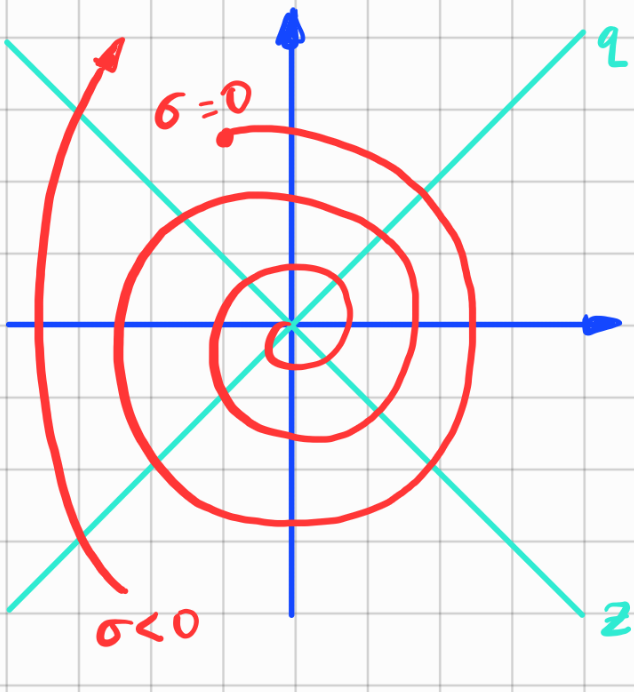
$$x(t) = \underbrace{\alpha_1^0 e^{\lambda_1 t}}_{\alpha_1(t)} v_1 + \underbrace{\alpha_2^0 e^{\lambda_2 t}}_{\alpha_2(t)} v_2 + \dots$$

$$\Downarrow$$

$$\alpha_1(t) v_1 + \alpha_2(t) v_2 \in \text{span}\{q, z\}$$
$$= \beta_1(t) q + \beta_2(t) z$$

Example:

$\lambda = \sigma + j\omega$, $\quad e^{\sigma t}\cos(\omega t + \phi)$



Eigenspace

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \qquad x(0) = x_0$$

$$\boxed{x(t) = \phi(t)\,x_0 \qquad\qquad \phi(t) = \begin{cases} e^{At} & \text{for C-T systems} \\ A^t & \text{for D-T systems}\end{cases}}$$

We consider the simplified case of $a_i = g_i$ (A diagonalisable). Let $(\lambda_i, \vec{v}_i)$, $i = 2, \ldots, n$, the set of eigenvalues and associated eigenvectors of A

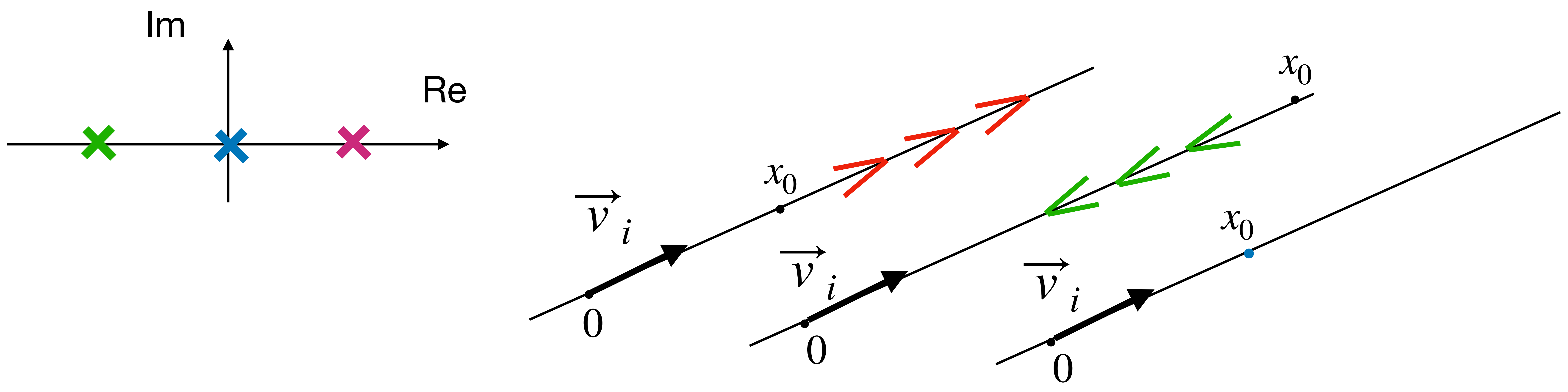
Eigenspace associated to $(\lambda_i, \vec{v}_i)$: $\mathscr{W}_i = \{x \in \mathbb{R}^n \ : \ x = c\vec{v}_i \ \ c \in \mathbb{R}\}$

Let x_0 be an initial condition on $\mathscr{W}_i$ ($x_0 = \bar{c}\vec{v}_i$). Then, $x(t) = e^{\lambda_i t}x_0$ for C-T systems and $x(t) = \lambda_i^t x_0$ (for DT systems).

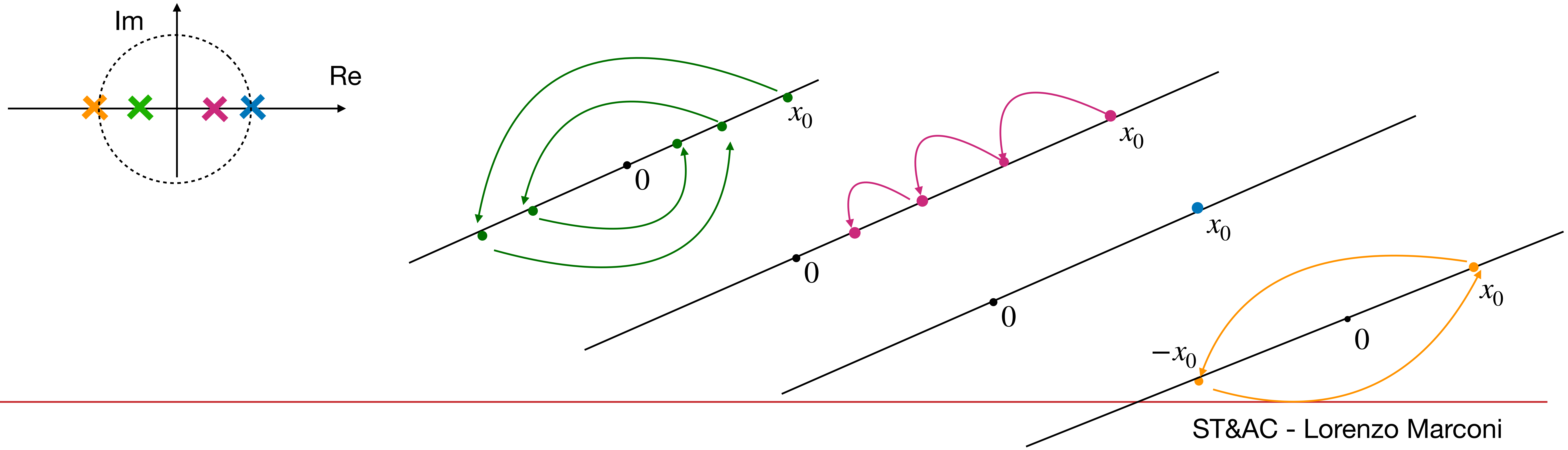
$\boxed{\text{Eigenspaces are invariant and the internal dynamics is described by the associated eigenvalue}}$

Real Case

C-T case



D-T case

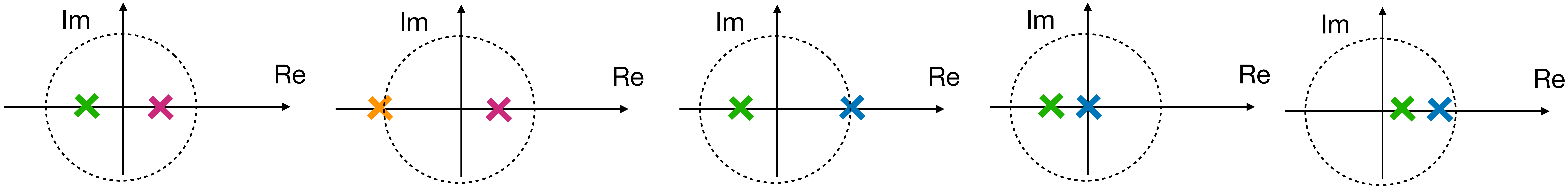


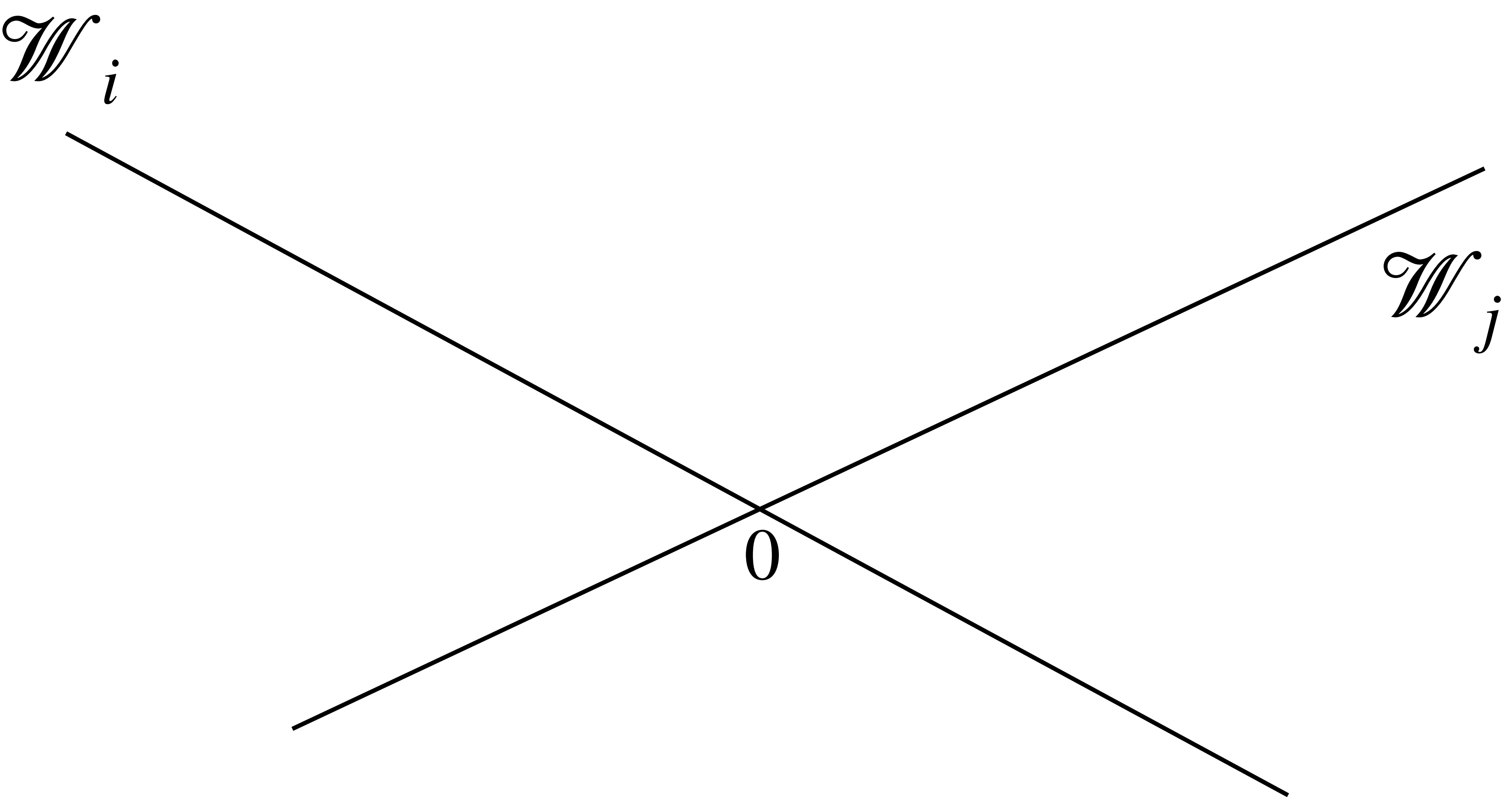
Geometry of trajectories (C-T case)

Pick now two eigenvalues. By the the previous reasonings, an invariant plane $\mathscr{W}_i \oplus \mathscr{W}_j \subset \mathbb{R}^n$ can be identified with the geometry of trajectories that is induced by the projection on the two eigenspaces

Geometry of trajectories (D-T case)

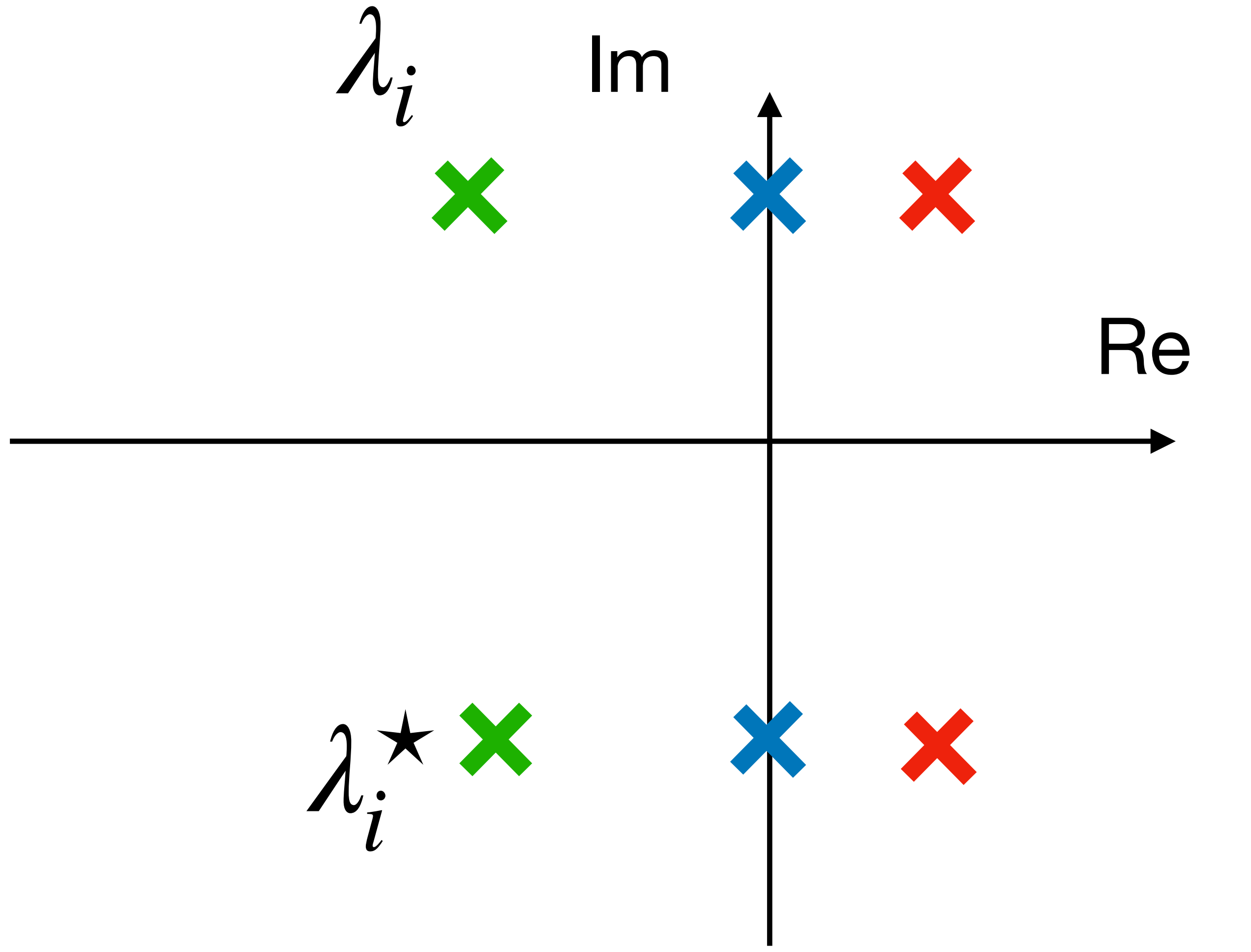
Homework: Plot the geometry of the trajectories in the cases below





Geometry of trajectories (C-T case, complex conjugate eqigenvalues)

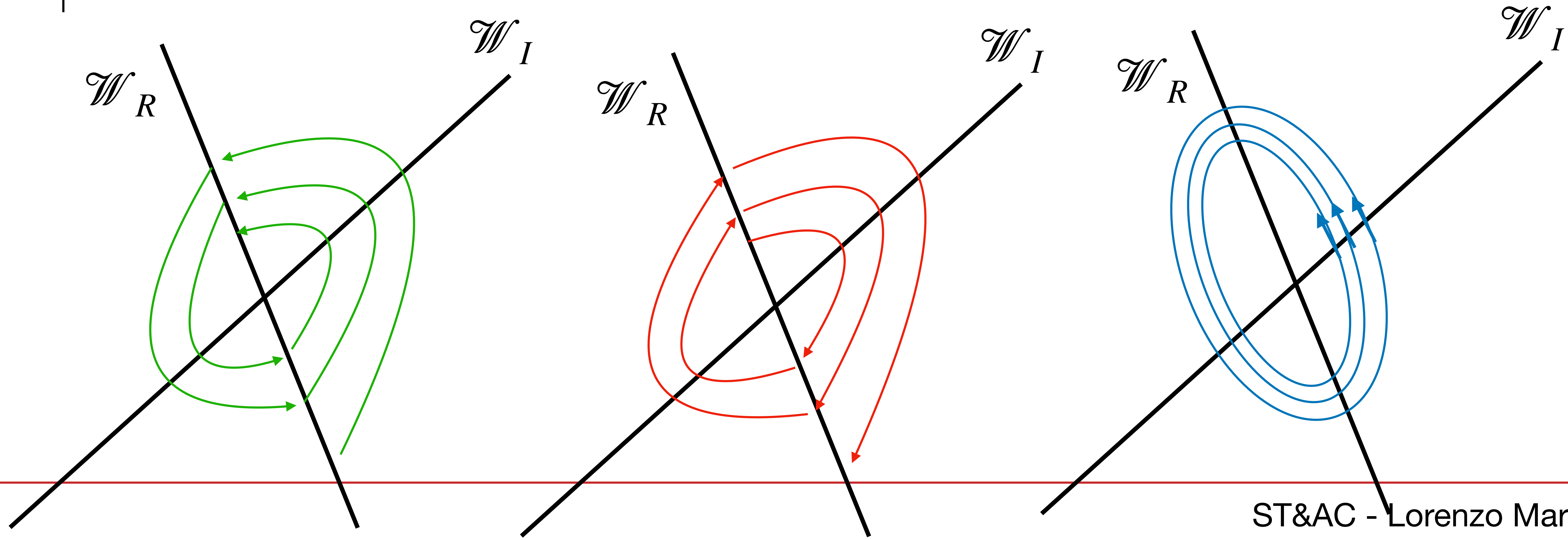
In case of complex conjugate eigenvalues



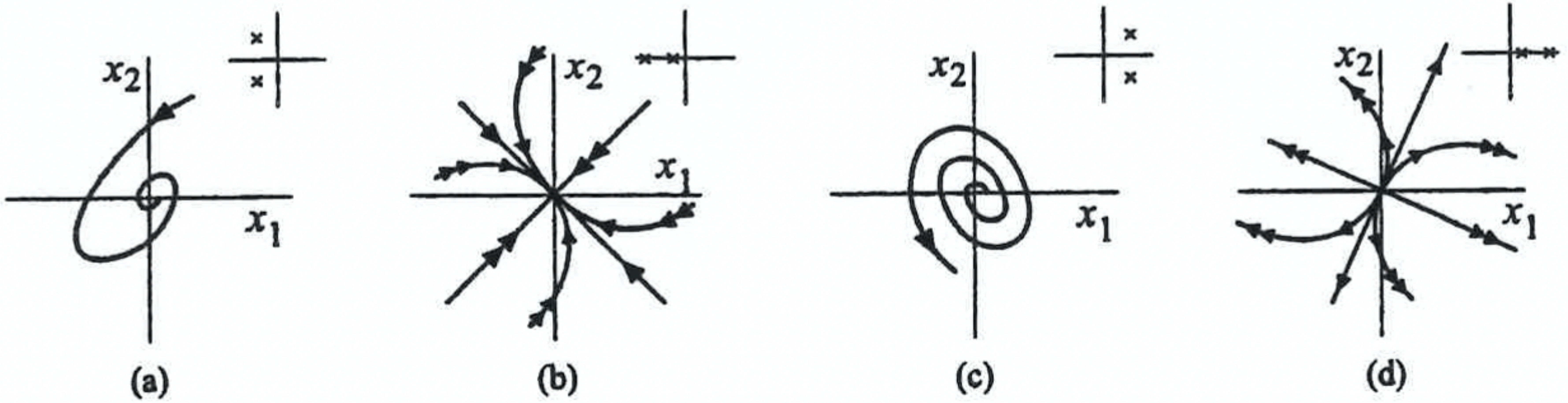
$$\lambda_i = \lambda_{iR} + j\lambda_{iI} \qquad \vec{v}_i = \vec{v}_{iR} + j\,\vec{v}_{iI} \qquad \mathscr{W}_R = \{x \in \mathbb{R}^n \;:\; x = c\,\vec{v}_{iR} \;\; c \in \mathbb{R}\}$$

$$\lambda_i^\star = \lambda_{iR} - j\lambda_{iI} \qquad \vec{v}_i^{\,\star} = \vec{v}_{iR} - j\,\vec{v}_{iI} \qquad \mathscr{W}_I = \{x \in \mathbb{R}^n \;:\; x = c\,\vec{v}_{iI} \;\; c \in \mathbb{R}\}$$

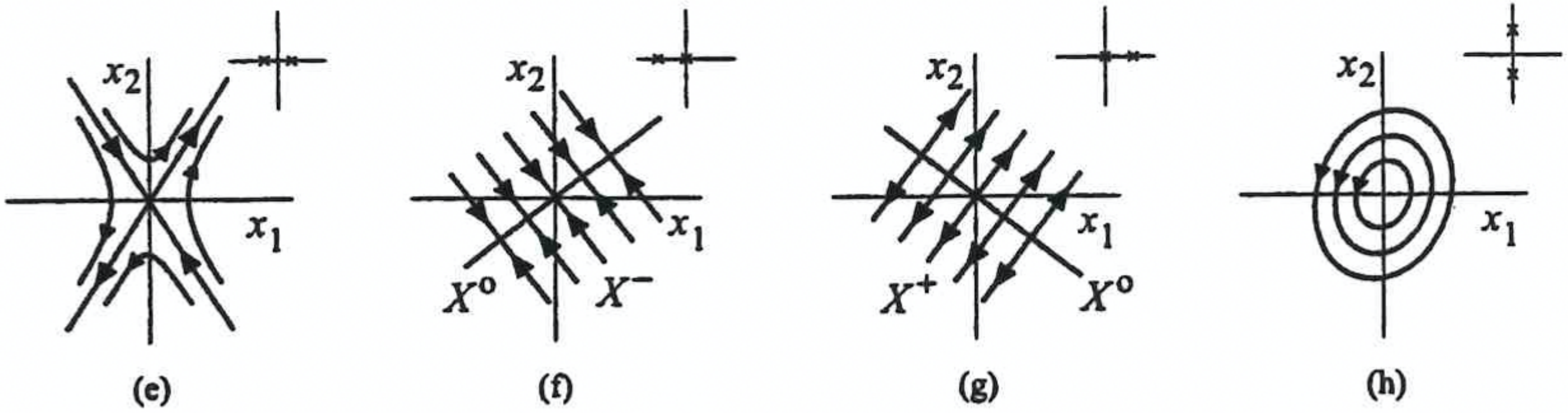
$\boxed{\mathscr{W}_R \oplus \mathscr{W}_I \text{ is an invariant plane and the trajectories spiral within it}}$



Nomenclature for C-T systems ($n = 2$)

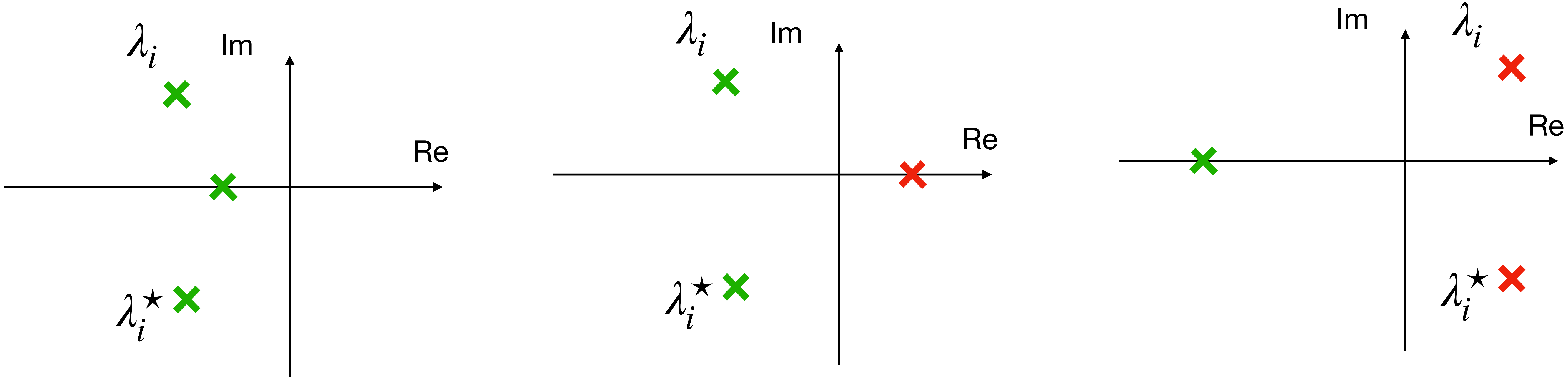


Stable Focus | Stable Node | Unstable Focus | Unstable Node

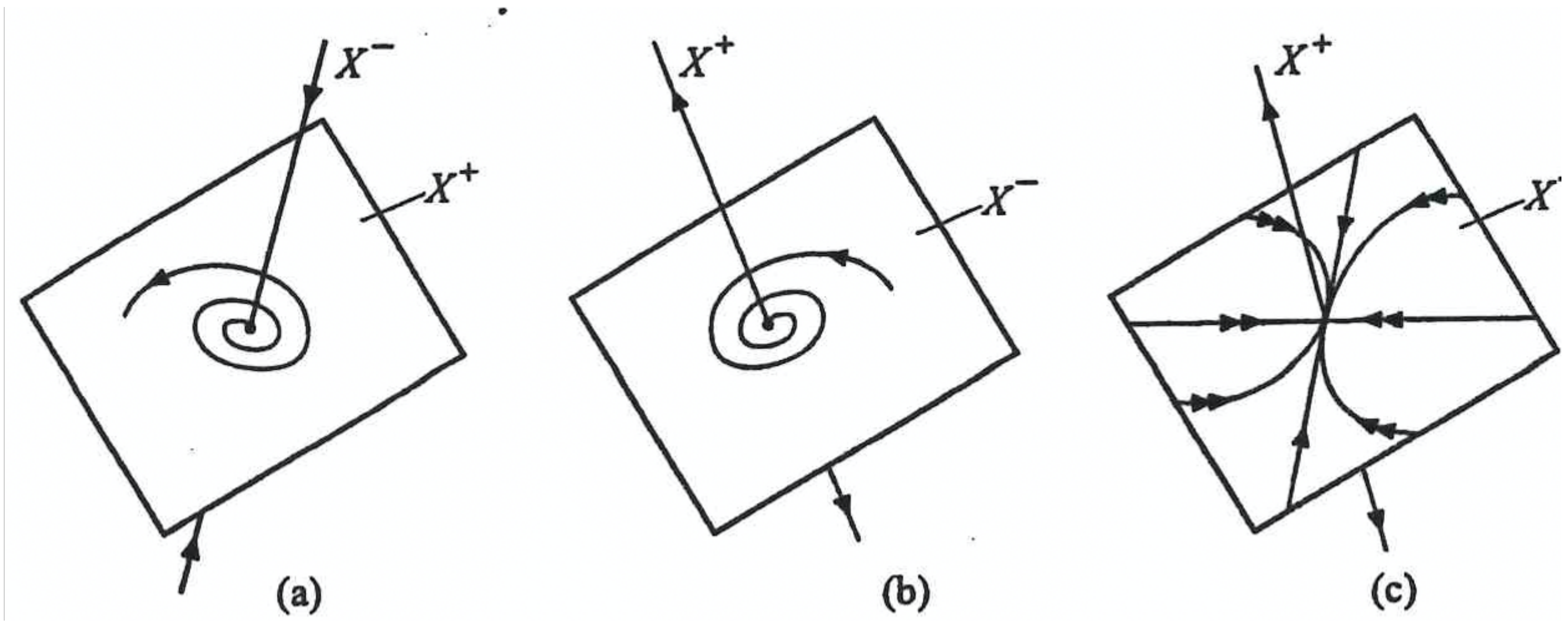


Saddle | Center | Center | Center

Combining CC and real eigenvalues (C-T case)



Nomenclature for C-T systems ($n = 3$)



Saddle-Focus **Saddle-Focus** **Real Saddle**

Combining CC and real eigenvalues (C-T case)

The previous arguments show that the whole state space $\mathbb{R}^n$ can be "foliated" in many invariant subspaces (eigenspaces) with "internal dynamics" that are governed by the specific eigenvalues. More in detail we can cluster $\mathbb{R}^n$ in three subspaces each governed by eigenvalues with positive real part, zero real part, positive real part (respectively within, on the boundary and outside the unitary circle)

eig with $Re < 0$ # eig with $Re = 0$

$$n = n^- + n^+ + n^0$$

eig with $Re > 0$

$$\mathbb{R}^n = X^- \oplus X^+ \oplus X^0$$

Stable manifold (dim n^-)
Unstable manifold (dim n^-)
Center manifold (dim n^-)

Hyperbolic Systems: systems without the centre manifold ($X^0 = 0$)

0