Mathematical Methods for Automation Engineering M
– Exercises on Combinatorics/Probability –

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Part I

Exercises on Combinatorics/Probability

Example 1.

In how many ways can we assign 5 open positions for which 8 candidates apply, assuming that the order matters?

$$N = D_{8,5} = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6\,720$$

The possible rankings are as many as the 5-permutations of 8 elements

Example 2.

How many games are played in a soccer all-play-all (round-robin) tournament with 20 teams?

$$N = D_{20,2} = \frac{20!}{(20-2)!} = 19 \cdot 20 = 380$$

The number of games equals the number of ordered pairs that can be built from a set of 20 elements

1. In how many ways can we assign 5 open positions for which 8 candidates apply, assuming that the order matters?

5 open positions → k, 8 candidates → n

It's a permutation without repetition

$n = 8, \quad k = 5$

$$N = D_{n,k} = D_{8,5} = \frac{8!}{(8-5)!} = \frac{8!}{3!}$$

$$= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = 6,720 \text{ ways}$$

2. How many games are played in a soccer all-play-all (round-robin) tournament with 20 teams?

20 teams → n

Order matters (permutation) with size of 2 from 20 teams

2 → k, 20 → n

$\boxed{A \text{ vs } B}$, $\boxed{B \text{ vs } A}$ } 2 matches

$$N = D_{n,k} = D_{20,2} = \frac{20!}{(20-2)!} = \frac{20 \cdot 19 \cdot \cancel{18!}}{\cancel{18!}}$$

$$= 380 \text{ matches}$$

Example 1.

In how many ways can 7 persons take seats at a round table if

1. everybody can seat next to anybody else
2. two of the seven persons do not want to seat next to each other

In the first case

$$N_1 = P_7^c = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$$

We can think of letting one person choose his/her seat, and then give seats to the other 6 persons

Exercises on Combinatorics/Probability

Example 1.

In how many ways can 7 persons take seats at a round table if

1. everybody can seat next to anybody else
2. two of the seven persons do not want to seat next to each other

In the second case

$$N_2 = P_7^c - P_6^c \cdot 2! = 720 - 120 \cdot 2 = 480$$

We can first compute the number of ways the two persons can seat one next to the other. Treating these two persons as a single individual, the cyclic permutations of 6 people are $P_6^c = 5! = 120$. But the two persons can seat one next to the other in $2! = 2$ ways. So, $P_6^c \cdot 2!$ is the number of arrangements in which the two persons happen to be one next to the other

1. In how many ways can 7 persons take seats at a round table if

1. everybody can seat next to anybody else
2. two of the seven persons do not want to seat next to each other

First case:

A cyclic permutation

$$N_1 = P_7^c = (7-1)! = 6! = \underline{\underline{720}} \text{ ways}$$

Second case:

It's easier to assign these two persons sit next to each other, then substract this number to the total number of possibilities.

2 possible ways to sit next i.e.

$$N' = P_6^c \cdot 2! = (6-1)! \cdot 2! = 240$$

we consider A-B as an unseparable pair

$$N_2 = N_1 - N' = 720 - 240 = \underline{\underline{480}} \text{ ways}$$

Example 1.

There is a group of 20 people. How many shaking-hands are there, if anybody shakes anybody else's hand?

$$N = C_{20,2} = \binom{20}{2} = \frac{20!}{2! \cdot 18!} = \frac{19 \cdot 20}{2} = 190$$

This is just the number of unordered pairs (namely, *combinations*) that can be built with a set of 20 elements

$n \quad k=2$

1. There is a group of 20 people. How many shaking-hands are there, if anybody shakes anybody else's hand?

There are no repetitions since nobody shakes their own hand and the order does not matter.

$n = 20, \quad k = 2$

$$N = C_{20,2} = \binom{20}{2} = \frac{20!}{2! \cdot 18!} = \frac{20 \cdot 19}{2}$$

$= \underline{190}$ ways

Exercises on Combinatorics/Probability

Example 1.

A committee of 3 mathematicians and 3 engineers is to be selected from a group of 5 mathematicians and 7 engineers. In how many ways can the committee be appointed if

1. no restrictions are imposed
2. one of the engineers must be in the committee
3. two of the engineers pull back

In the first case

$$N_1 = C_{5,3} \cdot C_{7,3} = \binom{5}{3} \cdot \binom{7}{3} = \frac{5!}{3! \cdot (5-3)!} \cdot \frac{7!}{3! \cdot (7-3)!} = 10 \cdot 35 = 350$$

According to the basic principle of counting, the 3 mathematicians can be chosen in $\binom{5}{3}$ ways, and the 3 engineers can be chosen in $\binom{7}{3}$ ways

Exercises on Combinatorics/Probability

Example 1.

A committee of 3 mathematicians and 3 engineers is to be selected from a group of 5 mathematicians and 7 engineers. In how many ways can the committee be appointed if

1. no restrictions are imposed
2. one of the engineers must be in the committee
3. two of the engineers pull back

In the second case

$$N_2 = C_{5,3} \cdot C_{6,2} = \binom{5}{3} \cdot \binom{6}{2} = \frac{5!}{3! \cdot (5-3)!} \cdot \frac{6!}{2! \cdot (6-2)!} = 10 \cdot 15 = 150$$

The 3 mathematician can be chosen in $\binom{5}{3}$ ways; 2 engineers can be chosen out of 6 (the third one is imposed) in $\binom{6}{2}$ ways

Example 1.

A committee of 3 mathematicians and 3 engineers is to be selected from a group of 5 mathematicians and 7 engineers. In how many ways can the committee be appointed if

1. no restrictions are imposed
2. one of the engineers must be in the committee
3. two of the engineers pull back

In the third case

$$N_3 = C_{5,3} \cdot C_{5,3} = \binom{5}{3} \cdot \binom{5}{3} = \left(\frac{5!}{3! \cdot (5-3)!} \right)^2 = 10^2 = 100$$

The 3 engineers can now be chosen among 5

1. A committee of 3 mathematicians and 3 engineers is to be selected from a group of 5 mathematicians and 7 engineers. In how many ways can the committee be appointed if

1. no restrictions are imposed
2. one of the engineers must be in the committee
3. two of the engineers pull back

First case:
The order does not matter, and repetition is not allowed.

Mathematicians $\longrightarrow C_{5,3} = \binom{5}{3} = \frac{5!}{3! \cdot 2!}$

$= \frac{5 \cdot 4}{2} = \underline{10}$

Engineers $\longrightarrow C_{7,3} = \binom{7}{3} = \frac{7!}{3!\, 4!}$

$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = \underline{35}$

So, $N_1 = C_{5,3} \times C_{7,3} = 10 \times 35 = 350$ ways

Second case:
Since one of the engineer must be included, we only consider 6 engineers to be selected in 2 positions $(C_{6,2}) = 15$

So, $N_2 = C_{5,3} \times C_{6,2} = 10 \times 15$

$= \underline{\underline{150}}$ ways

Third case :

$N_3 = C_{5,3} \times C_{5,3} = \underline{\underline{100}}$ ways

Example 1.

Given a deck of 52 cards, determine the number of ways in which we can consecutively extract 3 cards under the following conditions

1. inserting each card back in the deck before extracting the next one
2. without inserting the card back in the deck

In the first case, the ordered samples are

$$N = D'_{52,3} = 52^3 = 140\,608$$

In the second case, the ordered samples are

$$N = D_{52,3} = \frac{52!}{(52-3)!} = 52 \cdot 51 \cdot 50 = 132\,600$$

1. Given a deck of 52 cards, determine the number of ways in which we can consecutively extract 3 cards under the following conditions

$\rightarrow n$ $\rightarrow k$

1. inserting each card back in the deck before extracting the next one
2. without inserting the card back in the deck

First case :
Permutation because the order matters and considering repetition because we put back the card before extracting the card again

$$N_1 = D'_{52,3} = 52^3 = 140,608$$

Second case :
Permutation because of no repetition.

$$N_2 = D_{52,3} = \frac{52!}{(52-3)!} = \frac{52!}{49!} = 52 \cdot 51 \cdot 50$$

$$= 132,600$$

Part II

Example ①

In the Italian *Lotto* game, 5 numbers are randomly drawn from a pool of 90.

- Choosing 3 numbers, what is the probability that they are all amongst the 5 drawn numbers?
- Choosing 5 numbers, what is the probability of guessing all the drawn numbers?

These kind of games are neat examples of equally likely sample spaces, so probabilities can be calculated making use of the **classical definition** (ratio of *favorable cases* and *total cases*)

In the first case (guess of 3 numbers)

$$p = \frac{\binom{5}{3}}{\binom{90}{3}} = \frac{10}{117\,480} = \frac{1}{11\,748} \simeq 8.512 \cdot 10^{-5}$$

Exercises on Combinatorics/Probability

Example

In the Italian *Lotto* game, 5 numbers are randomly drawn from a pool of 90.

- Choosing 3 numbers, what is the probability that they are all amongst the 5 drawn numbers?
- Choosing 5 numbers, what is the probability of guessing all the drawn numbers?

These kind of games are neat examples of equally likely sample spaces, so probabilities can be calculated making use of the **classical definition** (ratio of *favorable cases* and *total cases*)

In the second case (guess of all the 5 numbers)

$$p = \frac{\binom{5}{5}}{\binom{90}{5}} = \frac{1}{43\,949\,268} \simeq 2.28 \cdot 10^{-8}$$

equally likely

1. In the Italian *Lotto* game, 5 numbers are randomly drawn from a pool of 90.

- Choosing 3 numbers, what is the probability that they are all amongst the 5 drawn numbers?
- Choosing 5 numbers, what is the probability of guessing all the drawn numbers?

First case:

favorable cases

$$P = \frac{M}{N}$$

all possible events

the order is irrelevant

$$N = \binom{n}{k} = \binom{90}{3}$$

$$= \frac{90!}{3!\,87!} = \frac{90 \cdot 89 \cdot 88}{6}$$

$$= 117{,}480$$

Let's say we have number

3 21 32 40 63

And the players asked to enseir ther nubers:

$$M = \binom{n}{k} = \binom{5}{3} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 10$$

$$P = \frac{10}{117{,}480}$$

Second case:

$$P = \frac{M}{N} = \frac{\binom{5}{5}}{\binom{90}{5}} = \frac{1}{43{,}949{,}268} = 2.28 \cdot 10^{-8}$$

Example 1.

In the Italian Lotto game, what is the probability of guessing 3 of the 5 drawn numbers, betting on 5 numbers?

The number of "possible bets" is $\binom{90}{5} = 43\,949\,268$

The number of "winning bets" (sets of 5 numbers containing 3 of the 5 drawn numbers, and 2 of the 85 non-drawn numbers) is calculated combining the number of sets of 5 numbers containing 3 of the 5 drawn numbers, and 2 of the 85 remaining numbers: the 3 winning numbers can be chosen in $\binom{5}{3}$ ways; the other 2 numbers can be chosen in $\binom{85}{2}$ ways

$$p = \frac{\binom{5}{3} \cdot \binom{85}{2}}{\binom{90}{5}} = \frac{10 \cdot 3\,570}{43\,949\,268} \simeq \frac{1}{1\,231} \simeq 8.12 \cdot 10^{-4}$$

2. In the Italian Lotto game, what is the probability of $\boxed{\text{guessing 3}}$ $\boxed{\text{of the 5 drawn numbers}}$ betting on 5 numbers?

exactly 3 numbers, the other 2 are not

$$P = \frac{M}{N} = \frac{\binom{5}{3} \cdot \binom{85}{2}}{\binom{90}{5}} = \frac{10 \cdot 3{,}570}{43{,}949{,}268}$$

$\binom{85}{2} \rightarrow$ exclusion

$$= 8{,}12 \cdot 10^{-4}$$

it's a product because we have to choose 3 winning and 2 losing numbers

Example (A different approach to the previous exercises)

In the Italian Lotto game, what is the probability of guessing 3 of the 5 drawn numbers, betting on 3 numbers?

Instead of reasoning in terms of combinations, like we did earlier, we may reason in terms of k-permutations as follows:

- In how many ways, n, can we choose our 3 numbers, assuming that the order matters?

$$n = D_{90,3} = \frac{90!}{(90-3)!} = 90 \times 89 \times 88$$

- In how many ways, m, can our 3 chosen numbers match 3 of the 5 extracted numbers?

$$m = D_{5,3} = \frac{5!}{(5-3)!} = 5 \times 4 \times 3$$

Therefore:

$$P = \frac{m}{n} = \frac{5 \times 4 \times 3}{90 \times 89 \times 88} = \frac{1}{11\,748}$$

Exercises on Combinatorics/Probability

Example (A different approach to the previous exercises)

Likewise, if we choose 5 numbers, what is the probability that they match all the 5 extracted numbers?

- In how many ways, n, can we choose our 5 numbers, assuming that the order matters?

$$n = D_{90,5} = \frac{90!}{(90-5)!} = 90 \times 89 \times 88 \times 87 \times 86$$

- In how many ways, m, can our 5 chosen numbers match all the 5 extracted numbers?

$$m = D_{5,5} = \frac{5!}{(5-5)!} = 5 \times 4 \times 3 \times 2 \times 1$$

Therefore:

$$P = \frac{m}{n} = \frac{5 \times 4 \times 3 \times 2 \times 1}{90 \times 89 \times 88 \times 87 \times 86} = \frac{1}{43\,949\,268}$$

Example (A different approach to the previous exercises)

Finally, what is the probability of guessing 3 of the 5 extracted numbers, betting on 5 numbers?

- In how many ways, n, can we choose our 5 numbers?

$$n = D_{90,5} = \frac{90!}{(90-5)!} = 90 \times 89 \times 88 \times 87 \times 86$$

- In how many ways, m, can we choose 3 number from the "extracted numbers" and 2 among the "non-extracted numbers"? (Note that the $C_{5,3} = \binom{5}{3}$ is due to the fact that the five numbers can be shuffled!)

$$m = (D_{5,3} \cdot D_{85,2}) \cdot C_{5,3} = (5 \times 4 \times 3) \times (85 \times 84) \times \binom{5}{3}$$

Therefore:

$$P = \frac{m}{n} = \frac{5 \times 4 \times 3 \times 85 \times 84 \times 10}{90 \times 89 \times 88 \times 87 \times 86} = \frac{1}{1\,231}$$

Example 3.

What is the probability of having two aces when drawing simultaneously two cards from a deck of 52?

The N possible ways of picking 2 cards from a deck of 52 are

$$N = \binom{52}{2} = 1\,326$$

The *favorable cases* (two aces) are

$$n = \binom{4}{2} = 6$$

so

$$p = \frac{n}{N} = \frac{6}{1\,326} = \frac{1}{221} \simeq 0.0045$$

3. What is the probability of having two aces when drawing simultaneously two cards from a deck of 52?

N possible ways to pick 2 cards from 52 cards

$$N = \binom{52}{2} = \frac{52!}{2!\ 50!} = \frac{52 \cdot 51}{2} = 1,326$$

n possible ways to pick 2 aces from 4 other aces

$$n = \binom{4}{2} = \frac{4!}{2!\ 2!} = 6$$

$$p = \frac{n}{N} = \frac{6}{1,326} = 0.045$$

Exercises on Combinatorics/Probability

Example

P.

Randomly choosing 2 balls from a box containing 4 red balls and 3 blue balls, what is the probability that the extracted balls are of different color?

The number of possible sets of 2 balls that can arranged from a box of 7 is

$$N_{total} = \binom{7}{2} = 21$$

The *favorable events* (one red ball and one blue ball) are

$$N_{success} = \binom{4}{1} \cdot \binom{3}{1} = 12$$

and so

$$p = \frac{N_{success}}{N_{total}} = \frac{12}{21} \simeq 0.57$$

4. Randomly choosing 2 balls from a box containing 4 red balls and 3 blue balls, what is the probability that the extracted balls are of different color?

picking 2 balls from all 7 balls

$$N_{total} = \binom{7}{2} = \frac{7!}{5!\,2!} = \frac{7 \cdot 6}{2} = \underline{\underline{21}}$$

$$N_{success} = \binom{4}{1} \cdot \binom{3}{1} = 4 \cdot 3 = \underline{\underline{12}}$$

$$p = \frac{12}{21} \simeq 0.57$$

Example

Randomly choosing 2 balls from a box containing 4 red balls and 3 blue balls, what is the probability that the extracted balls are of different color?

Alternate solution The probability that the first drawn ball is blue is $3/7$, and then that the second drawn ball is red is $4/6$ (after extracting the blue ball, we are left with 6 ball, 4 of which are red). Analogously, the probability that the first extracted ball is red and the second is blue are, respectively, $4/7$ and $3/6$. Therefore

$$p = \frac{3}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{3}{6} = \frac{12}{21}$$

Exercises on Combinatorics/Probability

Example

If 20 people are present in a room, what is the probability that at least two of them celebrate their birthday on the same day of the year?

Let E_{20} be the event "at least two people have b-day on the same day"

$$\mathcal{P}\left(E_{20}\right) = 1 - \mathcal{P}\left(\overline{E_{20}}\right) \qquad \mathcal{P}\left(\overline{E_{20}}\right) = \frac{\left|\overline{E_{20}}\right|}{|\Omega|}$$

Neglecting leap years

$$|\Omega| = D'_{365,20} = 365 \cdot 365 \cdot \ldots \cdot 365 = 365^{20}$$

$$\left|\overline{E_{20}}\right| = D_{365,20} = 365 \cdot 364 \cdot \ldots \cdot 346 = \frac{365!}{(365-20)!} = \frac{365!}{345!}$$

$$\Rightarrow \quad \mathcal{P}\left(E_{20}\right) = 1 - \mathcal{P}\left(\overline{E_{20}}\right) = 1 - \frac{365 \cdot 364 \cdot \ldots \cdot 346}{365^{20}} \simeq 0.411$$

5. If 20 people are present in a room, what is the probability that at least two of them celebrate their birthday on the same day of the year?

Say E_{20} is an event for two people have the same birthday.

Probability (p) of E_{20} = $1 - p(\bar{E}_{20})$

$$p(\bar{E}_{20}) = \frac{|\bar{E}_{20}|}{|\Omega|}$$

$|\Omega| \rightarrow$ How many possible ways for having a birthday

$$|\Omega| = 365 \cdot 365 \cdot 365 \cdot \ldots = 365^{20}$$

$$|\bar{E}_{20}| = 365 \cdot 364 \cdot 363 \cdot \ldots \cdot (\underbrace{365-20+1}_{346})$$

$$= \frac{365!}{345!}$$

$$P(E_{20}) = 1 - \frac{365!/345!}{365^{20}} \simeq 0.411$$

Example

If n people are present in a room, what is the probability that at least two of them celebrate their birthday on the same day of the year?

$$\mathcal{P}(E_n) = 1 - \frac{365!/(365-n)!}{365^n}$$

n	10	20	30	50	70
$\mathcal{P}(E_n)$	0.117	0.411	0.706	0.970	0.999

(Note that if $n > 365$, the probability is 1)