NONLINEAR CHANGES OF COORDINATES

In the first module we considered changes of coordinates of the form

$$x \mapsto z \doteq Tx$$

The the dynamic equations changed as:

$$\dot{x} = Ax + Bu \quad \longmapsto \quad \dot{z} = T(Ax + Bu)\Big|_{x = T^{-1}z} = TAT^{-1}z + TBu$$

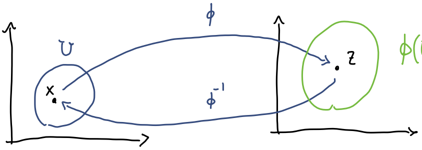
We now develop a similar concept for nonlinear systems

Let $U \subset \mathbb{R}^n$ be open. A function $\phi : U \to \mathbb{R}^n$ is called a DIFFEOMORPHISM if:

- It is INVERTIBLE : $\exists \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x \in U$, $\phi^{-1}(\phi(x)) = x$.
- Both ϕ and ϕ^{-1} are SMOOTH on U

partial derivatives of any order exist and are continuous

[this can be relaxed by asking ϕ and ϕ^{-1} to be just C^1]



$$\phi(U) = \bigcup_{u \in U} \phi(u)$$

If $U = \mathbb{R}^n$, then ϕ is called a GLOBAL DIFFEOMORPHISM

The following result gives sufficient conditions for $\phi : \mathbb{R}^n \to \mathbb{R}^n$ to be a diffeomorphism

RESULT (INVERSE FUNCTION THEOREM). Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be smooth, and let $\bar{x} \in \mathbb{R}^n$ be such that the Jacobian of g computed at $x = \bar{x}$, i.e.

$$\frac{d}{dx} g(x)\Big|_{x=\bar{x}} = \frac{dg}{dx}(\bar{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial g_1}{\partial x_n}(\bar{x}) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial g_n}{\partial x_n}(\bar{x}) \end{pmatrix}$$

is non-singular.

Then, there exists an open set $U \subset \mathbb{R}^n$ containing $\bar{x}$ such that the restriction $\phi : U \to \mathbb{R}^n$, $\phi(x) = g(x)$ of g on U is a diffeomorphism. (notice that g and ϕ are different functions as they have different domains)

Dealing with non-linear system. Say a non-linear system has $\begin{cases} \dot{x} = f(x, u). \\ y = h(x) \end{cases}$ Now we are going to construct non-linear canonical form.

$u, y \in \mathbb{R}$, single input

$x \in \mathbb{R}^{n = n_x}$ single output (SISO)

Changing coordinate in non-linear system:

$\phi : U \longrightarrow \mathbb{R}^n$ is defined as DIFFEOMORPHISM. It must hold if:

$U \subset \mathbb{R}^n$ open set

1) $\exists\, \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that each $\forall x \in U$, so it applies $\phi^{-1}(\phi(x)) = x$
2) ϕ and ϕ^{-1} are smooth.

If U ranges the whole $\mathbb{R}^n$ ($U = \mathbb{R}^n$), ϕ is GLOBAL DIFFEOMORPHISM.

INVERSE FUNCTION THEOREM

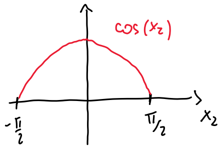
Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ SMOOTH and assume $\exists\, \bar{x} \in \mathbb{R}^n$ such that $\partial g / \partial x (\bar{x})$ is non-singular. Then, $\exists\, U \subset \mathbb{R}^n$ open such that $\bar{x} \in U$ and $\phi : U \longrightarrow \mathbb{R}^n$, $\phi(x) = g(x)$ is a DIFFEOMORPHISM. ($\phi = g|_U$) It's a local result.

EXAMPLE. Let

$$g(x) = \begin{pmatrix} x_1 + x_2 \\ \sin x_2 \end{pmatrix}$$

Its Jacobian is

$$\frac{\partial g}{\partial x}(x) = \begin{pmatrix} 1 & 1 \\ 0 & \cos x_2 \end{pmatrix}$$



For every $\bar{x} \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\frac{\partial g}{\partial x}(\bar{x})$ is non-singular

$\Rightarrow \exists\, U \subset \mathbb{R}^2$ s.t. $\phi : U \to \mathbb{R}^2$ (= restriction of g on U) is a diffeomorphism

In this case $U = \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

$$\phi^{-1}(z) = \begin{pmatrix} z_1 - \arcsin z_2 \\ \arcsin z_2 \end{pmatrix} \qquad \left(\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

CHANGE OF VARIABLES

Given a system of the form

$$\dot{x} = f(x, u) \quad , \quad y = h(x)$$

a diffeomorphism $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ induces a CHANGE OF VARIABLES

$$x \mapsto z \doteq \phi(x) \qquad \left(\text{conversely, } x = \phi^{-1}(z) \text{ on } \phi(U)\right)$$

In the new variables we have:

$$\begin{cases} \dot{z} = \tilde{f}(z, u), \\ y = \tilde{h}(z) \end{cases}$$

$$\boxed{\begin{aligned} \tilde{f}(z,u) &\doteq \left.\frac{d\phi}{dx}(x) \cdot f(x, u)\right|_{x = \phi^{-1}(z)} = \frac{d\phi}{dx}\left(\phi^{-1}(z)\right) \cdot f\left(\phi^{-1}(z), u\right) \\ \tilde{h}(z) &\doteq h\left(\phi^{-1}(z)\right) \end{aligned}}$$

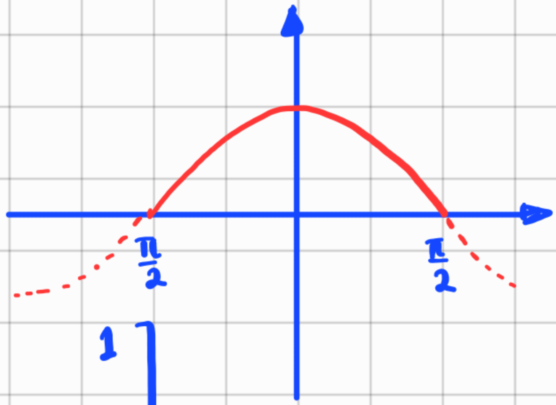
Indeed:

$$\dot{z} = \frac{d}{dt}\left(\phi(x)\right) = \frac{d\phi(x)}{dx} \cdot \dot{x} = \frac{d\phi(x)}{dx} \cdot f(x, u)$$

and $x = \phi^{-1}(z)$.

__Example__: $n=2$

$$g(x) = \begin{bmatrix} x_1 + x_2 \\ \sin x_2 \end{bmatrix}$$



Take a Jacobian $\dfrac{\partial g}{\partial x}(x) = \begin{bmatrix} 1 & 1 \\ 0 & \cos x_2 \end{bmatrix}$

$\dfrac{\partial g}{\partial x}(\bar{x})$ non-singular $\forall \bar{x} \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Say $z = g(x)$ $\begin{cases} x_1 + x_2 = z_1 \rightarrow x_1 = z_1 - \sin^{-1}(z_2) \\ \sin x_2 = z_2 \rightarrow x_2 = \sin^{-1}(z_2) \end{cases}$

$x \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $U = \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\phi(x) = g(x) \longrightarrow \phi^{-1}(z) = \begin{bmatrix} z_1 - \sin^{-1}(z_2) \\ \sin^{-1}(z_2) \end{bmatrix}$

$\hookrightarrow$ for $\forall x \in U$

Why we care DIFFEORMORPHISM? Because it acts like T in $z = Tx$ for linear system. So, $\boxed{z = \phi(x)}$ we can say $\phi(x) = Tx$, and the Jacobian would be $\partial \phi(x) / \partial x = T$.

Consider a generic system:

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases}$$

We map $x \longrightarrow z = \phi(x)$. So:

$$\dot{z} = \frac{\partial \phi}{\partial x}(x) \cdot \dot{x} \Big|_{x = \phi^{-1}(z)} = \frac{\partial \phi}{\partial x}(x)\, f(x,u) \Big|_{x=\phi^{-1}(z)}$$

$$\boxed{= \frac{\partial \phi}{\partial x}(\phi^{-1}(z)) \cdot f(\phi^{-1}(z), u) = \tilde{f}(z,u)}$$

$$y = h(x)\Big|_{x = \phi^{-1}(z)} = \boxed{h(\phi^{-1}(z))^{-1} = \tilde{h}(z)}$$

We can express in z-space:

$$\begin{cases} \dot{z} = \tilde{f}(z,u) \\ y = \tilde{h}(z) \end{cases}$$

In comparison with linear system:

$$\phi(x) = Tx = z \iff x = T^{-1}z = \phi^{-1}z \implies \dot{z} = \underbrace{\frac{\partial \phi}{\partial x}(\phi^{-1}(z))}_{T} \cdot \underbrace{f(\phi^{-1}(z), u)}_{A\underbrace{\phi^{-1}(z)}_{T^{-1}z} + Bu}$$

$$\Downarrow$$

$$f(x,u) = Ax + Bu$$

$$\frac{\partial \phi}{\partial x}(x) = T, \ \forall x$$

$$= TAT^{-1}z + TBu$$

REMARK. If T is invertible then $\phi(x) \doteq Tx$ is a global diffeom.

If $f(x,u) = Ax + Bu$ and $y = Cx$ (hence, we have an LTI system), the previous formula gives:

$$\tilde{f}(z,u) = \underbrace{\frac{d\phi(x)}{dx}}_{= T} \cdot f(x,u) \Big|_{x = \underbrace{\phi^{-1}(z)}_{= T^{-1}z}} = T \cdot f\left(T^{-1}z, u\right)$$

$$= TAT^{-1}z + TBu$$

$$\tilde{h}(z) = h\left(\phi^{-1}(z)\right) = CT^{-1}z$$

So we recover the linear formulas.

EXAMPLE.

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 u \\ \dot{x}_2 = x_1 u \end{cases}$$

change variable through previous $\phi(x) = \begin{pmatrix} x_1 + x_2 \\ \sin x_2 \end{pmatrix}$, $\phi^{-1}(z) = \begin{pmatrix} z_1 - \arcsin z_2 \\ \arcsin z_2 \end{pmatrix}$

$$\tilde{f}(z,u) = \frac{\partial\phi}{\partial x}(x)\, f(x,u) \Big|_{x = \phi^{-1}(z)}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & \cos x_2 \end{pmatrix} \begin{pmatrix} -x_1^3 + x_2 u \\ x_1 u \end{pmatrix} \Bigg|_{x = \phi^{-1}(z)} = \begin{pmatrix} -x_1^3 + (x_1 + x_2) u \\ \cos x_2 \cdot x_1 u \end{pmatrix} \Bigg|_{x = \phi^{-1}(z)}$$

$$= \begin{pmatrix} -(z_1 - \arcsin z_2)^3 + z_1 u \\ (z_1 - \arcsin z_2) \cdot \cos(\arcsin z_2) \cdot u \end{pmatrix}$$

→ In the new coordinates:

$$\begin{cases} \dot{z}_1 = -(z_1 - \arcsin z_2)^3 + z_1 u \\ \dot{z}_2 = (z_1 - \arcsin z_2) \cdot \cos(\arcsin z_2) \cdot u \end{cases}$$

INPUT-AFFINE SYSTEMS

Systems of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad \longleftarrow$$

OUR FOCUS FROM NOW ON

are called "input-affine"

$\rightarrow$ given a general system of the form

$$\begin{cases} \dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}) \\ y = \bar{h}(\bar{x}, \bar{u}) \end{cases}$$

I can "extend" the input $\bar{u}$ by defining

$\dot{\bar{u}} = u$ $\qquad$ u = new input

$x = (\bar{x}, \bar{u})$ $\qquad$ x = new state

The resulting system with input u and state x is input-affine since

$$\begin{aligned} \dot{x} &= f(x) + g(x)\bar{u} \\ y &= h(x) \end{aligned} \qquad \text{with} \qquad \begin{cases} f(x) = \begin{pmatrix} \bar{f}(x) \\ 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ h(x) = \bar{h}(x) \end{cases}$$

$\Rightarrow$ Not a big loss of generality to focus on input-affine systems.

RESULT. "Input-affinit-ness" is preserved under changes of variables:

Proof. If

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

and $\phi : U \to \mathbb{R}$ a diffeom.

Then $z = \phi(x)$ satisfies

$$\dot{z} = \frac{\partial\phi}{\partial x}(x) \cdot \dot{x}\Big|_{x=\phi^{-1}(z)} = \underbrace{\frac{\partial\phi}{\partial x}(x)\, f(x)\Big|_{x=\phi^{-1}(z)}}_{\tilde{f}(z)} + \underbrace{\frac{\partial\phi}{\partial x}(x)\, g(x)\Big|_{x=\phi^{-1}(z)} \cdot u}_{\tilde{g}(z)\cdot u}$$

$$y = h(x) = \underbrace{h(\phi^{-1}(z))}_{\tilde{h}(z)}$$

$\square$

INPUT AFFINE SYSTEM

$\xrightarrow{} Au+b \to$ but with bias b

$\to$ affine on u

$$\begin{cases} \dot{x} = f(x) + \underbrace{g(x)\cdot u} \\ y = h(x) \end{cases}$$

$\to$ affined on u

Assume we have:

$$\begin{cases} \dot{x} = \tilde{f}(\tilde{x}, u) \\ y = \tilde{h}(\tilde{x}) \\ \dot{u} = v \end{cases} \longrightarrow x = \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}$$

we treat u as a state

$$\dot{x} = \begin{bmatrix} \tilde{f}(\overbrace{\tilde{x}, u}^{x}) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$\dot{x} = f(x) + g(x)v$$

$\longrightarrow$ Now, we can affine the system on the input in form of v

The structure of affined system is not changed when we change the coordinate.

$$z = \phi(x)$$

$$\dot{z} = \frac{\partial \phi}{\partial x}(x) \cdot f(x) + \frac{\partial \phi}{\partial x}(x)\, g(x) u \Big|_{x = \phi^{-1}(z)}$$

$$= \underbrace{\frac{\partial \phi}{\partial x}(\phi^{-1}(z)) \cdot f(\phi^{-1}(z))}_{\tilde{f}(z)} + \underbrace{\frac{\partial \phi}{\partial x}(\phi^{-1}(z))\, g(\phi^{-1}(z)) u}_{\tilde{g}(z)u}$$

$$\boxed{\begin{cases} \dot{z} = \tilde{f}(z) + \tilde{g}(z)u \\ y = h(\phi^{-1}(z)) = \tilde{h}(z) \end{cases}}$$

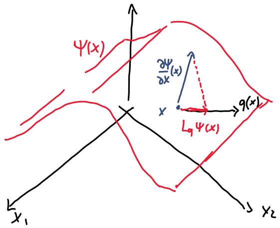
LIE DERIVATIVES

Let $n \in \mathbb{N}$, $q : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}$. The <u>LIE DERIVATIVE of ψ along q</u> is the function

$$L_q \psi : \mathbb{R}^n \to \mathbb{R}$$

defined as

$$L_q \psi(x) = \frac{\partial \psi}{\partial x}(x) \cdot q(x) = \left(\frac{\partial \psi}{\partial x_1}(x) \; \cdots \; \frac{\partial \psi}{\partial x_n}(x) \right) \begin{pmatrix} q_1(x) \\ \vdots \\ q_n(x) \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(x)\, q_i(x)$$



$$L_q \psi(x) = \text{projection of } \frac{\partial \psi}{\partial x}(x) \text{ on } q(x)$$

REMARKS

1) $L_q \psi : \mathbb{R}^n \to \mathbb{R}$ $\Rightarrow$ we can define $L_q(L_q \psi)$ as

$$L_q(L_q \psi)(x) = \frac{\partial}{\partial x}\left(L_q \psi(x) \right) \cdot q(x)$$

we use the notation

$$L_q^2 \psi \doteq L_q(L_q \psi)$$

we can as well define

$$L_q^3 \psi \doteq L_q(L_q^2 \psi), \quad L_q^4 \psi \doteq L_q(L_q^3 \psi), \quad \ldots \quad \boxed{L_q^k \psi = L_q L_q^{k-1} \psi}$$

2) If $d : \mathbb{R}^n \to \mathbb{R}^n$ is another function, we can also "mix" Lie derivatives:

$$L_d L_q \psi = \frac{\partial L_q \psi}{\partial x}(x) \cdot d(x)$$

$$L_d^2 L_q^3 \psi, \quad L_q L_d L_q \psi, \quad \ldots$$

LIE DERIVATIVE → the results are linear

↳ constructed on two function

$$q : \mathbb{R}^n \to \mathbb{R}^n, \quad \psi : \mathbb{R}^n \to \mathbb{R}$$

The LIE DERIVATIVE of ψ along q:

$$L_q \psi : \mathbb{R}^n \to \mathbb{R}$$

$$L_q(L_q \psi) = L_q^2 \psi$$
$$L_q^k \psi$$

$$L_q \psi(x) = \frac{\partial \psi}{\partial x}(x)\, q(x)$$

↳ der. on q direction

$$= \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(x)\, q_i(x)$$

→ it's a directional derivative

It simplifies the derivative in changed coordinate

$$\dot{y} = \frac{\partial h}{\partial x}(x)\, \dot{x} = \frac{\partial h}{\partial x}(x) \cdot f(x) + \frac{\partial h}{\partial x}(x)\, g(x) u$$

$$\boxed{\dot{y} = L_f h(x) + L_g h(x) u}$$

↳ LIE DERIVATIVE of f

↳ LIE DERIVATIVE of g

If we take further derivative

$$\ddot{y} = L_f^2 h(x) + L_f(L_g h(x)) u + \ldots\ldots$$

We can conclude $q : \mathbb{R}^n \to \mathbb{R}^n$, $\psi : \mathbb{R}^n \to \mathbb{R}$, a vector with same dimension of q, named $p : \mathbb{R}^n \to \mathbb{R}^n$

$$L_p(L_q \psi) = L_p L_q \psi$$
$$L_p^3 L_q^2 \psi$$

The linearity of LIE DERIVATIVE

$q : \mathbb{R}^n \to \mathbb{R}^n, \quad \psi_1 : \mathbb{R}^n \to \mathbb{R}$

$\psi_2 : \mathbb{R}^n \to \mathbb{R}$

$$\boxed{L_q(\alpha_1 \psi_1 + \alpha_2 \psi_2) = \alpha_1 L_q \psi_1 + \alpha_2 L_q \psi_2}$$

$\forall \alpha_1, \alpha_2 \in \mathbb{R}$

3) The operator $L_g : \psi \mapsto L_g \psi$ is LINEAR :

$$\forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ and } \psi_1, \psi_2 : \mathbb{R}^n \to \mathbb{R}, \quad L_g(\alpha_1 \psi_1 + \alpha_2 \psi_2) = \alpha_1 L_g \psi_1 + \alpha_2 L_g \psi_2$$

4) Consider an input-affine system "SISO" (single-input-single-output) (namely $n_u = n_y = 1$)

$$\begin{cases} \dot{x} = f(x) + g(x) u \\ y = h(x) \end{cases}$$

Then we have

$$\dot{y} = \frac{d}{dt} h(x) = \frac{\partial h}{\partial x}(x) \cdot \dot{x} = \overbrace{\frac{\partial h}{\partial x}(x)\big(f(x) + g(x) u\big)}^{\text{Lie derivative of the output map along the dynamics}} = L_h f(x) + L_h g(x) \cdot u$$

RELATIVE DEGREE

Consider a SISO system $(x \in \mathbb{R}^n$ and $u, y \in \mathbb{R})$

$$\begin{cases} \dot{x} = f(x) + g(x) u \\ y = h(x) \end{cases}$$

and a point $\bar{x} \in \mathbb{R}^n$

To see how $u(t)$ affects the output $y(t)$ around $\bar{x}$, we start taking derivatives:

$$\dot{y} = L_f h(x) + L_f g(x) u$$

the input is "ONE DERIVATIVE AWAY"

If $L_f g(\bar{x}) \neq 0$ we stop and say that the system has RELATIVE DEGREE $r = 1$ at $\bar{x}$

If, instead, $L_f g(x) = 0$ $\forall x$ in an open set around $\bar{x}$, we keep going:

at $x(t) = \bar{x}$ we have

$$\ddot{y} = \frac{d}{dt}\Big(L_f h(x) + \underbrace{\cancel{L_f g(x) u}}_{=0}\Big) = L_f^2 h(x) + L_g L_f h(x) \cdot u$$

If $L_g L_f h(\bar{x}) \neq 0$ $\rightarrow$ we stop and say the RELATIVE DEGREE at $\bar{x}$ is $r = 2$

If $L_g L_f h(x) = 0$ in an open set around $\bar{x}$ we keep going

...

we stop (if possible) when we find $r > 0$ such that $L_g L_f^{r-1} h(\bar{x}) \neq 0$

Intuitively: RELATIVE DEGREE = number of times I need to differentiate the output to have the input appearing

RELATIVE DEGREE → depends on the state

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

First, fix $\bar{x} \in \mathbb{R}^n$, then we differentiate the output.

$$\dot{y} = \frac{\partial h}{\partial x} \dot{x} = \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h}{\partial x}(x) g(x) u$$

$$\dot{y} = L_f h(x) + L_g h(x) \cdot u$$

We verify:

1) $L_g h(\bar{x}) \neq 0$. Thus it affects the input u. So the relative degree $r = 1$ of $\bar{x}$. OR, goes to 2)

2) $L_g h(\bar{x}) = 0$
   - ↳ Case 1 : $L_g h(x) = 0$, not only on $\bar{x}$, but also in every $\forall x \in U$, where U is an open set, and $\bar{x} \in U$.
   - ↳ Case 2: Otherwise, we say the system does not have relative degree $r = 1$. We have to take a higher derivative. (We can't verify further/useless)

With case 1, we can take $\dot{y} = L_f h(x)$, $x \in U$.

$$\ddot{y} = \partial/_{\partial x} (L_f h(x))(f(x) + g(x)u)$$

In other word: $\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u$

We'll find that $L_g L_f h(\bar{x}) \neq 0$ in relative degree $r=2$. We should keep differenciating until condition 1) is fulfilled and find higher relative degree.

Conclusion:

The system has REL. DEGREE $r > 0$ at $\bar{x}$, if:

1) $L_g L_f^k h(x) = 0 \quad \forall k < r-1$

$\leq r-2$

$\forall x \in U$ open, $\bar{x} \in U$

2) $L_g L_f^{r-1} h(\bar{x}) \neq 0$

all $\mathbb{R}^n$

If system has REL. DEGREE r at $\bar{x} \in \mathbb{R}^n$, then it has GLOBAL REL. DEGREE r.

$\Downarrow$

RESULT. If system has REL. DEGREE r at $\bar{x}$, then $r \leq n$.

In linear system we have $f(x) = Ax$, $g(x) = B$, $h(x) = C(x)$. Then the system is:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

In non-linear system:

$L_f h(x) = CAx$, $L_f^2 h(x) = CA^2 x$, $L_f^k h(x) = CA^k x$,

$L_g L_f^k h(x) = CA^k B$. not dependent on x

Because of those properties, it implies the condition:

1) $CA^k B = 0 \quad \forall k = 0, \dots, r-2$ (not depend on x)

2) $CA^{r-1} B \neq 0$

Formally, the system has <u>RELATIVE DEGREE r AT $\bar{x}$</u> if

1) $L_g L_f^k h(x) = 0$ in an open set around $\bar{x}$ $\forall k = 0, \dots, r-2$

2) $L_g L_f^{r-1} h(\bar{x}) \neq 0$ (by continuity $L_g L_f^{r-1} h(x) \neq 0$ in an open set around $\bar{x}$)

the fact that these properties must hold in an open set around $\bar{x}$ means that having a given relative degree is a "ROBUST" property: if I move a bit off $\bar{x}$ I do not lose it

<u>REMARKS.</u>

1) r depends on $\bar{x}$

2) r may not exist

<u>RESULT.</u> If r exists, then $r \leq n$

3) extension to MIMO systems non-trivial

4) for LTI systems: $h(x) = Cx$, $L_f^k h(x) = CA^k x$, $L_g L_f^k h(x) = CA^k B$

↳ the system has relative degree r at any $\bar{x}$ if

1) $CA^k B = 0$ $\forall k = 0, \dots, r-2$

2) $CA^{r-1} B \neq 0$

↳ r equals the difference between the order of the numerator and the denominator of the transfer function $G(s) = C(sI - A)^{-1} B$

↳ r does not depend on $\bar{x}$ (we can speak of "RELATIVE DEGREE of the system")

↳ If (A, B) is controllable then r exists

↳ If $\mathcal{R}^+ \cap (\mathcal{E}^+)^\perp \neq 0$, then r exists

If a system has relative degree r at every $\bar{x} \in \mathbb{R}^n$ we say it has <u>GLOBAL REL. DEGREE r</u>

EXAMPLES (VAN DER POL OSCILLATOR)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + (1-x_1^2)\,x_2 + u \end{cases} \quad \leftarrow \quad \dot{x} = f(x) + g(x)u \quad \text{with}$$

$$f(x) = \begin{pmatrix} x_2 \\ -x_1 + (1-x_1^2)x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we consider two outputs:

CASE I) $y = x_1$

CASE II) $y = \sin x_2$

CASE I $\quad h(x) = x_1$

- $L_g h(x) = \frac{\partial h}{\partial x}(x) \cdot g(x) = (1 \quad 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$r \neq 1 \quad \forall \bar{x} \in \mathbb{R}^n$. Keep going:

- $L_f h(x) = \frac{\partial h}{\partial x}(x) f(x) = (1 \quad 0)\begin{pmatrix} x_2 \\ -x_1 + (1-x_1^2)x_2 \end{pmatrix} = x_2$
- $L_g L_f h(x) = (0 \quad 1) \cdot g(x) = 1$

$\Rightarrow$ the system has GLOBAL REL. DEGREE $r = 2$

CASE II $\quad (h(x) = \sin x_2)$

$$L_g h(x) = \frac{\partial h}{\partial x}(x)\, g(x) = (0 \quad \cos x_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos x_2$$

$\Rightarrow r = 1$ at every $\bar{x} \in \left\{ x \in \mathbb{R}^2 : x_2 \neq k\frac{\pi}{2},\ k \in \mathbb{Z} \right\}$

If $\bar{x} = \frac{\pi}{2}$ we CANNOT PROCEED since for every open set $U \ni \bar{x}$ there exists $x \in U$ such that $L_g h(x) \neq 0$

$\hookrightarrow$ The system does not have relative degree at $\bar{x} = k\frac{\pi}{2}$, $k \in \mathbb{Z}$

Example: (Van der Pol Oscillation)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + (1 - x_1^2) x_2 + u \end{cases}$$

CASE 1 : $y = x_1$
CASE 2 : $y = \sin x_2$

} Let's determine the relative degree!

CASE 1

$$\dot{x} = f(x) + g(x) u$$

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + (1 - x_1^2) x_2 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad h(x) = x_1$$

$$L_g h(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \quad \text{for every } x \in \mathbb{R}^n$$

$$L_f h(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 + (1 - x_1^2) x_2 \end{bmatrix} = x_2$$

$$L_g L_f h(x) = \frac{\partial}{\partial x} (L_f h(x)) \cdot g(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

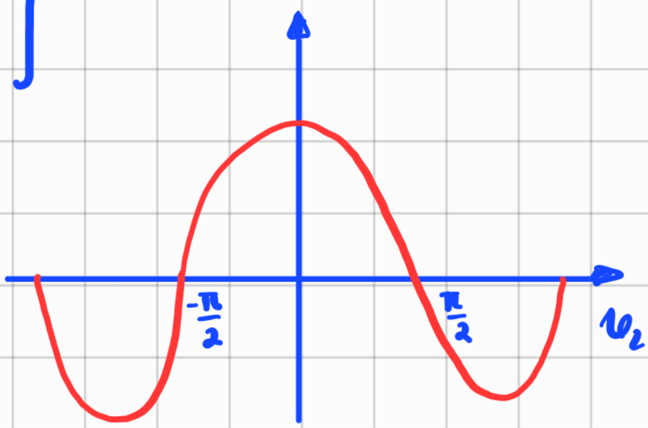
$= 1$ for every $x \in \mathbb{R}^n$

is $\neq 0$

CASE 2

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + (1 - x_1^2)x_2 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad h(x) = \sin x_2$$

$$L_g h(x) = \begin{bmatrix} 0 & \cos x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cos x_2$$



$r = 1$ at every $\bar{x}$ such that $\bar{x}_2 \neq k\frac{\pi}{2}$, k odd

What if we take $\bar{x} : \bar{x}_2 = \frac{\pi}{2}$,

$L_g h(\bar{x}) = 0$. Should we go on?

$r = 2$, $L_g h(x) = 0$, $\forall x \in U$ open $\ni \bar{x}$

$L_g L_f h(\bar{x}) \neq 0$

See reference Isidori, Ch.4

We consider

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

assumably has REL. DEGREE $r \leq n$ at $\bar{x} \in \mathbb{R}^n$. The result:

$$\frac{\partial}{\partial x} \underbrace{\begin{bmatrix} h(\bar{x}) \\ L_f h(\bar{x}) \\ L_f^2 h(\bar{x}) \\ \vdots \\ L_f^{r-1} h(\bar{x}) \end{bmatrix}}_{n} \left.\vphantom{\begin{bmatrix} a\\a\\a\\a\\a \end{bmatrix}}\right\} r = r \text{ (full rank)}$$

We can say the coord. change matrix ϕ

$\phi_1(x) = h(x)$

$\phi_2(x) = L_f h(x)$

$\vdots$

$\phi_r(x) = L_f^{r-1} h(x)$

$$\Rightarrow \frac{\partial}{\partial x}\begin{bmatrix}\phi_1(\bar{x})\\ \vdots \\ \phi_r(\bar{x})\end{bmatrix} = \text{full rank}$$

Find $\phi_{r+1}(x), \dots, \phi_n(x)$ such that:

- the function $\bar{\phi}(x) = \begin{bmatrix}\phi_1(x)\\ \vdots \\ \phi_r(x)\\ \phi_{r+1}(x)\\ \vdots \\ \phi_n(x)\end{bmatrix}$

- $\text{rank } \frac{\partial}{\partial x}\bar{\phi}(\bar{x}) = n \iff \frac{\partial}{\partial x}\bar{\phi}(\bar{x})$ non singular

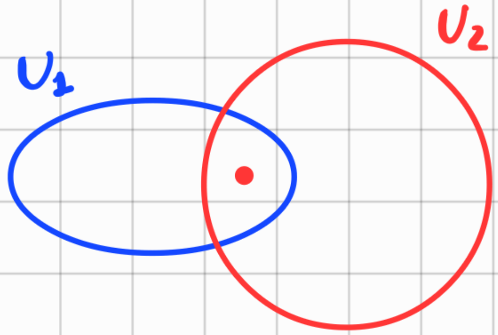
- $\forall i = r+1, \dots, n, \ L_g \phi_i(x) = 0, \ \forall x$ in some open $\bar{U} \ni \bar{x}$

RESULT: such a choice of $\phi_{r+1}, \dots, \phi_n$ always exists

(ANOTHER)
RESULT: There exists $U \subset \mathbb{R}^n$ open and containing $\bar{x}$ such that $\phi = \underbrace{\phi|_U : U \to \mathbb{R}^n}_{\text{inverse function theorem}}$ is a DIFFEOMORPHISM.

$$\forall x \in U, \quad \phi(x) = \bar{\Phi}(x) = \begin{bmatrix} \phi_r(x) \\ \\ \phi_n(x) \end{bmatrix} \quad \text{and}$$

$$\forall x \in U, \quad L_g \phi_i(x) = 0 \quad \forall i = r+1, \dots, n$$



We now change the coordinate :

$$x \longrightarrow z = \phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \\ \hline \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \xi \in \mathbb{R}^n \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \eta \in \mathbb{R}^{n-r} \end{array}$$

$$\downarrow$$

$$z = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$\phi_1(x) = h(x), \quad y = \xi_1$$

Let's see what component that satisfies

$$\dot{\xi}_1 = \dot{\phi}_1(x) = \frac{\partial}{\partial x} \phi_1(x) \left(f(x) + g(x) u \right)$$

$$= L_f \phi_1(x) + L_g \phi(x) u$$

$$= \underbrace{L_f h(x)}_{\phi_2(x)} + \underbrace{\cancel{L_g h(x)}}_{=0} u = \phi_2(x) = \xi_2$$

We have $\phi_1(x) = h(x)$, $\phi_2(x) = L_f h(x)$, $\dots$

$\phi_i(x) = L_f^{i-1} h(x)$ $\to$ $i \leq r$

$$\dot{\xi}_2 = \dot{\phi}_2(x) = \frac{\partial}{\partial t}\left(L_f h(x)\right) = \underbrace{L_f^2 h(x)}_{\phi_3} + \underbrace{\cancel{L_g L_f h(x)}}_{=0} u$$

$$= \phi_3(x) = \xi_3$$

It implies that $\dot{\xi}_i = \xi_{i+1} \quad \forall i = 1, \dots, r-1$

We arrive at $\dot{\xi}_r = \dot{\phi}_r(x) = \dfrac{d}{dt}\left(L_f^{r-1} h(x)\right)$

$$= L_f^r h(x) + L_g L_f^{r-1} h(x) u$$

$$= \underbrace{L_f^r h(\phi^{-1}(\xi, \eta))}_{q(\xi,\eta)} + \underbrace{L_g L_f^{r-1} h(\phi^{-1}(\xi, \eta))}_{b(\xi,\eta)} u$$

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3$$
$$\vdots$$
$$\dot{\xi}_{r-1} = \xi_r$$
$$\boxed{\dot{\xi}_r = q(\xi,\eta) + b(\xi,\eta) u}$$

$$\eta_1 = \phi_{r+1}(x)$$
$$\dot{\eta}_1 = \dot{\phi}_{r+1}(x) = L_f \phi_{r+1}(x) + L_g \phi_{r+1}(x) u$$

$\forall i = r+1, \dots, n \quad \dot{\eta}_i = L_f \phi_i(x) + \underbrace{\cancel{L_g \phi_i(x)}}_{=0} u$

$L_g \phi_i(x) = 0 \ \forall x \in U$, $i = r+1, \dots, n$

We now have a generic equation:

$$\boxed{\dot{\eta}_i = L_f \phi_i(\phi^{-1}(\xi, \eta))} \to \psi_i(\xi, \eta)$$

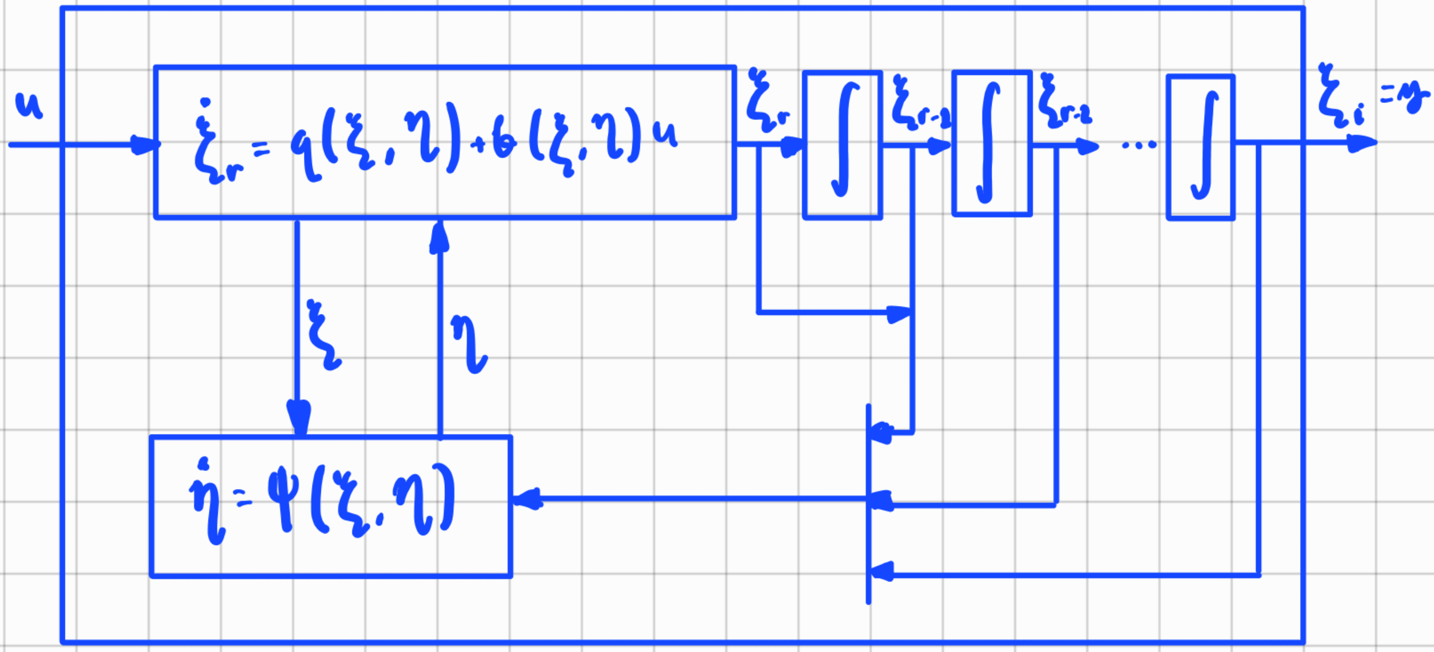
Together with $\boxed{\dot{\xi}_r = q(\xi, \eta) + b(\xi, \eta)u}$

$$\boxed{y = \xi_1}$$

THE NORMAL FORM

$$\boxed{\dot{\eta} = \psi(\xi, \eta)}$$

$$\xi_1 = y$$
$$\xi_2 = \dot{\xi}_1 = \dot{y}$$
$$\xi_3 = \dot{\xi}_2 = \ddot{y}$$
$$\vdots$$
$$\xi_r = y^{(r-1)}$$



NORMAL FORM

Consider the system (SISO)

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad x \in \mathbb{R}^n, \qquad u, y \in \mathbb{R}$$

RESULT. If the system has relative degree $r \leq n$ at $\bar{x}$, then

$$\operatorname{rank} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} h(\bar{x}) \\ \frac{\partial}{\partial x} L_f h(\bar{x}) \\ \vdots \\ \frac{\partial}{\partial x} L_f^{r-1} h(\bar{x}) \end{bmatrix}}_{\in \mathbb{R}^{r \times n}} = r$$

we can define

$$\phi_1(x) \doteq h(x), \quad \phi_2(x) \doteq L_f h(x), \quad \dots, \quad \phi_r(x) = L_f^{r-1} h(x)$$

By the above result, the matrix

$$\frac{\partial}{\partial x} \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \end{bmatrix} \in \mathbb{R}^{r \times n}$$

has full rank r at $x = \bar{x}$.

Let us find $\Phi_{r+1}, \dots, \Phi_n$ such that: (always possible)

1) The matrix

$$\frac{\partial}{\partial x} \begin{bmatrix} \Phi_1(x) \\ \vdots \\ \Phi_r(x) \\ \Phi_{r+1}(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix} \in \mathbb{R}^{n \times n}$$

has full rank $= n$ at $x = \bar{x}$ ($\Leftrightarrow$ it is nonsingular)

2) $\exists\, \bar{U} \subset \mathbb{R}^n$ open and containing $\bar{x}$ such that

$$L_g \Phi_i(x) = 0 \quad , \quad \forall x \in \bar{U} \quad , \quad \forall i = r+1, \dots, n$$

↑ This is always possible (see ISIDORI, Nonlinear Control Systems)

Then we have constructed a function $\bar{\Phi} : \bar{U} \to \mathbb{R}^n$ as

$$\bar{\Phi}(x) = \left(\phi_1(x), \dots, \phi_n(x) \right)$$

that satisfies

$$\operatorname{rank} \frac{\partial}{\partial x} \Phi(\bar{x}) = n$$

By the implicit function theorem $\exists\, U \subset \bar{U}$ open such that $\bar{x} \in \bar{U}$ and $\phi : U \to \mathbb{R}^n$ defined by

$$\phi(x) = \left(\phi_1(x), \dots, \phi_n(x) \right) \qquad \left(= \text{restriction of } \bar{\Phi} \text{ on } U \right)$$

is a DIFFEOMORPHISM

↓

We now change coordinates by using this ϕ : (only valid for $x \in U$)

$$x \longmapsto z = \phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_r(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} = \xi \in \mathbb{R}^r \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} = \eta \in \mathbb{R}^{n-r} \end{array}$$

In the new coordinates we have:

$$y = \xi_1 \quad , \quad \xi = (\xi_1, \dots, \xi_r) \quad , \quad \eta = (\eta_1, \dots, \eta_{n-r})$$

and ξ satisfies the following equations:

$$\dot{\xi}_1 = \dot{h}(x) = \frac{\partial h(x)}{\partial x}\left(f(x)+g(x)u\right) = \overset{\xi_2}{\overset{\shortparallel}{L_f h(x)}} + \overset{=0}{\cancel{L_g h(x)}}\cdot u = \xi_2$$

$$\dot{\xi}_2 = \ddot{h}(x) = \frac{d}{dt}\left(L_f h(x)\right) = \overset{\xi_3}{\overset{\shortparallel}{L_f^2 h(x)}} + \overset{=0}{\cancel{L_g L_f h(x)}}\, u = \xi_3$$

$$\vdots$$

$$\dot{\xi}_{r-1} = \frac{d^{(r-1)}}{dt^{r-1}} h(x) = \overset{\xi_r}{\overset{\shortparallel}{L_f^{r-1} h(x)}} + \overset{=0}{\cancel{L_g L_f^{r-2} h(x)}}\, u = \xi_r$$

$$\dot{\xi}_r = L_f^r h(x) + L_g L_f^{r-1} h(x)\, u$$

$$= \underbrace{L_f^r h\left(\phi^{-1}(\xi,\eta)\right)}_{\doteq\, q(\xi,\eta)} + \underbrace{L_g L_f^{r-1} h\left(\phi^{-1}(\xi,\eta)\right)}_{\doteq\, b(\xi,\eta)}\cdot u$$

$$= q(\xi,\eta) + b(\xi,\eta)\cdot u$$

Instead, η satisfies:

$$\forall i = r+1, \dots, n, \qquad \dot{\eta}_i = \dot{\phi}_i(x) = L_f \phi_i(x) + \overset{=0 \text{ by construction}}{\cancel{L_g \phi_i(x)}}\, u$$

$$= \underbrace{L_f \phi_i\left(\phi^{-1}(\xi,\eta)\right)}_{\doteq\, \psi_i(\xi,\eta)}$$

Therefore we obtain

$$\boxed{\begin{aligned} \dot{\xi}_i &= \xi_{i+1} \qquad i = 1,\dots,r-1 \\ \dot{\xi}_r &= q(\xi,\eta) + b(\xi,\eta)\, u \\ \dot{\eta} &= \psi(\xi,\eta) \\ y &= \xi_1 \end{aligned}}$$

NORMAL FORM

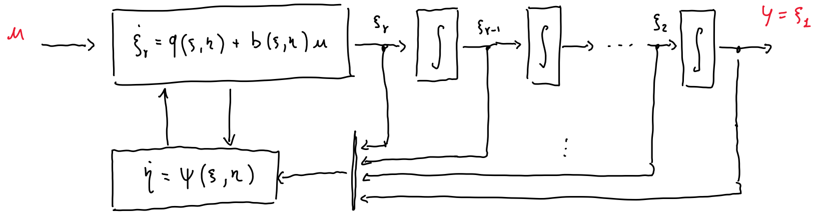
In summary, we have shown that if

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

locally, around $\bar{x}$

has relative degree $= r$ at $x = \bar{x}$, then it is diffeomorphic to a normal form

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} & i = 1, \dots, r-1 \\ \dot{\xi}_r = q(\xi,\eta) + b(\xi,\eta)u \\ \dot{\eta} = \psi(\xi,\eta) \\ y = \xi_1 \end{cases}$$



EXAMPLE

$$\dot{x} = \underbrace{\begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix}}_{g(x)} u$$

$$y = \underbrace{x_3}_{h(x)}$$

Let us compute (if it exists) the diffeomorphism ϕ bringing the system to a normal form

$$\phi_1(x) = h(x)$$

we have:

$$L_g h(x) = \frac{\partial h}{\partial x}(x)\, g(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow$ we keep going:

$$\phi_2(x) = L_f h(x) = \frac{\partial h}{\partial x}(x)\, f(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix} = x_2$$

EXAMPLE

$$f(x) = \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} \qquad g(x) = \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \qquad h(x) = x_3$$

The Jacobian from the Lie Derivative:

$$L_g h(x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} = 0 \quad \forall x \in \mathbb{R}^3$$

zero for whole $\mathbb{R}^3$

$$L_f h(x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} = x_2$$

$$L_g L_f h(x) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} = 1, \quad \forall x \in \mathbb{R}^3$$

global relative degree $(r=1)$

We now can have

$$\bar{\phi}(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \hdashline \phi_3(x) \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ \phi_3(x) \end{bmatrix}$$

$$\hookrightarrow \frac{\partial}{\partial x} \bar{\phi}(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{\partial \phi_3}{\partial x_1}(x) & \frac{\partial \phi_3}{\partial x_2}(x) & \frac{\partial \phi_3}{\partial x_3}(x) \end{bmatrix}$$

non-singular?

$\frac{\partial \phi_3}{\partial x_3}(x) \neq 0 \quad \forall x$

$\hookrightarrow L_g \phi_3(x) = 0$

$$= \begin{bmatrix} \frac{\partial \phi_3(x)}{\partial x_1} & \frac{\partial \phi_3(x)}{\partial x_2} & \frac{\partial \phi_3(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{\partial \phi_3(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi_3(x)}{\partial x_2}$$

$\phi_3(x) = x_1 - e^{x_2} + \text{constant}$

1) $\frac{\partial \phi_3(x)}{\partial x_1} = 1$

2) $L_g \phi_3(x) = 1 \cdot e^{x_2} - e^{x_2} = 0$

We have $r = 2$ (global)

$$\bar{\phi}(x) = \begin{bmatrix} x_3 \\ x_2 \\ \hdashline x_1 - e^{x_2} \end{bmatrix} \qquad z = \phi(x) = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$x \longrightarrow \left.\begin{aligned} \xi_1 &= x_3 \\ \xi_2 &= x_2 \\ \eta &= x_1 - e^{x_2} \end{aligned}\right\} \quad \begin{aligned} x_1 &= \eta + e^{\xi_2} \\ x_2 &= \xi_2 \\ x_3 &= \xi_1 \end{aligned} \implies \phi^{-1} \text{ is identified on all } \mathbb{R}^3$$

We can conclude $\phi = \bar{\phi}$ is a global diffeomorphism

Does it satisfy the normal form? $\qquad \dot{x} = f(x) + g(x)u$

$\dot{\xi}_1 = \dot{x}_3 = x_2 = \xi_2$

$\dot{\xi}_2 = \dot{x}_2 = x_1 x_2 + u \rightarrow$ not satisfied

$\quad = (\eta + e^{\xi_2})\xi_2 + u$

$$\dot\eta = \dot x_1 - \dot x_2 e^{x_2} = -x_1 + \cancel{e^{x_2} u}$$
$$= -(x_1 x_2 + \cancel{u}) e^{x_2}$$
$$= -x_1 - x_1 x_2 = -x_1(1 + x_2 e^{x_2})$$
$$= -(\eta + e^{\xi_1})(1 + \xi_2 e^{\xi_1})$$

Collect all of the equations above

$$\begin{cases} \dot\xi_1 = \xi_2 \\ \dot\xi_2 = \underbrace{(\eta + e^{\xi_1})}_{q(\xi,\eta)} \xi_2 + u \\ \dot\eta = -\underbrace{(\eta + e^{\xi_1})(1 + \xi_2 e^{\xi_1})}_{\psi(\xi,\eta)} \end{cases}$$

Why we need these?

$$\begin{cases} \dot x = f(x) + g(x) u' \\ y = h(x) \end{cases}$$

Assume we have $\boxed{r = n}$ relative degree

$$\dot\xi_1 = \xi_2 \qquad \hookrightarrow \bar\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$
$$\vdots$$
$$\dot\xi_{r-1} = \xi_r$$
$$\dot\xi_r = q(\xi) + b(\xi) u$$

$\rightarrow \neq 0$ because rel. degree $r = n$

What if $u = \frac{1}{b(\xi)}(-q(\xi) + v) \Rightarrow \dot\xi_r = v \xrightarrow{} \boxed{\text{System}} \xrightarrow{} y = \xi_1$

$$\begin{cases} \dot\xi_1 = \xi_2 \\ \vdots \\ \dot\xi_r = v \end{cases} \qquad \begin{aligned} \dot\xi &= A\xi + Bv \\ y &= C\xi \end{aligned}$$

The result is similar to the linear canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

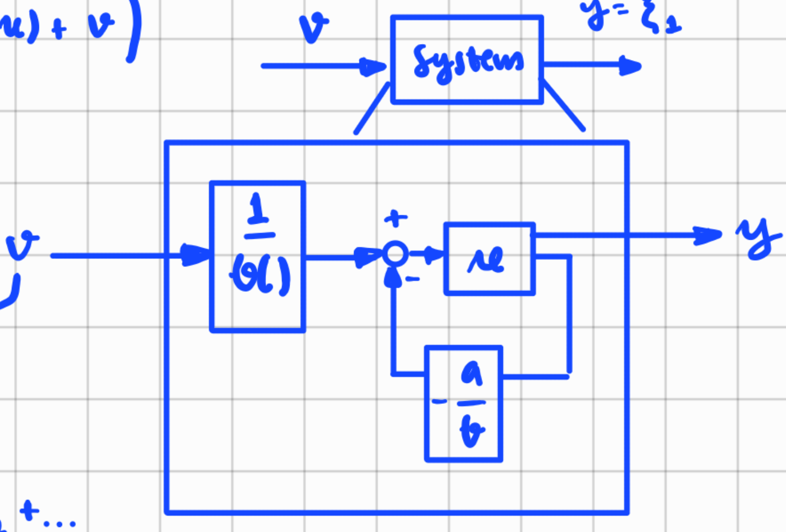
$$C = \begin{bmatrix} 1 & 0 & \cdots\cdots & 0 \end{bmatrix}$$

It is called FEEDBACK LINEARIZATION

$$a(\xi) = L_f^n h(x) \Big|_{x = \phi^{-1}(\xi)}$$

$$b(\xi) = L_g L_f^{n-1} h(x) \Big|_{x = \phi^{-1}(\xi)}$$

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v \right)$$



$$v = K\xi = K_1\xi_1 + K_2\xi_2 + \ldots + K_n\xi_n$$

$$\dot{\xi} = (A + BK)\xi \qquad \xi \to 0$$

The form in original coord. $v = K_1 h(x) + K_2 L_f h(x) + \ldots + K_n L_f^{n-1} h(x) = K\phi(x)$

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + K\phi(x) \right) = \frac{1}{b(\xi)} \left(-q(\xi) + K\xi \right)$$

Back to $r = n$

$$\phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

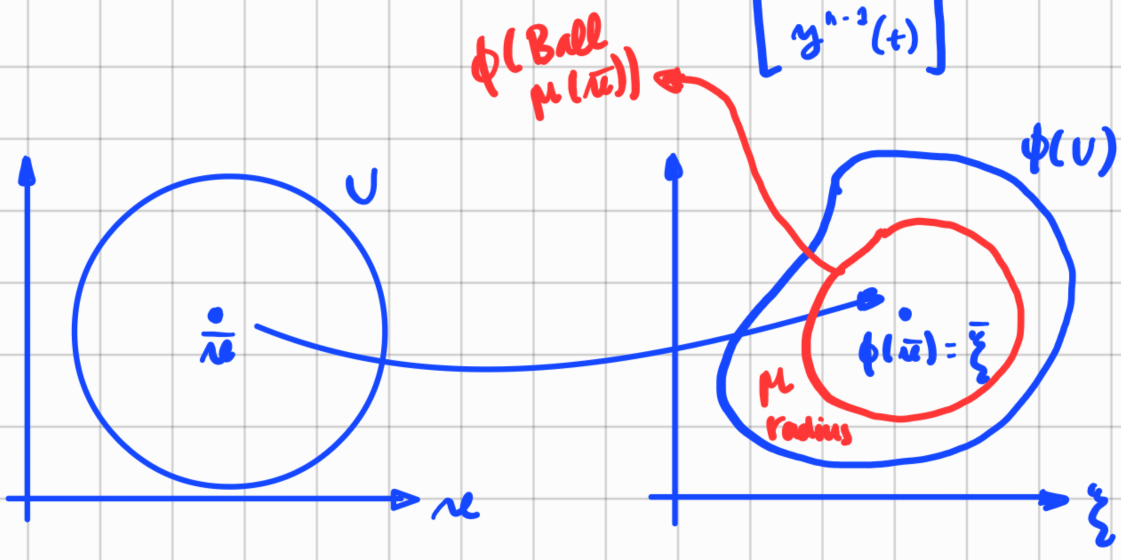
$$\dot{x} = \underbrace{Ax}_{f(x)} + \underbrace{Bu}_{g(x)}$$

$$y = \underbrace{Cx}_{h(x)}$$

$$L_f h(x) = \frac{\partial}{\partial x}(Cx) \cdot Ax = CAx$$

$$\phi(x) = \begin{bmatrix} Cx \\ CAx \\ CA^2x \\ \vdots \\ CA^{n-1}x \end{bmatrix} = Ox = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

$$\phi(x(t)) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{n-1}(t) \end{bmatrix}$$



We can't do anything on ξ because when the traj. leaves $\phi(U)$, there's no correlation

RESULT:

ϕ diffeomorphism $\Rightarrow \forall C \subset U,\ \phi(C)$ open

If $\xi(t) \in Ball_{\mu}(\bar{\xi})$ then $x(t) = \phi^{-1}(\xi(t)) \in U$

and

$$L_g L_f h(x) = \frac{\partial L_f h}{\partial x}(x) \cdot g(x) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} = \underline{\underline{1}} \qquad \forall x$$

$\Rightarrow$ the system has global rel. degree $r = 2$

We miss ϕ_3 $\rightarrow$ we now have to find ϕ_3 such that $\left\{ \frac{\partial}{\partial x}\phi_1(x), \frac{\partial}{\partial x}\phi_2(x), \frac{\partial \phi_3}{\partial x}(x) \right\}$ is linearly independent and $L_g \phi_3(x) = 0$ in some open $U \subset \mathbb{R}^3$

Imposing $L_g \phi_3(x) = 0$ means

$$0 = \frac{\partial}{\partial x}\phi_3(x) \cdot g(x) = \frac{\partial}{\partial x}\phi_3(x) \cdot \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} = e^{x_2} \frac{\partial}{\partial x_1}\phi_3(x) + \frac{\partial}{\partial x_2}\phi_3(x)$$

A solution is

$$\phi_3(x) = 1 + x_1 - e^{x_2}$$

indeed

$$e^{x_2} \frac{\partial}{\partial x_1}\phi_3(x) + \frac{\partial}{\partial x_2}\phi_3(x) = e^{x_2} \cdot 1 - e^{x_2} = 0$$

Moreover,

$$\frac{\partial}{\partial x}\phi_3(x) = \begin{pmatrix} 1 & -e^{x_2} & 0 \end{pmatrix} \Rightarrow \text{it is linearly independent from } \frac{\partial}{\partial x}\phi_1(x) \text{ and } \frac{\partial}{\partial x}\phi_2(x)$$

The diffeomorphism reads

$$\phi(x) = \begin{pmatrix} x_3 \\ x_2 \\ 1 + x_1 - e^{x_2} \end{pmatrix}$$

It's easy to see that

$$\phi^{-1}(z) = \begin{pmatrix} z_3 - 1 + e^{z_2} \\ z_2 \\ z_1 \end{pmatrix}$$

Since ϕ and ϕ^{-1} are defined and smooth on $U = \mathbb{R}^n$, ϕ is a global diffeomorp.

using

$$f(x) = \begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix} \quad \text{and} \quad g(x) = \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix}$$

in the new coordinates $z = \phi(x) = \begin{pmatrix} x_3 \\ x_2 \\ 1 - x_1 + e^{x_2} \end{pmatrix} \begin{matrix} \}\ \xi \\ \\ \}\ \eta \end{matrix}$ we get

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = x_1 x_2 + u = \underbrace{(\eta - 1 + e^{\xi_2})\,\xi_2}_{q(\xi,\eta)} + \underbrace{u}_{b(\xi,\eta) = 1}$$

$$\dot{\eta} = \underbrace{(\eta - 1 + e^{\xi_2})}_{\psi(\xi,\eta)}$$

EXAMPLE

$$\dot{x} = \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} u$$

$$y = x_3$$

Let's build the diffeomorphism :

- $\phi_1(x) = h(x) = x_3$

- $L_g h(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} = 0$

→ we keep going

- $\phi_2(x) = L_f h(x) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} = x_1 - x_2$

- $L_g L_f h(x) = \frac{\partial}{\partial x}(L_f h)(x) \cdot g(x) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} = 0$

→ we keep going:

- $\phi_3(x) = L_f^2 h(x) = \frac{\partial L_f h}{\partial x}(x) \cdot f(x) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} = -x_1 - x_2^2$

- $L_g L_f^2 h(x) = \frac{\partial}{\partial x}(L_f^2 h)(x) \cdot g(x) = \begin{pmatrix} -1 & -2x_2 & 0 \end{pmatrix} \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} = -e^{x_2}(1 + 2x_2)$

$\to$ $L_g L_f^2 h(x) \neq 0 \qquad \forall x \in \mathcal{O} \doteq \{ x \in \mathbb{R}^3 : \ x_2 \neq -\frac{1}{2} \}$ (open but NOT connected)

$\to$ Since $\mathcal{O}$ is open, the system has relative degree $r=3$ at every $x \in \mathcal{O}$

Define the function $\tilde{\phi} : \mathcal{O} \to \mathbb{R}^3$ as

$$\tilde{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 - x_2 \\ -x_1 - x_2^2 \end{pmatrix}$$

question: does it exist an open set $U \subset \mathcal{O}$ such that the restriction ϕ of $\tilde{\phi}$ on U is a diffeomorphism ?

- $\tilde{\phi}$ is smooth on $\mathcal{O}$
- Thus we only need an open set $U \subset \mathcal{O}$ such that $\tilde{\phi}|_U$ is invertible on U and its inverse is smooth

$\downarrow$

Let us try to compute $\tilde{\phi}^{-1}$. We have to solve, in the unknown x,

$$\begin{cases} x_3 = z_1 \\ x_1 - x_2 = z_2 \\ -x_1 - x_2^2 = z_3 \end{cases} \to \begin{cases} x_3 = z_1 \\ x_1 = z_2 + x_2 \\ x_2^2 + x_2 + z_2 + z_3 = 0 \end{cases} \longrightarrow \begin{cases} x_3 = z_1 \\ x_1 = z_2 + x_2 \\ x_2 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - (z_2 + z_3)} \end{cases} \quad (*)$$

we can write:

$$\mathcal{O} = \underbrace{\{ x \in \mathbb{R}^3 : \ x_2 < -\tfrac{1}{2} \}}_{= \mathcal{O}_1} \cup \underbrace{\{ x \in \mathbb{R}^3 : \ x > -\tfrac{1}{2} \}}_{= \mathcal{O}_2}$$

Equations $(*)$ tell us that we can get $\tilde{\phi}|_{\mathcal{O}_1}^{-1}$ with the choice

$$\tilde{\phi}|_{\mathcal{O}_1}^{-1}(z) = \begin{pmatrix} z_2 - \frac{1}{2} - \sqrt{\frac{1}{4} - (z_2 + z_3)} \\ -\frac{1}{2} - \sqrt{\frac{1}{4} - (z_2 + z_3)} \\ z_1 \end{pmatrix}$$ (this is smooth)

and we can get $\tilde{\phi}|_{\mathcal{O}_2}^{-1}$ as

$$\tilde{\phi}\big|_{\mathcal{O}_2}^{-1}(z) = \begin{pmatrix} z_2 - \frac{1}{2} + \sqrt{\frac{1}{4} - (z_2+z_3)} \\ -\frac{1}{2} + \sqrt{\frac{1}{4} - (z_2+z_3)} \\ z_1 \end{pmatrix}$$

(smooth as well)

Notice that $\sqrt{\frac{1}{4} - (z_2+z_3)}$ is always real on $\tilde{\phi}(\mathcal{O})$ since

$$z_2 + z_3 < \frac{1}{4} \iff -(x_2 + x_2^2) < \frac{1}{4} \iff \frac{1}{4}(x_2+1) > 0 \iff x_2 \neq -\frac{1}{2}$$

Therefore, given any point $x \in \mathcal{O}$, we can find the diffeomorphism $\phi : U \to \mathbb{R}^3$ where

$$U = \begin{cases} \mathcal{O}_1 & \text{if } x_2 < -\frac{1}{2} \\ \mathcal{O}_2 & \text{if } x_2 > -\frac{1}{2} \end{cases} \qquad \phi(x) \doteq \tilde{\phi}\big|_U(x) = \begin{pmatrix} x_3 \\ x_1 - x_2 \\ -x_1 - x_2^2 \end{pmatrix}$$

In the new variables we have

$$\left[\dot{x} = \begin{pmatrix} 0 \\ x_1 + x_2^2 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \\ 0 \end{pmatrix} u \right]$$

$$z = \phi(x) = \xi$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\dot{\xi}_3 = -2x_2(x_1 + x_2^2) - (1+2x_2)e^{x_2} u$$

$$= q(\xi) + b(\xi) u$$

where

$$q(\xi) = \left[-2x_2(x_1+x_2^2)\right]\Big|_{x=\phi^{-1}(\xi)}, \quad b(\xi) = -\left[e^{x_2}(1+2x_2)\right]_{x=\phi^{-1}(\xi)}$$

In case we are in $U = \mathcal{O}_1$ (hence $x_2 < -\frac{1}{2}$) we have

$$\phi^{-1}(z) = \begin{pmatrix} z_2 - \frac{1}{2} - \sqrt{\frac{1}{4} - (z_2+z_3)} \\ -\frac{1}{2} - \sqrt{\frac{1}{4} - (z_2+z_3)} \\ z_1 \end{pmatrix} = \begin{pmatrix} \xi_2 - \frac{1}{2} - \sqrt{\frac{1}{4} - (\xi_2+\xi_3)} \\ -\frac{1}{2} - \sqrt{\frac{1}{4} - (\xi_2+\xi_3)} \\ \xi_1 \end{pmatrix}$$

Hence

$$q(\xi) = 2\left(-\frac{1}{2} - \sqrt{\frac{1}{4} - (\xi_2+\xi_3)}\right) \cdot \xi_3$$

$$b(\xi) = -2 \cdot e^{\left(-\frac{1}{2} - \sqrt{\frac{1}{4} - (\xi_2+\xi_3)}\right)} \cdot \sqrt{\frac{1}{4} - (\xi_2+\xi_3)}$$

Notice that $b(\xi) = 0$ iff

$$\xi_2 + \xi_3 = \frac{1}{4}$$

that is iff

$$x_2 = -\frac{1}{2}$$

FEEDBACK LINEARIZATION

ASSUMPTION. There exists an open set $U \in \mathbb{R}^n$ such that the system has relative degree $r = n$ in U and it has a normal form on U

$\downarrow$

The diffeomorphism $\phi : U \to \mathbb{R}^n$ bringing the system to its normal form reads as

$$\phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}$$

The normal form only has the ξ-variables: $(z = \xi)$

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = q(\xi) + b(\xi)u \end{cases} \qquad \begin{aligned} q(\xi) &= L_f^n h(x) \Big|_{x=\phi^{-1}(\xi)} \\ b(\xi) &= L_g L_f^{n-1} h(x) \Big|_{x=\phi^{-1}(\xi)} \end{aligned}$$

Consider the controller

$$u = \frac{1}{b(\xi)} \left(-q(\xi) + v \right) \qquad (v = \text{AUXILIARY INPUT})$$

$$= \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v \right)$$

Then, we have:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_{n-1} = \xi_n \\ \dot{\xi}_n = v \end{cases} \quad \rightarrow \quad \text{IT IS } \underline{\text{LINEAR}}:$$

→ The closed-loop system with input $v(t)$ and output $y(t)$ is <u>LINEAR</u> and controllable

$$\begin{cases} \dot{\xi} = A\xi + Bv \\ y = C\xi \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ & & & \ddots & \\ & & & & 1 \\ 0 & 0 & \cdots & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

We can design on the above linear system a control law of the kind

$$v(t) = \gamma(\xi(t), t)$$

to accomplish the control task we desire. The total control law is

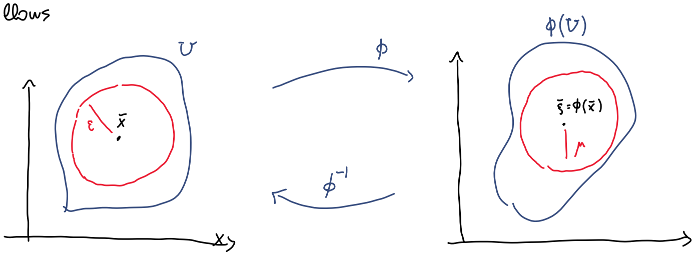
$$u(t) = \frac{1}{L_g L_f^{n-1} h(x)} \left(-L_f^n h(x) + v(\xi, t) \Big|_{\xi = \phi^{-1}(x)} \right)$$

→ The only thing we should guarantee is that $x(t)$ stays in U

↓

- If $U = \mathbb{R}^n$ → NO PROBLEM → This will be our assumption from now on since we are interested in global solutions

- Otherwise, we can map the constraint $x \in U$ to a constraint on ξ as follows



Ball of radius ε centered at $\bar{x}$

- Since U is open, given any $\bar{x} \in U$, exists $\varepsilon > 0$, such that $\underbrace{\mathcal{B}_\varepsilon(\bar{x})} \subset U$

- Since ϕ is a diffeomorphism $\exists \mu > 0$ such that $\mathcal{B}_\mu(\bar{\xi}) \subset \phi(\mathcal{B}_\varepsilon(\bar{x}))$ where $\bar{\xi} := \phi(\bar{x})$

- If v is able to ensure that

$$\|\xi(t) - \bar{\xi}\| < \mu \quad \forall t \geq 0 \qquad \left(\Leftrightarrow \ \xi(t) \in \mathcal{B}_\mu(\bar{\xi}) \right)$$

then for all $t \geq 0$

$$\phi(x(t)) = \xi(t) \in \mathcal{B}_\mu(\bar{\xi}) \Rightarrow x(t) = \phi^{-1}(\xi(t)) \subset \phi^{-1}\left(\mathcal{B}_\mu(\bar{\xi})\right) \subset \phi^{-1}\left(\phi\left(\mathcal{B}_\varepsilon(\bar{x})\right)\right)$$
$$\subset \mathcal{B}_\varepsilon(\bar{x}) \subset U$$

Namely, $\underline{x(t) \in U, \forall t \geq 0}$.

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