Observability and Reconstructability

Master degree in Automation Engineering

Prof. Lorenzo Marconi

Dipartimento di Ingegneria dell'Energia Elettrica e dell'Informazione (DEI)
Università di Bologna

lorenzo.marconi@unibo.it
Tel. +39 051 2093788

Observabilty/Reconstructabilty

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \qquad x(0) = x_0$$

initial cond.

$$y = Cx$$

$$y(t) = C\,\boxed{\phi(t)\,x(t_0)}$$

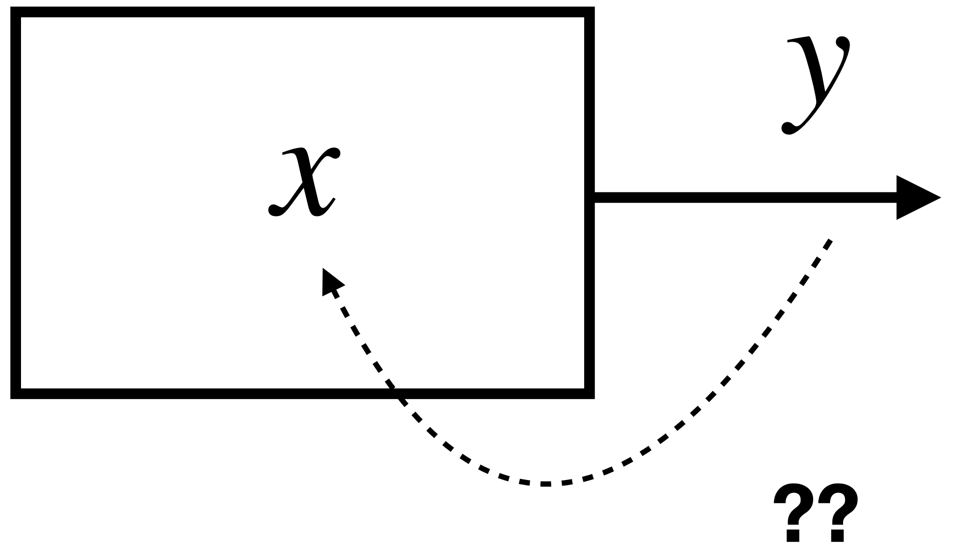
No inputs involved in the following analysis

Main question to be answered: given the system in free evolution can we reconstruct the state trajectory by just reading the output trajectory in an appropriate time interval? Are there some “regions” of $\mathbb{R}^n$ that are not observable, namely such that different state trajectories generate “identical” output trajectories?

Observability: Can we reconstruct the initial state by processing the future output trajectory ? —> $\boxed{\mathscr{E}^+}$

Reconstructabilty: Can we reconstruct the actual state by processing the past output trajectory ? —> $\boxed{\mathscr{E}^-}$

reverse the time-axis



Observability

$\mathcal{E}_c$ +

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \qquad x(0)=x_0$$

$$y = Cx$$

$$\boxed{y(t) = C\,\phi(t)\,x(t_0)}$$

No inputs involved in the following analysis

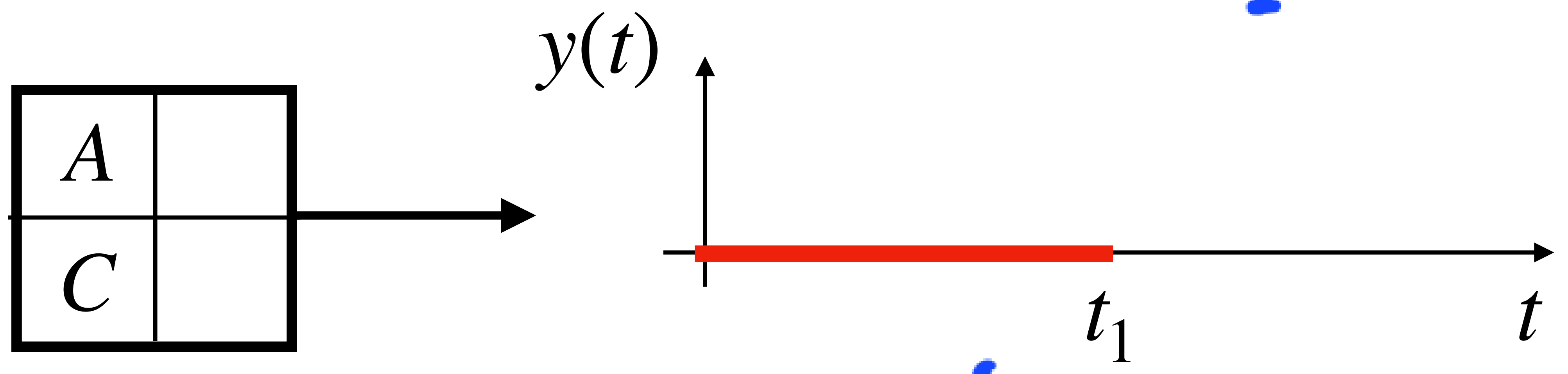
$\boxed{\forall\, t \in \{0, 1, \ldots, t_1 - 1\}}$ for D-T systems

non-observable

Set of unobservable states in $[0,t_1)$: $\quad \mathcal{E}^+_{NO}(t_1) = \{x \in \mathbb{R}^n \;:\; C\,\phi(t)\,x = 0 \;\; \forall\, t \in [0,t_1)\}$

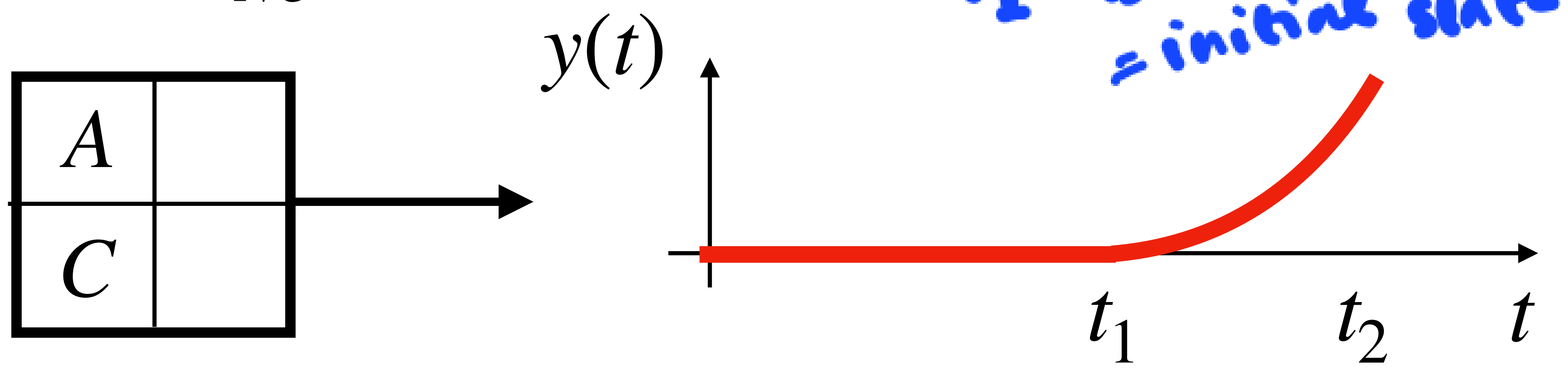
interval from 0 to t_1

If $t_2 > t_1$ then $\mathcal{E}^+_{NO}(t_2) \subseteq \mathcal{E}^+_{NO}(t_1)$



$x(0) \in \mathcal{E}^+_{NO}(t_1)$

Suppose $t_2 = t_1$, the output = initial state



$x(0) \in \mathcal{E}^+_{NO}(t_1),\; x(0) \notin \mathcal{E}^+_{NO}(t_2)$

Unobservable set: $\mathcal{E}^+_{NO} := \mathcal{E}^+_{NO}(\infty)$

Definition: The system is said to be Completely Observable if $\mathcal{E}^+_{NO} = \{0\}$

Observability

contains the origin too!

Result: the unobservable set $\mathscr{E}^+_{NO}$ is a subspace of $\mathbb{R}^n$ (much more than a set!)

Theorem : $\mathscr{E}^+_{NO} = \text{Ker}\, O$ $\quad O := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Observability matrix

Dimension = $p\, n \times n$: $\begin{cases} \text{“Tall” matrix if } p > 1, \\ \text{“Square” matrix if } p = 1 \end{cases}$

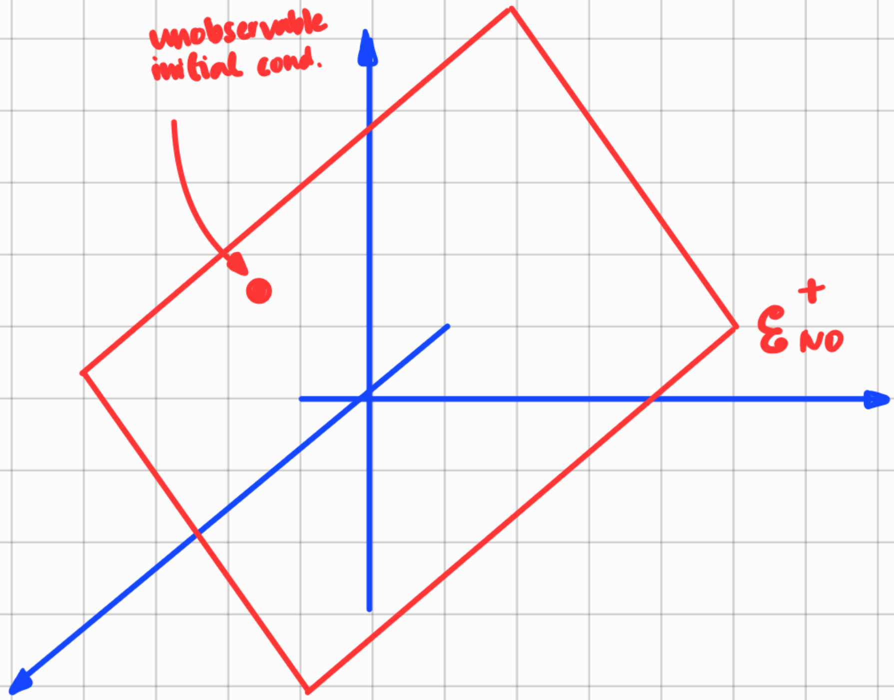
Proof … (for D-T systems and intuition for C-T systems)

Remarks:

- If rank$(O) = n$ (full column rank) then $\mathscr{E}^+_{NO} = \{0\}$ (all initial states in $\mathbb{R}^n \setminus \{0\}$ produce a free output trajectory that is not identically zero)
- For C-T (D-T) systems if an initial state “generates” a nonzero output then $y^{(i)}(0) \neq 0$ for some $i < n$ ($y(i) \neq 0$ for some $i < n$)

index

Consider a $\mathbb{R}^3$,



$$\underset{\text{observ. matrix}}{\mathcal{O}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{n} \begin{matrix} \}\, p \rightarrow \text{number \# outputs} \\ \\ \}\, n \cdot p \end{matrix}$$

$$\mathcal{E}^+_{NO} = \text{kernel } \mathcal{O} = \{v \in \mathbb{R}^n : \mathcal{O}v = 0\}$$

If $\text{rank } \mathcal{O} = n \Rightarrow \mathcal{E}^+_{NO} = \{0\} \Rightarrow$

Discrete-time:

$x(t+1) = Ax(t) \qquad x(0) = x_0$

$y(t) = Cx(t)$

NB: $x(1) = Ax(0)$

$y(0) = Cx_0$

$y(1) = CAx_0$

$y(2) = CA^2x_0$

$$\begin{pmatrix} y(0) \\ \vdots \\ y(t+1) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{pmatrix} x_0$$

$O = O_n$

$$\underbrace{\begin{pmatrix} y(0) \\ \vdots \\ y(t-1) \end{pmatrix}}_{y_0^{t-1}} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{pmatrix}}_{O_t} x_0 \Rightarrow \boxed{y_0^{t-1} = O_t x_0}$$

$x_0 \in \mathcal{E}^+_{NO}(1)$

$x_0 \in \text{kernel } O_1 = \text{kernel } C$

$x_0 \in \mathcal{E}^+_{NO}(2)$

$x_0 \in \text{kernel } O_2 = \text{kernel} \begin{pmatrix} C \\ CA \end{pmatrix}$

bigger set

$\text{kernel } O_1 \supseteq \text{kernel } O_2$

$\mathcal{E}^+_{NO}(1) \supseteq \mathcal{E}^+_{NO}(2)$

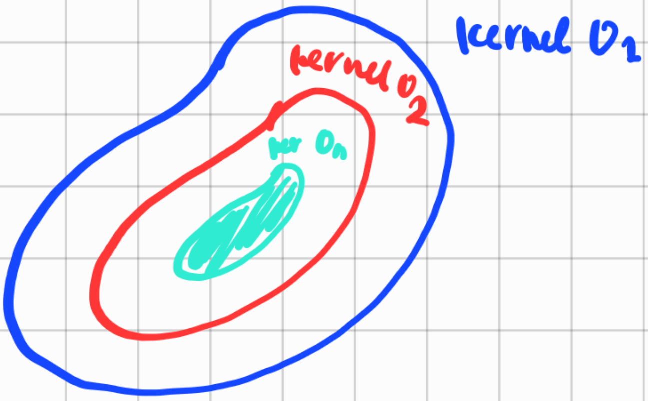
$x_0 \in \mathcal{E}^+_{NO}(n)$

$$x_0 \in \text{kernel } O_n = \text{kernel } O = \text{kernel} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

$$\text{kernel } O_1 \supseteq \text{kernel } O_2 \supseteq \dots \supseteq \text{kernel } O_n \equiv \text{kernel } O_t \ (t > n)$$

$$\text{kernel } O_n = \text{kernel } O \supseteq \mathcal{E}^+_{NO}(n)$$

$x(t) = R_t u_0^{t-1}$



$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = \text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \\ CA^n \end{pmatrix}$$

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0$$
$$y(t) = Cx(t)$$

$$\downarrow$$

$$y(0) = Cx_0$$
$$y^{[1]}(0) = CAx_0$$
$$y^{[k]}(0) = CA^k x_0$$

$$y_0^{k-1} = \begin{bmatrix} y(0) \\ y^{[1]}(0) \\ \vdots \\ y^{[k-1]}(0) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{O_k} x_0$$

$$x_0 \in \text{kernel } O_n \Rightarrow x_0 \in \text{kernel } O_k \quad k > n$$

Observability

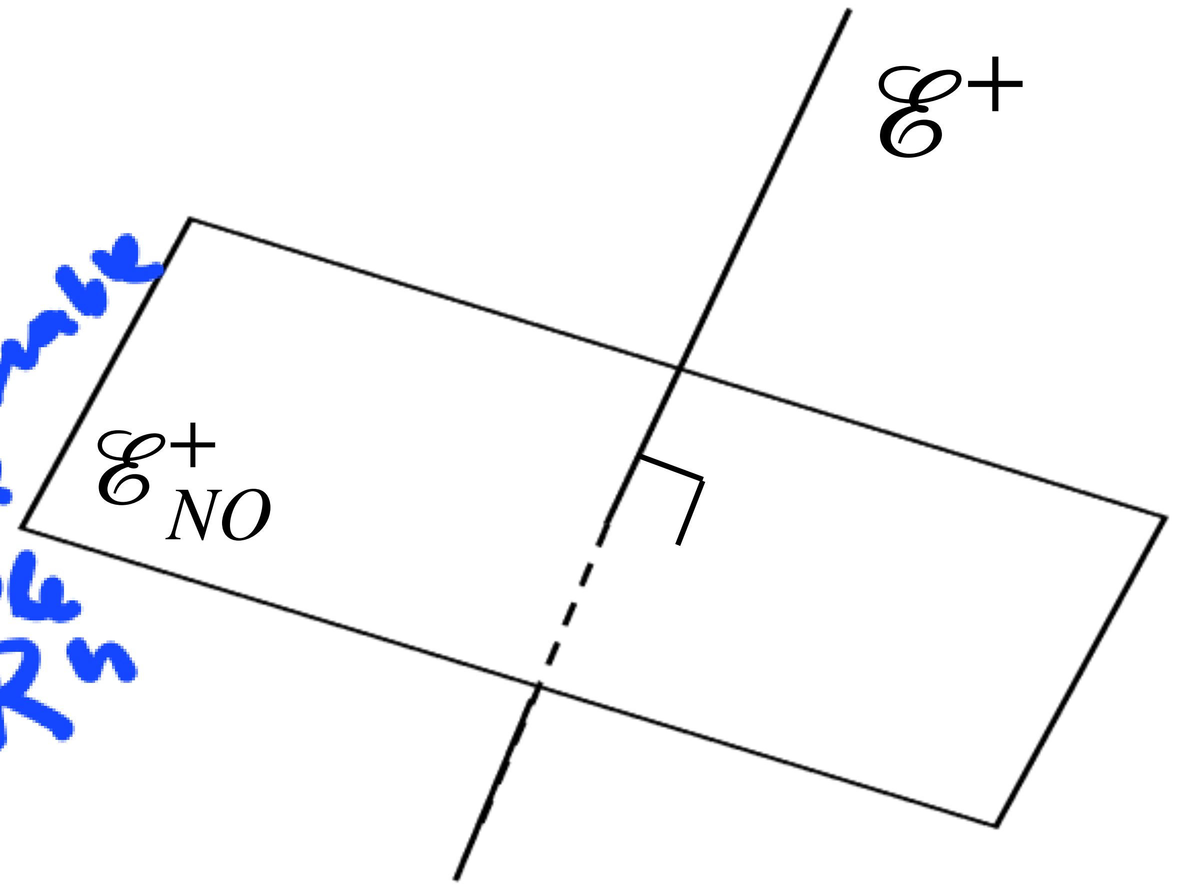
Orthogonal complement of $\mathscr{E}^+_{NO}$

Definition

Observable Set: $\mathscr{E}^+ = [\mathscr{E}^+_{NO}]_\perp$

non-observable is at the origin

observable is the whole $\mathbb{R}^n$



Remark: if the system is completely observable then $\mathscr{E}^+_{NO} = \{0\}$ and $\mathscr{E}^+ = \mathbb{R}^n$

Remark: From linear algebra we know that $\operatorname{Ker} M = (\operatorname{Im} M^T)_\perp$ with M a generic matrix. Hence:

kernel of M is the image of M^T in orthogonal complement.

$$\mathscr{E}^+ = \operatorname{Im}[O^T] \qquad O^T := \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix}$$

PBH Test

Popov - Belevitch- Hautus test: it presents a sufficient and necessary condition for complete observability not requiring the computation of the observability matrix

The pair (A, C) is completely observable ($\text{rank}O = n$, $\mathscr{E}^{+} = \mathbb{R}^{n}$) **iff**

$$\textbf{rank} \begin{bmatrix} \overbrace{\lambda I - A}^{n} \\ C \end{bmatrix} = n \qquad \forall \lambda \; (\in \sigma(A))$$

full column-rank (if λ is not in $\sigma(A)$ the condition is always fulfilled)

Proof (rank$O = n \quad \Rightarrow$ rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all λ **)**

(A, C) is completely observable if the kernel $\mathcal{O} = \{0\}$ and rank $\mathcal{O} = n$.

$$\underbrace{\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}}_{n} \begin{array}{l} \}\, n \\ \}\, p \end{array}$$

non-singular for every $\lambda \notin \sigma(A)$

In PBH test:

$$[\lambda I - A \mid B] \quad \begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

Reconstructing the state from the output - The D-T case

How to practically compute the (initial) state by processing the associated output trajectories?

We know that $y_0^{t-1} = O_t\, x(0)$ with $\underset{t \cdot p \times n}{O_t} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}$ and $y_0^{t-1} = [y(0)\ y(1)\ \cdots\ y(t-1)]^T$

- Case $\mathscr{E}^+ = \mathbb{R}^n$ (all initial states of $\mathbb{R}^n$ are observable and the output becomes not zero in at most n steps).

$t \geq n,\ p \geq 1$

$$y_0^{t-1} = O_t\, x(0) \qquad x(0) = \boxed{(O_t^T\, O_t)^{-1}\, O_t^T}\ y_0^{t-1}$$

"Tall" matrix (full column rank)

Left inverse of O_t

Result: Given any full column rank matrix M, the square matrix $M^T M$ is nonsingular

Special case $t = n, p = 1$

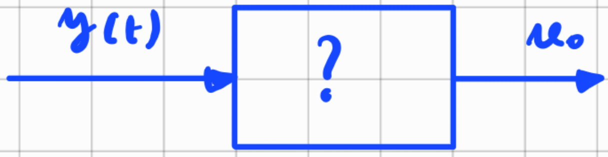
$$y_0^{n-1} = O_n\, x(0) = O\, x(0) \qquad x(0) = O^{-1}\, y_0^{n-1}$$

square (not singular)

$$x(t+1) = Ax(t),$$
$$y(t) = Cx(t)$$

(A, C) is completely observable.

$$\{y(0), y(1), y(2), \dots, y(t-1)\}$$
$t \geq n$



$y^{t-1} = O_t x_0$ if $p = 1$, $t = n$, $O_n = O$ ↪ square $n \times n$ and non-singular

pick $x_0 = O^{-1} y_0^{n-1}$

if $p \geq 1$, $t \geq n$, O_t is a tall matrix

$$[O_t^T O_t]^{-1} O_t^T \begin{bmatrix} y_0^{t-1} \end{bmatrix} = [O_t^T O_t]^{-1} \boxed{O_t^T \begin{bmatrix} O_t \end{bmatrix}} \begin{bmatrix} x_0 \end{bmatrix}$$

$n \times n$
non-singular

Reconstructing the state from the output - The D-T case

- In all the other cases (namely $t < n$ or $\mathscr{E}^{+} \subset \mathbb{R}^{n}$) the initial state $x(0)$ is not generic but, to be observable, must fulfi $y_0^{t-1} \in \mathrm{Im}\, O_t$. In these cases the equation $y_0^{t-1} = O_t\, x(0)$ can be solved for $x(0)$:

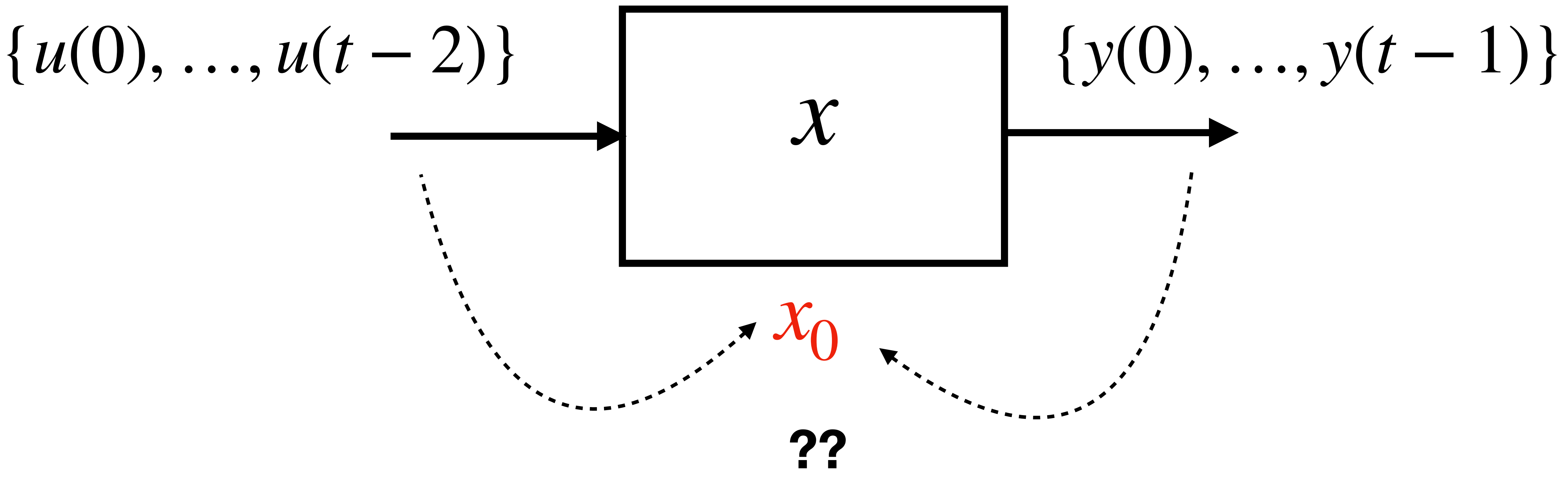
$$\boxed{x(0) = O_t^{\dagger}\, y_0^{t-1}}$$

Generalized (Moore Penrose) inverse of O_t

General case: How to compute the initial state by processing the output and input samples when the system is driven by an input $u(\,\cdot\,)$?

$$y(t) - \sum_{i=0}^{t-1} C A^{t-i-1} B\, u(i) = C A^{t}\, x(0)$$

Generalised output $\bar{y}(t)$



Reconstructing the state from the output - The C-T case

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx\end{aligned} \qquad x(0) = x_0 \qquad y(t) = C\, e^{At}\, x_0$$

How to compute the initial condition x_0 by processing the output $y(s),\, s \in [0,t)$?

Observability Gramian at time t

$$V(t) = \int_0^t e^{A^T \overset{s}{\cancel{(t-s)}}}\, C^T\, C\, e^{A \overset{s}{\cancel{(t-s)}}}\, ds$$

Theorem: The Gramian is not singular for each $t > 0$ **iff** $\mathscr{E}^+ = \mathbb{R}^n$

Proof…

Result: Assume that the system is completely observable. Let $t > 0$ be an arbitrary time. then the initial state that originated an output trajectory $y(s),\, s \in [0,t)$ is

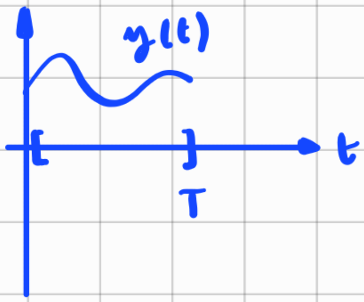
$$x(0) = V^{-1}(t) \int_0^t e^{A^T s}\, C^T\, y(s)\, ds$$

$\dot{x}(t) = Ax(t) \qquad x(0) = x_0$

$y(t) = Cx(t)$

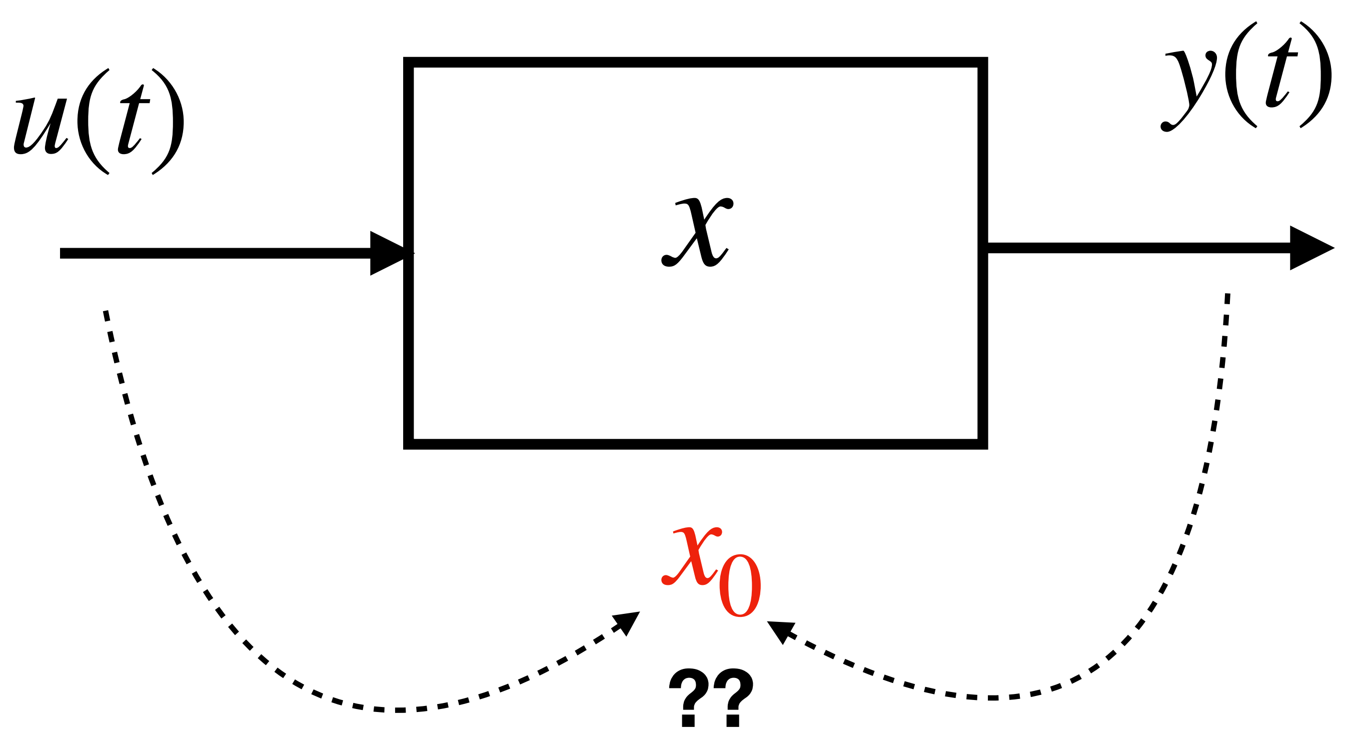
$$\int_0^T e^{A^T t} C^T y(t)\,dt = \overbrace{\int_0^T e^{A^T t} C^T C e^{At}\,dt}^{\text{observ. Gramian at } T \;\; V(T)} \; x_0$$

?



Reconstructing the state from the output - The C-T case

General case: How to compute the initial state by processing the output and input trajectory when the system is driven by an input $u(\cdot)$?



$$Ce^{At}x_0 = y(t) - \int_0^t e^{A(t-s)}\, B\, u(s)\, ds$$

Because of linearity, the problem can be cast as the previous one but considering a fictitious output trajectory $\bar{y}(t) := y(t) - \int_0^t e^{A(t-s)}\, B\, u(s)\, ds$:

$$\boxed{x(0) = V^{-1}(t) \int_0^t e^{A^T s}\, C^T \left[y(s) - \int_0^s e^{A(s-s')}\, B\, u(s')\, ds' \right] ds}$$

Reconstructability

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \quad x(0)=x_0$$

$$y = Cx$$

$$\boxed{y(t) = C\,\phi(t)\,x(t_0)}$$

No inputs involved in the following analysis

Set of states not reconstructable in $[0,t_1)$:

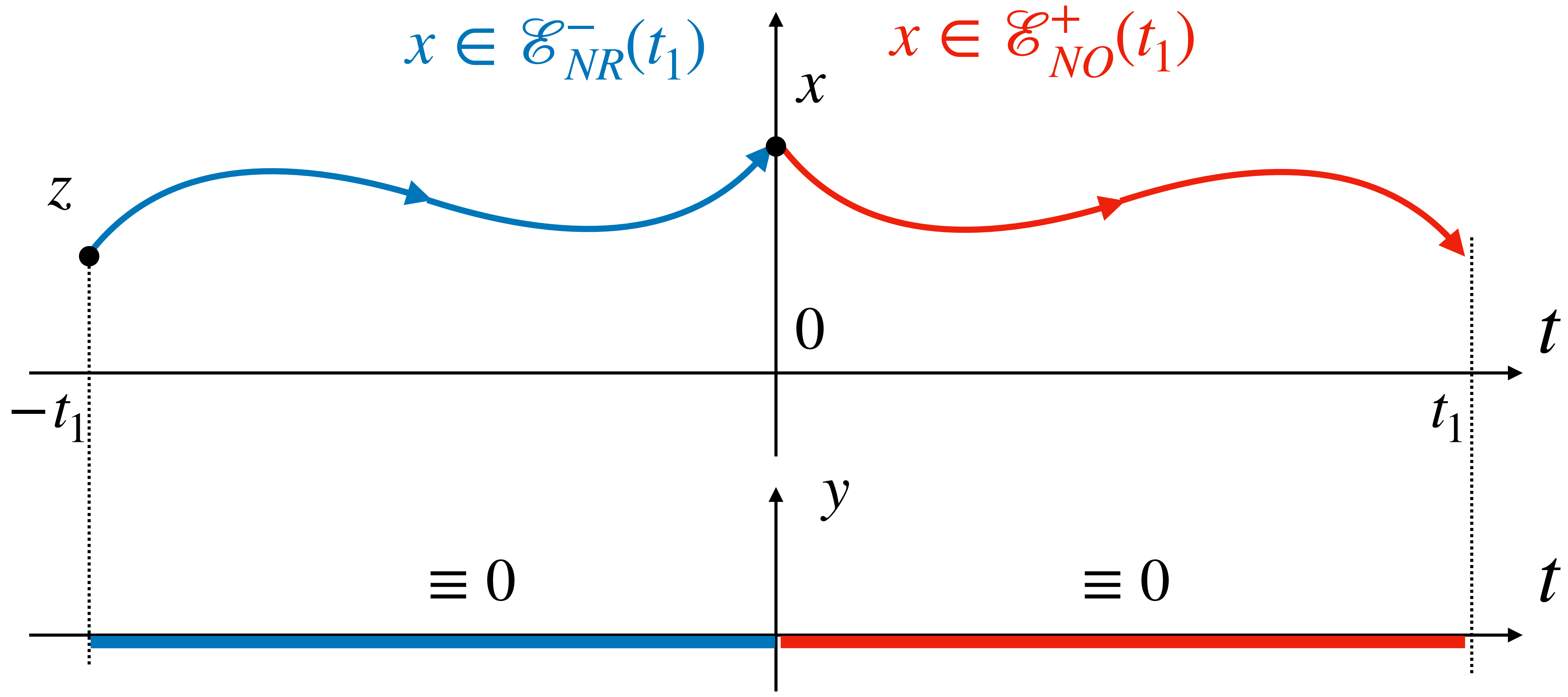
$$\mathcal{E}^-_{NR}(t_1) = \{x \in \mathbb{R}^n \ : \quad x = \phi(t_1)z \qquad C\phi(t)z = 0 \ \ \forall\, t \in [0,t_1) \ \text{ for some } z \in \mathbb{R}^n\}$$

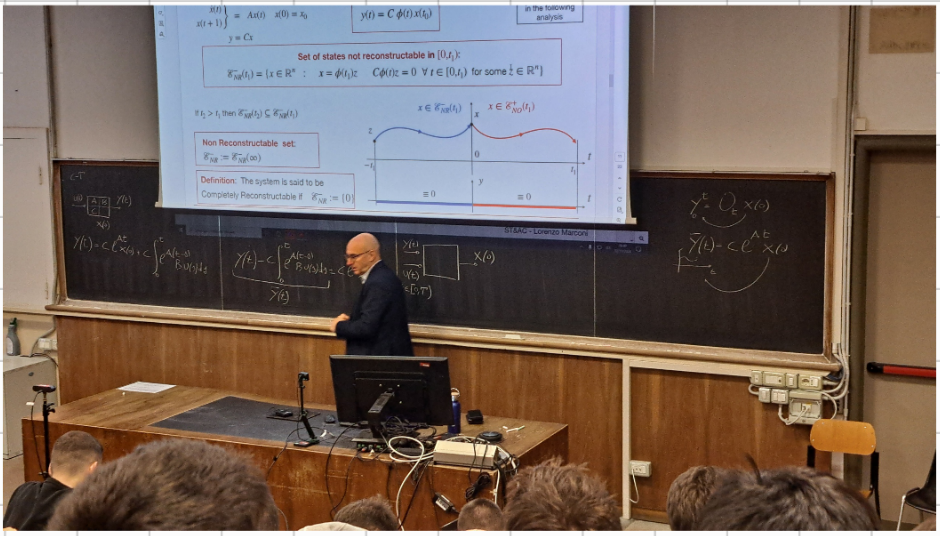
If $t_2 > t_1$ then $\mathcal{E}^-_{NR}(t_2) \subseteq \mathcal{E}^-_{NR}(t_1)$

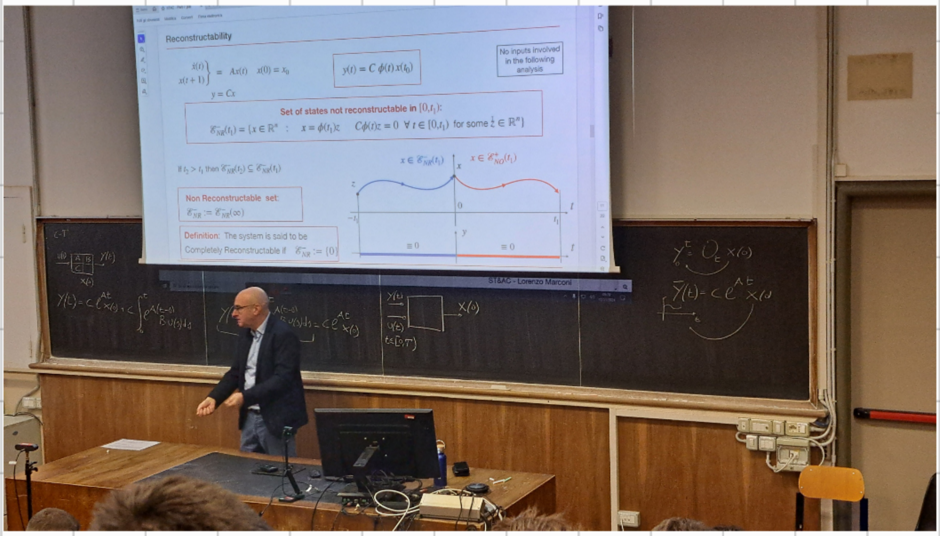
Non Reconstructable set:

$$\mathcal{E}^-_{NR} := \mathcal{E}^-_{NR}(\infty)$$

Definition: The system is said to be Completely Reconstructable if $\mathcal{E}^-_{NR} := \{0\}$




in the following analysis
Set of states not reconstructable in [0,t1):
Non Reconstructable set:
Definition: The system is said to be
Completely Reconstructable if
ST&AC - Lorenzo Marconi


Reconstructability
No inputs involved in the following analysis
Set of states not reconstructable in [0,t1):
Non Reconstructable set:
Definition: The system is said to be
Completely Reconstructable if
ST&AC - Lorenzo Marconi

Reconstructability

Result: the non reconstructable set $\mathscr{E}^-_{NR}$ is a subspace of $\mathbb{R}^n$ (much more than a set!)

Theorem : $\mathscr{E}^-_{NR} \subseteq \mathscr{E}^+_{NO}$ in general. $\mathscr{E}^-_{NR} = \mathscr{E}^+_{NO}$ if the transition matrix is not singular for all t (reversible systems)

Remark: There could be systems with state that are not observable but that can be reconstructed. **Complete Observability ⇒ Complete Reconstructability**

Remark: All continuous-time systems are reversible, and thus $\mathscr{E}^-_{NR} = \mathscr{E}^+_{NO}$. However there could be discrete time systems for which the inclusion holds true

Example:

$$\begin{aligned} x(t+1) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned} \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} \star \\ 0 \end{pmatrix} \qquad x \in \mathscr{E}^+_{NO} \quad x \notin \mathscr{E}^-_{NR}$$

Remark: D-T sampled-data systems are reversible and thus $\mathscr{E}^-_{NR} = \mathscr{E}^+_{NO}$

Remark: From now on observability and reconstructability will be confused

Suppose a discrete-time system with

$$\begin{cases} x(t+1) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

part of non-obsv. set

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} * \\ 0 \end{bmatrix} \begin{matrix} \in \mathcal{E}^{+}_{NO} \\ \notin \mathcal{E}^{-}_{NR} \end{matrix}$$

$x_1(t+1) = x_2(t)$

$x_2(t+1) = 0$

$y(t) = x_2(t)$

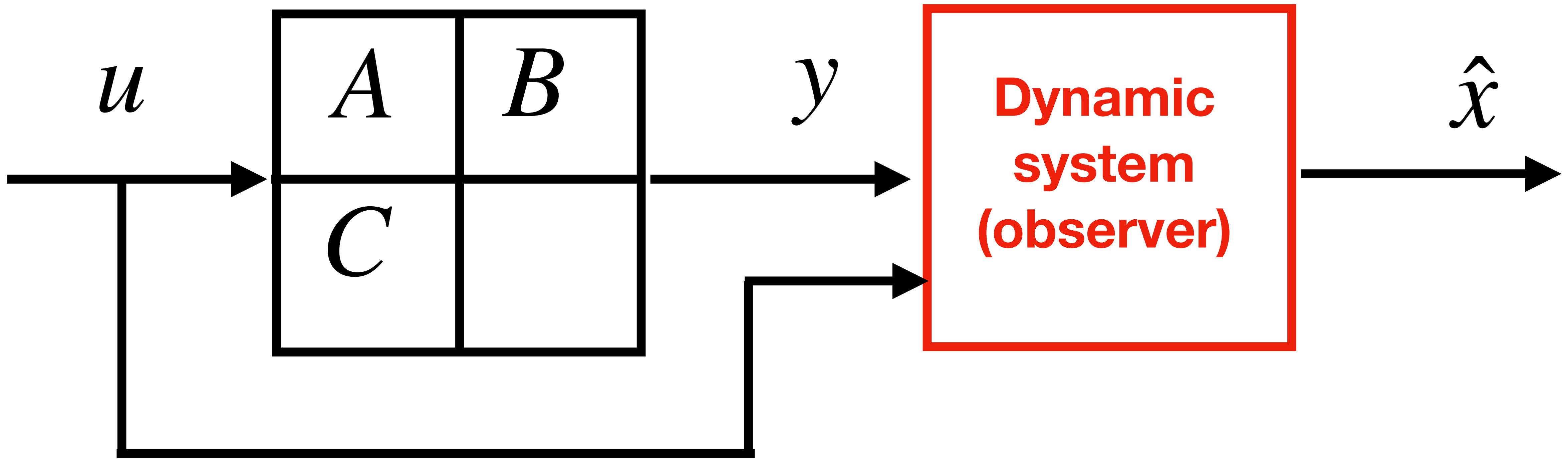
Observability and Asymptotic Observers

Observability and asymptotic observers

dynamic observer

Up to now we focused on computing the initial condition responsible of an output trajectory. If the system is completely observable we can reconstruct, without ambiguity, the initial condition (also in an arbitrary time for C-T systems). Now we are interested to link the observability property of a system to the ability of designing an "observer" able to asymptotically reproduce the state of the system

Not computing initial states but reproducing dynamics



$$\lim_{t\to\infty} \hat{x}(t) - x(t) = 0$$

Can we design observers with estimation error dynamics arbitrarily assignable?

Observability and asymptotic observers

Question: Are observability properties affected by change of coordinates? For instance, if (A, C) is completely observable, is a "similar pair" $(\tilde{A}, \tilde{C}) = (TAT^{-1}, CT^{-1})$ also completely observable?

change the coord. to design the obs.

in original coordinate

$$\tilde{O} = \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{pmatrix} \qquad \tilde{O} = O\,T^{-1} \qquad O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

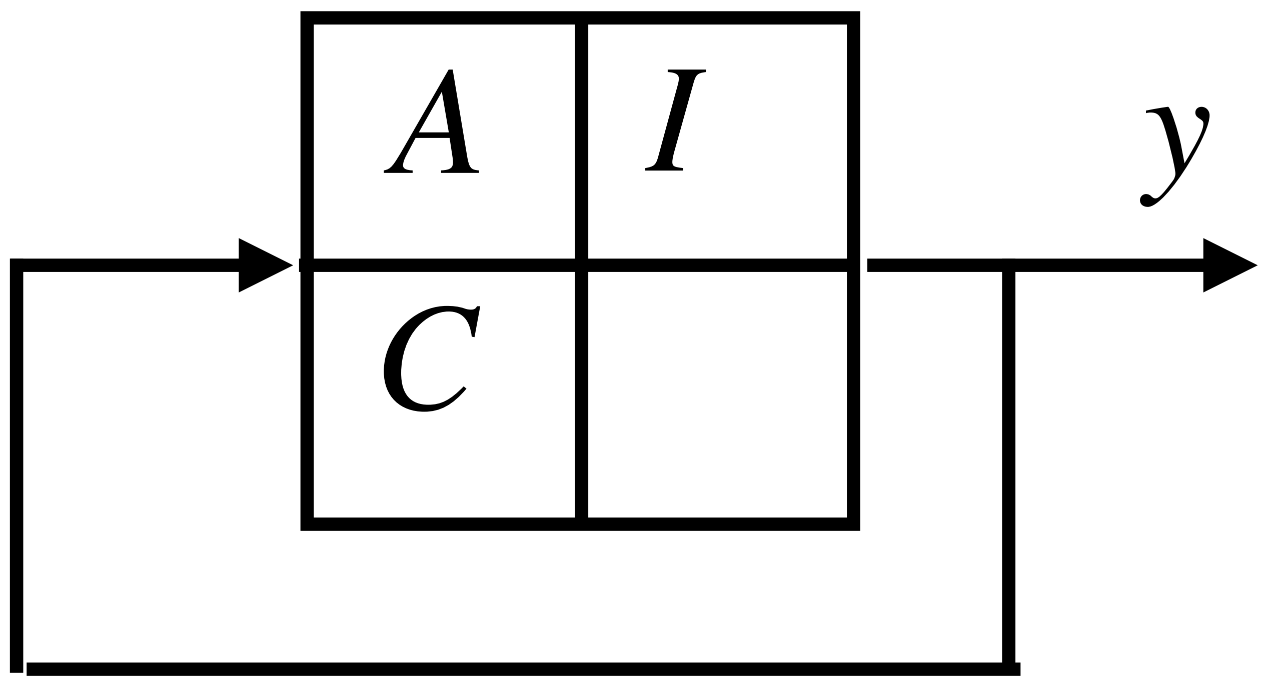
$$\begin{bmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \\ \vdots \\ CT^{-1}TAT^{-1}\dots TAT^{-1} \end{bmatrix} = \begin{bmatrix} CT^{-1} \\ CAT^{-1} \\ \vdots \\ CA^{n}T^{-1} \end{bmatrix}$$

$\Rightarrow \;\; \text{rank} O = \text{rank} \tilde{O}$

Observability/reconstructability is a "structural property", not affected by the coordinate framework used to describe the system

Observability/reconstructability of the system

Output injection



Question: Are Observability properties affected by output injection ?

(A,B) comp. controllable $\Rightarrow$ $(A+BK, B)$ comp. controllable $\forall K$

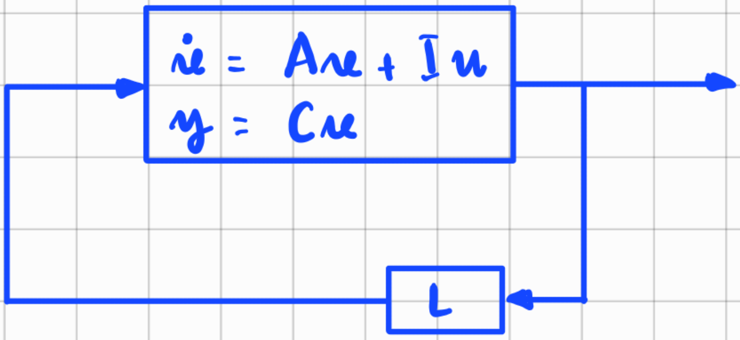
Result

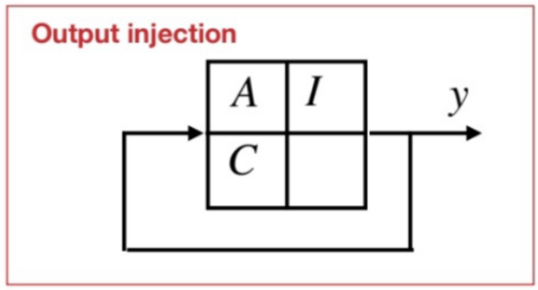
(A, C) completely observable $\Rightarrow$

$(A + LC, C)$ completely observable $\forall\, L$

Homework: prove it

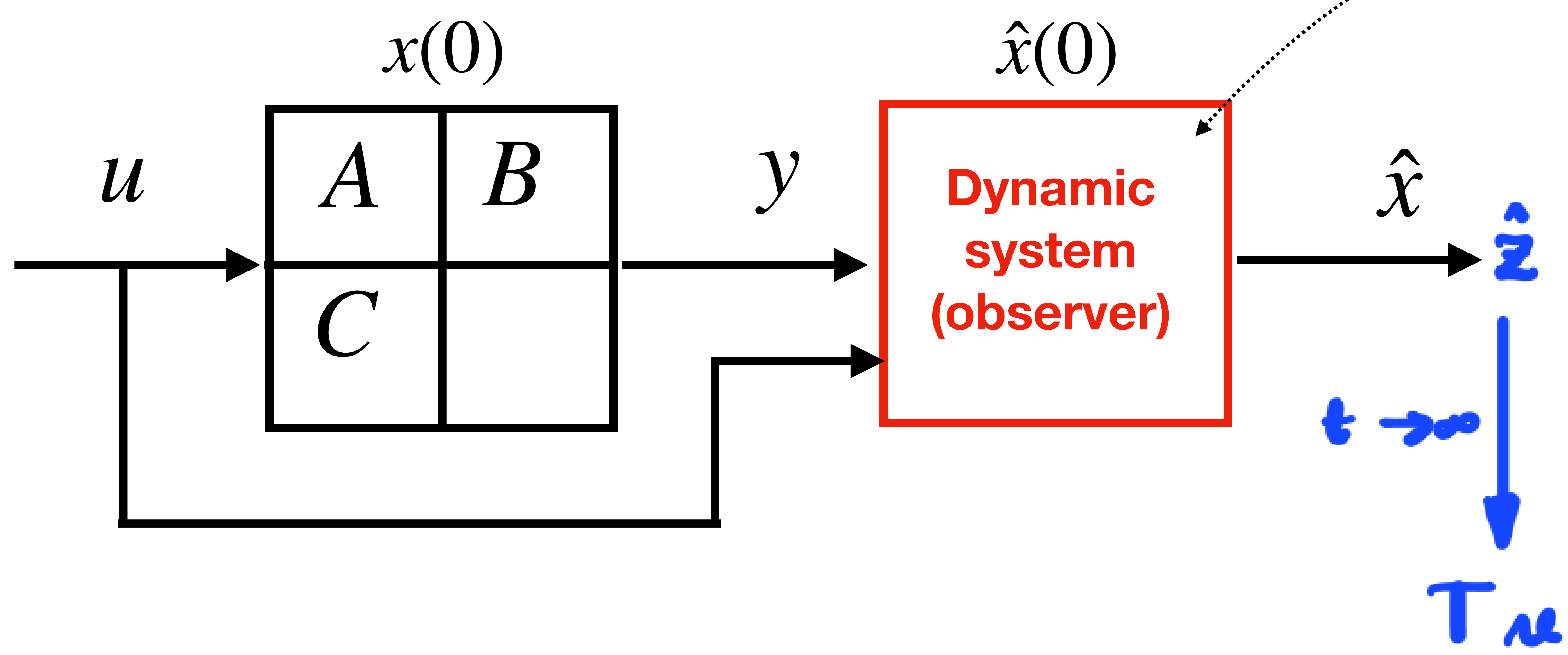
Suppose $B = I$,





Luenberger (identity) Observer

"copy"



estimation of the output

$$\hat{y}(t) = C\hat{x}(t)$$

$$\left.\begin{array}{c}\dot{\hat{x}}(t)\\ \hat{x}(t+1)\end{array}\right\} = A\hat{x}(t) + Bu(t) + {\color{red}L}\,\underbrace{(\hat{y}(t) - y(t))}_{\text{"Innovation"}}$$

Output injection matrix (dof)

If $\hat{x}(0) = x(0)$ then $x(t) \equiv \hat{x}(t)$ for all $t \geq 0$. What happen for different initial conditions?

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \to \begin{pmatrix} x \\ e := \hat{x} - x \end{pmatrix} = T \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

cascade

$$\left.\begin{array}{c}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t)$$

$$\left.\begin{array}{c}\dot{\hat{x}}(t)\\ \hat{x}(t+1)\end{array}\right\} = (A + LC)\,\hat{x}(t) - LCx(t) + Bu(t)$$

$$\xrightarrow{T}$$

parallel

$$\left.\begin{array}{c}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t)$$

$$\left.\begin{array}{c}\dot{e}(t)\\ e(t+1)\end{array}\right\} = {\color{red}(A + LC)}\, e(t)$$

becomes autonomous system

Can L be designed so that $A + LC$ is Hurwitz/Schur??

Suppose the system has defined A, B, and $u(t)$

$$\boxed{\dot{x}(t) = A\underline{x(t)} + Bu(t)} \qquad \underline{x(0)}$$

unknown unknown

observed sys

We create a replica of the system:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t),$$

$\hat{x}(0) = \cancel{x(0)}$ UNKNOWN

$\downarrow$ estimation $\qquad \Downarrow$ $\hat{x}(t) \equiv x(t)$

what Luenberger does

$n \times p$ (D.O.F.)

observer

$$\boxed{\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C\hat{x}(t)\end{aligned}}$$

estimation

where $\dim \hat{x} = \dim x$

$$\boxed{y(t) = C\underline{x(t)}}$$

UNKNOWN

known

observed sys

If we "know" the initial state better, pick any of $\hat{x}(0)$. So that the error eq:

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \to \begin{pmatrix} x \\ e \end{pmatrix} = \overset{T}{\begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

$$e(t) \triangleq \hat{x}(t) - x(t)$$

$$\dot{e}(t) = \underbrace{A\hat{x} + Bu + L(C\hat{x} - Cx)}_{\dot{\hat{x}}} - Ax - Bu$$

Observability and asymptotic observers

Can we identify a coordinate framework where the design of L is easier ?!

Observability canonical form $p = 1$)

$$A_o = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{pmatrix}$$

non-zero

$$\varphi_A(\lambda) = \det(\lambda I - A)$$
$$= \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n$$

$$C_o = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$p = 1$, single output

Theorem ($p = 1$)

A pair (A, C) is similar to (A_o, C_o) $\boxed{\text{iff the system is Completely Observable}}$

$$O_o = \begin{pmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n-1} \end{pmatrix} \qquad T = T_0 := \boxed{O_o^{-1}\, O} \qquad O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Changing the coordinate using T_0:

$$T_0(A+LC)T_0^{-1} = \underbrace{T_0 A T_0^{-1}}_{A_0} + \underbrace{T_0 L}_{L_0}\,\underbrace{C T_0^{-1}}_{C_0}$$

$$= \sigma(A_0 + L_0 C_0) = \sigma(A + LC)$$

$$\underset{A_0}{\begin{bmatrix} 0 & 0 & \cdots & -\alpha_n \\ 1 & 0 & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\alpha_1 \end{bmatrix}} + \underset{L_0}{\begin{bmatrix} l_n \\ \vdots \\ \vdots \\ \vdots \\ l_1 \end{bmatrix}} \underset{C_0}{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}} = A_0 + L_0 C_0$$

$$\begin{bmatrix} 1 & 0 & \underbrace{-\alpha_n + l_n}_{-\alpha_n^+} \\ & \ddots & \\ 0 & 1 & \underbrace{-\alpha_1 + l_1}_{-\alpha_1^+} \end{bmatrix}$$

Observability and asymptotic observers

eigenvalues assignment theorem

Theorem ($p \geq 1$)

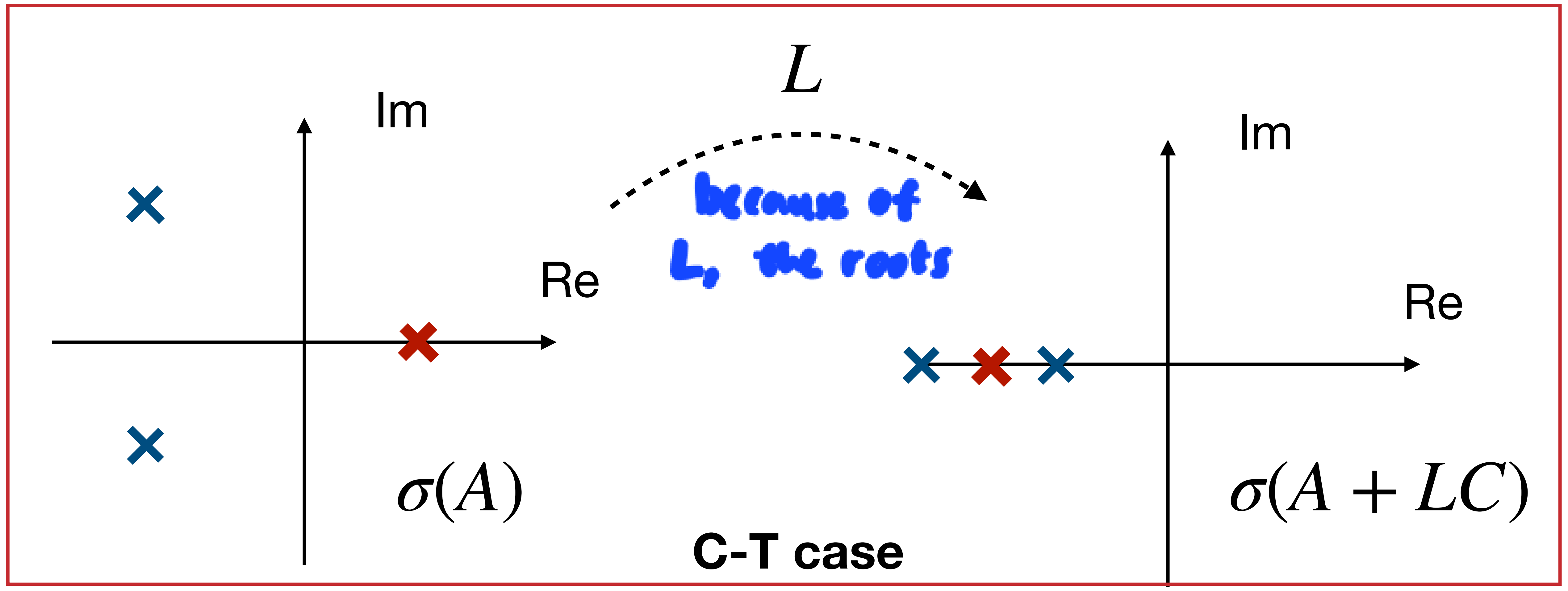
A pair (A, C) is completely Observable **iff** for all $\{\lambda_1^\star, \ldots, \lambda_n^\star\}$ (set of desired eigenvalues) there exists a L such that $\sigma(A + LC) = \{\lambda_1^\star, \ldots, \lambda_n^\star\}$

Constructive proof of the if part ($p = 1$):

- Let $(\alpha_1^\star, \ldots, \alpha_n^\star)$ be such that $\{\lambda_1^\star, \ldots, \lambda_n^\star\}$ are roots of $\lambda^n + \alpha_1^\star \lambda^{n-1} + \ldots + \alpha_{n-1}^\star \lambda + \alpha_n^\star = 0$

  $(\lambda - \lambda_1^*)(\lambda - \lambda_2^*) \cdots (\lambda - \lambda_n^*) = 0$
- Let $L_o = (\alpha_n - \alpha_n^\star \quad \cdots \quad \alpha_1 - \alpha_1^\star)^T$
- Pick $L = T_o^{-1} L_o$

Full authority in eigenvalues assignment



Corollary

A completely observable system can be always observed asymptotically (…and much more!)

Start from $A+LC$ in original coordinate. Then determine the eigval

$$\sigma(A+LC) = \sigma(T_o(A+LC)T_o^{-1}) = \sigma(A_o + L_o C_o)$$

$$L_o = T_o L \qquad L = T_o^{-1} L$$

$$\underbrace{\begin{bmatrix} 0 & \cdots & 0 & -\alpha_n \\ 1 & & 0 & -\alpha_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -\alpha_1 \end{bmatrix}}_{A_o} + \underbrace{\begin{bmatrix} l_n \\ l_{n-1} \\ \vdots \\ l_1 \end{bmatrix}}_{L_o} \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}}_{C_o}$$

$A_o + L_o C_o$

$$\begin{bmatrix} 0 & \cdots & 0 & -\alpha_n + l_n \\ 1 & & 0 & -\alpha_{n-1} + l_{n-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -\alpha_1 + l_1 \end{bmatrix}$$

and $l_i = \alpha_i - \alpha_i^*$ is to be designed

Using the notion:

$$\sigma(M) = \sigma(M^T)$$

$$\Updownarrow$$

$$\sigma(A+BK) = \sigma(A^T + K^T B^T)$$

in MATLAB:

$$K = \text{place}(A, B, \{\lambda^*\})$$

$$\Updownarrow$$

$$\sigma(A+BK) = \{\lambda^*\}$$

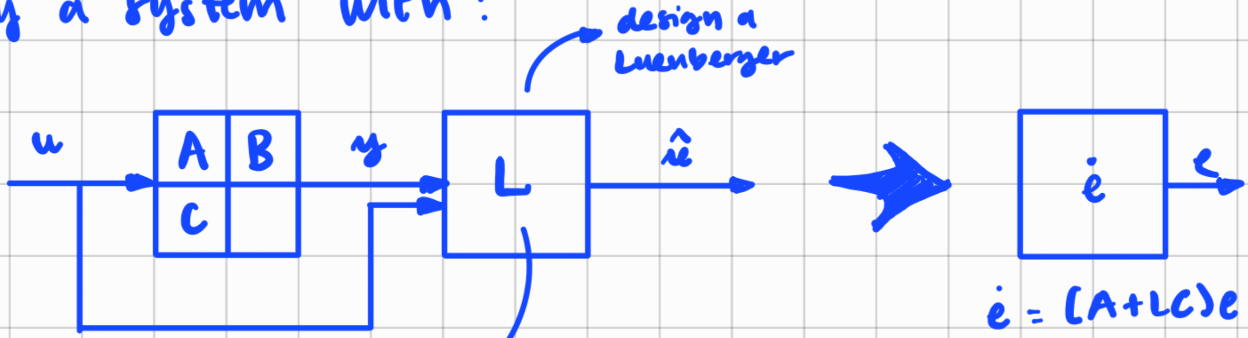
$$L^T = \text{place}(A^T, C^T, \{\lambda^*\})$$

$$\sigma(A^T + C^T L^T) = \{\lambda^*\}$$

$$\|$$

$$\sigma(A+LC)$$

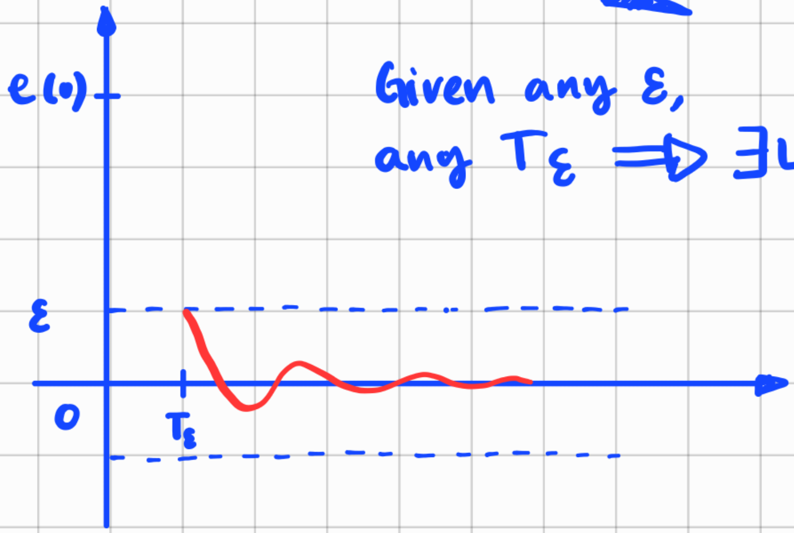
Say a system with:



$\dot{e} = (A+LC)e$

$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y)$

$|e(t)| \leq c(K^*)e^{-K^*t}$



Given any ε,
any $T_\varepsilon \Rightarrow \exists L$

Kalman decomposition

not fully-rank

Suppose now the (A, C) is not completely observable, namely $\text{rank}O = n_o < n$ ($\dim\mathscr{E}^+ = n_o < n$)

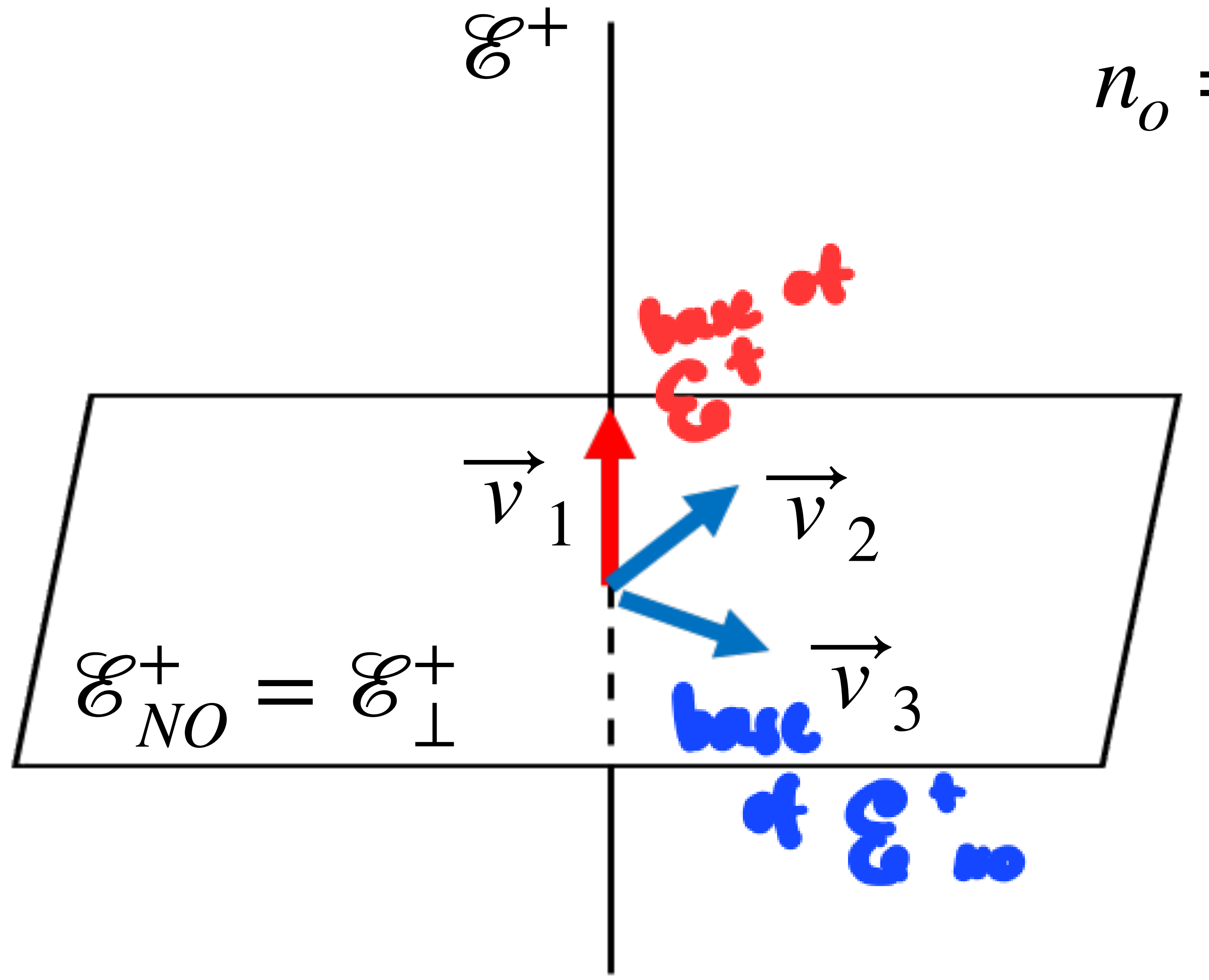
Let $\mathscr{E}^+_\perp$ be the orthogonal complement of $\mathscr{E}^+$. It turns out that $\dim\mathscr{E}^+_\perp = n - n_o$

Remark: $\mathscr{E}^+_\perp = \mathscr{E}^+_{NO}$

Let $\{\vec{v}_1, \ldots, \vec{v}_{n_o}\}$ be a base of $\mathscr{E}^+$ and let $\{\vec{v}_{n_o+1}, \ldots, \vec{v}_n\}$ be a base of $\mathscr{E}^+_\perp$.
The two sets of vectors are all linearly independent

$n = 3$

$n_o = 1$

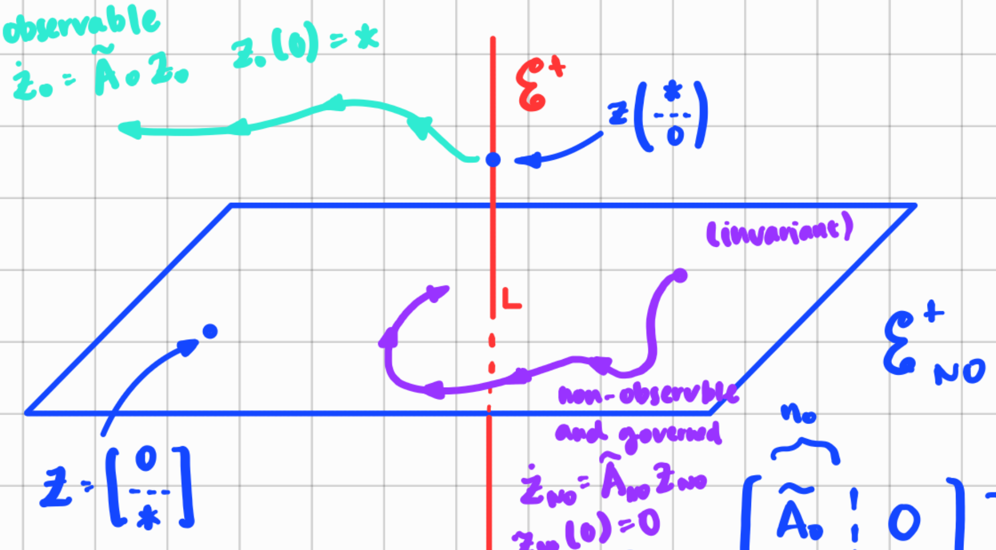


Consider the change of variables $T_K^{-1} = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_{n_o} & \vec{v}_{n_o+1} & \ldots & \vec{v}_n \end{bmatrix}$

$$z = T_K\, x = \begin{pmatrix} z_o \\ z_{no} \end{pmatrix} \qquad x = T_K^{-1} z$$

It turns out that $x \in \mathscr{E}^+ \;\Rightarrow\; z = \begin{pmatrix} \star \\ 0 \end{pmatrix} \qquad x \in \mathscr{E}^+_\perp \;\Rightarrow\; z = \begin{pmatrix} 0 \\ \star \end{pmatrix}$

We have $\mathcal{E}^+_{NO} \supset \{0\}$ $\quad \mathcal{E}^+ = (\mathcal{E}^+_{NO})_\perp \subset \mathbb{R}^n$



$$z = T_K u$$
$$u = T_K^{-1} z$$

$$\tilde{A}_K = \begin{array}{c} \overbrace{\quad}^{n_o} \\ \left[\begin{array}{c|c} \tilde{A}_o & 0 \\ \hline \tilde{A}_J & \tilde{A}_{NO} \end{array}\right] \end{array} \begin{array}{l} \}\, n_o \\ \}\, n - n_o \end{array}$$

$$\tilde{C}_K = \left[\begin{array}{c|c} \tilde{C}_J & 0 \end{array}\right]$$

$$\dot{z}_o = \tilde{A}_o z_o \qquad z_o(0) = 0$$
$$z_o(t) \equiv 0, \ \forall t \geq 0$$

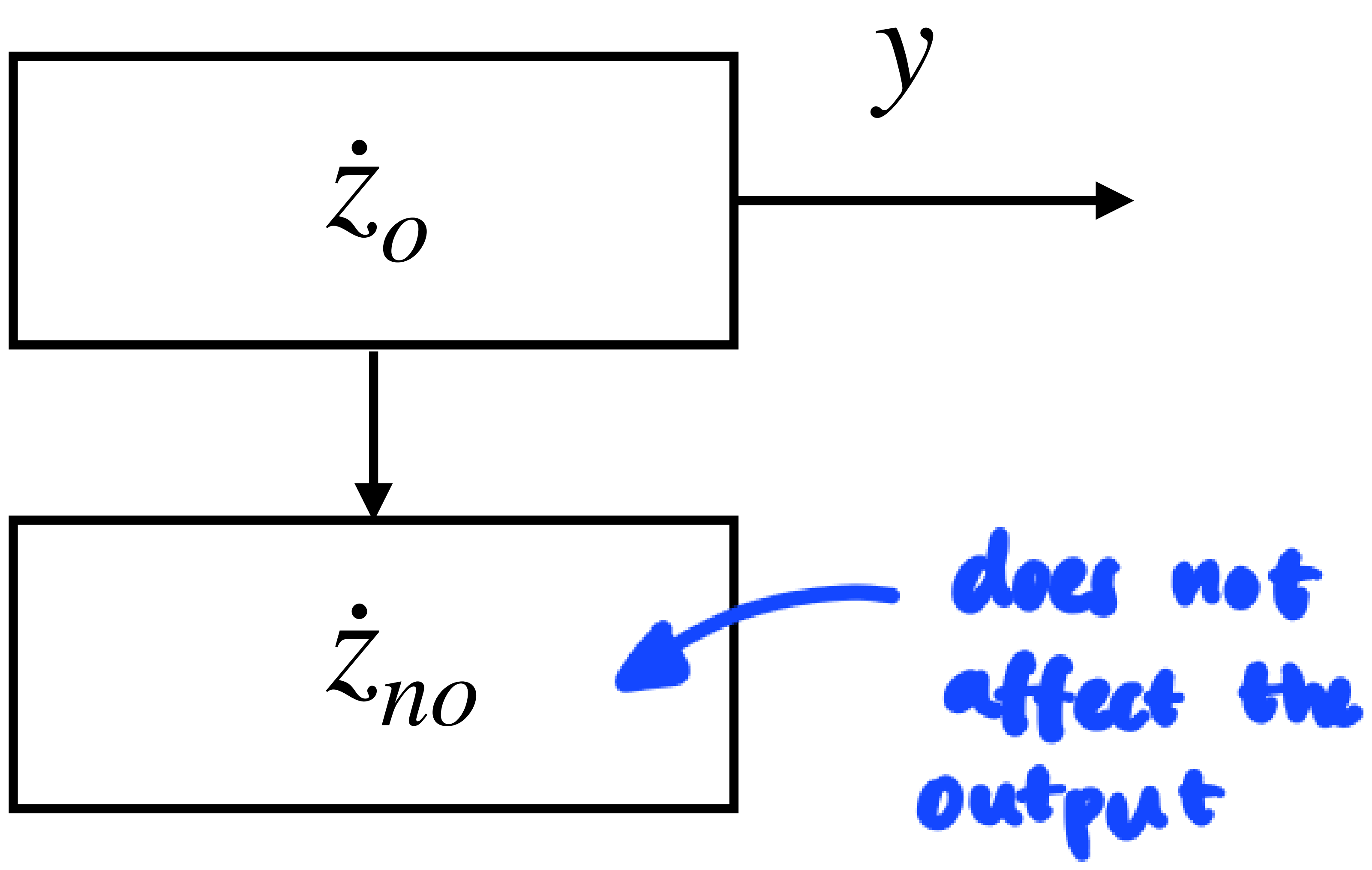
Kalman decomposition

Result (by Kalman)

$$\tilde{A}_K = T_K\, A\, T_K^{-1} = \begin{array}{c} n_o \times n_o \\ \begin{pmatrix} \tilde{A}_O & 0 \\ \tilde{A}_J & \tilde{A}_{NO} \end{pmatrix} \\ \qquad\qquad n - n_o \times n - n_o \end{array} \qquad \tilde{C}_K = C\, T_K^{-1} = \begin{pmatrix} \tilde{C}_O & 0 \end{pmatrix}$$

$\boxed{(\tilde{A}_O, \tilde{C}_O) \quad \text{Completely observable}}$

$$\begin{aligned} \dot{x}/x^+ &= A\,x \\ y &= C\,x \end{aligned} \quad \Rightarrow \quad \begin{cases} \dot{z}_o/z_o^+ = \tilde{A}_O\, z_o \\ \dot{z}_{no}/z_{no}^+ = \tilde{A}_{NO}\, z_{no} + \tilde{A}_J\, z_o \\ y = \tilde{C}_O\, z_o \end{cases}$$



- The set $\mathscr{E}_\perp^+$ is forward invariant for the system dynamics. The "internal dynamics" are not observable

- If $\tilde{A}_{NO}$ is Hurwitz then trajectories starting inside $\mathscr{E}_\perp^+$ asymptotically converge to zero

Detectability

Definition

A system (A, C) is said to be **detectable** if there exists a L such that $A + LC$ is Hurwitz (Schur)

Remark

implies

(A, C) completely observable $\Rightarrow$ (A, C) detectable

$\cancel{\Leftarrow}$

but not vice versa

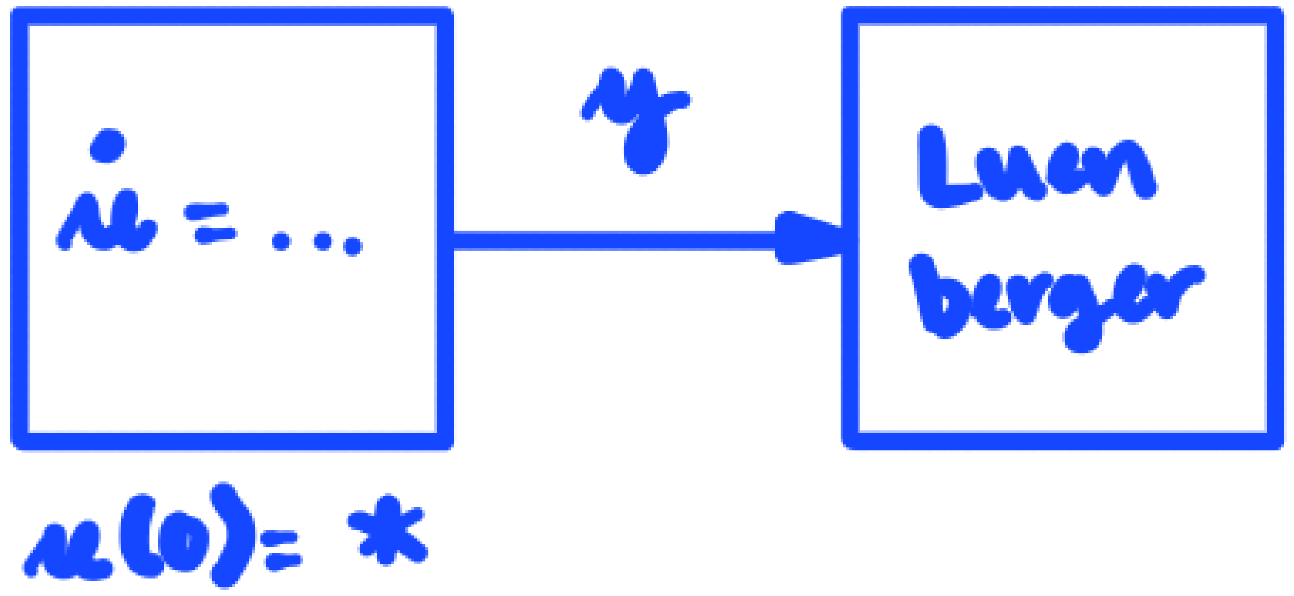
Theorem

(A, C) is detectable **iff** $\tilde{A}_{NO}$ is Hurwitz (Schur)

if unobsv. dynamics is stable, it's detectable

Constructive proof ($\tilde{A}_{no}$ Hurwitz/Schur $\Rightarrow \exists\, L \;:\; (A + LC)$ is Hurwitz/Schur

The selection of L is: $\quad L = T_K^{-1}\, L_K \qquad L_K = [L_{Ko}\ \star]$ with K_{Ko} so that $\tilde{A}_O + K_{Ko}\tilde{C}_O$ is Hurwitz/Schur



Known $\tilde{A}_{NO}$ Hurwitz / Schur

? : L : $A + LC$ is Hurwitz / Schur

$$T_K (A + LC) T_K^{-1} = \tilde{A}_K + \underset{\overset{\|}{T_K L}}{L_K} \tilde{C}_K$$

$$\underbrace{\left[\begin{array}{c|c} \tilde{A}_O & 0 \\ \hline \tilde{A}_J & \tilde{A}_{NO} \end{array}\right]}_{\tilde{A}_K} + \underbrace{\left[\begin{array}{c} L_{KO} \\ \hline L_{KNO} \end{array}\right]}_{L_K} \left[\begin{array}{c|c} \tilde{C}_O & 0 \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_O + L_{KO}\tilde{C}_O & 0 \\ \hline \tilde{A}_J + L_{KNO}\tilde{C}_O & \tilde{A}_{NO} \end{array}\right]$$

Good (Hurwitz/Schur)

$$\Downarrow$$

$$\sigma(\tilde{A}_K + L_K \tilde{C}_K) = \sigma(\underbrace{\tilde{A}_O + L_{KO}\tilde{C}_O}_{\text{arbitrarily assignable}}) \cup \sigma(\tilde{A}_{NO}) \rightarrow \text{Good}$$

$$L_K = \begin{bmatrix} L_{KO} \\ \cdots\cdots \\ * \end{bmatrix} \begin{array}{l} \} n_o \\ \\ \} n - n_o \end{array}$$

Observability of a cascade

$x \in \mathbb{R}^n$

$$\dot{x} = Ax \qquad y_x = Cx$$

$y_x \in \mathbb{R}$

$z \in \mathbb{R}^\ell$

$$\dot{z} = Fz + Gy_x \qquad y = Hz$$

$y \in \mathbb{R}$

(A, C), (F, H) observable

$\Downarrow$??

$\begin{pmatrix} A & 0 \\ GC & F \end{pmatrix}, (0 \quad H)$ observable

Result

The cascade is observable iff the pairs (A, C), (F, H) are observable and

$$\text{rank} \begin{pmatrix} \lambda I - F & G \\ H & 0 \end{pmatrix} = n + 1 \qquad \forall\, \lambda \in \sigma(A)$$

Non resonance condition