Joining Controllability and Observability: Separation Principle and Ultimate Kalman

Master degree in Automation Engineering

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Dynamic output feedback

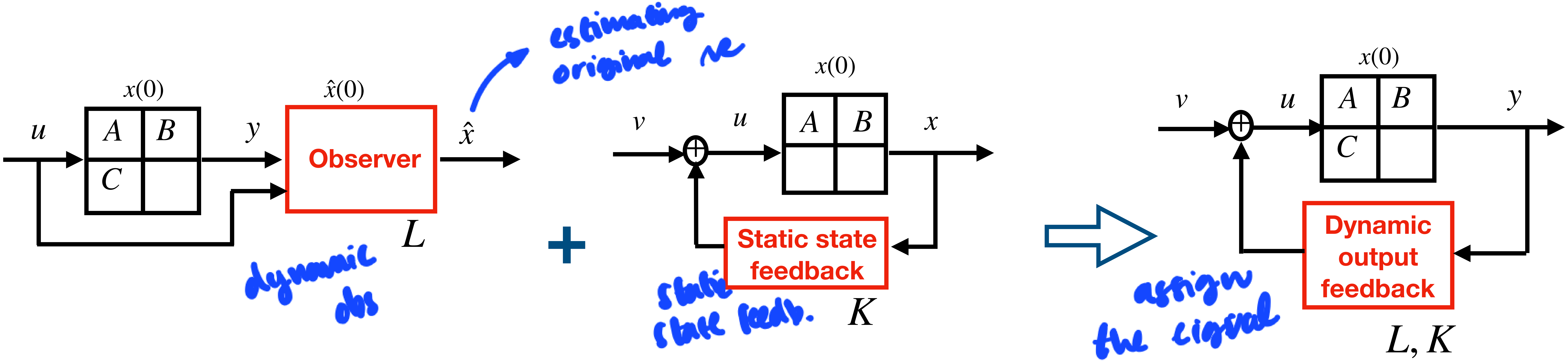
$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t) \qquad x(0) = x_0$$

$$y = Cx(t)$$

generic $u(t)$

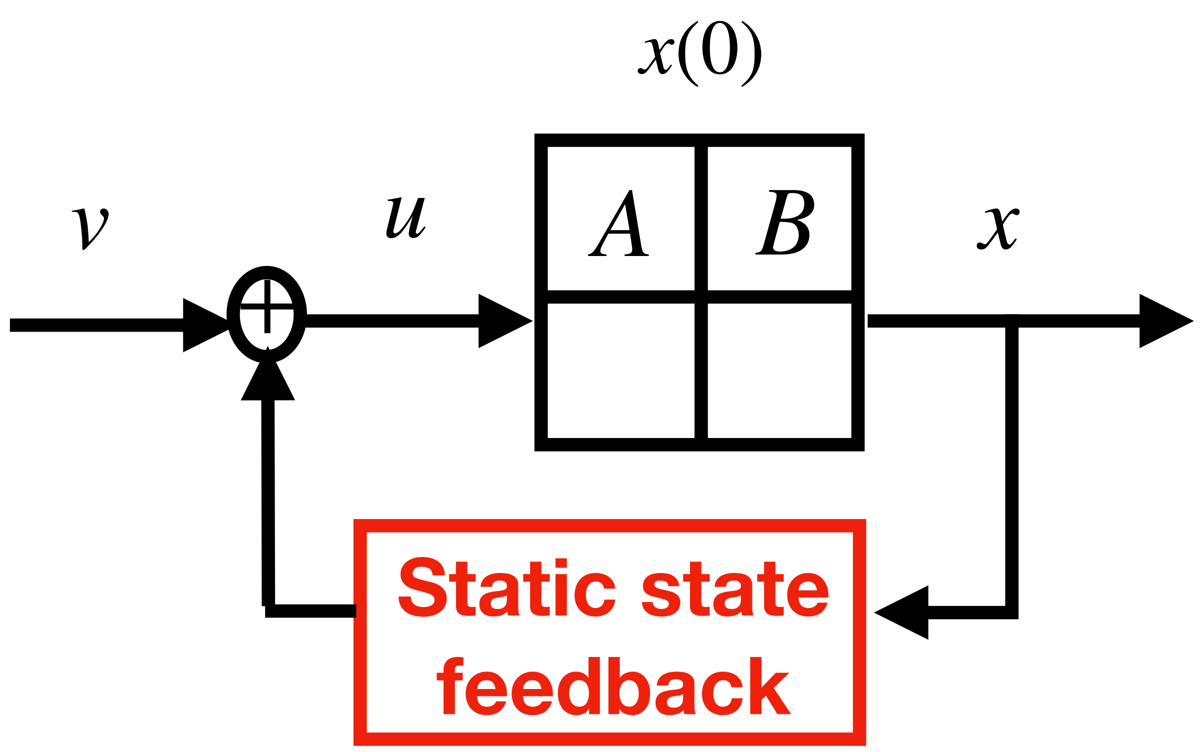
Complete system, with (control) inputs and outputs

Main question to be answered: suppose that the system (A, B) is **completely controllable** (if the state were accessible we could assign arbitrary dynamics by static state feedback) and **completely observable** (by processing the output we can reproduce the internal state with arbitrary dynamics and for all possible inputs), can we control the system by output feedback by arbitrarily assigning the closed-loop dynamics ?

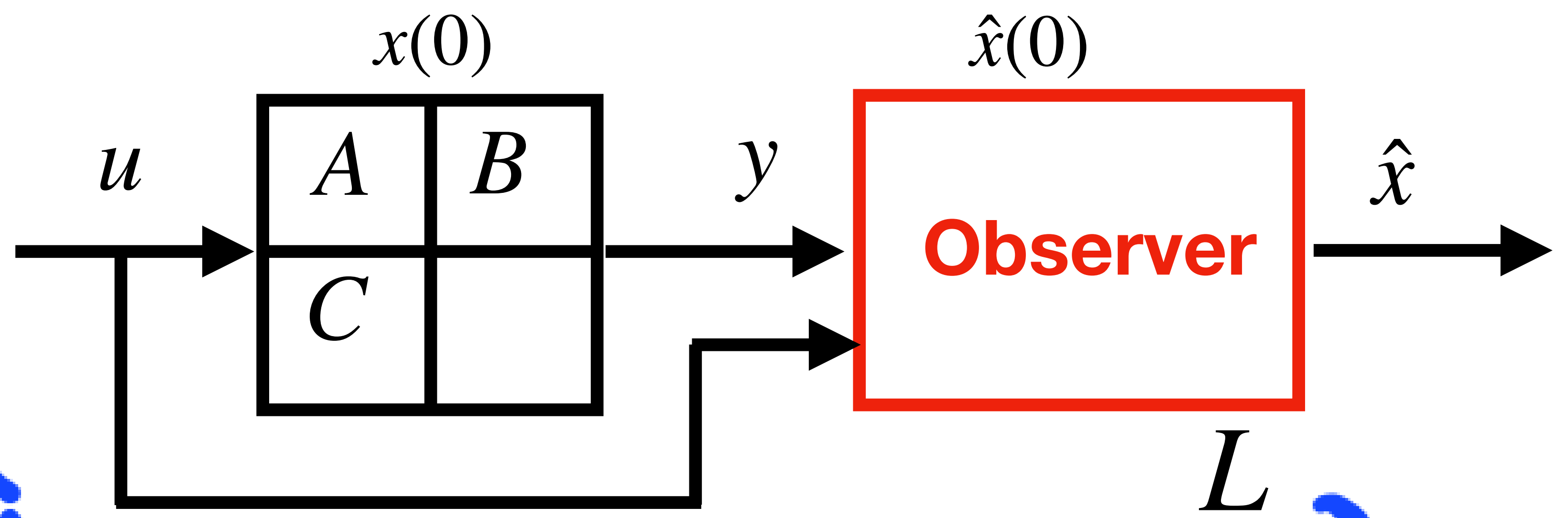


Separation principle

The most natural solution is to adopt the static state feedback designed as x were available and then substitute it with the estimate provided by an identity observer



K



L

replace with $\hat{x}$

$x \to \hat{x}$

$$u = K\,x + v$$

x is unknown

$$\left.\begin{array}{c}\dot{\hat{x}}(t)\\ \hat{x}(t+1)\end{array}\right\} = A\hat{x}(t) + Bu(t) + L\,(\hat{y}(t) - y(t))$$

Dynamic output feedback

y → , v - - → , → u

$$\left.\begin{array}{c}\dot{\hat{x}}(t)\\ \hat{x}(t+1)\end{array}\right\} = A\hat{x}(t) + B[K\hat{x} + v] + L\,(\hat{y}(t) - y(t))$$

$$u = K\,\hat{x} + v$$

$$\dot{x} = Ax + B\left(K\hat{x} + v\right)$$
$$\dot{\hat{x}} = A\hat{x} + B\left(K\hat{x} + v\right) + L\left(C\hat{x} - Cx\right)$$

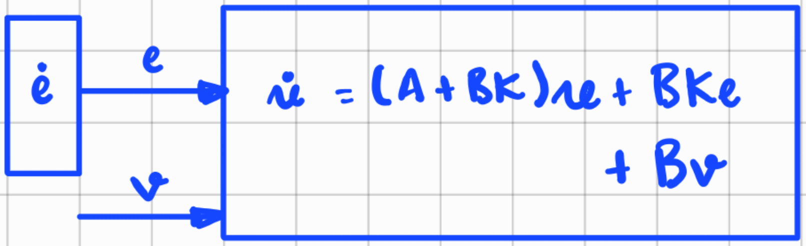
$$\hat{x} \to e \overset{\text{defined}}{\triangleq} \hat{x} - x$$

$$\dot{x} = Ax + BK(\overbrace{\underbrace{x+e}_{u}}^{\hat{x}}) + Bv$$

$$\dot{e} = (A+LC)e$$

$$\dot{x} = (A+BK)x + BKe$$
$$\dot{e} = (A+LC)e$$

Cascade system:



What's the eigval of this closed-loop system?

$$\dot{x} = (A+BK)x + BKe + Bv$$
$$\dot{e} = (A+LC)e$$

$$\underbrace{\left[\begin{array}{c|c} A+BK & BK \\ \hline 0 & A+LC \end{array}\right]\begin{bmatrix} B \\ 0 \end{bmatrix}}$$

$$\boxed{\sigma = \sigma(A+BK) \cup \sigma(A+LC)}$$

Separation principle

Changing coordinates as $\begin{pmatrix} x \\ \hat{x} \end{pmatrix} \to \begin{pmatrix} x \\ e := \hat{x} - x \end{pmatrix} = T \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$ the closed-loop is

e is arbitrarily small
with very small time

$$\boxed{\left.\begin{matrix} \dot{e}(t) \\ e(t+1) \end{matrix}\right\} = (A + LC)\, e(t)} \xrightarrow{e} \boxed{\left.\begin{matrix} \dot{x}(t) \\ x(t+1) \end{matrix}\right\} = (A + BK)x(t) + BKe(t) + Bv}$$

$$\boxed{\sigma \begin{pmatrix} A + BK & BK \\ 0 & A + LC \end{pmatrix} = \sigma\,(A + BK) \cup \sigma\,(A + LC)}$$

- Under stabilisability and detectability assumption it is possible to design K and L to that the closed-loop system is Hurwitz/Schur

- If the pair (A, C) is completely observable then the error $e(t)$ can be steered to arbitrarily small values in arbitrarily small time by thus recovering the “ideal” state feedback dynamics

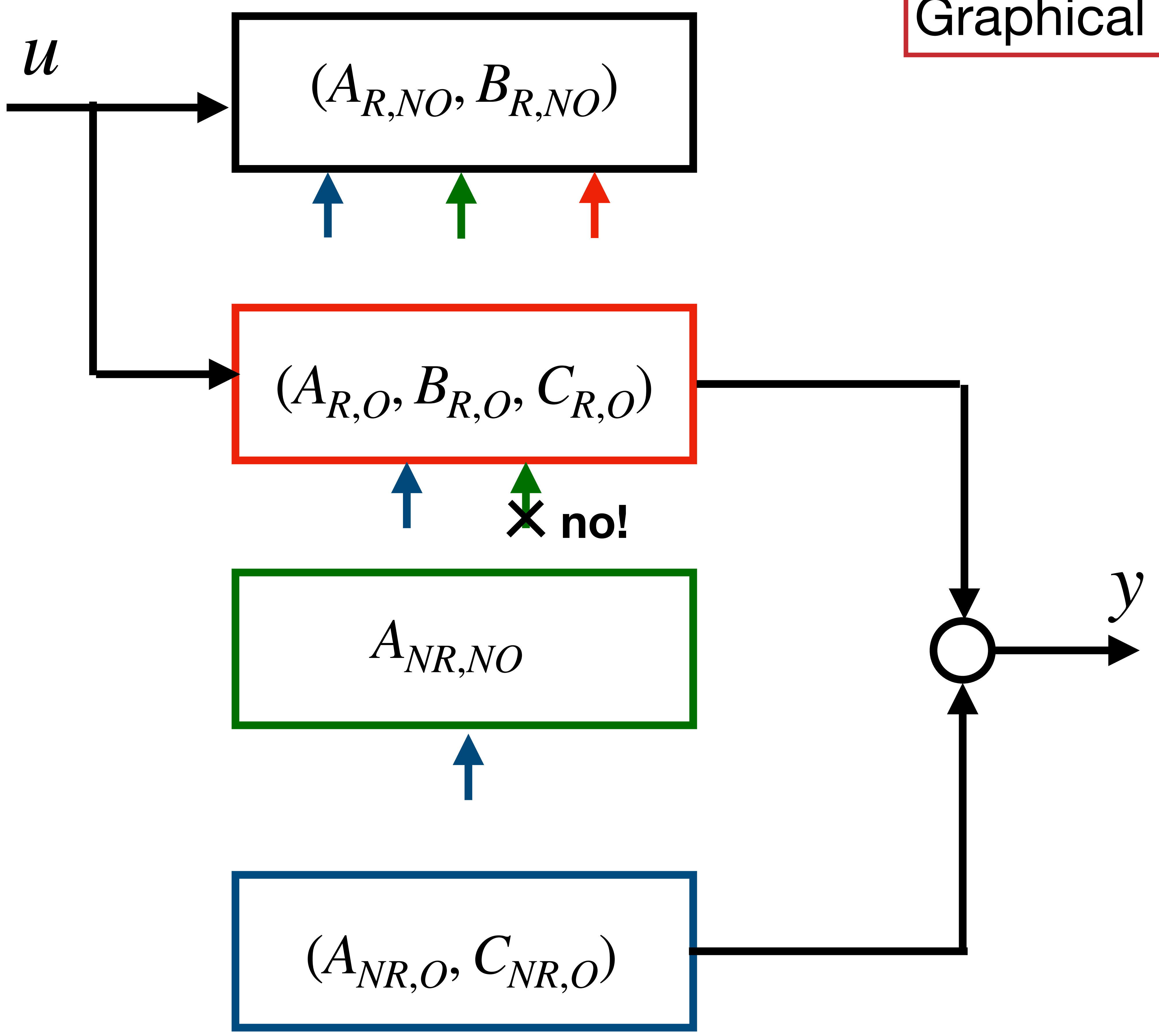
- Peaking phenomena make “aggressive” solutions not “optimal”

Ultimate Kalman decomposition

Suppose that (A, B) is not completely controllable ($\text{rank} R \leq n$) and (A, C) is not completely observable $\text{rank} O \leq n$.
Can we indetify a change of coordinates that put the system in a form suitable to isolate dynamics that are

- controllable and not observable
- controllable and observable
- non controllable and non observable
- non controllable and observable

Graphical intuition



“Arrows” from top to bottom are not allowed (they would “destroy” the properties of the single systems (graphically…)

“Arrows” from bottom to top are (almost all) allowed

Ultimate Kalman decomposition

$$\mathcal{R}^+ = \mathrm{Im} R$$

$$\mathcal{E}^+_{NO} = \mathrm{Ker} O \qquad \mathcal{E}^+ = \mathrm{Im} O^T$$

Theorem: Given (A, B, C)The following T

$$T^{-1} := \left[\overbrace{\mathcal{R}^+ \cap \mathcal{E}^+_{NO}}^{\text{base of}} \;\Big|\; \overbrace{\mathcal{R}^+ \cap \mathcal{E}^+}^{\text{base of}} \;\Big|\; \overbrace{\mathcal{R}^+_{NR} \cap \mathcal{E}^+_{NO}}^{\text{base of}} \;\Big|\; \overbrace{\mathcal{R}^+_{NR} \cap \mathcal{E}^+}^{\text{base of}} \right]$$

makes the system in the following **Kalman form**

$$\tilde{A} = TAT^{-1} := \begin{pmatrix} A_{R,NO} & A'_{R,NO} & A''_{R,NO} & A'''_{R,NO} \\ 0 & A_{R,O} & 0 & A'_{R,O} \\ 0 & 0 & A_{NR,NO} & A'_{NR,NO} \\ 0 & 0 & 0 & A_{NR,O} \end{pmatrix} \qquad \tilde{B} = TB := \begin{pmatrix} B_{R,NO} \\ B_{R,O} \\ 0 \\ 0 \end{pmatrix} \qquad \tilde{C} = CT^{-1} := \begin{pmatrix} 0 & C_{R,O} & 0 & C_{NR,O} \end{pmatrix}$$

where

$(A_{R,NO}, B_{R,NO})$ completely controllable

$(A_{R,O}, C_{R,O})$ completely observable

$\left(\begin{pmatrix} A_{R,NO} & A'_{R,NO} \\ 0 & A_{R,O} \end{pmatrix}, \begin{pmatrix} B_{R,NO} \\ B_{R,O} \end{pmatrix} \right)$ completely controllable

$\left(\begin{pmatrix} A_{R,O} & A'_{R,O} \\ 0 & A_{NR,O} \end{pmatrix}, \begin{pmatrix} C_{R,O} & C_{NR,O} \end{pmatrix} \right)$ completely observable