Linear Systems - Coordinates Change

Master degree in Automation Engineering

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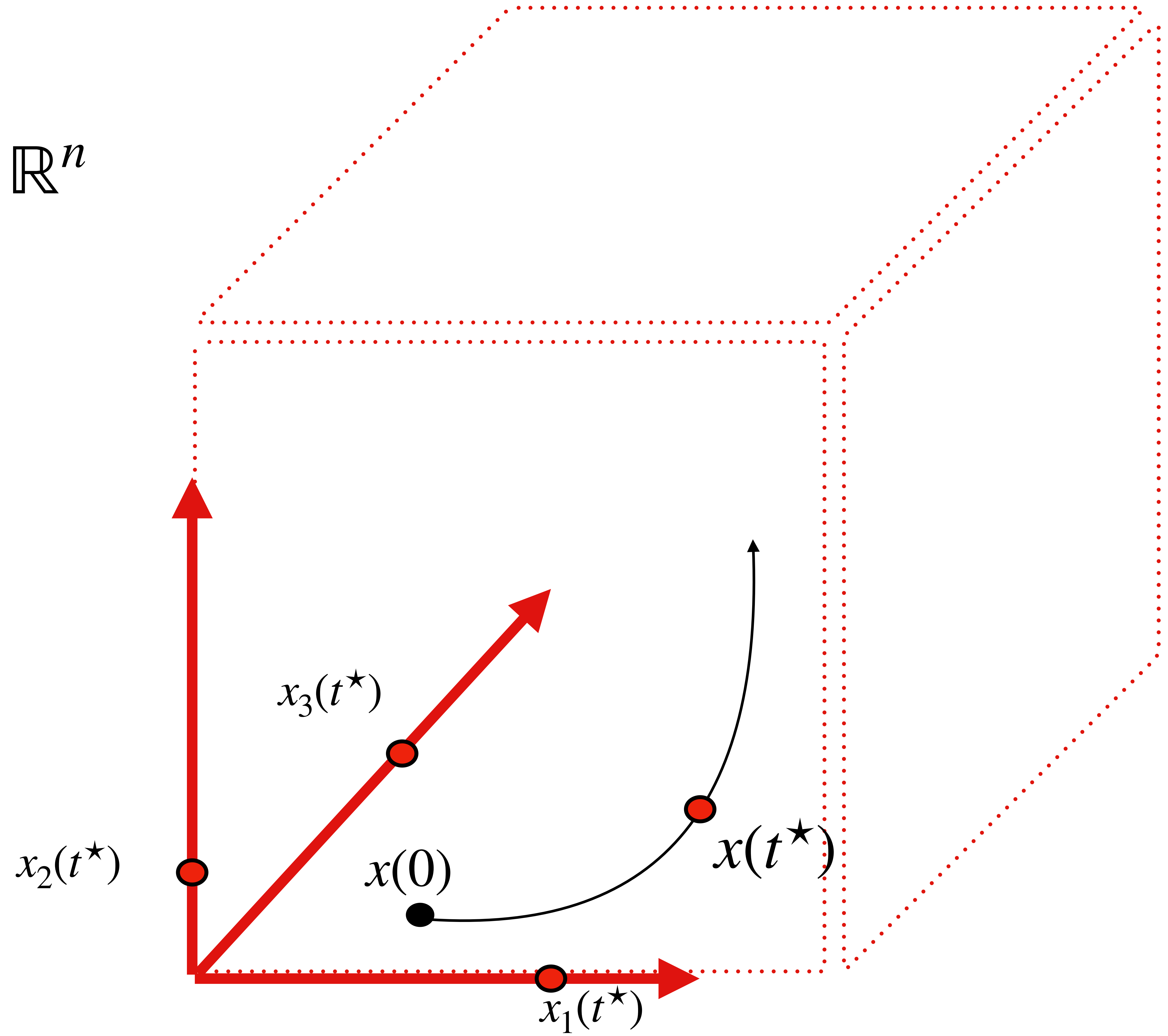
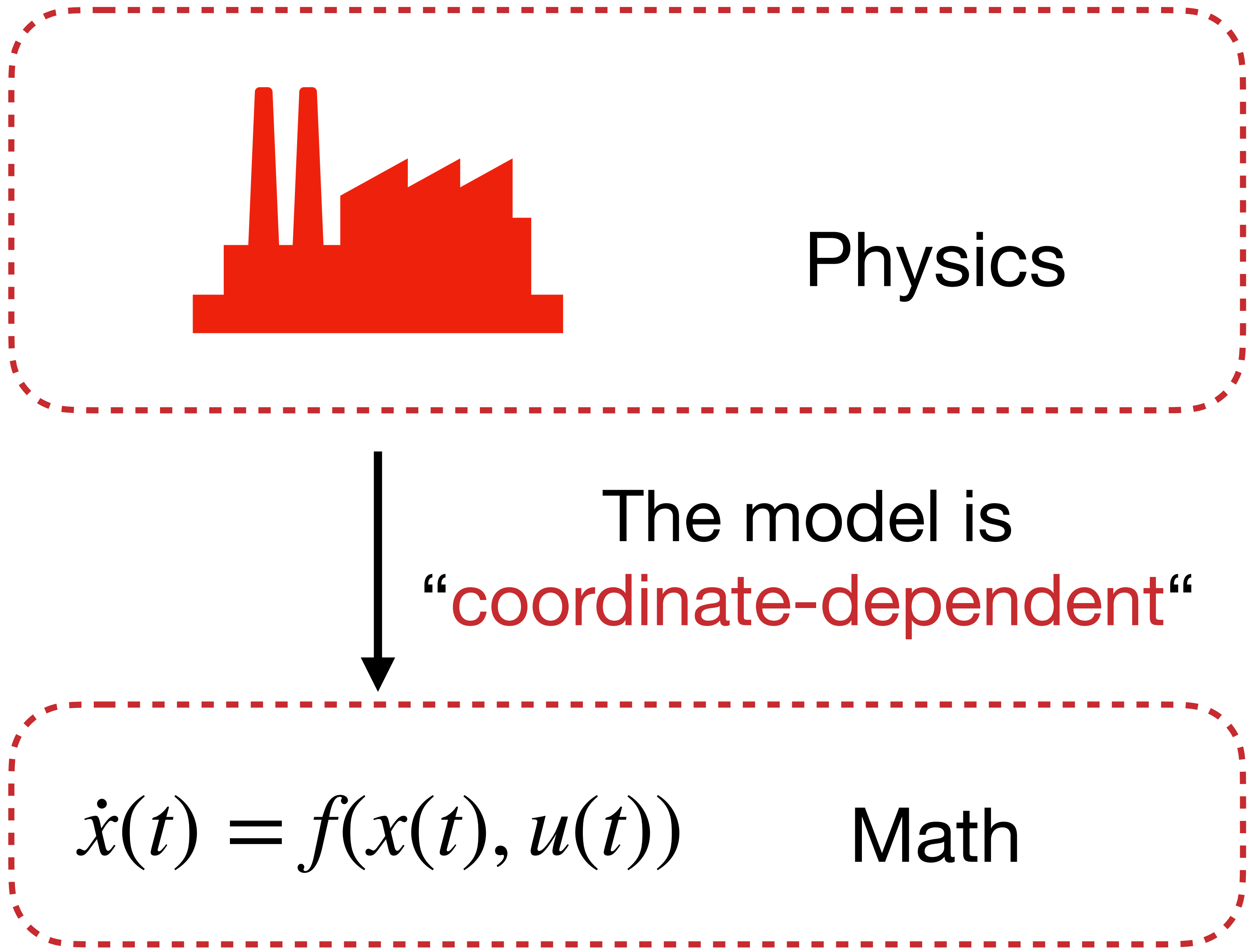
Coordinates Change

A given SS model implicitly "hides" a coordinate framework in $\mathbb{R}^n$ with respect to which the vector x is described





Coordinates Change

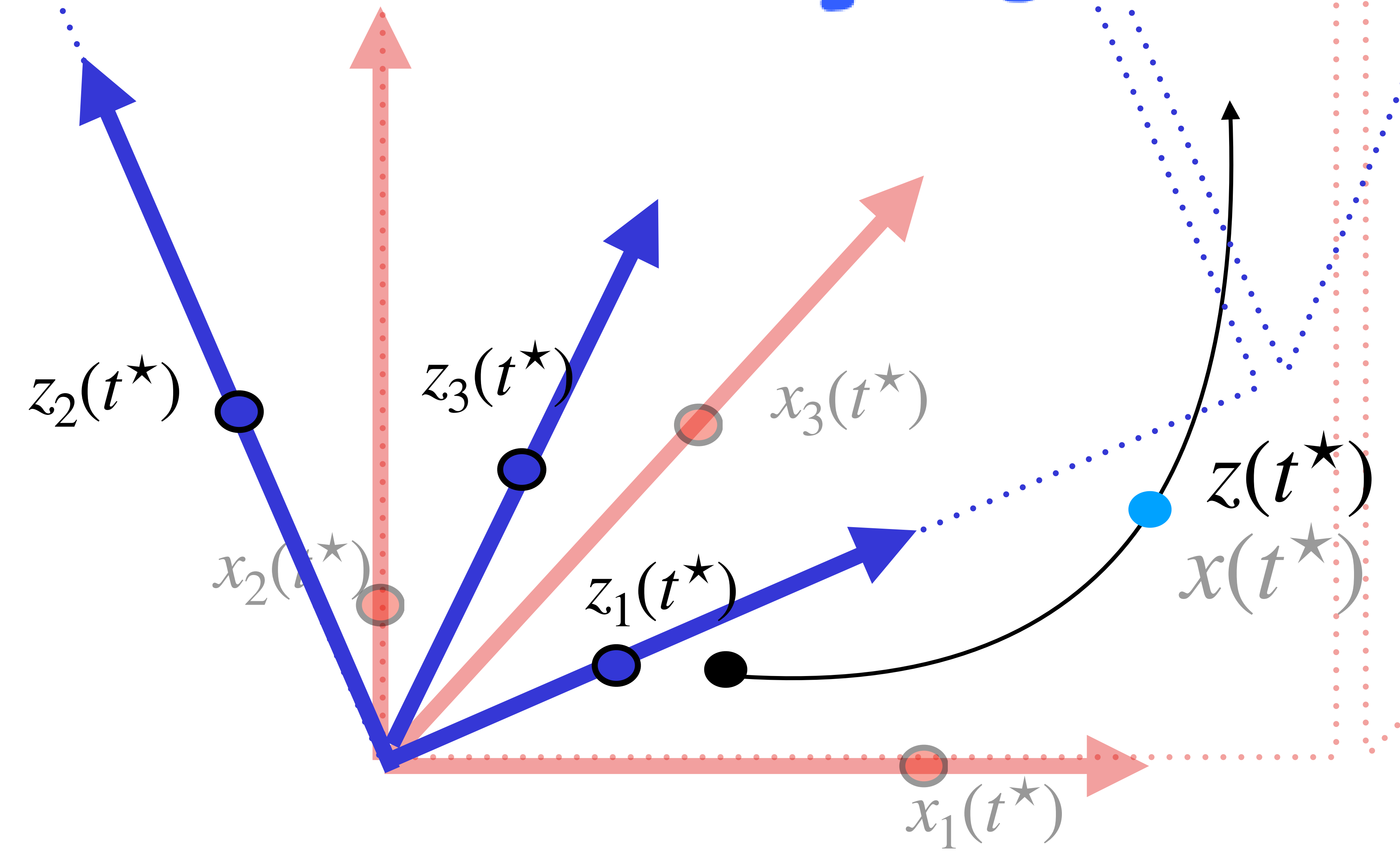
Idea: why not changing coordinate framework to obtain a description “better” than the original one?

Looking at the same system but w/ new perspective

$\mathbb{R}^n$

$$[z] = [T]_{n \times n} [x]$$

$$x = T^{-1} z$$



change the mathematic representation

Physics

The model is “coordinate-dependent“

$$\dot{x}(t) = f(x(t), u(t))$$

Math

construct a map

Inverse

$$x(t) = \phi^{-1}(z(t))$$

Coordinate-change

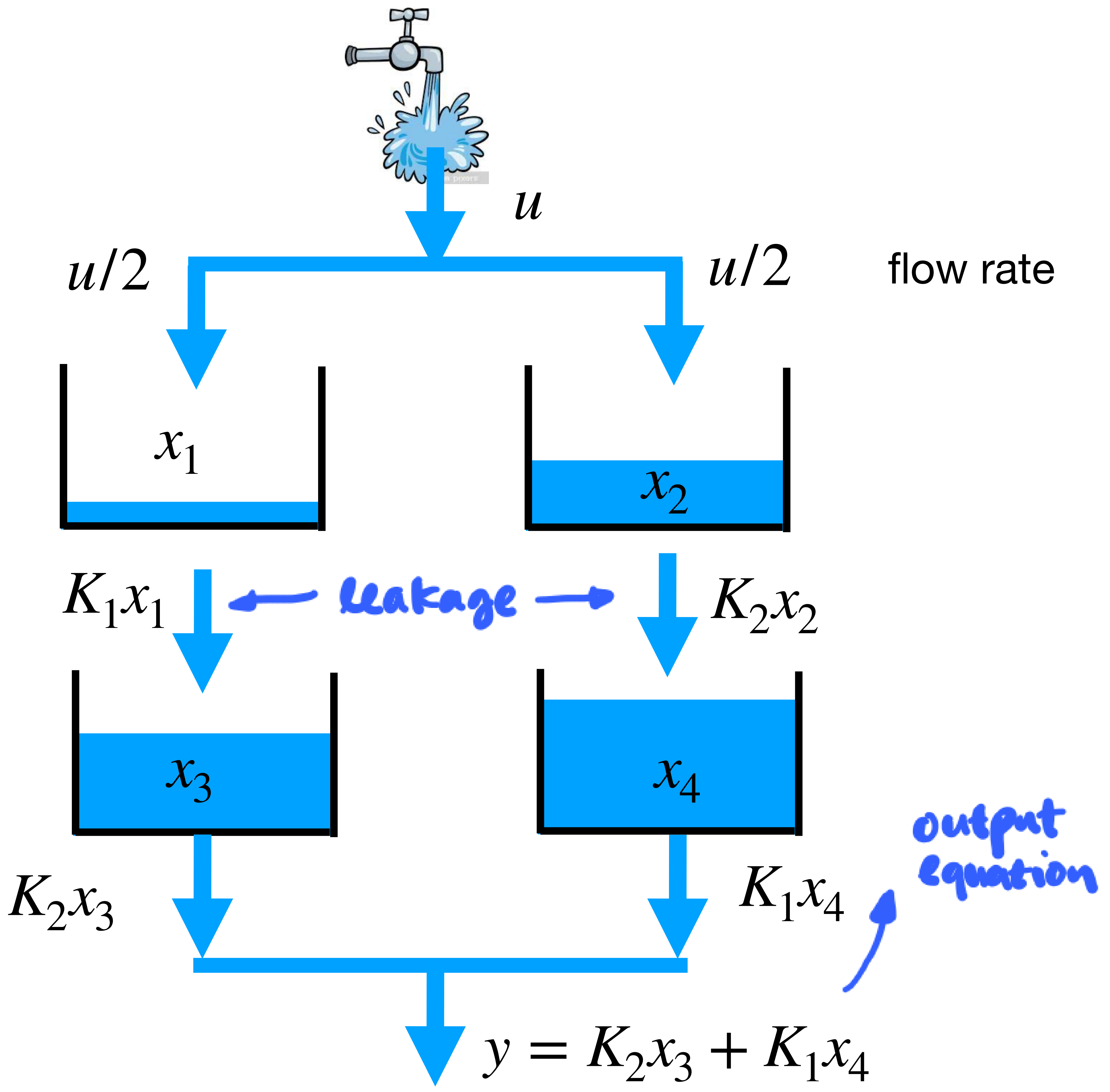
$$z(t) = \phi(x(t))$$

$$\dot{z}(t) = \varphi(z(t), u(t))$$

Coordinates Change

What “better” means?

Motivating example: Hydraulic system



COORDINATE CHANGE : HYDRAULIC CASE

$$\left.\begin{aligned}
\dot{x}_1 &= \frac{u}{2} - k_1 x_1 \\
\dot{x}_2 &= \frac{u}{2} - k_2 x_2 \\
\dot{x}_3 &= k_1 x_1 - k_2 x_3 \\
\dot{x}_4 &= k_2 x_2 - k_1 x_4
\end{aligned}\right\}
\quad A = \begin{bmatrix} -k_1 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ k_1 & 0 & -k_2 & 0 \\ 0 & k_2 & 0 & -k_1 \end{bmatrix}$$

$$y = k_2 x_3 + k_1 x_4$$

$$B = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & k_2 & k_1 \end{bmatrix}$$

$$\begin{aligned}
z_1 &= y \\
z_2 &= k_2 x_3 - k_1 x_4 \\
z_3 &= \dot{y} \\
z_4 &= \dot{z}_2
\end{aligned}
\qquad T = \begin{bmatrix} 0 & 0 & k_2 & k_1 \\ 0 & 0 & k_2 & -k_1 \\ k_1 k_2 & k_1 k_2 & -k_2^2 & -k_1^2 \\ k_1 k_2 & -k_1 k_2 & -k_2^2 & -k_1^2 \end{bmatrix}$$

non-singular T^{-1}

$$z = T^{-1} x$$

$$\dot{z} = T\dot{x} = T(Ax + Bu)$$

$\# \leftarrow T^{-1} z$

$$\dot{z} = \boxed{\underbrace{TAT^{-1}}_{\tilde{A}} z} + \underbrace{TB}_{\tilde{B}} u$$

$$y = Cx = \underbrace{CT^{-1}}_{\tilde{C}} z$$

$\# \rightarrow$ diag. matrix

$$\tilde{A} = \left[\begin{array}{c|c} \tilde{A}_1 & 0_{2\times 2} \\ \hline 0_{2\times 2} & \tilde{A}_1 \end{array}\right] \qquad \tilde{A}_1 = \begin{bmatrix} 0 & -1 \\ -k_1k_2 & -(k_1k_2) \end{bmatrix}$$

$$\tilde{B} = \left[\begin{array}{c} \tilde{B}_1 \\ \hline 0_{2\times 1} \end{array}\right] \qquad \tilde{B}_1 = \begin{bmatrix} 0 \\ k_1k_2 \end{bmatrix}$$

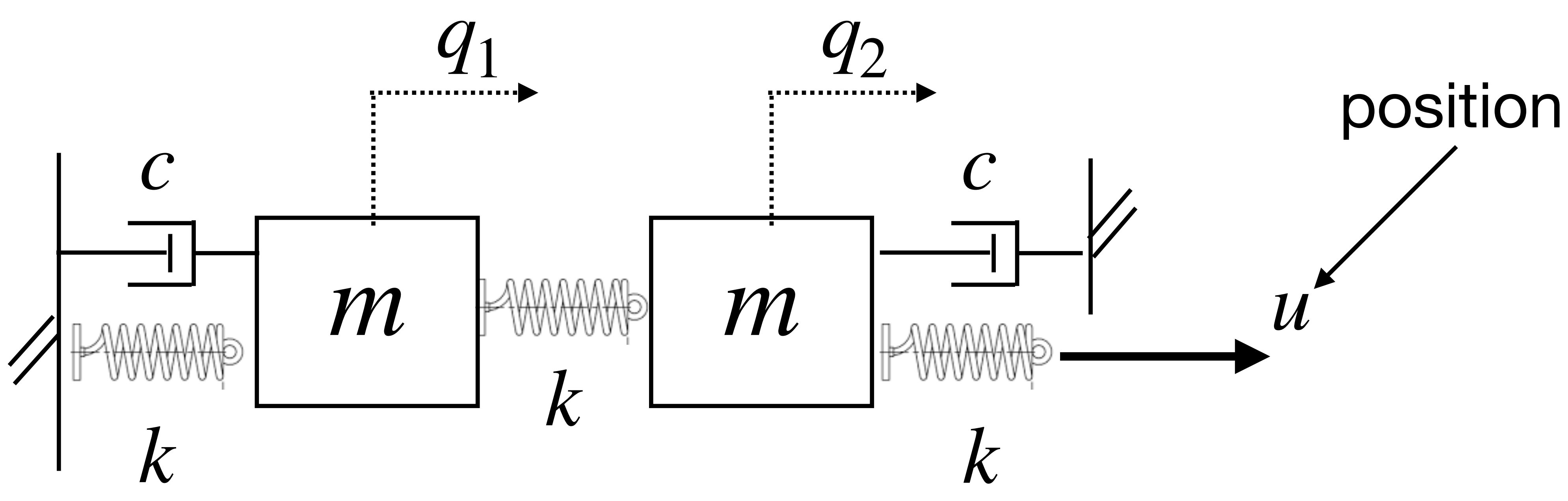
$$\tilde{C} = \left[\begin{array}{c|c} \tilde{C}_1 & 0_{1\times 2} \end{array}\right] \qquad \tilde{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$u \longrightarrow \begin{array}{|c|c|} \hline \tilde{A}_1 & \tilde{B}_1 \\ \hline \tilde{C}_1 & \\ \hline \end{array} \longrightarrow y$$

$$\begin{array}{|c|c|} \hline \tilde{A}_1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

Coordinates Change

Other motivating example: mechanical system



$$m_1\ddot{q}_1 = -2kq_1 - c\dot{q}_1 + kq_2$$
$$m_2\ddot{q}_2 = kq_1 - 2kq_2 - c\dot{q}_2 + ku$$

$$x := \begin{pmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{pmatrix}^T$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{pmatrix}$$

Homework: Compute and interprete the system by using the change of coordinates:

$$z_1 = \frac{1}{2}(x_1 + x_2), \quad z_2 = \frac{1}{2}(x_3 + x_4), \quad z_3 = \frac{1}{2}(x_1 - x_2), \quad z_4 = \frac{1}{2}(x_3 - x_4)$$

Coordinates Change - The Linear Case

Linear change of coordinates $T : \mathbb{R}^n \to \mathbb{R}^n$:

$$z = Tx$$

T is a $n \times n$ nonsingular matrix

$$x = T^{-1}z$$

Geometric interpretation of the column of T :

Equivalent / Similar

$$\left.\begin{array}{c}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$z = Tx$ ↓ ↑ $x = T^{-1}z$

$$\left.\begin{array}{c}\dot{z}(t)\\ z(t+1)\end{array}\right\} = \tilde{A}z(t) + \tilde{B}u(t)$$

$$y(t) = \tilde{C}z(t) + \tilde{D}u(t)$$

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}, \quad \tilde{D} = D$$

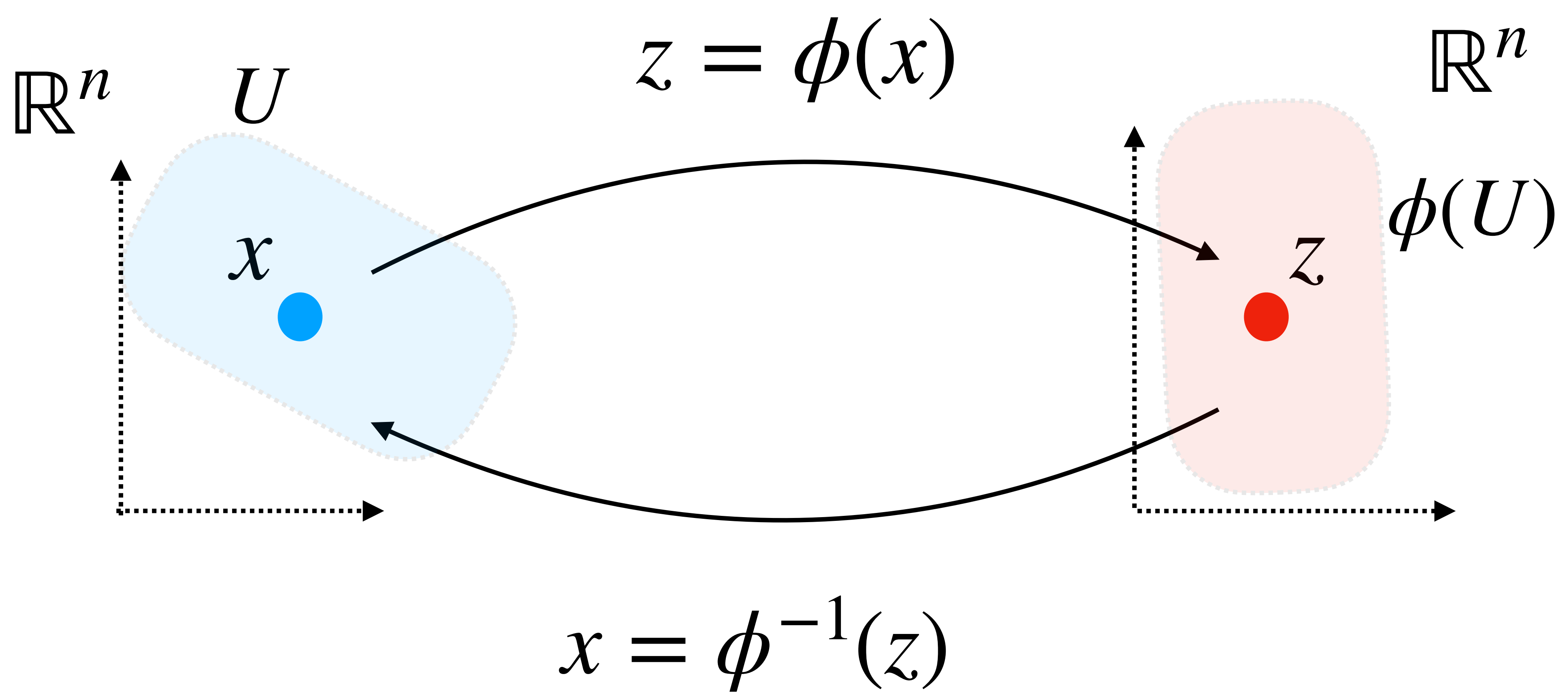
Coordinates Change - The Nonlinear case

postponed

$\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a **global diffeomorphism** if

- it is smooth ("differentiable a sufficiently high number of times")
- there exists a $\phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi^{-1}(\phi(x)) = x$ for all $x \in \mathbb{R}^n$

Local diffeomorphism defined similarly but with the previous definitions and properties that hold only in a set $U \subset \mathbb{R}^n$



Relevant class of **input-affine** systems

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Diffeomorphic

$$z = \phi(x) \downarrow \qquad \uparrow x = \phi^{-1}(z)$$

$$\dot{z} = \tilde{f}(z) + \tilde{g}(z)u$$
$$y = \tilde{h}(z)$$

$$\tilde{f}(z) := \frac{d\phi(x)}{dx} f(x)\Big|_{x=\phi^{-1}(z)} \qquad \tilde{g}(z) := \frac{d\phi(x)}{dx} g(x)\Big|_{x=\phi^{-1}(z)}$$

$$\tilde{h}(z) := h(\phi^{-1}(z))$$

For the time being not developed

The Linear Systems case

- Diagonalization of a matrix ✓
- Jordan form
- Schur decomposition
- Brunowsky canonical form

Brief Recall

- Characteristic polynomial linked to A
- Eigenvalues and Eigenvectors
- Geometric and Algebraic Multiplicity
- Generalised Eigenvectors
- Eigenspace

CHARACTERISTIC POLYNOMIAL

$A_{n \times n}$

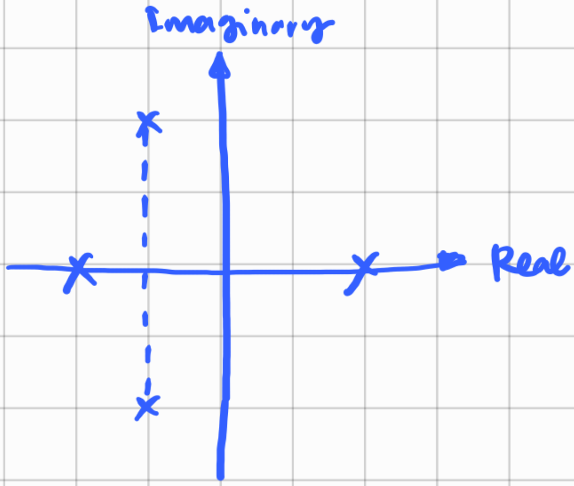
$$\varphi_A(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + a_n$$
$$\overset{\Delta}{=} \det(\lambda I_n - A)$$

characteristic polynomial

$\varphi_A(\lambda) = 0 \leftarrow$ characteristic equation

DEFINITION

The λ_i^* : $\varphi_A(\lambda_i^*) = 0$ are eigenvalues when $\det(\lambda_i^* I - A) = 0$, $i = 1, 2, \dots, n$



$$\varphi_A(\lambda) = (\lambda - \lambda_1^*)(\lambda - \lambda_2^*) \cdots (\lambda - \lambda_n^*)$$

DEFINITION

Algebraic multiplicity a_i of λ_i^* $\equiv$ multiplicity of λ_i^* as root of $\varphi_A(\lambda) = 0$

$$\varphi_A(\lambda) = (\lambda - \lambda_1^*)^{a_1} (\lambda - \lambda_2^*)^{a_2} \cdots (\lambda - \lambda_r^*)^{a_r}$$

r is the number of distinct Eigenvalue

$$\sum_{i=1}^{r} a_i = n$$

CAYLEY - HAMILTON THEOREM

$$\varphi(A) = A^n + \alpha_1 A^{n-1} + \dots + \alpha_{n-1} A + \alpha_n I_n$$
$$= 0$$

DEFINITION

Minimum polynomial of A $\varphi_A(\lambda)$ of minimum order $(\leq n)$ $\varphi_A(A) = 0$

with order n or lower

Example:

$A = I_n$

$$\det(\lambda I - A) = \begin{bmatrix} \lambda - 1 & & & 0 \\ & \lambda - 1 & & \\ & & \ddots & \\ 0 & & & \lambda - 1 \end{bmatrix}$$

$\varphi_t(\lambda) = (\lambda - 1)^n$, $\quad \lambda_1^* = 1$

$a_1 = n$ ⇝ algeb. mult.

$\psi_t(\lambda) = (\lambda - 1)$

EIGENVECTORS (v_i^*) linked to a certain eigenvalue λ_i^*

$\mathbb{R}^n$ (vector)

By the definition of eigenvalue $\det(\lambda_i^* I - A) = 0$ the kernel $(\lambda_i^* I - A)$ is non trivial

$\Downarrow$

DEFINITION of kernel

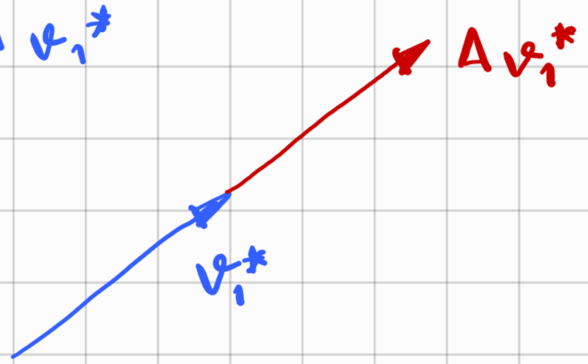
Suppose $M_{r \times c}$, the kernel $M = \{ v \in \mathbb{R}^c : Mv = 0 \}$

map to

$\exists v_1^* : \boxed{(\lambda_i^* I - A)\, v_i^* = 0}$

$\Downarrow$

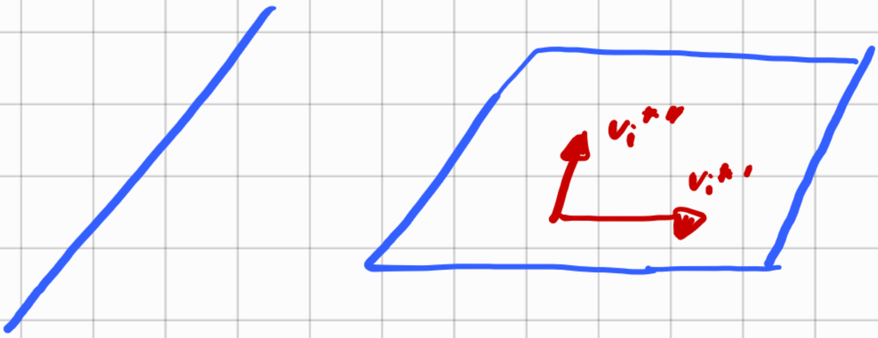
$\lambda_i^* v_1^* = A v_1^*$



The kernel of $(\lambda_i^* I - A)$ is a subspace

DEFINITION

The geometric multiplicity (g_i) linked to λ_i^* is the dimension of kernel $(\lambda_i^* I - A)$



Example:

Suppose $A = I$, $\lambda_1^* = 1$, $a_1 = n$

$$(\lambda_1^* I - \underset{A}{I}) = \left(0_{n\times n} \right)$$

The kernel of $0_{n\times n} = \mathbb{R}^n$, the geometric multiplicity $g_1 = n$.

The result : $g_i \leq a_i$.

If there are $\lambda_i^* \neq \lambda_j^*$. They are linearly independent.

$$\overbrace{v_{i,1}^* \cdots v_{i,g_i}}^{a_i,\, g_i} \qquad \overbrace{v_{j,1} \cdots v_{j,g_j}}^{a_j,\, g_j}$$

Hence, $\sum_{i=1}^{r} a_i = n$, $\sum_{i=1}^{r} g_i \leq n$

Suppose an assumption $g_i = a_i$, $\forall i = 1, 2, \dots, r$

$$A \underbrace{\begin{bmatrix} \underbrace{v^*_{11} \dots v^*_{1g_1}}_{\lambda^*_1} & \vdots & \underbrace{v^*_{21} \dots v^*_{2g_2}}_{\lambda_2^*} & \vdots & \dots & \vdots & \underbrace{v^*_{r1} \dots v^*_{rg_r}}_{\lambda_r^*} \end{bmatrix}_{n \times n}}_{:= T^{-1}} = \begin{bmatrix} T^{-1} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \lambda_1^* & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_1^* & & & & \\ & & & \lambda_2^* & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2^* & \\ 0 & & & & & & \lambda_r^* \ddots \lambda_r^* \end{bmatrix}}_{\text{Diagonal matrix}}$$

g_1 times

g_2 times

g_r times

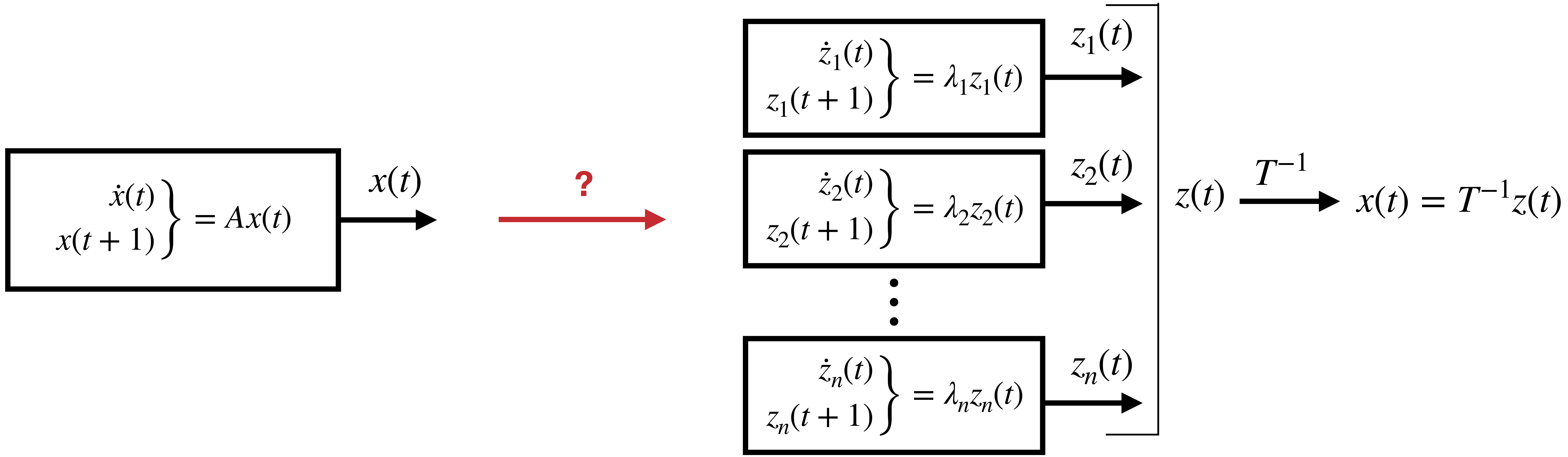
$$\boxed{\begin{aligned} &\begin{bmatrix} Av^*_{11} & Av^*_{12} & \dots \end{bmatrix} = \begin{bmatrix} \lambda_1^* v^*_{11} & \lambda_1^* v^*_{12} & \dots \end{bmatrix} \\ &TAT^{-1} = \text{diagonal matrix} \end{aligned}}$$

Diagonalization of a Matrix

Problem: Given a generic matrix $A \in \mathbb{R}^n \times \mathbb{R}^n$, is there a change of variable T making the matrix TAT^{-1} diagonal?

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \xrightarrow{\ ?\ } \tilde{A} = TAT^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

System perspective: Given a generic linear system is it possible to see it as the "parallel" of n scalar subsystems?



Diagonalization of a Matrix

set of A's eigenvalues

Result: Let $\{\lambda_1, \cdots, \lambda_r\}$ be set of distinct (real or complex conjugate) eigenvalues of $A \in \mathbb{R}^n \times \mathbb{R}^n$ and let $g_i \geq 1,\ i = 1,\ldots,r$, be the geometric multiplicity of λ_i . If

number of distinct eigenvalues

$$\sum_{i=1}^{r} g_i = n$$

then the matrix A is similar to a diagonal matrix.

a "completely" diagonal matrix

$$T^{-1} = [\overbrace{\vec{v}_{11} \quad \ldots \quad \vec{v}_{1g_1}}^{\text{relative to } \lambda_1} \quad \cdots \quad \overbrace{\vec{v}_{r1} \quad \ldots \quad \vec{v}_{rg_r}}^{\text{relative to } \lambda_r}]$$

$n \times n$ non singular

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} \lambda_r & 0 & \cdots & 0 \\ 0 & \lambda_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r \end{pmatrix} \end{pmatrix}$$

$g_1 \times g_1$

$g_r \times g_r$

$n \times n$

Remark: λ_i could be real or complex. If complex, there necessarily exists a λ_j that is it s conjugate. In that case, the blocks linked to (λ_i, λ_j) can be compacted by obtaining a real representation (see next)

Diagonalization of a Matrix

complex

Suppose that $\boxed{\lambda_i = a_i + jb_i}$ and $\boxed{\lambda_{i+1} = a_i - jb_i}$ (complex conjugate pair, $a_i, b_i \in \mathbb{R}$). Let $\vec{v}_i = \vec{u}_i + j\vec{w}_i$, $\vec{u}_i, \vec{w}_i \in \mathbb{R}^n$, and $\vec{v}_{i+1}$ be the two linearly independent eigenvectors relative to $(\lambda_i, \lambda_{i+1})$. It turns out that $\boxed{\vec{v}_{i+1}} = \vec{u}_i - j\vec{w}_i$ (complex conjugate of $\vec{v}_i$) and that $\vec{u}_i$ and $\vec{w}_i$ are linearity independent vectors (also linearly independent to all the others eigenvectors $\vec{v}_j$)

eigenvectors are a complex conjugate pair

eigenvalues

$$A\vec{v}_i = \lambda_i \vec{v}_i \longrightarrow A(\vec{u}_i + j\vec{w}_i) = (a_i + jb_i)(\vec{u}_i + j\vec{w}_j) \longrightarrow \begin{cases} A\vec{u}_i = a_i\vec{u}_i - b_i\vec{w}_i \\ A\vec{w}_i = a_i\vec{w}_i + b_i\vec{u}_i \end{cases}$$

eigenvector

$$(a_i + jb_i)(\vec{u}_i + j\vec{w}_i) = \underbrace{a_i\vec{u}_i + b_i\vec{w}_i}_{\text{real part}} + j\underbrace{(b_i\vec{u}_i + a_i\vec{w}_i)}_{\text{imag. part}}$$

$$A\vec{v}_{i+1} = \lambda_{i+1}\vec{v}_{i+1} \longrightarrow A(\vec{u}_i - j\vec{w}_i) = (a_i - jb_i)(\vec{u}_i - j\vec{w}_j)$$

the same

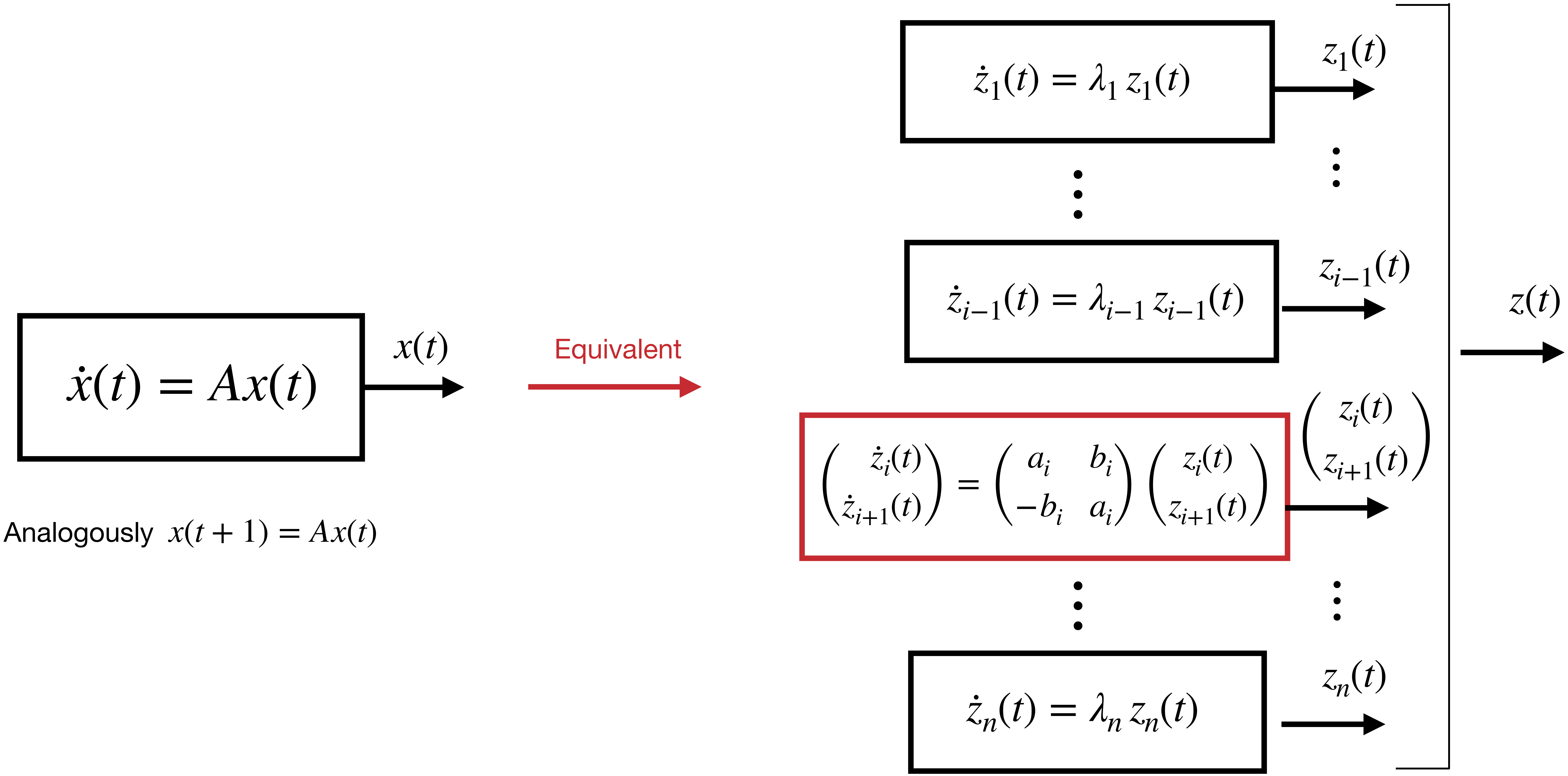
the same

2x2

$$\underset{n\times n}{A}\ \underset{n\times 2}{[\vec{u}_i \ \ \vec{w}_i]} = \underset{n\times 2}{[\vec{u}_i \ \ \vec{w}_i]} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} \lambda_1 & 0 & \cdot & \cdots & 0 \\ 0 & \lambda_2 & \cdot & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \longleftarrow \text{Use } [\vec{u}_i \ \vec{w}_i] \text{ in place of } [\vec{v}_i \ \vec{v}_{i+1}]$$

Diagonalization of a Matrix



Analogously $x(t+1) = Ax(t)$

GENERALIZED EIGENVECTOR (λ_i^*) OF ORDER $k \geq 1$

$$v_{i,k}^* : (\lambda_i^* I_n - A)^{\ell} v_{i,k}^* \neq 0 \text{ for } \ell = 0, 1, \dots, k-1$$

$$(\lambda_i^* I_n - A)^k v_{i,k}^* = 0$$

order, not a power

rank

$$\lambda_1^* (a_1, g_1) \qquad \lambda_2^* (a_2, g_2) \qquad \lambda_r^* (a_r, g_r)$$

$$v_{11}^{*1} \dots v_{1g_1}^{*1} \qquad v_{21}^{*1} \dots v_{2g_2}^{*1} \qquad v_{r1}^{*1} \dots v_{rg_r}^{*1}$$

$\boxed{r \text{ is number of distinct eigenvalues}}$

$$\Rightarrow \sum_{i=1}^{r} g_i < n$$

The procedure for constructing "chains" or generalized eigenvector:

$$v^* \neq 0 \qquad v_i^* \in \text{kernel}(\lambda_i^* I - A)$$

equals to 0

Assumption: suppose that $v_i^* \in$ image $I_n(\lambda^* I - A)$

span

SPAN OF A MATRIX: $M_{R \times c}$ span $M = \left\{ v \in \mathbb{R}^R : v = M w \right\}$ for some $w \in \mathbb{R}^c$

$$\exists\, v_1^{*2} : v_i^{*1} = (\lambda_i^* I - A) v_1^{*2}$$

is not 0

also not 0

a generalized eigenvector order 2

also, $(\lambda_i^* I - A)\underbrace{(\lambda_i^* I - A)\, v_i^{*2}}_{v_i^{*1}} = 0$

$\lambda_1^*(a_1, g_1)$ $\quad$ $\lambda_2^*(a_2, g_2)$ $\quad$ $\lambda_r^*(a_r, g_r)$

$\overbrace{v_{11}^{*1} \dots v_{1g_1}^{*1}}$ $\quad$ $\overbrace{v_{21}^{*1} \dots v_{2g_2}^{*1}}$ $\quad$ $\overbrace{v_{r1}^{*1} \dots v_{rg_r}^{*2}}$

if $v_{11}^{*1} \in \text{image}(\lambda_i^* I - A)$

v_{11}^{*2} $\quad$ $v_{1g_1}^{*2}$

if $v_{11}^{*2} \in \text{image}(\lambda_i^* I - A)$

v_{11}^{*3}

$v_i^{*(k-1)} : (\lambda_i^* I_n - A)^l v_i^{*l} \neq 0$

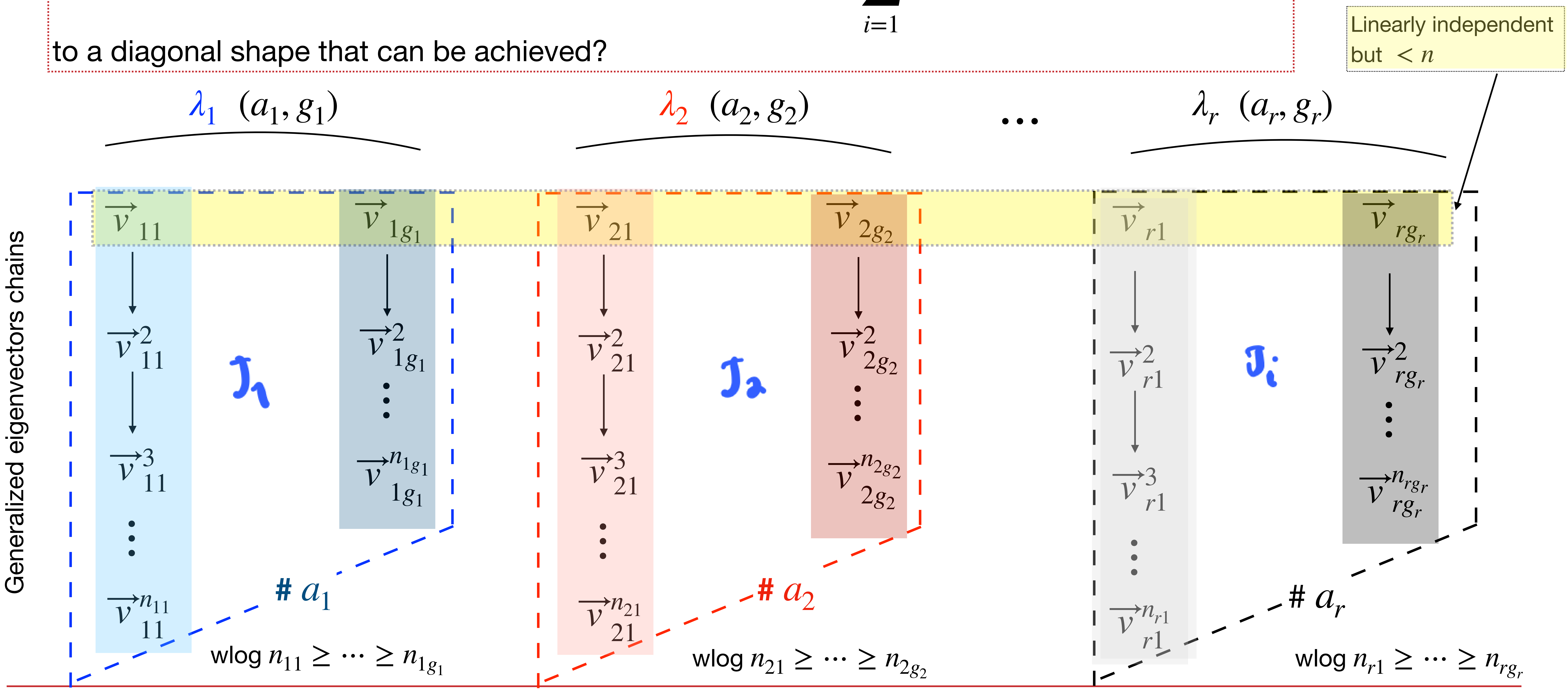
where $l = 0, 1, \dots, k-2$

$(\lambda_i^* I_n - A)^{(k-1)} v_i^{*(k-1)} = 0$

Assume if $v_i^{*(k-1)} \in \text{image}(\lambda_i^* I_n - A)$, there exists $\exists\, v_i^{*k}$, $v_i^{*(k-1)} = (\lambda_i^* I_n - A)\, v_i^{*k}$

Jordan Form

Suppose now that the matrix A is not diagonalisable, namely $\sum_{i=1}^{r} g_i < n$. What is the closest to a diagonal shape that can be achieved?



Result: The whole set is linearly independent and $= n$

Jordan Form

$$T^{-1} = [\vec{v}_{11} \quad \vec{v}^{\,2}_{11} \quad \cdots \quad \vec{v}^{\,n_{11}}_{11} \quad \cdots \quad \vec{v}_{1g_1} \quad \cdots \quad \vec{v}^{\,n_{1g_1}}_{1g_1} \quad \vec{v}_{21} \quad \cdots \quad \vec{v}^{\,n_{21}}_{21} \quad \cdots \quad \vec{v}_{2g_2} \cdots \quad \vec{v}^{\,n_{2g_2}}_{2g_2} \quad \cdots \quad \vec{v}^{\,n_{rg_r}}_{rg_r}]$$

zoom it!

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_r \end{pmatrix} \qquad J_i = \begin{pmatrix} J_{i1} & 0 & 0 & \cdots & 0 \\ 0 & J_{i2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{ig_i} \end{pmatrix} \qquad J_{ik} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

$n \times n$ $\qquad$ $a_i \times a_i \quad i = 1 \ldots r$ $\qquad$ $n_{ik} \times n_{ik} \quad k = 1 \ldots g_i$

J could be scalars or matrices

Jordan blocks $\qquad$ **Mini Jordan blocks**

Nested block diagonal form following the structure of the generalised eigenvectors chain

"Almost"diagonal

Jordan Form - Example

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is it diagonalizable or in Jordan form?

1. Determine $\varphi_A(\lambda) = \det(\lambda I - A) = \det \begin{vmatrix} \lambda-1 & -2 & 0 & -1 \\ 0 & \lambda-2 & 0 & 0 \\ 0 & 1 & \lambda-1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

$\hookrightarrow$ characteristic polynomial of A

$$= (\lambda-1) \underbrace{\det \begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix}}_{(\lambda-2)\,\underbrace{\det \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix}}_{(\lambda-1)^2}}$$

$$= (\lambda-1)^3(\lambda-2)$$

2. Determine the algebraic multiciplity, distinct eigenvalues, and rank

$$(\lambda - 1)^3 (\lambda - 2)$$

$\to a_1 = 3,\ g_1 = ?$

$\to r = 2$ (distinct eigen values)

$\to n = 4$

$\to a_2 = 1,\ g_2 = 1$

Determine g_1

$$(\lambda_1^* I - A) = \begin{pmatrix} 0 & -2 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\lambda_2^* I - A) = \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. What is the kernel $(\lambda_1^* I - A)$?

$$\boxed{\text{Remember: } \lambda_1^* = 1\ (a_1 = 3,\ g_1 = ?) \quad \lambda_2^* = 2\ (a_2 = g_2 = 1)}$$

$$v_{11}^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_{12}^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow g_1 = 2$$

4. What is the kernel $(\lambda_2^* I - A)$?

$$v_{21}^* = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

→ no way to generate

5. Is $v_{11}^* \in$ image $(I - A)$? Yes...

$$(I - A)\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v_{11}^*$$

$$v_{11}^{*2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

6. So the Jordan form of T^{-1}...

$$T^{-1} = \begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\qquad\quad v_{11}^* \quad v_{11}^{*2} \quad v_{12}^* \quad v_{21}^*$$

7. What is $\tilde{A}$?

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} (J_1)_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & (J_2)_{1\times1} \end{bmatrix}$$

Remember:

$\lambda_1^* = 1$

$\lambda_2^* = 2$

$(J_2)_{1\times1} = 2$

$$(J_1)_{3\times3} = \begin{pmatrix} (J_{11})_{2\times2} & \\ & (J_{12})_{1\times1} \end{pmatrix}$$

$$(J_{11})_{2\times2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (J_{12})_{1\times1} = 1$$

Schur decomposition

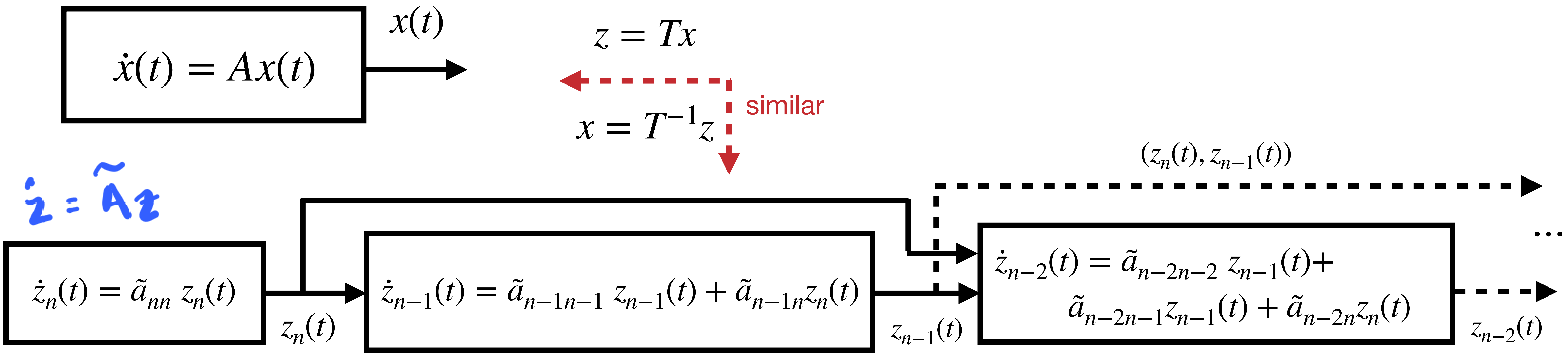
(or lower)

All the matrices are similar to upper (lower) triangular matrices $\begin{bmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$

Given $A \in \mathbb{R}^n \times \mathbb{R}^n$ there always exists a change of variable T such that $\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \tilde{a}_{1n} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \tilde{a}_{2n} \\ 0 & 0 & \tilde{a}_{33} & \cdots & \tilde{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{a}_{nn} \end{pmatrix}$

no assumptions like Jordan Form

System perspective: Given a generic linear system is it possible to see it as the "cascade" of n scalar subsystems



Construction of T

skipped

Brunowsky canonical form

always computable, no assumption

Definition: The triplet $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$ (**SISO**) has relative degree $r > 0$ if the following holds:

single input, single output

- $CA^kB = 0$ for all $k = 0,\ldots, r-2$
- $CA^{r-1}B \neq 0$

Result: The relative degree $r > 0$ always exists and $r \leq n$

Dynamic System interpretation:

- Continuous time systems:
  - ✓ $y^{(k)}(t) = CA^k x(t)$ for all $t \in \mathbb{R}$ and for all $k = 0,\ldots r-1$;
  - ✓ $y^{(r}(t) = CA^r x(t) + \underbrace{CA^{r-1}B}_{\neq 0}\, u(t)$ for all $t \in \mathbb{R}$
- Discrete-time systems
  - ✓ $y(t+k) = CA^k x(t)$ for all $t \in \mathbb{Z}$ and for all $k = 0,\ldots r-1$;
  - ✓ $y(t+r) = CA^r x(t) + \underbrace{CA^{r-1}B}_{\neq 0}\, u(t)$ for all $t \in \mathbb{Z}$

For continuous time systems, r is the minimum number to times the output must be derived in order to see the input explicitly appearing.
For discrete time systems, r is number of time delays after which the input shows up on the output.

Homework: With $G = C(sI - A)^{-1}B$ given by

$$G(s) = \frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

prove that $r = n - m$ → poles - zeros

$$G(s) = \frac{\text{numerator}(s)}{\text{denumerator}(s)}$$

$$r = \underbrace{\text{order of den}(s)}_{\text{n. of poles}} - \underbrace{\text{order of num}(s)}_{\text{n. of zeros}}$$

$r \rightarrow$ relative degree

$$G(s) = C(sI - A)^{-1} B + \cancel{D}$$

$$1\begin{array}{c} n \\ \begin{bmatrix} \quad C \quad \end{bmatrix} \end{array} \quad n\begin{array}{c} n \\ \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \end{array} \quad n\begin{array}{c} 1 \\ \begin{bmatrix} \\ B \\ \\ \end{bmatrix} \end{array}$$

Reinterpreting the relative degree (r)

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{y} = C\dot{x} = C(Ax + Bu)$$
$$= CAx + \boxed{CBu}$$

If the relative degree $r > 1$, $CB = 0$,

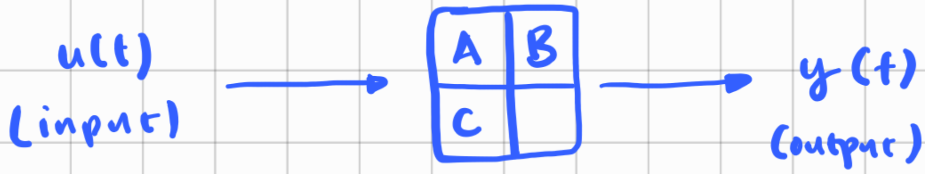
computing second derivative $\ddot{y}$:

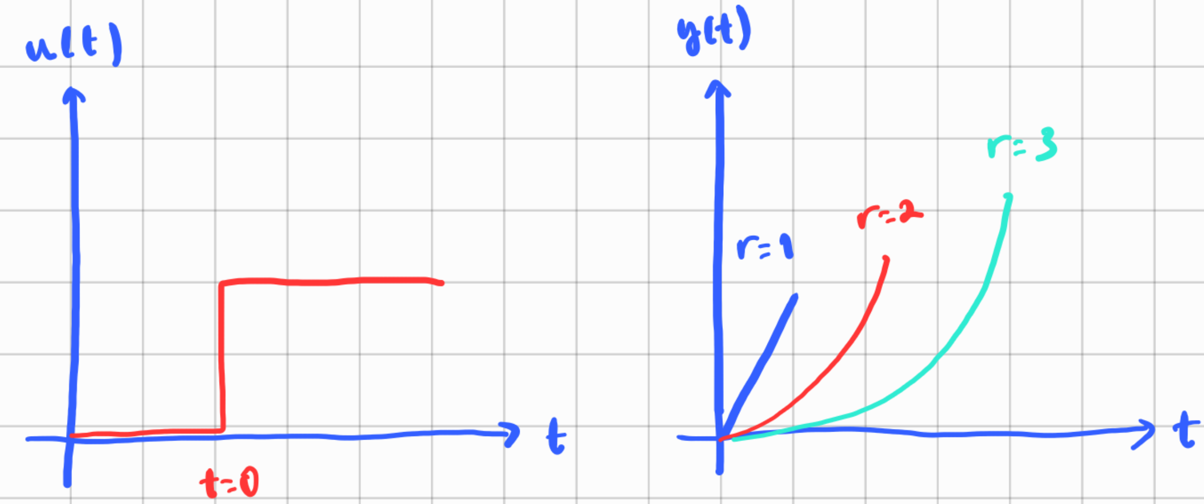
$$\ddot{y} = CA\dot{x} = CA(Ax + Bu)$$
$$= CA^2x + \boxed{CABu}$$

If $r > 2$, besides $CB = 0$, $CAB = 0$ too

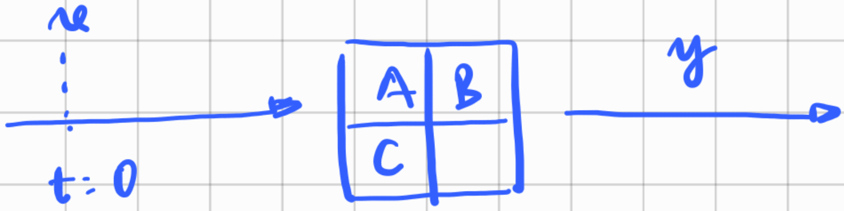
Hence, $y^{(r)} = CA^r x + \underbrace{\boxed{CA^{r-1}Bu}}_{\neq 0}$

Suppose a system:





For discrete-time systems:



Brunowsky canonical form

Result. rank $\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix}_{r\times n} = r$, namely all the rows of that matrix are linearly independent

$r \leq n$

$$\begin{bmatrix} [\quad C \quad] \\ [\quad CA \quad] \\ [\ CA^{r-1}\ ] \end{bmatrix}_{r \times n}$$

more columns than rows

Choose $T = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{n-r} \end{pmatrix}_{n\times n} := \begin{pmatrix} T_\xi \\ T_\eta \end{pmatrix}$ with the rows ϕ_i chosen so that T is not singular

$r \times n$

$(n-r) \times n$

add ϕ to make square matrix

$$[\phi_i][B] = 0$$

Remark:
(Always) possible choice: $T_\eta\, B = 0$

Brunowsky change of variable

$$z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} T_\xi\, x \\ T_\eta\, x \end{pmatrix} = Tx$$

$$T = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \\ \phi_1 \\ \vdots \\ \phi_{n-r} \end{bmatrix} = \begin{bmatrix} T_\xi \\ \hline T_\eta \end{bmatrix}$$

$$z = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \begin{matrix} \}\, r \\ \}\, n-r \end{matrix}$$

$$z = Tx$$
$$\xi = T_\xi x$$
$$\eta = T_\eta x$$

$$\xi_1 \triangleq Cx = y$$
$$\xi_2 \triangleq CAx$$

$$\dot{z} = \overbrace{TAT^{-1}}^{\tilde{A}} + \overbrace{TB}^{\tilde{B}}u$$

$$y = \underbrace{CT^{-1}}_{\tilde{C}} z$$

$$\dot{\xi}_1 = \overset{\xi_2}{\cancel{CAx}} + \cancel{CBu} \quad \text{if } r > 1$$

$$\dot{\xi}_2 = \underbrace{\overset{\xi_3}{\cancel{CA^2x}} + CABu}_{CA\dot{x}} \quad \text{if } r > 2$$

$$\dot{\xi}_r = \boxed{CA^r x} + \underbrace{\boxed{CA^{r-1}}}_{\neq 0} Bu$$

$$CA^r x = Q_\xi \xi + Q_\eta \eta$$

$$T^{-1}z = T^{-1}(\xi_\eta)$$

Hence if $\dot{\xi}_1 = \xi_2$

$$\vdots$$

$$\dot{\xi}_{r-1} = \xi_r \Rightarrow \dot{\xi}_r = Q_\xi \xi + Q_\eta \eta + \Delta u$$

Calculating η :

$$\dot{\eta} = T_\eta \dot{x} = T_\eta (Ax + Bu)$$

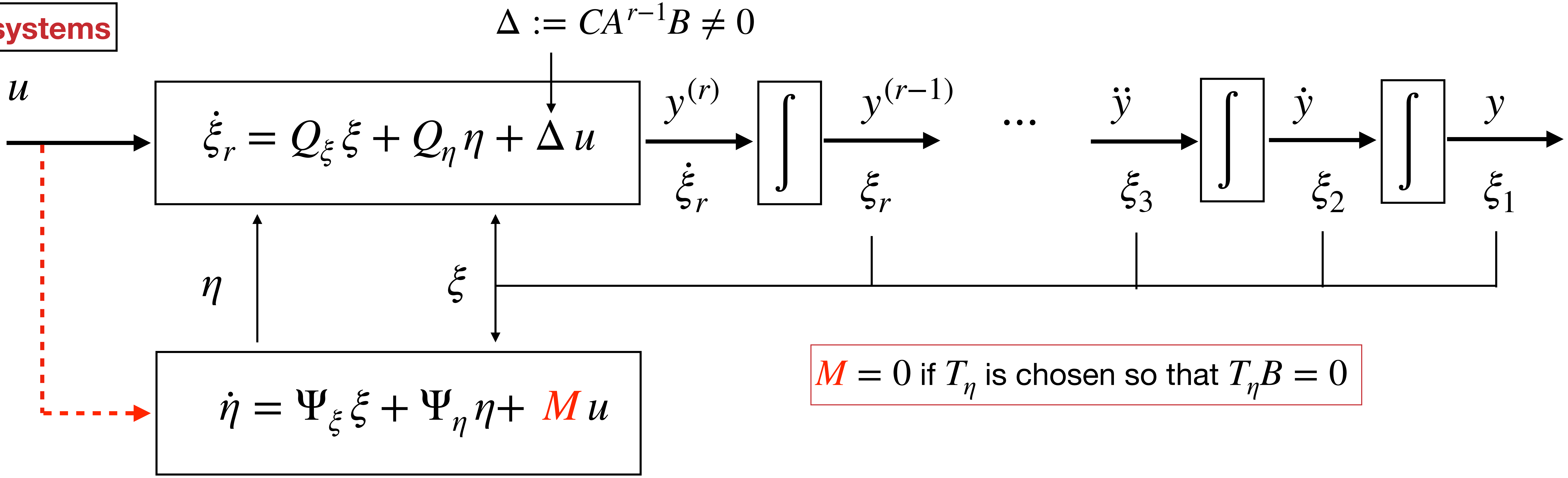
$$= T_\eta A T^{-1} z + T_\eta B u$$

$$= \underbrace{T_\eta A T^{-1} \begin{bmatrix} \xi \\ \eta \end{bmatrix}}_{\psi_\xi \xi + \psi_\eta \eta} + \cancel{T_\eta B u} \quad \text{if } \phi_i B = 0$$

$$\longrightarrow \quad \begin{aligned} y &= \xi_1 \\ \dot{\eta} &= \psi_\xi \xi + \psi_\eta \eta \end{aligned}$$

Brunowsky canonical form

C-T systems



$M = 0$ if T_η is chosen so that $T_\eta B = 0$

Always

$$
\begin{array}{rcl}
\dot{\xi}_1 & = & \xi_2 \qquad y = \xi_1 \\
\dot{\xi}_2 & = & \xi_3 \\
 & \vdots & \\
\dot{\xi}_{r-1} & = & \xi_r \\
\dot{\xi}_r & = & Q_\xi\xi + Q_\eta\eta + \Delta u \\
\dot{\eta} & = & \Psi_\xi\xi + \Psi_\eta\eta + M\,u
\end{array}
$$

When $T_\eta B = 0$

$$
\begin{array}{rcl}
\dot{\xi}_1 & = & \xi_2 \qquad y = \xi_1 \\
\dot{\xi}_2 & = & \xi_3 \\
 & \vdots & \\
\dot{\xi}_{r-1} & = & \xi_r \\
\dot{\xi}_r & = & Q_\xi\xi + Q_\eta\eta + \Delta u \\
\dot{\eta} & = & \Psi_\xi\xi + \Psi_\eta\eta
\end{array}
$$

If $r = n$

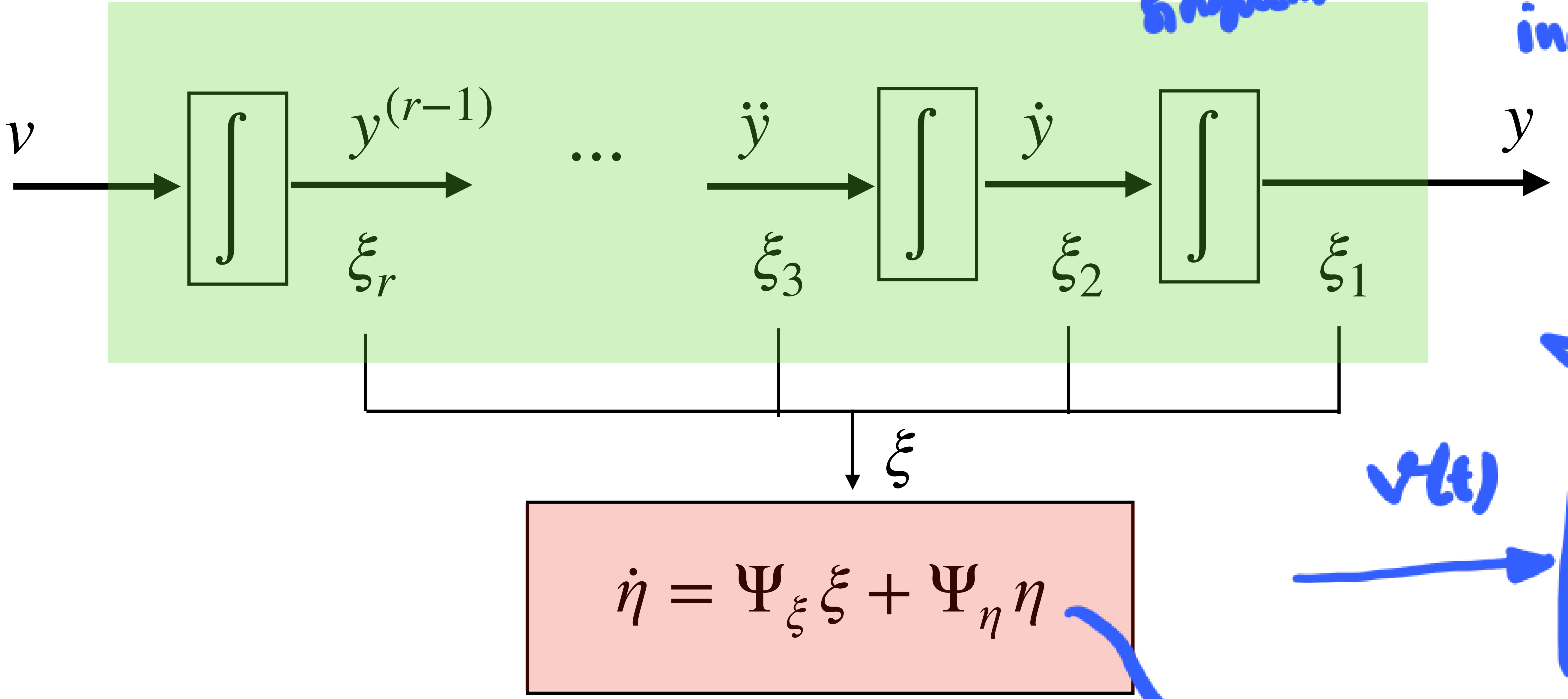
$$
\begin{array}{rcl}
\dot{\xi}_1 & = & \xi_2 \qquad y = \xi_1 \\
\dot{\xi}_2 & = & \xi_3 \\
 & \vdots & \\
\dot{\xi}_{n-1} & = & \xi_n \\
\dot{\xi}_n & = & Q_\xi\xi + Q_\eta\eta + \Delta u
\end{array}
$$

Brunowsky canonical form

If the input $u(t)$ is chosen as the state feedback: $u(t) = \frac{1}{\Delta}(-Q_\xi \xi(t) - Q_\eta \eta(t) + v(t))$ with $v(t)$ a new residual input ($T_\eta B = 0$)

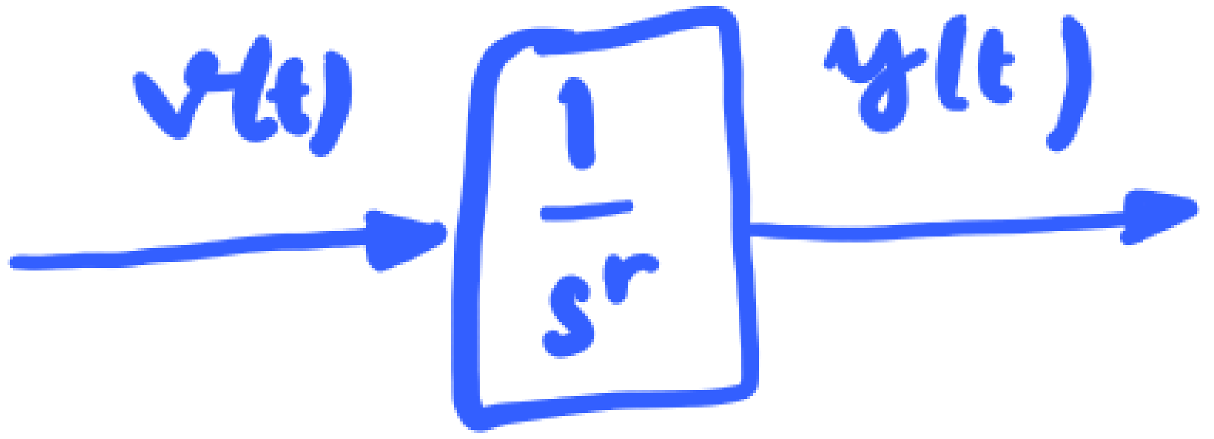
$\neq 0$, non singular

residual input



$y^{(r)}(t) = v(t)$

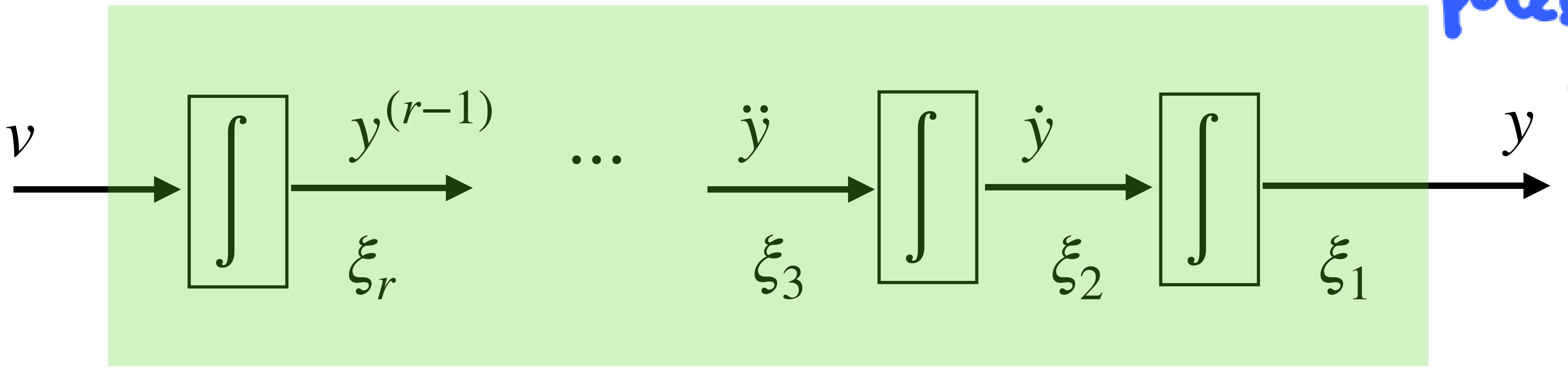
$\dot{\eta} = \Psi_\xi \xi + \Psi_\eta \eta$



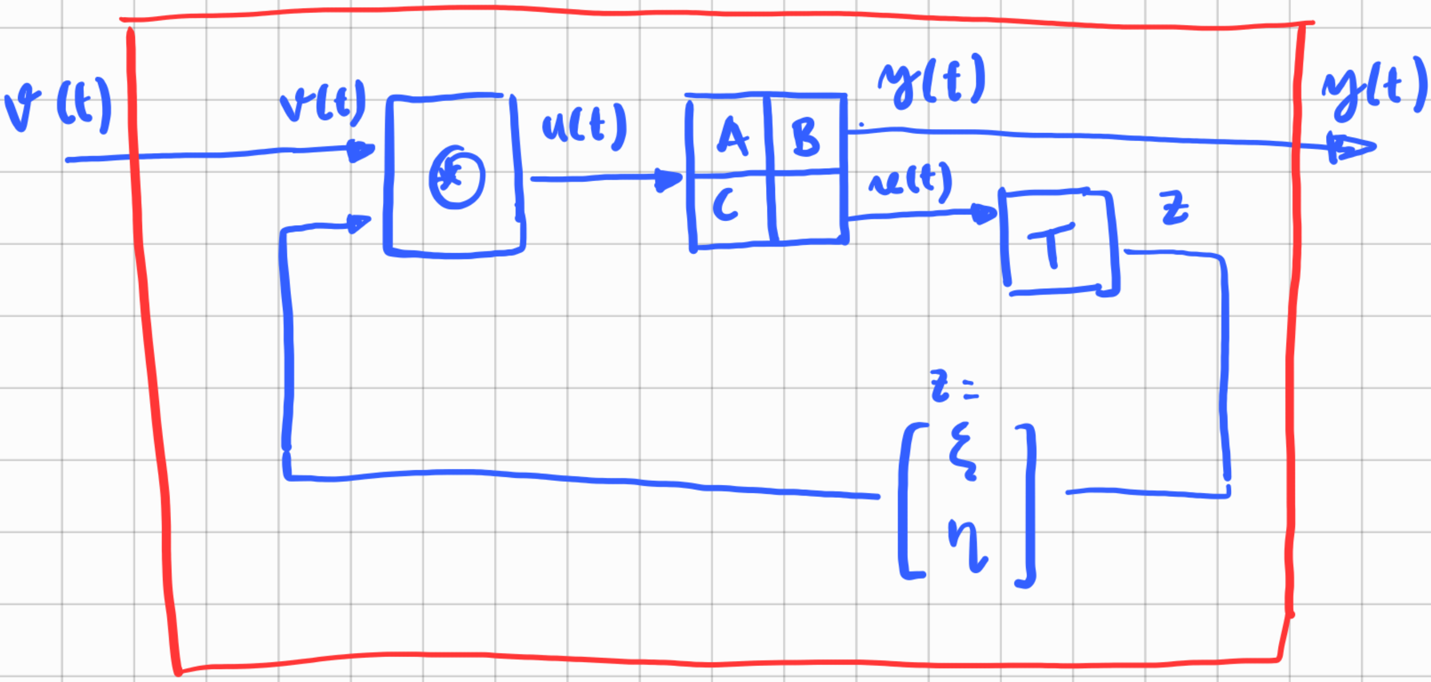
related to zeros and poles at the transfer function

The input output relation of all linear SISO systems can be transformed, under state feedback and change of coordinates, into a I/O chain of r integrators. If $r < n$ an "hidden" η -dynamics arise….

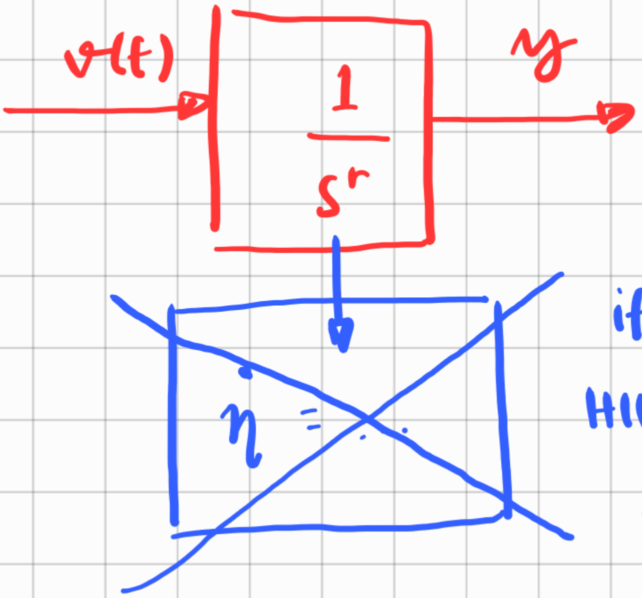
If $r = n$



$y^{(r)}(t) = v(t)$



r integrator



$y^{(r)} = v$

if $r = n$

HIDDEN

$\dot{\eta} = \psi_\xi \xi + \psi_\eta \eta$

Brunowsky canonical form

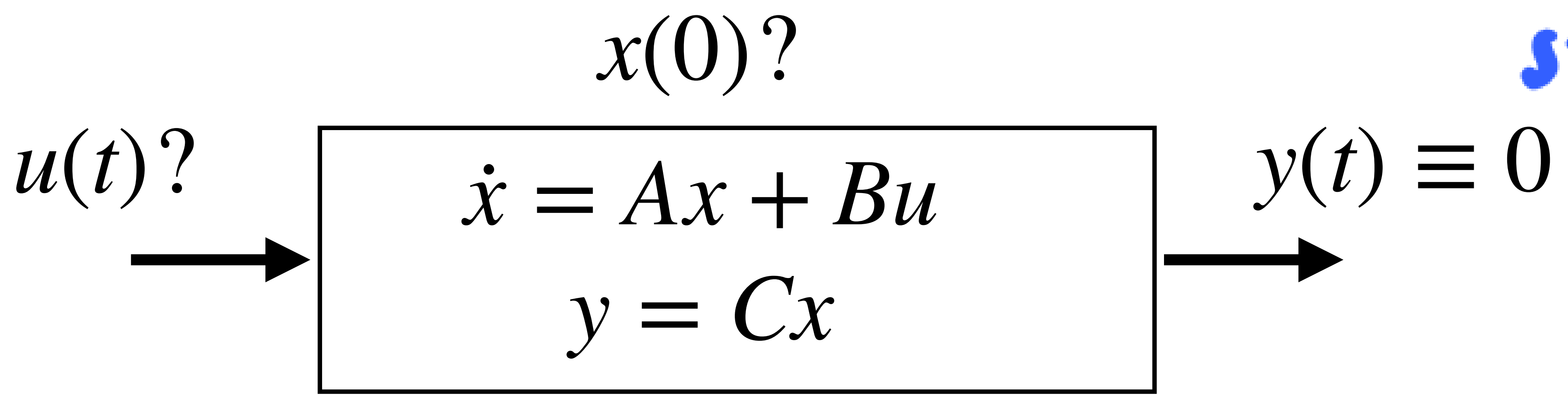
input

Experiment/problem: Given a linear system (A, B, C) compute the $x(0)$ and the $u(t)$ such that the output trajectory $y(t) \equiv 0$ for all $t \geq 0$

initial state

output $=0$

Trivial solution: do nothing while $x(0)=0$



In the Brunowsky coordinate the problem has an immediate solution:

$$x(0) = \begin{pmatrix} T_\xi \\ T_\eta \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \eta_0 \end{pmatrix}$$

$$u(t) = -\frac{Q_\eta \eta(t)}{\Delta} \qquad \text{with } \eta_o \in \mathbb{R}^{n-r} \text{ arbitrary}$$

$$\dot{\eta} = \Psi_\eta \eta \qquad \eta(0) = \eta_0$$

scalar

The "internal dynamics" $\dot{\eta} = \Psi_\eta \eta$ (dynamics of the system that are compatible with a trajectory of the output that is identically zero) is said to be the "**zero dynamics**" of the system. It can be shown to be related to the "zeros" of the transfer function between u and y:

$$\sigma(\Psi_\eta) = \text{zeros}(\underbrace{C(sI - A)^{-1}B}_{n-r})$$

sets of eigenvalues

Solution of output-zeroing problem:

$$\begin{cases} \xi(0) = 0 & y = \xi_1 \\ & \dot{\xi}_r = Q_\xi \xi + Q_\eta \eta + \Delta u = y^{(r)} \\ v(t) = \frac{1}{\Delta}\left(-Q_\eta \eta(t)\right) & \end{cases}$$

$\hookrightarrow \xi(t) \equiv 0$

$$\dot{\eta}(t) = \Psi_\eta \eta(t) \quad \forall \eta(0)$$

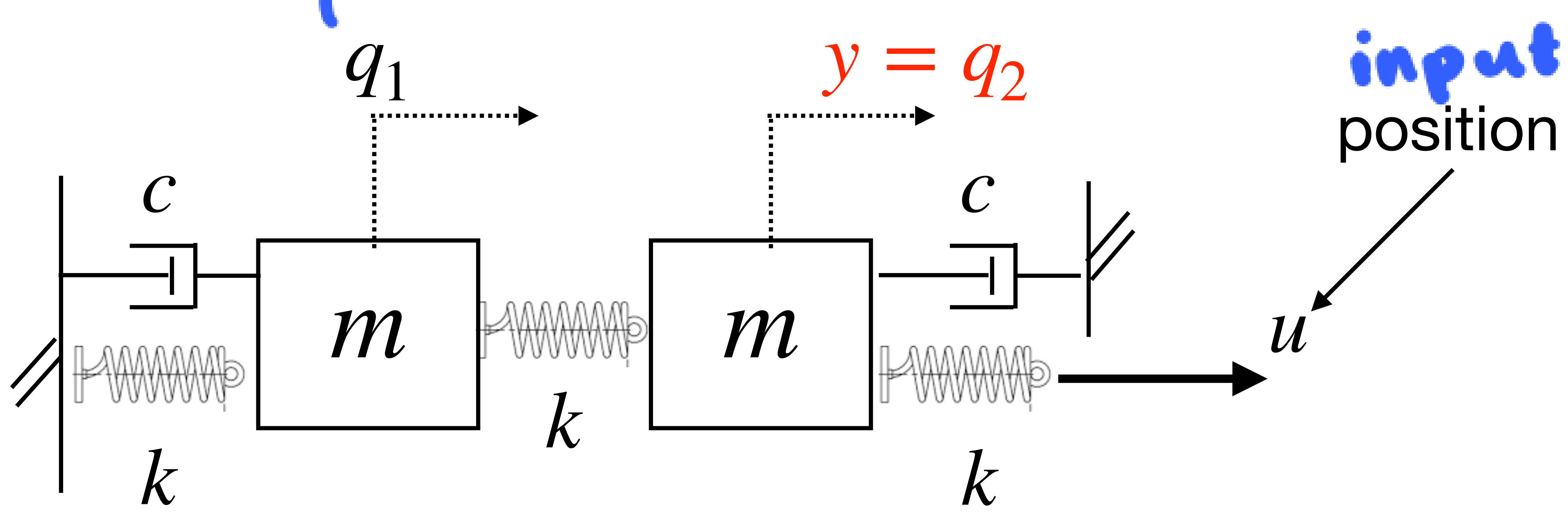
The non-trivial solution is not "straightforward":

$$\boxed{\begin{aligned} x(0) &= T^{-1}\begin{pmatrix} 0 \\ \vdots \\ * \end{pmatrix} \quad \text{generic } \eta(0) \\ u(t) &= \underbrace{\frac{1}{\Delta}}_{\text{non-zero}}\left[-Q_\eta T_\eta x(t)\right] \end{aligned}}$$

Brunowsky canonical form

Example:

position of mass



$$x := \begin{pmatrix} q_1 & q_2 & \dot q_1 & \dot q_2 \end{pmatrix}^T$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} C &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \qquad & CB &= 0 \\ CA &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \qquad & CAB &= \frac{k}{m} \end{aligned} \qquad \Rightarrow \quad r = 2$$

$$T_\xi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$ One possible choice fulfilling $T_\eta B = 0$

$$\dot\xi_1 = \xi_2$$

$$\dot\xi_2 = -\frac{2k}{m}\xi_2 + \frac{k}{m}\xi_1 - \frac{c}{m}\eta_2 + \frac{k}{m}u$$

$$\dot\eta_1 = \eta_2$$

$$\dot\eta_2 = -\frac{2k}{m}\eta_1 - \frac{c}{m}\eta_2 + \frac{k}{m}\xi_1$$

zero dynamics

Homework: Compute the Brunowsky canonical form in case $y = q_1$

Relative degree of the system:

$r = 2$, if $\underline{CB = 0} \xrightarrow{} \text{fulfilled}$ and $CAB \neq 0$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{pmatrix}$$

$$C = (0 \quad 1 \quad 0 \quad 0)$$

$$[0\ 0\ 0\ 1] \begin{bmatrix} 0 \\ 0 \\ 0 \\ k/m \end{bmatrix} = \frac{k}{m} \neq 0$$

fulfilled

According to Brunowsky canonical form:

$$T = \begin{bmatrix} C \\ CA \\ \hline \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} T_\xi \\ \hline T_\eta \end{bmatrix}$$

$$\boxed{\phi_i B = 0}$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = -\frac{2k}{m}\xi_2 + \frac{k}{m}\xi_1 - \frac{c}{m}\eta_2 + \frac{k}{m}u$$

$$\dot{\eta}_1 = \eta_2$$

$$\dot{\eta}_2 = -\frac{2k}{m}\eta_1 - \frac{c}{m}\eta_2 + \frac{k}{m}\xi_1$$

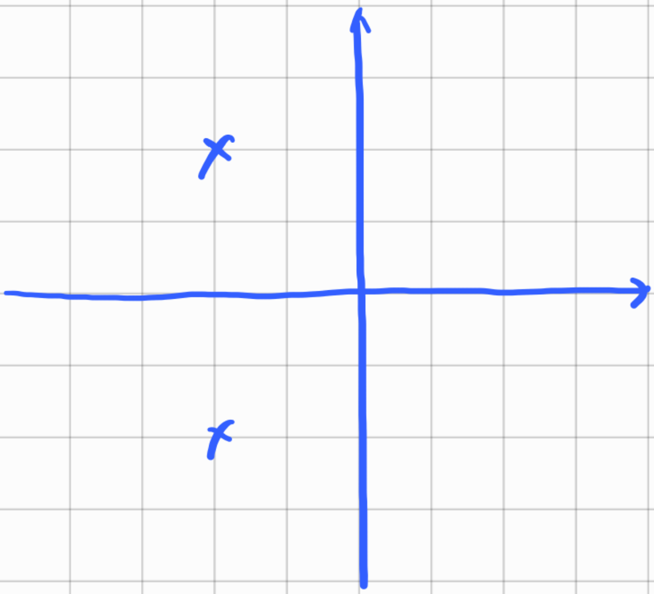
zero dynamics

$$Q_\xi = (k/m \quad -2k/m)$$

$$Q_\eta = (0 \quad c/m)$$

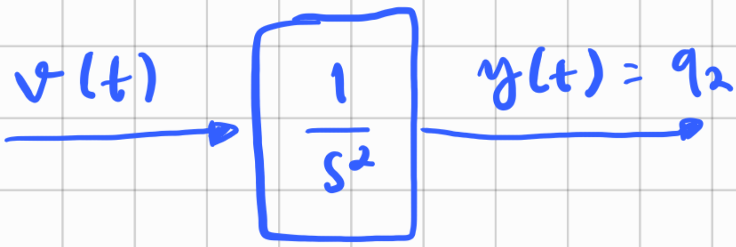
$$\Delta = k/m$$

$$\Psi_\eta = \begin{bmatrix} 0 & 1 \\ -2k/m & -c/m \end{bmatrix}$$



$$u(t) = \frac{m}{k} + \left(\frac{2k}{m}\xi_2(t) - \frac{k}{m}\xi_1(t) + \boxed{\frac{c}{m}\eta_2(t)} + v(t) \right)$$

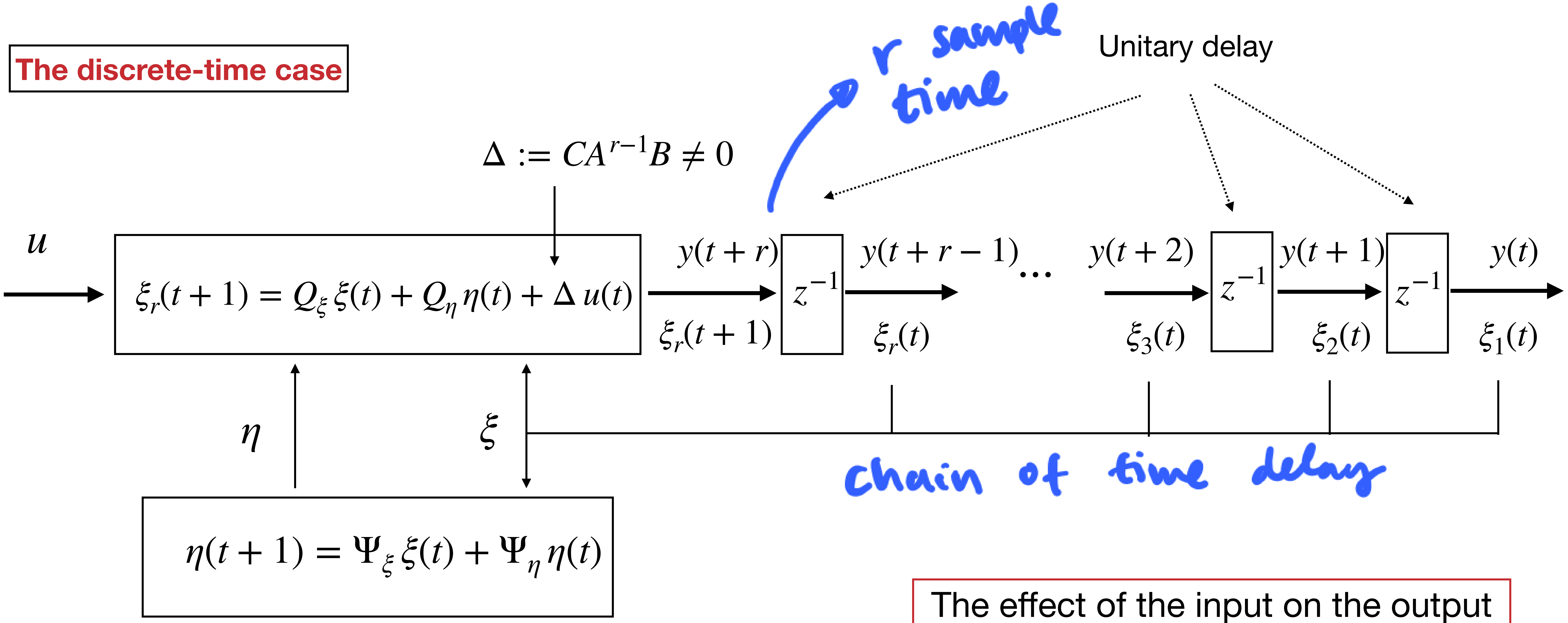
$\dot{y}(t)$ $\quad$ $y(t)$



$$\dot{\eta} = \Psi_\eta \eta + \begin{bmatrix} 0 \\ k/m \end{bmatrix} y(t)$$

Brunowsky canonical form

The discrete-time case



The effect of the input on the output is delayed by r samples