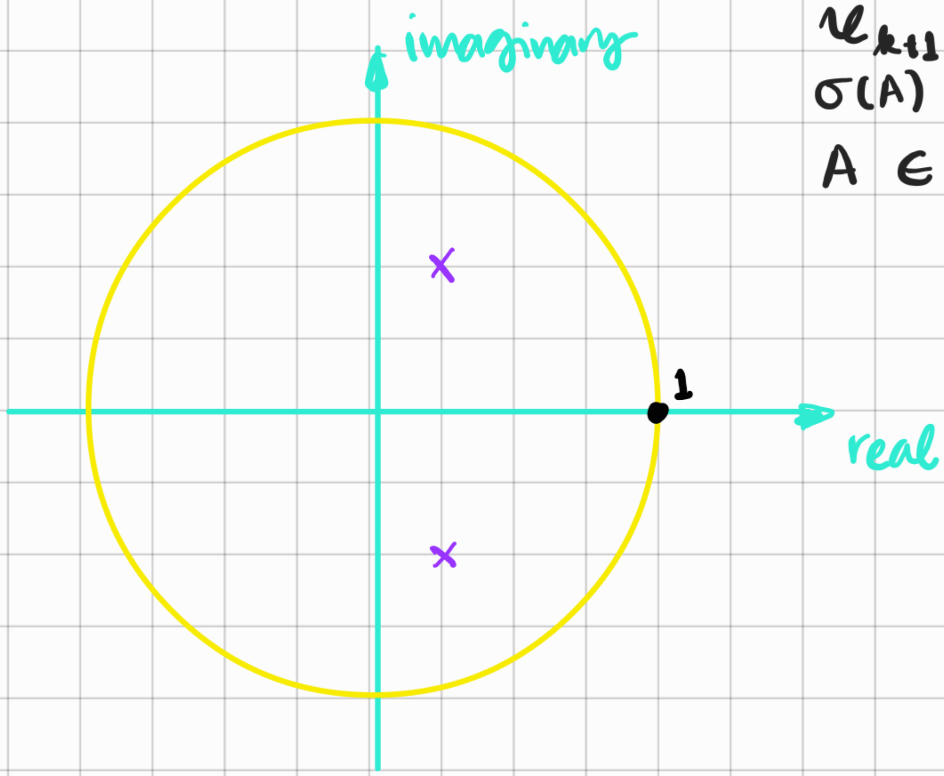
MATLAB EXERCISE #2

Discrete-time case:



$x_{k+1} = A x_k$

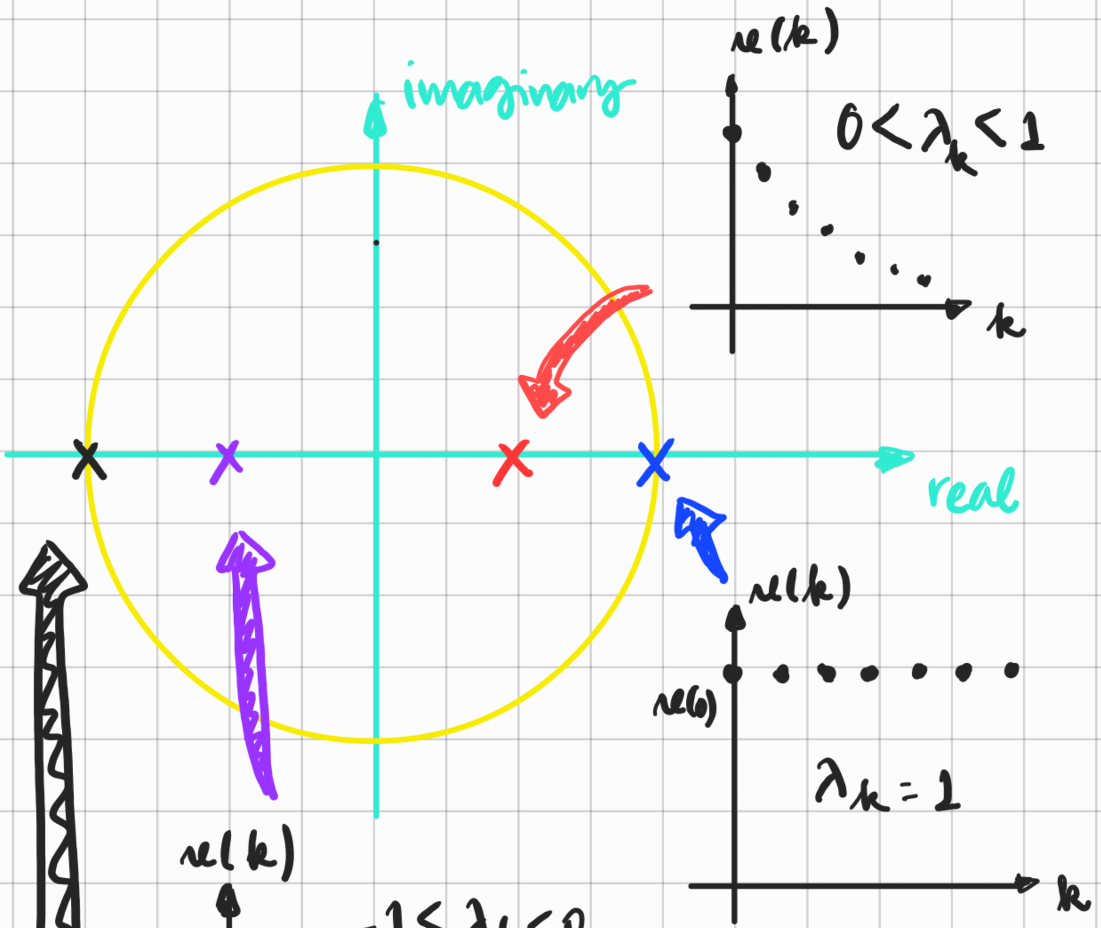
$\sigma(A) = \{\lambda_1, \dots, \lambda_m\}$

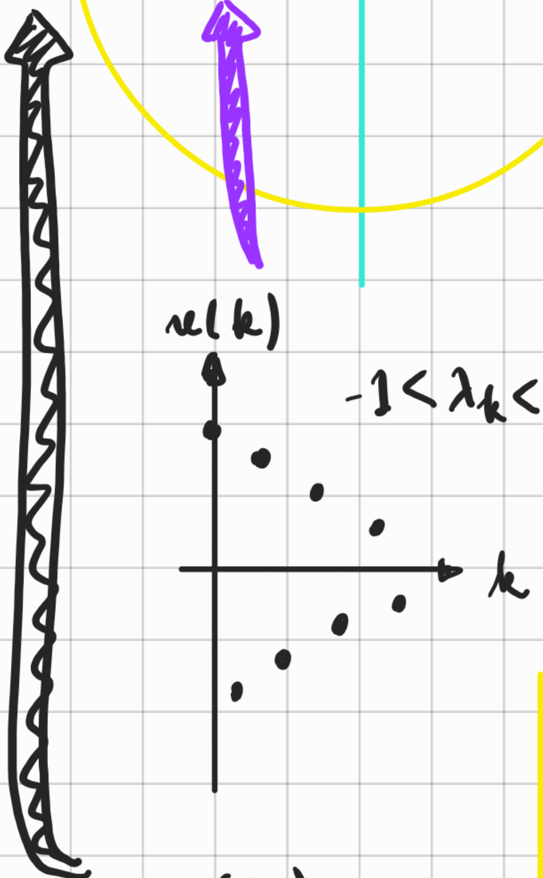
$A \in \mathbb{R}^{n \times n}$

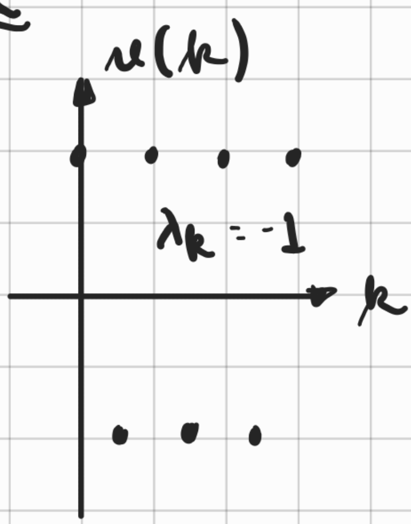
1. Asymptotically stable iff. $|\lambda_i| < 1$ $\forall i = 1, \dots, m$

2. Marginal stability $|\lambda_i| \leq 1$ $\forall i = 1, \dots m$, $\lambda_k = 1$, $a_k = g_k$

3. Unstable if $|\lambda_k| > 1$







Remember that:

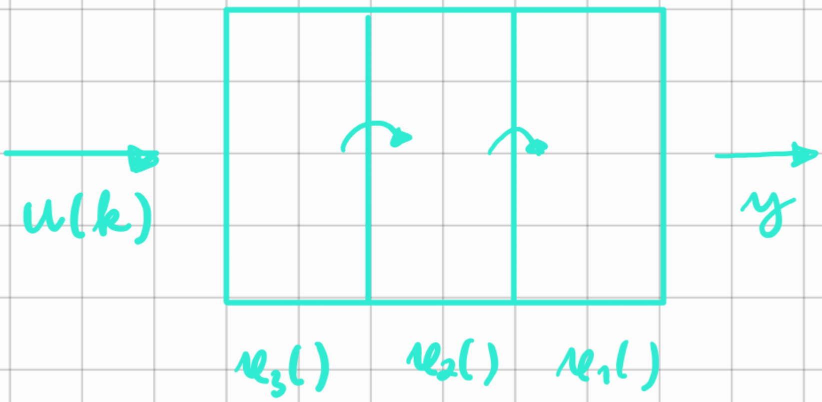
For the cont. system:

$$\dot{x} = Ax \rightsquigarrow A = [-a]$$

$$\dot{x} = -ax$$

$$x(t) = e^{-at}x_0$$

Bemporad:



$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} \leftarrow x_{k+1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} x_k +$$

$$\cancel{\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(k)}$$

Lyapunov Stability:

$\dot{x} = f(x), \quad \bar{x} = 0$

Say Lyapunov function $V: \Omega \to \mathbb{R}$,
where $\Omega \subset \mathbb{R}^4$,

- $V(x) > 0, \quad \forall x \in \Omega / \{0\}$
- $V(0) = 0$: (positive definite)
  - ↳ $\dot{V} \leq 0 \quad \forall x \in \Omega$ (stable)
  - ↳ $\dot{V} < 0 \quad \forall x \in \Omega / \{0\}$ (assym. stable)

↓

$$\dot{V} = \nabla V \, f(x)$$

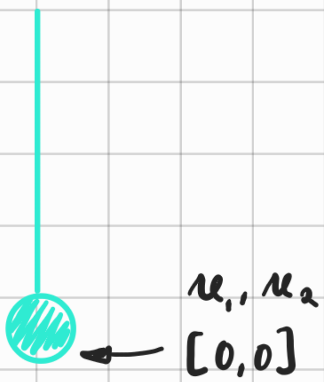
$$= \underbrace{\begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}}_{n} \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

< 0

Pendulum example

$\dot{x}_1 = x_2, \quad m = 1$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{b}{l} x_2$$



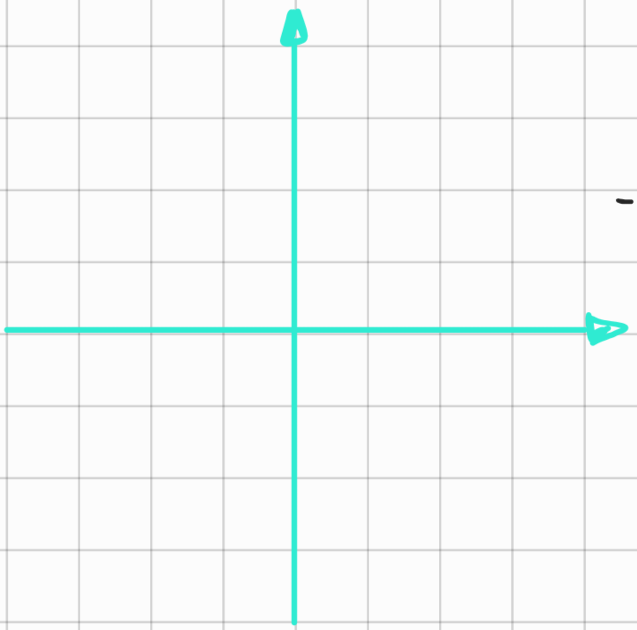
Book: Khalil

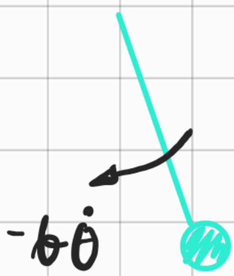
$$V(x) = \underbrace{\frac{1}{2} x_2^2}_{\text{kinetic}} + \underbrace{\frac{g}{l}(1 - \cos x_1)}_{\text{potential}}$$

$$\dot{V} = x_2 \dot{x}_2 + \frac{g}{l} (\sin x_1) \dot{x}_1$$

$$= \cancel{x_2 \left(-\frac{g}{l} \sin x_1\right.} \left. - \frac{b}{l} x_2 \right) + \cancel{\frac{g}{l} \sin x_1 x_2}$$

$$= -\frac{b}{l} x_2^2 \leq 0$$





Variable Gradient Method (VGM)

$$V(x) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2} x^T P x$$

$$V(x) = 0 \iff P > 0 \iff \begin{cases} \lambda_k > 0 \\ P = P^T \\ D_k > 0 \end{cases}$$

$$\downarrow$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

$$D_1 = \det(P_{11}) = P_{11} > 0$$

$$D_2 = \det(P) = P_{11} P_{22} - P_{12} > 0$$

$$\downarrow$$

$$P_{22} > 0$$

$$\boxed{P_{11} > 0,\ P_{22} > 0,\ P_{11} P_{22} - P_{12}^2 > 0}$$

Back to VGM:

$$V = \frac{\lambda}{2} [x_1 \quad x_2] \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{l}(1-\cos x_1)$$

$$= \frac{\lambda}{2}\left(P_{11} x_1^2 + P_{12} x_1 x_2 + P_{12} x_2 x_1 + P_{22} x_2^2\right) + \frac{g}{l}(1 - \cos x_1)$$

$$= P_{11} x_1 \dot{x}_1 + P_{12} x_1 \dot{x}_2 + P_{12} \dot{x}_1 x_2 + P_{22} x_2 \dot{x}_2 + \frac{g}{l} \sin x_1 \dot{x}_1$$

$$\dot{V} = \left(P_{11} x_1 + P_{12} x_2 + \frac{g}{l} \sin x_1\right) x_2 + (P_{12} x_1 + P_{22} x_2)\left(-\frac{g}{l} \sin(x_1) - \frac{b}{l} x_2\right)$$

$$= P_{11} x_1 x_2 + P_{12} x_2^2 + \frac{g}{l} P_{12} x_1 \sin x_1 - \frac{g}{l} P_{12} x_1 \sin x_1 - \frac{b}{l} P_{12} x_1 x_2 + \cdots$$

Final eq.

$$\dot{V} = \left(P_{12} - \frac{b}{l} P_{22}\right) x_2^2 + \left(P_{11} - \frac{b}{l} P_{12}\right) x_1 x_2 + \left(\frac{g}{l} - \frac{g}{l} P_{22}\right) x_2 \sin x_1 - \frac{g}{l} P_{12} x_1 \sin x_1$$

So, according to Lyapunov, the system is stable if:

1.) $P_{12} - \frac{b}{l} P_{22} < 0 \Longrightarrow 0 < P_{12}^{*} < \frac{b}{l}$

2.) $P_{11} - \frac{b}{l} P_{12} = 0 \Longrightarrow P_{11} = \frac{b}{l} P_{12}^{*}$

3.) $\frac{g}{l}(1 - P_{22}) = 0, \quad P_{22} = 1$

$$\dot{V} = \Big(\overbrace{P_{12} - \frac{b}{l} P_{22}}^{P_{12}^{*}}\Big) x_2^2 + \Big(P_{11} - \frac{b}{l} P_{12}\Big) x_1 x_2 + \Big(\frac{g}{l} - \frac{g}{l} P_{22}\Big) x_2 \sin x_1 - \frac{g}{l} P_{12} x_1 \sin x_1$$

$$x \in D = \{x \in \mathbb{R}^2 : x_1 \in (-\pi, \pi)\}$$

$$V = -\underbrace{P_{12}^{*}}_{\leq 0} x_2^2 - \frac{g}{l} P_{12}^{*} \underbrace{x_1 \sin x_1}_{< 0}$$