MATLAB #4 Reachability, Controllability

- Input → discrete - time
  → continuous - time
  → Optimality

- Controllability canonical form

- Kalman decomposition

If a system is completely reachable:

$$\boxed{\mathcal{R}^+ \subseteq \mathcal{R}^- \qquad \text{completely reachable} \;\underset{\nLeftarrow}{\Rightarrow}\; \text{completely controllable}}$$

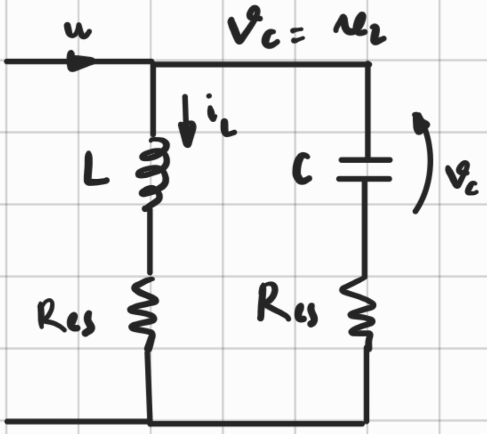
The inverse holds only for <u>reversible system</u>:

1.) discrete → sampled data
2.) continuous → reversible

$i_L = x_1$

$v_c = x_2$

Example: (RLC)



$$A = \begin{bmatrix} -2\,R_{es}/L & 1/L \\ -1/C & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} R_{es}/L \\ 1/C \end{bmatrix}$$

$$R = \begin{bmatrix} Res/L & -2Res^2/L^2 + 1/LC \\ \underbrace{-1/C}_{B} & \underbrace{-Res/LC}_{AB} \end{bmatrix} \quad \text{rank}(R) = 1$$

If rank $(R) = 2$, $\mathcal{R}^+ \subset \mathbb{R}^2$.

For $\sigma(A) = \lambda$ with $a = 2$ and $g = 1$, the Jordan block would be:

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} e^{\lambda t} x_1(0) + t e^{\lambda t} x_2(0) \\ e^{\lambda t} x_2(0) \end{bmatrix} \begin{matrix} x_1(t) \\ x_2(t) \end{matrix}$$

Discrete-time control input

$$x_t = A^t x_0 + R_t \begin{bmatrix} u_{t-1} \\ \vdots \\ u_0 \end{bmatrix}$$

new origin

$$x_t = -\lambda^t x_0 \in \underline{\text{image}(R_t)}$$

linear variety

a subspace

a point $\in \mathbb{R}^n$

1) $x_{t_f} - A^{t_f} x_0 = R_{t_f} \underline{u}_{t_f - 1}$ (particular solution)

2) $0 = R_{t_f} \cdot v$ (homogenous solution)

$$\Downarrow$$

Solution:

1) $(R_{t_f} R_{t_f}^T)\eta = x_{t_f} - A^{t_f} x_0$

2) $\underline{u}^* = R_{t_f}^T \eta$

$\forall \eta_1, \eta_2$ solution of 1)

$\eta_1 - \eta_2 \in \text{kernel}(R_{t_f} R_{t_f}^T)$

$$\boxed{R_{t_f}^T \eta_1 = R_{t_f}^T \eta_2}$$

Let $x_0 \longrightarrow x_f$ as "minimum energy input"

$$\|\underline{u}\| = \langle \underline{u}, \underline{u} \rangle^{1/2} = \left(\sum_{k=0}^{t_f - 1} u_k^T u_k \right)^{1/2}$$

<u>Proof</u> (Minimum energy) $y = Ax$

$$\underline{u}^* = R_{t_f}^T \eta \in \text{image}(R_{t_f}^T)$$
$$\equiv \text{kernel}(R_{t_f})^\perp$$

Every other solution $\underline{u}^* + v$, $v \in \text{kernel}(R_{tf})$

$$\boxed{\|\underline{u}^* + v\|^2 = \|u^*\|^2 + \|v\|^2}$$

Chickens, Wolves, Bears

Let $A = \begin{bmatrix} 0.8 & -0.2 & -0.1 \\ 0.1 & 0.5 & 0.1 \\ 0.1 & -0.1 & 0.5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & -0.1 & 0 \\ 0 & 0.3 & -0.4 & -0.5 \\ 0.5 & 0.2 & -0.5 & -0.4 \end{bmatrix}$

For a discrete-time system: $R_t = n$,

$R_1 = [B] = n$

$R_3 : [\cdots \; AB \; A^{t-1}] = n$

continuous-time system: $R_t : n$

finite

Kalman Decomposition

$A_c = \begin{bmatrix} 1 & \ddots & 0 \\ 0 & \ddots & 1 \\ -\alpha_0 & \cdots & \alpha_{n-1} \end{bmatrix}$ $B_c = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

$\sigma(A_c) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0$

In cont.-time system, it is completely reachable <=> completely controllable ⇒ stabilizable.

$\exists K : (A+BK)$ is Hurwitz, $\forall \lambda \in \sigma(A+BK)$,
real part of $\{\lambda\} < 0$

K = place(A, B, [ ])
↳ obs. eigenval.

$\boxed{u = Kx}$ $\boxed{y = 1x}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad u = Kx$$

$$R = \begin{array}{c} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} B & AB & A^2B \end{matrix} \end{array}$$

$\det(R) = -2$
$\text{rank}(R) = 3$

1) $\sigma_A(\lambda) = \sigma_{Ac}(\lambda)$

$$\det(\lambda I - A) = \underset{\alpha_0}{1} - \underset{\alpha_1}{\lambda} - \underset{\alpha_2}{\lambda^2} + \lambda^3$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

2) $T_c = R_c R^{-1}$

$\sigma(A + BK) = \{-2, -2, -5\}$

$\det(\lambda \hat{I} - A - BK) = \{K_i\}$

Create a char. polynom.

$P_{(A+BK)}(\lambda) = (\lambda+2)^2(\lambda+5)$

$= 20 + 24\lambda + 9\lambda^2 + \lambda^3$

$K_c = [K_{c0} \quad K_{c1} \quad K_{c2}]$ $\qquad -20 = -1 + K_{c0} \qquad -9 = 1 + K_{c2}$

$$A_c + B_c K_c = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 + K_{c0} & 1 + K_{c1} & 1 + K_{c2} \end{bmatrix} \\ \begin{array}{ccc} -d_0 & \quad -d_1 & \quad -d_2 \end{array} \end{array} \qquad B_c = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

$K_{c0} = -19$

$K_{c1} = -25$

$K_{c2} = -10$

$K_c = [-19 \quad -25 \quad -10]$

$$\boxed{\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= -2 \\ \lambda_3 &= -5 \end{aligned}}$$

$\rightarrow$ eigvals in $(A+BK)$

$K = K_c T_c$