Stability

Master degree in Automation Engineering

Prof. Lorenzo Marconi

Dipartimento di Ingegneria dell’Energia Elettrica e dell’Informazione (DEI)
Università di Bologna

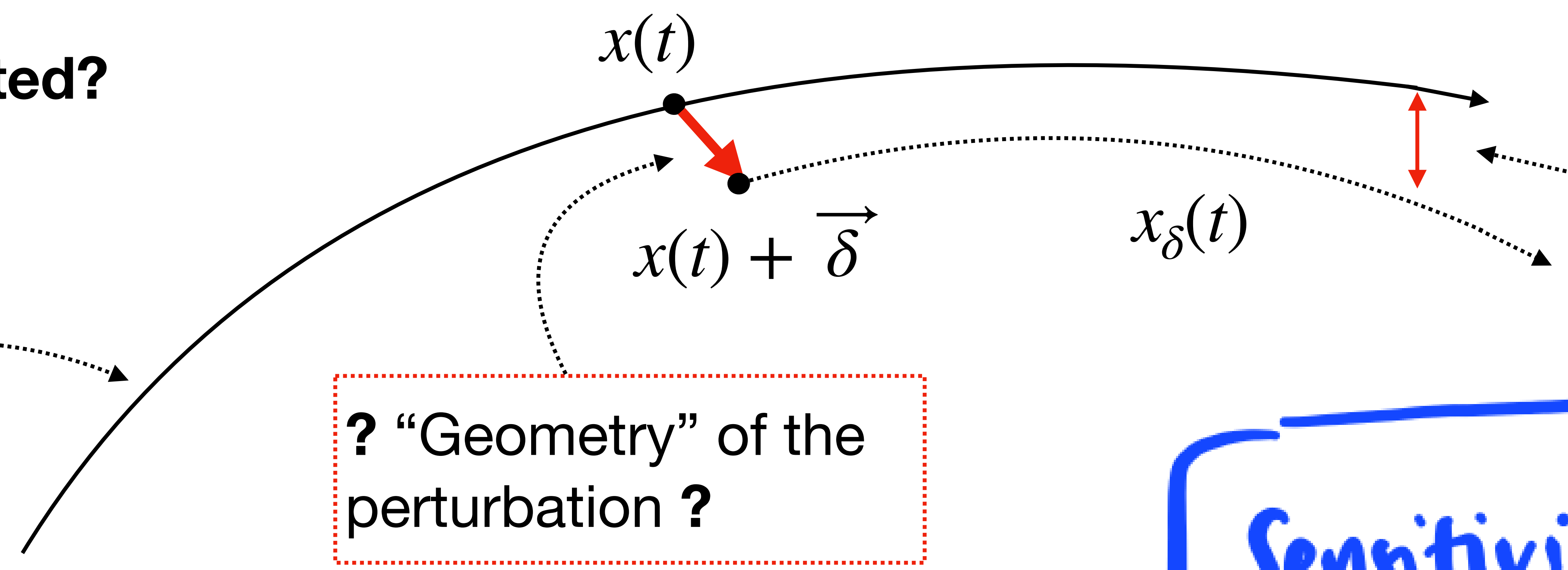
lorenzo.marconi@unibo.it
Tel. +39 051 2093788

Stability → how the system's trajectory behaves when being introduced a perturbation in its input

Very important area in Systems Theory aiming to study the behaviour of trajectories when a perturbation is applied.

How perturbed trajectories are related?

$$\dot{x} = f(x)$$



? Properties dependent on the "starting" "nominal trajectory**?**

? "Geometry" of the perturbation **?**

? remain close (same order of the magnitude od he), diverge, asymptotically converge **?**

Sensitivity (to initial conditions) analysis

$$\dot{x} = f(x(t))$$

? what about perturbation introduced by inputs**?**

$\dot{x} = f(x)$, $\bar{x}$ $f(\bar{x}) = 0$, $x(t) \equiv \bar{x}$

Starting point: autonomous systems (no inputs) and "reference" trajectory (the one that is perturbed) is an equilibrium point

$\epsilon - \delta_\epsilon$ Stability

Let's study the behaviour of the system close to an equilibrium point when the initial state is perturbed. Without loss of generality we take the equilibrium point to be the origin ($f(0) = 0$). n any (not necessarily planar or 3-D systems).

Stability: the magnitude of the perturbed trajectory is of the same order as the perturbation.

$\bar{x} \simeq$ Origin

ε is any number

The equilibrium $x = 0$ is stable if for all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that for all initial conditions satisfying $\|x(0)\| \leq \delta_\epsilon$ the resulting trajectory satisfies $\|x(t)\| \leq \epsilon$ for all $t \geq 0$.

Asymptotic stability: stability and the effect of the perturbation is vanishing.

stability + attractivity

The equilibrium $x = 0$ is asymptotically stable if it is stable and there exists a set $\mathscr{A} \supset \{0\}$ such that $x(0) \in \mathscr{A}$ implies $\lim_{t \to \infty} x(t) = 0$ (attractivity property). Terminology: $\mathscr{A}$ is said to be "Domain of Attraction".

Instability: lack of stability.

The equilibrium $x = 0$ is unstable if it is not stable (not joking!).

$\dot{x} = f(x)$, $\bar{x}$ $f(\bar{x}) = 0$, $x(t) \equiv x$

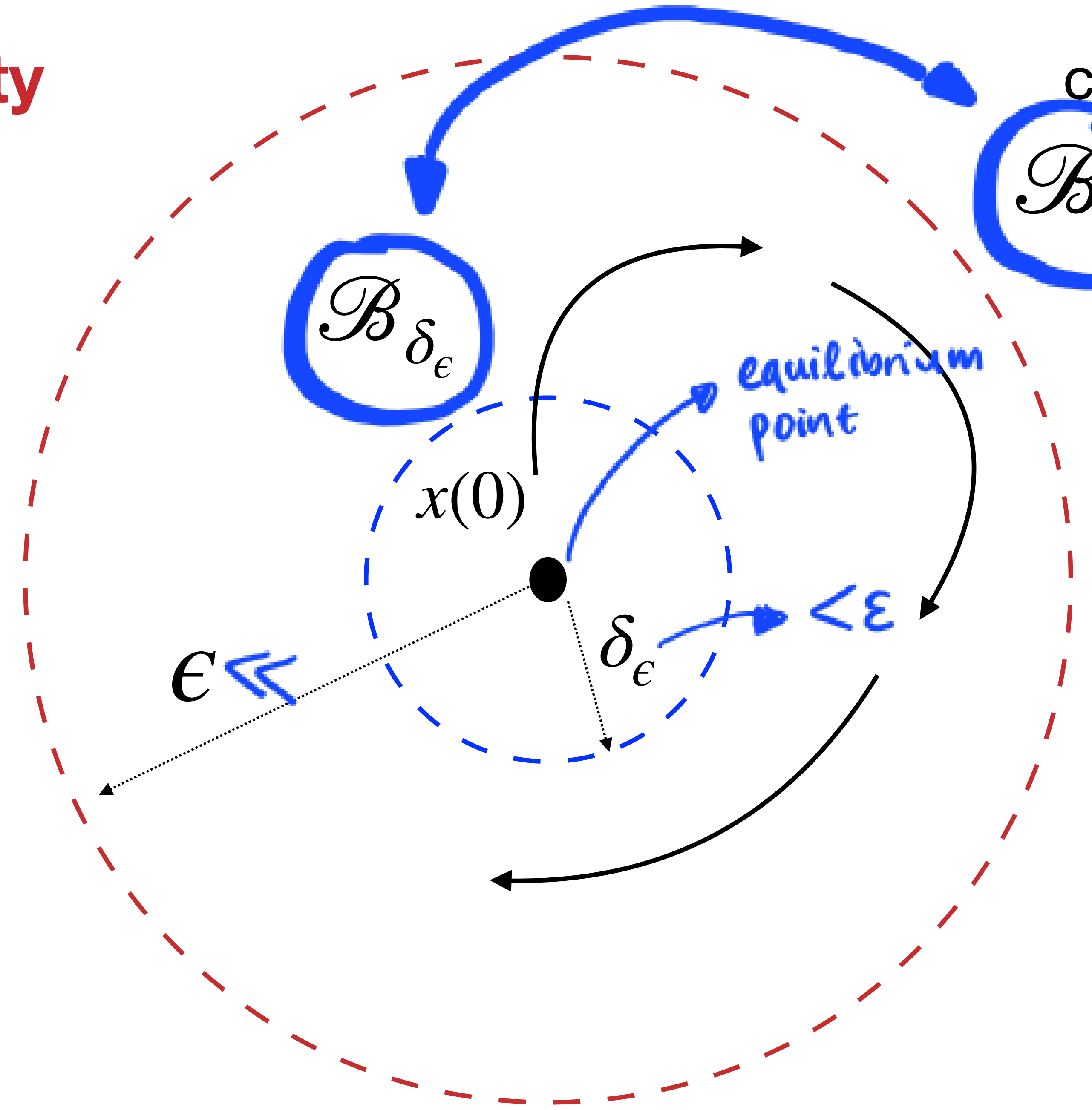
it's the origin

$\tilde{x} = x - \bar{x}$

$\dot{\tilde{x}} = f(\tilde{x} + \bar{x})$

$\epsilon - \delta_\epsilon$ stability notion

Stability



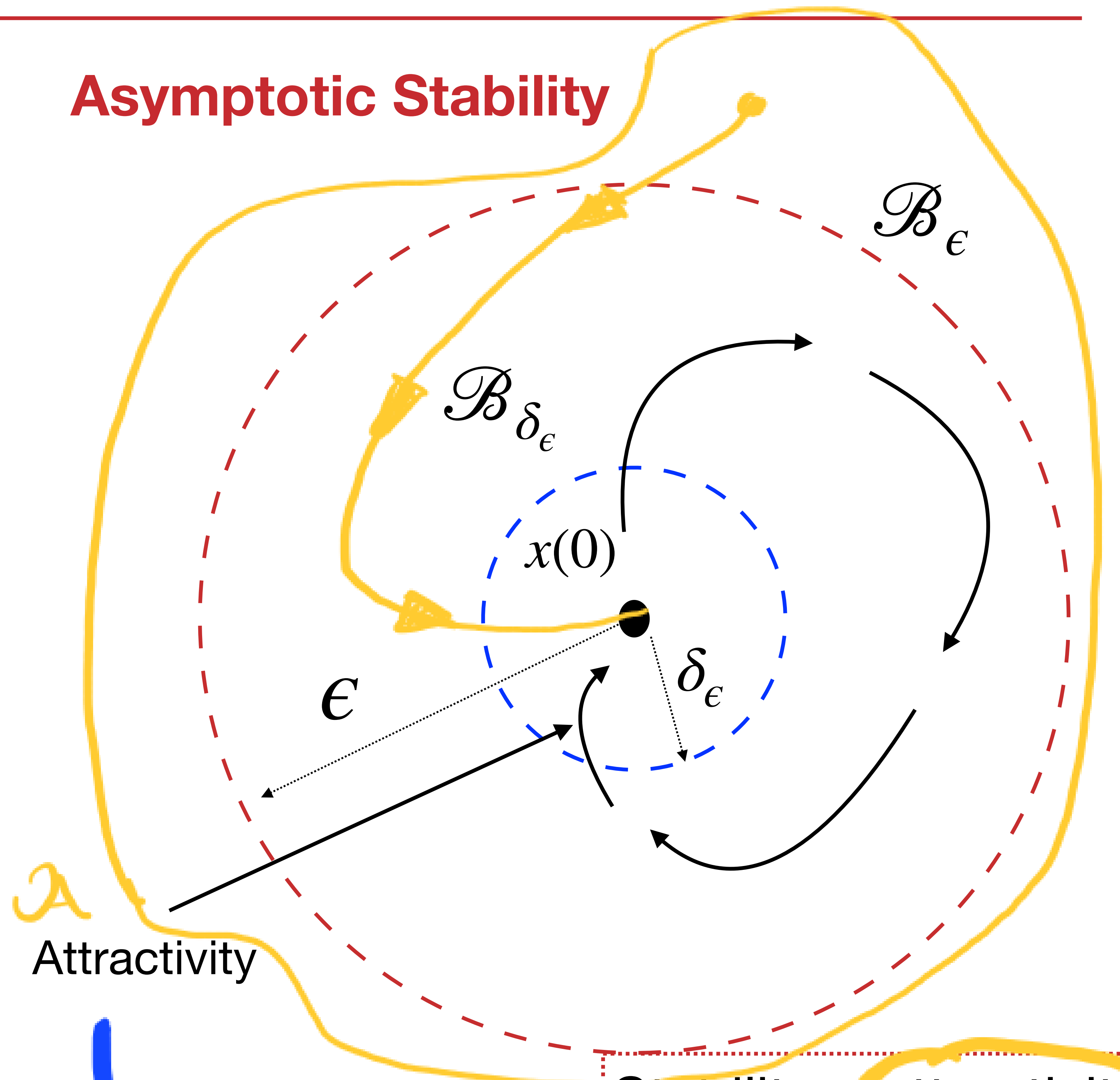
circle in the plane, sphere in the space, …

$$\mathcal{B}_\epsilon := \{x \in \mathbb{R}^n \ : \ \|x\| \leq \epsilon\}$$

ϵ is given (**any**, potentially very small), there must exist a δ_ϵ (smaller..) to have stability of the equilibrium point.

$x(0)$ is not chosen (for all initial conditions in the ball of radius δ_ϵ ….

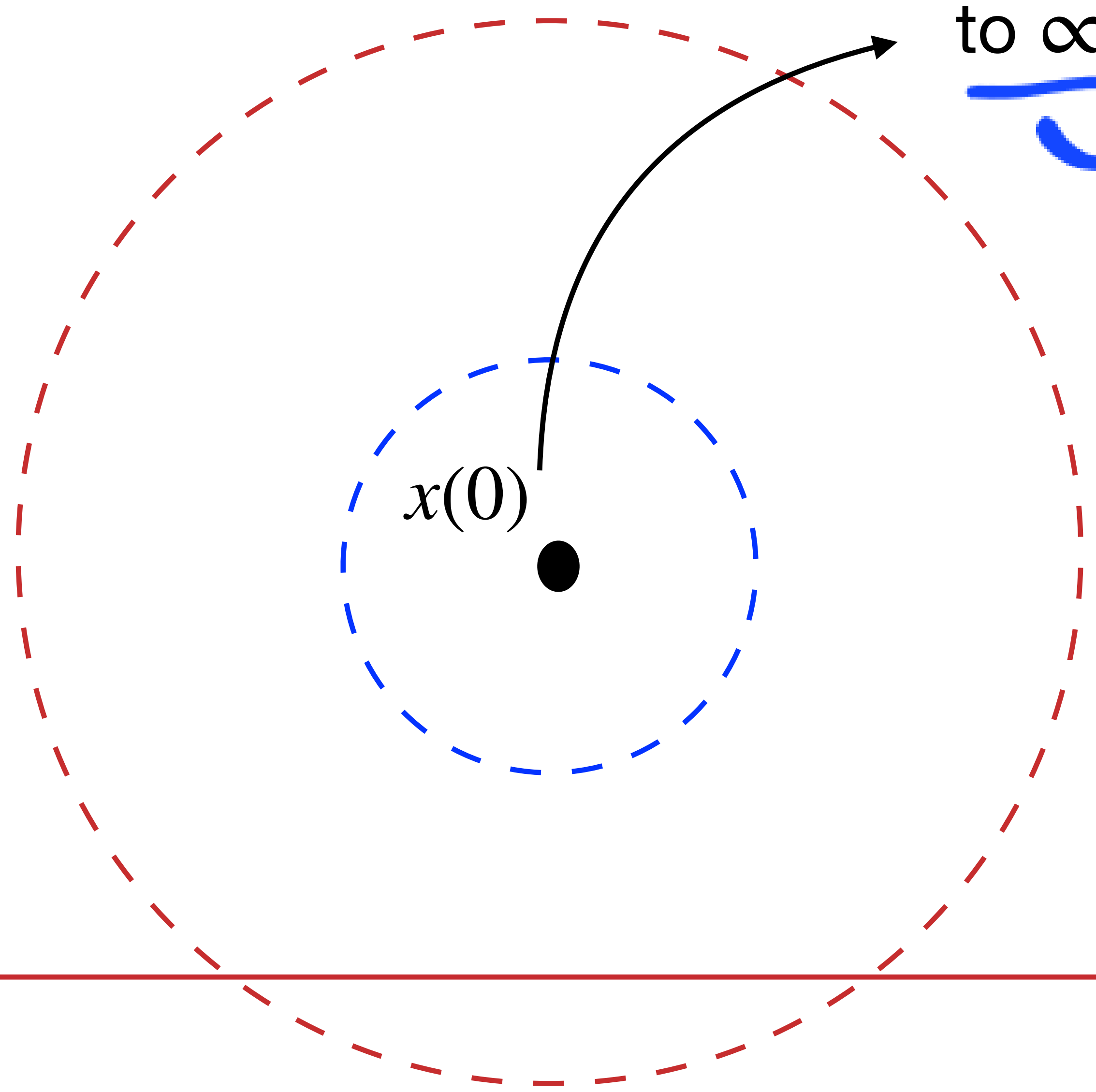
Asymptotic Stability



Stability + attractivity

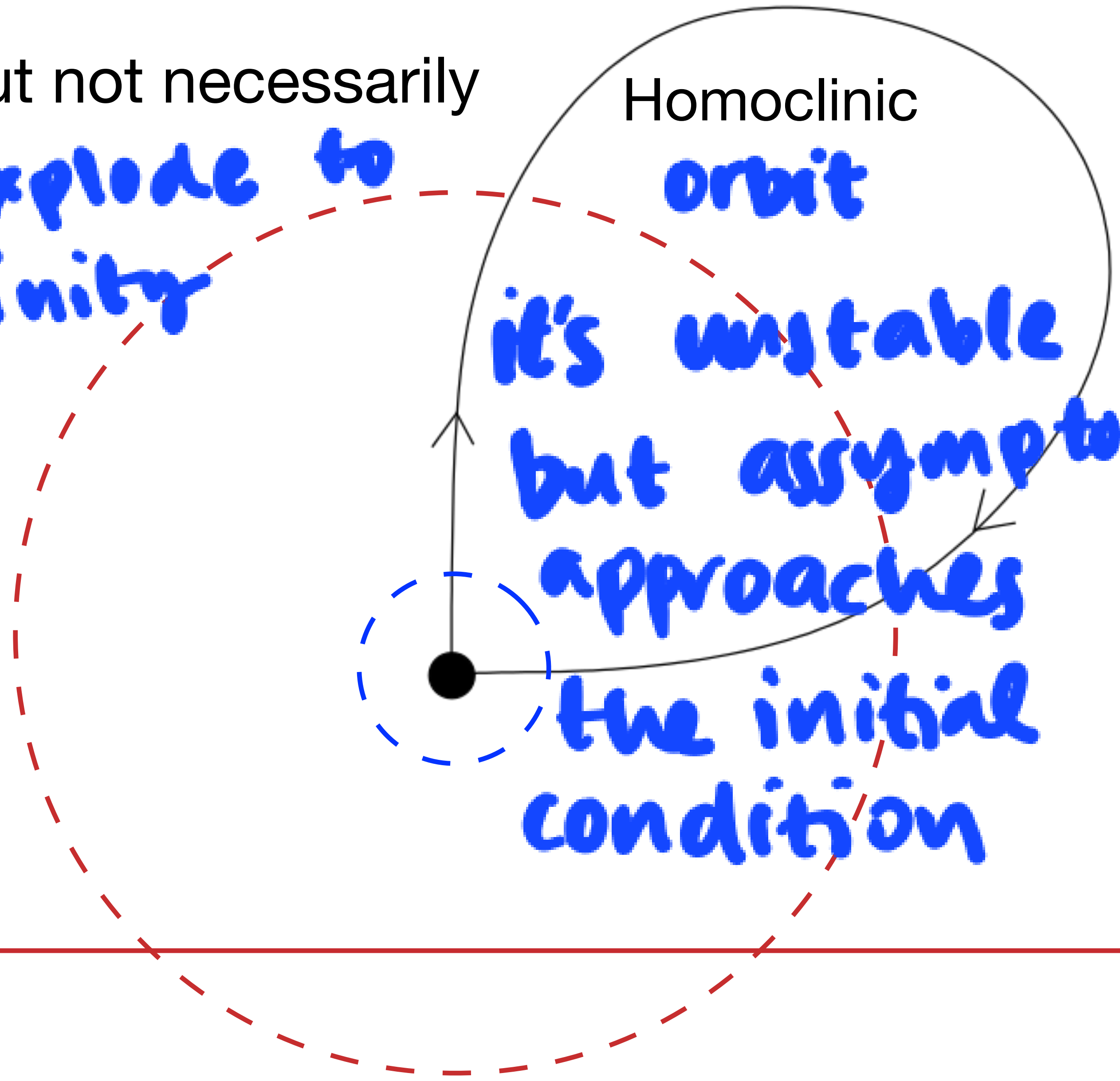
does not imply stability

Instability



to ∞ … but not necessarily

explode to infinity



it's unstable but asymptot. approaches the initial condition

No matter which δ_ϵ is taken, at least one trajectory exits $\mathcal{B}_\epsilon$

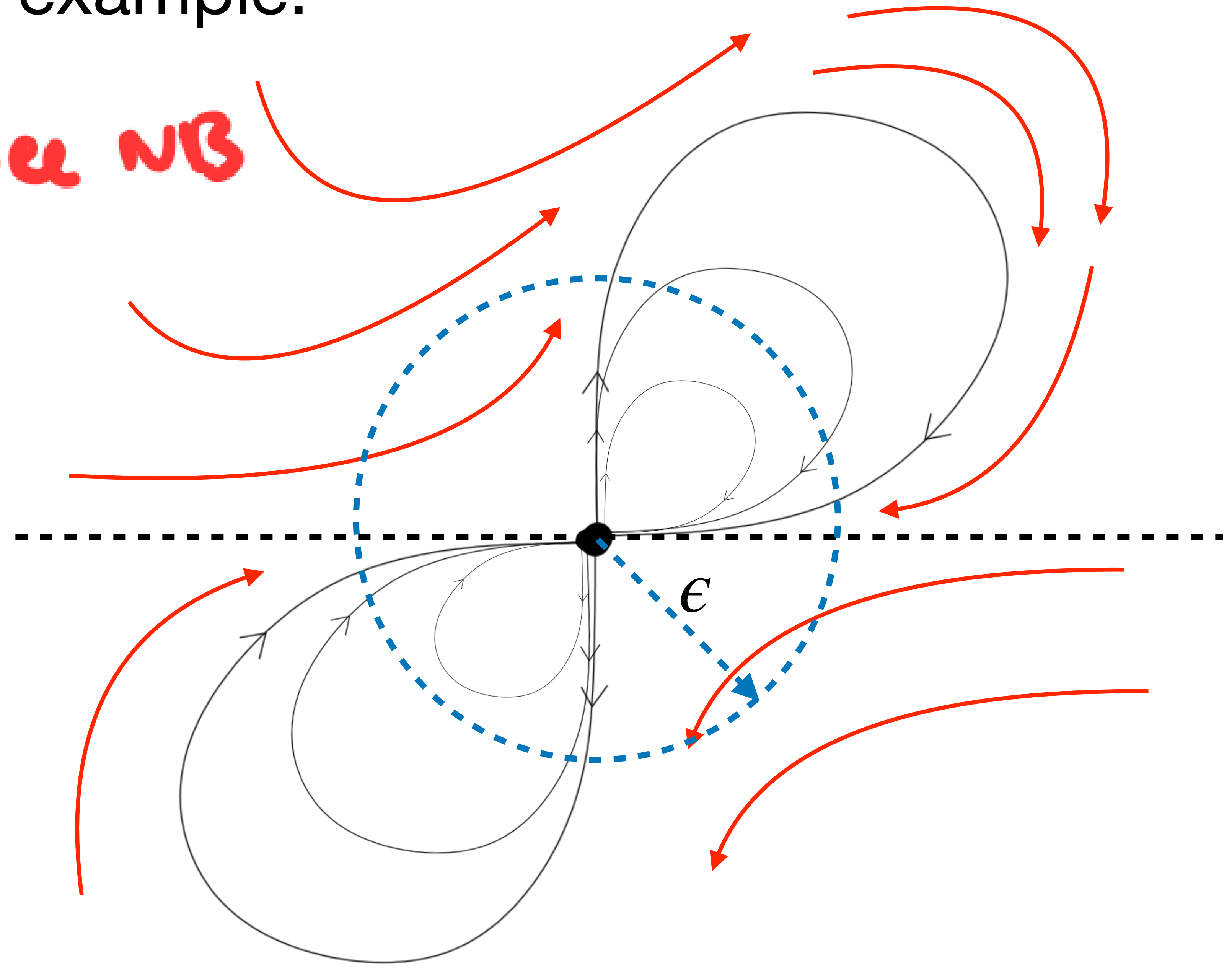
Attractivity does not imply stability (in a general nonlinear setting)

Remarks

- The lack of stability in a nonlinear system is not necessarily associated to perturbed trajectories that diverges from the equilibrium. Homoclinic example:

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\right\} \text{ see NB}$$

Say that the perturbation aligns w/ the orbit, $\|x(0)\| \approx 10^{-10}$, and $\|x(t)\| \approx 1$, there is no δ_ε



Given a circle of radius ϵ with the latter taken so that the circle intercept the homoclinic orbit, a δ_ϵ doesn't exist… and the origin is "globally" attractive

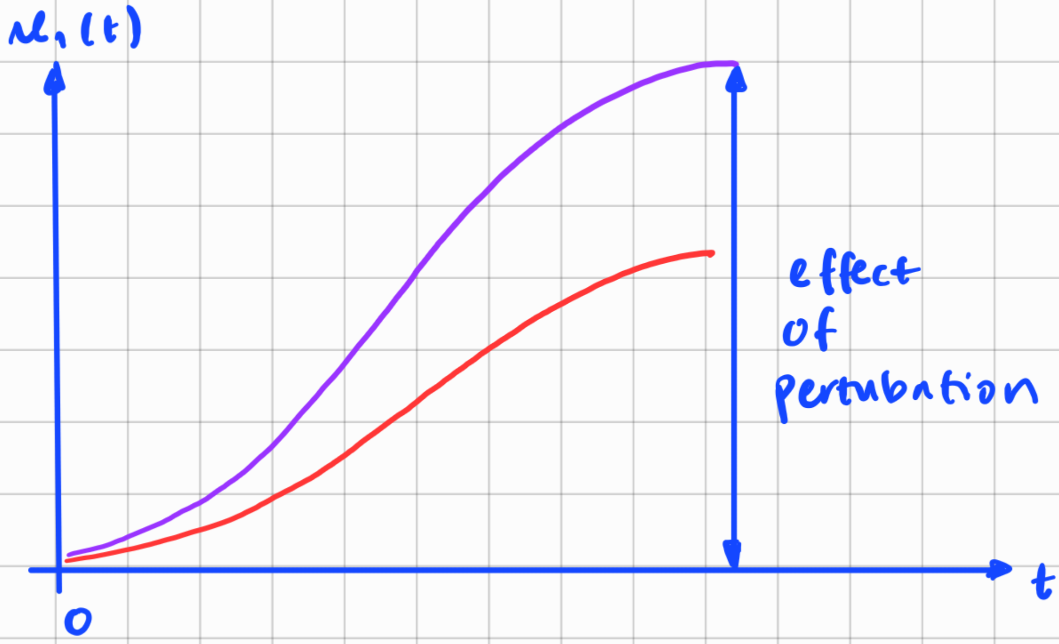
- The property of stability refers to an equilibrium point. Different equilibria of the same (nonlinear) system might have different stability properties. That is, the property of stability is not linked to the system but to a specific equilibrium

- For linear systems the notion of stability can be referred to the system and can be extended to generic trajectories (not necessarily equilibria)

NB:

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)} \quad \circledast$$

$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{\circledast}$$



LINEAR SYSTEM STABILITY

$$\dot{x} = Ax$$

$$x(t+1) = A\,x(t), \quad \text{where } x(t) = \phi(t)\cdot x(0)$$

$$x_{ref}(0) = 0$$

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 : \forall x(0)$$

$$\|x(0)\| \le \delta_\varepsilon \Rightarrow \|x(t)\| \le \varepsilon, \ \forall t \ge 0$$

$$\|x(t) - 0\| = \|\phi(t)\ x(0)\| \le \|\phi(t)\| \cdot \|x(0)\| \le \varepsilon$$

$\le \delta_\varepsilon$?

given

If $\phi(t)$ is bounded for $\forall t \ge 0$ (there exists $\exists M$: $\|\phi(t)\| \le M$ for $\forall t \ge 0$):

$$\boxed{\delta_\varepsilon = \frac{\varepsilon}{M}}$$

$$\le M\|x(0)\|$$

The case of linear systems

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \qquad x(0)=x_0$$

$$x(t) = \phi(t)\,x_0 \qquad\qquad \phi(t) = \begin{cases} e^{At} & \text{for C-T systems} \\ A^t & \text{for D-T systems}\end{cases}$$

For all $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that for all x_0 fulfilling $\|x_0\| \le \delta_\epsilon$ then $\|x(t)\| = \|\phi(t)\,x_0\| \le \epsilon$ for all $t \ge 0$

Since $\|\phi(t)\,x_0\| \le \|\phi(t)\|\,\|x_0\| \le \|\phi(t)\|\,\delta_\epsilon$ it turns out that:

If $\exists\, M > 0 :\ \|\phi(t)\| \le M\ \forall\, t \ge 0$ then the system is stable (pick $\delta_\epsilon = \dfrac{\epsilon}{M}$)

If $\exists\, M > 0 :\ \|\phi(t)\| \le M\ \forall\, t \ge 0$ and $\phi(t) \to 0$ as $t \to \infty$ then the system is asymptotically stable

Theorem: A C-T (D-T) linear system is stable **iff** all the eigenvalues of A have a nonpositive real part (amplitude ≤ 1) and possible eigenvalues with zero real part (amplitude $= 1$) have a geometric multiplicity equal to the algebraic one. It is asymptotically stable **iff** the eigenvalues of A have all negative real part (amplitude < 1)

The case of linear systems

Notation:

C-T asymptotically stable systems $\iff$ A **Hurwitz** → in negative real value

D-T asymptotically stable systems $\iff$ A **Schur** → within circle

For linear systems the notion and the properties can be extended to "generic trajectories"

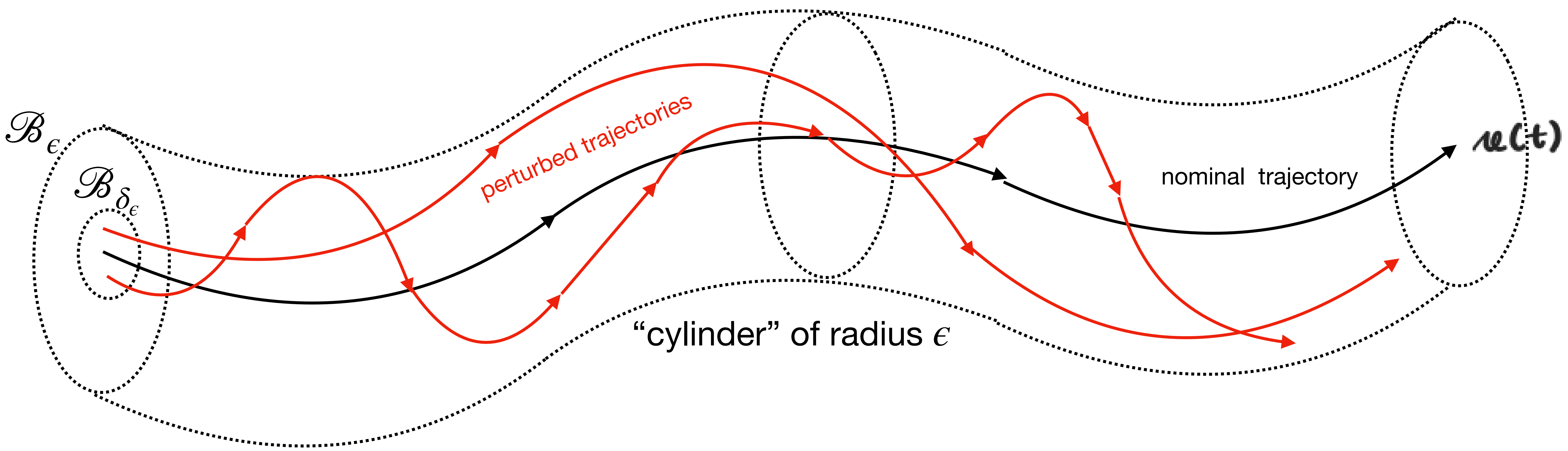
generic

$$x(t) = \phi(t)\, x_0 \; + \; \Psi(t)\, u([0,t))$$

$$x(t) = \phi(t)\, (x_0 + \delta) \; + \; \Psi(t)\, u([0,t))$$

$$\|x(t) - x(t)\| = \|\phi(t)\, \delta\| \leq \|\phi(t)\|\; \|\delta\|$$

If $\phi(t)$ is bounded then the ϵ/δ_ϵ condition is fulfilled



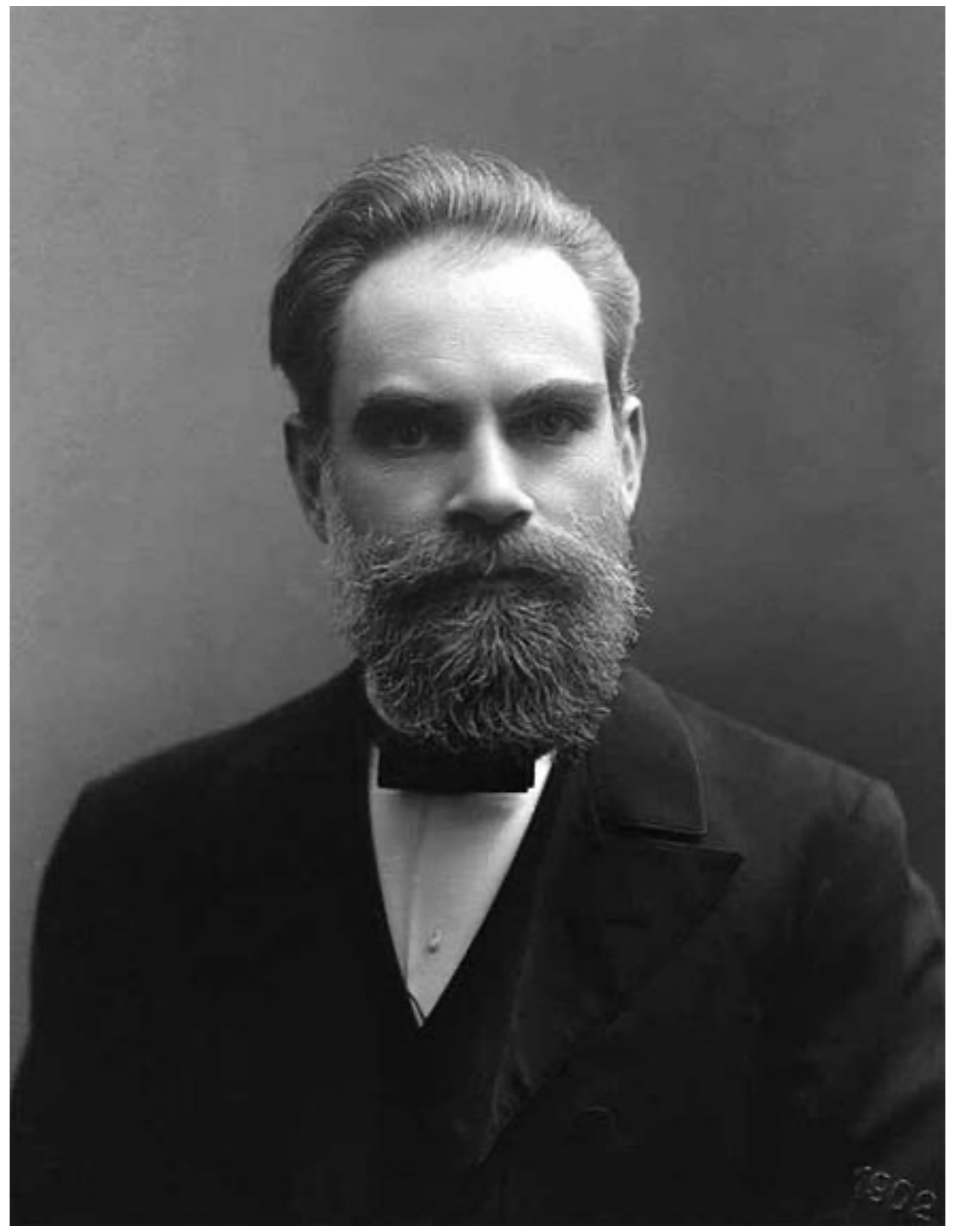
Furthermore, A Hurwitz then the trajectory generated by any bounded input $u(t)$ is also bounded (BIBS) alternative definition of $\varepsilon - \delta$ stability

The $\epsilon - \delta_\epsilon$ stability notion is often said also "Lyapunov Stability" because of a result, due to A. M. Lyapunov, providing sufficient and necessary conditions under which an equilibrium point of a nonlinear dynamic system is stable/asymptotically stable.

integrate to get $x(t)$

While the $\epsilon - \delta_\epsilon$ stability notion involves, in principle, trajectories $x(t)$ and, thus, would involve solving the differential equation $\boxed{\dot{x} = f(x)}$, the Lyapunov result is rather "geometric" in the sense that it involves only checking state-wise a real-valued function defined in the state space.

The result has a nice "energy" interpretation: if an equilibrium point is stable then there exists a form of "energy" function that is not increasing along the system trajectories. In case of an asymptotically stable equilibrium point such a energy function is also decreasing reaching a minimum at the equilibrium.



Aleksandr M. Lyapunov, 1857 - 1918

the idea why this theorem developed is because plotting $x(t)$ could be in $\mathbb{R}^n$

we lose the t variable, but we inspect the geometry

Lyapunov Direct Theorem

open set where V is defined

real-valued function

associated with real number

Definition. A function $V : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be **positive definite** with respect to $\bar{x} \in \mathscr{D}$ if $V(\bar{x}) = 0$ and $V(x) > 0$ for all $x \in \mathscr{D} \backslash \{\bar{x}\}$. Notation: $V > 0$ ($\bar{x}$ implicit from the context).

Definition. A function $V : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be **positive semi-definite** with respect to $\bar{x} \in \mathscr{D}$ if $V(\bar{x}) = 0$ and $V(x) \geq 0$ for all $x \in \mathscr{D}$. Notation: $V \geq 0$ ($\bar{x}$ implicit from the context).

Definition. Negative definite and **semi-definite** functions can be similarly defined in a straightforward way.

Examples in $\mathbb{R}^2$

$$V(x) = x_1^2 + x_2^2$$

> 0 wrt the origin

$sg(s) > 0$

$$V(x) = x_1^2 + \int_0^{x_2} g(s)\, ds$$

> 0 wrt the origin

$$V(x) = x_1^2 + (1 - \cos x_2)$$

> 0 wrt the origin

$\mathscr{D} = \{(x_1, x_2) \ : \ |x_2| < \pi/2\}$

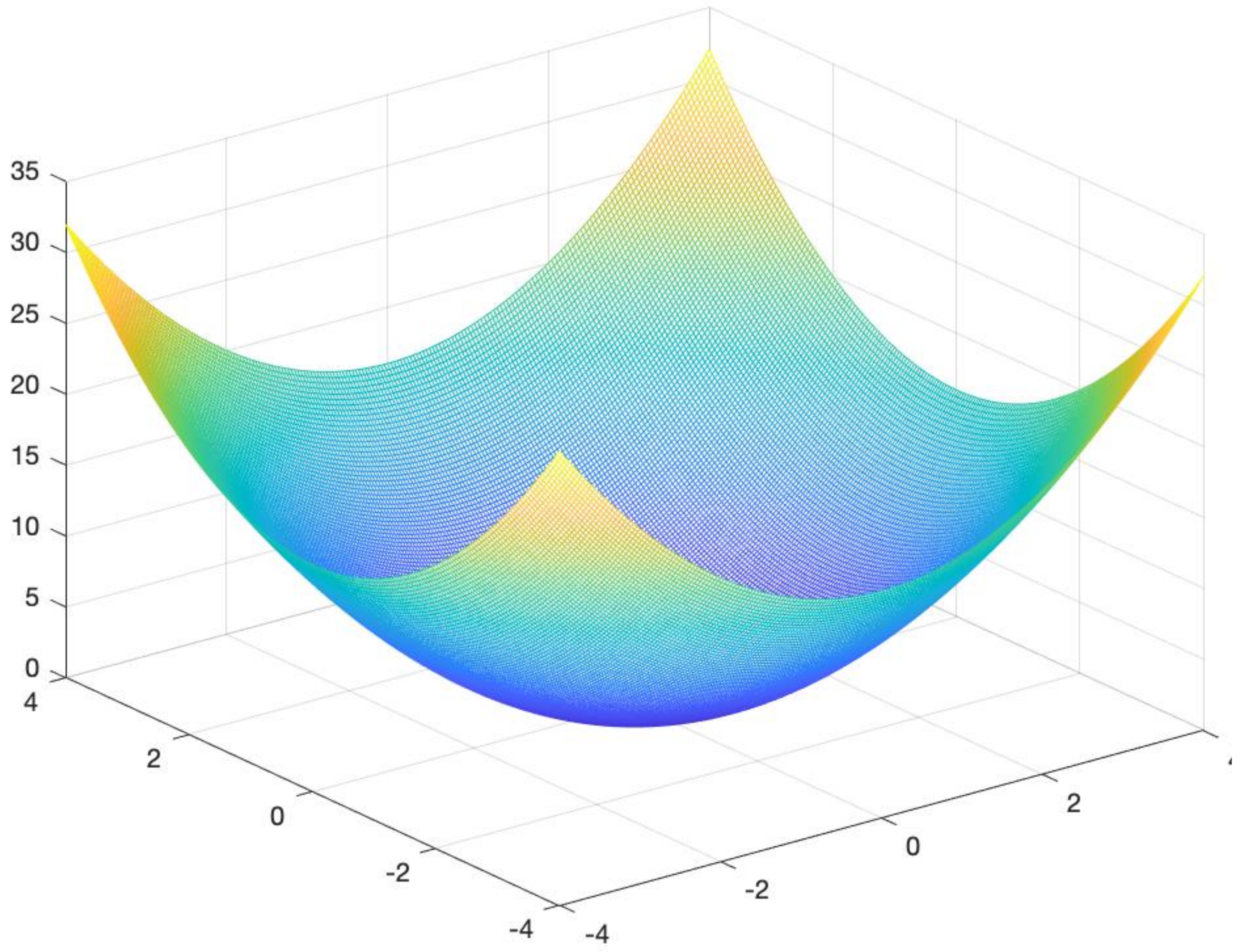
$$V(x) = x_1^4$$

≥ 0 wrt any point in $\{(x_1, x_2) : x_1 = 0\}$

Examples with Matlab

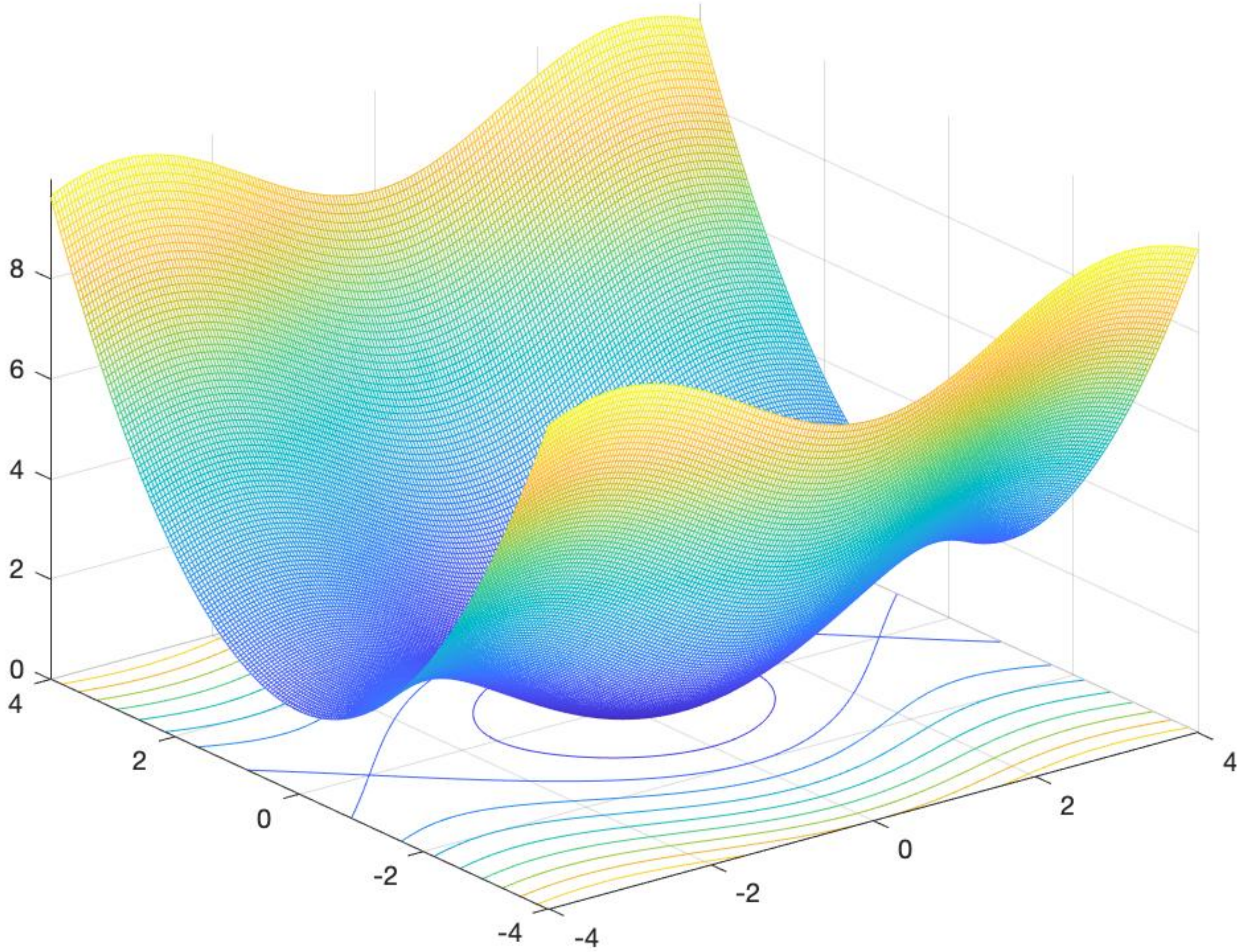
$$V(x,y) = x^2 + y^2$$

```
x=[-4:.04:4];
y=x;
[X,Y]=meshgrid(x,y);
z=X.^2 + Y.^2;
mesh(X,Y,z)
```



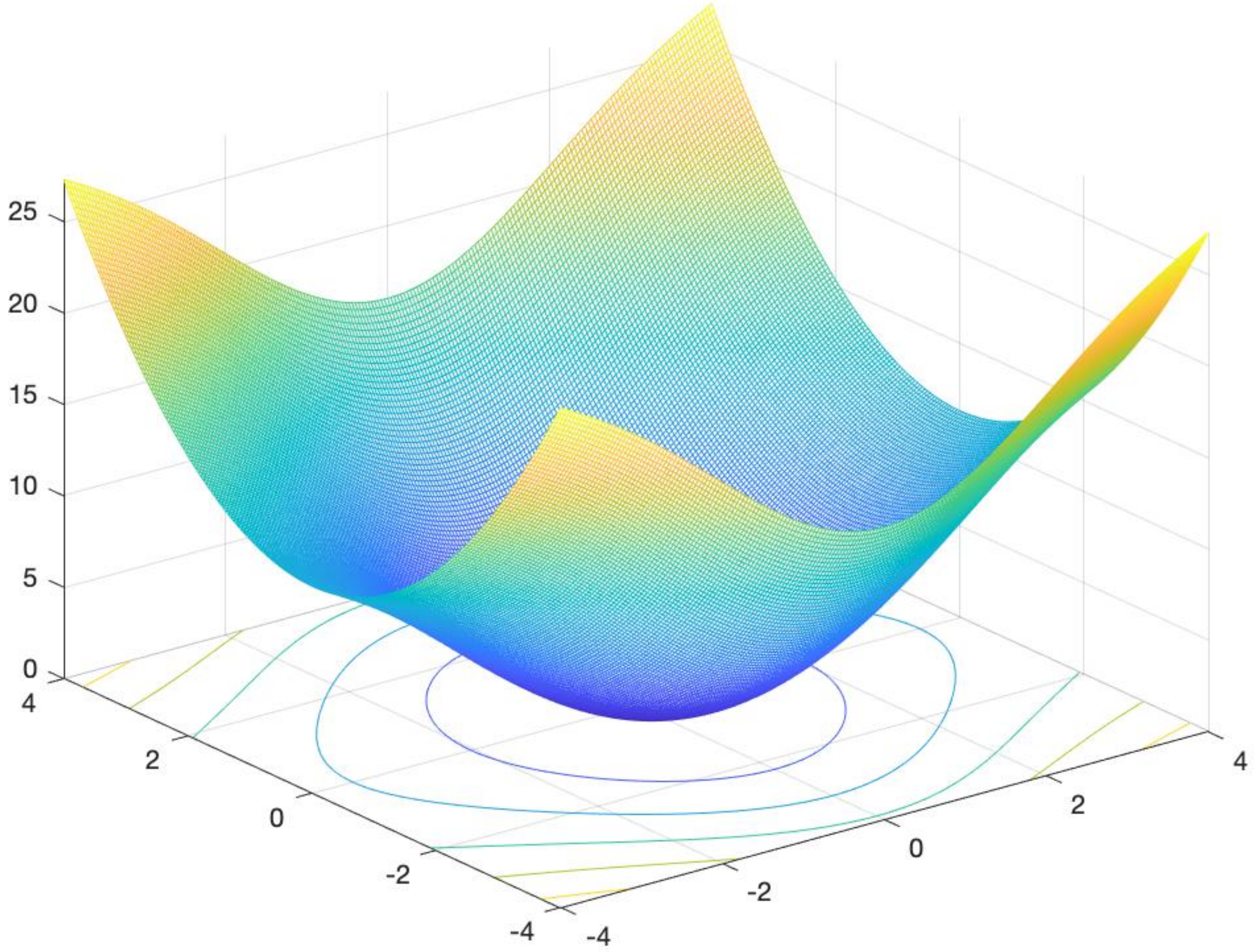
$$V(x,y) = 1 - \cos(x) + \frac{1}{2}y^2$$

```
x=[-4:.04:4];
y=x;
[X,Y]=meshgrid(x,y);
z=(1-cos(X))+(Y.^2)/2;
meshc(X,Y,z)
```



$$V(x,y) = 2(1 - \cos(x)) + \frac{1}{2}y^2 + \frac{1}{2}(x^2 + y^2)$$

```
x=[-4:.04:4];
y=x;
[X,Y]=meshgrid(x,y);
z=2*(1-cos(X))+(X.^2 + Y.^2)/2 + (Y.^2)/2;
meshc(X,Y,z)
```



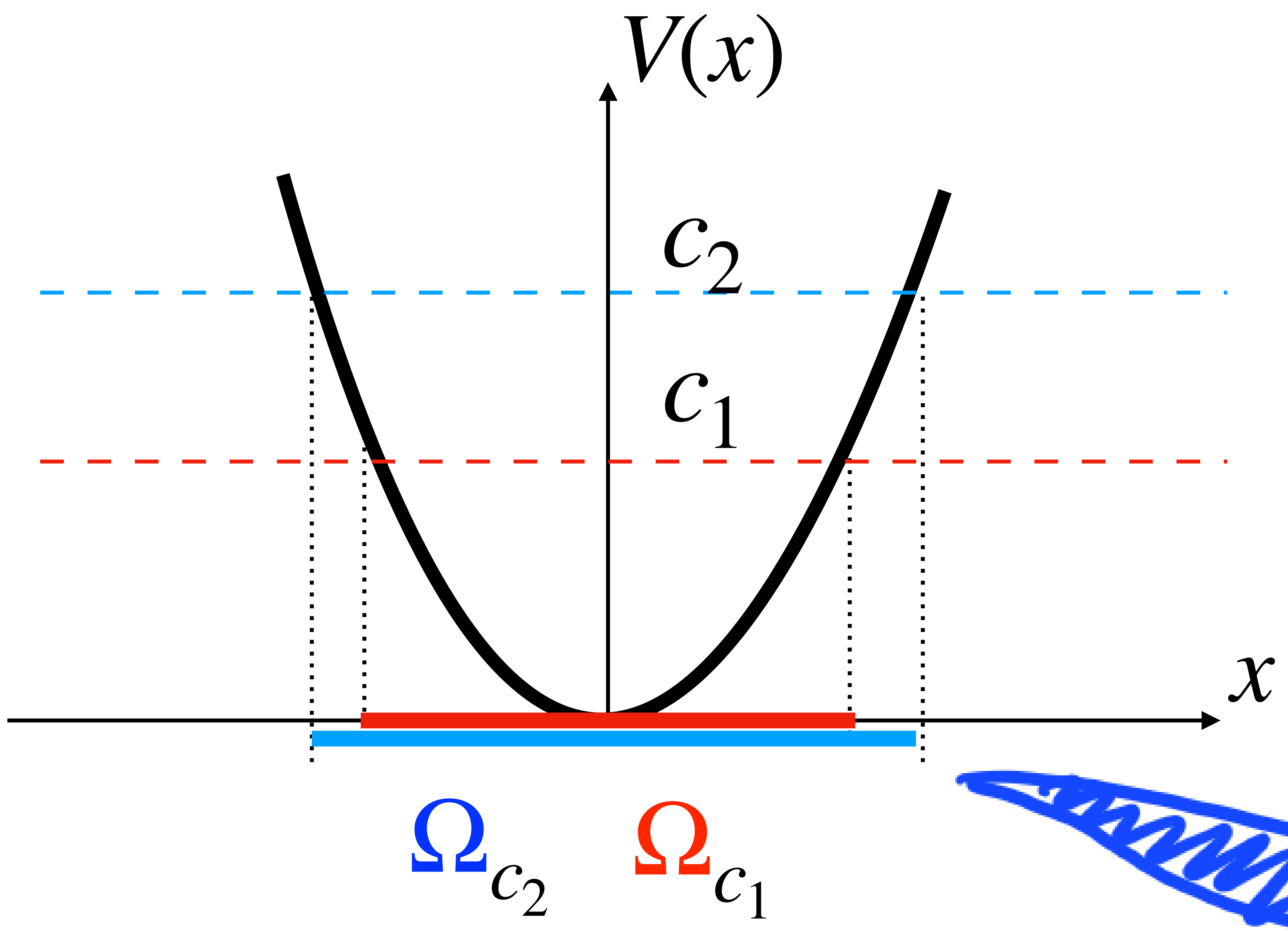
Lyapunov Direct Theorem

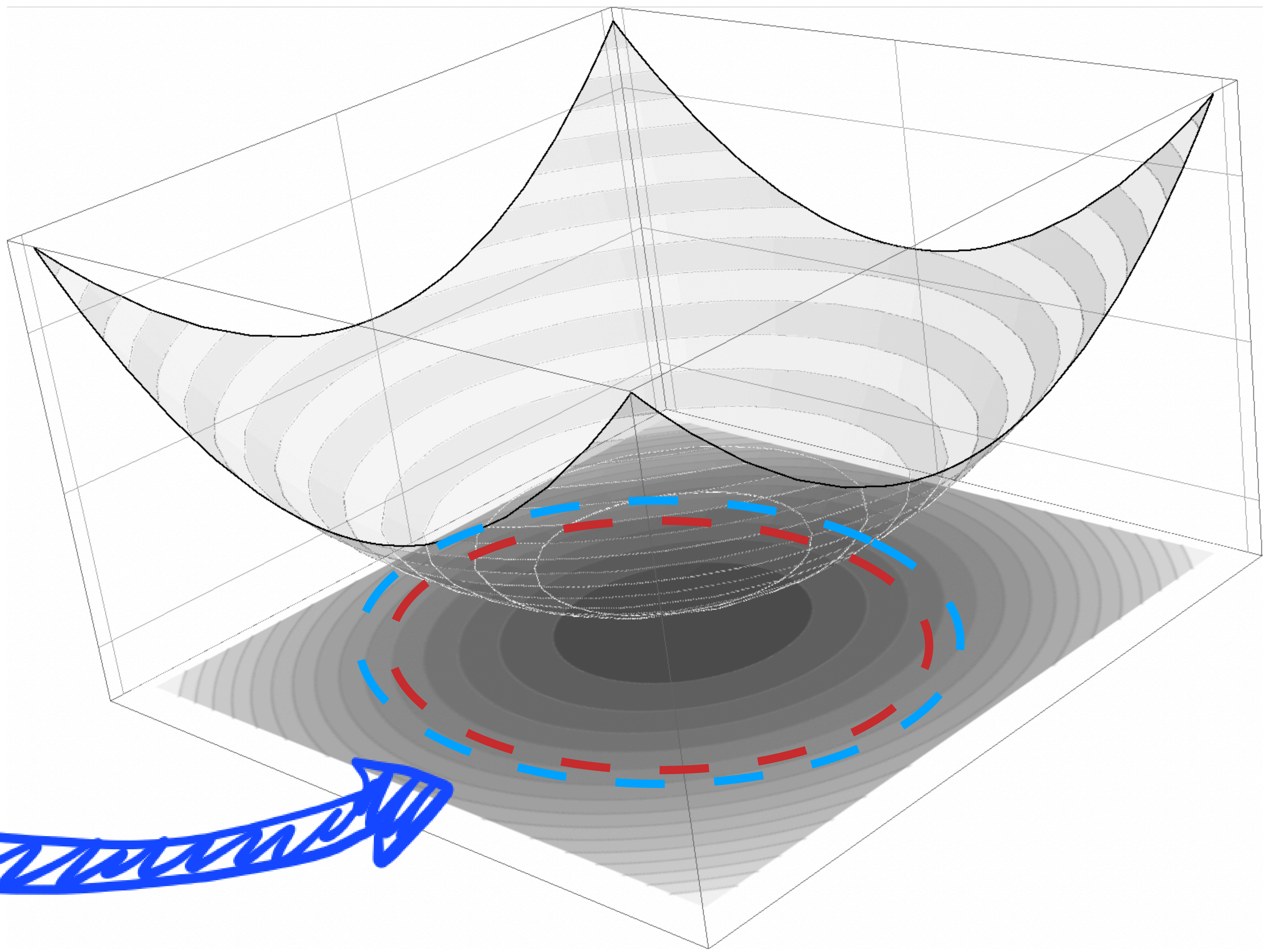
in $\mathbb{R}^n$

Particularly relevant in the analysis that will come is the notion of level set of $V(x)$

a set

$$\Omega_c := \{x \in \mathbb{R}^n \quad : \quad V(x) \leq c\}$$





$$c_1 < c_2 \quad \Rightarrow \quad \Omega_{c_1} \subset \Omega_{c_2}$$

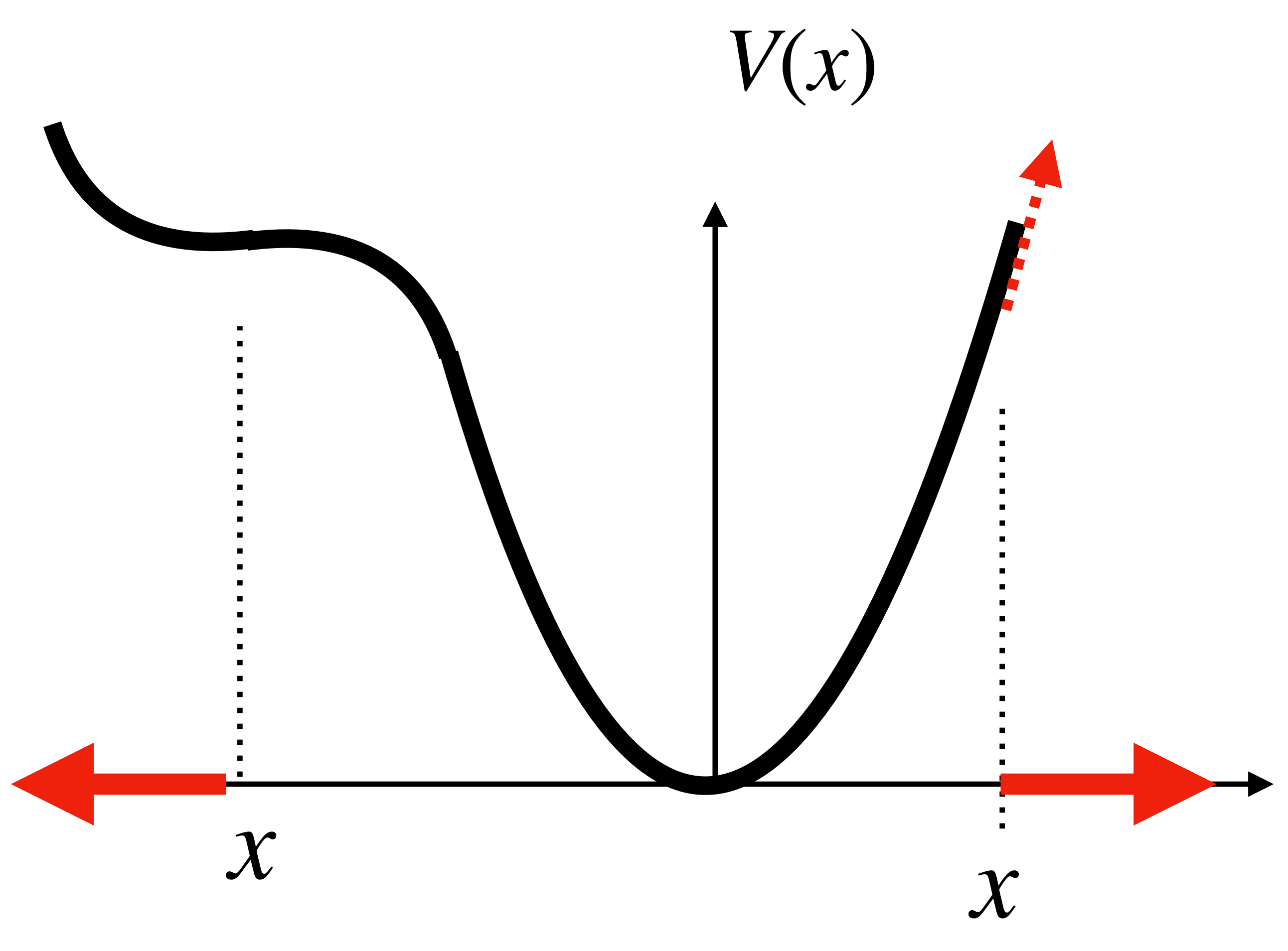
Lyapunov Direct Theorem

vector of partial derivative of function V

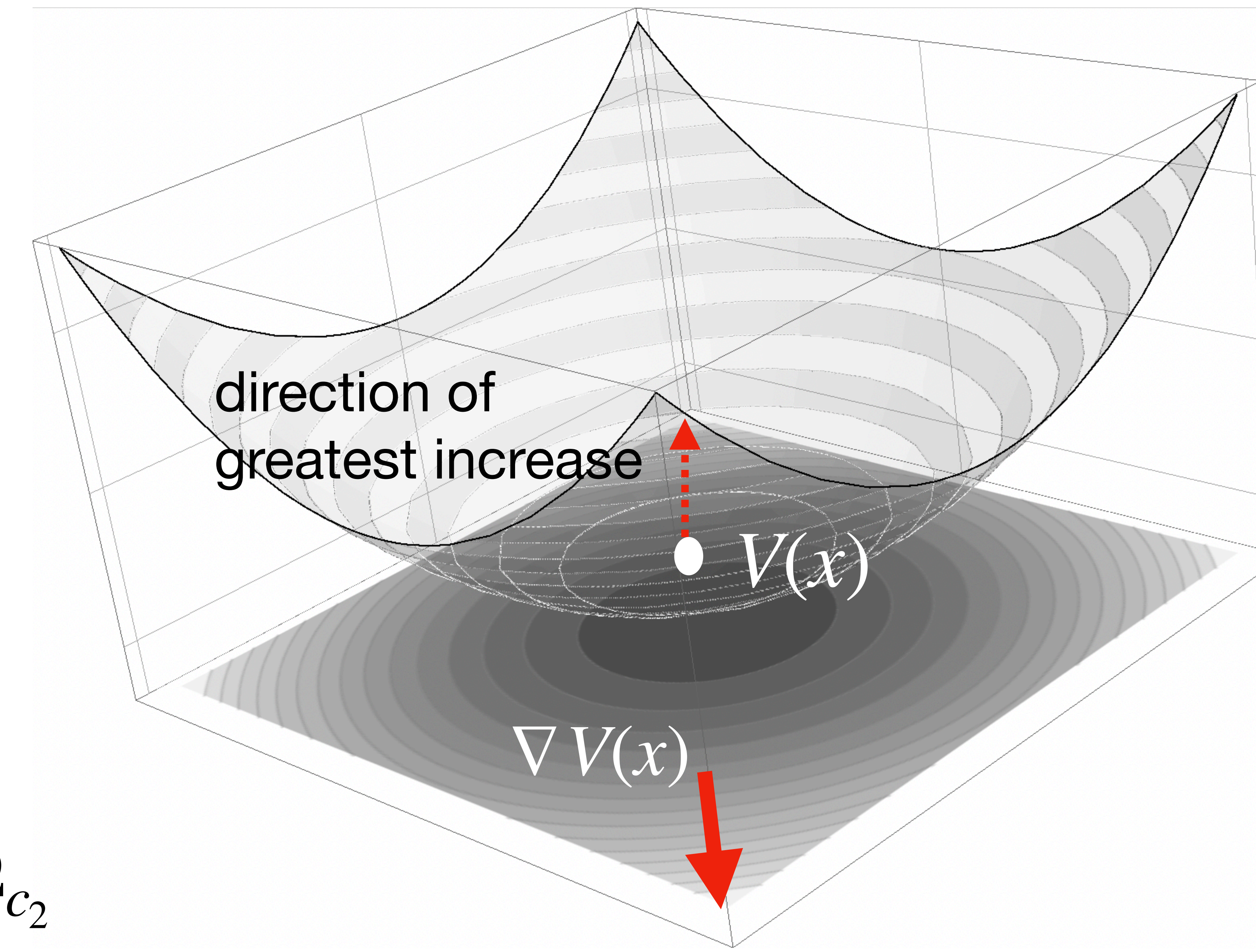
… and the notion of divergence of $V(x)$ at the point x:

$$\nabla V(x) := \frac{dV(x)}{dx} = \begin{pmatrix} \frac{\partial V(x)}{\partial x_1} & \frac{\partial V(x)}{\partial x_2} & \cdots & \frac{\partial V(x)}{\partial x_n} \end{pmatrix}$$

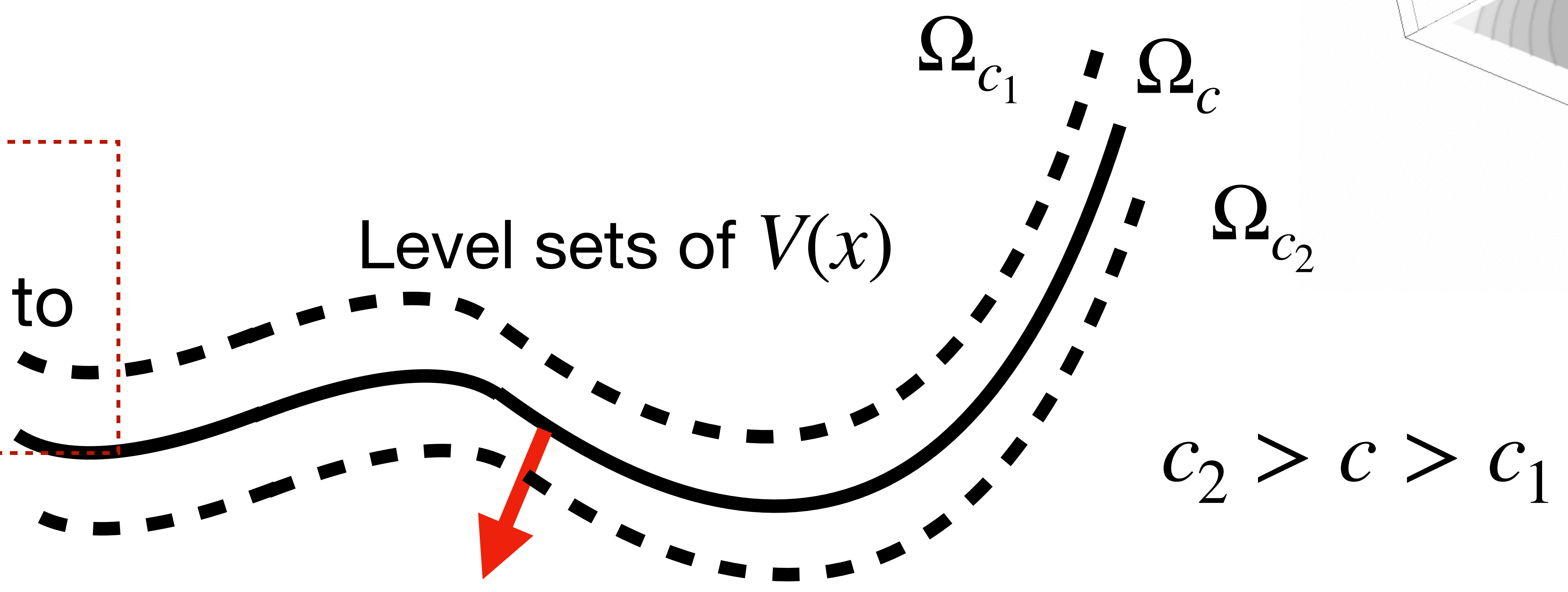
nabla of V   Jacobian of V



$\nabla V(x)$ always directed toward the greatest increase in $V(x)$



$\nabla V(x)$, for $x \in \Omega_c$, always points toward level sets linked to higher c



The special case of quadratic forms

$n \times n$ matrix!

$$V(x) = x^T P x$$ **Quadratic form**

$V : \mathbb{R}^n \longrightarrow \mathbb{R}$

$x \longrightarrow x^T P x$

→ not all P is positive-definite

Divergence of a quadratic form: $\nabla V(x) = 2x^T P$

P is necessarily equal to P^T or symetric

Definitions

- The matrix P is **positive semi-definite** ($P \geq 0$) if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$
- The matrix P is **positive definite** ($P > 0$) if $x^T P x > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$
- The matrix P is **negative semi-definite** ($P \geq 0$) if $x^T P x \leq 0$ for all $x \in \mathbb{R}^n$
- The matrix P is **negative definite** ($P < 0$) if $x^T P x > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$

Remark. In quadratic forms the matrix P can be taken symmetric ($P = P^T$) without loss of generality

Result. A matrix P is positive definite iff all its leading principal minors D_k are positive for all $k = 1, \ldots, n$

Definition. D_k is the determinant of the matrix obtained by eliminating the last $n - k$ rows and columns from P

not necessarily symmetric

Say $P \neq P^T$, and $V = x^T P x$

$P = P_S + P_A$

$P_S = \dfrac{P + P^T}{2} = P_S^T$

$P_A = \dfrac{P - P^T}{2} = -P_A^T$

Let's embed P in quadratic form

$V = x^T P x = x^T (P_S + P_A) x = \underbrace{x^T P_S x}_{\text{scalar}} + \boxed{\underbrace{x^T P_A x}_{\text{scalar}}}$

it's always $= 0$

pick the transpose

$(wMv)^T = v^T M^T w^T$

$\underbrace{x^T P_A^T x}_{\text{scalar}} = \underbrace{-x^T P_A x}_{\text{scalar}}$

the only scalar that equals to its neg. value is $\underline{\underline{0}}$

So, $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is positive definite

$D_1 = \det 1 = 1$

$D_2 = \det P = 1$

$\dfrac{d(x^T P x)}{dx} = (2x^T P)$

The special case of quadratic forms

What's the geometry of $\Omega_c = \{x \in \mathbb{R}^n \ : \ x^T P x \le c\}$?

not necessarily positive definite

Properties of any symmetric positive definite matrix $\overbrace{P = P^T}^{\text{symmetric matrix}} > 0$

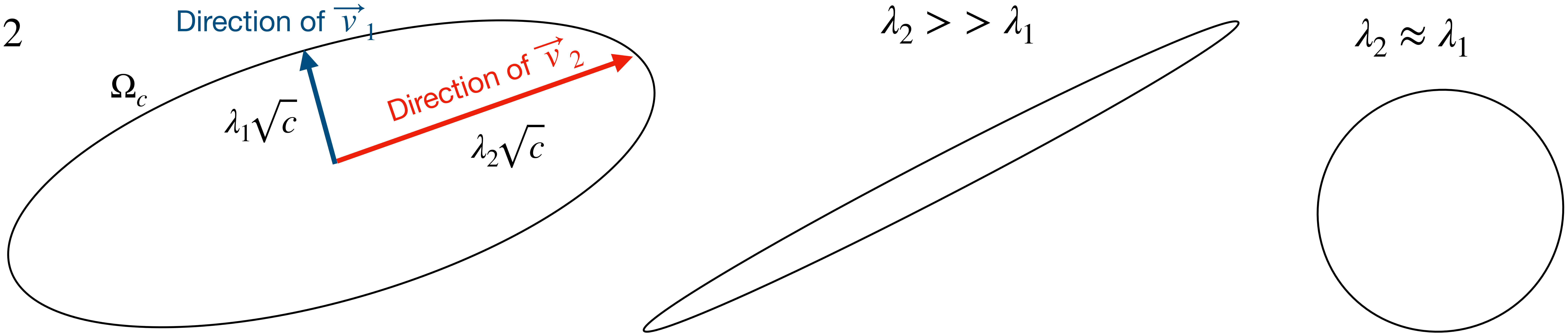
- $\boxed{P \text{ is diagonalisable}}$ $(a_i = g_i)$
- The eigenvalues λ_i and eigenvectors $\vec{v}_i$ are real, $i = 1,\ldots,n$
- $\lambda_i > 0$, $i = 1,\ldots,n$, and for each pair $(\vec{v}_i, \vec{v}_j)$ of eigenvectors $\boxed{< \vec{v}_i, \vec{v}_j > = 0}$ (orthogonal)

cross-product is orthogonal

eigval is positive

Ω_c are ellipsoids with principal axes direct as the eigenvalues of P and amplitude proportional to the relative eigenvalues

Example $n = 2$



Lyapunov Direct Theorem (C-T systems)

eq. point (maybe at the origin)

Direct Lyapunov Theorem. Let $\bar{x} \in \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a positive definite function with respect to $\bar{x} \in \mathscr{D}$ and consider the real-valued function $V'(x) : \mathscr{D} \to \mathbb{R}$ defined as $V'(x) := \nabla V(x) f(x)$. The following holds:

semi-negative definite

(a) if $V'(x) \leq 0$ for all $x \in \mathscr{D}$, then $\bar{x}$ is a stable equilibrium point of the system;

(b) if $V'(x) < 0$ for all $x \in \mathscr{D} \backslash \{0\}$, then $\bar{x}$ is an asymptotically stable equilibrium point of the system with a certain domain of attraction $\mathscr{A}$

negative definite

divergence

$V'(x) = \nabla V(x)\, f(x)$ scalar product between the divergence and the vector field

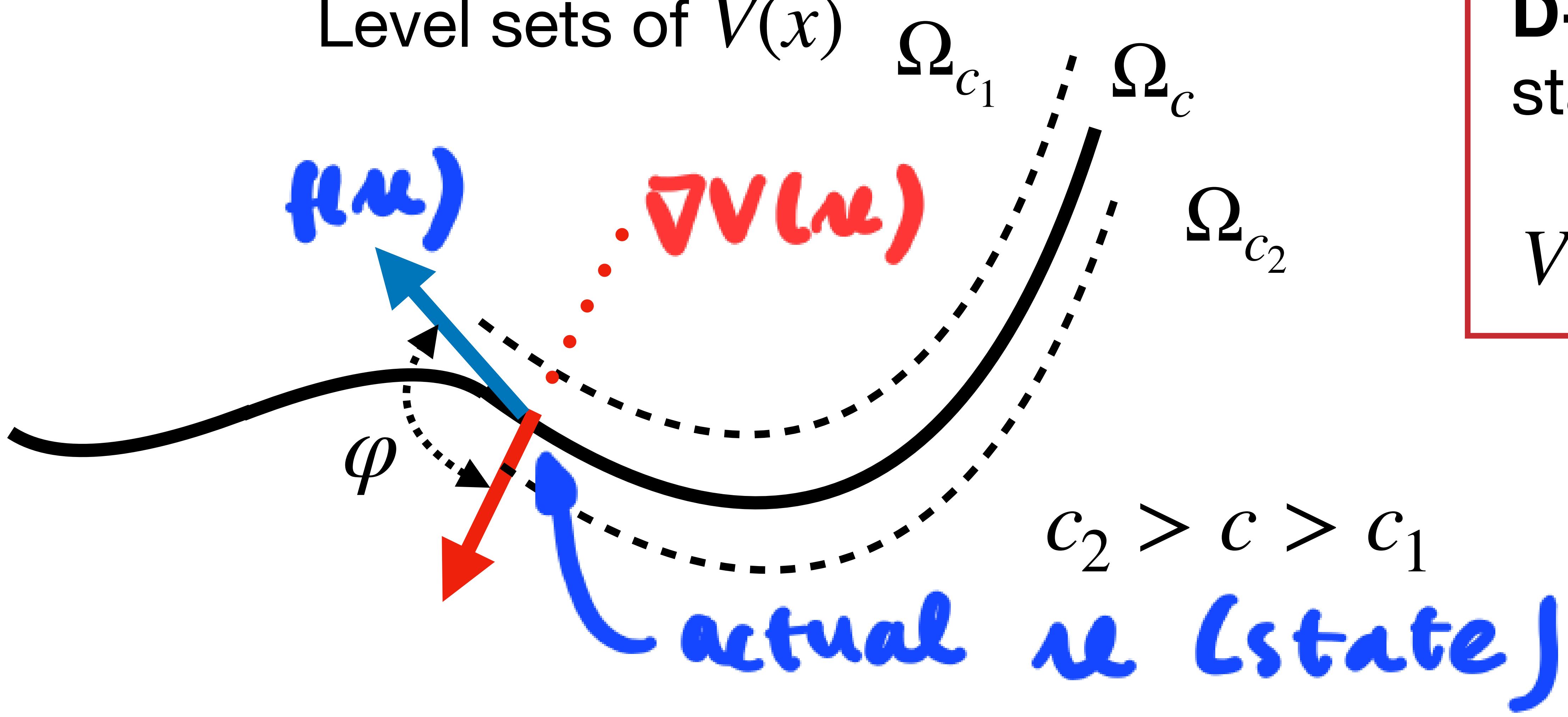
angle

$$\nabla V(x)\, f(x) < 0 \iff |\varphi| > \frac{\pi}{2}$$

def. negative

$$\nabla V(x)\, f(x) = 0 \iff \varphi = \pm \frac{\pi}{2}$$

Level sets of $V(x)$

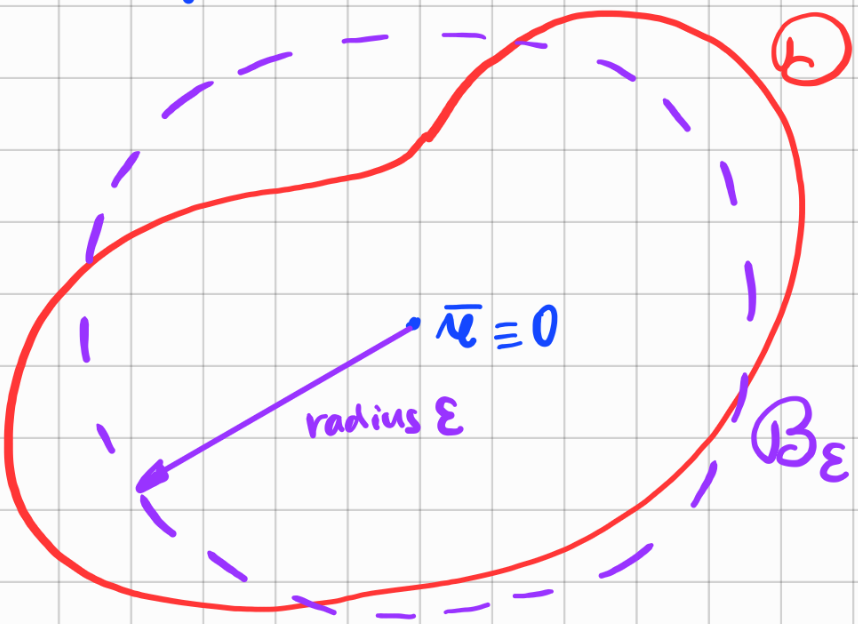


D-T systems: Same statement but with

$$V'(x) := V(f(x)) - V(x)$$

The condition is equivalent to ask that the vector field points inward (tangent) the level set

Say $\bar{x} \equiv$ Origin, $n = 2$



$\mathcal{D}$ domain of Lyapunov ("stable region")

Lyapunov claims:

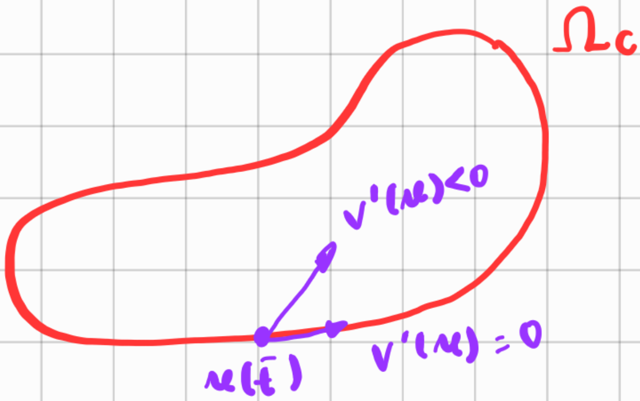
it implies that for $\forall x \in \mathcal{D}$ is "forward invariant" for $\dot{x} = f(x)$

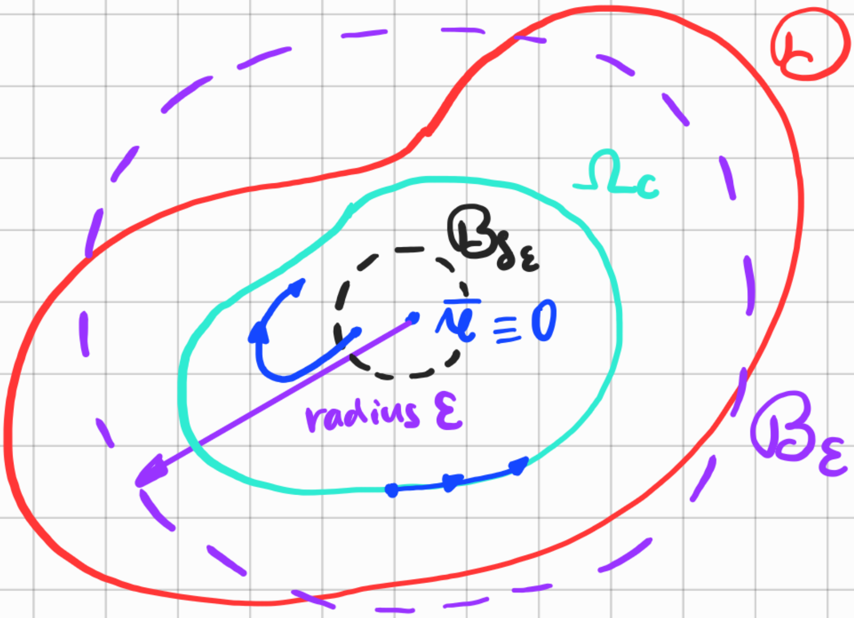
a.) $V'(x) = \nabla V(x) f(x) \leq 0, \quad \forall x \in \mathcal{D}$

$\Rightarrow \forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 : \forall \|x(0)\| \leq \delta_\varepsilon$

$\Rightarrow \|x(t)\| \leq \varepsilon \quad \forall t \geq 0.$

It implies that $V'(x) \leq 0 \Rightarrow \forall \ c > 0, \ x(\bar{t}) \in \Omega_c,$
$x(t) \in \Omega_c \quad \forall t \geq \bar{t}$



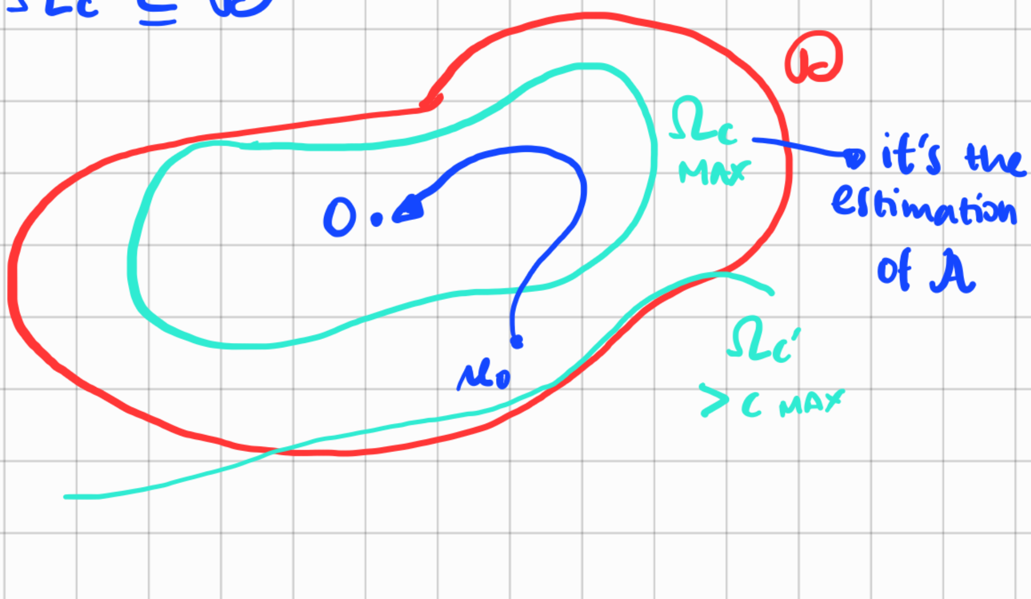


$\boxed{\text{STEP 1}}$ Pick $c > 0$: $\Omega_c \subseteq D$; $\Omega_c \subseteq B_\varepsilon$

$\boxed{\text{STEP 2}}$ $\delta_\varepsilon > 0$: $B_{\delta_\varepsilon} \subseteq \Omega_c$

$\dot{V}(x) < 0 \quad \forall x \in D \setminus \{0\}$.

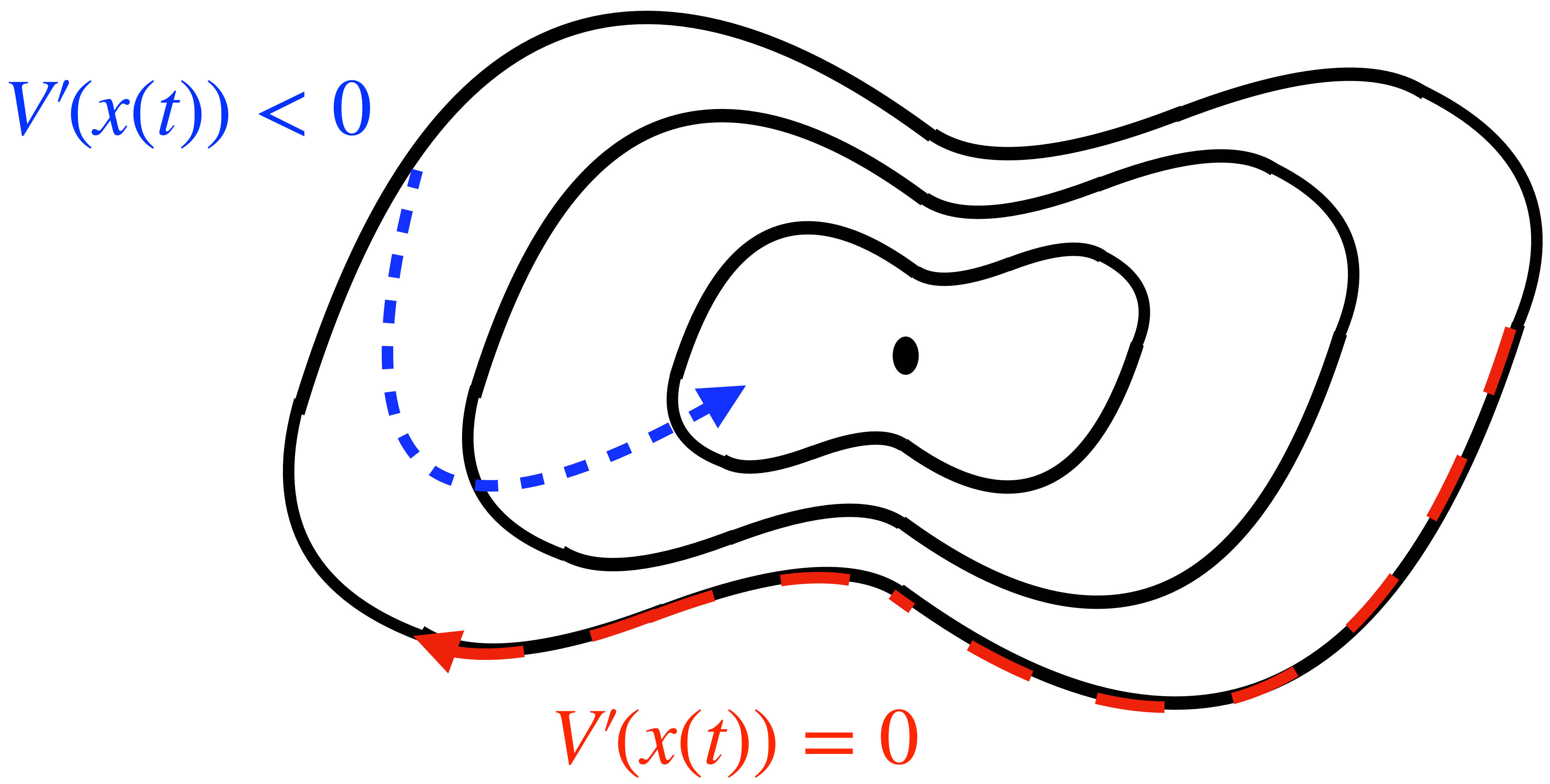
Guess the set of attraction A? — It is the largest $\Omega_c \subseteq D$

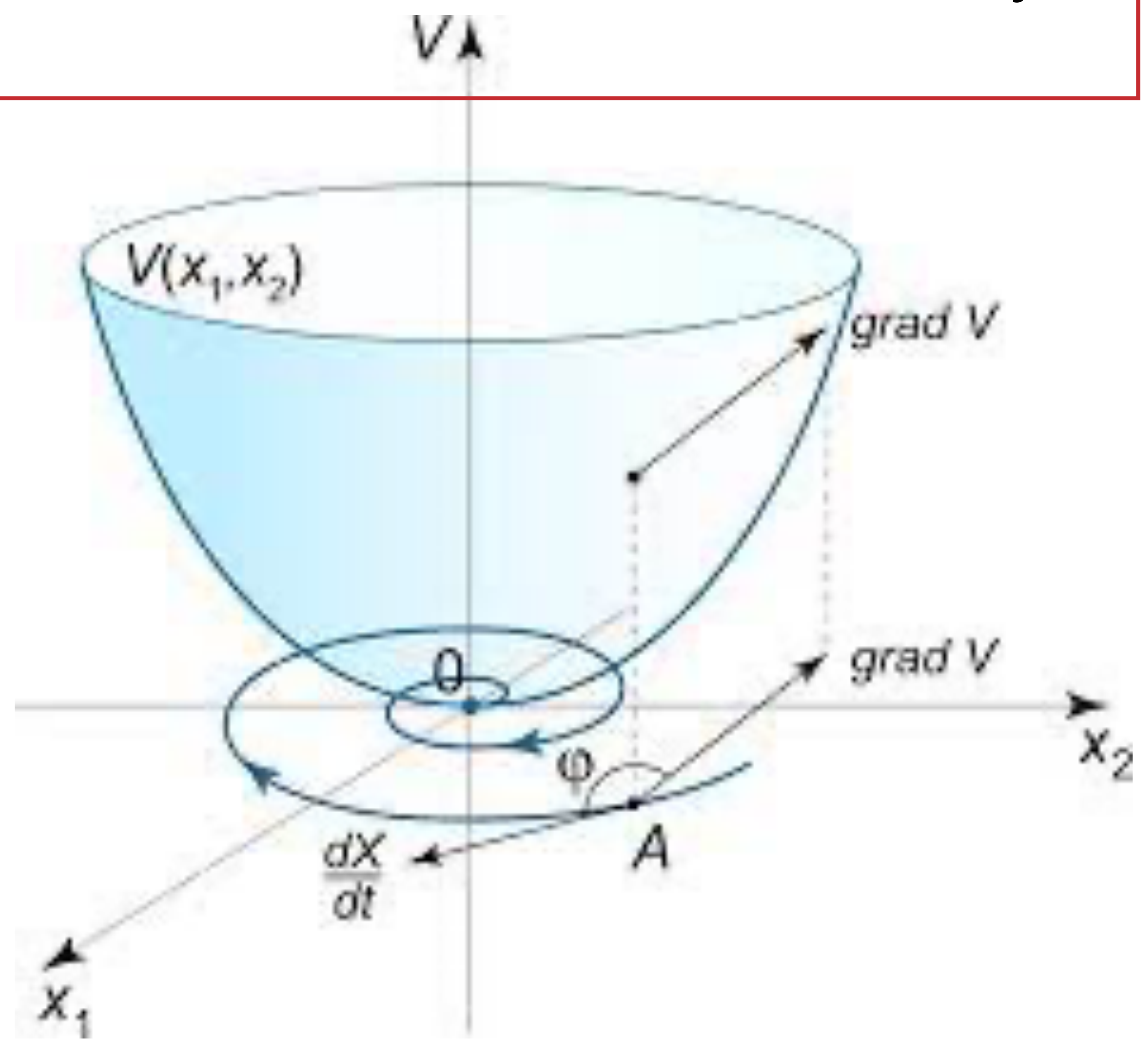


Lyapunov Direct Theorem

$$\frac{dV(x(t))}{dx} = V'(x(t)) = \nabla V(x(t))\, f(x(t))$$ derivative of $V(x(t))$ along the solution of $\dot{x}(t) = f(x(t))$

If the condition $V'(x(t)) \leq 0$ is fulfilled then the level sets are “**invariant**” (*namely trajectories starting in a level set do not escape it*). If $V'(x'(t)) = 0$ trajectories run over the boundary of the the level set. Otherwise, they point toward inner level sets.

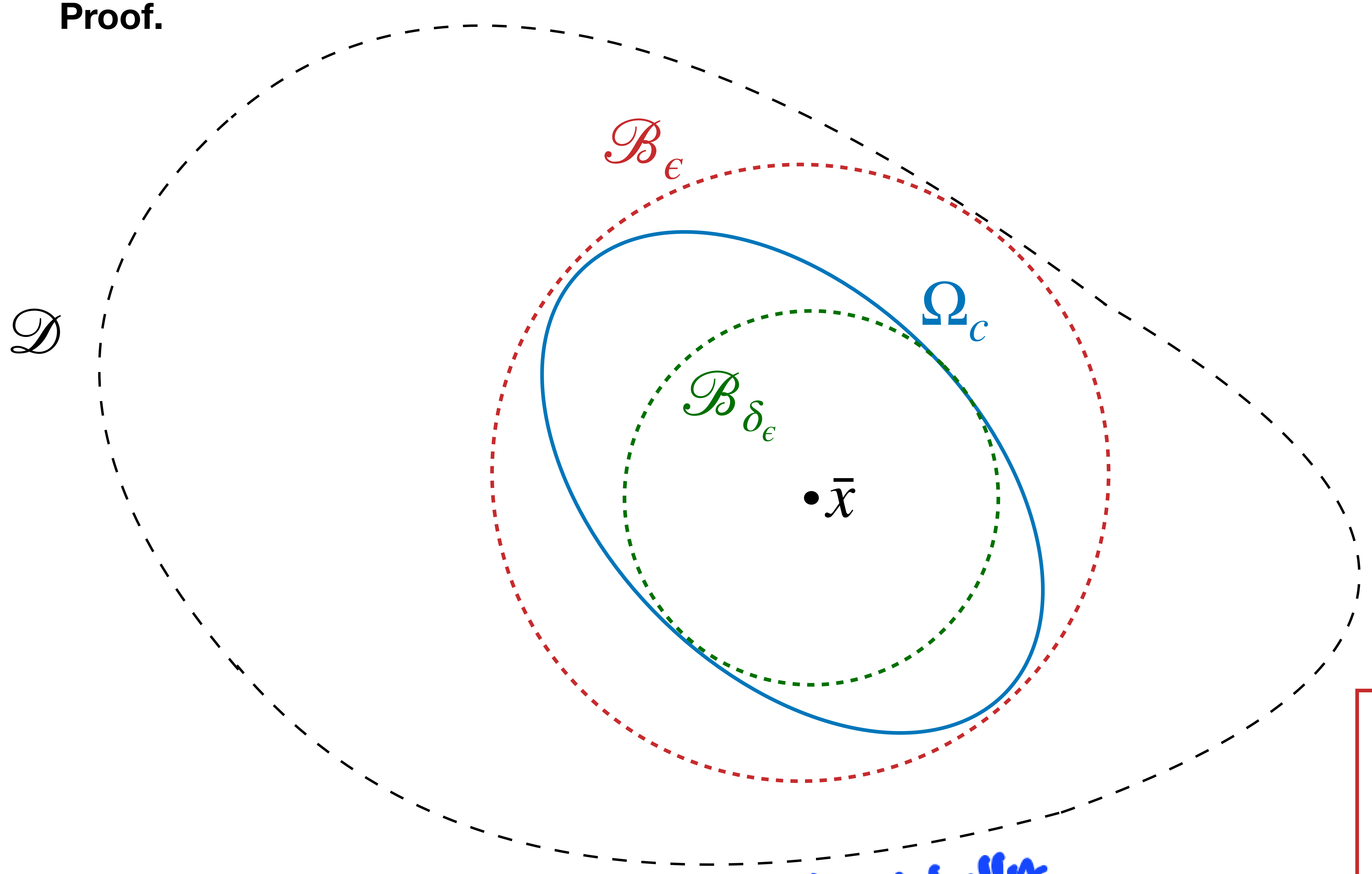




Same considerations if the system is D-T. The condition $V'(x) := V(f(x)) - V(x) \leq 0$ means that the trajectory jumps into inner level sets (or remain on the actual one if $V'(x) = 0$).

Lyapunov Direct Theorem

Proof.



Remark. In case of asymptotic stability the largest level set of V included in $\mathscr{D}$ is a (conservative) estimation of the domain of attraction $\mathscr{A}$

Terminology

LAS (Local Asymptotic Stability) if $\mathscr{A} \subset \mathbb{R}^n$

GAS (Global Asymptotic Stability) if $\mathscr{A} = \mathbb{R}^n$

is globally as. stable if the attraction is the whole nat. number

Converse Lyapunov Theorem

The direct Lyapunov theorem presents a sufficient condition for the (asymptotic) stability of an equilibrium point. If a candidate Lyapunov function V (i.e. a smooth function that is positive definite wrt the equilibrium) fails to fulfil $V' \leq 0$ the equilibrium point is not necessarily unstable (simply the V is not a Lyapunov function). However, if an equilibrium point is stable (asymptotically stable) a Lyapunov function fulfilling the above conditions necessarily exists.

Converse Lyapunov Theorem. Let $\bar{x} \in \mathbb{R}^n$ be a stable (asymptotically stable) equilibrium point of $\dot{x} = f(x)$ (with domain of attraction $\mathscr{A}$). Then there exists a differentiable $V : \mathscr{D} \to \mathbb{R}$ that is positive definite wrt $\bar{x}$ (with $\mathscr{D} \supset \mathscr{A}$) and fulfilling $V'(x) \leq 0$ ($V'(x) < 0$) for all $x \in \mathscr{D}$

Unfortunately the proof of the theorem is not constructive…

Searching for a Lyapunov function is not easy. Energy-related considerations are good starting points in general. When you fail to prove that the derivative of a certain V is negative could be either that the equilibrium under study is unstable or simply that the chosen V is wrong

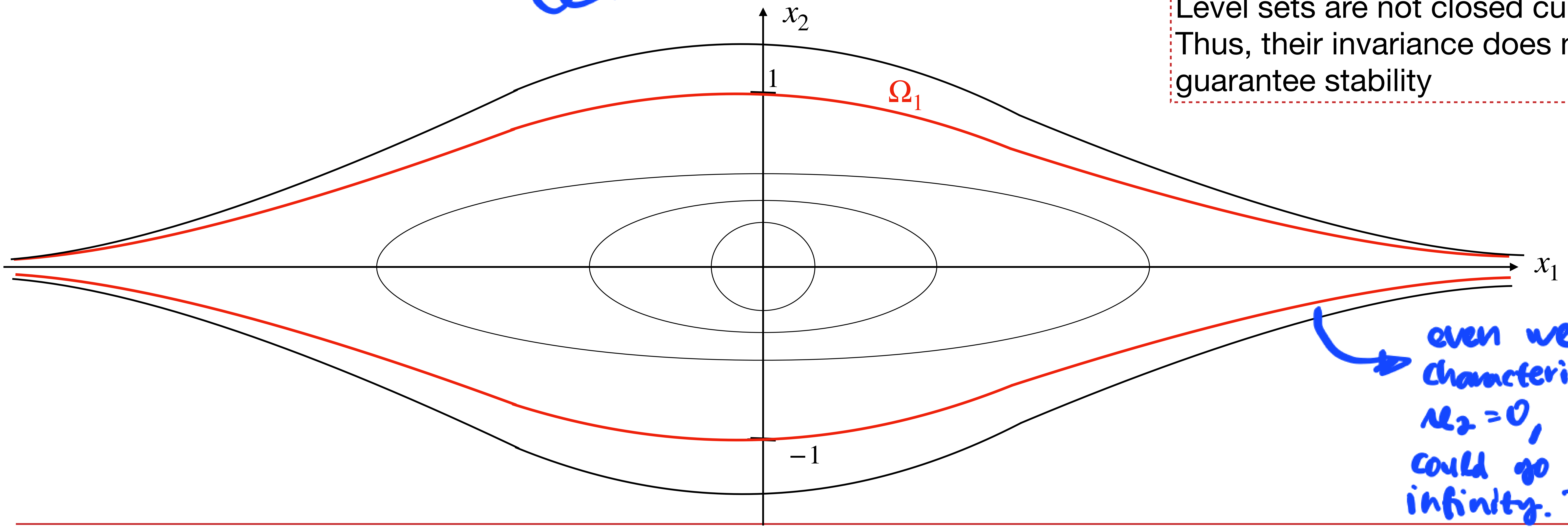
Global Asymptotic Stability

If $\mathscr{D} = \mathbb{R}^n$ and $V'(x) < 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$ $\implies$ $x = 0$ GAS ? **NO !!!**

Example $(n = 2)$ $V = \dfrac{x_1^2}{1 + x_1^2} + x_2^2$ and $f(x)$ any function so that $V'(x) < 0$ for all $x \in \mathbb{R}^2 \backslash \{0\}$

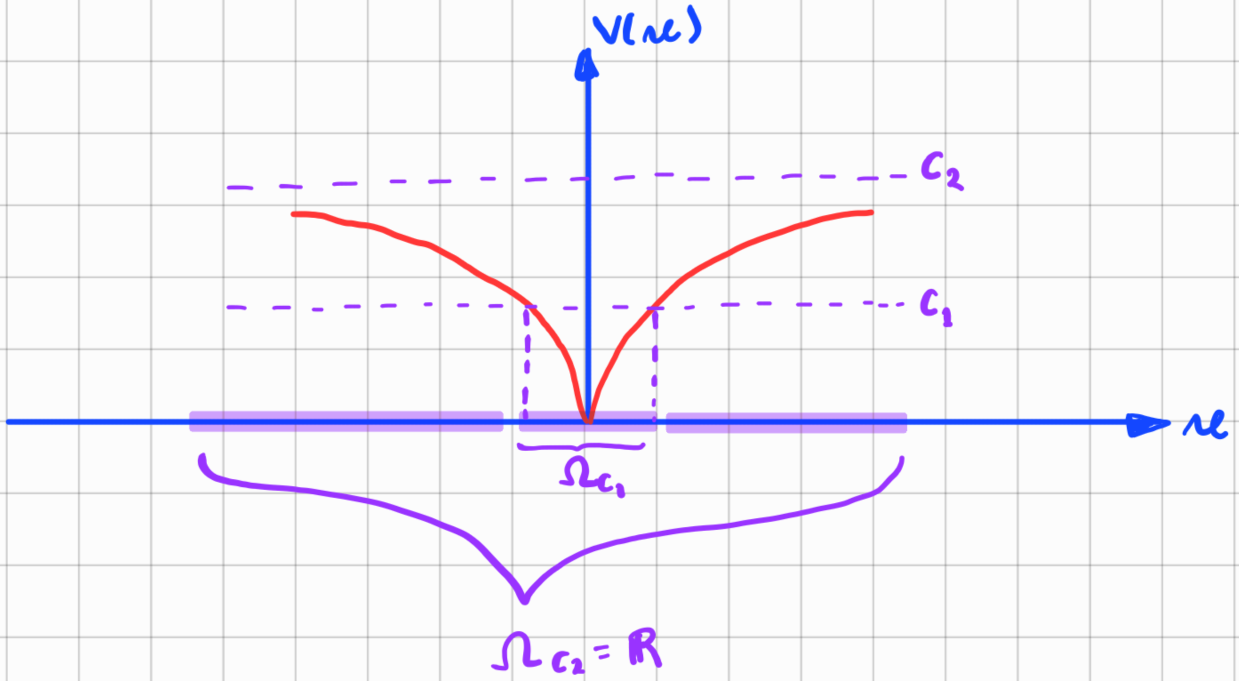
always < 1

Level sets are not closed curves. Thus, their invariance does not guarantee stability



even we characterize $x_2 = 0$, x_1 could go to infinity. It's open set

Suppose we take, $n = 1 \longrightarrow V(x) = \arctan(x)$



defined everywhere

Direct Lyapunov Theorem (GAS). Let $\bar{x} \in \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite function with respect to $\bar{x}$ (eq. point) which is **radially unbounded** ($\|x\| \to \infty \implies V(x) \to \infty$) and consider the real-valued function $V'(x) : \mathbb{R}^n \to \mathbb{R}$ defined as $V'(x) := \nabla V(x) f(x)$. Then if $V'(x) < 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$, then $\bar{x}$ is **GAS**

if we push the x to infinity, V(x) also goes to infinity

negative definite

The special case of linear systems $\longrightarrow$ using the Lyapunov Stability

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \qquad x(0) = x_0$$

$$V(x) = x^T \underbrace{P}_{n \times n} x$$

Quadratic Lyapunov Function

Theorem

- The system is **Hurwitz** (**Schur**) iff there exist $P = P^T > 0$ and $Q = Q^T > 0$ solution of the **Lyapunov matrix equation**

$$PA + A^T P = Q \qquad\qquad A^T PA - P = Q$$

- If there exists a solution (P, Q) then there exists an infinite number of other solutions one for each $Q = Q^T > 0$. That is, Q is arbitrary

- If the Lyapunov matrix equations are fulfilled with $Q = Q^T \geq 0$ then the system is stable

Proof…

Hurwitz system:

$\dot{x} = f(x) = Ax$

$V(x) = \underbrace{x^T P x}_{\text{quadratic form}}$

$P = P^T > 0$?

$V'(x) = \underbrace{(2x^T P)}_{\nabla V(x)} \underbrace{A x}_{f(x)}$

$= x^T (2PA) x = x^T (PA + A^T P) x$

not necessarily symetric

symetric part

$\dfrac{\cancel{2}PA + \cancel{2}A^T P^T}{\cancel{2}}$ $P^T = P$, because it's symetric

Suppose that there exists $\exists Q \geq 0$ ($n \times n$)

$x^T (2PA) x = x^T (PA + A^T P) x = -x^T Q x, \forall x$

$\underbrace{PA}_{>0} + \underbrace{A^T P}_{>0} = \underbrace{-Q}_{\geq 0} \overset{\text{this implies}}{\Longrightarrow} x = 0$ is stable

If $\exists Q > 0$, $-x^T Q x < 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$,

$Q > 0 \Longrightarrow x = 0$ is asimptotically stable

Schur System:

$x(t+1) = Ax(t)$

$V'(x) = V(\underbrace{f(x)}_{A(x)}) - V(x)$

$V(x) = x^T P x$

$x^T \underbrace{(A^T P A - P)}_{\text{symmetric}} x = V'(x) = \underbrace{(Ax)^T P (Ax)}_{V(f(x))} - x^T P x$

$= -Q$

If $\exists\, P = P^T > 0$, $Q = Q^T > 0$, $PA + A^T P = -Q$

proof? $\Uparrow$ $\Downarrow$ (by Lyapunov)

A is Hurwitz (all λ_i Real$(\lambda_i) < 0$)

pick $P = \int_0^\infty e^{A^T s} Q e^{As}\, ds$, Q is any

well-defined (because A Hurwitz)

Lyapunov Matrix Eq. (LME):

$$\int_0^\infty A^T e^{A^T s} Q e^{As} + e^{A^T s} Q e^{As}\, ds = \int_0^\infty \frac{d}{ds} e^{A^T s} Q e^{As}\, ds$$

$$⊛ = \left[e^{A^T s} Q e^{As} \right]_0^\infty = 0 \boxed{- Q}$$

<u>First claim</u>:

P solves the LME for every $\forall Q$

$$P^T = \int_0^\infty (e^{As})^T Q^T (e^{A^T s})^T ds = P, \text{ if } Q \text{ is symmetric}$$

<u>Second claim</u>:

If $Q = Q^T > 0$, then $P = P^T > 0$

$$x^T P x = \int_0^\infty x^T e^{A^T s} Q \underbrace{e^{As} x}_{= z(s)} ds$$

$$= \int_0^\infty \underbrace{z^T(s) Q z(s)}_{> 0 \ \forall s} ds > 0$$

$$\boxed{PA + A^T P = -I}$$

Indirect Lyapunov Theorem

Taylor expansion around $\bar{x}$

$$\dot{x} = f(x) = f(\bar{x}) + A(x-\bar{x}) + g(x-\bar{x}) \qquad A = f(x) = \frac{df(x)}{dx}\bigg|_{x=\bar{x}} = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1}\Big|_{x=\bar{x}} & \cdots & \frac{\partial f_1(x)}{\partial x_n}\Big|_{x=\bar{x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1}\Big|_{x=\bar{x}} & \cdots & \frac{\partial f_n(x)}{\partial x_n}\Big|_{x=\bar{x}} \end{pmatrix}$$

$= 0$ if $\bar{x}$ is an equilibrium point of $f(x)$

Higher order terms in $x - \bar{x}$

If x ranges in a small neighbourhood of $\bar{x}$ the higher order term are negilible and $\dot{x} = f(x) \approx A(x-\bar{x})$

Question: if A is Hurwitz can we conclude that $\bar{x}$ is LAS for $\dot{x} = f(x)$? Yes!

Similar reasonings if the system is D-T

Indirect Lyapunov Theorem

- Suppose that there exists a $P = P^T > 0$ and $Q = Q^T > 0$ solution of the Lyapunov matrix equation. Then $x = \bar{x}$ is LAS for $\dot{x} = f(x)$ with a certain domain of attraction ($V = (x-\bar{x})^T P(x-\bar{x})$ is a possible Lyapunov function);
- Suppose that A has at least one eigenvalue with positive real part. Then $\bar{x}$ is unstable for $\dot{x} = f(x)$.

Proof…

$$\dot{x} = f(x) = f(0) + \overset{A}{\boxed{\left.\frac{\partial f}{\partial x}\right|_{x=0} x}} + \text{hot}(x)$$

hot: higher order term

$$f(0) = 0$$

$$A = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_0 & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_0 \\ & & \\ \left.\frac{\partial f_n}{\partial x_1}\right|_0 & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_0 \end{bmatrix}$$

Suppose $\dot{x} = f(x)$

$\quad \approx \; x \approx 0$

$\dot{x} = Ax \implies x = 0$ is locally assym. stable for $\dot{x} = f(x)$

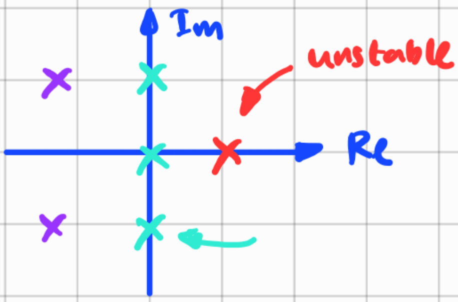
$\quad \longrightarrow$ is A Hurwitz?

A is Hurwitz $\implies \exists\, P = P^T > 0, \begin{cases} PA + A^T P = -Q \\ Q = Q^T > 0 \end{cases}$

$V(x) = x^T P x, \quad V'(x) < 0$

$\uparrow$

$\underbrace{\nabla V(x)}_{\frac{\partial V}{\partial x}} Ax \implies \bar{x} = 0$ is Lyapunov Asympt. Stable for $x = f(x)$



$\dot{x} = f(x) = \alpha x^3$

$\bar{x} = 0 \,, \quad A = \dfrac{\partial f}{\partial x} = 3\alpha x^2 \Big|_0 = 0$

Scenario a.) $\alpha < 0 \Rightarrow \bar{x} = 0$ is globally as. stable $V(x) = \frac{1}{2} x^2$

$V'(x) = x \dot{x} = \alpha x^4 < 0$

b.) $\alpha > 0 \Rightarrow \exists x_0 : x(t) \rightarrow \infty$, unstable

Let's compute $V'(x) = \nabla V(x) \underbrace{f(x)}_{Ax + g(x)}$

higher order term (HOT)

$$= \frac{\partial V}{\partial x}\big(Ax + g(x)\big) = \underbrace{2x^T P A x}_{-x^T Q x < 0 \;\; \forall x \in \mathbb{R}^n \setminus \{0\}} + \underbrace{2x^T P g(x)}_{?}$$

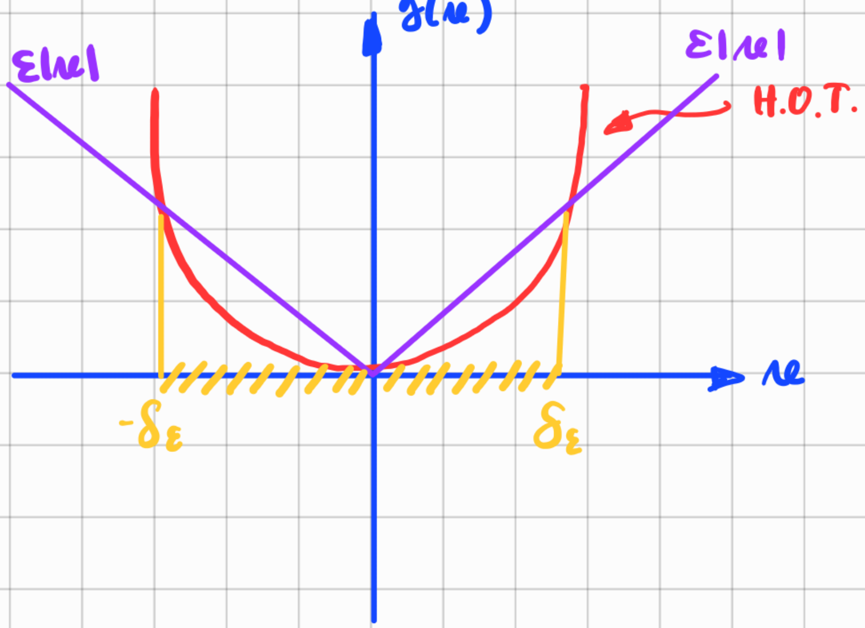
$\frac{\partial V}{\partial x} = 2X^T P$

what happens inside HOT?

? $g(x)$ H.O.T

$$\forall \varepsilon > 0, \quad \exists \delta_\varepsilon > 0, \quad \|g(x)\| \le \varepsilon \|x\| \quad \forall x : \|x\| \le \delta_\varepsilon$$

small number



$$V'(x) \le -x^T Q x + 2\|x\|\,\|P\|\,\|g(x)\|$$

Pick $\varepsilon, \delta_\varepsilon$. $\boxed{\text{If } \|x\| \le \delta_\varepsilon}$

$$V'(x) \le \boxed{-x^T Q x} + 2\|P\| \, \varepsilon \, \|x\|^2$$

$$\underbrace{\underline{\lambda} \|x\|^2}_{} \le x^T Q x \le \bar{\lambda} \|x\|^2$$

min. eigval of Q

max eigval of Q

It implies then, $-x^T Q x \le -\underline{\lambda} \|x\|^2$:

$$V'(x) \le -(\underline{\lambda} - 2\|P\| \varepsilon) \|x\|^2$$

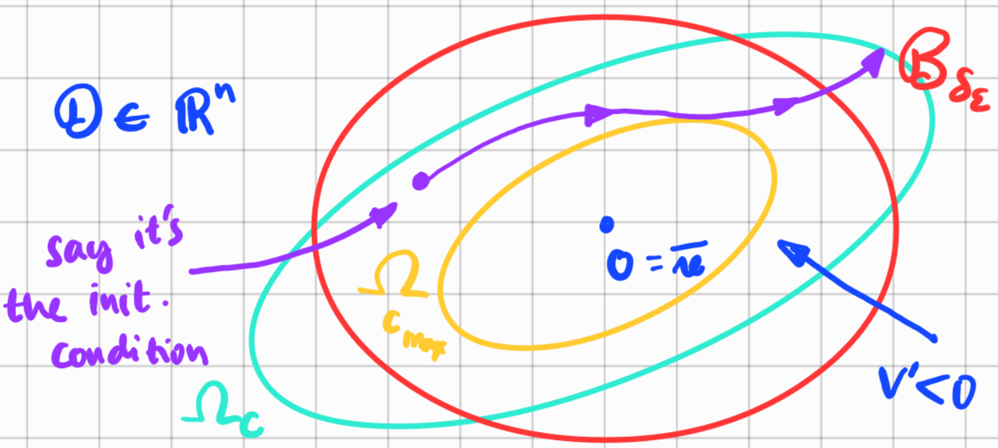
Pick $\varepsilon = \dfrac{\underline{\lambda}}{4\|P\|}$ and δ_ε accordingly:

$$\|x\| \le \delta_\varepsilon, \; V'(x) \le -\frac{\underline{\lambda}}{4} \|x\|^2$$

$\Downarrow$ by direct Lyapunov theory

$\bar{x} = 0$ is Lyapunov assymp. stable for $\dot{x} = f(x)$

What is the attraction domain A? $\Longrightarrow$ A is the largest level set of $x^T P x$ contained in $\mathcal{B}_{\delta_\varepsilon}$



Indirect Lyapunov Theorem

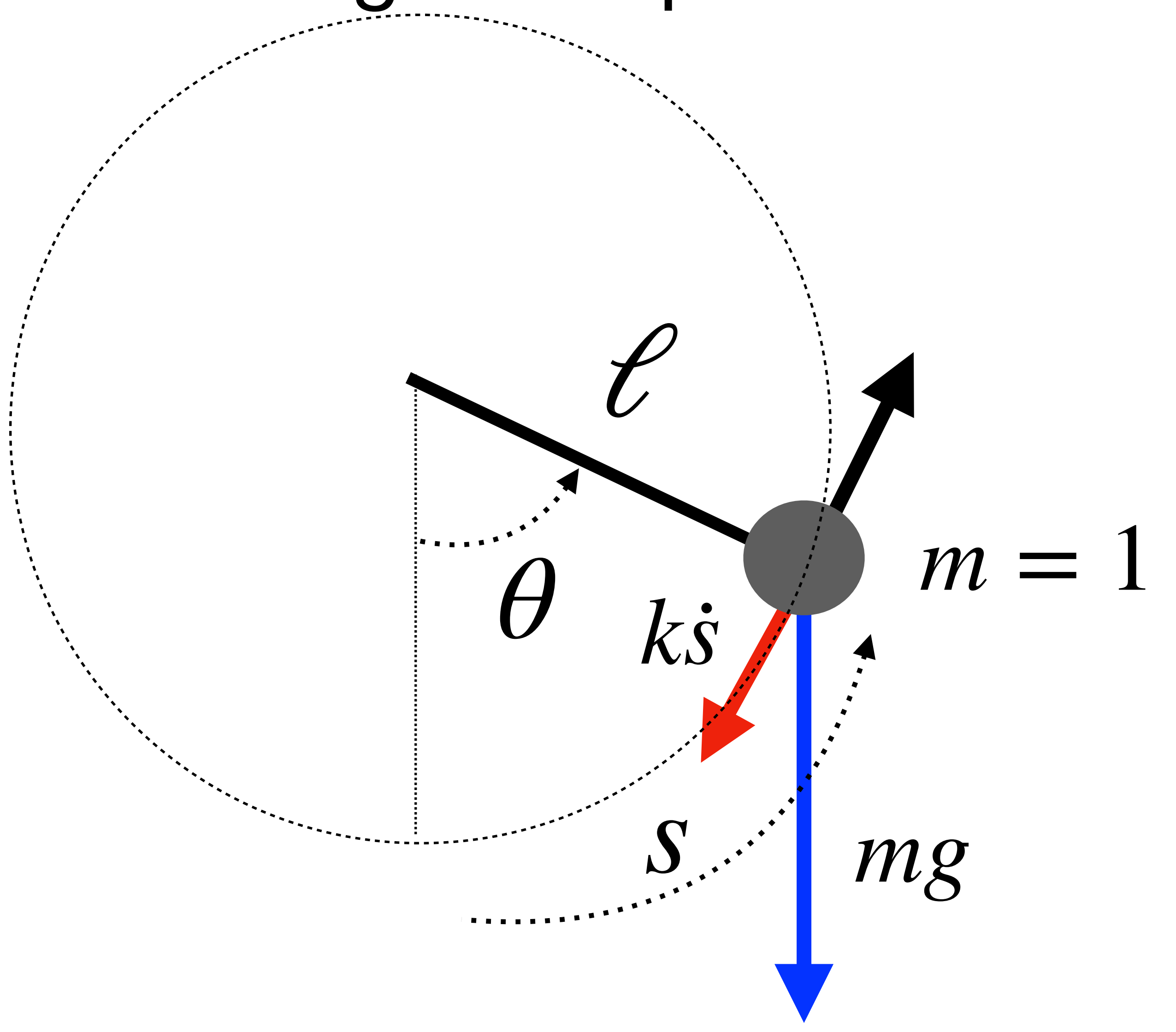
The theorem is not conclusive when A has eigenvalues on the imaginary axis. In this case the nonlinear system is said to be **non hyperbolic** at $\bar{x}$

In the non-hyperbolic case the equilibrium could be either stable, asymptotically stable or unstable for the non linear dynamics. The high-order terms play a role to determine the stability properties of the system

Example. $\dot{x} = \alpha\, x^3$ with α an arbitrary constant.

Krasovski-La Salle Criterion

Motivating example



$x_1 := \theta,\ x_2 := \dot{\theta}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - k\ell x_2$$

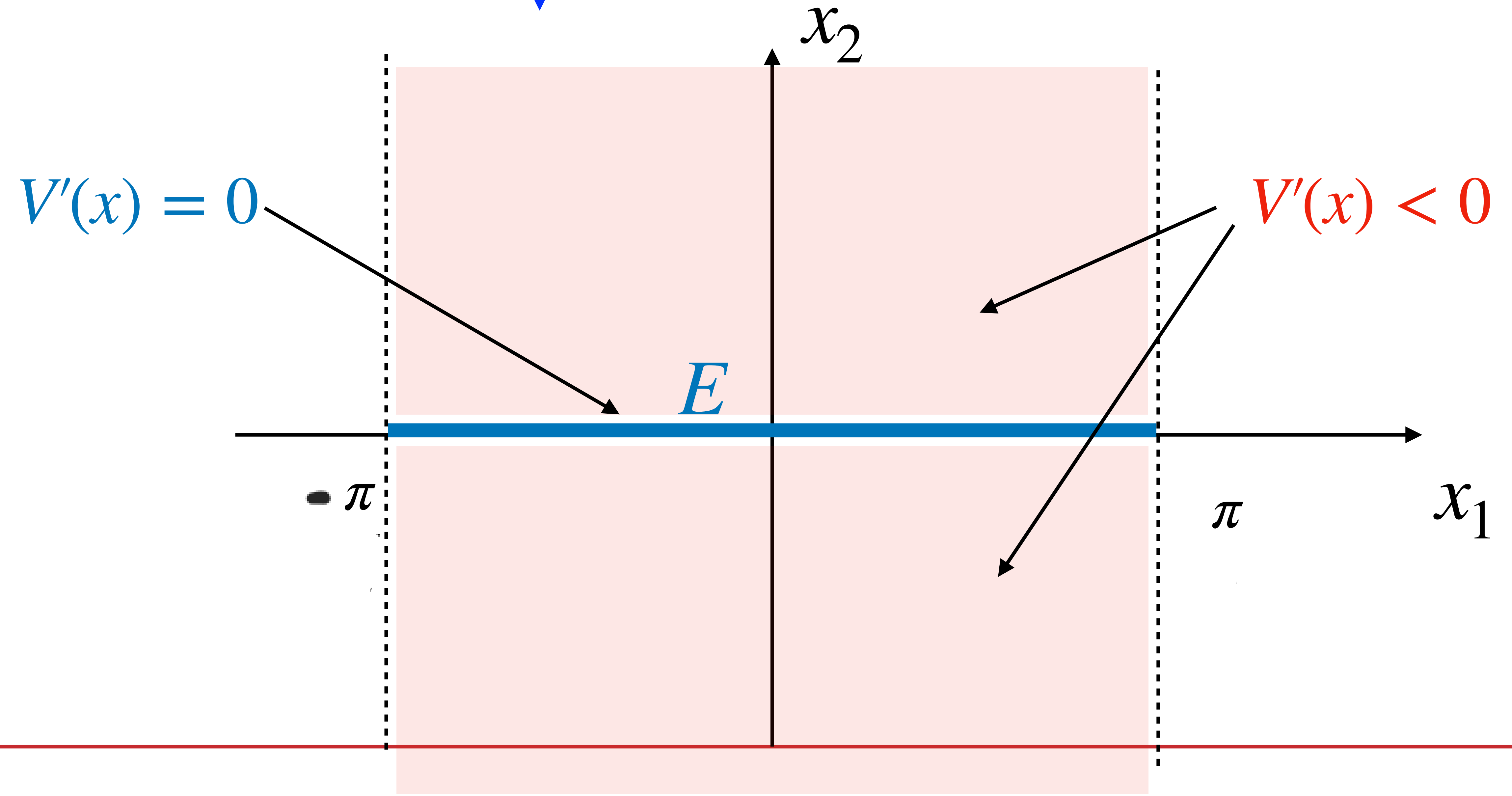
We know that the origin is LAS.
Let's try to prove it by Lyapunov

Candidate Lyapunov Function: Energy

$$V(x) = \frac{1}{2}x_2^2 + \frac{g}{\ell}(1 - \cos x_1)$$

Kinetic Energy   Potential Energy

Positive definite with respect to $x_1 = x_2 = 0$ with $\mathscr{D} = \{(x_1, x_2)\ :\ |x_1| < \pi/2\}$



The energy is always decreasing except when $x \in E := \{(x_1, x_2 \in \mathscr{D}\ :\ x_2 = 0\}$. Thus $V'(x)$ is only semidefinite negative and we cannot conclude asymptotic stability (just stability).

The set E, however, is only a "transition" set….

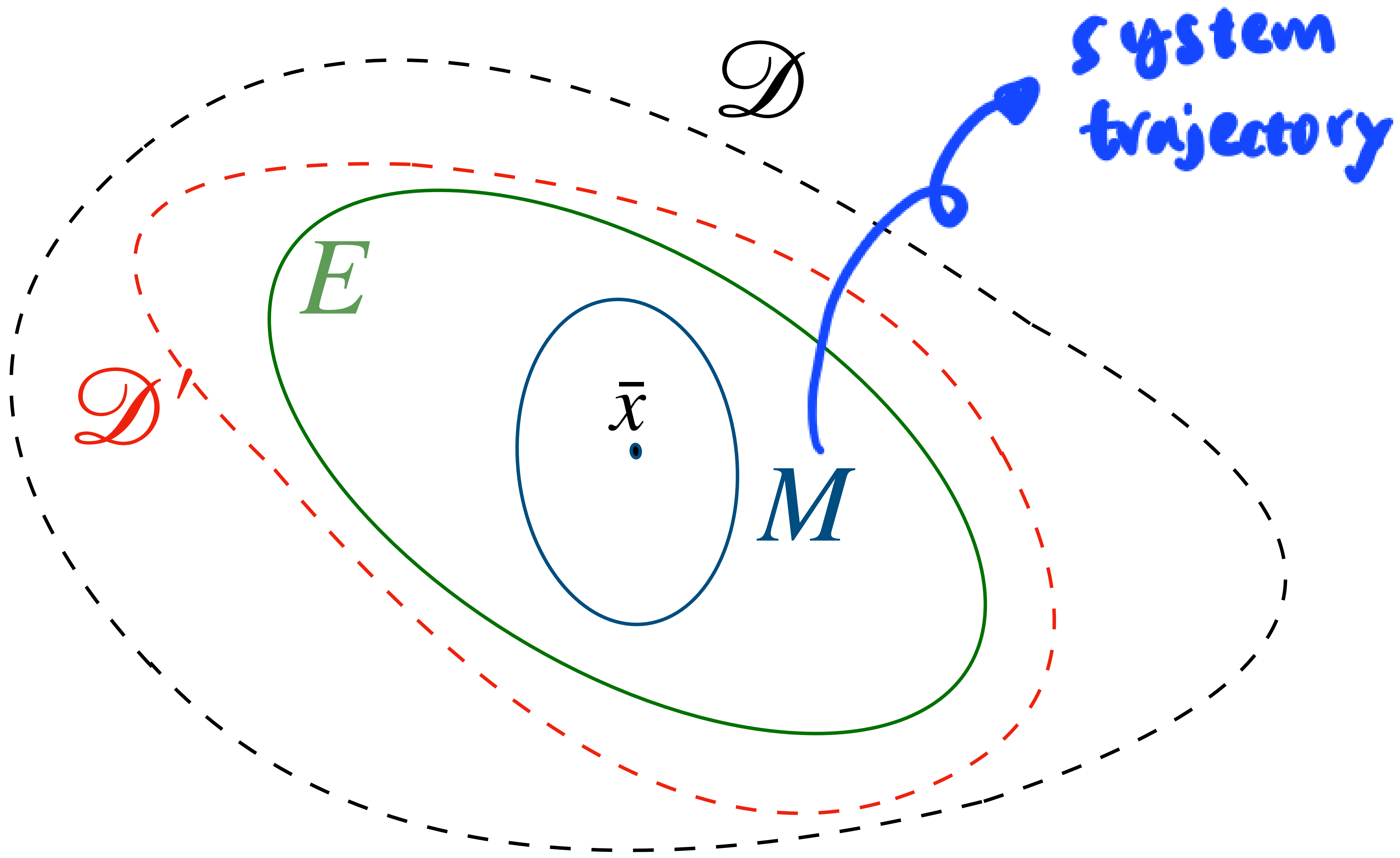
$$V' = \frac{\partial V}{\partial x} f(x) = -k\ell x_2^2 \leq 0$$

$$\begin{bmatrix} -\frac{g}{\ell}\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - k\ell x_2 \end{bmatrix}$$

Krasovski-La Salle Criterion

Let $V : \mathscr{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a positive definite function relative to $\bar{x}$, an equilibrium point of the system $\dot{x} = f(x)$. Suppose that $V'(x) \leq 0$ for all $x \in \mathscr{D}' \subseteq \mathscr{D}$. Let $E \subseteq \mathscr{D}'$ be a set where $V'(x) = 0$ and let M the largest set contained in E which is invariant for the trajectories of the system. Then:

- the equilibrium $\bar{x}$ is stable (classical Direct Lyapunov result)
- the set M is attractive for the trajectories of the system, namely $\lim_{t \to \infty}$ **dist**$(\,x(t), M\,) = 0$



restricted set

Definition. A set M is invariant for $\dot{x} = f(x)$ if, having denoted by $x(t, x_o)$ the trajectory at time t with initial condition x_0 at $t = 0$, then

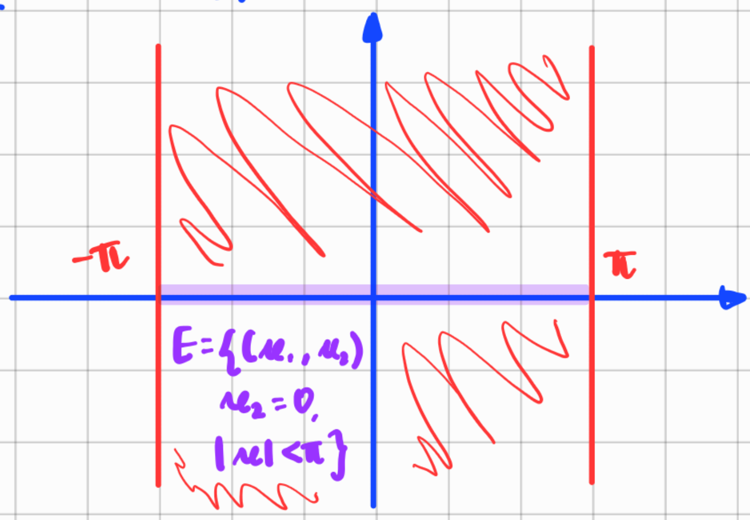
$$x_0 \in M \implies x(t, x_0) \in M \quad \forall\, t \in \mathbb{R}$$

Special case. if $M = \bar{x}$ then the equilibrium point is asymptotically stable

$$V(x) = \frac{1}{2} x_2^2 + \frac{g}{l}(1 - \cos x_1)$$

$D = \{(x_1, x_2) : |x_1| < \pi\}$

$D' \equiv D$



What is M? $\subseteq E$

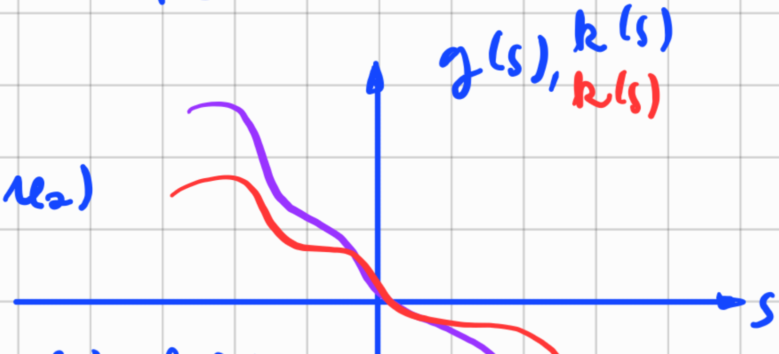
- Obs # 1 $\Rightarrow$ All trajectory of the system evolving in E have $x_2(t) \equiv 0$, $\dot{x}_2(t) \equiv 0$.

- Obs # 2 $\Rightarrow$ $\sin(x_1(t)) \equiv 0$

- Obs # 3 $\Rightarrow$ The only trajectory of the sys. in E consistent with obs # 1 and # 2 is $(x_1, x_2) \equiv 0$

Example : "Generalized Pendulum"

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g(x_1) + k(x_2) \end{cases}$$



$g(0) = k(0) = 0$

$s g(s) < 0 \ \forall s$

$s k(s) < 0 \ \forall s$

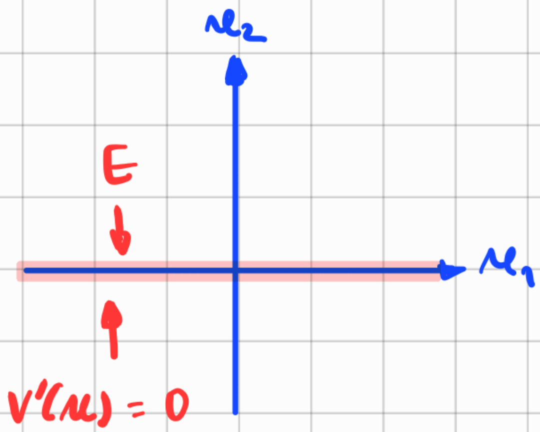
Can we conclude $\bar{x} = (x_1, x_2) = (0,0)$ is global as. stable?

positive def. stable

$$V(x) = \frac{1}{2} x_2^2 - \int_0^{x_1} g(s)\, ds$$

$$V'(x) = \underbrace{\begin{pmatrix} g(x_1) & x_2 \end{pmatrix}}_{\nabla V} \underbrace{\begin{pmatrix} x_2 \\ g x_1 + k x_2 \end{pmatrix}}_{f(x)}$$

$$= x_2 k(x_2) \le 0$$



Examples

- Pendulum case with “variable-gradient method”
- Pendulum case with La Salle
- Generalised Pendulum
- Adaptive (to control direction) control systems

EXAMPLE: Adaptive Control

$\dot{y} = ay + u$, $a \in \mathbb{R}$ unknown, ← State equation

The goal: design $u(y)$: $y \xrightarrow[t \to \infty]{} 0$

Trivial Scenario:

$\bar{a} > 0$ known

$|a| \leq \bar{a}$

$u = -ky$ ⟶ $u = -ky$

$k > \bar{a}$ $\qquad \dot{k} = \gamma y^2$

$\dot{y} = -\underbrace{(k - a)}_{>0} y$

$x_1 = y \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_2 = k$

$$\boxed{\begin{aligned} \dot{x}_1 &= a x_1 - x_1 x_2 \\ \dot{x}_2 &= \gamma x_1^2 \end{aligned}}$$

$$\nabla V = \begin{bmatrix} 2x_1 & \frac{2}{\gamma}(x_2 - b) \end{bmatrix}$$

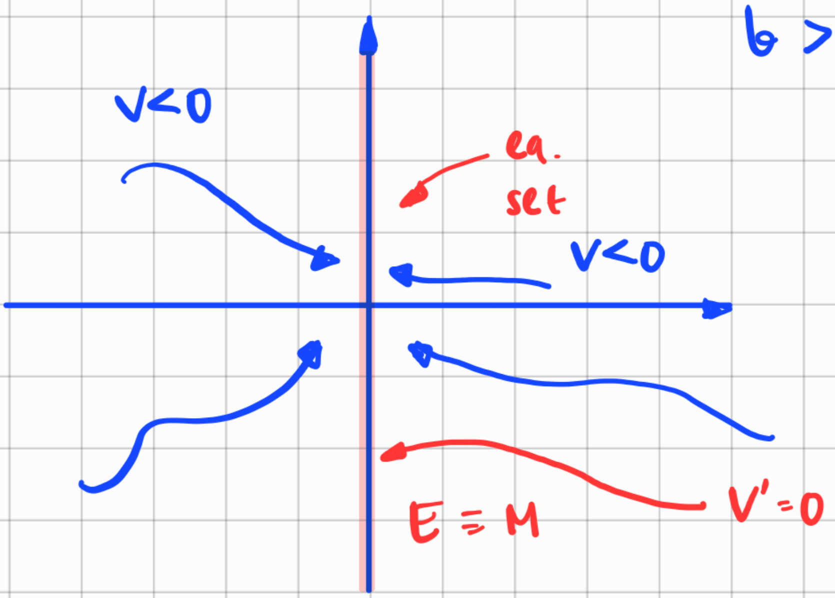
$$V' = \nabla V \cdot f = 2a x_1^2 - \cancel{2x_1^2 x_2} + 2(x_2 - b) x_1^2$$

$$= -2\underbrace{(b - a)}_{>0} x_1^2 \leq 0$$

$$V(x) = x_1^2 + \frac{1}{\gamma}(x_2 - b)^2$$

$V(x) > 0$ with respect to $\boxed{(x_1, x_2) = (0, b)}$

$b > |a|$



$$\lim_{t \to \infty} (x_1, x_2) \to E = M$$

Example: active noise cancellation

Measured

$n(t)$

injected noise

Measured

$w(t) - z(t) := e(t)$

injecting extra noise

$z(t)$ = propagation of the external noise into the headphone

$w(t)$ = generated noise ideally opposite in phase wrt $z(t)$

unknown

$$\dot z(t) = \boxed{a_0} z(t) + \boxed{b_0} n(t)$$

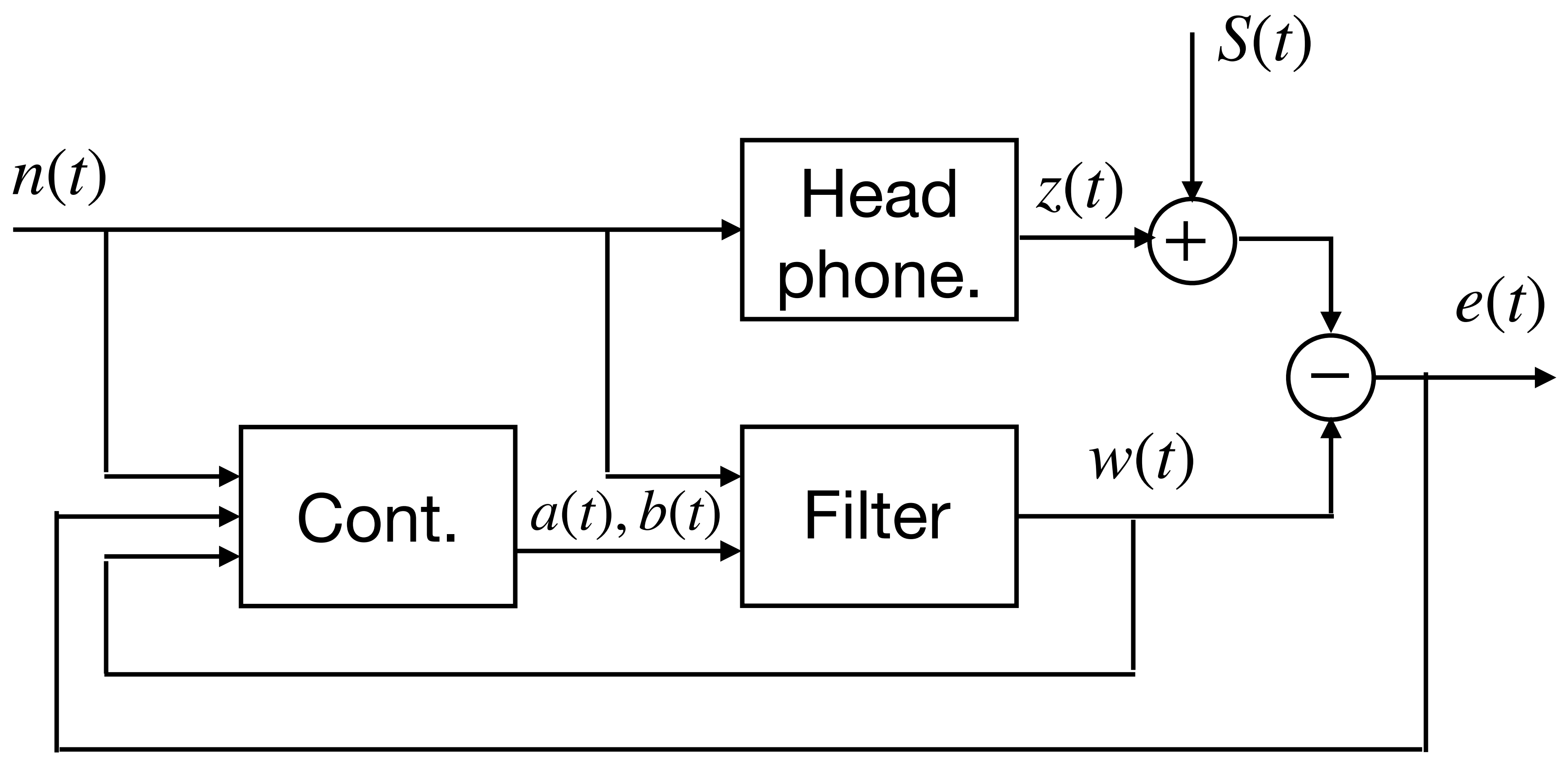
Model, with (a_0, b_0) unknown

$$\dot w(t) = a(t)w(t) + b(t)n(t)$$

$(a(t), b(t))$ to be adapted so that $e(t) \to 0$

generated by controller

Known "ideal" sound



$$e = w - z \qquad\qquad a_0 < 0$$

$$\dot{e} = aw + bn - az - bn$$

$\pm\, a\,w \longrightarrow$ add and subtract

$$(a - a_0)\,w + a_0 \overbrace{(w - z)}^{e} + (b - b_0)\,n$$

$$\left.\begin{aligned} \dot{a} &= v_a \\ \dot{b} &= v_b \end{aligned}\right\} \text{ to be designed}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e \\ a - a_0 \\ b - b_0 \end{bmatrix}$$

$$\dot{x}_1 = x_2\, w(t) + a_0\, x_1 + x_3\, n(t)$$

error

known — known from $e(t)$ — known

$$\dot{x}_2 = v_a$$
$$\dot{x}_3 = v_b$$

Lyapunov Equation:

$$V(x) = \frac{1}{2}\left(x_1^2 + \frac{1}{\gamma_a} x_2^2 + \frac{1}{\gamma_b} x_3^2\right)$$

good term since x_1 is always > 0 and $a_0 < 0$

$$\nabla V \cdot f(x) = \cancel{x_1 x_2 w} + a_0 x_1^2 + x_1 x_3 n + \cancel{\frac{1}{\gamma_a} x_2 v_a} + \frac{1}{\gamma_b} x_3 v_b \le 0$$

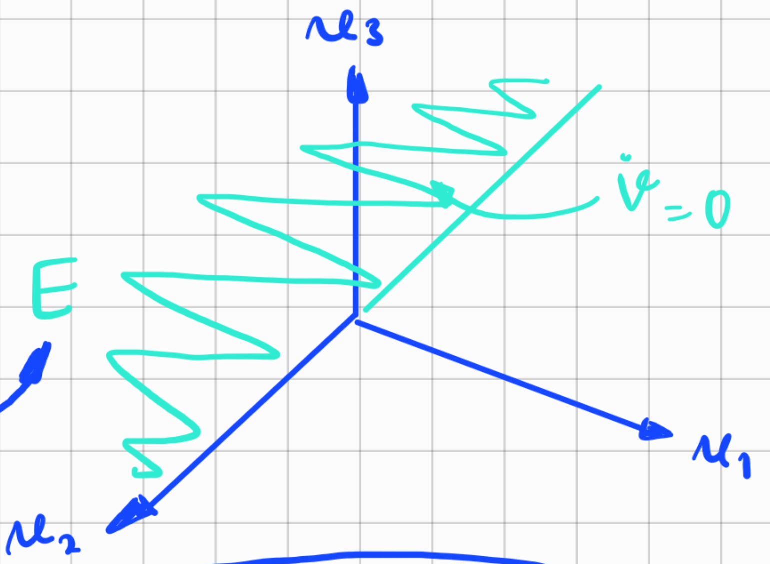
$v_a = -\gamma_a x_1 w$

$v_b = -\gamma_b x_3 n$

$v_a = \dot{x}_2$

$v_b = \dot{x}_3$

M?



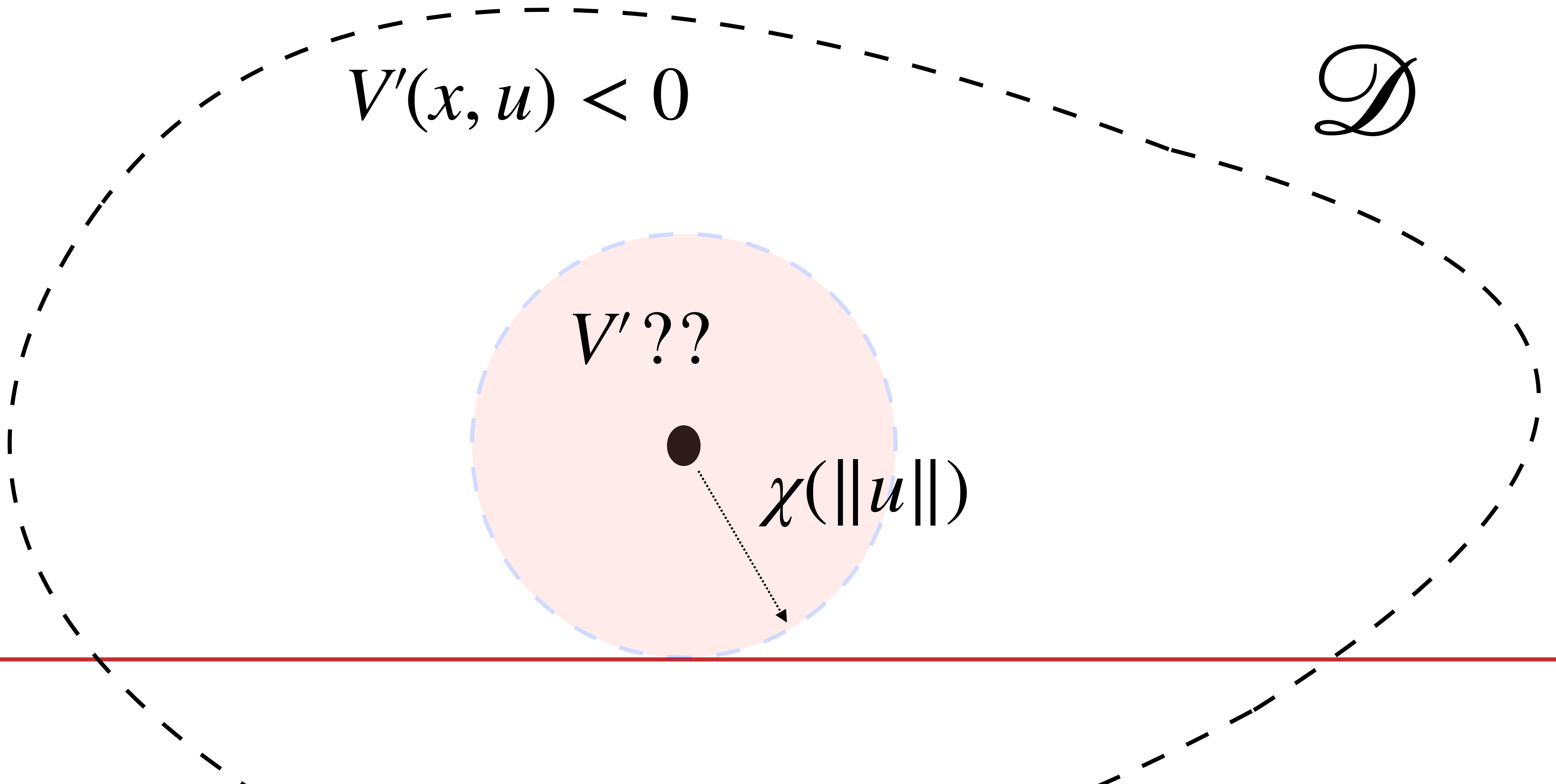
$$\boxed{\begin{gathered} 0 = x_2 w(t) + x_3 n(t) \\ \dot{x}_2 = 0, \quad \dot{x}_3 = 0 \end{gathered}}$$

Systems with inputs (and outputs)

When the system has (generic) inputs, the notion of stability gets more involved. As seen, an input can substantially change properties of the input-free system. A well known notion in literature to express a "stability" property in presence of inputs is the one called "**Input-to-State Stability**" (**ISS**). It still refers to an equilibrium point and, in Lyapunov terms, can be expressed as follow.

A system $\dot{x} = f(x, u)$ fulfilling $f(0,0) = 0$ is ISS with respect to $\bar{x} = 0$ if there exist a positive definite real valued function $V : \mathscr{D} \to \mathbb{R}$ (**ISS Lyapunov function**) and a **strictly increasing** function $\chi : \mathbb{R} \to \mathbb{R}$ fulfilling

$$\|x\| \geq \chi(\|u\|) \quad \Longrightarrow \quad V'(x, u) = \nabla V(x) f(x, u) < 0$$

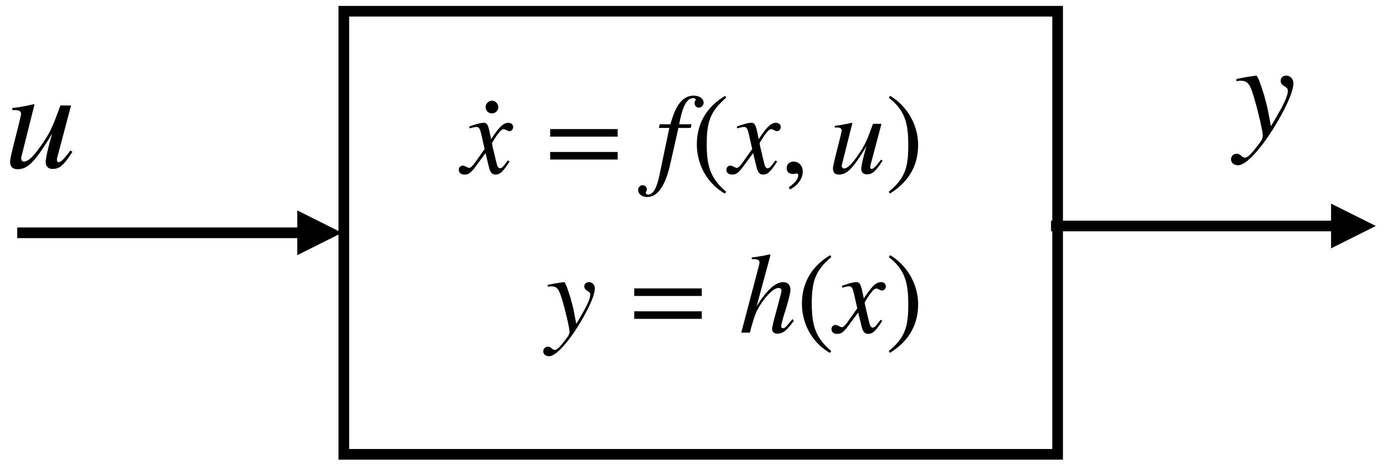


In a nutshell: when the state is "large" with respect to the input then derivative of V is decreasing

When $u = 0$ we obtain the classical stability notion

Systems with inputs (and outputs)

For systems with input and output an important (in the field of engineering) property related to stability and Lyapunov analysis is the one of "**passivity**"

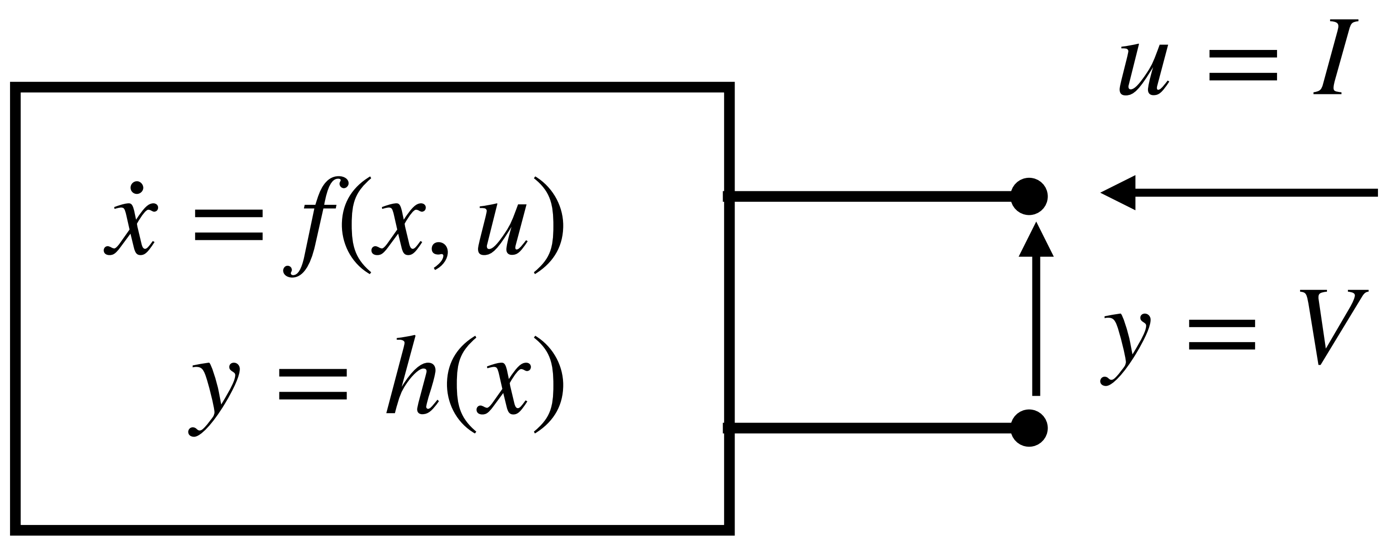


The system aside is passive if there exist a positive definite real valued function $V : \mathscr{D} \to \mathbb{R}$ (**storage function**) fulfilling

$$\dot{V}(x(t)) = V'(x,u) = \nabla V(x) f(x,u) \leq u^T y$$

Interpretation (circuit perspective):

$$\int_0^T \dot{V}(x(t))dt = \underbrace{V(x(T)) - V(x(0))}_{\text{Energy stored in the system in the interval } [0,T]} \leq \underbrace{\int_0^T u(t)^T y(t)dt}_{\text{Energy supplied to the system in the interval } [0,T]}$$



The system dissipates energy