Mathematical Methods for Automation Engineering M
– *Continuous Random Variables* –

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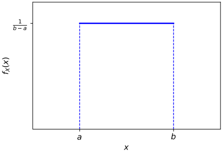
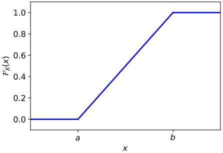
Continuous Random Variables

Definition (Continuous uniform random variable)

The continuous random variable X, defined on an interval $[a, b]$, is a *uniform random variable* if its probability density function is constant

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{elsewhere} \end{cases} \quad \Rightarrow \quad \mathcal{F}_X(x) = \begin{cases} 0 & x \leqslant a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geqslant b \end{cases}$$





Theorem (Expected value & Variance of the continuous uniform r. v.)

Expected value and variance of the continuous uniform random variable are

$$\mu_X = \frac{a+b}{2} \qquad \sigma_X^2 = \frac{(b-a)^2}{12}$$

Definition (Rayleigh random variable)

The continuous random variable X is a *Rayleigh random variable* of parameter $\boxed{\sigma}$ if its probability density function is

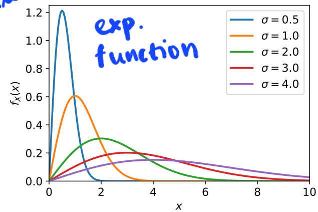
not a variance

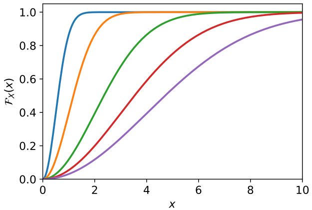
CDF

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \quad \Rightarrow \quad \mathcal{F}_X(x) = \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

$\int_0^\infty f_X(x)\,dx = 1$

exp. function





Theorem (Expected value and Variance of the Rayleigh r. v.)

Expected value and variance of a Rayleigh random variable are

$$\mu_X = \sigma\sqrt{\pi/2} \simeq 1.253\,\sigma, \qquad \sigma_X^2 = (2 - \pi/2)\,\sigma^2 \simeq 0.429\,\sigma^2$$

Continuous Random Variables

Example (1)

paralayang

The distance (in meters) of the point of landing of a paratrooper from the center of a the target landing area is a Rayleigh random variable X of parameter $\sigma = 10$ m

1. Determine the probability that the paratrooper will land within a distance of 10 m from the center of the target landing area
2. Determine the distance r such that there is a probability of 25% that the paratrooper will land at a distance greater than r from the center of the landing area

Recalling that $\mathcal{F}_X(x) = 1 - e^{-\frac{x^2}{2\sigma^2}} \quad (x \geqslant 0)$, we have

1. $\mathcal{P}(X \leqslant 10) = \mathcal{F}_X(10) = 1 - e^{-100/200} = 1 - e^{-0.5} \simeq 0.393$
2. $\mathcal{P}(X > r) = 1 - \mathcal{P}(X \leqslant r) = 1 - \mathcal{F}_X(r) = e^{-r^2/200} = 0.25$
   and then: $r^2 = -200 \ln 0.25 \quad \Rightarrow \quad r = \sqrt{200 \ln 4} \simeq 16.65\,\mathrm{m}$

Example (1):

The paratrooper will land within a distance of 10m from the center : $P(X \leq 10)$

$$P(X \leq 10) = F_X(10) = 1 - e^{-100/200} \simeq$$

A probability 25% of the paratrooper will land over than r meters radius

$$P(X > r) = 1 - P(X \leq r) = e^{-r^2/200}$$

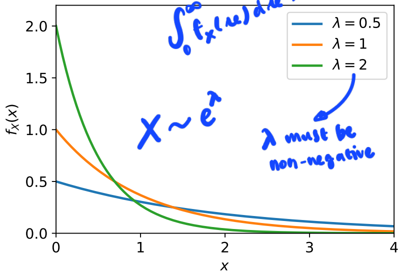
$$= 0.25 \longrightarrow \text{find } r \text{ in } \ln$$

$$r \simeq 26.65$$

Definition (Exponential random variable)

A random variable X is an *exponential random variable* with parameter λ, $X \sim Exp(\lambda)$, if its probability density function and CDF are

$$f_X(x) = \begin{cases} \lambda\, e^{-\lambda x} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \quad \Rightarrow \quad \mathcal{F}_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

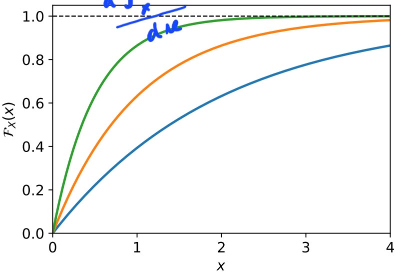


$\int_0^\infty f_X(x)\,dx = 1$

$X \sim e^\lambda$

λ must be non-negative

$\frac{d\mathcal{F}_X(x)}{dx} = f_X(x)$



Continuous Random Variables

Theorem (Expected value and Variance of the exponential r. v.)

The main statistics of the exponential random variable are

$$\boxed{\mu_X = \frac{1}{\lambda}} \qquad \boxed{\sigma_X^2 = \frac{1}{\lambda^2}}$$

and

$$\mathcal{M} = 0 \qquad \mathcal{M}_e = \frac{\ln 2}{\lambda} \simeq \frac{0.693}{\lambda} \qquad \gamma_1 = 2$$

Proof. (expected value)

$$\mu_X = \int_0^\infty x \underbrace{\lambda\, e^{-\lambda x}}_{f_X(x)} dx = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda}$$

- Let X be a random variable representing the duration of life of a device
- Let $\mathcal{F}_X(x)$ be its CDF $\quad F_X(x) = P(X \leq x)$

Definition (Reliability or survival function)

We call *reliability function* (or *survival function*), $\boxed{R(x)}$, the probability that the life of the device will last at least $\boxed{x}$

life of the device $= X > x$
$$R(x) = 1 - \mathcal{F}_X(x)$$

$$R(x) = P(X > x) = 1 - P(X \leq x) = 1 - F_X(x)$$

The **hazard function** (or **instantaneous failure rate**, or **failure intensity**), represents the probability that a device that has survived until time $\boxed{x}$, will fail in the "next instant of time" dx

still working at time x but stop at dx

$P(x \le X \le x+dx)$ → conditional probability (given $X > x$)

$$h(x)\,dx = \mathcal{P}\left(\underbrace{X \in [x,\, x+dx]}_{A} \,\underbrace{|}_{|}\, \underbrace{X > x}_{B}\right) = \frac{\mathcal{P}\left(X \in [x,\, x+dx],\, X > x\right)}{\mathcal{P}\left(X > x\right)}$$

$$= \frac{\mathcal{P}\left(X \in [x,\, x+dx]\right)}{1 - \mathcal{F}_X(x)} = \frac{f_X(x)\, dx}{1 - \mathcal{F}_X(x)}$$

$P(A|B) = P(A \cap B)$

prob. of A given B

dx is very small

Definition (Hazard function, or failure rate)

The *hazard function*, $h(x)$, is the probability density that a device which has survived until x will fail in the next dx

x represents time

$$\boxed{h(x) = \frac{f_X(x)}{R(x)} = \frac{f_X(x)}{1 - \mathcal{F}_X(x)}}$$

Continuous Random Variables

$X \sim e^{\lambda}$

$\mu = \frac{1}{\lambda} \quad \sigma^2 = \frac{1}{\lambda^2}$

For the exponential random variable we find

$$h(x) = \frac{f_X(x)}{1 - \mathcal{F}_X(x)} = \frac{\lambda\, e^{-\lambda x}}{e^{-\lambda x}} = \lambda$$

it indicates the failure rate is constant!

so the **exponential random variable** is associated to a **constant hazard function** $h(x) = \lambda \quad \rightarrow \quad$ **process without memory**

- λ is the **failure rate** ✓ (e.g. 0.00025 failures/h)
- $1/\lambda$ is the **MTBF** ✓ *(mean time between failures)* (e.g. 4 000 h)
- $1 - e^{-8760\lambda}$ is the **AFR** ✓ *(annualized failure rate)* (e.g. 0.9 failures/1st yr)

The exponential distribution is largely used in the study of random processes in which *duration processes* are involved (lifetime of components, waiting time in queues. . .)

Continuous Random Variables

Example (1)

A given class of electronic components has a constant failure rate λ over time.
Find the failure rate λ knowing that experience has shown that 30% of these components last longer than 10^4 hours.

Being the failure rate constant, the random variable X representing the duration of life of the components has an exponential distribution.

$$\begin{aligned}\mathcal{P}\left(X > 10^4\right) &= 1 - \mathcal{P}\left(X \leqslant 10^4\right) = 1 - \mathcal{F}_X\left(10^4\right) \\ &= 1 - \left(1 - e^{-10^4\lambda}\right) = e^{-10^4\lambda} = 0.3\end{aligned}$$

therefore $-10^4\lambda = \ln 0.3 \simeq -1.2 \quad \Rightarrow \quad \lambda \simeq 1.2 \times 10^{-4}$ hours^{-1}

Example (1)

- The failure rate is constant λ.
- 30% of the components last > 10^4 hours.
- What is λ?

Known $P(x > 10^4) = 30\%$

$$\downarrow$$

$$1 - P(x \le 10^4) = 1 - F(10^4) = e^{(-\lambda \cdot 10^4)}$$

$$e^{-10^4\lambda} = 0.3$$

$$-10^4\lambda = \ln 0.3$$

$$\lambda \simeq 1.2 \cdot 10^{-4} / \text{hour}$$

Given a **non constant hazard function** $\boxed{h(x)}$, how do we calculate the corresponding cumulative distribution function?

$$h(x) = \frac{f_X(x)}{1 - \mathcal{F}_X(x)} = \frac{\mathcal{F}'_X(x)}{1 - \mathcal{F}_X(x)} = -\frac{d}{dx} \ln(1 - \mathcal{F}_X(x))$$

and then

$$\begin{aligned} \int_0^x h(t)dt &= -\ln(1 - \mathcal{F}_X(x)) + \ln(1 - \mathcal{F}_X(0)) \\ &= -\ln(1 - \mathcal{F}_X(x)) \end{aligned}$$

the hazard function can recover the CDF and also the reliability function

finally

$$\boxed{\mathcal{F}_X(x) = 1 - \exp\left(-\int_0^x h(t)dt\right)} \Rightarrow \boxed{R(x) = \exp\left(-\int_0^x h(t)dt\right)}$$

Continuous Random Variables

Let us assume a **linearly time-varying failure rate**

how if the failure rate increases over the time? (linear)

$$h(t) = a + bt$$

The corresponding cumulative distribution function is

$$\mathcal{F}_X(x) = 1 - \exp\left(-\int_0^x h(t)dt\right) \quad \Rightarrow \quad \mathcal{F}_X(x) = 1 - e^{-ax - bx^2/2}$$

from which we obtain

$$\boxed{f_X(x) = (a + bx)\, e^{-ax - bx^2/2}}$$

0 failure at $t=0$

note that, for $a = 0$, this corresponds to the *Rayleigh distribution* of parameter $\sigma = b^{-1/2}$

Definition (Gamma random variable)

X is a *gamma random variable* of parameters α and λ $(\alpha, \lambda > 0)$ if its probability density function is

$$f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}}{\Gamma(\alpha)} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \quad \Rightarrow \quad \mathcal{F}_X(x) = \begin{cases} \frac{\gamma(\alpha, \lambda x)}{\Gamma(\alpha)} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

gamma random variable

$\Gamma(\alpha)$

where $\Gamma(\alpha)$ and $\gamma(\alpha, \lambda x)$ are, respectively, the *Euler gamma function* and the *Euler incomplete gamma function* $\gamma(\alpha, y)$

$$\boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx} \qquad \boxed{\gamma(\alpha, y) = \int_0^y x^{\alpha-1} e^{-x}\, dx} \qquad (\alpha,\ y > 0)$$

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \quad (\alpha > 0) \quad \Rightarrow \quad \Gamma(k+1) = k! \qquad (k \in \mathbb{N}^+)$

Definition (Gamma random variable)

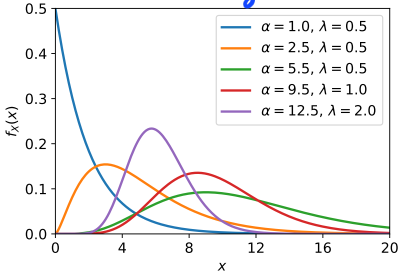
X is a *gamma random variable* of parameters α and λ ($\alpha, \lambda > 0$) if its probability density function is

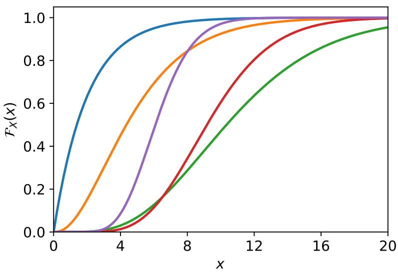
shape param.

scaling param.

$$f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}}{\Gamma(\alpha)} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \quad \Rightarrow \quad \mathcal{F}_X(x) = \begin{cases} \frac{\gamma(\alpha, \lambda x)}{\Gamma(\alpha)} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

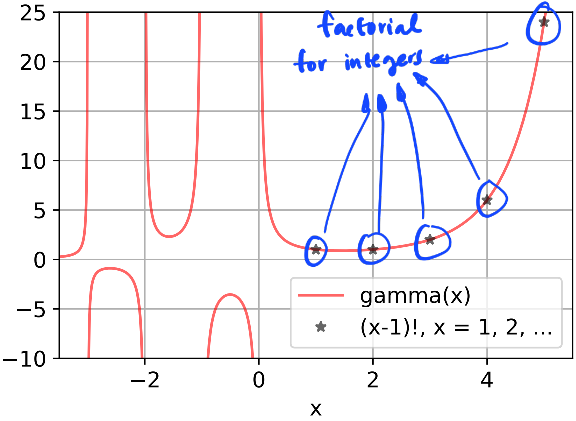
generalization of exponential





Euler gamma function

$$\Gamma\left(\alpha\right)=\int_0^{\infty}x^{\alpha-1}e^{-x}\,dx$$



Continuous Random Variables

Theorem (Expected value and Variance of the gamma r. v.)

The main statistics of the gamma random variable are

$$\boxed{\mu_X = \frac{\alpha}{\lambda}} \qquad \boxed{\sigma_X^2 = \frac{\alpha}{\lambda^2}}$$

and

$$\boxed{\mathcal{M} = \frac{\alpha - 1}{\lambda} \quad (\alpha \geqslant 1)} \qquad \boxed{\gamma_1 = \frac{2}{\sqrt{\alpha}}}$$

(the median $\mathcal{M}_e$ of the gamma random variable does not have an analytical expression)

- When $\alpha = 1$, the exp distribution (of parameter λ) is recovered
- The Gamma distribution is associated to the **hazard function**

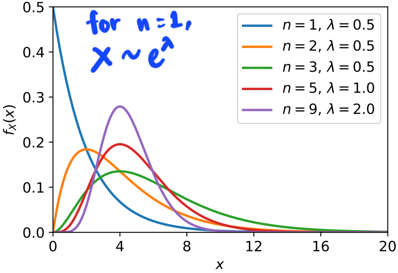
$$\boxed{h(x) = \frac{(\lambda x)^{\alpha-1}\,\lambda e^{-\lambda x}}{\Gamma(\alpha) - \gamma(\alpha,\, \lambda x)}}$$

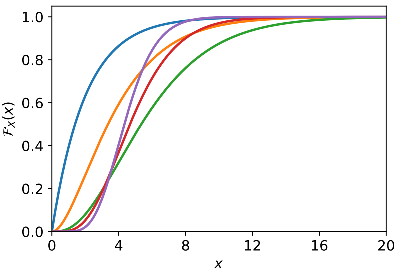
Continuous Random Variables

Definition (Erlang random variable)

The *Erlang random variable* is the special case of the Gamma random variable obtained when the parameter $\boxed{\alpha}$ (now denoted as n) is an $\boxed{\text{integer}}$
Probability density function and CDF are $\Gamma(n) = (n-1)!$

$$f_X(x) = \begin{cases} \frac{(\lambda x)^{n-1}\lambda e^{-\lambda x}}{(n-1)!} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \qquad \mathcal{F}_X(x) = \begin{cases} 1 - e^{-\lambda x}\sum_{k=0}^{n-1}\frac{(\lambda x)^k}{k!} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$



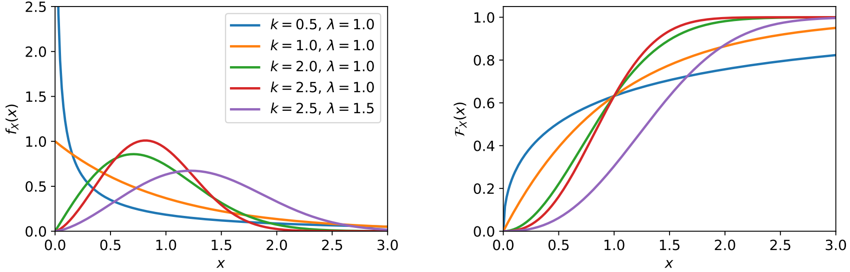


Definition (Weibull random variable)

X is a *Weibull random variable* of parameters λ and k ($\lambda, k > 0$) if its probability density function is

$$f_X(x) = \begin{cases} \underline{k\lambda^k x^{k-1} e^{-(\lambda x)^k}} & x \geqslant 0 \\ 0 & x < 0 \end{cases} \Rightarrow \mathcal{F}_X(x) = \begin{cases} 1 - e^{-(\lambda x)^k} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

generalization of exponential



Continuous Random Variables

Theorem (Expected value and Variance of the Weibull r. v.)

The mean value and variance of the Weibull random variable are

$$\mu_X = \frac{1}{\lambda}\Gamma\left(1+\frac{1}{k}\right) \qquad \sigma_X^2 = \frac{1}{\lambda^2}\left(\Gamma\left(1+\frac{2}{k}\right) - \Gamma\left(1+\frac{1}{k}\right)^2\right)$$

being $\Gamma(x)$ *the Euler gamma function*

- When $k = 1$, the exponential distribution (of parameter λ) is recovered
- When $k = 2$, the Rayleigh distribution (of parameter $\sigma = 1/\left(\lambda\sqrt{2}\right)$) is recovered
- The Weibull distribution is associated to a (non constant) **hazard function**

$$\boxed{h(x)} = \frac{k\,\lambda^k\,x^{k-1}e^{-(\lambda x)^k}}{1-\left(1-e^{-(\lambda x)^k}\right)} = \boxed{k\,\lambda^k\,x^{k-1}}$$

Continuous Random Variables

The **exponential random variable** (1 parameter), the Rayleigh and – above all – the **Weibull random variable** (2 parameters) are very popular when modeling the lifetime and failure rates of electronic/mechanical components

- **exponential distribution** $\rightarrow$ $h(x)$ constant
- **Rayleigh distribution** $\rightarrow$ $h(x) \propto x$
- **Weibull distribution** $\rightarrow$ $h(x) \propto x^s$

modify λ to make a product more reliable

The **Weibull random variable** has 2 parameters:

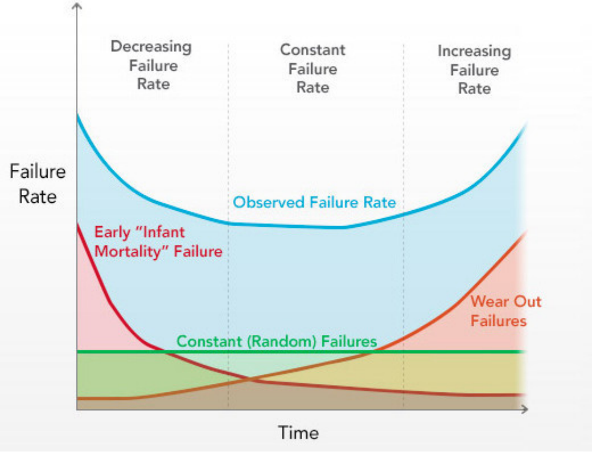
- $\boxed{\lambda\text{: } \textit{scale parameter}}$; usually depends on product design, and it can be adjusted to meet a reliability requirement
- $\boxed{k\text{: } \textit{shape parameter}}$; depends on the material property which is fixed given the type of the material producing the component

Other relevant distributions: **logistic distribution**, **lognormal distribution**...

Continuous Random Variables

How does the hazard function look like, in reality? The **Bathtub curve** is a very well-known and successful model. Such a failure rate can be modelled by means of **modified Weibull distributions**

$h(x) \propto x^s$



(from BackBlaze website)

Continuous Random Variables

Definition (Normal (Gaussian) random variable)

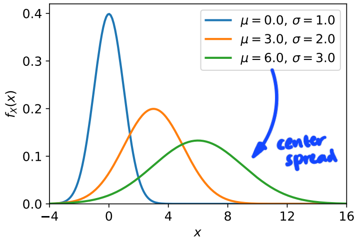
Given a random variable X, we say that X is a *normal (or Gaussian) random variable* of parameters μ, σ^2 ($\mu \in \mathbb{R}$, $\sigma > 0$), denoted by

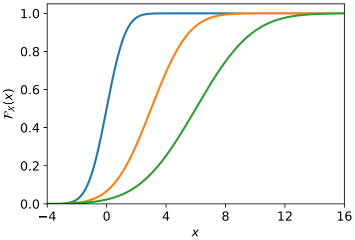
$$X \sim N\left(\mu, \sigma^2\right)$$

symetry at $x = \mu$

if its probability density function is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$





Continuous Random Variables

We may check that the Gaussian is actually a **probability density function**

- Clearly, $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} > 0 \quad \forall x$
- Let us check that $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$\frac{x-\mu}{\sigma\sqrt{2}} = t \qquad \Rightarrow \qquad x = \mu + \sigma\sqrt{2}\, t \qquad dx = \sigma\sqrt{2}\, dt$$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = 1$$

where it has been taken into account that (*Euler-Poisson integral*)

$$\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}$$

Theorem (Main statistics of the normal random variable)

The main statistics of the normal random variable are

$$\mu_X = \mathcal{M} = \mathcal{M}_e = \mu \qquad \sigma_X^2 = \sigma^2 \qquad \gamma_1 = 0$$

Proof. (expected value and variance) Letting $(x-\mu)/(\sigma\sqrt{2}) = t$

$$\begin{aligned}
\mu_X = \int_{-\infty}^{\infty} x\, f_X(x)\; dx &= \int_{-\infty}^{\infty} x\, \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\; dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\mu + \sigma\sqrt{2}\, t\right) e^{-t^2}\; dt \\
&= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\; dt + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} te^{-t^2}\; dt \\
&= \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} + 0 = \mu
\end{aligned}$$

$$\begin{aligned}
\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\; dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2\; e^{-\frac{(x-\mu)^2}{2\sigma^2}}\; dx \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2}\; dt \\
&= \frac{\sigma^2}{\sqrt{\pi}} \left(\left[-te^{-t^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{+\infty} e^{-t^2}\; dt \right) \\
&= \frac{\sigma^2}{\sqrt{\pi}} \left(0 + \sqrt{\pi}\right) = \sigma^2
\end{aligned}$$

Continuous Random Variables

Cumulative distribution function

$$\mathcal{F}_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}}\, d\xi$$

Laplace function

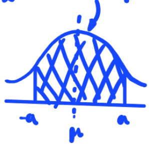
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{\zeta^2}{2}}\, d\zeta \qquad (\xi = \mu + \sigma\zeta)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

where Φ is the **Laplace function** (*special function*, tabulated)

- $\Phi(a)$ is an increasing function, $\quad \Phi(a) \xrightarrow{a\to-\infty} 0, \quad \Phi(a) \xrightarrow{a\to\infty} 1$
- $\Phi(-a) + \Phi(a) = 1$

$\int_{-\infty}^{a} f_X(x)\, dx = \phi(a)$



Therefore

$$\boxed{\mathcal{P}(\alpha \leqslant X \leqslant \beta) = \Phi\left(\frac{\beta-\mu}{\sigma}\right) - \Phi\left(\frac{\alpha-\mu}{\sigma}\right)}$$

$$\mathcal{P}(\mu - \Delta \leqslant X \leqslant \mu + \Delta) = 2\,\Phi\left(\frac{\Delta}{\sigma}\right) - 1$$

Definition (Standardized normal random variable)

Given a normal random variable $X \sim N\left(\mu, \sigma^2\right)$ with expected value μ and variance σ^2, the so-called *standardized normal random variable* Z is defined as follows

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

It turns out that

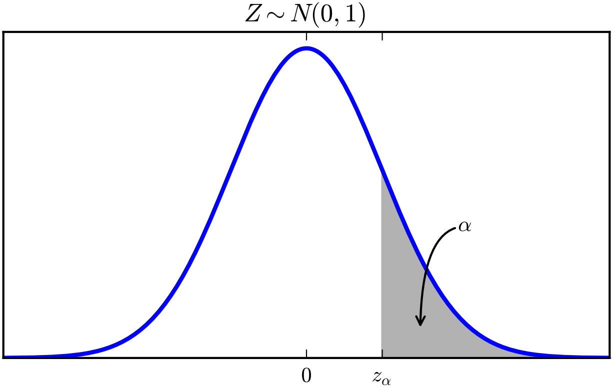
$$\boxed{Z \sim N(0, 1)}$$

For a standardized normal random variable Z, we have

$$\mathcal{P}\left(\alpha \leqslant Z \leqslant \beta\right) = \Phi\left(\beta\right) - \Phi\left(\alpha\right)$$

Therefore, **the Laplace function is the CDF of the standardized normal random variable Z**

$$Z \sim N(0,1) \qquad \mathcal{P}\left(Z > z_\alpha\right) = \alpha \qquad \Phi\left(z_\alpha\right) = \mathcal{P}\left(Z < z_\alpha\right) = 1 - \alpha$$



Example

[900, 1100] acceptable range

A production line manufactures 1 000 Ω resistors with a maximum tolerance of 10%. Let X be the resistance of the devices. Assuming that X has a normal distribution with parameters $\mu = 1\,000\,\Omega$ and $\sigma_X^2 = 2\,500\,\Omega^2$, find the probability that a resistor chosen at random is rejected

$$= \Phi\left[\frac{900-1000}{\sqrt{2500}}\right] = \Phi(-2)$$

$$p = \mathcal{P}(X \leqslant 900) + \mathcal{P}(X > 1100) = \mathcal{F}_X(900) + (1 - \mathcal{F}_X(1100))$$

$$\left(1 - \Phi\left(\frac{1100-1000}{\sqrt{2500}}\right)\right)$$

where X is a normal r.v. with $\mu_X = 1000$ and $\sigma_X^2 = 2500$ $(\sigma_X = 50)$

$$\mathcal{F}_X(900) = \Phi\left(\frac{900-1000}{50}\right) = \Phi(-2) = 1 - \Phi(2)$$

$$\mathcal{F}_X(1100) = \Phi\left(\frac{1100-1000}{50}\right) = \Phi(2)$$

From the tables: $\Phi(2) \simeq 0.9772$, therefore $p = 2(1 - \Phi(2)) \simeq 0.045$

Example

observed without theoretical reason

The weight of the components manufactured in a production line is normally distributed. Knowing that 4% of the components is lighter than $x_1 = 490$ g and that 5% is heavier than $x_2 = 500$ g, determine the expected value and the variance of the weight of the components

Introducing the standardized normal random variable Z, we have

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{490 - \mu}{\sigma} \qquad z_2 = \frac{x_2 - \mu}{\sigma} = \frac{500 - \mu}{\sigma}$$

$$\mathcal{P}\left(Z \leqslant z_1\right) = 0.04 \qquad \Phi(z_1) = 0.04$$
$$\mathcal{P}\left(Z > z_2\right) = 0.05 \qquad 1 - \Phi(z_2) = 0.05$$

Using a mathematical SW (or the tables): $z_1 \simeq -1.751$, $z_2 \simeq 1.645$, than

$$\frac{490 - \mu}{\sigma} \simeq -1.751 \quad \frac{500 - \mu}{\sigma} \simeq 1.645 \quad \Rightarrow \quad \mu \simeq 495.156 \quad \sigma \simeq 2.945$$

Given :

$\phi(z_1) = 0.04$

$1 - \phi(z_2) = 0.05$

μ? σ^2?

We cannot find $\phi(z_1) = 0.04$, so we will find $\phi(z_1) = 1 - 0.04 = 0.96$, and it will be the the negative of that z.

$z_1 = -1.75$.

Look for $\phi(z_2) = 0.95$

$z_2 = 1.645$ (interpolation)

$1.645 = \dfrac{500 - \mu}{\sigma} \iff 1.645\sigma = 500 - \mu$

$-1.75 = \dfrac{490 - \mu}{\sigma} \iff -1.75\sigma = 490 - \mu$

$\sigma(1.645 + 1.75) = 10$

$\sigma = 2.95$

$\mu = 495.9$