Mathematical Methods for Automation Engineering M
– Combinatorial Analysis –

Andrea Mentrelli

Department of Mathematics &
Alma Mater Research Center on Applied Mathematics, AM^2
University of Bologna

andrea.mentrelli@unibo.it

A.Y. 2024/25

Combinatorial Analysis

we need to compute the cardinality using combinatorics

Problem. In many contexts, it is necessary to compute **the number of possible outcomes** of an experiment

Some examples

- How many triples are possible in the lottery?
- In how many different ways can 10 persons sit at a round table?
- From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed, knowing that 2 of the men are feuding and refuse to serve on the committee together?

Combinatorial Analysis

Definition (Combinatorial analysis)

Combinatorial analysis is the branch of mathematics dealing with computing the cardinality (i.e. the number of elements) of sets made from available objects *following some prescribed rules*

Theorem (The basic principle of counting)

If an experiment E can be ideally decomposed into M sub-experiments, the k^{th} of which has n_k possible outcomes, and different sequences of the M sub-experiments result in differentiated final outcomes, then the total number n of outcomes of the experiment E is

$$n = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_M$$

$n_1 \times n_2 \times n_3 \times \ldots \times n_M$

Example

How many passwords are available on a login system allowing for sequences of 2 letters and 3 digits, like "AB 123"?

$$n = n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot n_5 = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676\,000$$

This "experiment" is divided into 5 "sub-experiments". The first two have 26 possible outcomes (each); the others have 10 possible outcomes (each)

Example

In how many ways can a candidate answer the admission test to the School of Engineering? (80 quizzes, each with 5 available answers)

$$n = n_1 \cdot n_2 \cdot \ldots \cdot n_{80} = 5 \cdot 5 \cdot \ldots \cdot 5 \ = 5^{80} \ \approx 8.27 \times 10^{55}$$

(80 "sub-experiments", each of which has 5 possible outcomes)

Example

In how many ways can the podium of a competition to which 10 athletes take part be composed?

$$n = n_1 \cdot n_2 \cdot n_3 = 10 \cdot 9 \cdot 8 = 720$$

The reasoning goes as follows: The first position can be taken by any of the 10 athletes; the second position by any of the remaining 9 athletes; the third position by any of the remaining 8 athletes

Example

Given a classroom of 20 students, in how many ways a pair of student delegates can be chosen?

$$n = n_1 \cdot n_2 = 20 \cdot 19 = 380\,? \quad \textbf{WRONG!}$$

We only count A-B and B-A once!

If we merely apply this formula, we end up with the result $n = 20 \cdot 19 = 380$, which is wrong. Here, the way in which the two students are selected does not matter (this is not a "ordered pair"). The 380 outcomes are composed by pairs (A, B) and (B, A) which have to be counted only once

The key observation is that the condition **"different sequences of the M sub-experiments result in differentiated final outcomes"** does not hold in this case arrangement is important

Definition (Permutations)

We call *permutations of n elements*

A B C these are different

B C A

P_n the number of possible permutation of n objects

the ordered groups of the n elements, which differ one from the other *only* on the basis of the order of the elements

Theorem

The number of permutations of n elements is

$$P_n = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$$

By definition: $0! = 1$ it's just a convention without repetition

Definition (k-permutations)

We call *k-permutations of n elements ($k \leqslant n$)*

$$D_{n,k}$$

the ordered groups of k of the n given elements, which differ one from the other on the basis of the identity of the elements and their order

Theorem

The number of k-permutations of n elements ($k \leqslant n$) is

i.e. k is number of required pairs/group

$$D_{n,k} = n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-k+1) = \frac{n!}{(n-k)!}$$

Combinatorial Analysis

Example

How many ordered pairs (tuples) N can be built from a set of 100 elements?

$$N = D_{100,2} = \frac{100!}{(100-2)!} = \frac{100!}{98!} = 100 \cdot 99 = 9\,900$$

For computational reasons, it is convenient first to simplify the ratio, rather than evaluating the factorials and subsequently compute the ratio. In fact

$$100! \simeq 0.933 \cdot 10^{158} \qquad 98! \simeq 0.943 \cdot 10^{154}$$

Remark: Working in *double precision* floating-point arithmetics (8 bytes), the largest factorial that can be stored in 64 bits is
$170! \simeq 7.257415615308 \times 10^{306}$

Example

In a lottery, 100 000 tickets are sold. How many different permutations of tickets winning the first three prizes are possible?

$$N = D_{100\,000,3} = \frac{100\,000!}{(100\,000-3)!} = \frac{100\,000!}{99\,997!}$$
$$= 100\,000 \cdot 99\,999 \cdot 99\,998 \simeq 100\,000^3 = 10^{15}$$

We don't even attempt a direct evaluation of the two factorials. Also, here we have done a very convenient appoximation

Combinatorial Analysis

Definition (Permutations with repetition)

We call *permutations with repetition of n elements*

$$P'_n$$

the groups of all the n elements, n_1 of which are indistinguishable one from the other, n_2 of which are indistinguishable one from the other $\ldots$ n_r of which are indistinguishable one from the other $(n = n_1 + n_2 + \ldots + n_r)$

Theorem

The number of permutations with repetition of n elements $(n = n_1 + n_2 + \ldots + n_r)$ is

$$P'_n = \binom{n}{n_1,\ n_2,\ \ldots n_r} \equiv \frac{n!}{n_1!\, n_2!\, \ldots n_r!}$$

swapable letters

Example

How many anagrams (non necessarily meaningful) can be extracted from the Italian name “Marinella”?

A appears twice
L appears twice

$n=9$

$$N = P'_9 = \frac{9!}{1!\,1!\,1!\,1!\,1!\,2!\,2!} = \frac{362\,880}{2 \cdot 2} = 90\,720$$

M, A, R, I, N, E, L

$$(n = 9, \quad n_M = n_R = n_I = n_N = n_E = 1, \quad n_A = n_L = 2)$$

now it is allowed

Definition (k-permutations with repetition)

We call *k-permutations with repetition of n elements taken k-by-k*

k times

$$D'_{n,k}$$

the groups made up with n elements taken k-by-k

Theorem

The number of k-permutations with repetition of n elements taken k-by-k is

$$\boxed{D'_{n,k} = n^k}$$

Example

The passwords of length 8 that can be made with the extended-ASCII code are a neat example of 8-permutations with repetitions of n elements ($n = 256$, in this case). How many of these passwords are there?

n k

$$N = D'_{256,8} = 256^8 = 2^{64} \simeq 1.84 \cdot 10^{19}$$

$D'_{n,k}$

256 256 256 256 256 256 256 256

Remark: The ASCII code (American Standard Code for Information Interchange) is a 7-bit standard code proposed by ANSI in 1963 (then officially recognized in 1968). ASCII is the standard code for microcomputers and, being a 7-bit code, it covers $2^7 = 128$ symbols (integers from 0 to 127). Later, the code was extended to a 8-bit code ($2^8 = 256$ symbols). In the extended set, the symbols from 128 to 255 represent special characters, mathematical symbols...

Example

How many Internet addresses can be assigned using the IPv4 and the IPv6 protocols?

$\underbrace{192}_{8} . \underbrace{0}_{8} . \underbrace{0}_{8} . \underbrace{1}_{8}$ → 0 - 255 = 2^8

8 + 8 + 8 + 8 = 32 bits

Since IPv4 addresses are 32 bits, we have

$$N_{v4} = D'_{2,32} = 2^{32} = 256^4 = 4\,294\,967\,296 \simeq 4.3 \cdot 10^9$$

$D'_{n,k}$

→ a lot but not enough

IPv6 addresses are, instead, 128 bit addresses

$$\begin{aligned} N_{v6} &= D'_{2,128} = 2^{128} \\ &= 340\,282\,366\,920\,938\,463\,463\,374\,607\,431\,768\,211\,456 \\ &\simeq 3.4 \cdot 10^{38} \end{aligned}$$

orders matter

there's no first and last element

Definition (Cyclic permutations)

We call *cyclic permutations of n elements*

$$P_n^c$$

the permutations of n elements along a closed path (a circle, for instance)

$A \to B \to C \to D$

same cycle

$D \to A \to B \to C \to D$

Theorem

The number of cyclic permutations of n is

$$P_n^c = (n-1)!$$

A B C D
B C D A
C D A B
D A B C
A C B D
C B D A

$\frac{n!}{n} = (n-1)!$

Example

In how many different ways can 10 persons take a seat at a round table?

$$N = P_{10}^{c} = 9! = 362\,880$$

Combinatorial Analysis

Given a box filled with n balls we extract k balls (**sample of size** k)

- **Sampling with repetitions (Bernoullian sampling)**. Before extracting a ball, the previously extracted ball is put back in the box. In this case, how many possible **ordered** samples of size k are there?

$$\boxed{N = D'_{n,k} = n^k}$$

- **Sampling without repetitions (cluster sampling)**. After extraction, balls are not put back in the box. In this case, how many possible **ordered** samples of size k are there?

$$\boxed{N = D_{n,k} = \frac{n!}{(n-k)!}}$$

Combinatorial Analysis

Definition (Binomial coefficient)

natural number includes zero

Given $n, k \in \mathbb{N}_0$ with $n \geqslant k$, we define the *binomial coefficient* as follows

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}} = \frac{(n-k+1)\cdot(n-k+2)\cdots n}{1\cdot 2\cdots k}$$

(to be read: "n choose k")

$$0! = 1 \text{ (by definition)} \quad \Rightarrow \quad \binom{0}{0} = 1 \qquad \binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

Theorem (Binomial theorem)

The n^{th} power of a binomial can be written as follows

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k}$$

Definition (Combinations)

We call *combinations of size k of n elements*

$$C_{n,k}$$

the groups of size k of the n elements, no matter the order of appearance

Theorem

The number of combinations of size k of n elements is where $(k \leq n)$

permutation

$$\boxed{C_{n,k} = \frac{D_{n,k}}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}}$$

Combinations of size k are nothing but k-permutations for which the order of appearance does not matter

12345

Definition (Combinations with repetition)

We call *combinations with repetition of size k of n elements*

$$C'_{n,k}$$

the groups of n elements, taken k-by-k, with possible repetition of each element (up to k times, obviously), in which the order does not matter

less common

Theorem

The number of combinations with repetition of size k of n elements is

$$C'_{n,k} = C_{n+k-1,k} = \binom{n+k-1}{k}$$

Combinatorial Analysis

Example

How many groups of 4 elements can be built with the symbols X, Y and Z? how many groups of 2 elements?

$n = 3$

$k = 4$

In the first case, we have

$n = 3$

$k = 4$

$$N = C'_{3,4} = \binom{6}{4} = \frac{6!}{4!\,2!} = 15$$

$$N = \binom{n+k-1}{k} = \binom{3+4-1}{4} = \binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!\,2!} =$$

In the second case

$n = 3$

$k = 2$

$$N = C'_{3,2} = \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

$$N = \binom{n+k-1}{k} = \binom{3+2-1}{2} = \binom{4}{2}$$

Notice that k can be either greater, smaller, or equal to n

Summary

$\frac{n!}{n_1!\, n_2! \ldots n_r!}$

kind	differentiation	type	formulae
permutations	order	without repetitions	$P_n = n!$
		with repetitions	$P'_n = \binom{n}{n_1, n_2, \ldots n_r}$
		cyclic	$P^c_n = (n-1)!$
k-permutations	type and order	without repetitions	$D_{n,k} = \frac{n!}{(n-k)!}$
		with repetitions	$D'_{n,k} = n^k$
combinations	type	without repetitions	$C_{n,k} = \binom{n}{k}$
		with repetitions	$C'_{n,k} = \binom{n+k-1}{k}$