Mathematical Methods for Automation Engineering M
– Conditional Probability –

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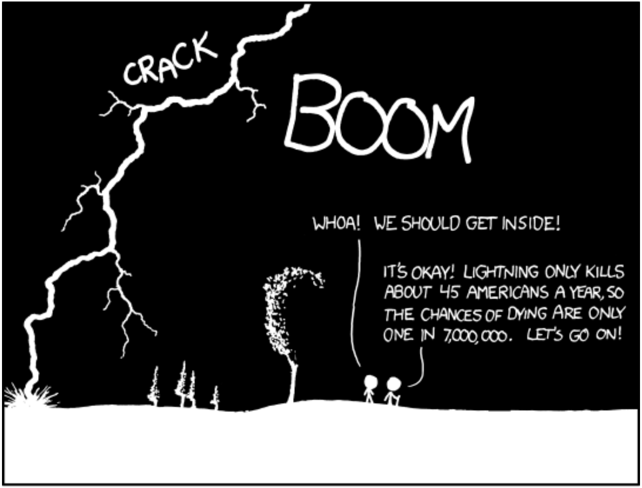
Conditional Probability

https://xkcd.com/795/



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

a priori

1: 7,000,000

updated

1:6

Conditional Probability

a probability conditoned after an event occurred

Conditional probability is essential in modern theory

- When we need to estimate the probability of some events, we want to **use the available information**
- As more evidence is available, we want to **update the probability** of those events
- Often, even when no partial information is available, conditional probabilities can be used to compute the desired probabilities more easily

Conditional Probability

Example

A machine fails whenever any of its six components fails (they all have the same probability of failure). Given that the machine is not working, we denote with ω_k the event "k-th component not working". The sample space is

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$$

a priori

Lacking any further information, we assign a probability $\boxed{1/6}$ of failure to each component

After some inspection, we conclude that the failure must be in the component No. 2, 4, or 6. How do we *update* the probabilities of failure?

Behind the scenes, what we actually do is to change the sample space: Ω is substituted by the *reduced* sample space

$$\Pi \equiv \{\omega_2, \omega_4, \omega_6\}$$

pi

Conditional Probability

Example

The point is that the events of the sample space are still equally likely, but the sample space has changed. The probability of an event E given Π (i.e. the *probability of E conditioned to Π*), will be assigned as follows

cardinality

$$\underbrace{\mathcal{P}(E \mid \Pi)}_{E \text{ conditioned to } \pi} = \frac{|E \cap \Pi|}{|\Pi|}$$

new/reduced sample space

namely, as the ratio between the cardinality of $E \cap \Pi$ *(favorable cases)* and the cardinality of Π *(possible cases)*

For instance $(\Pi \equiv \{\omega_2, \omega_4, \omega_6\})$

- $E = \{\omega_4\} \implies \mathcal{P}(E \mid \Pi) = 1/3$
- $E = \{\omega_1, \omega_2, \omega_3\} \implies \mathcal{P}(E \mid \Pi) = 1/3$

$$P = \frac{|\{\omega_4\} \cap \{\omega_2, \omega_4, \omega_6\}|}{|\{\omega_2, \omega_4, \omega_6\}|} = 1/3$$

$$P = \frac{|\{\omega_1, \omega_2, \omega_3\} \cap \{\omega_2, \omega_4, \omega_6\}|}{|\{\omega_2, \omega_4, \omega_6\}|} = 1/3 \, !$$

Conditional Probability

In the general case of any events A and B, for an **equally likely** space

no condition

$$\mathcal{P}\left(B \cap A\right) = \frac{|B \cap A|}{|\Omega|} \qquad \mathcal{P}\left(A\right) = \frac{|A|}{|\Omega|}$$

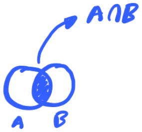
and then

$$\mathcal{P}\left(B \mid A\right) = \frac{|B \cap A|}{|A|} = \frac{|B \cap A|}{|\Omega|}\,\frac{|\Omega|}{|A|} = \boxed{\frac{\mathcal{P}\left(B \cap A\right)}{\mathcal{P}\left(A\right)}}$$

probab. of B given A

The latter can be generalized to **non equally likely** spaces

$\mathcal{P}\left(B \cap A\right) \equiv \mathcal{P}\left(A \cap B\right)$ is called **joint probability** of A and B



Definition (Conditional probability)

Given an event A such that $\mathcal{P}(A) > 0$, we call *conditional probability of B given A*, denoted by the symbol $\mathcal{P}(B \mid A)$, the probability of occurrence of the event B knowing that the event A has occurred

This probability is calculated as

$$\boxed{\mathcal{P}(B \mid A) = \frac{\mathcal{P}(B \cap A)}{\mathcal{P}(A)}}$$

Clearly, switching A and B (assuming that $\mathcal{P}(B) > 0$), we can also write

$$\mathcal{P}(A \mid B) = \frac{\mathcal{P}(A \cap B)}{\mathcal{P}(B)}$$

Example

A coin is flipped twice. Assuming that all outcomes of the sample space $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$ are equally likely, what is the conditional probability that both flips land on *heads*, given that

- the first flip is a head?
- at least one flip is a head?

given that the first flip is a head

$P(H,H) = \frac{1}{4}$

$E = \{H, H\}$

$P(E \mid \Pi)$

In the first case, the reduced sample space is

reduced sample

$$\Pi = \{(H, H), (H, T)\}$$

in this case $P(\{H,H\}) = P(E) = 1/4$

then

$$p = \frac{\mathcal{P}(\{H, H\} \cap \Pi)}{\mathcal{P}(\Pi)} = \frac{\mathcal{P}(\{H, H\})}{\mathcal{P}(\Pi)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$\frac{|\Pi|}{|\Omega|} = \frac{2}{4}$

Example

A coin is flipped twice. Assuming that all four points in the sample space $\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that

- the first flip is a head?
- at least one flip is a head?

In the second case, the *reduced* sample space is

$$\Pi = \{(H,H), (H,T), (T,H)\}$$

then

$$p = \frac{\mathcal{P}(\{H,H\} \cap \Pi)}{\mathcal{P}(\Pi)} = \frac{\mathcal{P}(\{H,H\})}{\mathcal{P}(\Pi)} = \frac{1/4}{3/4} = \frac{1}{3}$$

why is it lower than the first case?

$$P(E|\Pi) = \frac{P(E \cap \Pi)}{P(\Pi)} = \frac{P(H,H)}{P(\Pi)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Conditional Probability

Example

Is the last result surprising?

If you would guess that the probability is $1/2$, probably your reasoning goes as follows. Given that at least one flip is a head, there are two possible results: Either they are both heads or only one is. The mistake, however, is in assuming that these two possibilities are equally likely. For, initially, there are 4 equally likely outcomes. Because the information that at least one flip lands on head is equivalent to the information that the outcome is not (T, T), we are left with the 3 equally likely outcomes (H, H), (H, T), (T, H), only one of which results in both flips landing on heads

Conditional Probability

Theorem (Total probability)

sets of hypothesis

Let $H_1, H_2, \dots H_n$ be a partition of Ω (which we call "hypothesis"), the probability of any given event $E \subset \Omega$ can be written as

$$\boxed{\mathcal{P}(E) = \sum_{i=1}^{n} \mathcal{P}(E \mid H_i)\, \mathcal{P}(H_i)}$$

namely, the probability of E is the weighted sum of the conditional probabilities of E given the hypothesis, being the weights the probabilities of the hypothesis

Proof. Making use of the properties of a partition and the III axiom

$$E = E \cap \Omega = E \cap (H_1 \cup \dots \cup H_n) = (E \cap H_1) \cup \dots \cup (E \cap H_n)$$

$$\mathcal{P}(E) = \mathcal{P}((E \cap H_1) \cup \dots \cup (E \cap H_n)) = \sum_{i=1}^{n} \mathcal{P}(E \cap H_i) = \sum_{i=1}^{n} \mathcal{P}(E \mid H_i)\, \mathcal{P}(H_i)$$

Theorem (Bayes' Theorem)

sample space

Let $H_1, H_2, \dots H_n$ be a partition of Ω, for any given $E \subset \Omega$, the conditional probability of H_k given E is

Ω

P(E)

$$\boxed{\mathcal{P}(H_k \mid E) = \frac{\mathcal{P}(E \mid H_k)\,\mathcal{P}(H_k)}{\sum_{i=1}^{n} \mathcal{P}(E \mid H_i)\,\mathcal{P}(H_i)}} \qquad (k = 1, \dots n)$$

Say $E = \{H_2, H_4\}$

Proof. From the previous theorem

prob. of E given H_k

$$\mathcal{P}(E) = \sum_{i=1}^{n} \mathcal{P}(E \mid H_i)\,\mathcal{P}(H_i)$$

Prob. of H_k given E

Combining the latter with the definition of conditional probability, we have $(k = 1, \dots n)$

$$\mathcal{P}(H_k \mid E) = \frac{\mathcal{P}(H_k \cap E)}{\mathcal{P}(E)} = \frac{\mathcal{P}(E \mid H_k)\,\mathcal{P}(H_k)}{\sum_{i=1}^{n} \mathcal{P}(E \mid H_i)\,\mathcal{P}(H_i)}$$

$\square$

Conditional Probability

Remark. Bayes' rule allows to write the conditional probabilities $\mathcal{P}(H_k \mid E)$ in terms of the conditional probabilities $\mathcal{P}(E \mid H_k)$

$$\mathcal{P}(H_k \mid E) = \frac{\mathcal{P}(E \mid H_k)\,\mathcal{P}(H_k)}{\sum_{i=1}^{n} \mathcal{P}(E \mid H_i)\,\mathcal{P}(H_i)} \qquad k = 1, 2, \ldots n$$

If E is an event that can occur as a consequence of n *causes* H_k, each of which with probability $\mathcal{P}(H_k)$ (at least one greater than zero), then Bayes' formula provides the **posterior (updated) probability** $\mathcal{P}(H_k \mid E)$, namely the probability that having occurred the event E, this was generated by the cause H_k, in terms of the **prior probabilities** $\mathcal{P}(H_k)$ and of the **likelihood** $\mathcal{P}(E \mid H_k)$

it's not conditional

Conditional Probability

Example

A program crashes with a SEGFAULT, and it is presumed that this is due to a bug in any of 3 possible modules. Let $1 - \beta_k$ $(k = 1, 2, 3)$ denote the probability that the bug will be found upon a code review of the k^{th} module when the bug is, in fact, in that module. (The constants β_k are called *overlook probabilities*). What is the probability that the bug is in the k^{th} module, given that a review of module 1 is unsuccessful?

probability of bugs being found

Let: $H_k =$ "the bug is in the k^{th} module", and
$E =$ "code review of module 1 is unsuccessful"

$$\mathcal{P}(H_1 \mid E) = \frac{\mathcal{P}(E \mid H_1)\,\mathcal{P}(H_1)}{\sum_{k=1}^{3} \mathcal{P}(E \mid H_k)\,\mathcal{P}(H_k)} = \frac{\beta_1/3}{\beta_1/3 + 1/3 + 1/3} = \frac{\beta_1}{\beta_1 + 2}$$

$$\mathcal{P}(H_j \mid E) = \frac{\mathcal{P}(E \mid H_j)\,\mathcal{P}(H_j)}{\sum_{k=1}^{3} \mathcal{P}(E \mid H_k)\,\mathcal{P}(H_k)} = \frac{1/3}{\beta_1/3 + 1/3 + 1/3} = \frac{1}{\beta_1 + 2} \qquad (j = 2, 3)$$

For example: $\beta_1 = 0.4 \quad \Rightarrow \quad \mathcal{P}(H_1 \mid E) = 1/6, \quad \mathcal{P}(H_2 \mid E) = 5/12 \simeq 0.42$

Hypothesis of bugs being found: $\{H_1, H_2, H_3\}$

$H_k \longrightarrow$ the bug is in the k^{th} module

$\hookrightarrow$ is a partition

$E \longrightarrow$ code review of module #1 is unsuccessful

For this problem, $P(H_1 | E)$ is asked. Assume that the probability is equally likely, hence:
$\boxed{P(H_1)} = 1/3$, $P(H_2) = 1/3$, $P(H_3) = 1/3$.

Say β_1 is the probability of a bug in module #1 overlooked.

$$P(H_1 | E) = \frac{P(E|H_1) \cdot P(H_1)}{P(E|H_1)P(H_1) + P(E|H_2)P(H_2) + P(E|H_3)P(H_3)}$$

Assume for a moment that the bug is in mod. 1

$$P(H_1 | E) = \frac{\beta_1 \cdot 1/3}{\beta_1 \cdot 1/3 + 1 \cdot 1/3 + 1 \cdot 1/3} = \frac{\beta_1/3}{\beta_1/3 + 2/3} = \frac{\beta_1}{\beta_1 + 2}$$

If $\beta_1 = 0.2$, $P(H_1 | E) = \frac{0.2}{0.2 + 2} = \frac{0.2}{2.2} \simeq 0.1$

Keep in mind that H_k is an assumption that a bug is in k-th module. Hence, the probability of not finding a bug in module 1 given that the bug is actually in module 2 is 1.

Conditional Probability

Example

- Note that the updated (that is, the conditional) probability that the bug is in module j, given the information that a review of module 1 did not find it, is greater than the initial probability that it was in module j when $j \neq 1$ and is less than the initial probability when $j = 1$. This statement is certainly intuitive, since not finding the bug in module 1 would seem to decrease its chance of being in that module and increase its chance of being elsewhere
- Further, the conditional probability that the bug is in module 1 given an unsuccessful search of that module is an increasing function of the overlook probability β_1. This statement is also intuitive, since the larger β_1 is, the more it is reasonable to attribute the unsuccessful search to "bad luck" as opposed to the bug's not being there. Similarly, $\mathcal{P}\left(H_2 \mid E\right)$ and $\mathcal{P}\left(H_3 \mid E\right)$ are decreasing functions of β_1

Conditional Probability

Example

Let's assume that 40% of the devices are produced by line A, and 60% by other lines. It turns out that 25% of the devices produced by A are faulty, and only 7% of the others are faulty

1. What is the probability that a component chosen at random is faulty?
2. What is the probability that a faulty component is produced by line A?

Let A and $\overline{A}$ be the events "component produced by line A" and "component produced by some other line", respectively

In the first case, making use of the formula of the total probabilities

$$\mathcal{P}(F) = \mathcal{P}(F \mid A)\, \mathcal{P}(A) + \mathcal{P}\left(F \mid \overline{A}\right)\, \mathcal{P}\left(\overline{A}\right) = 0.25 \cdot 0.4 + 0.07 \cdot 0.6 = 0.142$$

In the second case, making use of the Bayes' formula

$$\mathcal{P}(A \mid F) = \frac{\mathcal{P}(F \mid A)\, \mathcal{P}(A)}{\mathcal{P}(F)} = \frac{0.25 \cdot 0.4}{0.142} \simeq 0.704$$

First case

Say A events : component produced by line A

$\bar{A}$ events : ——— " ——— other lines

F : faulty products

A and $\bar{A}$ form partitions.

$$P(F) = P(F|A)\,P(A) + P(F|\bar{A})\,P(\bar{A})$$
$$= 0.25 \cdot 0,4 + 0.07 \cdot 0.6 = 0.142$$

second case

$$P(A|F) = \frac{P(F|A) \cdot P(A)}{P(F)} = \frac{0.25 \cdot 0.4}{0.142} \simeq 0.704$$

Conditional Probability

Two events A and B are *independent* when the occurrence of either one does not influence the occurrence of the other one. Making use of the definition of conditional probability, we can rigorously define this property

Definition (Independent events)

The events A and B are said to be *independent* if

$$\mathcal{P}\left(B \mid A\right) = \mathcal{P}\left(B\right) \qquad\qquad \mathcal{P}\left(A \mid B\right) = \mathcal{P}\left(A\right)$$

It is also easily seen that

$$\mathcal{P}\left(B \mid A\right) = \frac{\mathcal{P}\left(A \cap B\right)}{\mathcal{P}\left(A\right)} \equiv \mathcal{P}\left(B\right) \qquad \Rightarrow \qquad \mathcal{P}\left(A \cap B\right) = \mathcal{P}\left(A\right)\mathcal{P}\left(B\right)$$

It can be useful to take the last relation as definition of independence (it does not need the assumption $\mathcal{P}\left(A\right), \mathcal{P}\left(B\right) > 0$)

The definition of independence of two events is generalized to the case of three events as follows

Theorem

Three events A, B and C are said to be independent if

$$\mathcal{P}(A \cap B \cap C) = \mathcal{P}(A)\,\mathcal{P}(B)\,\mathcal{P}(C)$$
$$\mathcal{P}(A \cap B) = \mathcal{P}(A)\,\mathcal{P}(B)$$
$$\mathcal{P}(A \cap C) = \mathcal{P}(A)\,\mathcal{P}(C)$$
$$\mathcal{P}(B \cap C) = \mathcal{P}(B)\,\mathcal{P}(C)$$

Definition (Dependent events)

When two or more events are not independent, they are said to be *dependent*

Conditional Probability

Example

In answering a question on a multiple-choice test, a student either knows the answer (probability p) or guesses (probability $1-p$). Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered correctly?

Let C and K denote, respectively, the events "the student answers the question correctly" and "the student actually knows the answer"

$$\begin{aligned}\mathcal{P}(K \mid C) &= \frac{\mathcal{P}(C \mid K)\,\mathcal{P}(K)}{\mathcal{P}(C \mid K)\,\mathcal{P}(K) + \mathcal{P}\left(C \mid \overline{K}\right)\,\mathcal{P}\left(\overline{K}\right)} \\ &= \frac{1 \cdot p}{1 \cdot p + (1/m)(1-p)} = \frac{mp}{1+(m-1)\,p}\end{aligned}$$

For example, if $m = 5$ and $p = 1/2$, then $\mathcal{P}(K \mid C) = 5/6$

Conditional Probability

Example (Diagnostic test, I)

In a population, one person out of 100 000 has a rare virus. An extensively experimented test can correctly diagnose the virus in 97% of the cases (namely, there is a 3% of false positives/negatives). If it turns out that a person is positive to the test, what is the probability that the person is really affected by the virus?

V: "the person is affected by the virus", $\mathcal{P}(V) = 0.00001$

$\overline{V}$: "the person is not affected by the virus", $\mathcal{P}\left(\overline{V}\right) = 0.99999$

T: "the test is positive", $\mathcal{P}(T \mid V) = 0.97$, $\mathcal{P}\left(T \;\middle|\; \overline{V}\right) = 0.03$

$$\mathcal{P}(V \mid T) = \frac{\mathcal{P}(T \mid V)\,\mathcal{P}(V)}{\mathcal{P}(T \mid V)\,\mathcal{P}(V) + \mathcal{P}\left(T \;\middle|\; \overline{V}\right)\,\mathcal{P}\left(\overline{V}\right)} \simeq 0.00032$$

Conditional Probability

The previous result might be at first surprising: When the test is positive, there is a very small probability that the person is actually affected by the virus. This seems to be in contrast with the fact that the test seems to be reliable (after all, it gives a wrong result in only 3% of the cases!). How is that possible?

This result has a very clear explanation. Even though the test is correct 97% of the times, there are only few people affected by the virus. This means that the vast majority of the people are healthy, and 3% of these people will be mistakenly diagnosed the virus. This “false positive” will be, in absolute terms, a large number, compared to the “true positive”, which are a small part of the population

This results is well-known in Statistics (and in Medical Science) and is part of the reason why rare diseases are difficult to diagnose

Conditional Probability

Example (Diagnostic test)

You wake up and feel a little sick. You go the doctor and she recommends a battery of tests. When results come back, it turns that you tested positive to a rare disease that affects 0.1% of the population. The doctor tells you that this test correctly identifies 99% of people that have the disease, and only incorrectly identifies 1% of people who don't have the disease.

Sounds pretty bad, but we already know that the probability that you have the disease is *not* 99%. Let's call H the hypothesis (having the disease), and E the event (positive test)

$$\mathcal{P}(H \mid E) = \frac{\mathcal{P}(E \mid H)\mathcal{P}(H)}{\mathcal{P}(E \mid H)\mathcal{P}(H) + \mathcal{P}\left(E \mid \overline{H}\right)\mathcal{P}\left(\overline{H}\right)} = \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.01 \cdot 0.999} \simeq 0.09$$

What to do? You decide to take another test! (Of course you go to a different lab, just to make sure that the results are independent)

Conditional Probability

Example (Diagnostic test)

So you get the test run again, and let's say that test also comes back as positive. Now what is the probability that you actually have the disease?

Well, you can use Bayes formula again

$$\mathcal{P}(H \mid E) = \frac{\mathcal{P}(E \mid H)\mathcal{P}(H)}{\mathcal{P}(E \mid H)\mathcal{P}(H) + \mathcal{P}\left(E \mid \overline{H}\right)\mathcal{P}\left(\overline{H}\right)} = \frac{0.99 \cdot 0.09}{0.99 \cdot 0.09 + 0.01 \cdot 0.91} \simeq 0.907$$

except this time for your **prior probability** that you have the disease, you have to put in the probability that we worked out before which is 9% (posterior probability), because you've already had one positive test. If you crunch those numbers, the new probability based on two positive tests is almost 91%

Conditional Probability

One of the many applications of Bayesian theory is in **Bayesian filters**

Example (Spam Filtering)

Traditional spam filters actually do a kind of bad job, there's too many false positives, too much of your email ends up in spam, but using a Bayesian filter, you can look at the various words that appear in e-mails, and use Bayes' Theorem to give a probability that the email is spam, given that those words appear

$$\mathcal{P}\left(\text{spam} \mid \text{word}\right) = \frac{\mathcal{P}\left(\text{word} \mid \text{spam}\right)\mathcal{P}\left(\text{spam}\right)}{\mathcal{P}\left(\text{word}\right)}$$

Example (Let's Make a Deal)

In the well known TV program "*Let's Make a Deal*", a contestant is shown three closed doors. Behind one of the doors there is **car**, and behind the other two doors there are **goats**. The contestant has to choose one of the doors, winning the corresponding item hidden behind that door. After the contestant has chosen one door – before opening it – the host of the game, Monty Hall (who knows where the car is), opens one of the remaining two doors, revealing one goat, and offers the contestant the possibility to change the selected door.

What is the better strategy to win the car? Should the contestant change door or keep the chosen one?

Conditional Probability

Example (Let's Make a Deal)

Let's assume, without loss of generality, that the contestant has chosen door 1, and Monty has opened door 3. The probability of winning the car after changing the door (from 1 to 2) is

$$\begin{aligned}\mathcal{P}(C_2 \mid M_3) &= \frac{\mathcal{P}(M_3 \mid C_2)\,\mathcal{P}(C_2)}{\mathcal{P}(M_3)} \\ &= \frac{\mathcal{P}(M_3 \mid C_2)\,\mathcal{P}(C_2)}{\mathcal{P}(M_3 \mid C_1)\,\mathcal{P}(C_1) + \mathcal{P}(M_3 \mid C_2)\,\mathcal{P}(C_2) + \mathcal{P}(M_3 \mid C_3)\,\mathcal{P}(C_3)}\end{aligned}$$

C_k = "the car is behind door k",
M_k = "Contestant chooses door 1; Monty opens door k"

$\mathcal{P}(C_k) = 1/3$ (*prior* probability is the same for all doors)

$\mathcal{P}(M_3 \mid C_1) = 1/2$ (Monty can choose amongst doors 2 and 3)

$\mathcal{P}(M_3 \mid C_2) = 1$ (Monty can only open door 3)

$\mathcal{P}(M_3 \mid C_3) = 0$ (Monty cannot open the door with the car)

Conditional Probability

Example (Let's Make a Deal)

Let's assume, without loss of generality, that the contestant has chosen door 1, and Monty has opened door 3. The probability of winning the car after changing the door (from 1 to 2) is

$$\mathcal{P}(C_2 \mid M_3) = \frac{\mathcal{P}(M_3 \mid C_2)\,\mathcal{P}(C_2)}{\mathcal{P}(M_3)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{2}{3}$$

C_k = "the car is behind door k", M_k = "Monty opens door k"

$\mathcal{P}(C_k) = 1/3$ (*prior* probability is the same for all doors)

$\mathcal{P}(M_3 \mid C_1) = 1/2$ (Monty can choose amongst doors 2 and 3)

$\mathcal{P}(M_3 \mid C_2) = 1$ (Monty can only open door 3)

$\mathcal{P}(M_3 \mid C_3) = 0$ (Monty cannot open the door with the car)

Conditional Probability

Example (Let's Make a Deal)

Alternate solution. There are three possible scenarios, each of which has probability $1/3$

I. The contestant chooses goat No.1. Monty reveals goat No.2. Changing door leads to the **car**

II. The contestant chooses goat No.2. Monty reveals goat No.1. Changing door leads to the **car**

III. The contestant chooses the car. Monty reveals one goat (no matter which). Changing door leads to one **goat**

Changing door leads to win the car 2/3 of the times!