Modes of Linear Systems

Master degree in Automation Engineering

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Dynamic modes of linear systems

By bearing in mind the Lagrange formula we observe that the exponential of the state matrix A for C-T systems and the power of it for D-T systems play a role in determining the state evolution.

$$x(t) = e^{A(t-t_0)}\, x(t_0) + \int_{t_0}^{t} e^{A(t-s)}\, B\, u(s)\, ds \qquad\qquad x(t) = A^{t-t_0}\, x(t_0) + \sum_{i=t_0}^{t} A^{t-1-i}\, B\, u(i)\, ds$$

Notation: Transition matrix $\phi(t) = \begin{cases} e^{At} & \text{for C-T systems} \\ A^t & \text{for D-T systems} \end{cases}$

$$\boxed{x(t) = \phi(t)\, x(t_0) \; + \; \Psi(t)\, u([0,t))}$$

Idea: can we obtain a “simple” insight about the transition matrix by playing with the coordinates changes?

$$e^{\tilde{A}t} := e^{TAtT^{-1}} = T\, e^{At}\, T^{-1}$$

$$e^{At} = T^{-1}\, e^{\tilde{A}t}\, T$$

$$\tilde{A}^t := (T A T^{-1})^t = (T A^t T^{-1})$$

$$A^t := (T^{-1} \tilde{A}^t T)$$

Previously, we say $\tilde{A} = TAT^{-1}$, where T is the matrix of coordinate change. How to proof that $e^{\tilde{A}t} \sim e^{At}$?

$$e^{\tilde{A}t} = I + \tilde{A}t + \frac{1}{2!}\tilde{A}^2 t^2 + \dots =$$

$$\underbrace{I}_{TT^{-1}} + TAT^{-1}t + \frac{1}{2!}(TA\underbrace{T^{-1})(T}_{I}AT^{-1})t^2 + \dots =$$

$$T\underbrace{\left(I + At + \frac{1}{2!}A^2t^2 + \dots\right)}_{e^{At}}T^{-1}$$

Then, the following equation is fulfilled:

$$e^{\tilde{A}t} := e^{TAtT^{-1}} = Te^{At}T^{-1}$$
$$e^{At} = T^{-1}e^{\tilde{A}t}T$$

The Continuous-Time case

Instead of describing the system with the A, B, C matrix, we describe it by its transfer func.

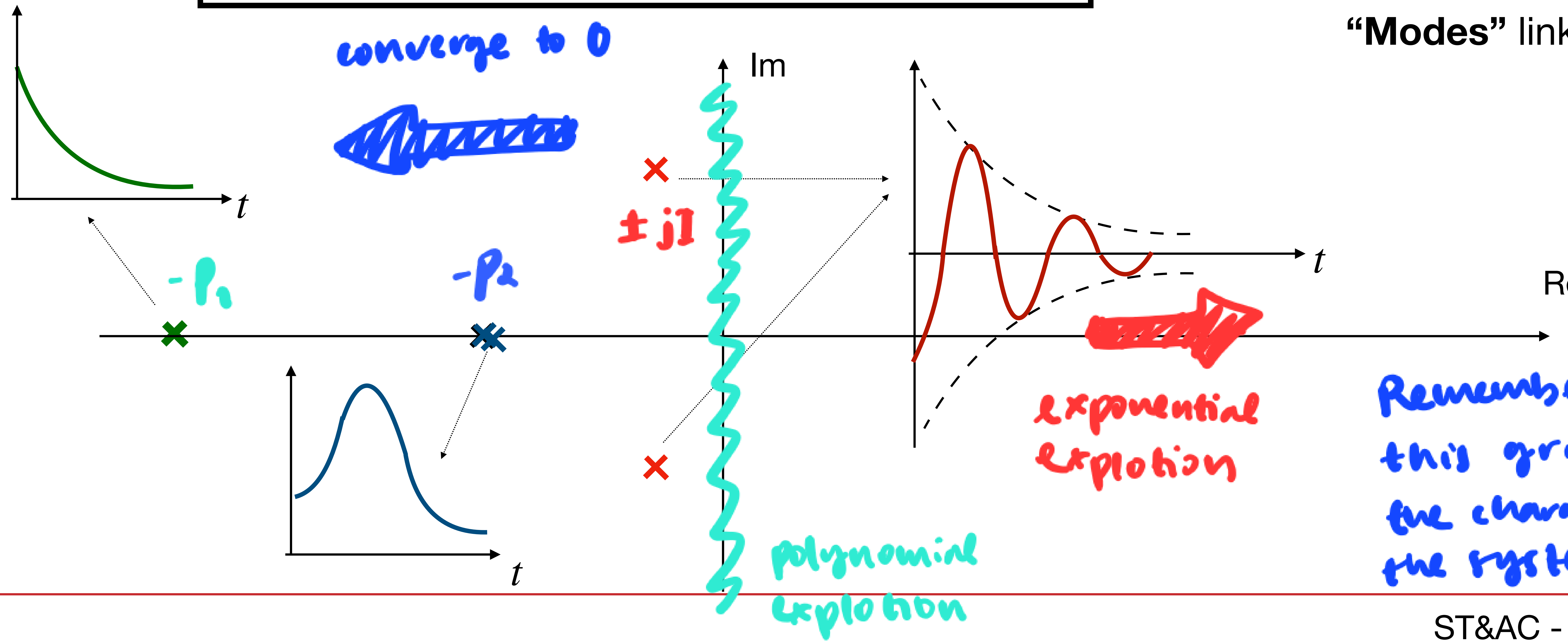
Quick recall: the transfer function case $G(s) = C(sI - A)^{-1}B$

= effect of initial condition

$$y(t) = c_1\, e^{-p_1 t} + (c_{22} + c_{23}t)\, e^{-p_2 t} + c_3 e^{-Rt} \sin(It + \varphi)$$

$$G(s) = \frac{N(s)}{(s+p_1)\,(s+p_2)^2\,(s+R+jI)(s+R-jI)}$$

"Modes" linked to the poles



Remember that this graph gives the characteristic of the system

The Continuous-Time case

How to identify “modes” for SS representations? **Idea**: let's diagonalise A

$$A = T^{-1}\tilde{A}T \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad \tilde{A} = TAT^{-1}$$

$$e^{\tilde{A}t} = T\,e^{At}\,T^{-1}$$
$$e^{At} = T^{-1}\,e^{\tilde{A}t}\,T$$

Remark: Eigenvalues are coordinates independent: $\sigma(A) = \sigma(\tilde{A})$

Consider first the case in which A is diagonalisable ($\sum_{i=1}^{r} g_i = n$) :

$\tilde{A} = \text{diag}(\lambda_1, \cdots, \lambda_n)$

with all of the eigenvalues

$$e^{\tilde{A}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$$

$$e^{At} = T^{-1}\,e^{\tilde{A}t}\,T$$

Each element of transition matrix in the original coordinates is a linear combination of the scalar exponential governed by the specific eigenvalues (elementary mode)

The Continuous-Time case

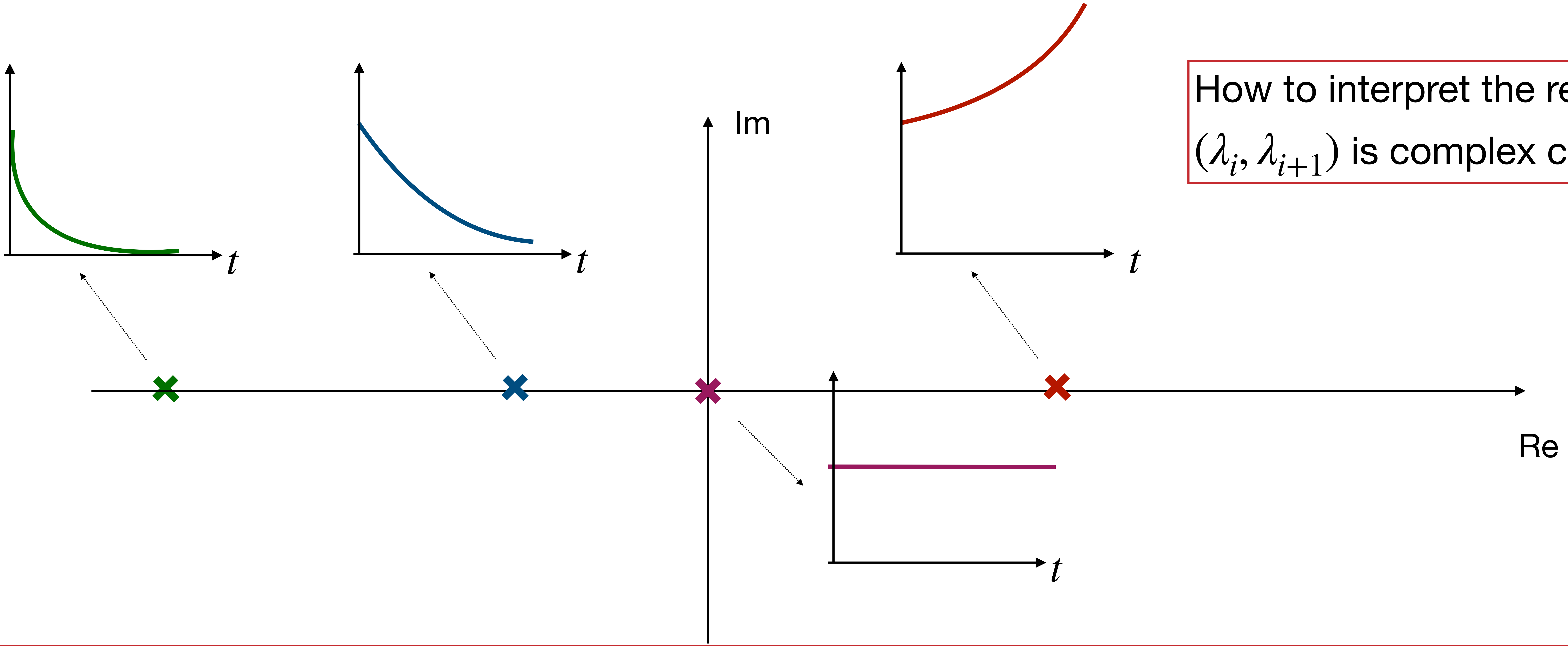
$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \qquad x(0) = x_0 \longrightarrow y(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}$$

dependent on x_0

consider that:
$x(t) = e^{At} x_0$

"Modes" linked to the eigenvalues of A

Remark: even if $a_i > 1$ the previous considerations apply (provided that $g_i = a_i$)



How to interpret the result if one pair $(\lambda_i, \lambda_{i+1})$ is complex coniugate ?

The Continuous-Time case

How to interpret the result if one pair $(\lambda_i, \lambda_{i+1})$ is complex conjugate ?

"Magically" cancel out in original coordinates (as an effect of pre/post multiplication by T/T^{-1})

$$e^{a_i t}(\cos(b_i t) + j\sin(b_i t)) \qquad e^{a_i t}(\cos(b_i t) - j\sin(b_i t))$$

By Euler formula

$$\tilde{A} = \text{diag}(\lambda_1, \cdots, \lambda_i, \lambda_i^\star, \cdots \lambda_n)$$

$$e^{\tilde{A}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & & \cdot & & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & e^{(a_i+jb_i)t} & 0 & & \cdot \\ & & 0 & e^{(a_i-jb_i)t} & & \\ \cdot & \cdot & & \cdot & & \cdot \\ 0 & 0 & & \cdot & & e^{\lambda_n t} \end{pmatrix}$$

complex

$$e^{At} = T^{-1} e^{\tilde{A}t} T$$

complex

real

$$e^{(a_i + jb_i)t} = e^{a_i t} \cdot e^{j b_i t} \Rightarrow \cos b_i t + j \sin b_i t$$

Example: $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

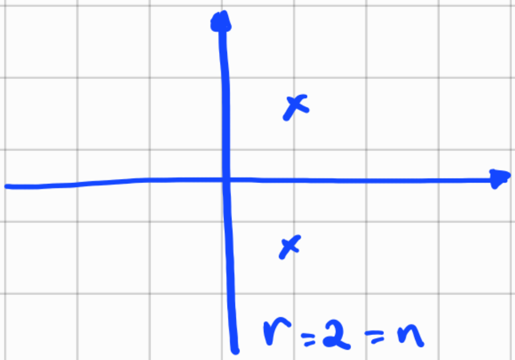
Suppose $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, what is $e^{At}x_0$?

Is it diagonalizable?

* $\det(\lambda I - A) = 0 \iff \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 + 1 = 0$

$\underbrace{\lambda_1 = 1 + j, \quad \lambda_2 = 1 - j}_{\text{two distinct eigenval.}}$

in complex conjugate



$r = 2 = n$

$a_1 = g_1 = 1$

$a_2 = g_2 = 1$

* $(\lambda_1 I - A)\, v_1 = 0 \iff \begin{bmatrix} j & 1 \\ 1 & j \end{bmatrix} \underbrace{\begin{bmatrix} j \\ -1 \end{bmatrix}}_{\substack{v_1 \text{ or} \\ \text{the elem.} \\ \text{of kernel}}} = 0$

* $(\lambda_2 I - A)\, v_2 = 0 \qquad v_2 = \begin{bmatrix} -j \\ -1 \end{bmatrix}$

* $T^{-1} = \begin{bmatrix} j & -j \\ -1 & -1 \end{bmatrix} \Longrightarrow \tilde{A} = TAT^{-1} = \begin{bmatrix} 1+j & 0 \\ 0 & 1-j \end{bmatrix}$

Homework, compute $T = (T^{-1})^{-1}$

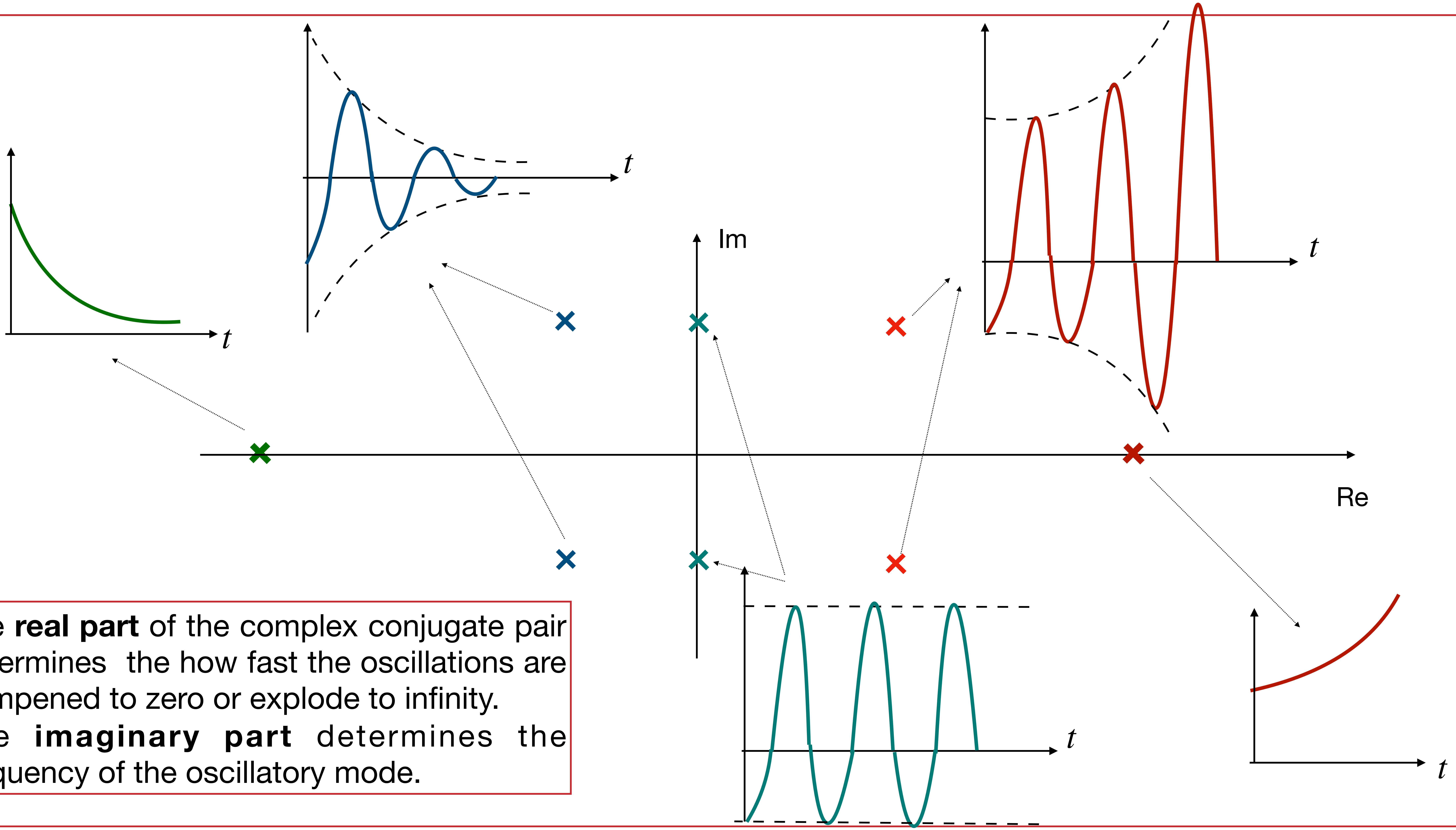
* $e^{\tilde{A}t} = \begin{bmatrix} \underset{\#}{e^{(1+j)t}} & 0 \\ 0 & \underset{\#}{e^{(1-j)t}} \end{bmatrix} \Longrightarrow \boxed{e^{At} = T^{-1} e^{\tilde{A}t} T}$

$\# = e^{t}(\cos(t) + j\sin(t))$

$\# = e^{t}(\cos(t) - j\sin(t))$

Eventually, $e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} e^{t}$

The Continuous-Time case



The **real part** of the complex conjugate pair determines the how fast the oscillations are dampened to zero or explode to infinity.
The **imaginary part** determines the frequency of the oscillatory mode.

The Continuous-Time case

Consider now the case in which A is not diagonalisable ($\sum_{i=1}^{r} g_i < n$) :

Jordan form

$$\tilde{A} = TAT^{-1} = \underbrace{\begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_r \end{pmatrix}}_{n \times n} \qquad J_i = \underbrace{\begin{pmatrix} J_{i1} & 0 & 0 & \cdots & 0 \\ 0 & J_{i2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{ig_i} \end{pmatrix}}_{a_i \times a_i} \quad i = 1 \ldots r$$

$$J_{ik} = \underbrace{\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}}_{n_{ik} \times n_{ik}} \quad k = 1 \ldots g_i$$

$$e^{\tilde{A}t} = \begin{pmatrix} e^{J_1 t} & 0 & \cdots & 0 \\ 0 & e^{J_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_r t} \end{pmatrix}$$

$$e^{J_i t} = \begin{pmatrix} e^{J_{i1} t} & 0 & \cdots & 0 \\ 0 & e^{J_{i2} t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{J_{ig_i} t} \end{pmatrix} \quad i = 1 \ldots r$$

$$e^{J_{ik} t} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n_{ik}-1}}{(n_{ik}-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_{ik}-2}}{(n_{ik}-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} e^{\lambda_i t} \quad k = 1 \ldots g_i$$

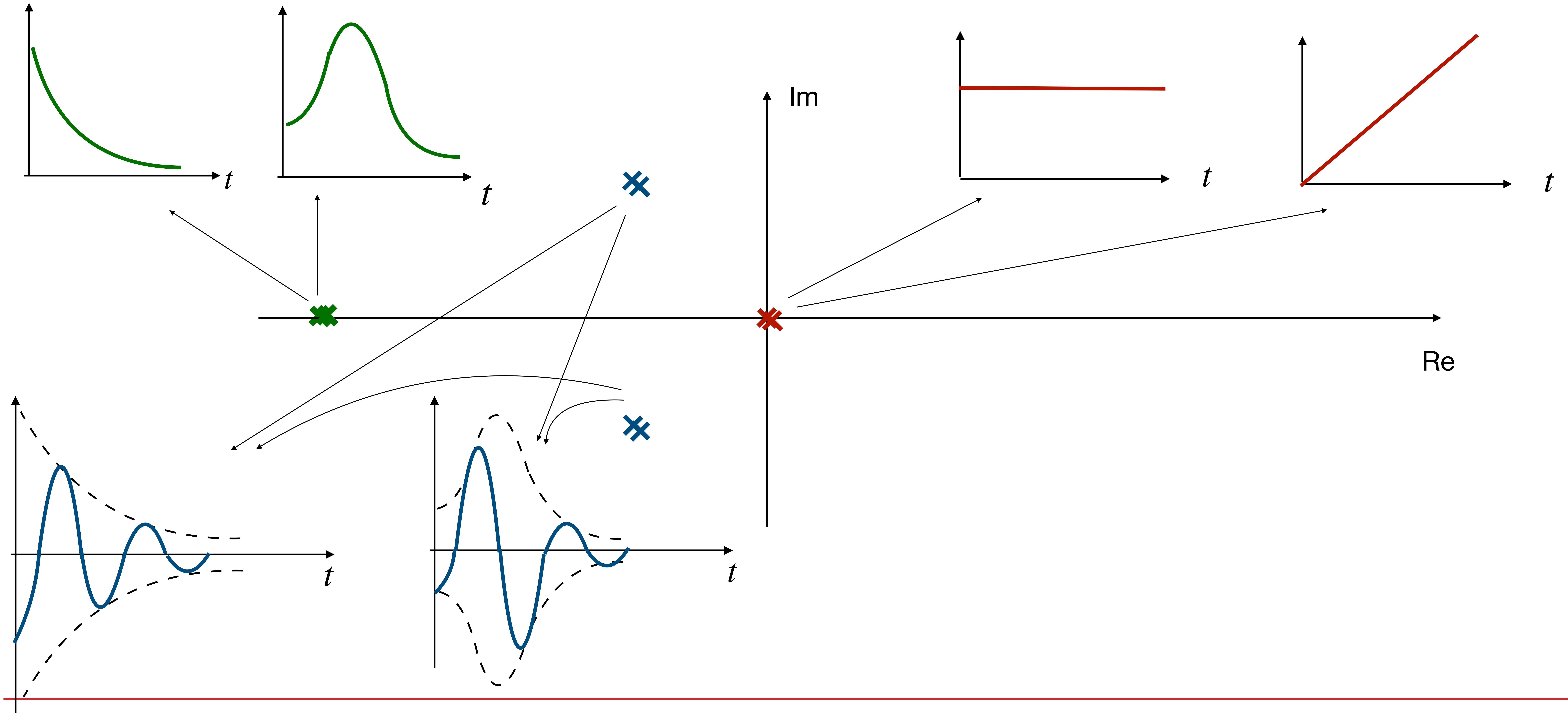
$$e^{At} = T^{-1} e^{\tilde{A}t} T$$

Suppose we have a mini Jordan block $n_{ik} = 2$.

$$e^{J_{ik} t} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{\lambda_i t}$$

The Continuous-Time case

$g_i = 1,\ a_i = 2$



Let's compute the eigenvalues from the last exercise. We got :

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\lambda_1 = 1$
- $\lambda_2 = 2$ $\rightarrow$ e^{2t} (from $e^{\lambda_2 t}$)
- $a_1 = 3$
- $a_2 = g_2 = 1$

- $g_1 = 2$ $\rightarrow$ $n_{11} = 2$ $\rightarrow$ $(1+t)e^{t}$
  $\rightarrow$ $n_{12} = 1$ $\rightarrow$ e^{t} (from $e^{\lambda_1 t}$)

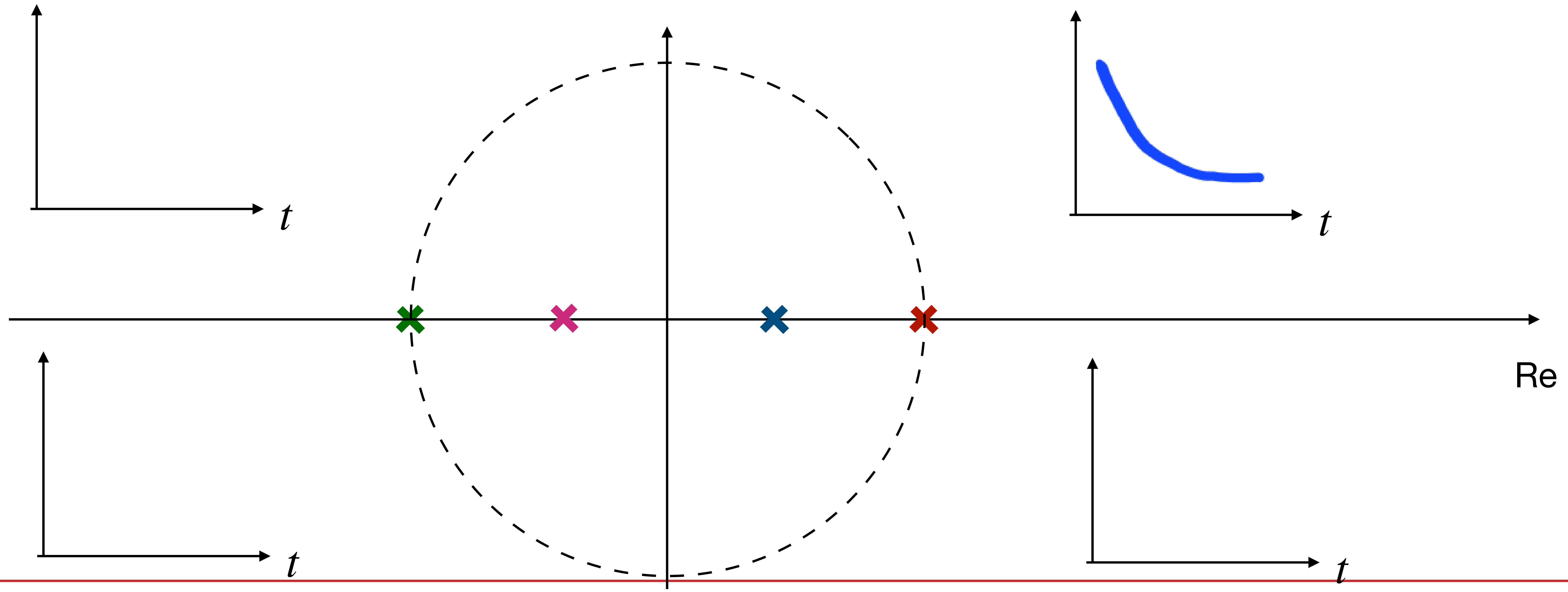
The Discrete -Time case

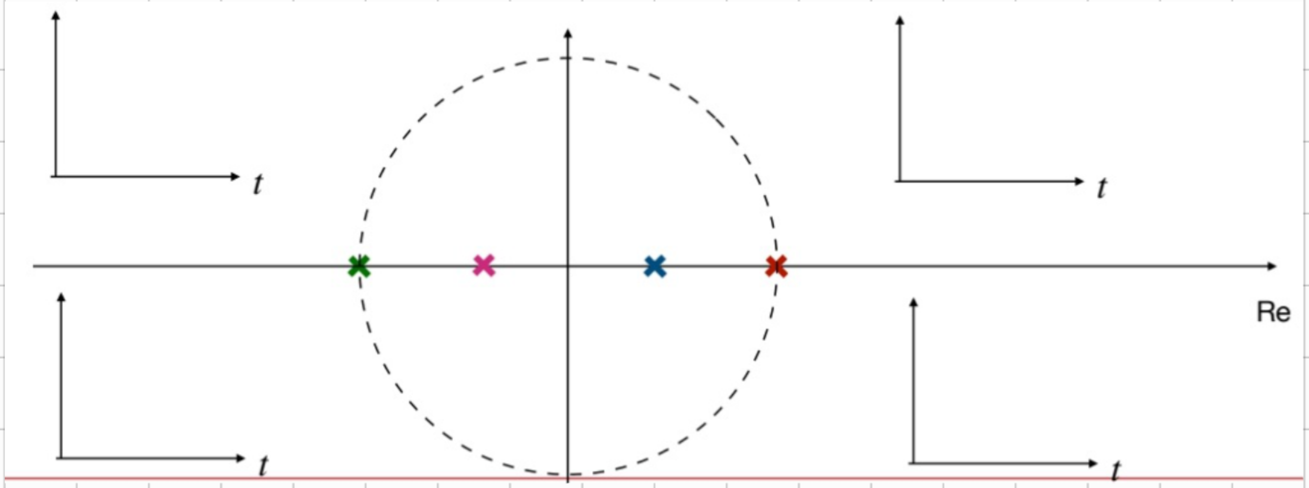
$$x(t+1) = Ax(t) \quad x(0) = x_0 \qquad\qquad x(t) = A^t x_0 \qquad\qquad A^t := (T^{-1}\tilde{A}^t T)$$

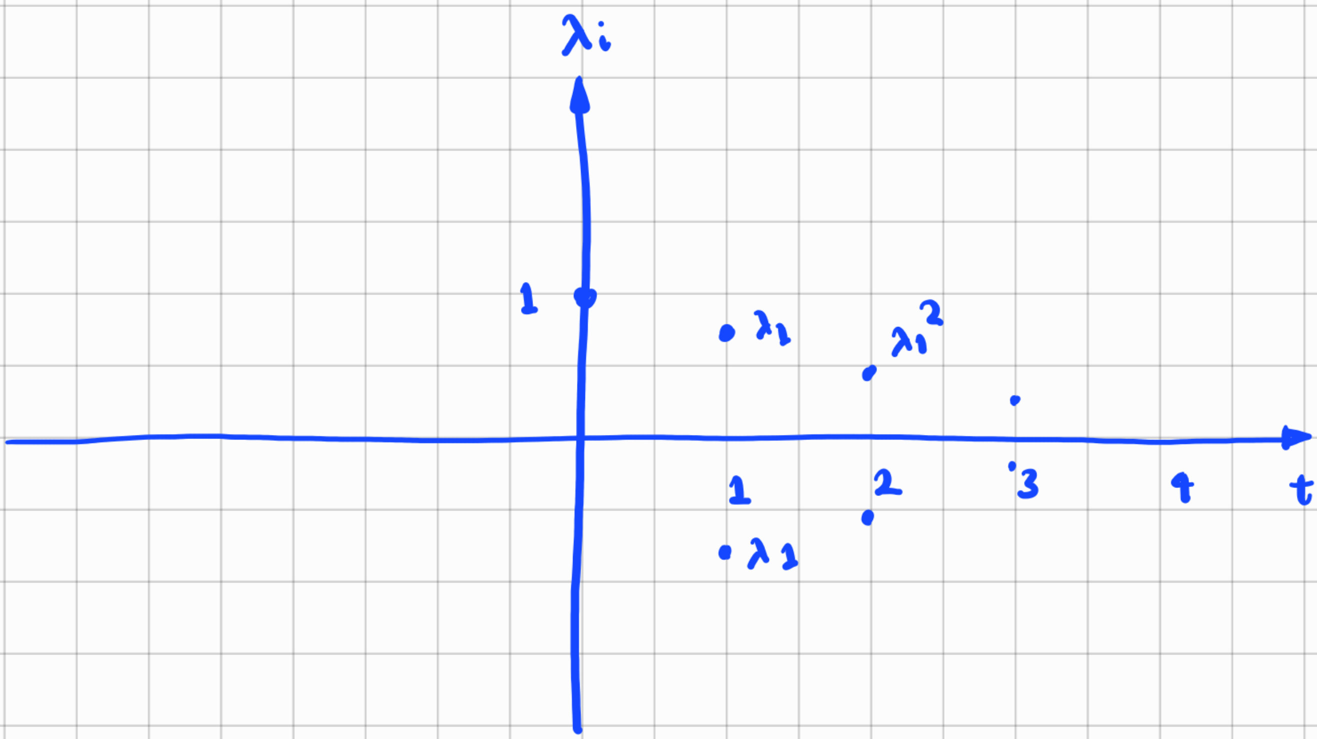
If A is diagonalisable:
$$\tilde{A}^t := \begin{pmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\cdots & \lambda_n^t \end{pmatrix}$$

Homework: develop and "plot" the case of complex conjugate eigenvalues and the one of eigenvalues with $g_i < a_i$ (non diagonalisable case)

Case of real λ_i:



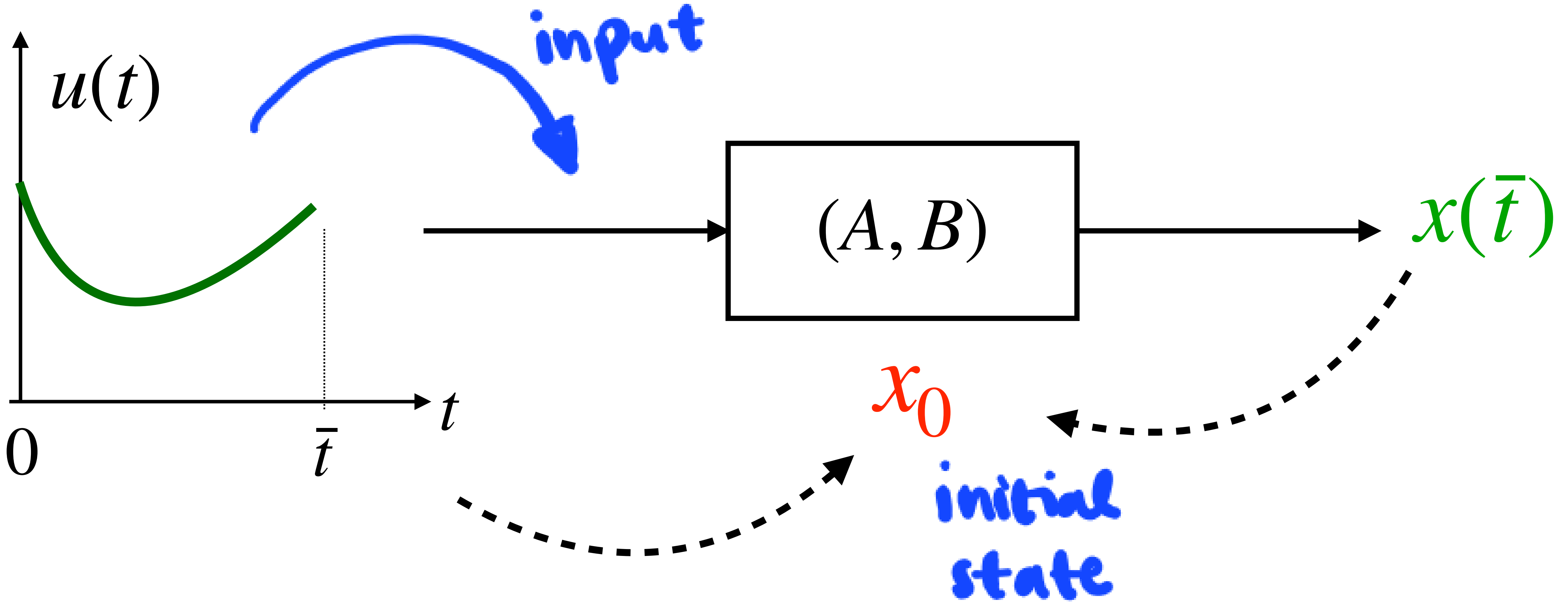

t
t
t
t
Re


λi
1
λ1
λ1²
1
2
3
4
t
λ1

Reversibility of a linear system

final state is known, and it is possible to find out the initial condition

Problem: Given a linear (C-T or D-T) system with initial condition x_0, is it possible to reconstruct the initial condition by knowing the state at a generic time $\bar{t} > 0$ and the applied input $u(t)$ in the interval $[0,\bar{t})$?



transition matrix

$$x(\bar{t}) = \phi(\bar{t})\, x(t_0) + \Psi(\bar{t})\, u([0,\bar{t}))$$

Known Unknown Known

Result: the system is reversible iff (*) the transition matrix $\phi(t)$ is invertible for all $t > 0$

e^{At} always invertible: $\boxed{(e^{At})^{-1} = e^{-At}}$

A^t could be singular

Result: All C-T systems are reversible. There could D-T systems not reversible.

Example $x_1(t+1) = x_2(t)$

$x_2(t+1) = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \forall\, t \geq 2$$

(*) iff = if and only if

$$x(t_0) = \underbrace{\phi^{-1}(\bar{t})}\left(x(\bar{t}) - \psi(t)\, u_{[0,\bar{t}]} \right)$$

transition matrix is invertible

where:

$$\phi(\bar{t}) = \begin{cases} e^{A\bar{t}} & \text{continuous-time} \\ A^{\bar{t}} & \text{discrete-time} \end{cases}$$

$$(e^{a\bar{t}})^{-1} = e^{-a\bar{t}} \iff (e^{M})^{-1} = e^{-M}$$

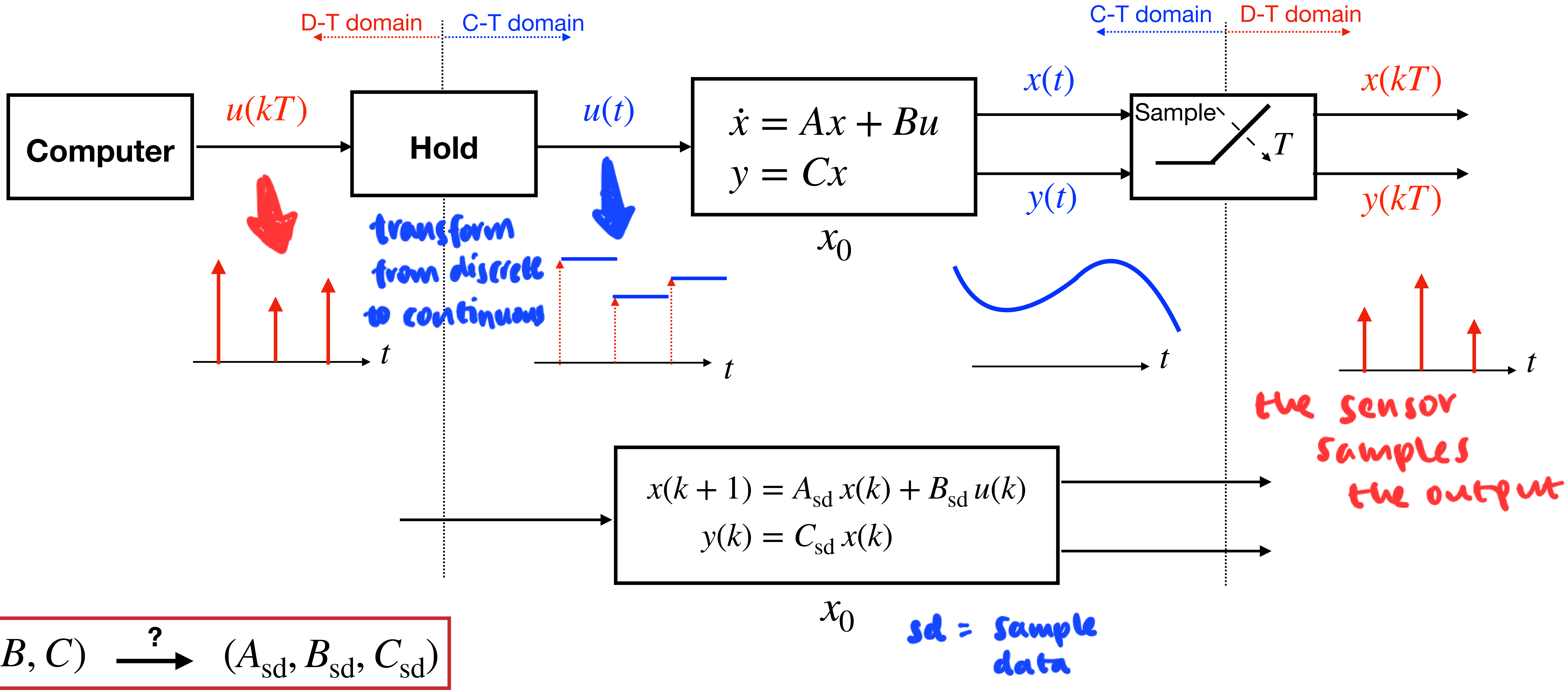
$$\Downarrow$$

$$e^{M} = I + M + \frac{1}{2!}M^2 + \dots$$

$$e^{-M} = I - M + \frac{1}{2!}M^2 - \dots$$

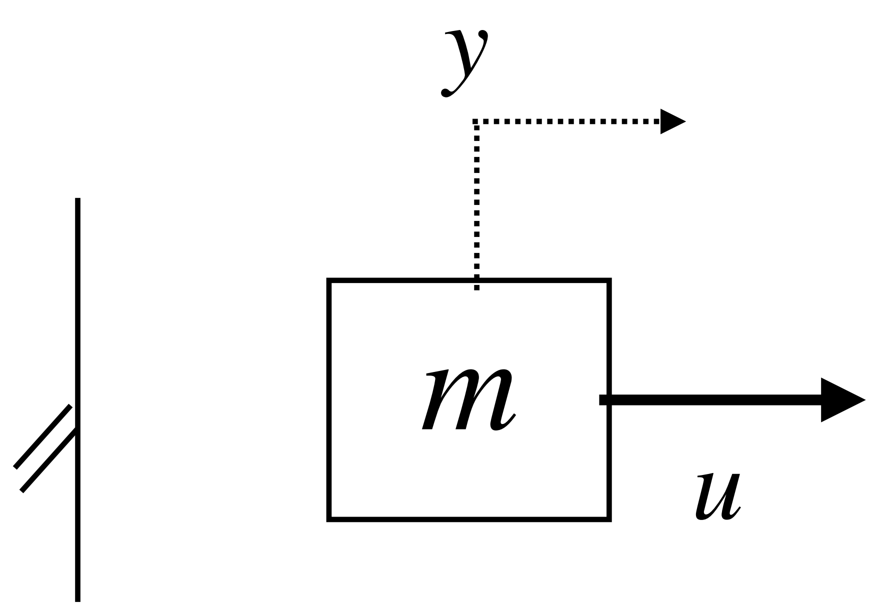
Sampled-data linear system

In the field of engineering there exists a relevant class of discrete time systems that is always reversible.



$$\boxed{(A, B, C) \xrightarrow{?} (A_{sd}, B_{sd}, C_{sd})}$$

Example:



Given A, B, C, T. How is the discrete time system always reversible?

$$\boxed{s' = \bar{t} + T - s}$$

$$x(t) = e^{A(t-\bar{t})} x(\bar{t}) + \underbrace{\int_{\bar{t}}^{t} e^{A(t-s)} B u(s) \, ds}_{\int_{\bar{t}}^{t} e^{A(t-s)} ds \, B u(\bar{t})}$$

where $t \in [\bar{t}, \bar{t}+T]$

$$\underbrace{x(\bar{t}+T)}_{x(k+1)} = \underbrace{\boxed{e^{AT}}}_{A_{sd}} \underbrace{x(\bar{t})}_{x(k)} + \underbrace{\boxed{\int_{\bar{t}}^{\bar{t}+T} e^{A(\underbrace{\bar{t}+T-s}_{s'})} \underbrace{ds}_{-ds'} \, B}}_{B_{sd}} \underbrace{u(\bar{t})}_{u(k)}$$

$y(t) = Cx(t)$

$y(\bar{t}) = Cx(\bar{t})$

$y(k) = \underbrace{C}_{C_{sd}} x(k)$

$$\boxed{\begin{aligned} A_{sd} &= e^{AT} \\ B_{sd} &= \int_0^T e^{As'}\,ds'\,B \\ C_{sd} &= C \end{aligned}}$$

$$\boxed{s' = \bar{t} + T - s}$$

$$\Downarrow$$

$$\phi(t) = \underbrace{e^{ATt}}_{A^t{}_{sd}} \implies \phi^{-1}(t) = e^{-ATt} \quad \forall t$$