Dynamic systems

Master degree in Automation Engineering

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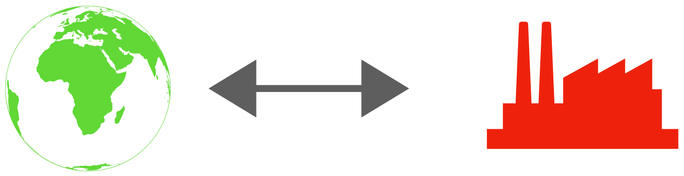
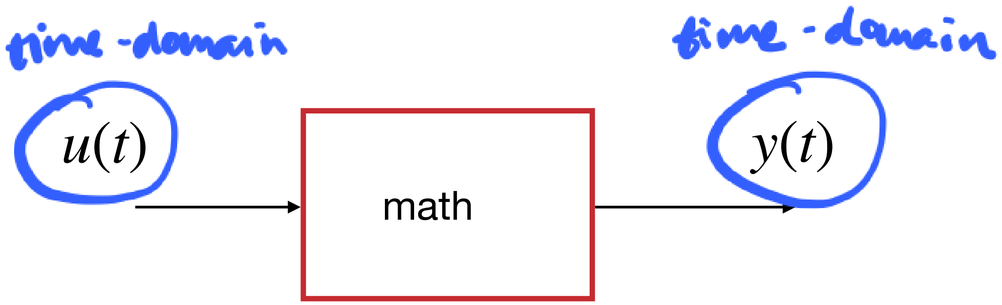
Dynamic systems representation

Context: We have a dynamic ("physical" or not) system (mechanical, electrical, hydraulic, social, economic..) and we look for a way to describe its behaviour by means of a mathematical model

The system interacts with the "surrounding" environment through **inputs** (through which the environment affects the systems) and **outputs** (through which the system affects the environment)



Abstraction, simplification



Dynamic…: the actual behaviour of the system depends on only on the actual input/state but also on the past data

Time domain: continuous-time, discrete-time

State-space representation

We shall adopt **state-space** representations to describe the dynamic system by means of **O**rdinary **D**ifferential **E**quations (ODEs) for continuous-time systems and **D**ifference **E**quations (DEs) for discrete-time systems

vector in n, m, or p dimension

Continuous-time systems ($t \in \mathbb{R}$)

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \qquad x(t_0) \in \mathbb{R} \quad \text{Typically assigned} \\ y(t) &= h(x(t), u(t), t) \qquad y \in \mathbb{R}^p \end{aligned}$$

n. input

n. output

Discrete-time systems ($t \in \mathbb{Z}$)

$$\begin{aligned} x(t+1) &= f(x(t), u(t), t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \qquad x(t_0) \in \mathbb{R} \quad \text{Typically assigned} \\ y(t) &= h(x(t), u(t), t) \qquad y \in \mathbb{R}^p \end{aligned}$$

- **Stationary** systems if the vector fields $f(\cdot)$ and $h(\cdot)$ do not depend on t ($t_0 = 0$ without loss of generality)
- **Linear** systems if the vector fields $f(\cdot)$ and $h(\cdot)$ are linear
- **Autonomous** systems if the vector fields $f(\cdot)$ and $h(\cdot)$ do not depend on independent variables $(u(t), t)$

Linear Systems

generically time-vary

ODEs $\dot{x}(t)$

DEs $x(t+1)$

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = A(t)x(t) + B(t)u(t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \qquad x(0) \in \mathbb{R}$$

$$y(t) = C(t)x(t) + D(t)u(t) \qquad y \in \mathbb{R}^p$$

Stationary linear systems

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

$$y(t) = Cx(t) + Du(t) \qquad y \in \mathbb{R}^p$$

In the course (system theory) we prevalently focus on stationary systems (mainly proper)

Proper linear systems

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) + Bu(t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

$$y(t) = Cx(t) \qquad y \in \mathbb{R}^p$$

Input/output not always present

Dynamic richness related to n

Autonomous linear systems

if there's no inputs affecting the output

$$\left.\begin{array}{r}\dot{x}(t)\\ x(t+1)\end{array}\right\} = Ax(t) \qquad x \in \mathbb{R}^n$$

$$y(t) = Cx(t) \qquad y \in \mathbb{R}^p$$

SISO - MIMO depending on m, p

Only ODEs, DEs? → focus on those two

ODEs, DEs could be not the appropriate math description tool to model certain physical systems:

Partial Differential Equations (PDEs)

$$\frac{\partial T(t,x)}{\partial x} = c\frac{\partial^2 T(t,x)}{\partial x^2}$$

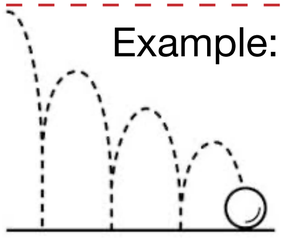
Example: heat transport dynamics

Hybrid systems (differential and difference equations) → mix of ODE and DE

sometimes two phenomenons couldn't be captured only with ODE

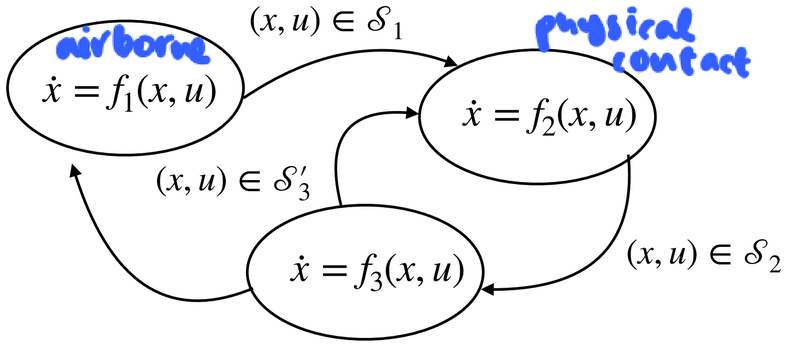
$$\begin{cases} \dot{x} = F(x,u) & (x,u) \in \mathscr{F} \\ x^+ = J(x,u) & (x,u) \in \mathscr{J} \end{cases}$$

Example: bouncing ball



Hybrid Automata

switching the state is possible



Example: drone airborne and in physical contact

Differential-Algebraic Equations (Descriptor systems)

$$\begin{cases} \dot{x} = F(x,u) \\ 0 = C(x,u) \end{cases}$$

Example: constrained mechanical systems

Non treated in the course!

FREE - EVOLUTION (Hybrid sys.)

$$x = \begin{pmatrix} x_1 & = \text{position} \\ x_2 & = \text{velocity} \end{pmatrix}$$

$$\dot{x}_1 = x_2$$

$$m\dot{x}_2 = -g$$

$\Rightarrow$ $\boxed{\text{when } x_1 \geq x_{1\,\text{floor}}, \text{ or when } x_1 = x_{1\,\text{floor}},\ x_2 > 0}$

Caligraphy F $\rightarrow$ $\mathcal{F}$

IMPACT

$$x^+ = x(t_{\text{impact}} + 1) = \begin{pmatrix} x_1^+ = x_1(t_{\text{impact}}) \\ x_2^+ = -c\, x_2(t_{\text{impact}}) \end{pmatrix}$$

where $c < 1$

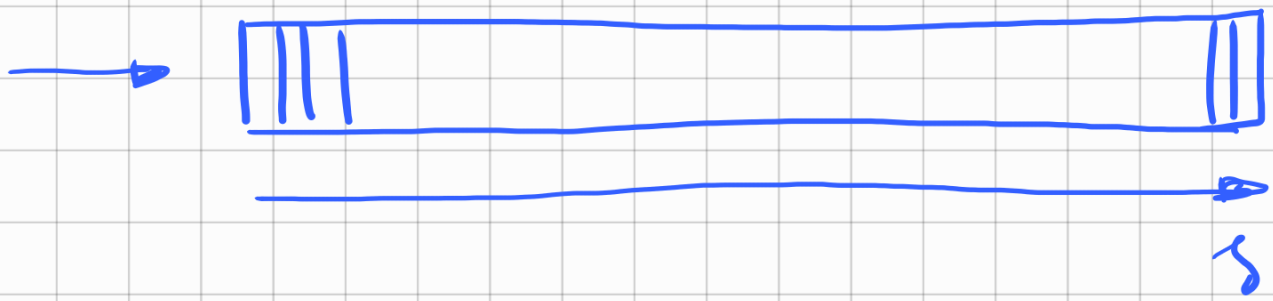
$\Rightarrow$ $\boxed{\text{when } x_1 = x_{1\,\text{floor}},\ x_2 < 0}$

caligraph I $\rightarrow$ $\mathcal{I}$

temperature funct.

Hes T

$T(s, t)$

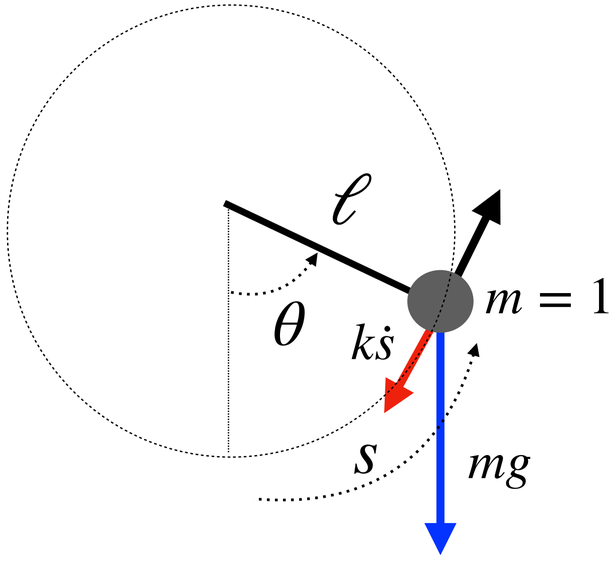


Where the system "lives"?

In the course we shall treat only systems whose state evolves on Euclidean spaces (distance -> Euclidean norm)

However, often the physics of the system implies that the state $x(t)$ does not evolve on an Euclidean space but rather on a "**manifold**" (topological space that locally resembles a an Euclidean space)

Example: Pendulum



Tangential force (gravity and friction)

$$F = ma$$

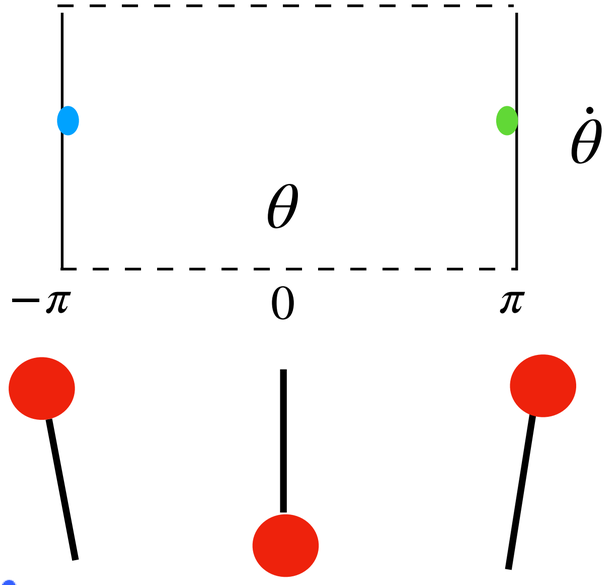
Tangential acceleration ($\ell\ddot{\theta}$)

$$x_1 := \theta,\ x_2 := \dot{\theta}$$

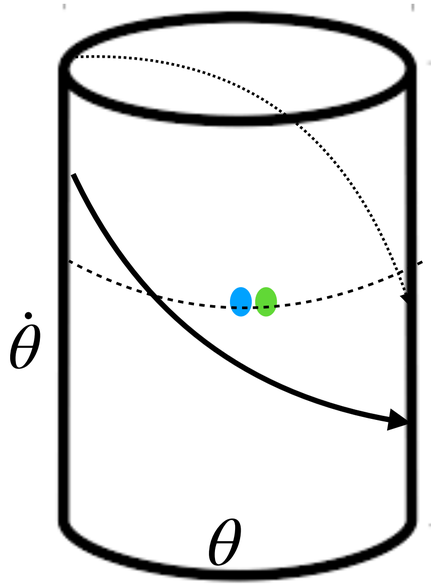
$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - k\ell x_2 \end{aligned}}$$

ℓ = length of the pendulum

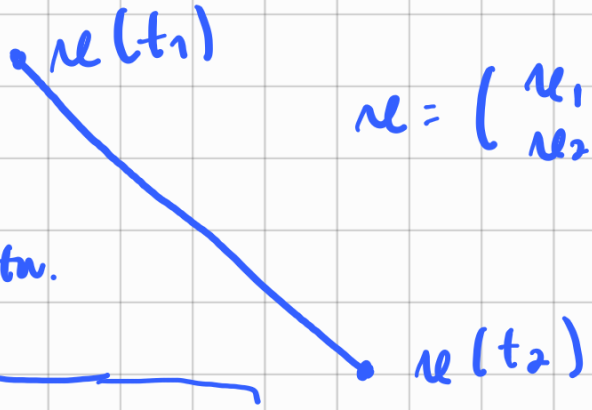
θ is not evolving on a euclidean space (distance between $\theta_1 = \pi - \epsilon$ and $\theta_2 = -\pi + \epsilon$ is not $2(\pi - \epsilon)$!)



The "supporting space" of the pendulum is the "cylinder" manifold $S^1 \times \mathbb{R}$



Euclidean Space



$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

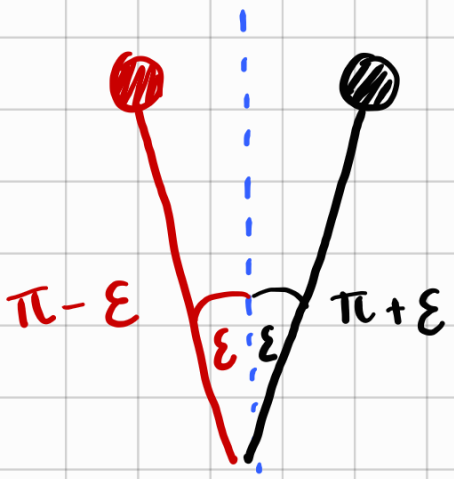
$n = 2$

the distance btw. two points =

$$\sqrt{(x_1(t_1) - x_1(t_2))^2 + (x_2(t_1) - x_2(t_2))^2}$$

Non-Euclidean space

Suppose the pendulum configuration

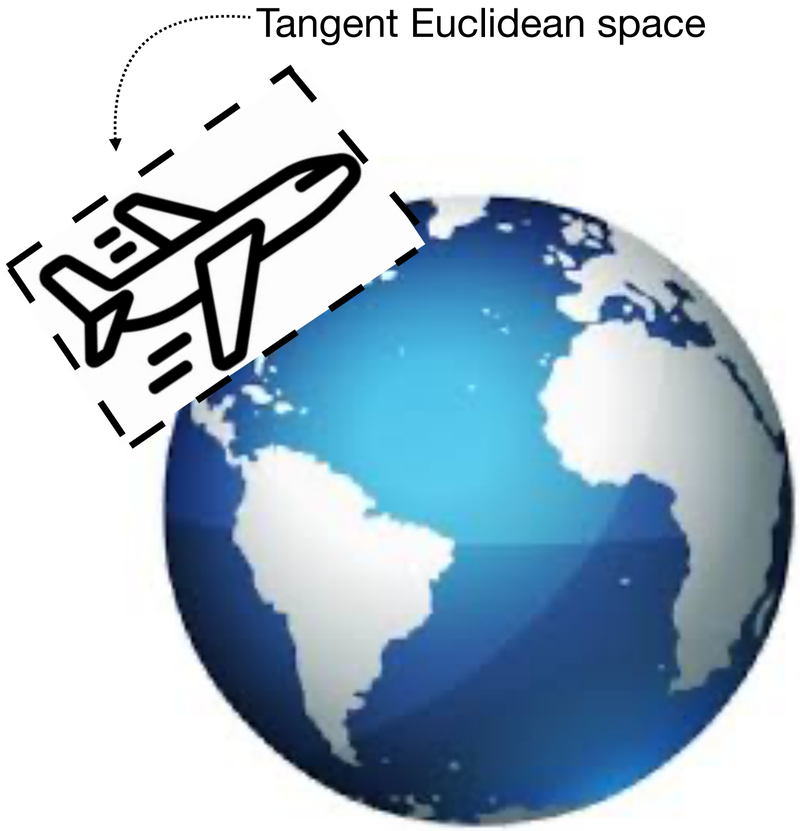


Maybe two configs are far away in Euclidian but the reality is NOT!

Manifold?

Where the system “lives”?

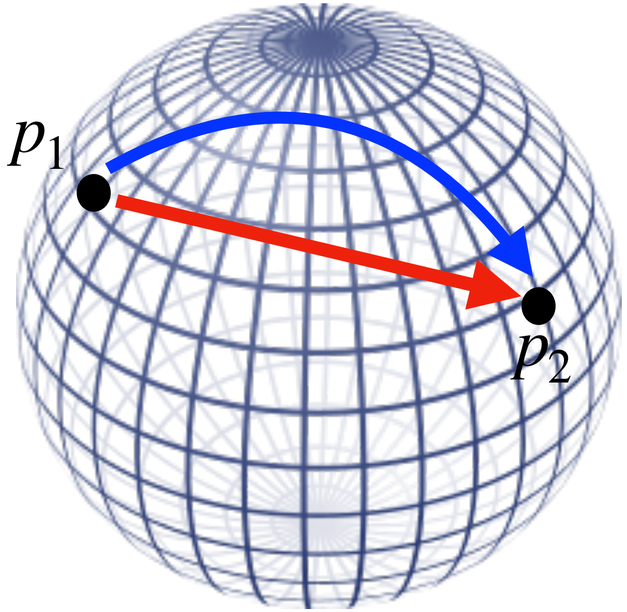
Example: aircraft



Sphere manifold

Euclidean distance (unfeasible trajectory)

Sphere distance (feasible trajectory)



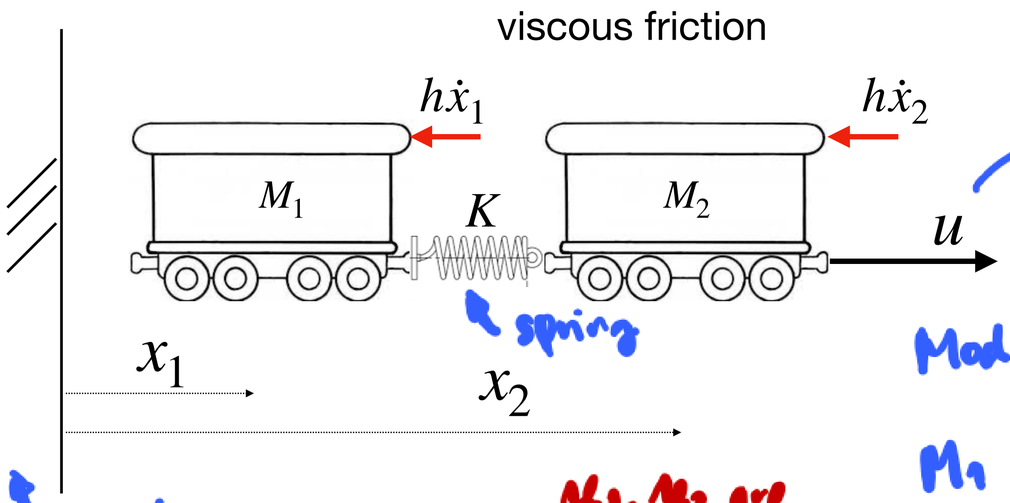
S^2 (2-D sphere)

In the course we shall treat only systems whose state evolves on Euclidean spaces (possibly local study)

“Examples”

Example: Mechanical system with two masses

Remember: this is an euclidean space!



force "u"

spring

fixed wall

x_1, x_2 are positions

where:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 = \dot{x}_1 \\ x_4 = \dot{x}_2 \end{pmatrix}$$

$n = 4$

Modelling the Newton Law

$x_3 = \dot{x}_1$

$$M_1 \ddot{x}_1 = M_1 \dot{x}_3 = \sum \text{FORCES ON } M_1 = -h x_3 + k(x_2 - x_1)$$

$$M_2 \ddot{x}_2 = M_2 \dot{x}_4 = \sum \text{FORCES ON } M_2 = -h x_4 + k(x_2 - x_1) + u$$

$x_4 = \dot{x}_2$

$$y = x_2 - x_1$$

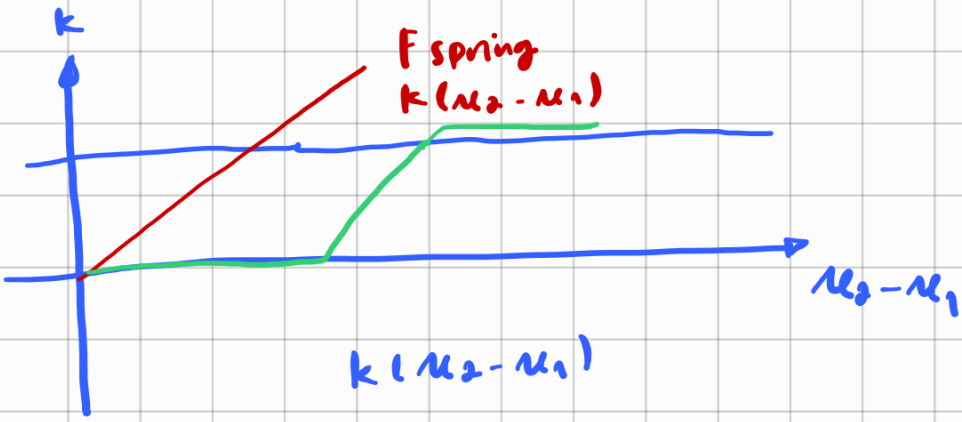
$\dot{x}_1 = x_3$

$\dot{x}_2 = x_4$

The A, B, C, D matrix:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/M_1 & k/M_1 & -h/M_1 & 0 \\ k/M_2 & -k/M_2 & 0 & -h/M_2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{pmatrix}$$

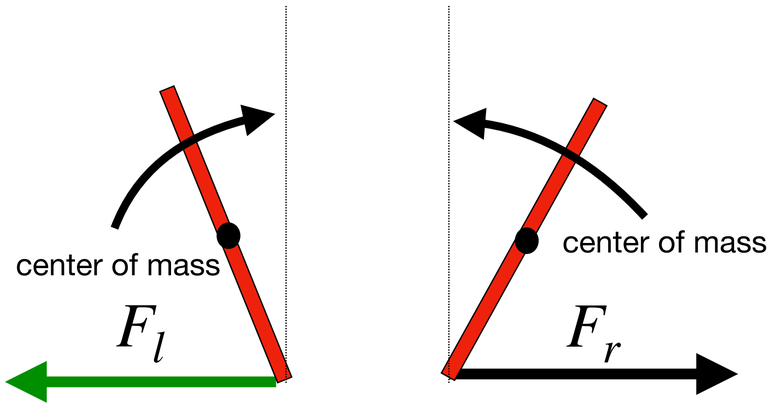
$$C = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix} \qquad D = 0$$



Example: Balance systems

would be the main focus of this course

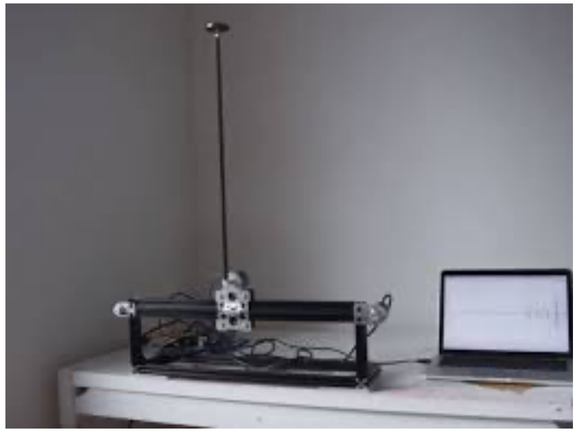
Class of mechanical systems in which the center of mass is balanced above a pivot point



Examples

Segway

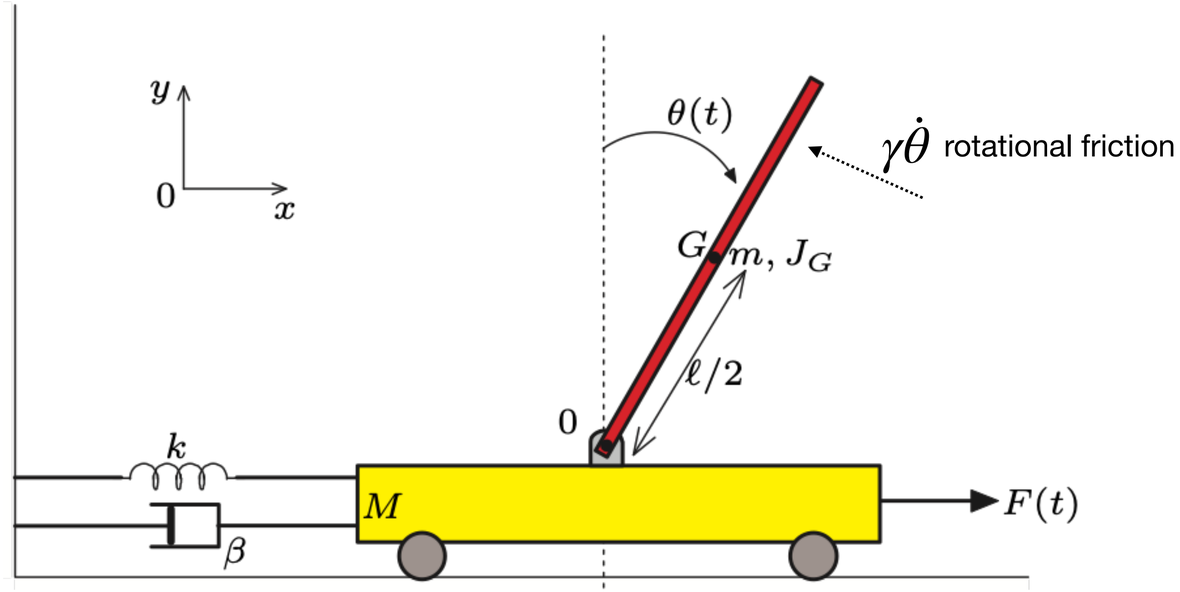
Rocket



Cart-pendulum

Example: Balance systems

Cart-pendulum



displacement of the cart

$$\begin{pmatrix} (M+m) & -m\ell\cos\theta \\ -m\ell\cos\theta & (J_G + m\ell^2) \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} \beta\dot{x} + kx + m\ell\sin\theta\,\dot{\theta}^2 \\ \gamma\dot{\theta} - mg\ell\sin\theta \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

angle of the pendulum

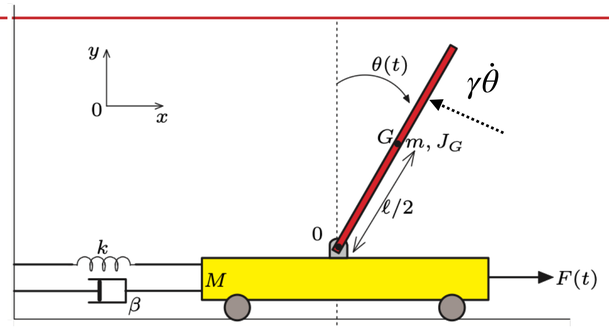
Homework: Using Lagrange compute/refresh the model of the cart-pendulum above

Example: Balance systems

Simplified case (**inverted pendulum**): if x is of no interest, we can just take the θ dynamics and confuse $m\ddot{x}$ with F

to ignore the x

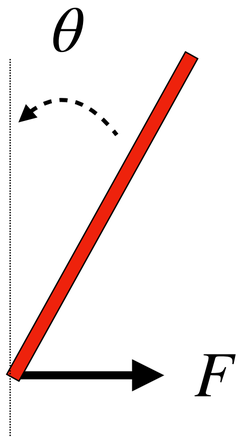
simplified



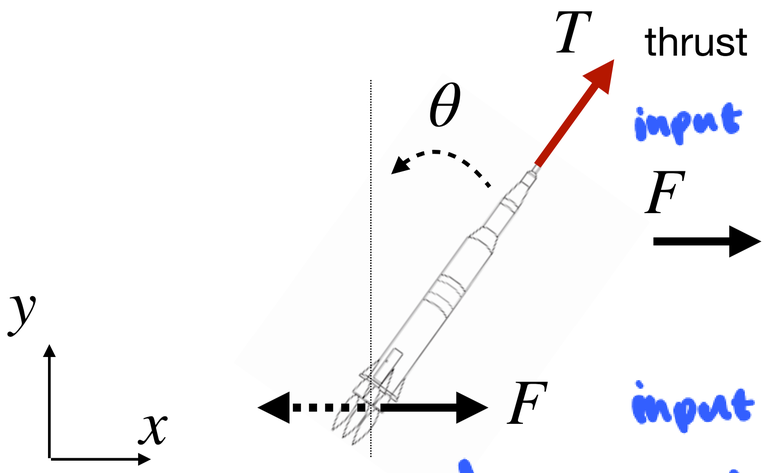
$$\begin{pmatrix} (M+m) & -m\ell\cos\theta \\ -m\ell\cos\theta & (J_G+m\ell^2) \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} \beta\dot{x} + kx + m\ell\sin\theta\,\dot{\theta}^2 \\ \gamma\dot{\theta} - mg\ell\sin\theta \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$F = m\ddot{x}$$

$$\dot{\theta}_1 = \theta_2$$
$$(J_G + m\ell^2)\,\dot{\theta}_2 = mg\ell\sin\theta - \gamma\dot{\theta} + \ell\cos\theta\, F$$



Example: Attitude control of a rocket



input

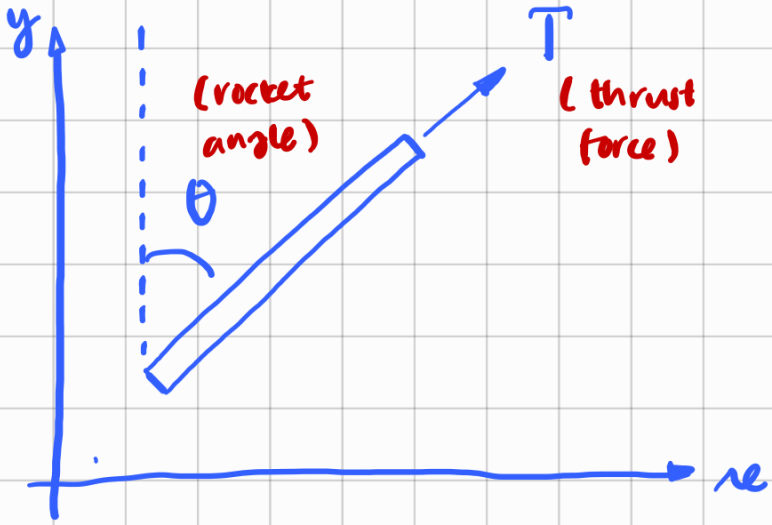
"Vector - Thrust"

$$F \longrightarrow \boxed{(\theta, \dot{\theta})} \xrightarrow{\theta} \boxed{\begin{aligned} \ddot{x} &= T\cos\theta \\ \ddot{y} &= T\sin\theta \end{aligned}} \longrightarrow$$

input = thrust

controlling left / right thruster

THE ROCKET THRUST CASE



suppose :

x_1 = x - position

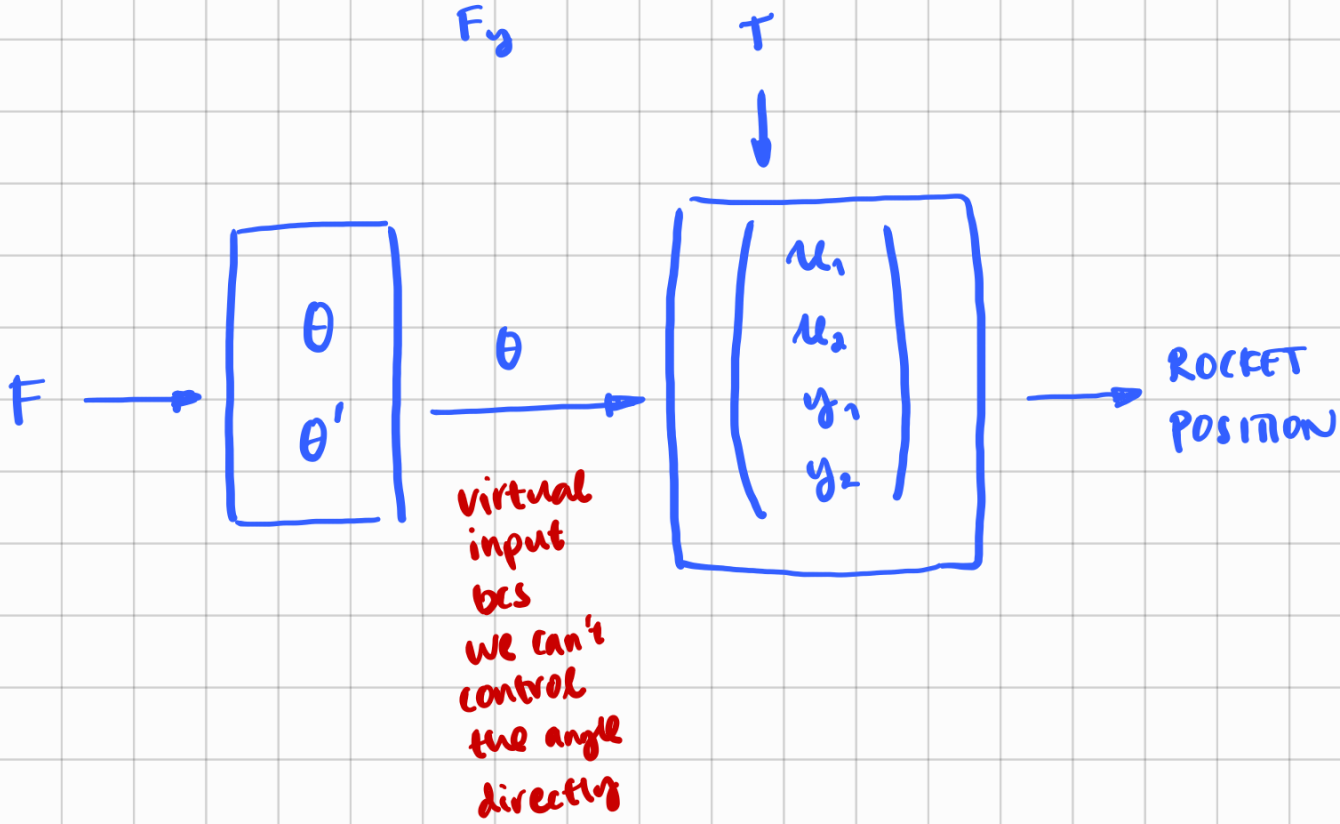
x_2 = horizontal speed $= \dot{x}_1$

y_1 = y - position

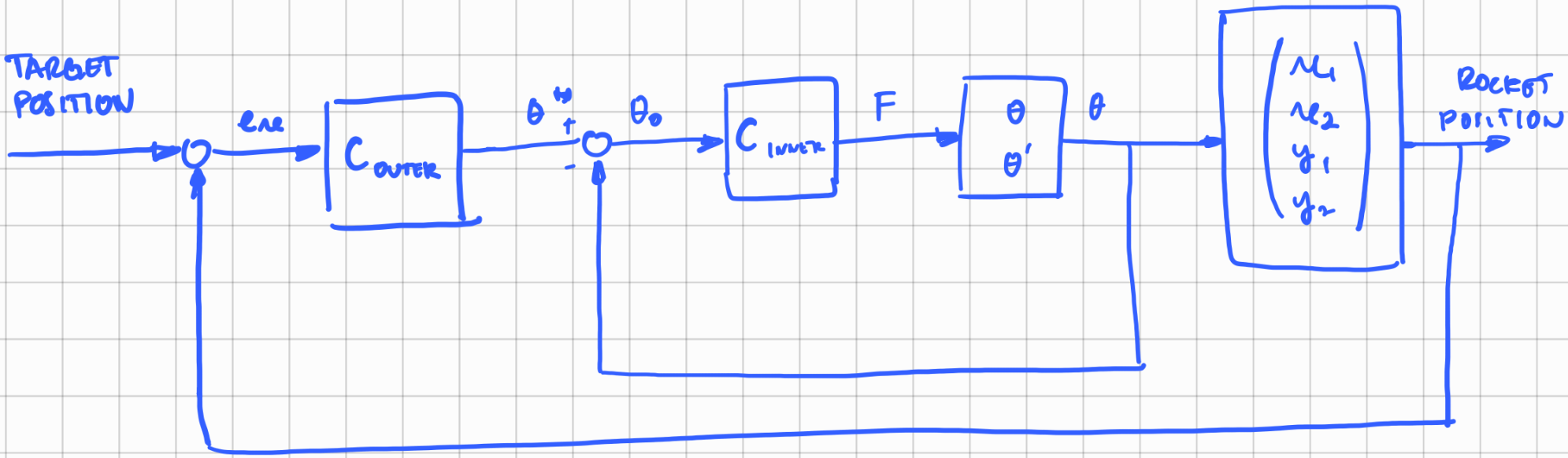
y_2 = vertical speed $= \dot{y}_1$

$$M\ddot{x}_1 = M\dot{x}_2 = \underbrace{T\sin\theta}_{F_x}$$

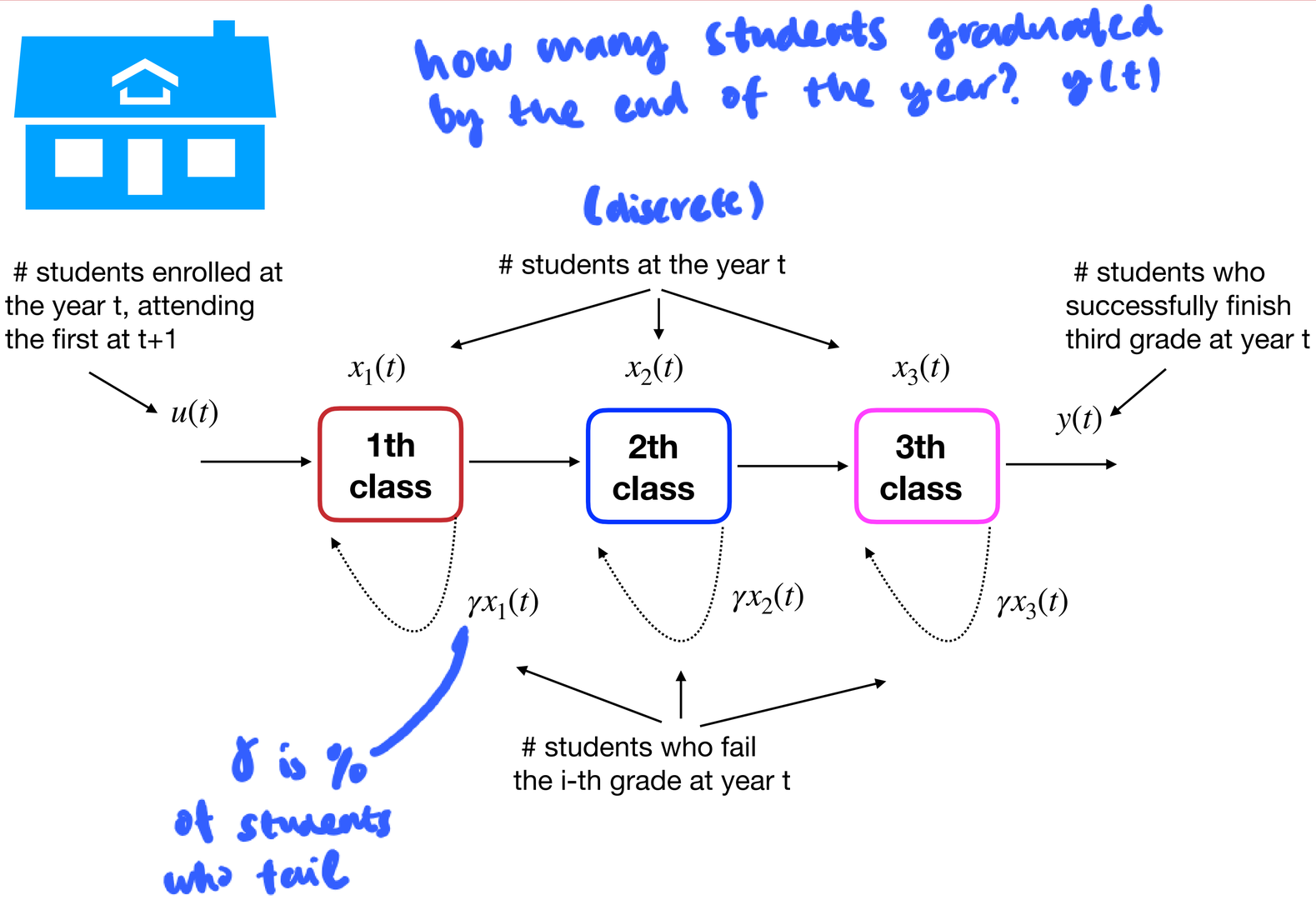
$$M\ddot{y}_1 = M\dot{y}_2 = \underbrace{T\cos\theta}_{F_y}$$



Block Diagram-Wise:



Example: Middle school students evolution



MODELLING STUDENTS EVOLUTION

$$x_1(t+1) = \gamma x_1(t) + u(t)$$
$$x_2(t+1) = (1-\gamma) x_1(t) + \gamma x_2(t)$$

* assume that there's no student attending 2nd and 3rd class without attending 1st

$$x_3(t+1) = (1-\gamma) x_2(t) + \gamma x_3(t)$$
$$y(t) = (1-\gamma) x_3(t)$$

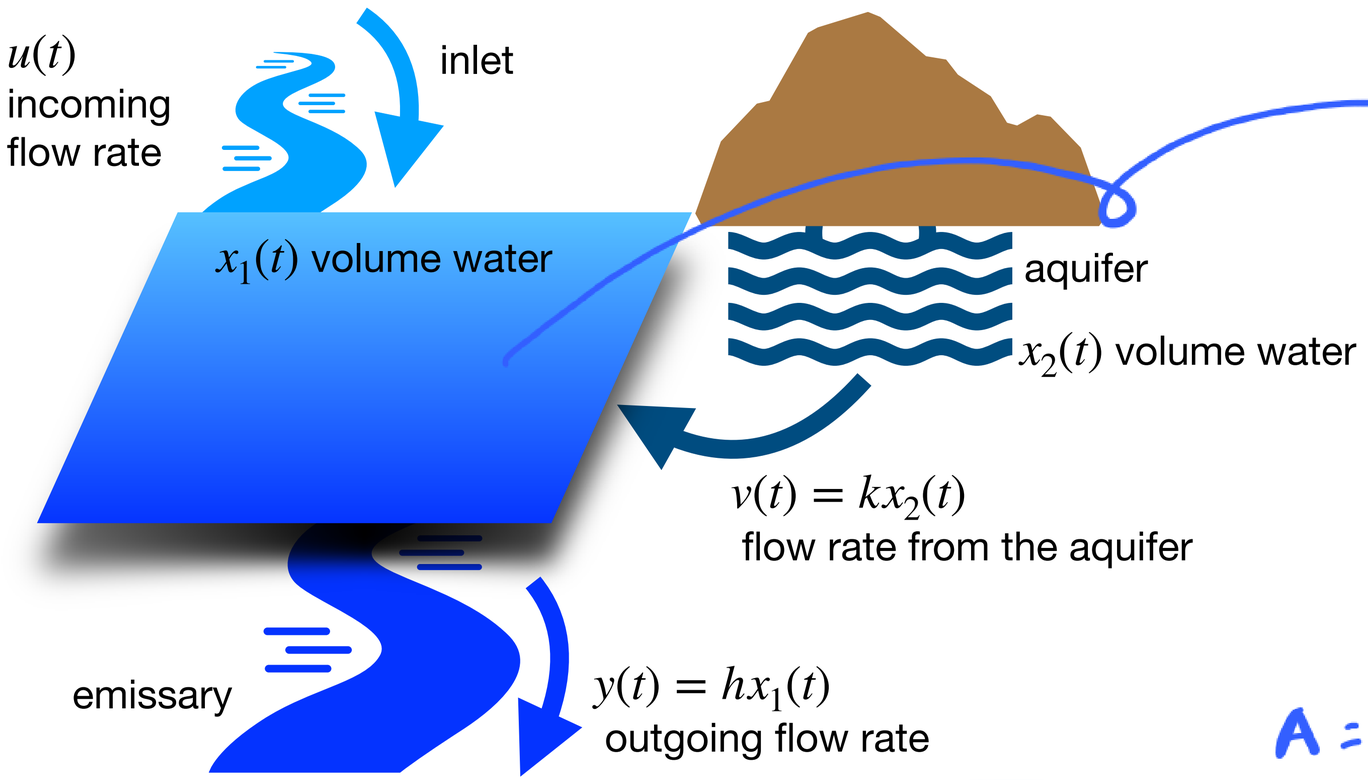
The A, B, C, D matrix:

$$A = \begin{pmatrix} \gamma & 0 & 0 \\ 1-\gamma & \gamma & 0 \\ 0 & 1-\gamma & \gamma \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 1-\gamma \end{pmatrix}$$

* assume the γ does not time-vary

Example: Hydraulic system (lake with underground aquifer)



$$\dot{x}_1 = u(t) - h x_1(t) + k x_2(t)$$

$$\dot{x}_2 = -k x_2(t)$$

$$y = h x_1(t)$$

$$A = \begin{pmatrix} -h & k \\ & -k \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} h & 0 \end{pmatrix}$$

2nd order system

MODELLING THE HYDRO SYSTEM

$$\dot{x}_1(t) = u(t) + k x_2(t) - h x_1(t)$$
$$\dot{x}_2(t) = -k x_2(t)$$
$$y(t) = h x_1(t)$$

The A, B, C, D matrix

$$A = \begin{pmatrix} -h & k \\ 0 & -k \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} h & 0 \end{pmatrix}$$

We'll try to do it in transfer function for this model (with 2nd ODE)

Example: Predator-Prey

AN AUTONOMOUS SYS SINCE NO INPUT

Lotka–Volterra equations (1910)

x is the number of prey (rabbits, hares, …)

y is the number of predator (foxes, lynches, …)

C-T

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \delta x y - \gamma y$$

rate of prey reproduction

rate of predation

rate of predator's increase

rate of pred. decay

D-T

$$x(t+1) = x(t) + \alpha x(t) - \beta x(t) y(t)$$
$$y(t+1) = y(t) + \delta x(t) y(t) - \gamma y(t)$$

- The prey are assumed to have an unlimited food supply and to reproduce exponentially, unless subject to predation (αx)
- The rate of predation is assumed to be proportional to the rate at which the predators and the prey meet ($-\beta xy$)
- Similarly, the growth of the predator population is assumed to be proportional to the rate at which the predators and the prey meet (δxy)
- Loss rate of the predators (death or emigration) exponentially decaying with their number in the absence of prey ($-\gamma y$).

Widely used in many contexts:

- Population dynamics (interactions, competition, disease, mutualism)
- Economic dynamics

Interaction between species having mutual benefits from the interaction

Possible "control input":
$\alpha(u)$ (feeding preys),
$\gamma(u)$ (suppressing predators)

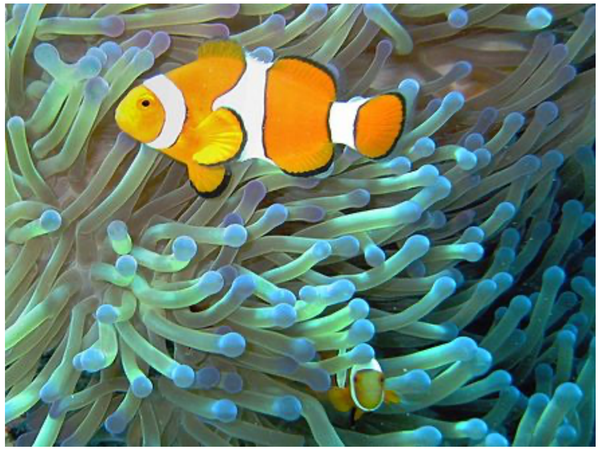
Example: Predator-Prey

Mutualism examples

Cleaning symbiosis

The red-billed oxpecker eats ticks on the impala's coat, in a cleaning symbiosis.

Service-service symbiosis



Ocellaris clownfish and Ritter's sea anemones live in a mutual service-service symbiosis, the fish driving off butterflyfish and the anemone's tentacles protecting the fish from predators.

Source: Wikipedia

Example: Predator-Pray

Lotka–Volterra equations (1910)

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \delta x y - \gamma y$$

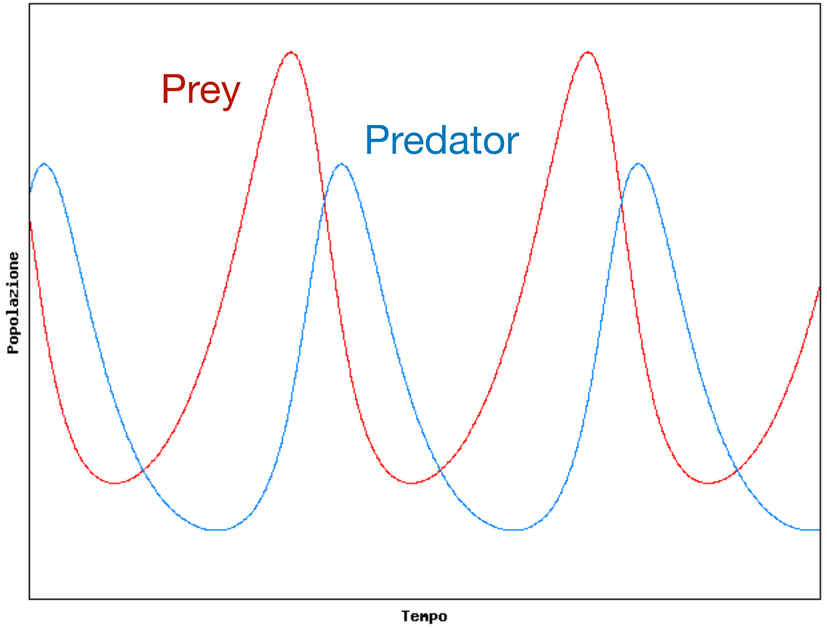
x is the number of prey

y is the number of predator

- Equilibria:
  - $(\bar{x}, \bar{y}) = (0,0)$
  - $(\bar{x}, \bar{y}) = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$
- $x(0) = 0, y(0) \neq 0$?
- $x(0) \neq 0, y(0) = 0$?
- $x(0) \neq 0, y(0) \neq 0$?

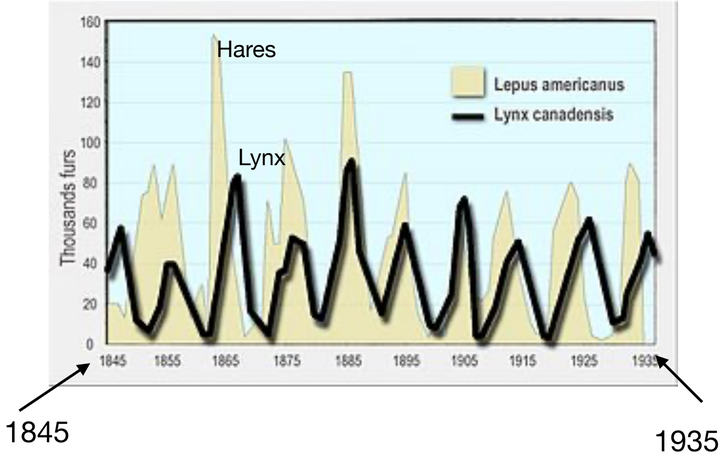
Amplitude of oscillations depends on the initial state

Simulation



Experimental data

Source: Canadian Rockies



In non-equilibrium situations, predators thrive when there is abundance of prey but, in the long run, there's not enough food and they die out. As the predator population decreases the prey population increases again; this dynamic continues in a cycle of growth and decline.

LOTKA-VOLTERRA EQUATIONS

$$x(\alpha - \beta y) = 0$$

$$\boxed{y = \bar{y} = \frac{\alpha}{\beta}}$$

} first eq.

$$y(\delta x - \gamma) = 0$$

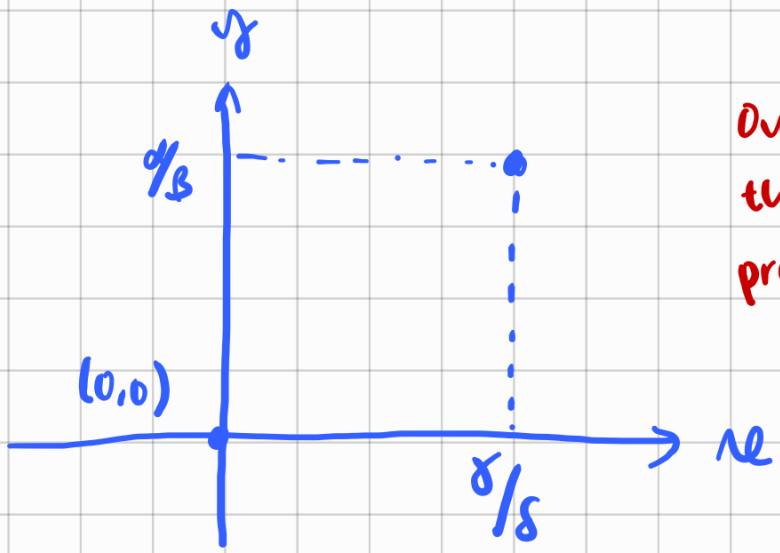
$$\boxed{x = \bar{x} = \frac{\gamma}{\delta}}$$

} second eq.

The equilibrium:

$(\bar{x}, \bar{y}) = (0,0)$ and $\left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right)$

$\bar{x}$ and $\bar{y}$ are the number of prey and predator to reach the equilibrium



Outside the equilibrium, the number of preys predators are oscillating.

Example: Predator-Prey

More sophisticated/realistic models consider α, β, δ as function of x according to the following rules:

to prevent addition non-linearity

$\alpha(x) = r\,(1 - \frac{x}{k})$ namely, the prey growth rate decreases (possibly negative) when x increases $(r, k > 0)$

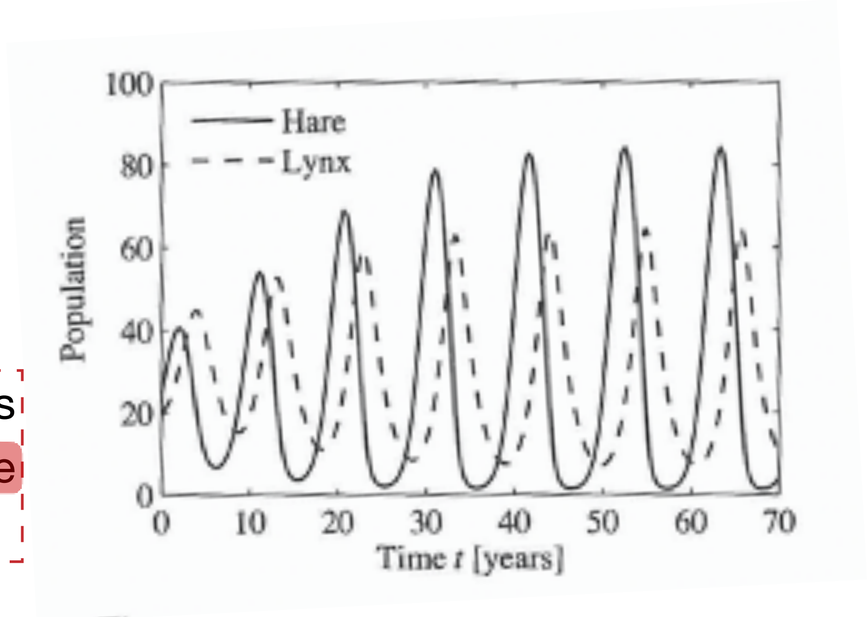
$\beta(x) = \frac{a}{c + x}$ namely, the predation rate decreases when x increases (effect of a population dominating on the other regardless the mutual strength) $(a, c > 0)$

$\delta(x) = b\,\beta(x)$ similarly as above $(b > 0)$

$$\dot{x} = r\,x\,(1 - \frac{x}{k}) - \frac{a\,x\,y}{c + x}$$

$$\dot{y} = b\frac{a\,x\,y}{c + x} - d\,y$$

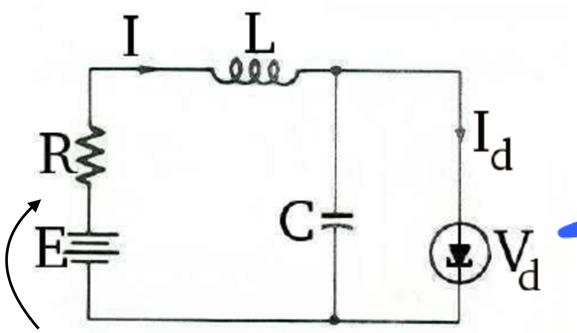
Amplitude of oscillations **does not** depend on the initial state



$a = 3.2, b = 0.6, c = 50, d = 0.56, k = 125, r = 1.6$

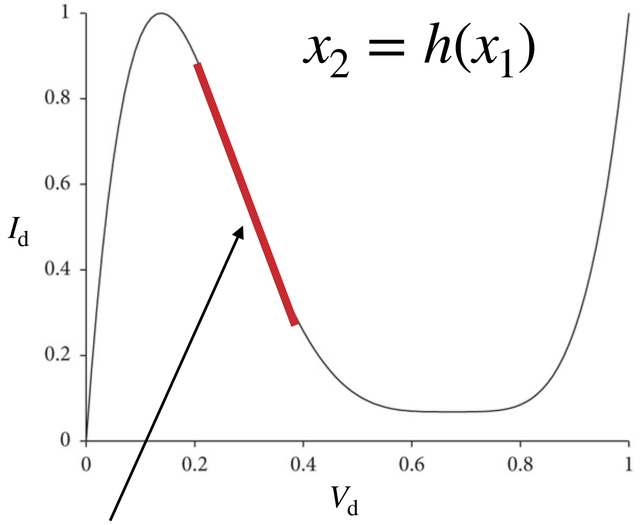
Example: Tunnel-Diode

$x_2 := I$



$x_1 := V_d$

$u = E$

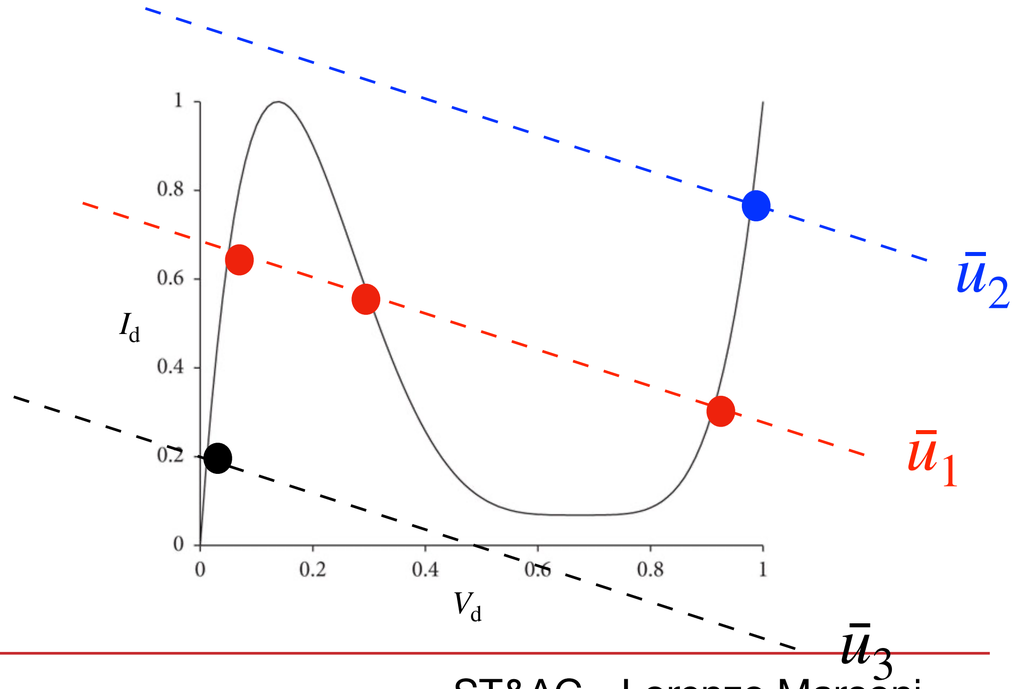


Negative resistance

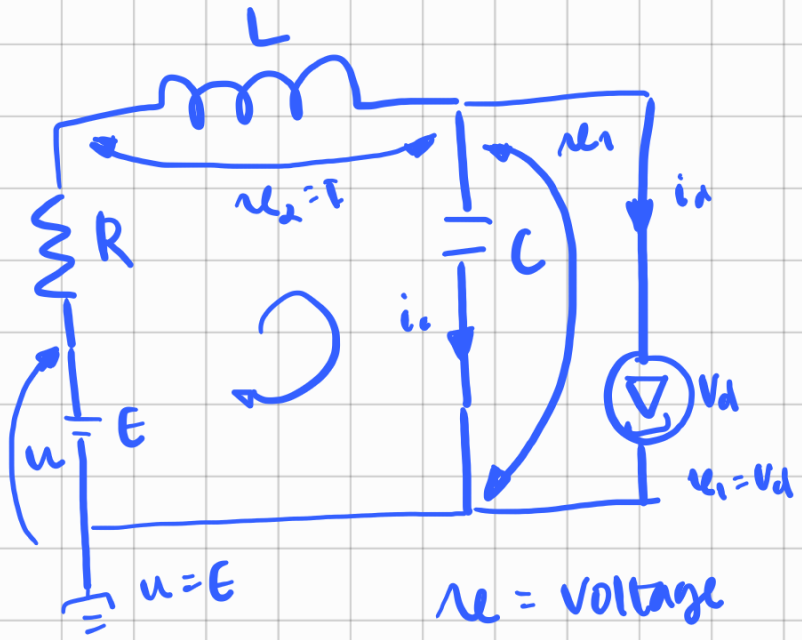
The "negative" differential resistance makes tunnel diodes key elements in circuits like oscillators and switching circuits.

Switching circuit

$$C\dot{x}_1 = x_2 - h(x_1)$$
$$L\dot{x}_2 = -x_1 - Rx_2 + u$$



TUNNEL DIODE



$$L \frac{dx_2}{dt} = V_L$$

$$C \frac{dx_1}{dt} = i_c$$

$$u - Rx_2 - V_2 - x_1 = 0$$

x = voltage

Kirchoff's 2nd Law:

$$u - Rx_2 - V_1 - x_1 = 0$$

$$\begin{cases} L \dfrac{dx_2}{dt} = V_L = -x_1 - Rx_2 + u \\ C \dfrac{dx_1}{dt} = i_c = x_2 - h(x_1) \end{cases}$$

Kirchoff's 1st Law:

$$x_2 - i_c - \underbrace{i_d}_{h(x_1)} = 0$$

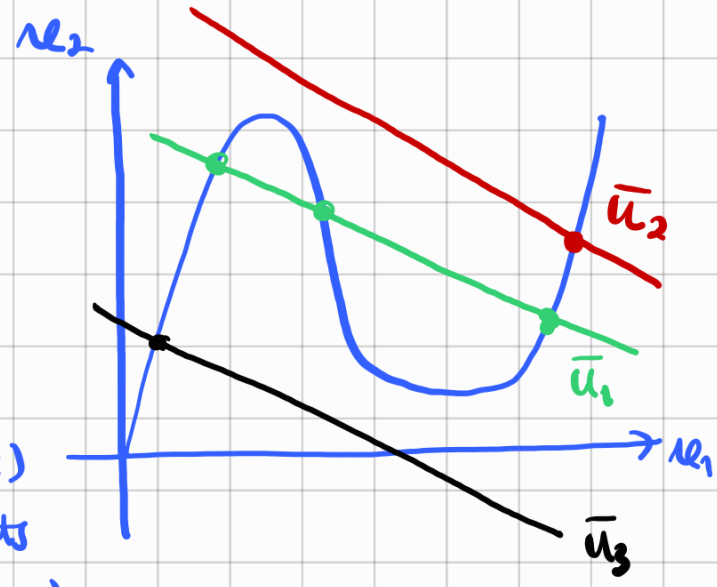
state-space model, 2nd order system

$$\begin{cases} \dot{x}_2 = -\dfrac{1}{L} x_1 - \dfrac{R}{L} x_2 + \dfrac{1}{L} u \\ \dot{x}_1 = \dfrac{1}{C} x_2 - \dfrac{1}{C} h(x_1) \end{cases}$$

stationary, non-linear system

Equilibrium (when u is constant)

$$x_2 = h(x_1)$$
$$x_2 = -\frac{1}{R} x_1 + \frac{1}{R} \bar{u}$$



- $\bar{u}_3$ intersects 1 eq. point (decreasing the current)
- $\bar{u}_2$ intersects 2 eq. points (increasing the voltage)
- $\bar{u}_1$ intersects 3 eq points

Generic linear system:

$$\dot{x} = Ax + Bu$$

pick $u = \bar{u} = 0 \Rightarrow$ autonomous linear system

$A\bar{x} = 0$ for equilibrium

if A is NON SINGULAR (A^{-1})

$\bar{x} = 0$

if A is SINGULAR (rank $= 1$) $n = 2$

$\bar{x} \in \ker(A)$

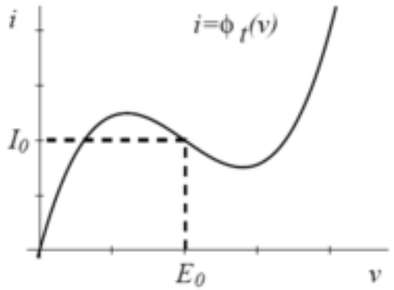
infinite eq. points in the $\ker(A)$

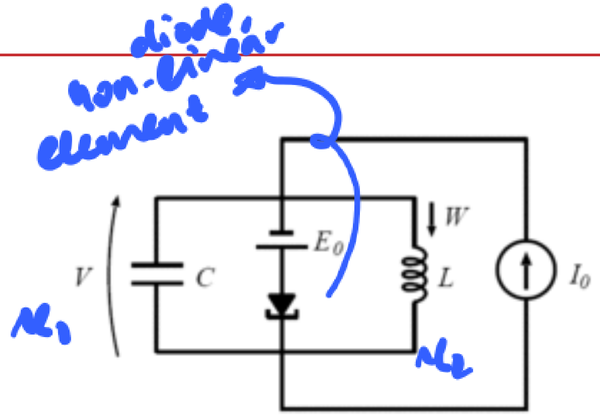
Example: Van der Pol oscillator

Van der Pol oscillator ($\mu > 0, \kappa > 0$)

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 \end{aligned}$$

constant





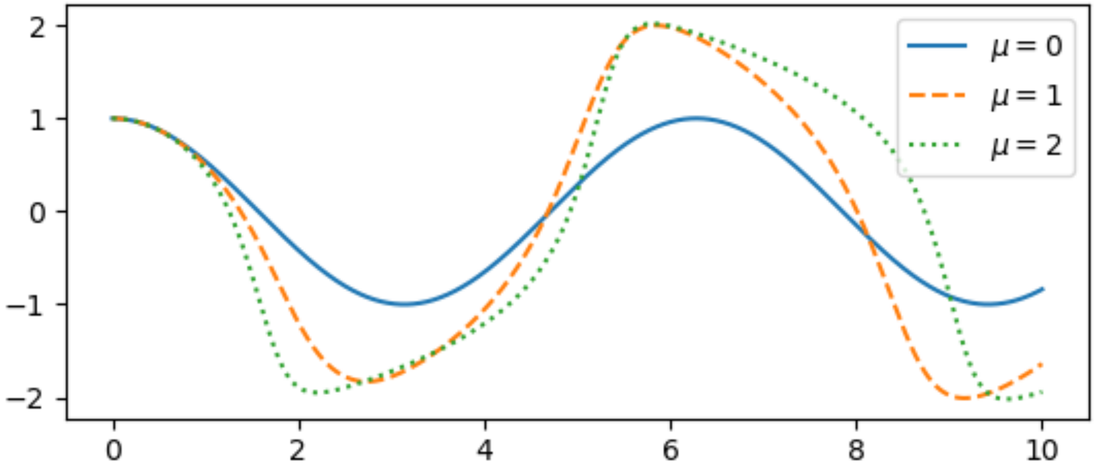
Balthasar van der Pol
(27 January 1889 – 6 October 1959)

Linear oscillator ($\mu = 0$)

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

Basic model for oscillatory processes in physics, electronics, biology, neurology, sociology and economics

Self-sustained oscillations whose amplitude is not dependent on initial conditions



The only equilibrium is the origin

VAN DER POL OSCILLATOR

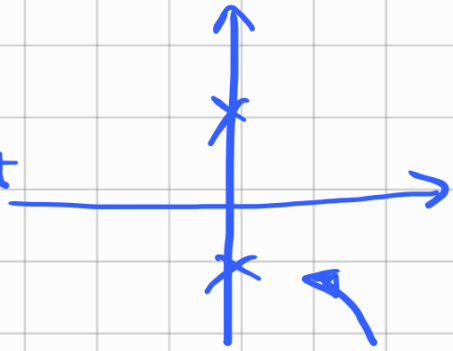
Set $\mu > 0$, $K > 0$:

$$\dot{x}_1 = K x_2$$
$$\dot{x}_1 = \mu(1 - x_1^2) x_2 - x_1$$

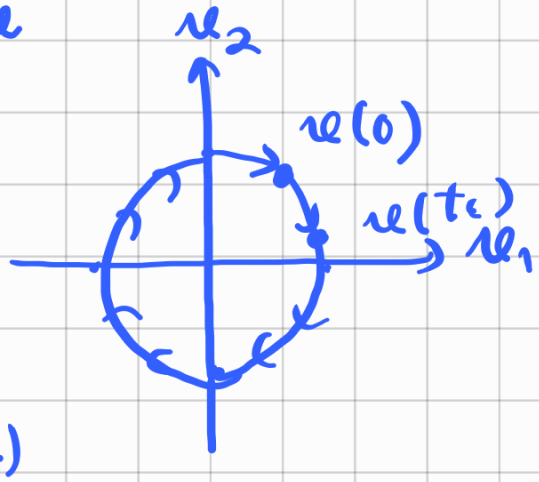
if $\mu = 0$ $\longrightarrow$ linear oscilator :

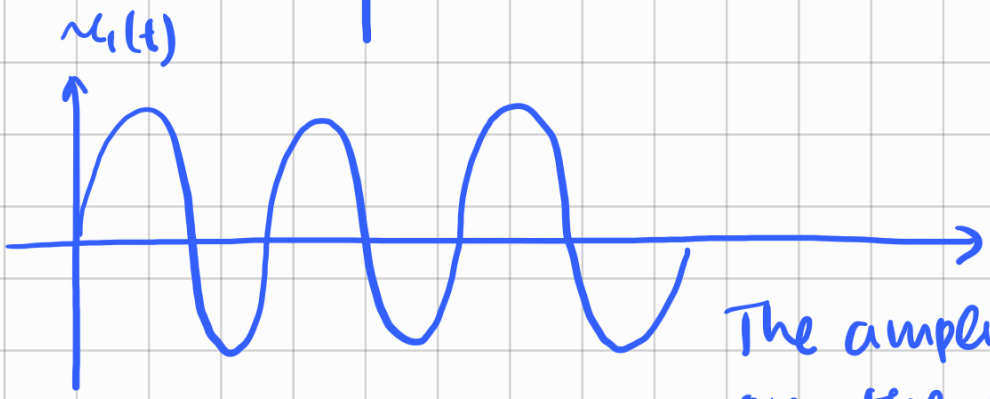
$$\begin{cases} \dot{x}_1 = K x_2 \\ \dot{x}_2 = -x_1 \end{cases} \qquad A = \begin{bmatrix} 0 & K \\ -1 & 0 \end{bmatrix}$$

$\bar{x} = 0$ is an eq. point
if we take x = constant
$= \bar{x}$



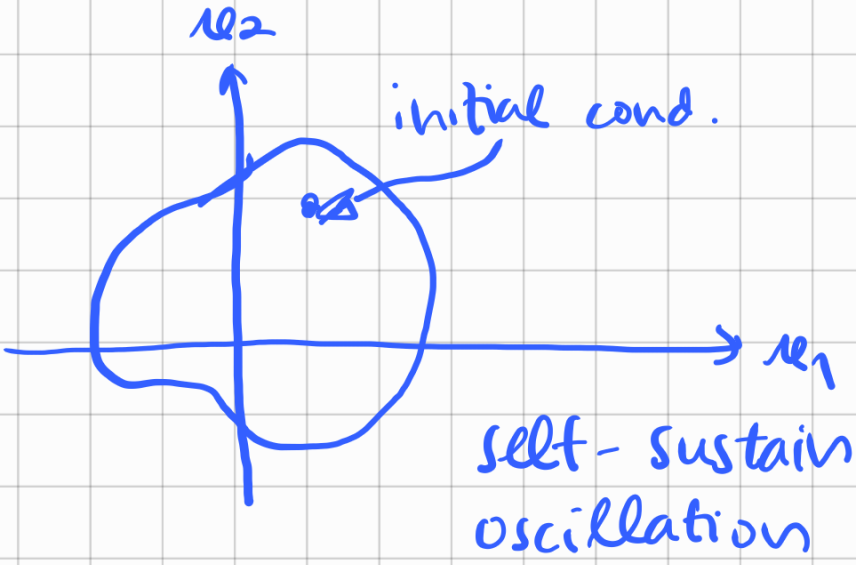
eigenval.





The amplitude depends on the ray (r) so on the initial condition.

When we choose a point that is not in an equilibrium condition, the amplitude does not depend on the initial condition.

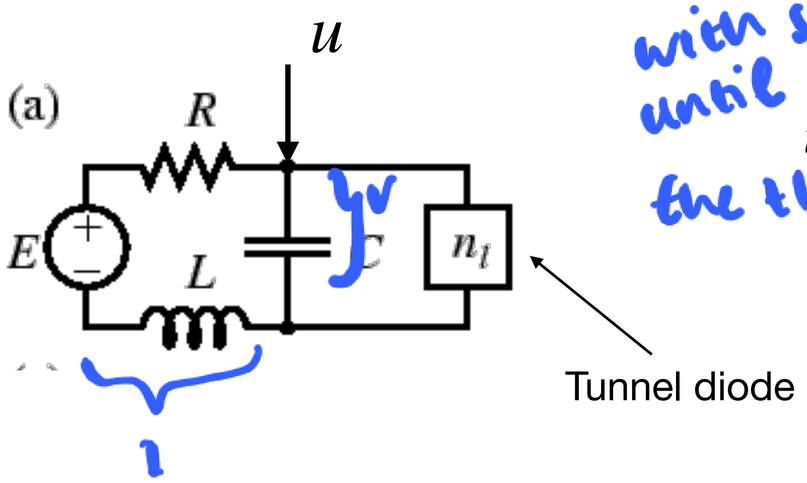


Example: FitzHugh-Nagumo circuit

Prototype of relaxation oscillator (when the activation input u exceeds a certain threshold the state variables exhibit a characteristic oscillatory behaviours). The circuit is a relevant benchmark able to capture typical **spike behaviours** observed in neurons after stimulation by an external input current

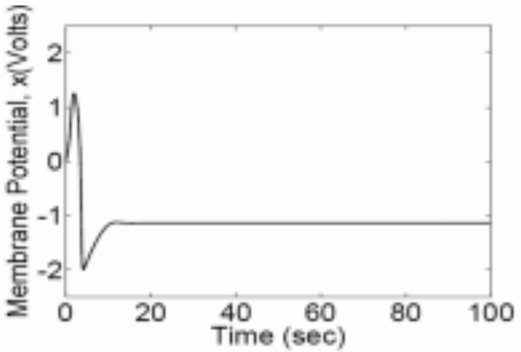
$$\begin{aligned} C\dot{V} &= \kappa V - \frac{V^3}{3} - I + u \\ L\dot{I} &= -I + RV \end{aligned}$$

activation input

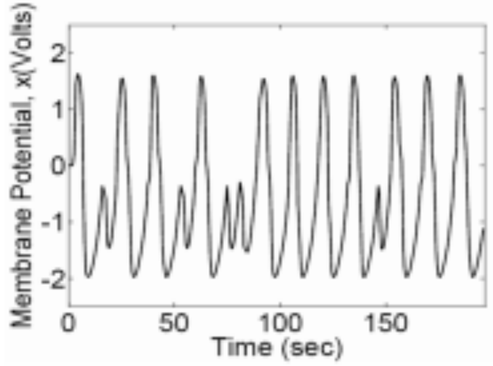


no effect with small u until exceed the threshold

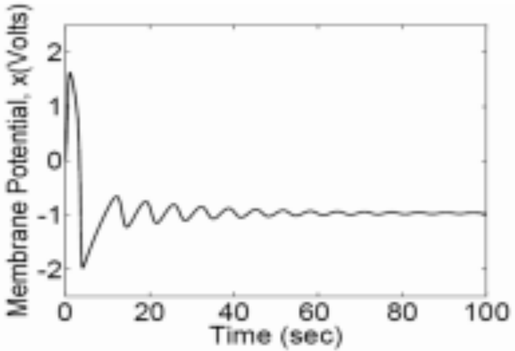
$u = 0.1$



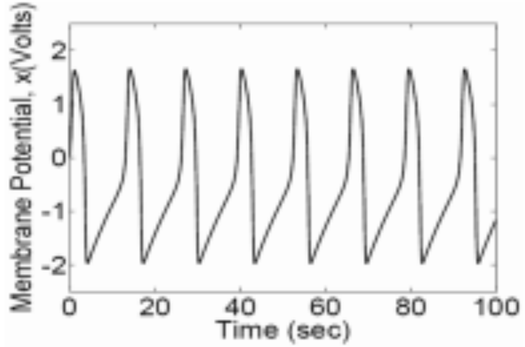
$u = 0.337$



$u = 0.33$



$u = 0.34$



Example: Susceptible-Infected-Resolved Model (COVID'19)

A simple model for epidemics spread (**SIR**):

probability of susceptible & infected meet each other

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \gamma I$$
$$\dot{R} = \gamma I$$

$$S(0) + I(0) + R(0) = 1$$

recovery rate

infection rate

where:

- S = percentage of **susceptible** individuals (not infected but that can be infected)
- I= percentage of **infected** individuals
- R = percentage of **resolved** individuals (dead or recovered)

and:

- β = average number of contacts per person per time $\times$ infection probability per contact
- γ = recovery rate (e.g., γ= (average duration of infection)$^{-1}$)

- from $dS = -(\beta SI)dt$ we conclude that βSI is the average percentage of susceptible individuals that become infected at each unit of time (taken from S and added to I)
- from $dI = (\beta SI)dt - \gamma Idt$ we conclude that γI is the average percentage of infected that heal or die per unit of time (taken from I and added to R)

$$\boxed{S, I} \xrightarrow{I} \boxed{R}$$

Epidemic indicators

Basic reproductive number: $R_0 \doteq \dfrac{\beta}{\gamma}$

Effective reproductive number: $R_{eff}(t) \doteq R_0 S(t) = \dfrac{\beta}{\gamma} S(t)$

focus on this

Example: Susceptible-Infected Model (COVID'19)

Let us neglect the R dynamics (it is only an accumulator, does not affect S or I)

lockdown reduces β

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \gamma I$$

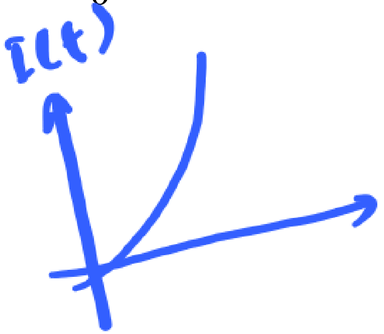
$$S(0) + I(0) = 1$$

$$\begin{bmatrix} \dot{S}(t) \\ \dot{I}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\beta I(t) & 0 \\ 0 & \beta S(t) - \gamma \end{bmatrix}}_{A(t)} \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$$

- The term in the first dynamics ,$-\beta I(t)$, is always negative (the susceptible population always decreases)

- The second is positive if $R_{eff}(t) > 1$; At the beginning of the infection $(S(0) \approx 1,\ I(0) \approx 0,\ R(0) = 0)$, $R_{eff}(0) \approx R_0 \rightarrow$ the disease spreads if $R_0 > 1$, otherwise it dies out! (in the early COVID case $R_0 \in [1.5,\ 4]$)

$\dot{I} = (>0)\, I$

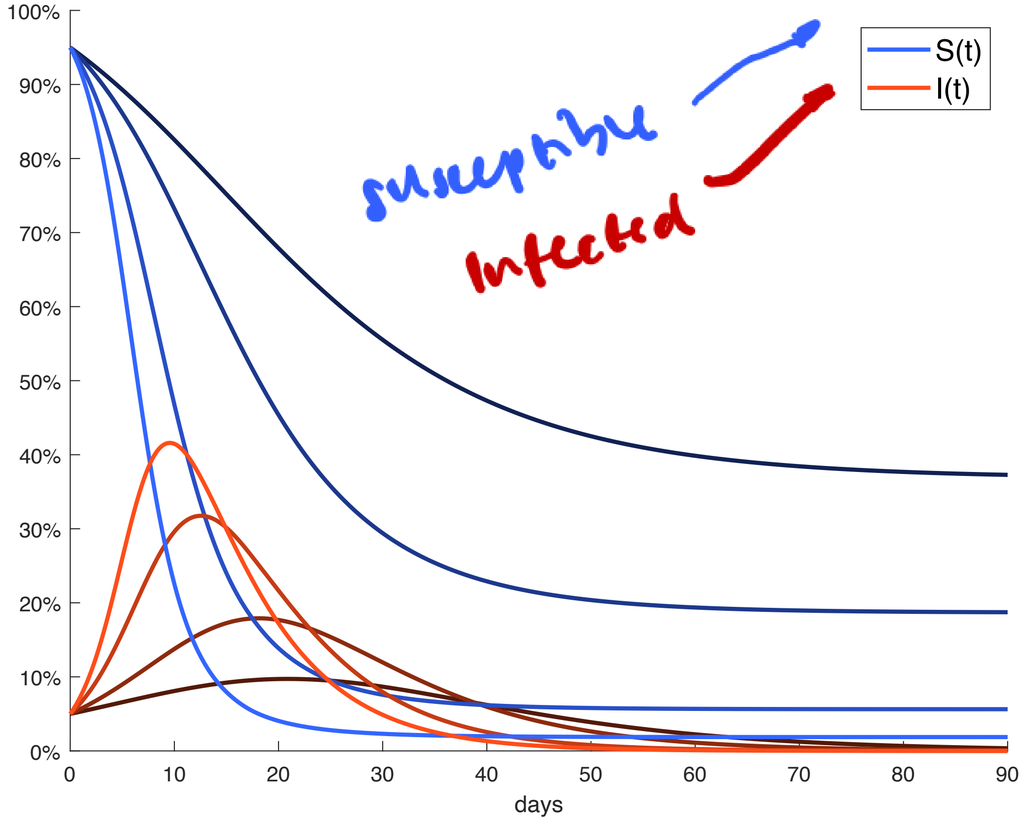


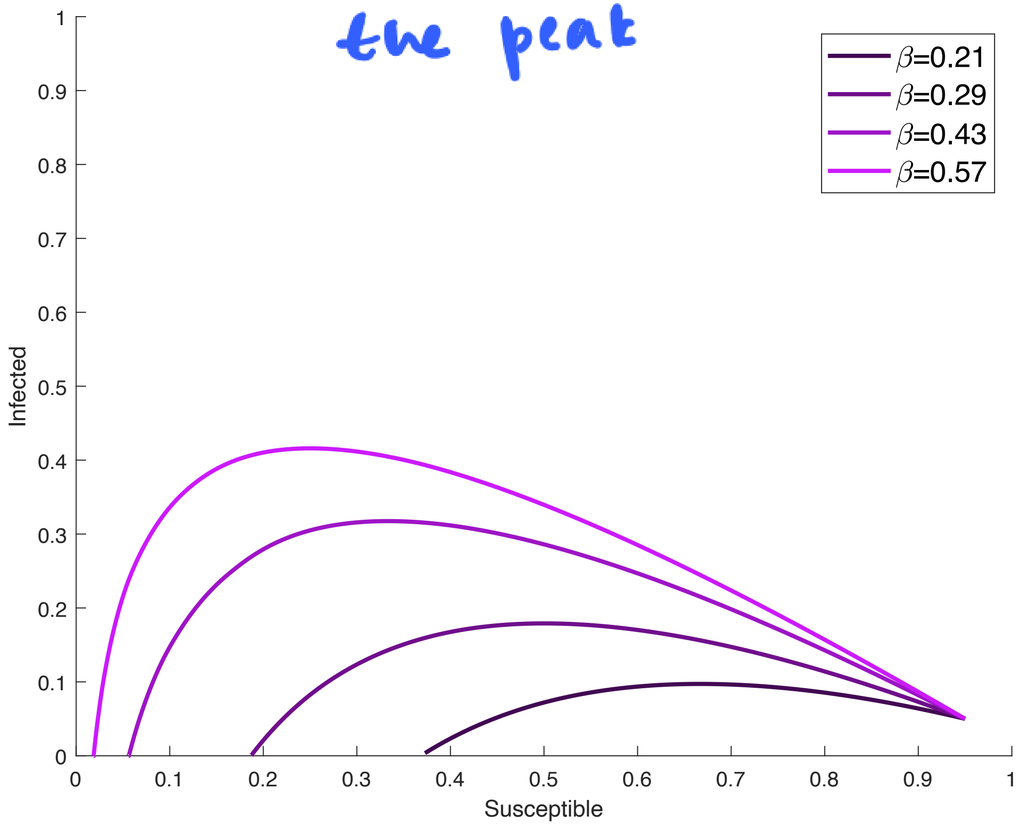
Example: Susceptible-Infected Model (COVID'19)

$$I(0) = 0.05, \;\; S(0) = 0.95 \qquad\qquad R_0 = \{1.5, 2, 3, 4\}$$

The equilibrium point reached depends on R_0 but $I_{eq} = 0$ always

Lower R_0 also lower the peak

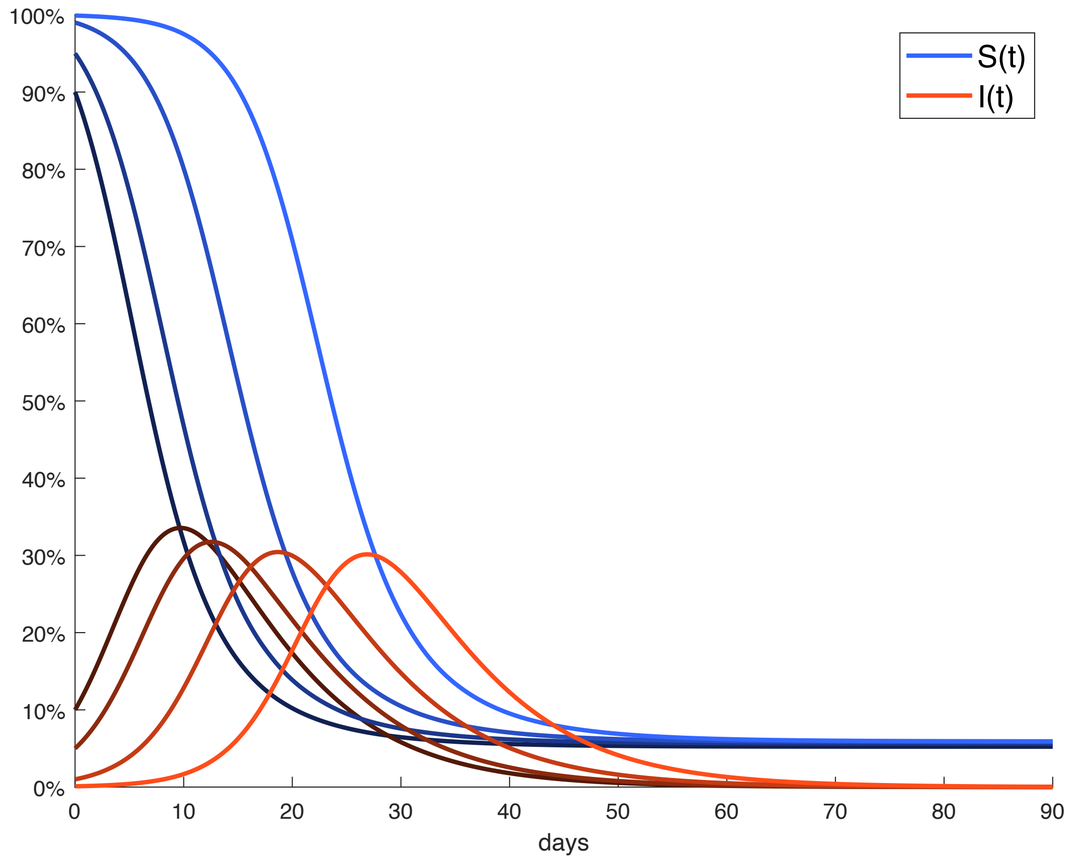




Example: Susceptible-Infected Model (COVID'19)

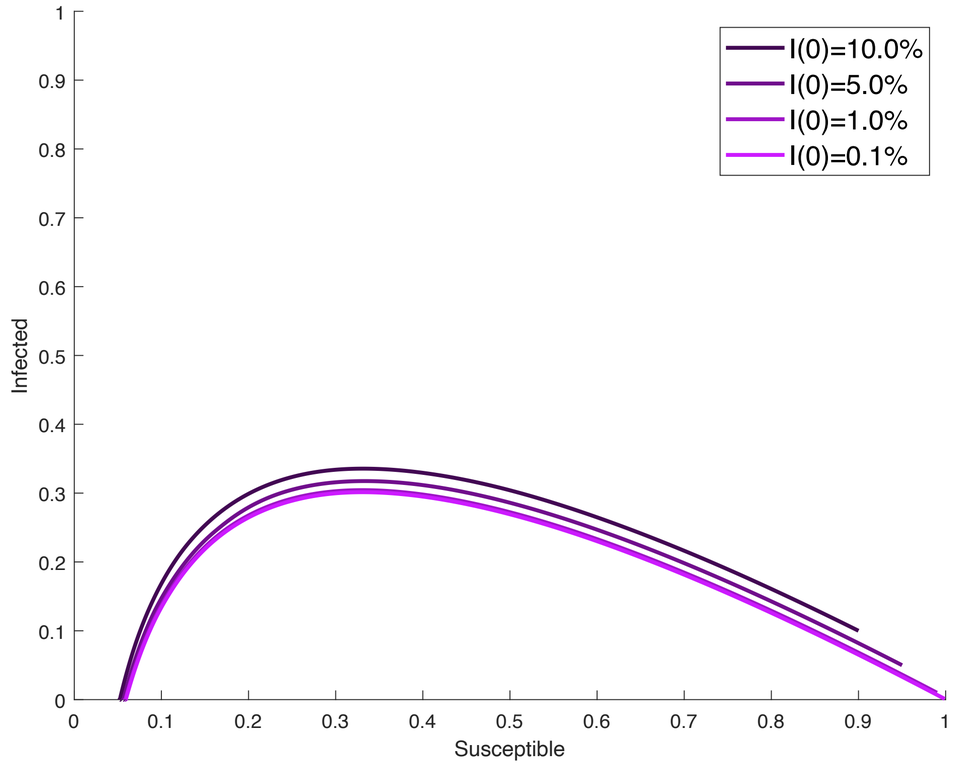
$$I(0) = \{0.1, 0.05, 0.01, 0.001\}$$
$$S(0) = 1 - I(0)$$

$$R_0 = 3$$



The equilibrium point reached depends does not depend on $I(0)$

Smaller $I(0)$ only delay the peak but its amplitude remains the same



Example: Susceptible-Infected Model (COVID'19)

could be time varying

A more realistic model (Reinfections and Variable Rate)

coefficient

$$\dot{S} = \beta(t)\, S\, I + \delta R$$
$$\dot{I} = \beta(t)\, S\, I - \gamma I$$
$$\dot{R} = \gamma I - \delta R$$

$$S(0) + I(0) + R(0) = 1$$

- We assume all infected heal ($I \to R$) and, with rate δ, become susceptible again ($R \to S$)
- $\beta(t)$ is periodic to model different infection rates in different periods of the year

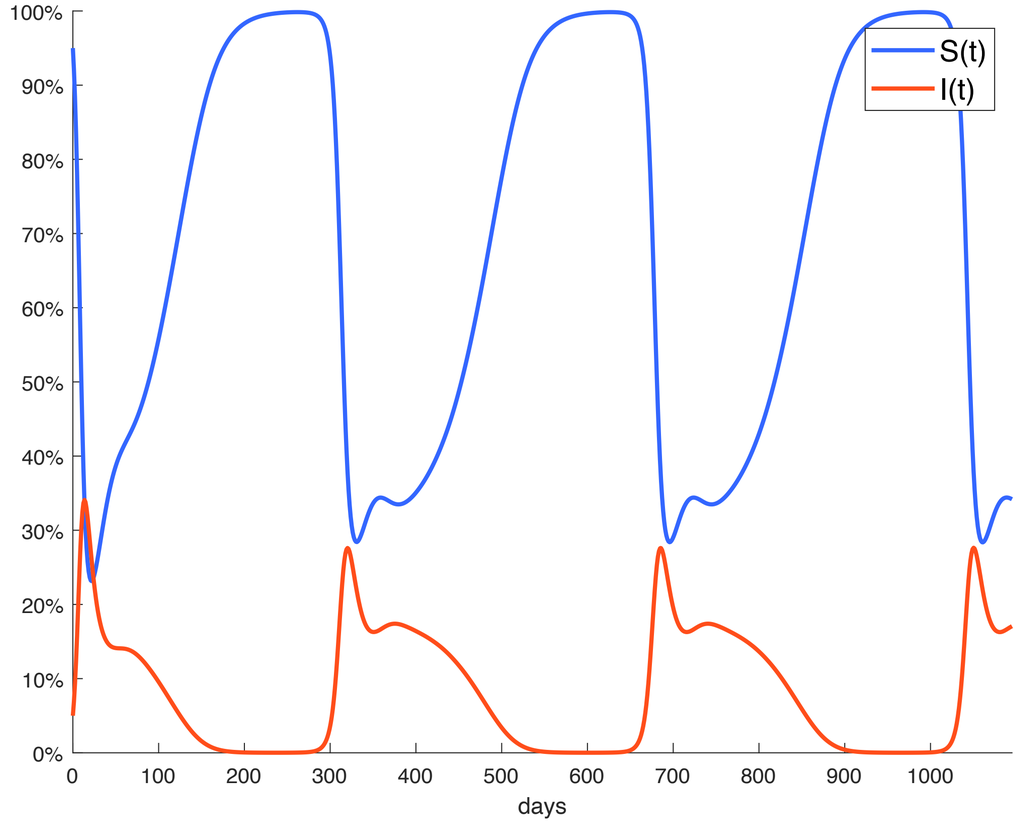
$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -\omega^2 z_1$$
$$\beta(t) = \beta^- + \frac{1 + z_1(t)}{2}\beta^+$$

with $z_1(0) = 1,\ z_2(0) = 0,\ \beta^+, \beta^- > 0,\ \ \omega \doteq \frac{2\pi}{365}$

System of order 5

Example: Susceptible-Infected Model (COVID'19)

Simulations with $\gamma = \frac{1}{7}, \delta = \frac{1}{20}, \beta^- = \frac{1}{2}\gamma, \beta^+ = 3\gamma$

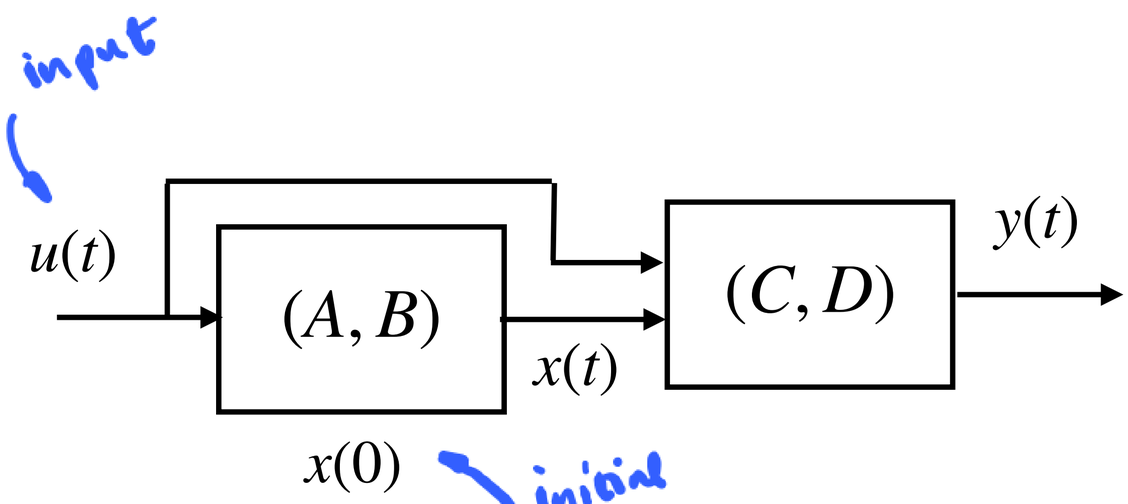


Input/Output description

I/O descriptions: transfer functions

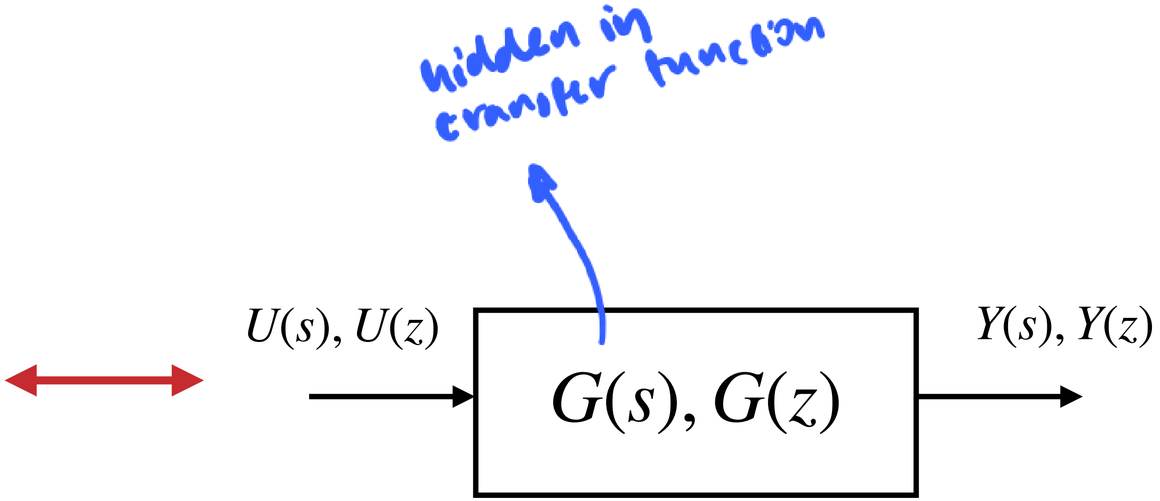
- Alternative ways to describe dynamical systems involve Input-output descriptions, such as **transfer functions**
- The concept of state is lost and only an input/output behaviour is described
- As such, I/O descriptions ignore certain "internal" dynamics (see example later)
- I/O descriptions become intractable in presence of (large) multiple input - multiple output systems
- ….as well as for nonlinear systems

there's no transfer function for non-linear systems



First-order differential equations

Algebraic state-output relations



Algebraic input-output relation in the Laplace/Z domains

I/O descriptions: transfer functions

state space
$A, B, C, D \longrightarrow G(s)$

How to systematically obtain an I/O model from a SS model? Using Laplace and Z transforms and their properties

Laplace-transformed of $u(t)$

Continuous-time ($t \in \mathbb{R}$)

$$X(s) = \mathscr{L}(x(t)), \quad U(s) = \mathscr{L}(u(t)), \quad Y(s) = \mathscr{L}(y(t))$$

$$\mathscr{L}(\dot{x}(t)) = sX(s) - x(0)$$

$$\mathscr{L}(Ax(t) + Bu(t)) = AX(s) + BU(s)$$

$$\mathscr{L}(Cx(t) + Du(t)) = CX(s) + DU(s)$$

algebraically eliminated the state

$$Y(s) = \underbrace{\boxed{\left(C(sI_n - A)^{-1}B + D\right)}}_{G(s)} U(s) \boxed{+ (sI_n - A)^{-1}x(0)}$$

$C(sI-A)^{-1}x(0) \longrightarrow p \times 1$ matrix

I/O relation — Initial state - output relation

Zeta-transformed of $u(t)$

Discrete-time ($t \in \mathbb{Z}$)

$$X(z) = \mathscr{Z}(x(t)), \quad U(z) = \mathscr{Z}(u(t)), \quad Y(z) = \mathscr{Z}(y(t))$$

$$\mathscr{Z}(x(t+1)) = zX(z) - x(0)$$

$$\mathscr{Z}(Ax(t) + Bu(t)) = AX(z) + BU(z)$$

$$\mathscr{Z}(Cx(t) + Du(t)) = CX(z) + DU(z)$$

$$Y(z) = \underbrace{\boxed{\left(C(zI_n - A)^{-1}B + D\right)}}_{G(z)} U(z) \boxed{+ (zI_n - A)^{-1}x(0)}$$

I/O relation — Initial state - output relation

I/O DESCRIPTIONS

$$\mathcal{L}[Ax(t) + Bu(t)] =$$

$$A \underbrace{\mathcal{L}(x(t))}_{X(s)} + B \underbrace{\mathcal{L}(u(t))}_{U(s)}$$

$$\dot{x} = Ax + Bu$$

$$\downarrow \mathcal{L}$$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + BU(s)$$

$$\underbrace{p\left\{\left[\overbrace{C}^{n}\right]\left[\overbrace{sI - A}^{n}\right]^{-1}\right\}n \left[\underbrace{B}_{m}\right]\Big\}n}_{p \times m} \oplus \left[\underbrace{D}_{m}\right]\Big\}p$$

↳ n. of output

$y = \mathbb{R}^p$

$x = \mathbb{R}^n$

$u = \mathbb{R}^m$

Suppose a matrix:

$$M_{n\times n} \longrightarrow M^{-1} = \underbrace{\frac{1}{\det M}}_{\text{SCALAR}} \underbrace{\operatorname{adj} M}_{n\times n}$$

$$(\operatorname{adj} M)_{i,j} = (-1)^{i+j} \det \underbrace{(M^T_{i,j})}_{n-1\times n-1}$$

So, $\det(sI - A)$ = polynomial in s of order n

element $\operatorname{adj}.(s - IA)$ are polynomial in s of order $n-1$

$$G(s) = (C)_{p\times n} \underbrace{(sI - A)^{-1}}_{n\times n} (B)_{n\times m} + (D)_{p\times m}$$

each element is a rational function $\frac{\text{numerator}(s)}{\text{denumerator}(s)}$

$$= \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}_{p\times m}$$

each element of this matrix is $\dfrac{\text{num}_{i,j}(s)}{\text{den}(sI-A)}$

order $\leq n-1$

order n

In scalar cases: $\frac{num(s)}{den(s)} + d$

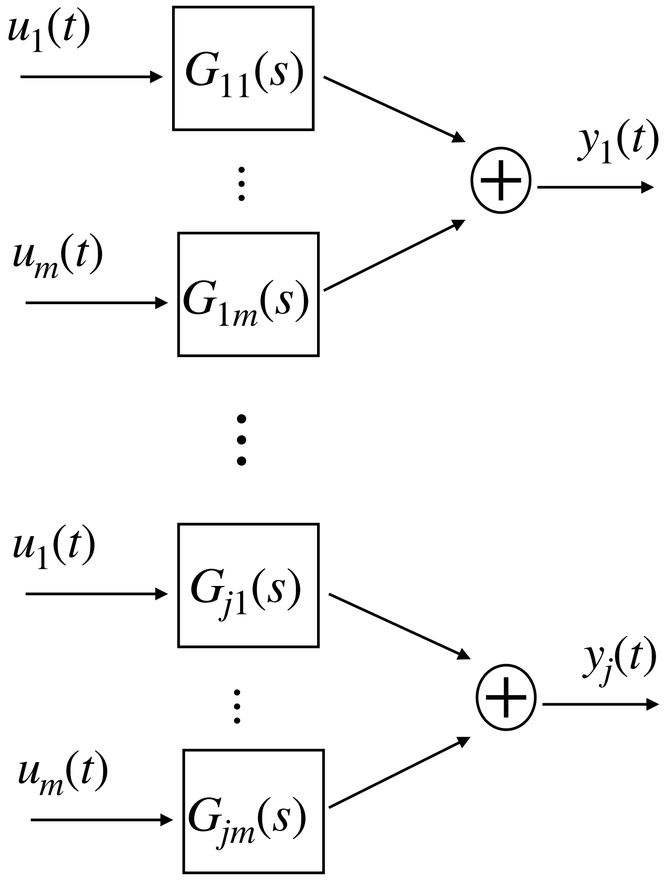
$$= \frac{num_{i,j}(s) + d \cdot den(s)}{den(s)}$$

I/O descriptions: transfer functions

$$\underset{p\times m}{G(s)} = \underset{p\times n}{C}\ \underbrace{\underset{n\times n}{(sI_n - A)^{-1}}}\ \underset{n\times m}{B} + \underset{p\times m}{D}$$

Transfer function matrix:

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pm}(s) \end{pmatrix}$$



I/O descriptions: transfer functions

$$G(s) = C \underbrace{(sI_n - A)^{-1}}_{\frac{1}{\underbrace{\textbf{det}\,(sI_n - A)}_{\text{polynomial of order } n}} \underbrace{\textbf{Adj}\,(sI_n - A)}_{\text{matrix } n \times n}} B + D$$

The (i, j) element of the Adj matrix is a polynomial in s of order $\leq n - 1$ of the form:

$$(-1)^{i+j}\,\textbf{det}\,(\underbrace{M_{ij}^T}_{\text{Minor (matrix of order } (n-1)\times(n-1) \text{ obtained by removing the } i\text{-row and } j\text{-column}})$$

$$\boxed{G_{ij}(s) = \frac{\text{Poly in } s \text{ of order } \leq n-1 \text{ if } D_{ij} = 0, \text{ of order } = n \text{ if } D_{ij} \neq 0}{\text{Poly in } s \text{ of order } n}}$$

- **Souriau** Method → computing in an algorithmic way to invert a matrix
- **Lake** Example

SOURIAU METHOD

minor of the matrix s

$$(sI - A)^{-1} = \frac{1}{\Delta_A(s)} M(s)$$

z

det. of $(sI - A)$

$$\Delta_A(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$$

$$M(s) = M_0 s^{n-1} + M_1 s^{n-2} + \dots + M_{n-1}$$

$n \times n$

$$\text{trace}(M) = \sum_{i=1}^{n} m_{n,n}$$

$$M_0 = I_{n \times n} \qquad \alpha_1 = -\text{trace}(M_0 A)$$

$$M_1 = M_0 A + \alpha_1 I_{n \times n} \qquad \alpha_2 = -\frac{1}{2}\text{trace}(M_1 A)$$

$$M_2 = M_1 A + \alpha_2 I_{n \times n} \qquad \alpha_3 = -\frac{1}{3}\text{trace}(M_2 A)$$

$$\vdots$$

$$M_{n-1} = M_{n-2} A + \alpha_{n-1} I_n$$

$$\alpha_n = -\frac{1}{n}\text{trace}(M_{n-1} A)$$

THE HYDRAULIC SYS PROBLEM

$$G(s) = (h \quad 0)(sI - A)^{-1} B \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(sI - A) = \begin{pmatrix} s+h & -k \\ 0 & s+k \end{pmatrix}$$

transposed:

$$\begin{pmatrix} s+h & 0 \\ -k & s+k \end{pmatrix}$$

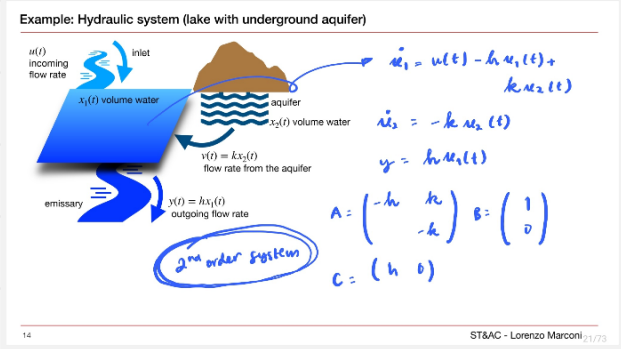
$$\det(sI - A) = (s+h)(s+k)$$

$$\text{adj.}(sI - A) = \begin{pmatrix} (-1)^2(s+k) & (-1)^3(-k) \\ (-1)^3(0) & (-1)^4(s+h) \end{pmatrix}$$

So, the transfer funct. of $G(s)$:

$$G(s) = \frac{h\cancel{(s+k)}}{(s+h)\cancel{(s+k)}} = \frac{h}{s+h}$$

the transfer function eliminates the dynamics that have no effects to the system. It only focuses on variables that affect the system.



Realization problem

How to systematically obtain an SS model from a TF? The problem is said "**Realization**". It's not unique, namely there are infinite SS models linked to the same TF

Assumptions: $D = 0\,,\ m = p = 1$

single input single output system

$$G(s) = \frac{\overbrace{\beta_1 s^{n-1} + \beta_2 s^{n-1} + \cdots + \beta_n}^{\text{order } n-1}}{\underbrace{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}_{\text{order } n}} \xrightarrow{\ ?\ } (A, B, C)$$

- **Control Canonical form**
  (suitable for control)

$$A_c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix} \qquad B_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad C_c = \begin{pmatrix} \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_2 \\ \beta_1 \end{pmatrix}^T$$

transposed

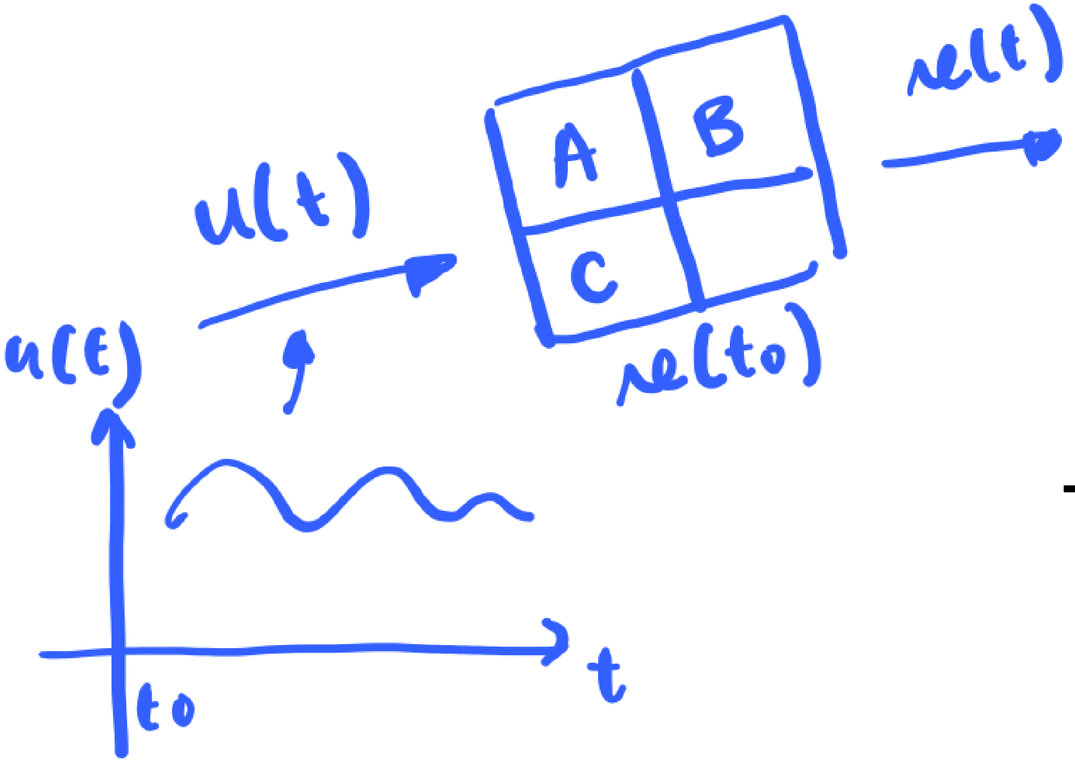
transposed

transposed

- **Observer Canonical form**
  (suitable for observation)

$$A_o = A_c^T \qquad B_o = C_c^T \qquad C_0 = B_o^T$$

Homework: double check!



Trajectories

Lagrange Formula (linear systems)

→ in notion of exponential of the matrix

The state and output trajectory $(x(t), y(t))$ depends on the initial state $x(t_0)$ and on the specific input applied to the system in the time interval $[t_0, t]$. For linear systems (A, B, C, D) the trajectory can be given an explicit form in which the effect of the initial state and of the input are decoupled.

$x(t_0)$ → initial condition

Continuous-time ($t \in \mathbb{R}$)

$$x(t) = e^{A(t-t_0)}\, x(t_0) + \int_{t_0}^{t} e^{A(t-s)}\, B\, u(s)\, ds$$

$$y(t) = Ce^{A(t-t_0)}\, x(t_0) + \int_{t_0}^{t} Ce^{A(t-s)}\, B\, u(s)\, ds + Du(t)$$

↳ state

→ generic time

→ initial time

s is integration variable

↳ output

Free state evolution | Forced state evolution | Free output evolution | Forced output evolution

Discrete-time ($t \in \mathbb{Z}$)

$$x(t) = A^{t-t_0}\, x(t_0) + \sum_{i=t_0}^{t} A^{t-1-i}\, B\, u(i)\, ds$$

$$y(t) = CA^{t-t_0}\, x(t_0) + \sum_{i=t_0}^{t} CA^{t-1-i}\, B\, u(i)\, ds + Du(t)$$

↳ the input affects not only at $t(0)$ but also the next t

let's say $n \times n$ dimension

$$e^M := I + M + \frac{1}{2!}M^2 + \ldots + \frac{1}{k!}M^k \ldots \; k \to \infty$$

↳ think of Taylor expansion

why is it relevant? rewatch the recording

Proof...

LAGRANGE FORMULA

$$\left(x(t)\right) = \underbrace{\left(e^{A(t-t_0)}\right)\overbrace{\left(x(t_0)\right)}^{\text{initial state}}}_{\text{free-state evolution}} + \underbrace{\int_{t_0}^{t} \left(e^{A(t-s)} \overset{n\times m}{B}\right) \cdot \overset{n\times n}{u(s)}\, ds}_{\text{forced-state evolution}}$$

$$\begin{pmatrix} u_1(s) \\ \vdots \\ u_m(s) \end{pmatrix}$$

Proof...

1.) $x(t_0) = x_0 \longrightarrow$ with t_0, the free-state evolution becomes I and forced-state becomes 0

2.) $\frac{d}{dt}(\text{log. form}) = A(\text{log. form}) + Bu(t)$

$$\frac{dx(t)}{dt} = \underbrace{\frac{d}{dt} e^{A(t-t_0)} x(t_0)}_{Ae^{A(t_1-t_0)}x(t_0)\ (*)} + \underbrace{\boxed{\frac{d}{dt}\int_{t_0}^{t} \dots}}_{(**)} = (***)$$

(*) remember:

$$\frac{d e^{at}}{dt} = a \cdot e^{at}$$

$$\frac{d}{dt}(e^{At}) = \frac{d}{dt}\Big(I + \underset{\substack{\uparrow \\ A}}{At} + \underset{\substack{\uparrow \\ A^2 t}}{\frac{1}{2!}A^2t^2} + \underset{\substack{\uparrow \\ \frac{1}{2!}\cdot\frac{3}{3!}A^3t^2}}{\frac{1}{3!}A^3t^3} + \dots\Big)$$

der. der. der.

$$\boxed{f(t,s) = e^{A(t-s)}Bu(s)}$$

(**)

$$\frac{d}{dt}\int_0^t f(t,s)\,ds =$$

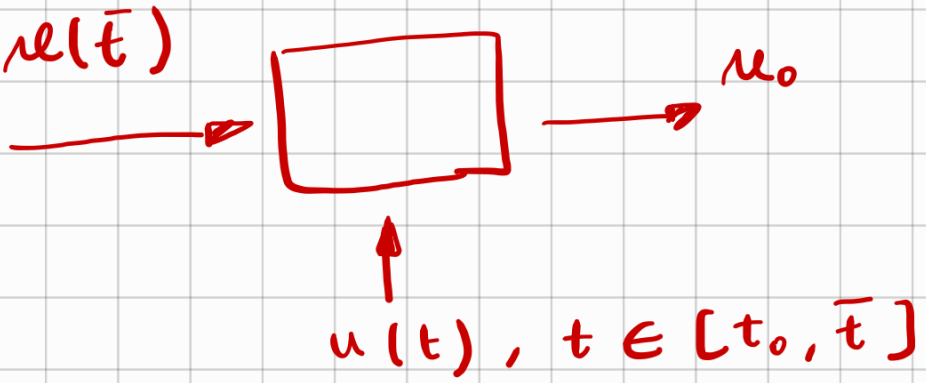
$$f(t,t) + \int_0^t \frac{d}{dt} f(t,s)\,ds$$

$$\Downarrow$$

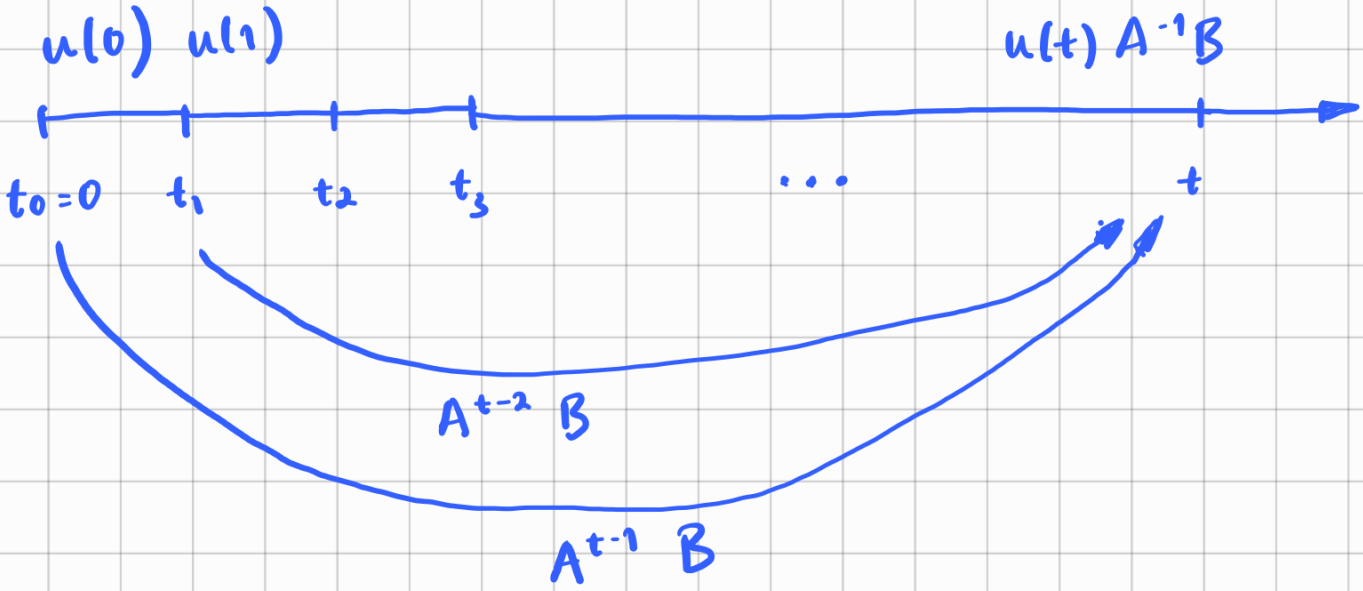
$$Bu(t) + \int_0^t A e^{A(t-s)} Bu(s)\,ds$$

(***)

$$\frac{dx(t)}{dt} = A\boxed{\left(e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-s)}Bu(s)\,ds\right)} + Bu(t)$$



DISCRETE-TIME ($t \in \mathbb{Z}$)



Lagrange Formula (linear systems)

Nice properties of linear (stationary) systems immediately visible from the Lagrange formula:

- Effects of (sum of) initial state(s) and (sum of) input(s) add up (**Additivity**)
- "Large"/"small" experiments differ only by scaling factors (**Homogeneity**)

Principle of superimposition

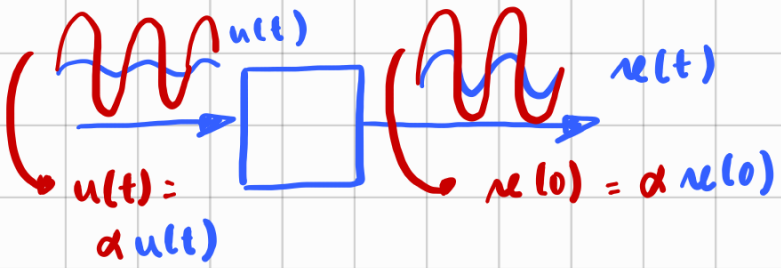
- The effects of a stimulus (initial state/input) applied at different times are the same, simply shifted in time (**shift-invariance**)
- Properties of the A affects the free state evolution, properties of A, B affects the state forced evolution
- C plays a role in the output free and forced evolution

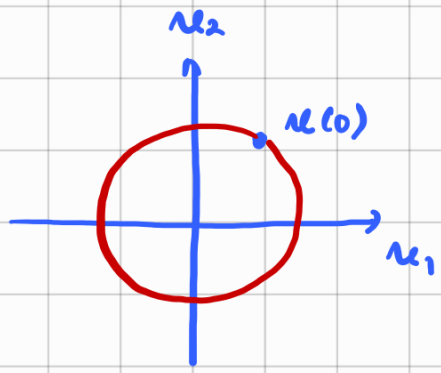
Principle of superimposition: property of linear systems (not necessarily stationary)

Shift-invariance: property of stationary systems → only holds for stationary systems

PROPERTIES OF LAGRANGE FORMULA

- Homogenity:





It shows the linearity by adding the scaling (α) factor

TRANSITION MARIX

$$\Phi(t) = \begin{cases} e^{At} & \text{for continuous time} \\ A^t & \text{for discrete time} \end{cases}$$

phi

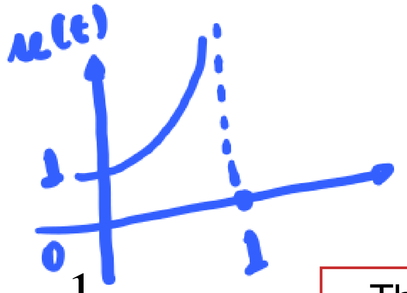
What about nonlinear systems?

Unfortunately, Lagrange-type formulas do not exist, in general, for nonlinear systems. The effect of the initial state and of the input are not "decoupled" but "mix-up".

Let's consider **autonomous systems** of the form $\dot{x} = f(x)$ (continuous-time without too much loss of generality).

Formally, we say that $x(t)$ is a solution for this ODE (trajectory for the system) with initial state x_0 at $t_0 = 0$ if

$x(0) = x_0$ and $\dot{x}(t) = f(x(t))$ for all $t > 0$



$\dot{x} = x$

$x(t) = e^t x_0$

For general nonlinear systems trajectories could be "pathological":

- **Finite Escape time**. Example: $\dot{x} = x^2 \quad x(0) = 1$ → non-linear

  $$x(t) = \frac{1}{1-t}$$

  Becomes a problem when the trajectory goes to infinite (the solution)

  The trajectory does not exist for $t \geq 1$

- **Uniqueness**. Example: $\dot{x} = 2\sqrt{x} \quad x(0) = 0$

  $$x(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq a \\ (t-a)^2 & \text{if} \quad t > a \end{cases}$$

  $a \geq 0$ any

  Becomes a problem when the solution is not unique (the solution)

  The trajectory is not unique

What about nonlinear systems?

Existence (for all t) and uniqueness are guaranteed if certain regularity properties on the vector field $f(\cdot)$ are assumed.

f(x) is Lipschitz continuous when its f(x)-f(y) is bounded by

y is another point

Lipschitz continuity: $\|f(x) - f(y)\| \leq c\|x - y\|$ for all (x, y) for some $c > 0$

$\dot{x} = f(x)$

Result: Uniform boundedness of the Jacobian $\dfrac{\partial F}{\partial x}(x)$ is a (strong) sufficient condition for existence and uniqueness of the solution

Continuity $\leq$ LIPSCHITZ CONTINUITY $\leq$ Differential Continuity

What about nonlinear systems?

it becomes catastrophic when the input changes

The effect of the input in nonlinear systems could be catastrophic. Systems with "nice" properties when $u = 0$ could have "deteriorated" behaviour when an input is applied. A clear distinction between free and forced evolution doesn't exist in general.

no input

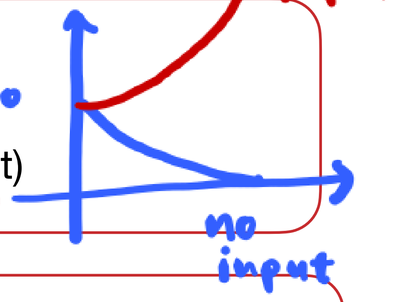
$u=0,\ x(t) = e^{-t}x(0)$

non-linear

$$\dot{x} = -x + xu$$

$\dot{x} = -x$ when $u = 0$ (exponential convergence to zero without input)

$\dot{x} = x$ when $u = 2$ (exponential explosion with (sufficiently large) input)



$$\dot{x} = -x + (1+x)xu$$

$\dot{x} = -x$ when $u = 0$ (exponential convergence to zero without input)

$\dot{x} = x^2$ when $u = 1$ (finite escape time with bounded input)

output goes to infinite in finite time

$$\dot{x} = -x + (x^2+1)u$$

$\dot{x} = -x$ when $u = 0$ (exponential convergence to zero without input)

$u(t) = 1/\sqrt{2t+2}$ and $x(0) = \sqrt{2}$ leads to $x(t) = \sqrt{2t+2}$ (explosion with vanishing input)

when the input $u(t) \to 0$

Superimposition properties are far to be true

Qualitative Behaviour of Trajectories

a tool for dealing w/ nonlinear systems

Let $u = 0$ and consider C-T systems

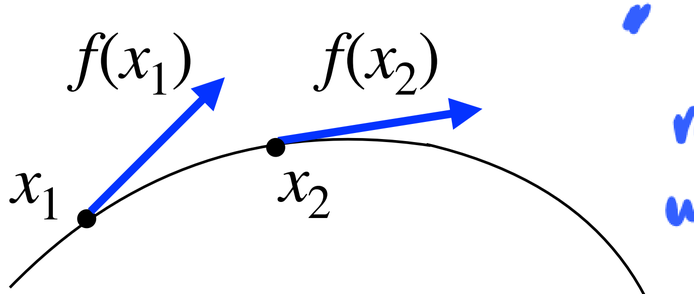
when the u is time-varing, it's useless

For dynamics of dimension $2, 3$ $(n = 2, 3)$, a qualitative behaviour of the trajectory can be obtained by plotting the "vector field" and the associated "phase portrait" ("phase plane" method)

A vector field is an assignment of a vector (which is $f(x)$) to each point x in the space/manifold

A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point

For **continuous-time systems**, at each $x \in \mathbb{R}^n$ $f(x)$ is a vector of $\mathbb{R}^n$, which is tangent to the trajectory (flow) passing through x. The geometry of the vector yields local information about the direction of the trajectory. The amplitude about the local "speed" $f(x)$ is the velocity vector at x.



" geometric representation w/o integrating "

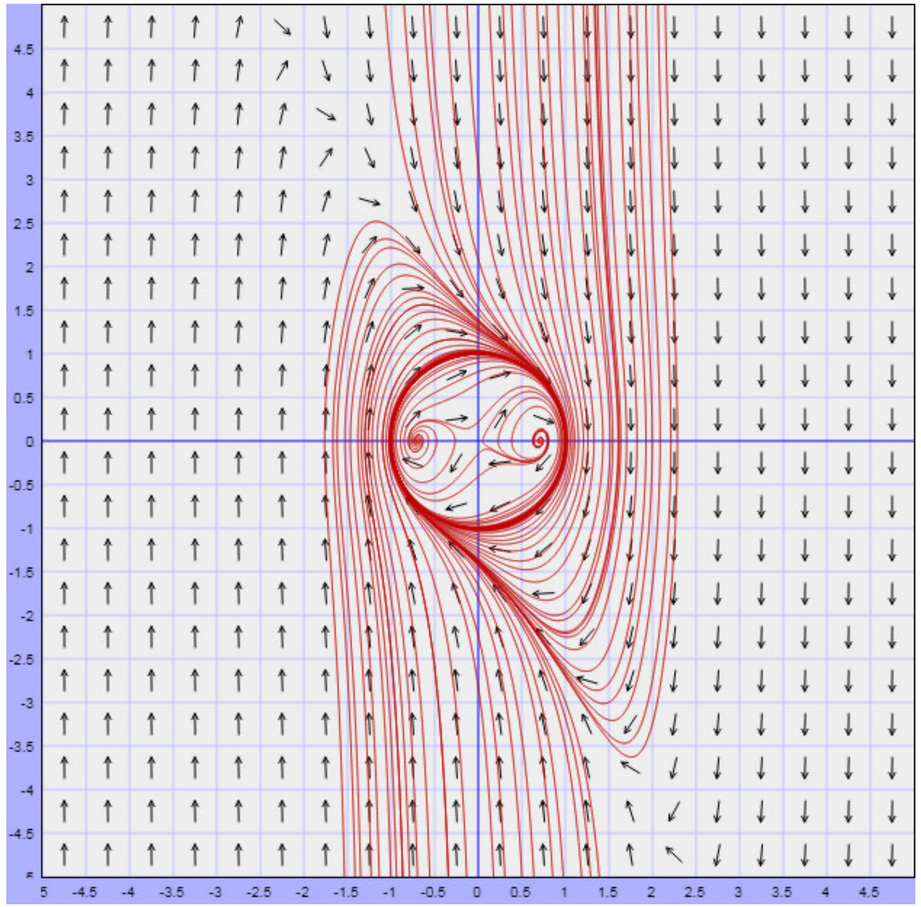
planar

A vector field is easy to plot. Covering $\mathbb{R}^2$ with vectors yield qualitative insight about the whole set of trajectories originating from different initial conditions. Up to $n = 3$ the tool is valuable.

Qualitative Behaviour of Trajectories

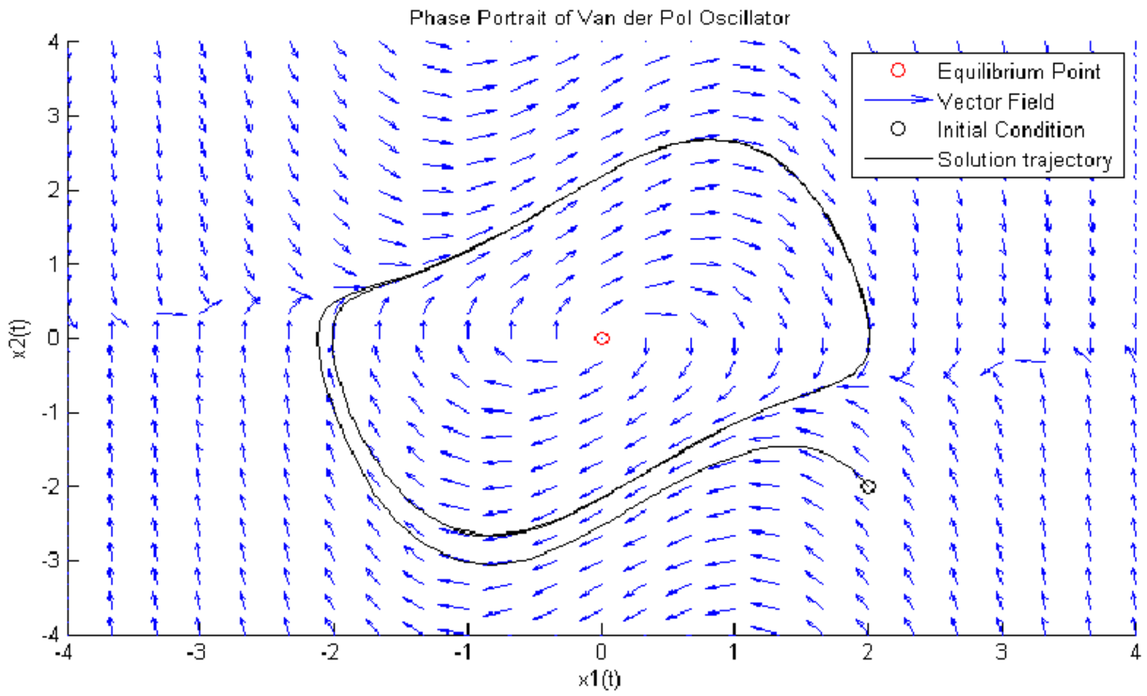
$n = 2$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2^3 - 2x_1^3 - x_1^2 x_2 - 2x_2^2 x_1 + x_1 + x_2$$



Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 \end{aligned}$$



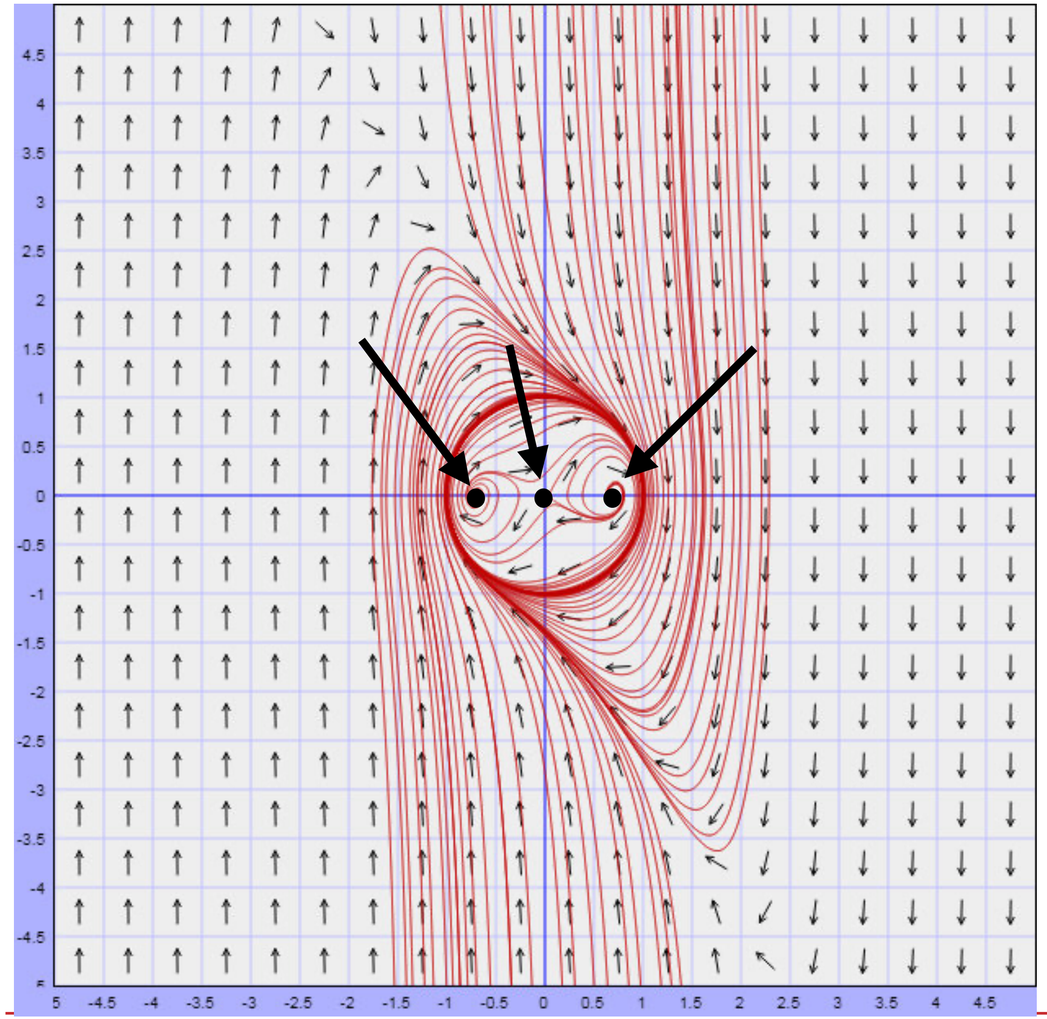
Qualitative Behaviour of Trajectories

Fixed Points (Equilibrium Points): points $x^\star \in \mathbb{R}^n$ such that $f(x^\star) = 0$

the first critical step is to compute its equilibria

$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ calculate its locus

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \underbrace{-x_2^3 - 2x_1^3 - x_1^2 x_2 - 2x_2^2 x_1 + x_1 + x_2}_{=0}$$

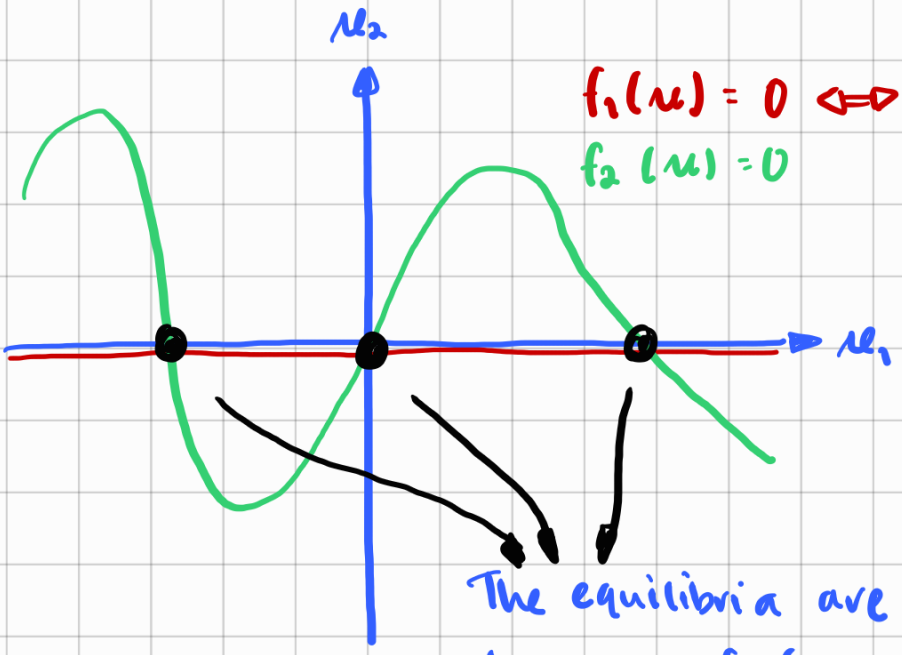


$$x_2^\star = 0, \;\; x_1^\star = 0, \;\; x_1^\star = \pm \frac{1}{\sqrt{2}}$$

Three isolated equlibria

Nonlinear systems could have **isolated** equilibrium points

Linear systems have a single equilibrium or a subspace of equilibria ($\{x^\star \;:\; x^\star \in \mathrm{Ker} A\}$)



$f_1(x) = 0 \iff x_2 = 0$

$f_2(x) = 0$

The equilibria are the intersections between $f_1(x)$ and $f_2(x)$.

$$-2x_1^3 + x_1 = 0$$

$$x_1(1 - 2x_1^2) = 0$$

$$x_1 = \pm\sqrt{\frac{1}{2}}$$

Example for linear systems

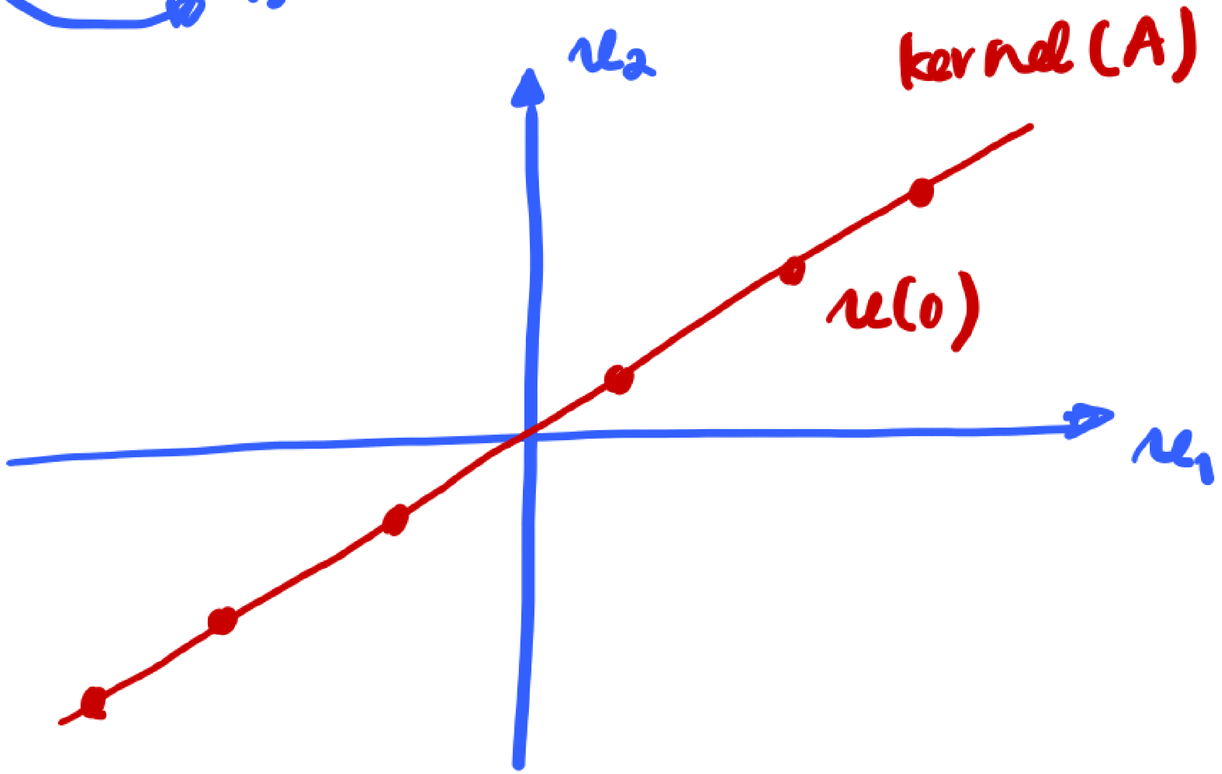
kernel of A

$$\dot{x} = Ax \qquad A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \textbf{Ker}\, A = \textbf{Im} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an image of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x^* : Ax^* = 0$

$x^* = \begin{pmatrix} x_1^* & \text{any} \\ x_1^* & \end{pmatrix}$

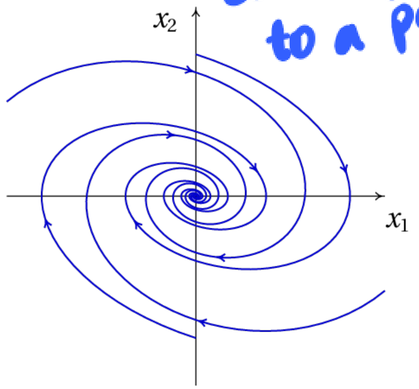


we'll find eq. along the line

Qualitative Behaviour of Trajectories

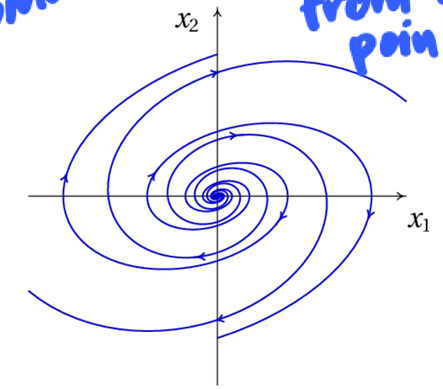
Equilibrium points can be **attractive** (all nearby trajectories converge to the point as $t \to \infty$), **repulsive** (all nearby trajectories converge to the point as $t \to -\infty$), "**saddles**" points with some directions attractive and others repulsive, they can be "anchor points" of "complex" orbits, such as **homoclinic orbits** (a trajectory hanged at the equilibrium), **heteroclinic orbits** (a trajectory connecting two isolated equilibria), or "**centers**" around which closed curves take place….

converge to a point



Attractive

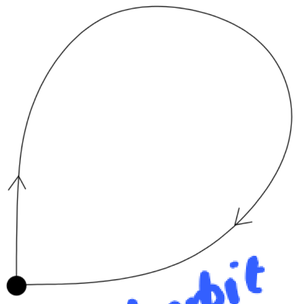
diverge from a point



Repulsive

at some points are attractive, the other are repulsive

Saddle

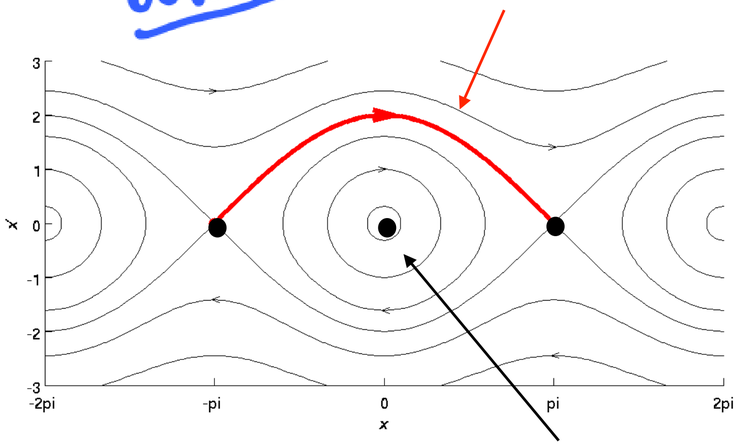


closed orbit

Homoclinic

closed orbit OUTSIDE of the equilibrium

Heteroclinic



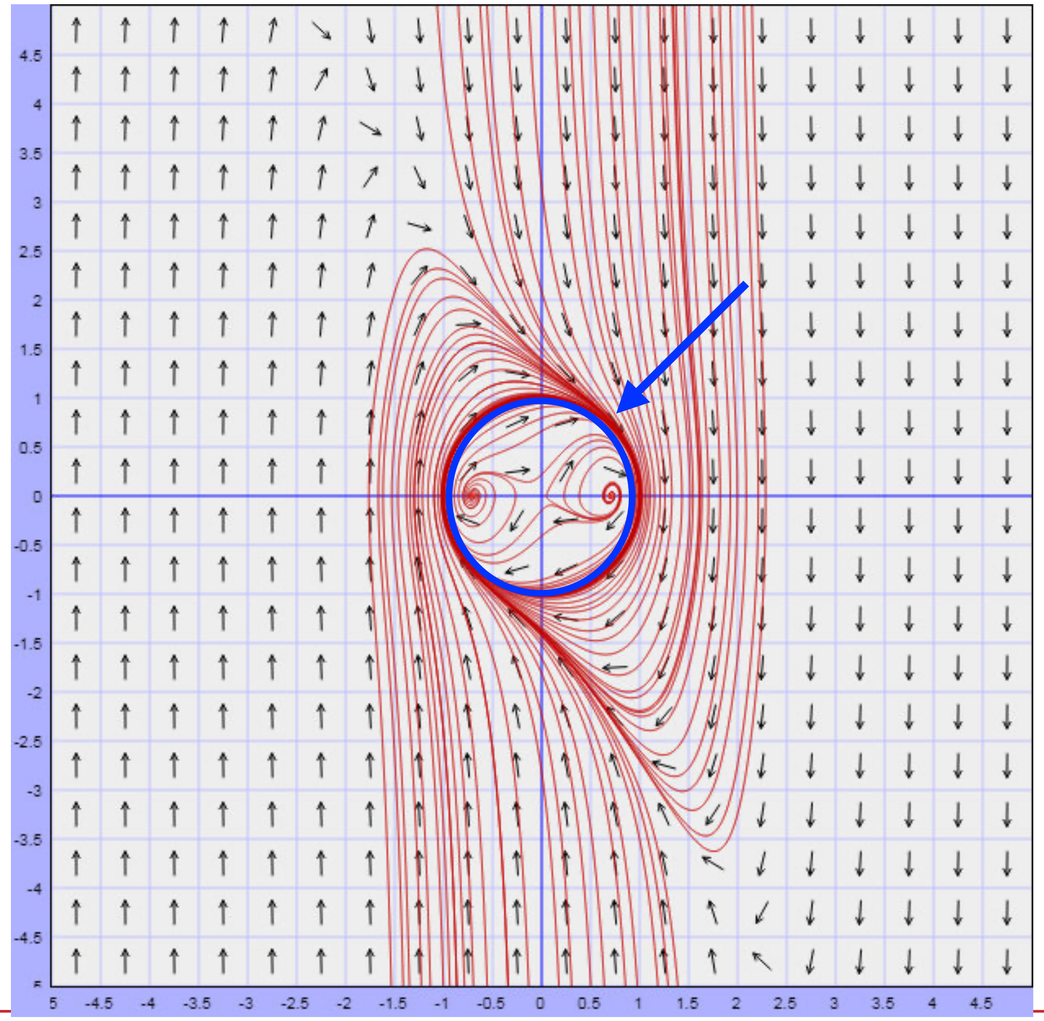
center of closed curves

Rich geometry!

Locally around the equilibrium point the "behaviour" of the trajectory can be studied by using the linearised dynamics (see later)

Qualitative Behaviour of Trajectories

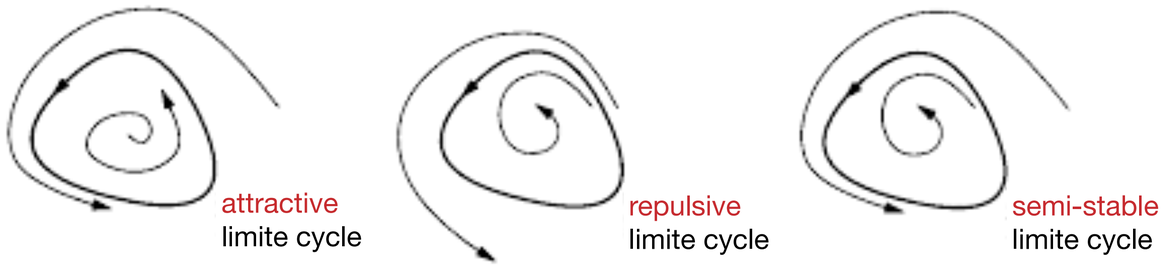
Limit cycle: **isolated** closed trajectory having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity



A limit cycle is a locus of periodic trajectories of the system

Period orbit with period T: $x(t) = x(t+T)\ \forall t \geq 0$

Limit cycles can be categorised according to their attractive/ repulsive properties:



All neighbourhood trajectories converge to the limit cycle as $t \to \infty$

All neighbourhood trajectories converge to the limit cycle as $t \to -\infty$

Convergence form one side, repulsive from the other

Qualitative Behaviour of Trajectories

not to integrate the equations!

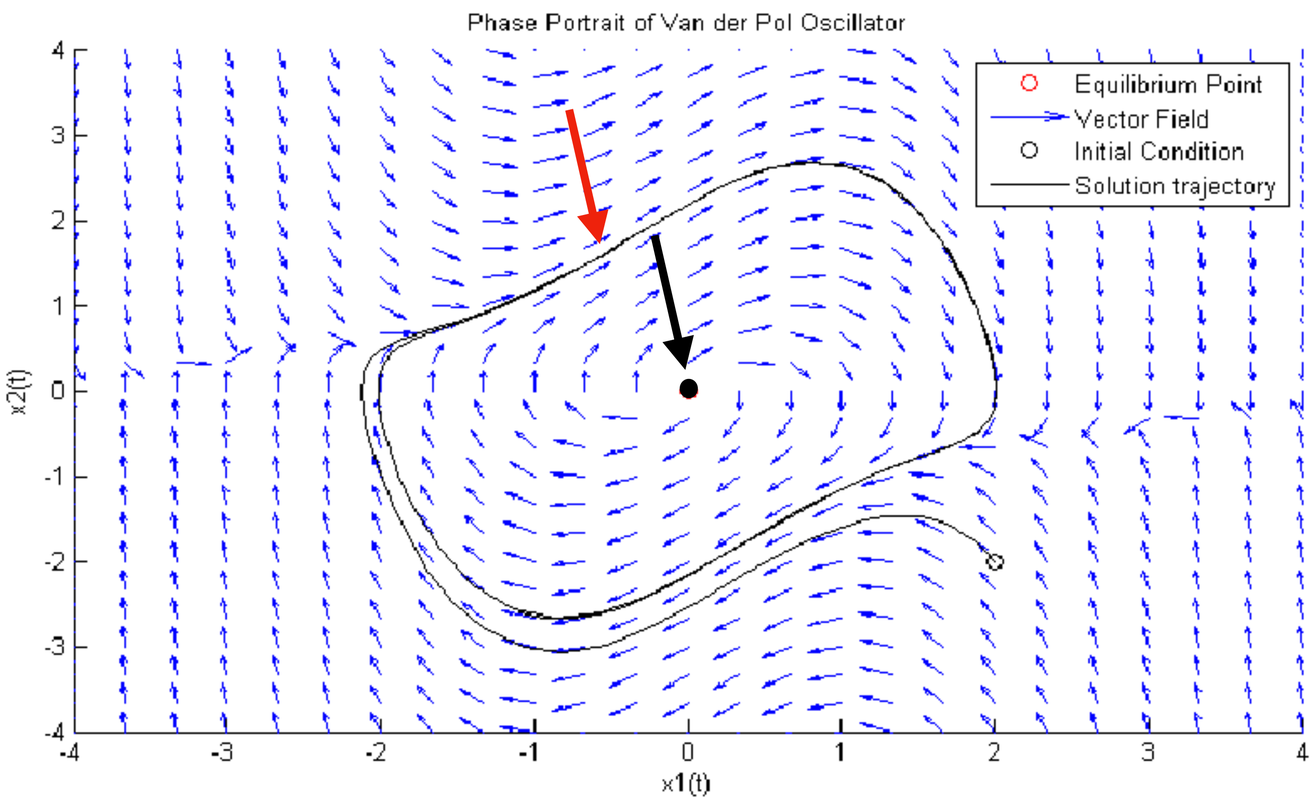
Van der Pol oscillator

$A = \begin{bmatrix} 0 & \kappa \\ -1 & 0 \end{bmatrix}$

$$\begin{aligned} \dot{x}_1 &= \kappa x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 \end{aligned}$$

$= 0$

$$x_1 = x_2 = 0$$



The origin is the only equilibrium point, which is repulsive. There exists a limit cycle that is attractive

MATLAB Friday

Homework: plot how the geometry of the limit cycle and the features of the periodic trajectory change when (μ, κ) change

Homework: plot how the phase portrait changes when $\mu = 0$ (linear oscillator)

Qualitative Behaviour of Trajectories

Pendulum

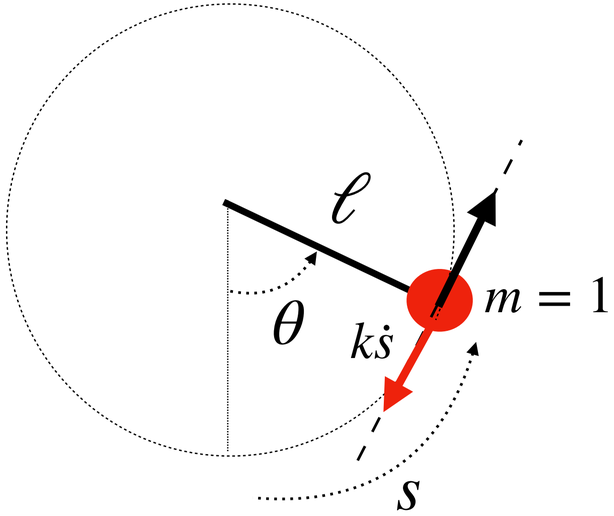
(mathematically tricky in state-space)

No friction

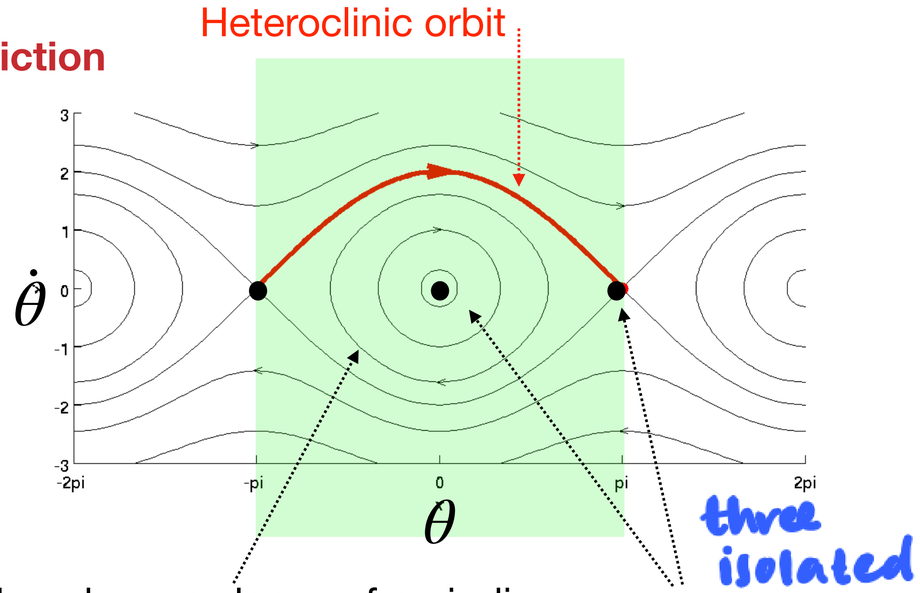
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

With friction

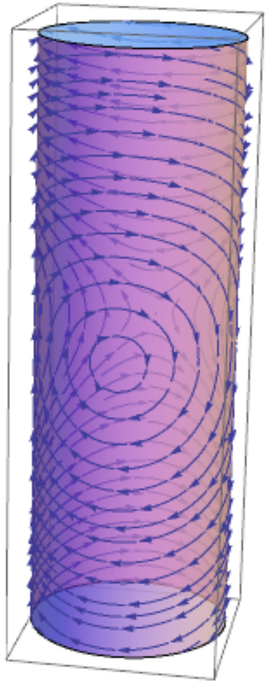
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - k\ell x_2$$



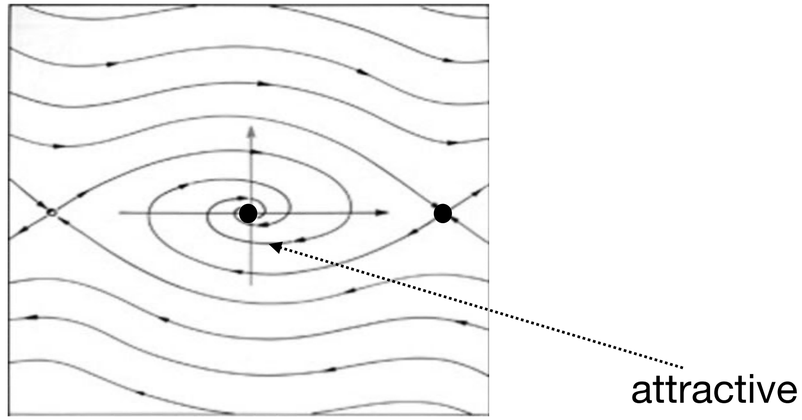
No friction



Phase portrait on the cylinder manifold



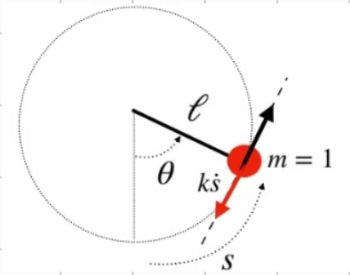
With friction



PENDULUM

$x_1 = \theta$

$x_2 = \dot{\theta} \iff \dot{x}_1 = x_2$



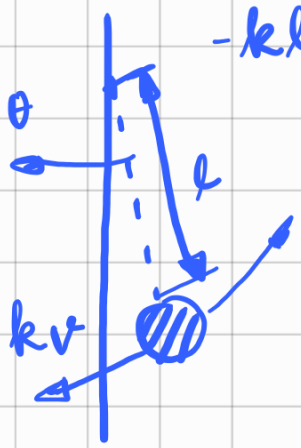
$$J\dot{x}_2 = \sum \text{torque}$$

$$= -\ell g \sin x_1$$

$J \to \ddot{\theta}$, $x_1 \to \theta$, $J = m\ell^2$

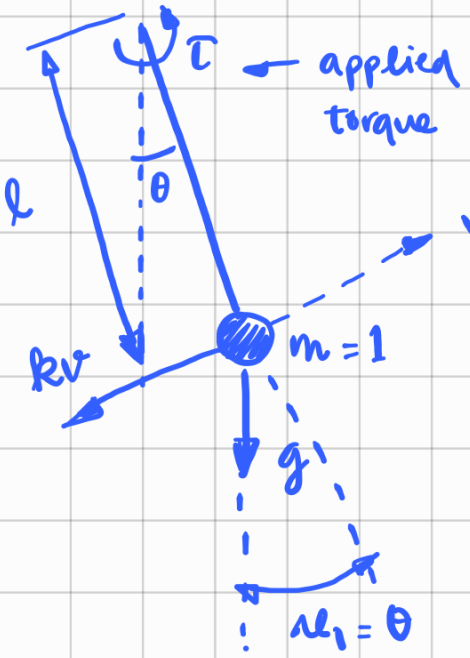
if friction included:

$-k\ell^2 x_2$



$v = \ell\dot{\theta}$

$= \ell x_2$



$v = \ell\dot{x}_2$

tangential velocity

$x_1 = \theta$

$x_2 = \dot{\theta}$

$\dot{x}_1 = x_2$

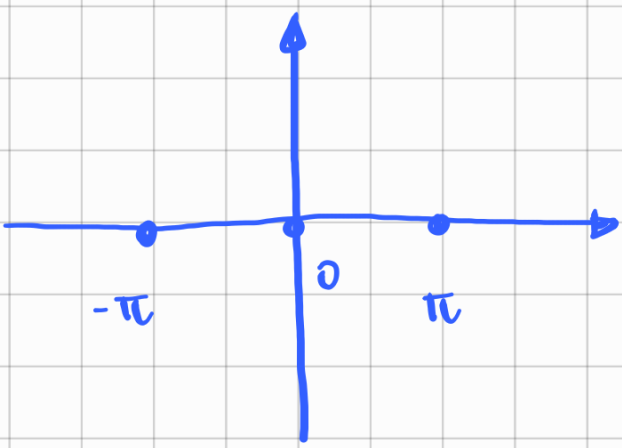
$J\dot{x}_2 = \sum \text{Torques}$

$= -g\ell \sin x_1 - k\ell \cdot \underbrace{\ell x_2}_{v} + \tau$

length of the arm

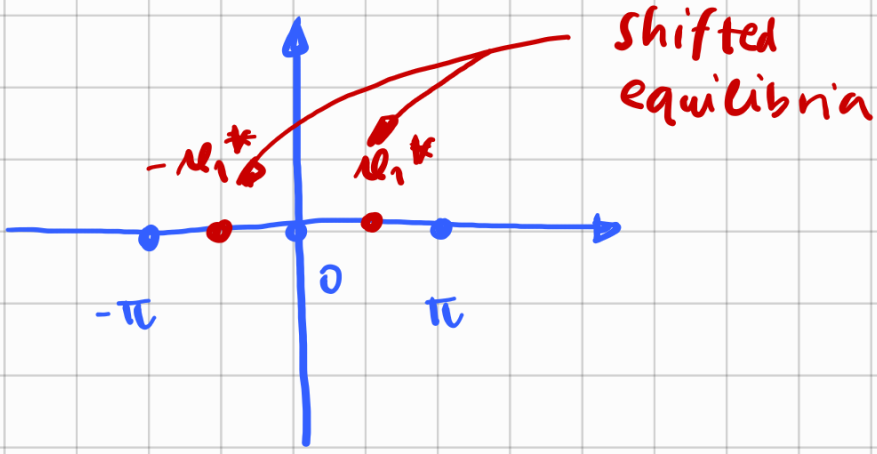
$J = \ell^2$

The equilibria for $\tau = 0$



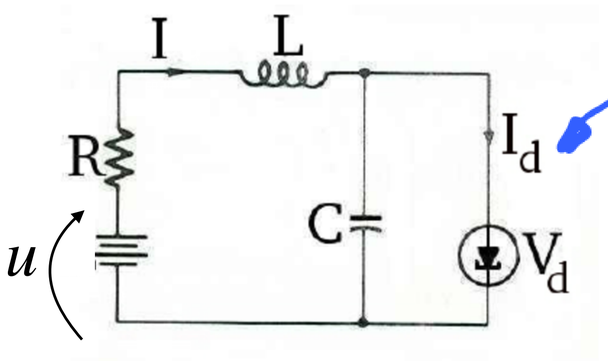
Once the $\tau \neq 0$ where τ is constant,

$$gl \sin u_1 = \bar{\tau}$$

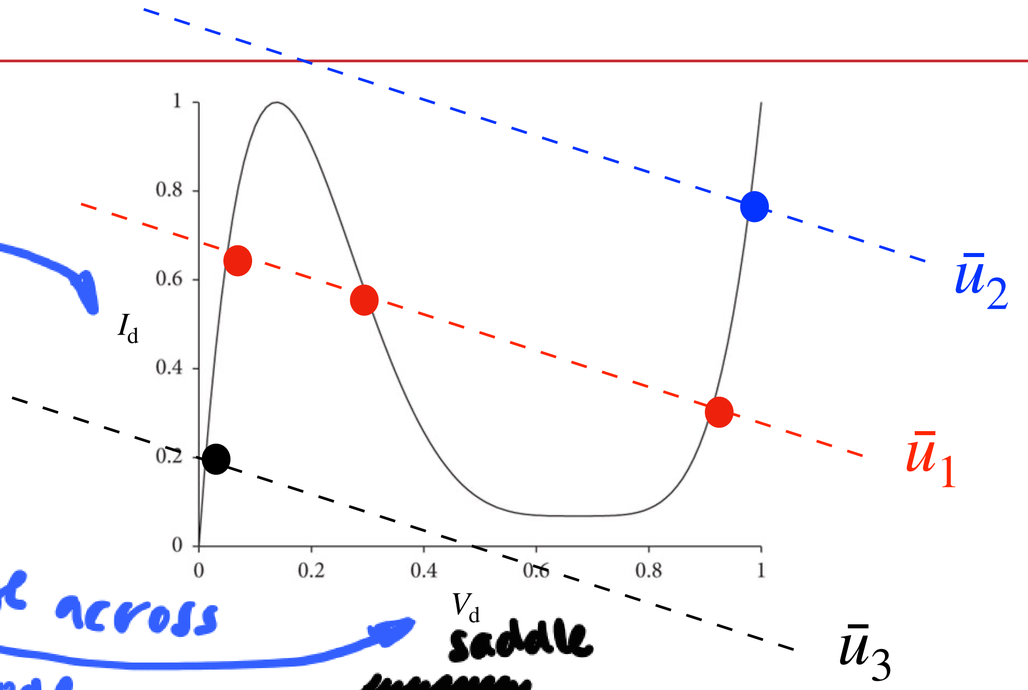


Qualitative Behaviour of Trajectories

Tunnel Diode example

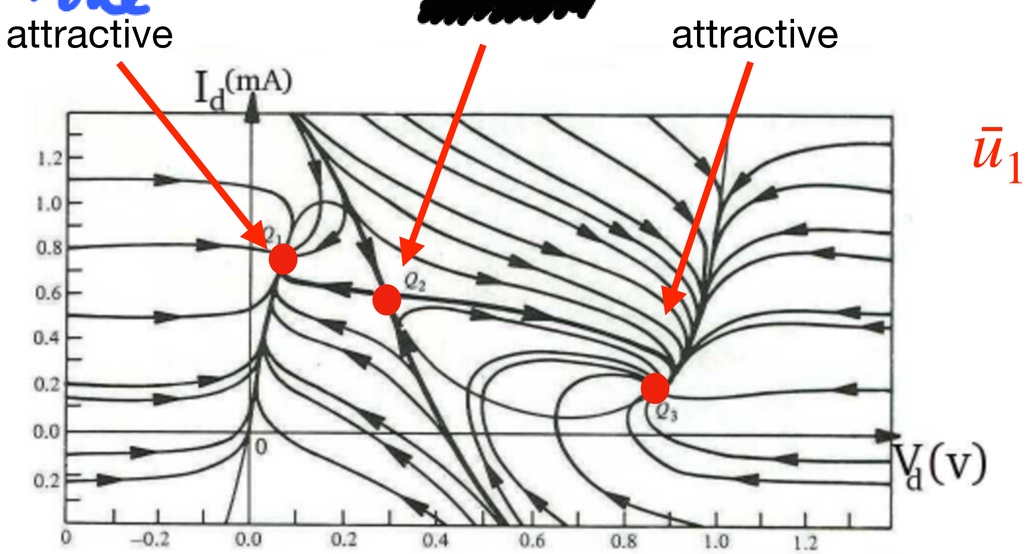


current



voltage across the diode

$$C\dot{x}_1 = x_2 - h(x_1)$$
$$L\dot{x}_2 = -x_1 - Rx_2 + u$$



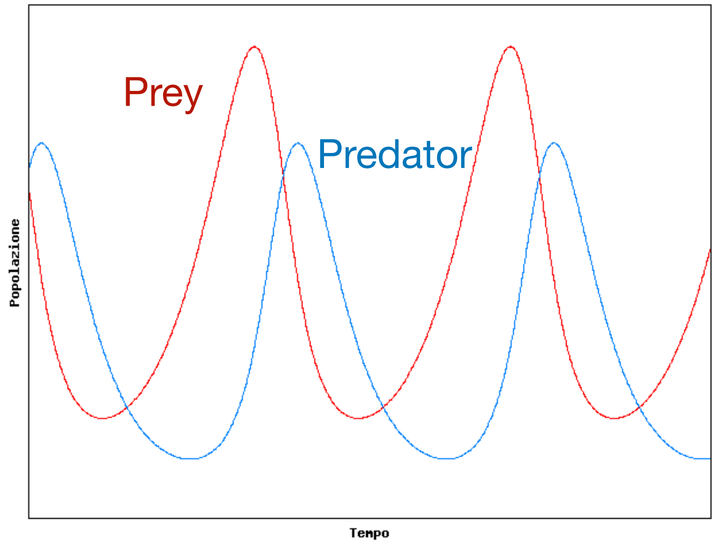
Homework: plot the phase portrait in the other two cases of $\bar{u}_2, \bar{u}_3$

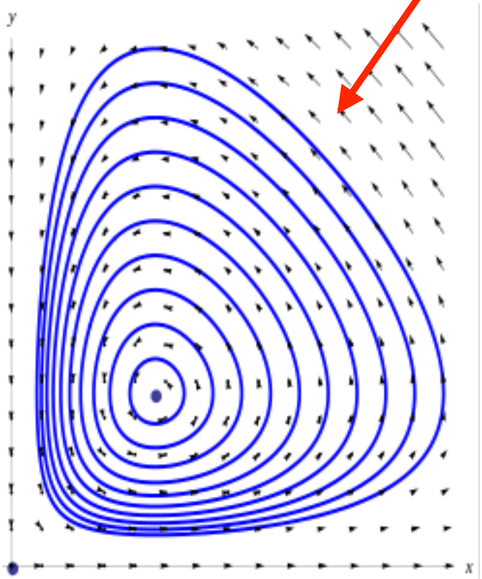
Qualitative Behaviour of Trajectories

Dense periodic trajectories
(they are not limit cycles)

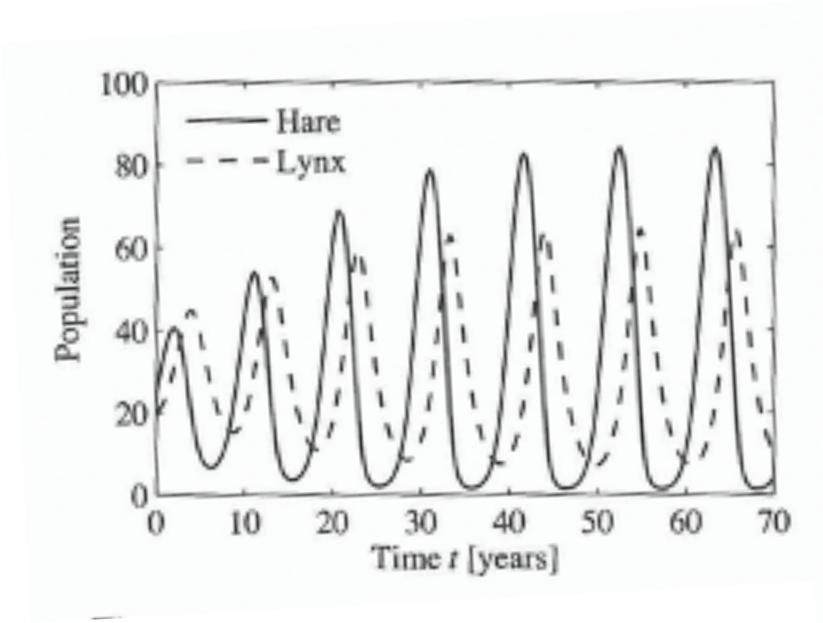
Predator-Prey system

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \delta x y - \gamma y$$

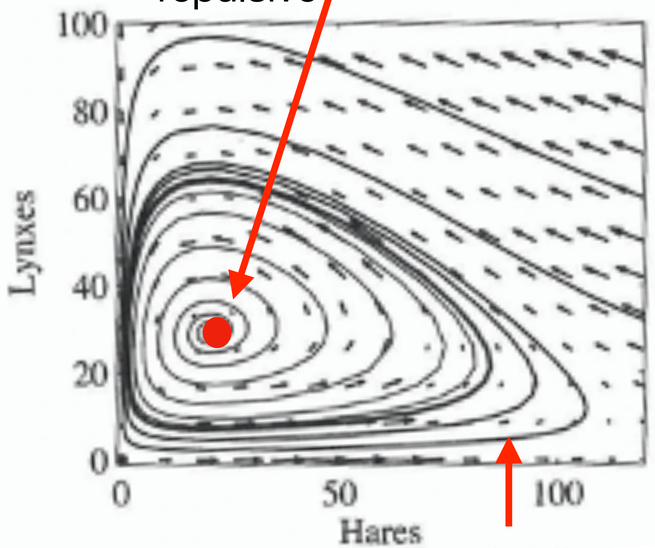




$$\dot{x} = r x (1 - \frac{x}{k}) - \frac{a x y}{c + x}$$
$$\dot{y} = b \frac{a x y}{c + x} - d y$$



repulsive



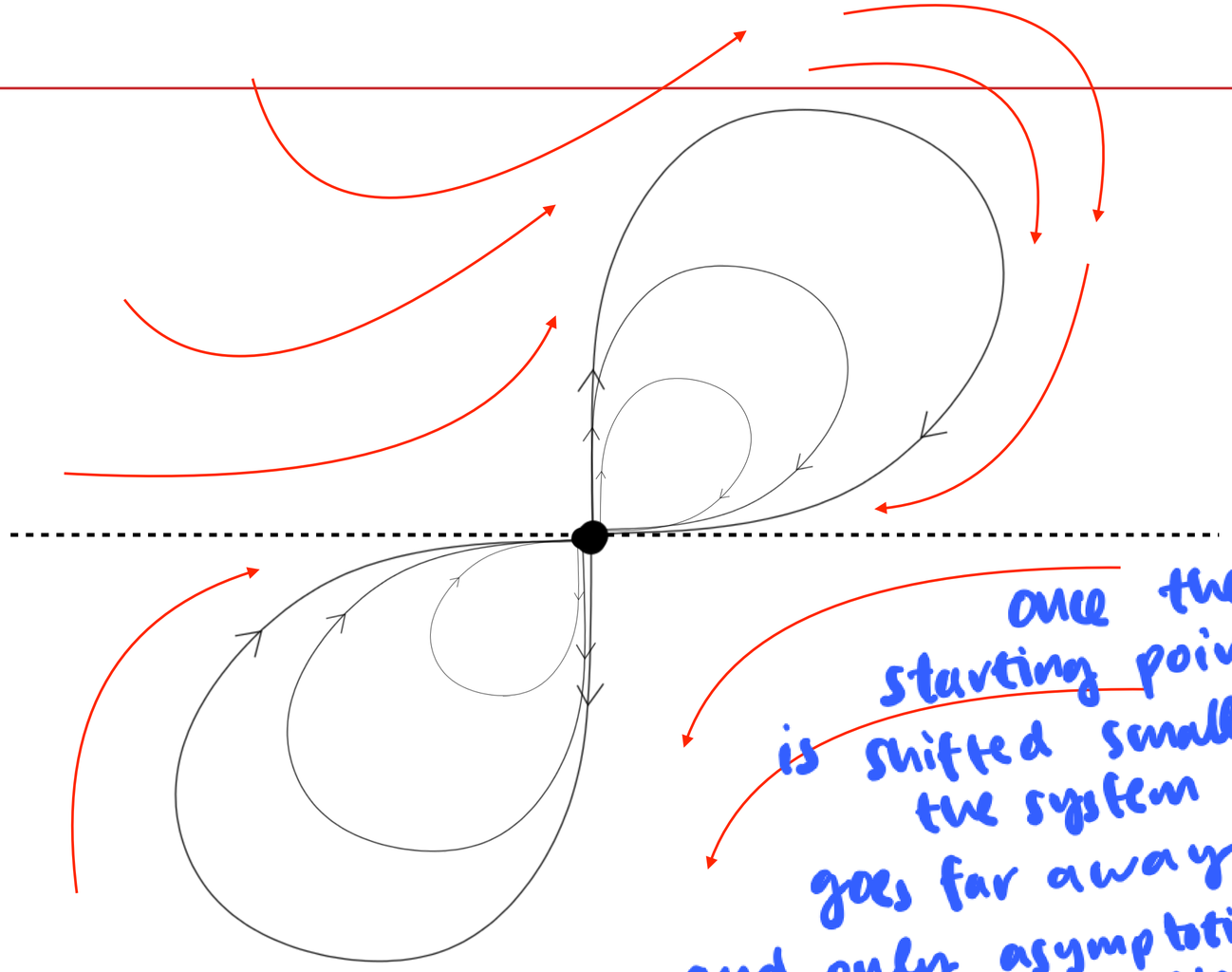
attractive limit cycle

Qualitative Behaviour of Trajectories

Example of a system with homoclinic orbit

$$\dot{x} = \frac{x^2(y-x)+y^5}{(x^2+y^2)(1+(x^2+y^2)^2)}$$

$$\dot{y} = \frac{y^2(y-2x)}{(x^2+y^2)(1+(x^2+y^2)^2)}$$



once the starting point is shifted small, the system goes far away and only asymptotically approach the eq.

The origin is the only equilibrium point. It is "globally attractive". From an engineering perspective such an equilibrium point is not "nice" (small perturbations of the initial state from the equilibrium lead to transients whose amplitude is not related to the amplitude of the perturbation)

Example: Susceptible-Infected Model (COVID'19)

Homework: plot and interpret the phase portrait of the SI model

$$\begin{bmatrix} \dot{S}(t) \\ \dot{I}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\beta I(t) & 0 \\ 0 & \beta S(t) - \gamma \end{bmatrix}}_{A(t)} \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$$

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \gamma I$$

$$\dot{x}_1 = -\beta x_1 x_2$$
$$\dot{x}_2 = \beta x_1 x_2 - \gamma x_2$$

Do it on MATLAB

Qualitative Behaviour of Trajectories

Finding limit cycles, in general, is a very difficult problem. There exist a many theoretical results predicting the existence or the absence of limit cycles of **two-dimensional** nonlinear dynamical systems. Examples:

has closed cycle

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Result

Every closed trajectory contains within its interior at least an equilibrium.

Bendixson's criterion

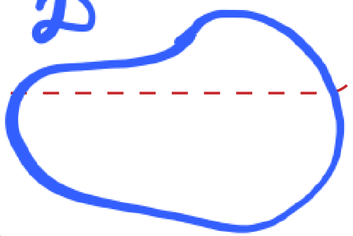
If on a simply connected region $\mathscr{D} \subset \mathbb{R}^2$ (i.e. $\mathscr{D}$ has no holes in it) the function

$$\nabla f(x) := \frac{\partial f_1(x_1, x_2)}{\partial x_1} + \frac{\partial f_2(x_1, x_2)}{\partial x_2}$$

is not identically zero and does not change sign, then the system has no closed orbits lying entirely in $\mathscr{D}$.

$\forall x \in \mathscr{D}, \nabla f(x)$

$\mathscr{D}$



Divergence of the vector field

Example

$$\dot{x}_1 = x_2 + x_1 x_2^2$$
$$\dot{x}_2 = -x_1 + x_1^2 x_2$$

$$\nabla f(x) = x_1^2 + x_2^2$$

No closed orbits

Homework. For the linear system $\dot{x}_1 = a_{11}x_1 + a_{12}x_2$, $\dot{x}_2 = a_{21}x_1 + a_{22}x_2$, find conditions on a_{ij} so that the system has not periodic solutions by using the Bendixson's criterion → inspect!

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2$$
$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2$$

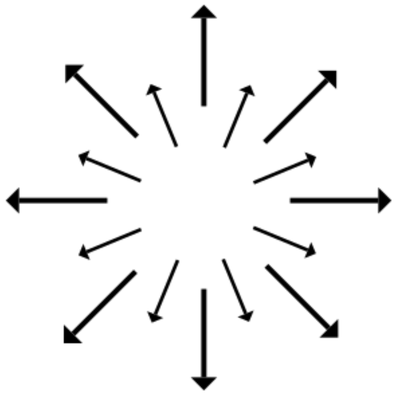
$$\nabla f(x) = a_{11} + a_{22}$$

$$a_{11} + a_{22} \neq 0$$

Qualitative Behaviour of Trajectories

Interpretation of divergence of a vector field at a point: the extent to which the vector field flux behaves like a source locally around that point (local measure of its “outgoingness”).

Source

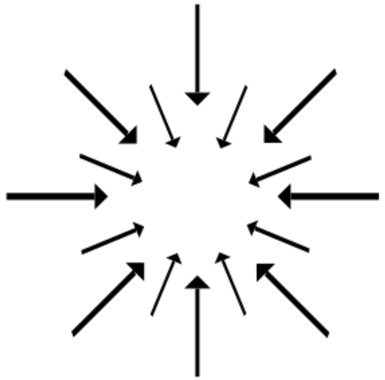


$$\frac{\partial f_1}{\partial x_1} > 0$$

$$\frac{\partial f_2}{\partial x_2} > 0$$

$$\nabla f > 0$$

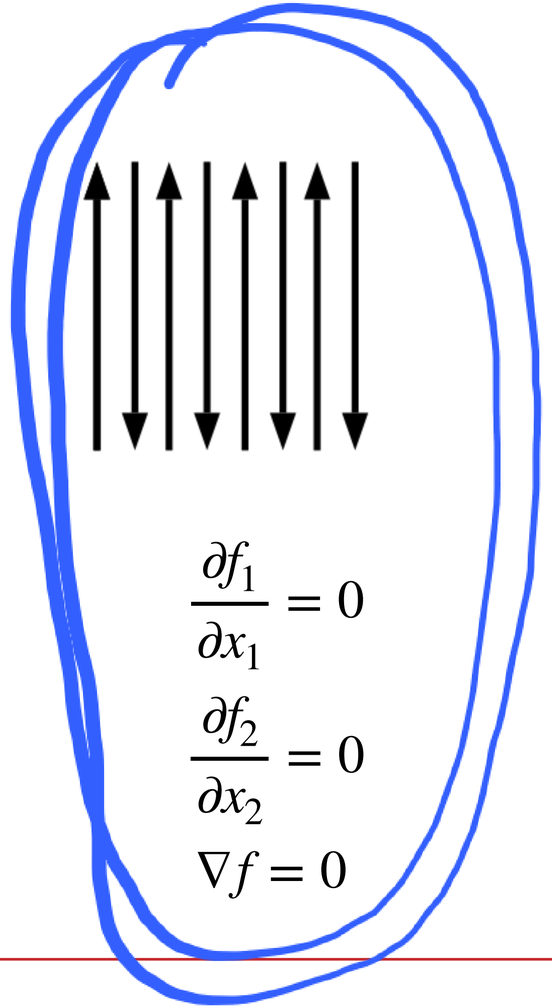
Sink



$$\frac{\partial f_1}{\partial x_1} < 0$$

$$\frac{\partial f_2}{\partial x_2} < 0$$

$$\nabla f < 0$$



$$\frac{\partial f_1}{\partial x_1} = 0$$

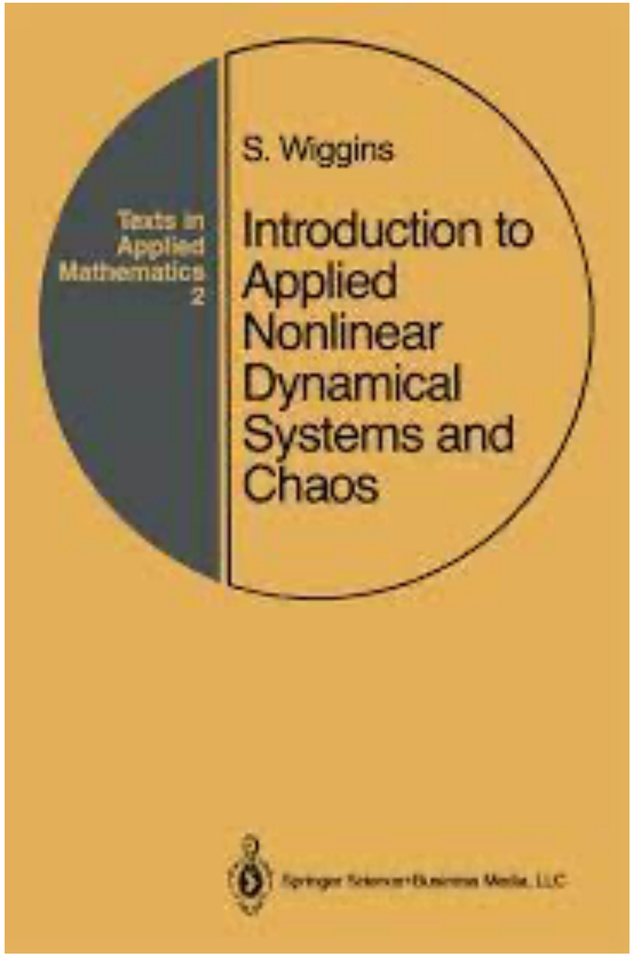
$$\frac{\partial f_2}{\partial x_2} = 0$$

$$\nabla f = 0$$

everything could happen

Qualitative Behaviour of Trajectories

Extremely rich area!



Linearization

We will see that the behaviour of trajectories of a nonlinear systems close to an equilibrium point can be often predicted by studying the linear approximation at the equilibrium point

C-T $\dot{x}(t) = f(x(t)), \;\; f(x^\star) = 0$

D-T $x(t+1) = f(x(t)), \;\; f(x^\star) = x^\star$

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^\star} = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}_{x=x^\star}$$