The Fourier transform

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1.1 Introduction

Roughly speaking, the Fourier transform is a "continuous" version of the Fourier series. Recall that the n^{th} Fourier coefficient of an integrable function $f\colon [0,1] \to \mathbb{C}$ is given by

$$a_n = \int_0^1 f(x)e^{-2\pi i n x}\, dx.$$

The Fourier transform can be thought of as indexing Fourier coefficients over $\mathbb{R}$ instead of $\mathbb{Z}$ (that is, allowing a_ξ for $\xi \in \mathbb{R}$).

Definition 1.1. *The **Fourier transform** of a function $f\colon \mathbb{R} \to \mathbb{C}$ is the function $\hat{f}\colon \mathbb{R} \to \mathbb{C}$ defined by*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi x}\, dx, \tag{1.1}$$

the integral being taken in the Riemann sense.

Of course, this definition is purely formal; it is not clear if the integral even converges. To make sense of this, we need some hypotheses on the smoothness and decay of f.

1.2 Schwartz functions

Definition 1.2. *A smooth function $f\colon \mathbb{R} \to \mathbb{C}$ is called a **Schwartz function** if f and all of its derivatives are rapidly decreasing, in the sense that*

$$\sup_{x\in\mathbb{R}} |x^k f^{(\ell)}(x)| < \infty, \qquad \forall k, \ell \geq 0. \tag{1.2}$$

*The collection of all such functions is called the **Schwartz space** on $\mathbb{R}$ and denoted by $\mathcal{S}$.*

Proposition 1.3. *The following are true about Schwartz space.*

(i) *$\mathcal{S}$ forms a vector space over $\mathbb{C}$. Moreover, $\mathcal{S}$ is closed under differentiation and multiplication by polynomials.*
(ii) $C_c^\infty(\mathbb{R},\mathbb{C}) \subsetneq \mathcal{S} \subsetneq C^\infty(\mathbb{R},\mathbb{C})$ *and* $\mathcal{S} \subsetneq \mathcal{M}$.

Proof. **(i)** This is mostly an application of the triangle inequality. Fix $k, \ell \geq 0$. Given $f, g \in \mathcal{S}$ and $a \in \mathbb{C}$, we have

$$\sup_{x\in\mathbb{R}} \left|x^k(af+g)^{(\ell)}(x)\right| \leq \sup_{x\in\mathbb{R}} \left(\left|x^k(af)^{(\ell)}(x)\right| + \left|x^k g^{(\ell)}(x)\right|\right) \leq |a| \sup_{x\in\mathbb{R}} |x^k f^{(\ell)}(x)| + \sup_{x\in\mathbb{R}} |x^k g^{(\ell)}(x)|.$$

Since $|x^k(f')^{(\ell)}(x)| = |x^k f^{(\ell+1)}(x)|$, Schwartz space is closed under differentiation. To show that $\mathcal{S}$ is closed under multiplication by polynomials, we show that for all $f \in \mathcal{S}$, we have $xf(x) \in \mathcal{S}$, and using this and the fact that $\mathcal{S}$ is a vector space, we obtain the result. We can show inductively that

$$(xf(x))^{(\ell)} = \ell f^{(\ell-1)}(x) + xf^{(\ell)}(x), \qquad \forall \ell \geq 0,$$

and hence,

$$\sup_{x\in\mathbb{R}} |x^k(xf(x))^{(\ell)}| \leq \ell \sup_{x\in\mathbb{R}} |x^k f^{(\ell-1)}(x)| + \sup_{x\in\mathbb{R}} |x^{k+1} f^{(\ell)}(x)| < \infty.$$

(ii) An example of a smooth function that is not Schwartz is given by e^x. The inclusion $C_c^\infty(\mathbb{R}, \mathbb{C}) \subseteq \mathcal{S}$ is a consequence of the extreme value theorem, and an example of a Schwartz function with non-compact support is the Gaussian

$$f(x) = e^{-ax^2}, \qquad a > 0.$$

To see this, observe that every derivative of f is of the form $f^{(\ell)}(x) = P_\ell(x)f(x)$ for some polynomial P_ℓ, and that

$$\sup_{x\in\mathbb{R}} |x^k f(x)| < \infty, \qquad \forall k \geq 0.$$

Finally, if $f(x)$ is bounded by α and $x^2 f(x)$ is bounded by β, then

$$(1+x^2)|f(x)| \leq \alpha + \beta,$$

hence $\mathcal{S} \subseteq \mathcal{M}$. On the other hand, the function $e^{-|x|}$ is moderately decreasing (in fact, rapidly decreasing) but not Schwartz because it is not differentiable at 0. ∎

Since Schwartz functions are moderately decreasing, their Fourier transform is well-defined, and it turns out that Schwartz functions play particularly nicely with the Fourier transform. In the following statements, the notation $f(x) \mapsto \hat{f}(\xi)$ means that $\hat{f}$ is the Fourier transform of f.

PROPOSITION 1.4. *Let $f \in \mathcal{S}$. The following are true:*

(i) If $h \in \mathbb{R}$, then

$$f(x+h) \mapsto \hat{f}(\xi)e^{2\pi i \xi h}$$

and

$$f(x)e^{-2\pi i h x} \mapsto \hat{f}(\xi + h).$$

(ii) If $\delta > 0$, then

$$f(\delta x) \mapsto \frac{1}{\delta}\hat{f}\left(\frac{\xi}{\delta}\right).$$

(iii) We have

$$f'(x) \mapsto 2\pi i \xi \hat{f}(\xi).$$

Furthermore, $\hat{f}$ is differentiable, and

$$-2\pi i x f(x) \mapsto (\hat{f})'(\xi).$$

Proof. Exercise. ∎