COMPUTATIONS OF MARGINS OF FUSIBLE NUMBERS

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The goal of this note is to derive formulas and bounds for the margins $m(3-2^{-n})$ using either Erickson's formula (EF) or the Main Conjecture (MC) (see my earlier paper *Survey on Fusible Numbers*). Some values of $m(3-2^{-n})$ under (EF) and bounds for $m(3-2^{-n})$ appeared in that paper without proof.

Let's fix some notations. Let $M(x) := -\log_2 m(x)$. Let $\langle n; n_1, n_2, \cdots, n_k\rangle$ denote the number $n - 2^{-n_1} - 2^{-n_2} - \cdots - 2^{-n_k}$, and let $\langle n; n_1, +n_2, +n_3, \cdots, +n_k\rangle$ denote $\langle n; n_1, n_1+n_2, n_1+n_2+n_3, \cdots, n_1+n_2+n_3+\cdots+n_k\rangle$.

In this setting, (EF) becomes

$$M(n; n_1, \cdots, n_k) = M(n; n_1, \cdots, n_k, M(n-1, n_1, \cdots, n_k)) + 1$$

and (MC) becomes (for integers $n \geq 0$, $0 < n_1 < n_2 < \cdots < n_k$ and $n' := M(n-1; n_1, \cdots, n_k)$)

$$M(n; n_1, \cdots, n_k) = \begin{cases} M(n; n_1, \cdots, n_k, n') + 1 & \text{if } n' > n_k \\ M(n; n_1, \cdots, n_{k-2}, n_{k-1}-1, n_k) + 1 & \text{if } n' = n_k \end{cases}$$

If $n' < n_k$, it is not hard to see that $2^{-n'} = \sum_{i=l+1}^{k} 2^{-n_i}$ for some $l < k$, and we have $M(n; n_1, \cdots, n_k) = M(n; n_1, \cdots, n_{l-1}, n_l - 1, n_{l+2} - 1, \cdots, n_k - 1)$ if $n > 0$.

(EF) and (MC) give exactly the same fusible numbers below 2, and M has the following simple formulas when $n = 2$, which can be proved easily by induction: for integers

$-2 \le n = n_1 < n_2 < \cdots < n_{k-1} \le n_k$,

$$M(2;n) = 2n+3$$

$$M(2;n_1,n_2,\cdots,n_k) = n_1+3-k+n_k \qquad \text{if } 2 \le k \le n_1+3$$

$$m(2;n_1,n_2,\cdots,n_k) = \sum_{i=n_1+3}^{k} 2^{-n_i} \qquad \text{if } k \ge n_1+3$$

Now let's consider $n=3$ and let's start with an example: the calculation of $M(3;1,5)$ under (MC).

$$\begin{aligned}
M(3;1,5) &= M(3;1,5,M(2;1,5))+1 & \\
&= M(3;1,\underline{5,7}) & +1 \\
&= M(3;1,5,7,8)+1 & +1 \\
&= M(3;1,5,6,8)+1 & +2 \\
&= M(3;1,5,5,8)+1 & +3 \\
&= M(3;1,\underline{4,8}) & +4 \\
&= M(3;1,4,8,9)+1 & +4 \\
&= M(3;1,4,4,9)+(8-4) & +5 \\
&= M(3;1,\underline{3,9}) & +9 \\
&= M(3;1,\underline{2,10})+(1+9-3) & +9 \\
&= M(3;1,\underline{1,11})+(1+10-2) & +16 \\
&= M(3;0,11) & +25 \\
&= 2\times 11+3+25 & = 50
\end{aligned}$$

It follows that $M(3;1) = M(3;1, M(2;1)) + 1 = M(3;1,5) + 1 = 51$. Now compare the reduction from $M(3;1,5,7) = M(3;1,5,+2)$ to $M(3;1,4,8) = M(3;1,4,+4)$ with the following reduction of $M(3;0,+2) = M(3;0,2)$ to $M(3;-1,+4) = M(3;-1,3)$:

$$\begin{aligned} M(3;0,2) &= M(3;0,2,M(1;2)) + 1 & \\ &= M(3;0,2,3) & +1 \\ &= M(3;0,1,3) + 1 & +1 \\ &= M(3;0,0,3) + 1 & +2 \\ &= M(3;-1,3) & +3 \end{aligned}$$

One sees that the steps exactly correspond. This is due to certain self-similarity of the set of fusible numbers below 2: the interval $[\langle 2;1,4\rangle, \langle 2;1,5\rangle]$ is simply a shrinked copy of $[\langle 2;-1\rangle, \langle 2;0\rangle]$. Similarly, for any $m \geq 1$ and $n \geq 0$, $[\langle 2;m,+n\rangle, \langle 2;m,+(n+1)\rangle]$ is a shrinked copy of $[\langle 2;m-2\rangle, \langle 2;m-1\rangle]$, hence a calculation in the latter interval is equivalent to a calculation in the former.

Now, for $m \geq -1$ and $n \geq 0$, let $h(m,n) := M(3;m,+n)$, let $g(m,n)$ be such that $\langle 3;m,+0,+g(m,n)\rangle = \langle 3;m-1,+(g(m,n)+1)\rangle$ is the first number below $\langle 3;m-1\rangle$ encountered in the calculation of $M(3;m,+n)$, and let $f(m,n)$ denote the number of steps needed to reduce $\langle 3;m,+n\rangle$ to $\langle 3;m-1,+(g(m,n)+1)\rangle$. Clearly

$$h(m,n) = f(m,n) + h(m-1, g(m,n)+1).$$

We shall exploit self-similarity to derive recursive formulas for f and g. Consider the calculation of $M(3;m,+n)$. If $n = 0$, it takes just one step to reduce to $\langle 3;m,m,2m+1\rangle =$

$\langle 3, m-1, +(m+2)\rangle$, hence

$$f(m,0) = 1, \quad g(m,0) = m+1.$$

If $n > 0$, we observe that $[\langle 2; m, +1\rangle, \langle 2; m, +(n+1)\rangle]$ is a shrinked copy of $[\langle 2; m, +0\rangle, \langle 2; m, +n\rangle]$; since it takes $f(m, n-1)$ steps to reduce $\langle 3; m, +(n-1)\rangle$ to $\langle 3; m, +0, +g(m, n-1)\rangle$, it takes the same number of steps to reduce $\langle 3; m, +n\rangle$ to $\langle 3; m, +1, +g(m, n-1)\rangle$. Now, because it takes $f(m-1, g(m, n-1))$ steps to reduce $\langle 3; m-1, +g(m, n-1)\rangle$ to $\langle 3; m-1, +0, +g(m-1, g(m, n-1))\rangle$, it also takes the same number of steps to reduce $\langle 3; m, +1, +g(m, n-1)\rangle$ to $\langle 3; m, +1, +0, +g(m-1, g(m, n-1))\rangle = \langle 3; m, +0, +(g(m-1, g(m, n-1)) + 1)\rangle$. Therefore

$$f(m,n) = f(m, n-1) + f(m-1, g(m, n-1)),$$

$$g(m,n) = g(m-1, g(m, n-1)) + 1.$$

For the initial values, one easily verifies that under (EF)

$$g(-1,n) = 0, \quad f(-1,n) = n+1, \quad h(-1,n) = n$$

and under (MC)

$$g(-1,n) = n, \quad f(-1,n) = 1, \quad h(-1,n) = n.$$

Clearly,

$$\begin{aligned} M_3(m) := M(3; m) &= M(3; m+1, +0) = h(m+1, 0) \\ &= f(m+1, 0) + h(m, g(m+1, 0) + 1) = 1 + h(m, m+3), \end{aligned}$$

hence $h(m-1,m+2) = M_3(m-1)-1$. Under (EF), the recursive formulas simplify significantly and we obtain

$$g(m,n) = m+1, \quad f(m,n) = 1 + nf(m-1,m+1),$$

$$h(m,n) = f(m,n) + h(m-1,m+2) = f(m,n) + M_3(m-1) - 1,$$

$$\begin{aligned} M_3(m) &= 1 + h(m,m+3) = 1 + f(m,m+3) + M_3(m-1) - 1 \\ &= M_3(m-1) + 1 + (m+3)f(m-1,m+1). \end{aligned}$$

Define $F(m) := f(m,m+2)$ and get

$$F(-1) = 2, \quad F(m) = 1 + (m+2)F(m-1),$$

$$M_3(m) = M_3(m-1) + 1 + (m+3)F(m-1) = M_3(m-1) + F(m) + F(m-1),$$

$$h(m,n) = nF(m-1) + M_3(m-1).$$

It is then easy to see that

$$F(m) = (m+2)! \sum_{i=0}^{m+2} \frac{1}{i!}$$

Clearly $\lim_{n\to\infty} F(n)/e(n+2)! = 1$. We show that also $\lim_{n\to\infty} F(n)/M_3(n) = 1$. First we establish the bound $M_3(n) \leq 2F(n)$. For $n = -1, 0$ this follows from direct computation. By induction, for $n \geq 1$ we then have $M_3(n) = 1 + F(n) + F(n-1) + M_3(n-1) \leq 1 + F(n) + F(n-1) + 2F(n-1) \leq 1 + (n+2)F(n-1) + F(n) = 2F(n)$. Therefore

$$\lim_{n\to\infty} \frac{1 + F(n-1) + M_3(n-1)}{M_3(n)} \leq \lim_{n\to\infty} \frac{3F(n-1)}{F(n)} = \lim_{n\to\infty} \frac{3}{n+2} = 0,$$

hence $\lim_{n\to\infty} F(n)/M_3(n) = 1$ and $\lim_{n\to\infty} M_3(n)/e(n+2)! = 1$.

We now establish lower bounds for $M_3(m)$ under (MC). Given the recursive formulas and initial values, it is easily shown that

$$g(0,n) = n+1, \quad f(0,n) = n+1, \quad h(0,n) = 2n+3,$$

$$g(1,n) = 2n+2, \quad f(1,n) = (n+1)^2, \quad h(1,n) = (n+3)^2+1,$$

$$g(2,n) = 3(2^{n+1}-1), \quad f(2,n) = 3(4^{n+1}-2^{n+3})+4n+13, \quad h(2,n) = 3(4^{n+2}-2^{n+2})+4n+15.$$

In particular, $g(2,n) > 2^n = 2 \uparrow^{2-1} n$ and $h(2,n) > 2^n = 2 \uparrow^{2-1} n$. If $m \geq 3$, then $g(m-1,n) > 2 \uparrow^{m-2} n$ for all $n \geq 0$ implies that

$$g(m,n) = (g(m-1,\cdot)+1)^{\circ n}(g(m,0)) > (2 \uparrow^{m-2})^n(m+1) \geq (2 \uparrow^{m-2})^n(4) = 2 \uparrow^{m-1} (n+2).$$

If in addition $h(m-1,n) > 2 \uparrow^{m-2} n$ for all $n \geq 0$, then

$$h(m,n) > h(m-1, g(m,n)+1) > 2 \uparrow^{m-2} (2 \uparrow^{m-1} (n+2)) = 2 \uparrow^{m-1} (n+3).$$

Therefore, for $m \geq 3$,

$$M_3(m) = 1 + h(m, m+3) > 2 \uparrow^{m-1} (m+6),$$

and there is also the worse but simpler bound

$$M_3(m) = h(m+1, 0) > 2 \uparrow^{m} 3 \quad (= 2 \uparrow^{m-1} 4).$$

The next goals would be finding an upper bound for M_3 and finding bounds for $M(4-2^{-n})$, or even $M(5-2^{-n})$, etc. This note will be updated if progress is made.