Extreme Value Analysis

The statistical analysis of low frequency, high severity events

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Extreme Value Analysis (EVA)

- EVA is a statistical methodology for making inferences about rare events (weather, finance, public health, materials, etc.)
  - It is also very often referred to as Extreme Value Theory (EVT)
- Disambiguation:
  - Extreme Value Theory (Analysis) has nothing to do with the Extreme Value Theorem, from elementary calculus.
- This talk will be limited to:
  - ”Classical” EVA (mostly)
  - Univariate, continuous probability distributions
  - Maxima, since $\min(X_1, X_2, \dots, X_n) = -\max(-X_1, -X_2, \dots, -X_n)$

But isn't "extrapolation" a Bad Thing?" Can't we just say, "No"?

- **"If people aren't given well-founded methods like EVT, they'll just use dubious ones instead**."
- "What EVT is doing is making the best use of whatever data you have about extreme phenomena."
- "EVT cannot do magic - but it can do a whole lot better than empirical curve-fitting and guesswork."

-- [Embrechts 1997]

North Sea Flood of 1953

Losses:

- 1836 people killed
- 72,000 people evacuated
- 49,000 houses and farms flooded
- 201,000 cattle drowned
- 500 km coastal defenses destroyed
- More than 200,000 ha flooded

Effect on Study of Extreme Events:

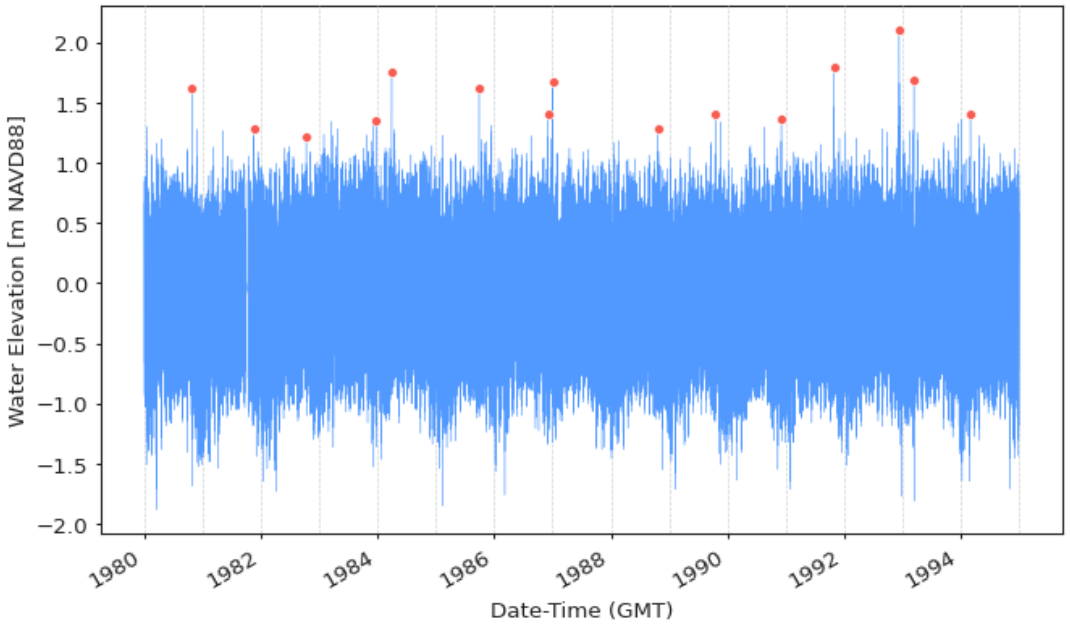
- Very little systematic statistical research w.r.t. height of the dikes was done before 1953
  - Flood of 1570 was mean-sea-level + 4m
- Gave EVA research a decisive push
- Needed height estimate **well outside range of existing data**
  - Van Dantzig report estimated p=1-10^{-4} quantile (one-in-ten-thousand-year surge height) of mean-sea-level + 5.14m

Source: [Embrechts 1997]

Netherlands, during 1953 North Sea Flood.
Viewed from a U.S. Army helicopter.

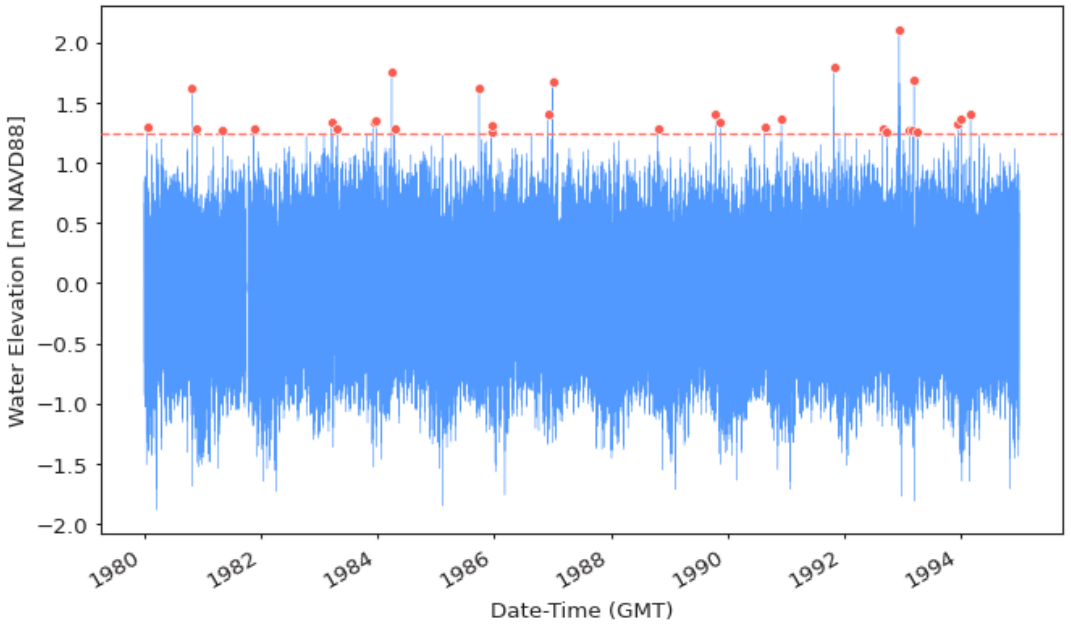
Source: https://en.wikipedia.org/wiki/North_Sea_flood_of_1953

Two Primary Approaches to EVA



Block Maxima (BM)

Divide the data into large/long blocks and use the maximum/minimum value in each block



Points Over Threshold (POT)

Use all data that exceeds a specific threshold

Source: PyExtremes User Guide

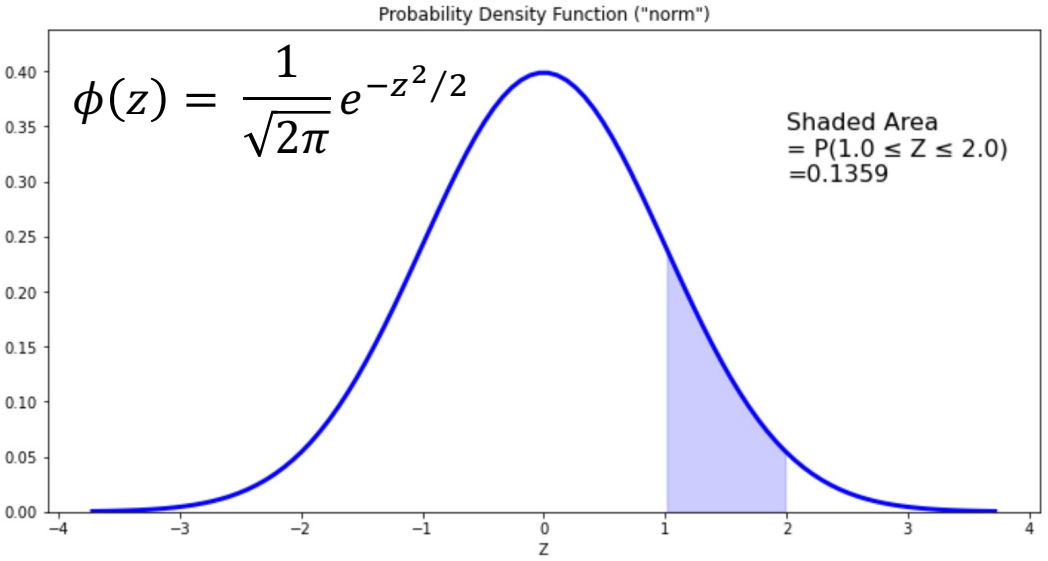
A Very Brief Refresher

Probability Theory: PDFs, CDFs, CLT, etc.

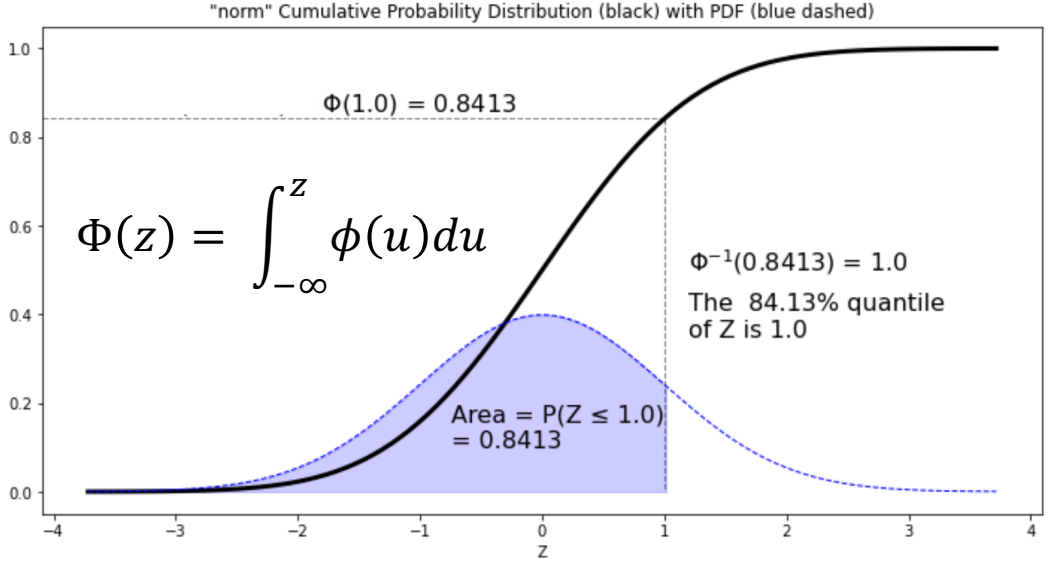
Continuous PDFs and CDFs

Standard Normal Distribution

Probability Density Function (PDF)



Cumulative Distribution Function (CDF)



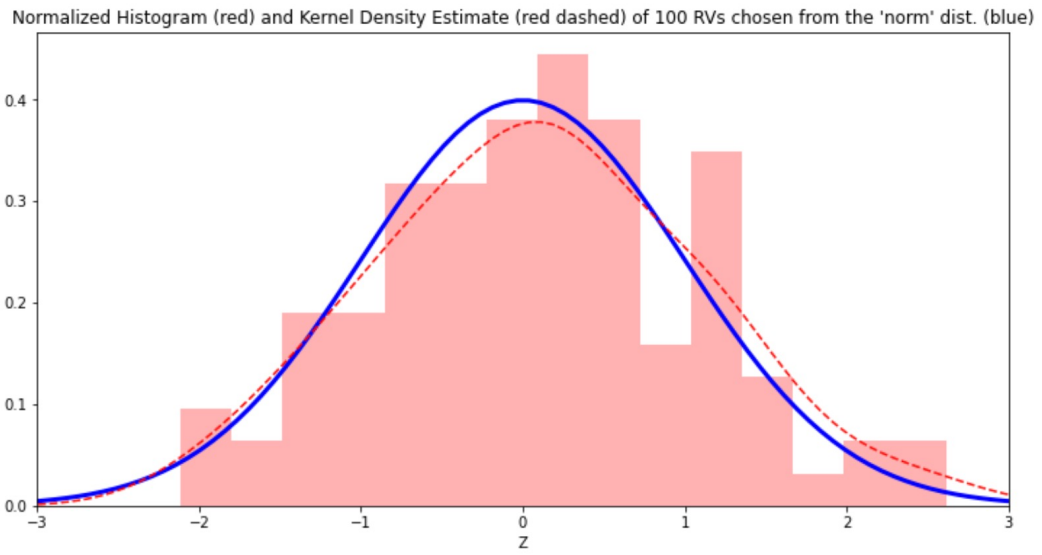
If $X \sim N(\mu, \sigma)$ and $Z = \frac{x-\mu}{\sigma}$,

then $F_X(x) = P(X \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, i.e., a location-scale family†

†A location-scale family is a family of distributions formed by translation and scaling of a *standard* family member.

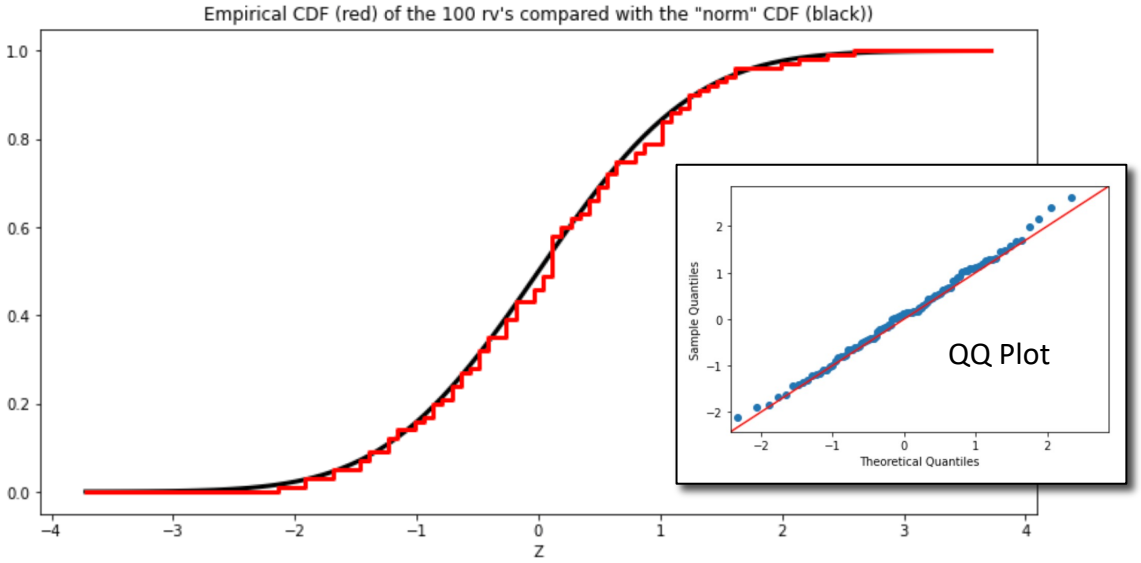
Histograms and Empirical CDFs

Normalized Histogram & Kernel Density Estimate of a Random Sample



- Histograms (normalized) are empirical estimates of PDFs, but their shape is sensitive to bin size
- Kernel density estimates are another form of PDF estimate, but their shape is sensitive to the type of kernel used

Empirical CDF of the Same Random Sample



- The Empirical CDF is less susceptible to subjective choices,
- so it is often used for model checking, for example, using Quantile-Quantile Plots.

Parameter Estimation

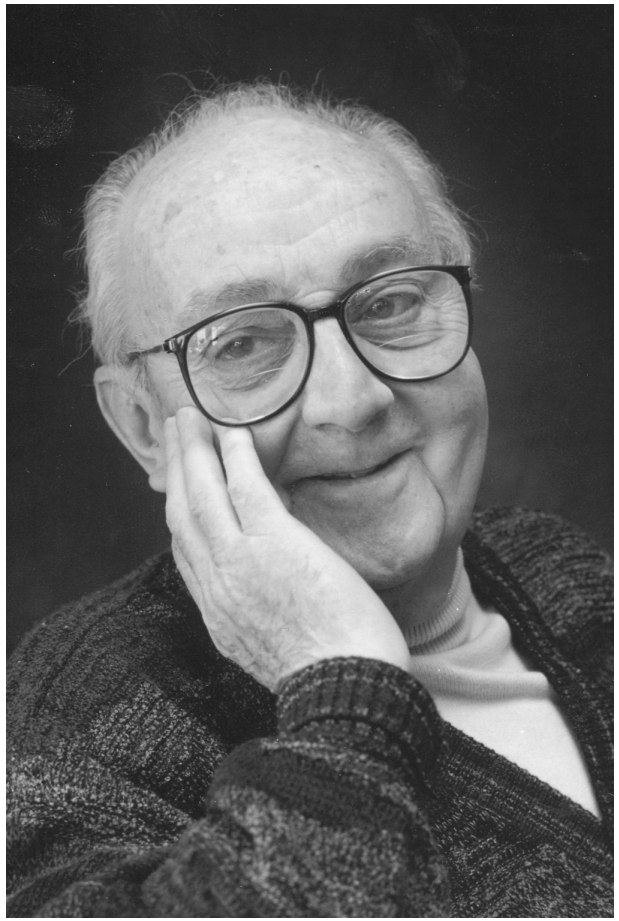
$X_1, \ldots, X_n \sim F(x; \underline{\theta})$ iid and $f = F'$

- Maximum Likelihood Est. (MLE)
  - $\underline{x} = (x_1, \ldots, x_n)^T$
  - $\mathcal{L}_n(\underline{\theta}; \underline{x}) = \prod_{i=1}^{n} f(x_i; \underline{\theta})$
  - $\underline{\hat{\theta}} = \underset{\underline{\theta}}{\operatorname{argmax}} \ln[\mathcal{L}_n(\underline{\theta}; \underline{x})]$
- Other Methods
  - MOM / PWM / L-Moments
  - Bayesian Parameter Estimation
  - …

- Confidence Intervals (CI)
  - Wald CI (”classical” method)
    - MLE: $\underline{\hat{\theta}} \dot{\sim} MVN_d(\theta, I^{-1}(\theta))$
  - Profile Likelihood CI
    - Likelihood ratio is asymptotically χ^2_{df}
  - Bootstrapping (resampling w/ replacement)
  - Credible Interval / Highest Posterior Density (HPD) Interval/Region (Bayesian)
  - …

“All models are wrong, but some are useful”

-- George E. P. Box



By DavidMCEddy at en.wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=115167166

Maximum Values

An experiment using pseudo-random numbers, along with some theory

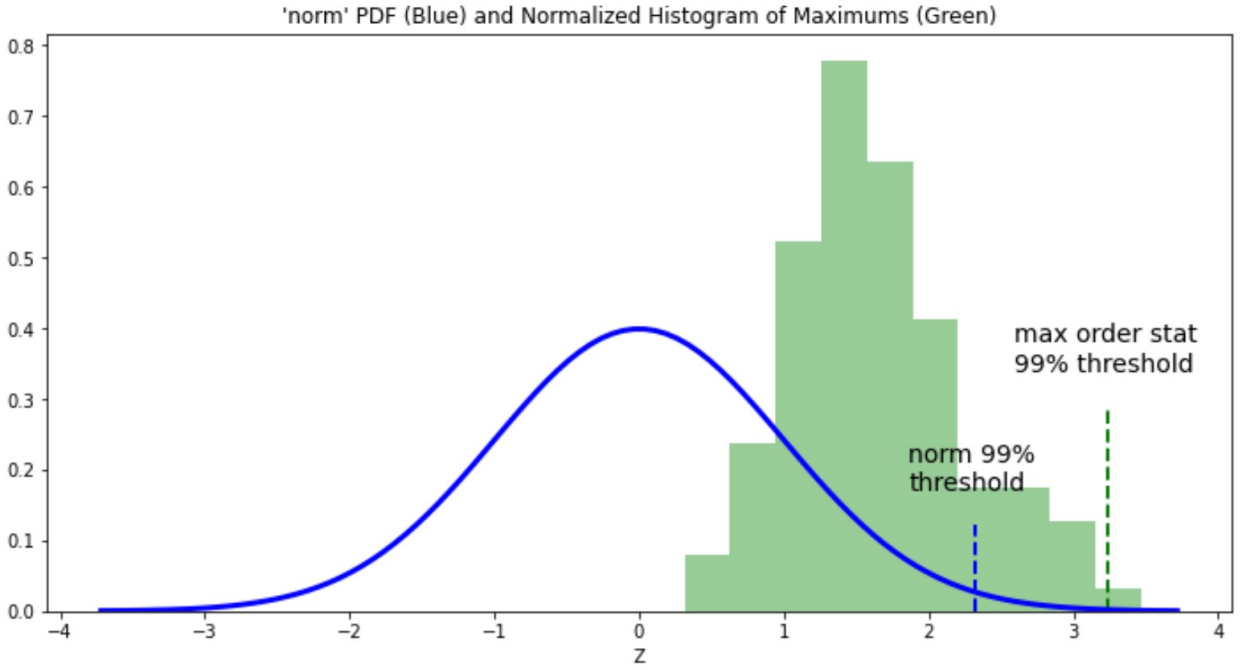
Order Statistics

- Let $X_1, X_2, X_3, \ldots, X_n$ be iid RVs
  - with CDF: $F(x)$
  - and PDF: $f(x) = \frac{d}{dx} F(x)$
- Also, let $Y_1 \leq Y_2 \leq Y_3 \leq \ \ldots \leq Y_n$ be the X's in ascending order
- The Y's are *Order Statistics* based on the X's
- We'll focus on the Maximum Order Statistic, Y_n

A Random Sample of Maximum Order Statistics (based on Std. Normal Dist.)

The figure at right depicts:

- Standard Normal PDF (blue)
- A normalized histogram of 200 maximum order statistics (green)
  - Where each maximum came from a random sample of 12 standard normal RVs
- Two quantiles ("thresholds"):
  - Standard normal 99% quantile (~2.33)
  - Empirical 99% quantile of the 200 maxs (~3.24)
- Note that there is almost a full N(0,1) standard deviation between the quantiles.



Distribution of the Maximum Order Statistic

The maximum order statistic …

- has CDF, $Y_n \sim G$, where $G(y) = [F(y)]^n$

$$
\begin{aligned}
G(y) &= P(Y_n \leq y) \\
&= P(X_1 \leq y, \ldots, X_n \leq y) \\
&= P(X_1 \leq y) \ldots P(X_n \leq y) \\
&= [F(y)]^n
\end{aligned}
$$

- The PDF is $g(y) = n[F(y)]^{n-1} f(y)$
- Note: If $F(x) < 1$, then $F^n(x) \to 0$, as $n \to \infty$

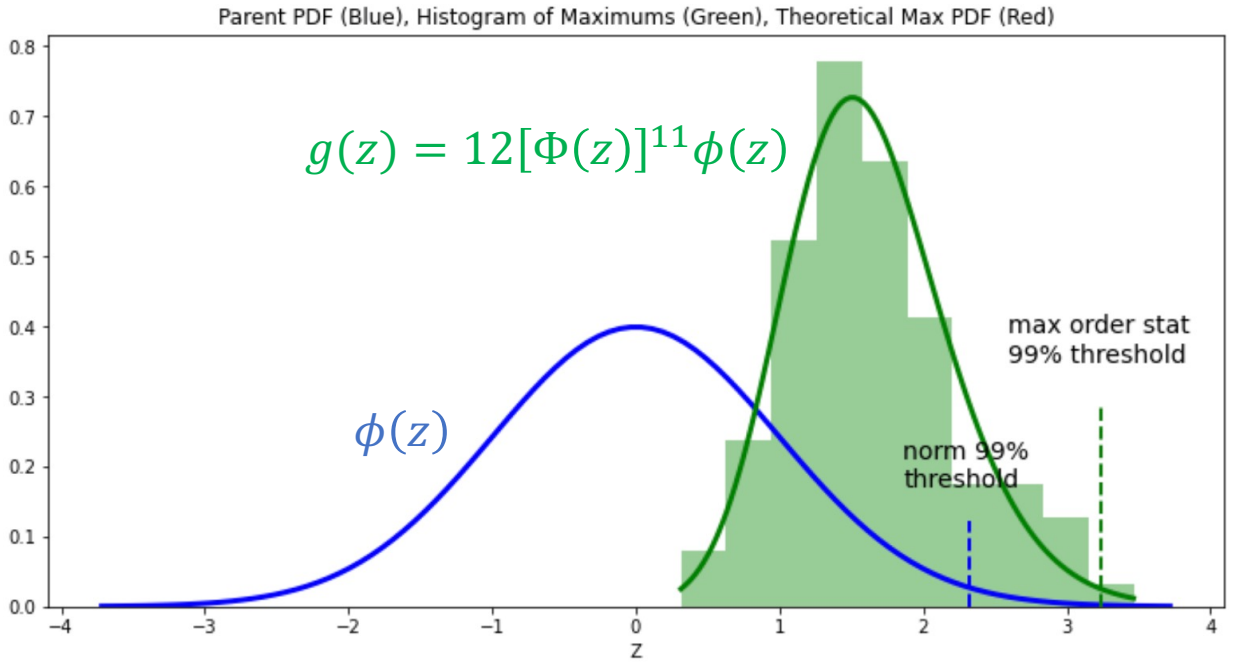
Max. Order Stat. Distribution (based on Std. Normal Dist.)

If $X_1, \ldots, X_n \sim N(0,1)$ iid and Y_n is the Maximum Order Statistic, then its CDF and PDF are as follows, resp.:

$$G(y) = [\Phi(y)]^n$$

$$g(y) = n[\Phi(y)]^{n-1}\phi(y)$$

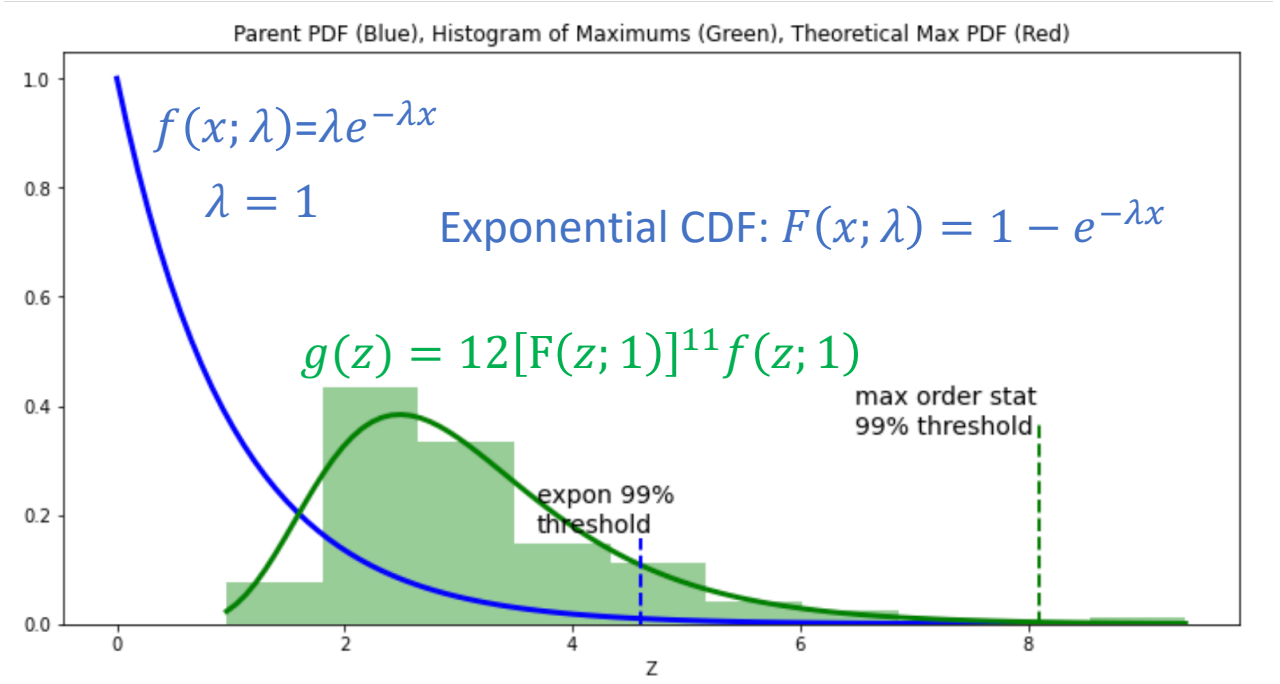
At right, the PDF, g, is plotted (in green) along with the histogram of maximums from 200 samples, each of size $n = 12$.



Max. Order Stat. Distribution (based on Exponential Dist.)

The figure at right depicts:

- Exponential PDF, $\lambda = 1$ (blue)
- A histogram of 200 maximum order statistics (green)
  - Where each maximum came from a random sample (iid) of 12 exponential RVs ($\lambda = 1$)
- Two quantiles ("thresholds"):
  - Exponential 99% quantile (~4.6)
  - Empirical 99% quantile of the 200 maxs (~8.1)



Extreme Value Theory (EVT)

The Generalized Extreme Value (GEV) Distribution and Theorem

Generalized Extreme Value (GEV) Distribution

CDF

$$G(z) = exp\left\{-\left[1+\xi\left(\frac{z-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

defined on the set $\left\{z: 1+\frac{\xi(z-\mu)}{\sigma} > 0\right\}$,

where $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \xi < \infty$.

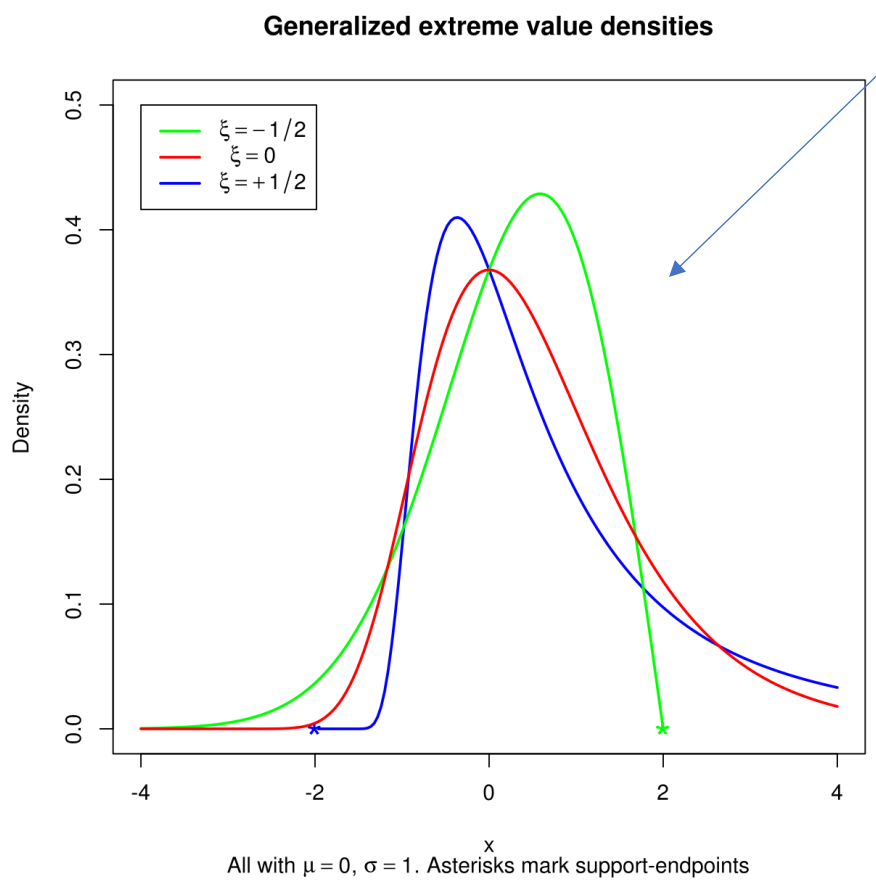
- If $\xi > 0$, then G is the **Fréchet** dist. (heavy tailed)
- If $\xi < 0$, then G is the **Weibull** dist. (upper-bounded)
- Taking the limit as $\xi \to 0$, obtains the **Gumbel** dist. (light-tailed)

$$G(z) = exp\left[-exp\left\{-\left(\frac{z-\mu}{\sigma}\right)\right\}\right], -\infty < z < \infty$$

Source: [Coles 2001]

NOTE: scipy.stats reverses the sign of ξ. $(c = -\xi)$

PDFs



Source: [Wikipedia: GEV]

Extreme Value Theorem

Let Y_n be the maximum order statistic of $X_1, \ldots, X_n$, a sequence of iid random variables with common CDF, F, then if there exists sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$P\left(\frac{Y_n - b_n}{a_n} \leq z\right) \to G(z) \text{ as } n \to \infty$$

for a non-degenerate distribution function G,

then G is a member of the GEV family (described on the previous slide).

GEV is a Location-Scale Family of Distributions

$P(Y_n \leq w)$ is approximated by another member, G^*, of the GEV location-scale family.

To see this, observe that the GEV distribution G is a family of distributions formed by translation and scaling of a standard family member.

The standard member is $H(x;\xi) = \exp\left[-(1+\xi x)^{-1/\xi}\right]$, for $\xi \neq 0$,

and so, $G(z) = H\left(\frac{z-\mu}{\sigma};\xi\right)$, for $\xi \neq 0$ and $\left\{z: 1+\frac{\xi(z-\mu)}{\sigma} > 0\right\}$

For $\xi = 0$, define $H(x;0) = \exp[-\exp(-x)]$

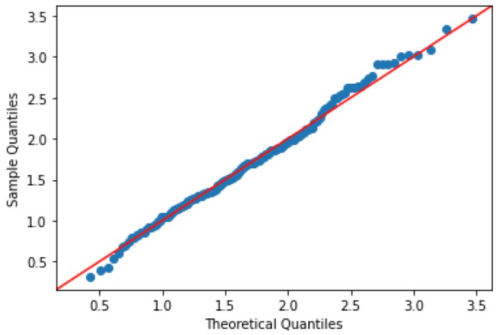
Furthermore, $P\left(\frac{Y_n - b_n}{a_n} \leq z\right) \approx H\left(\frac{z-\mu}{\sigma};\xi\right)$ for large n, which, after some algebraic manipulation, can be written as

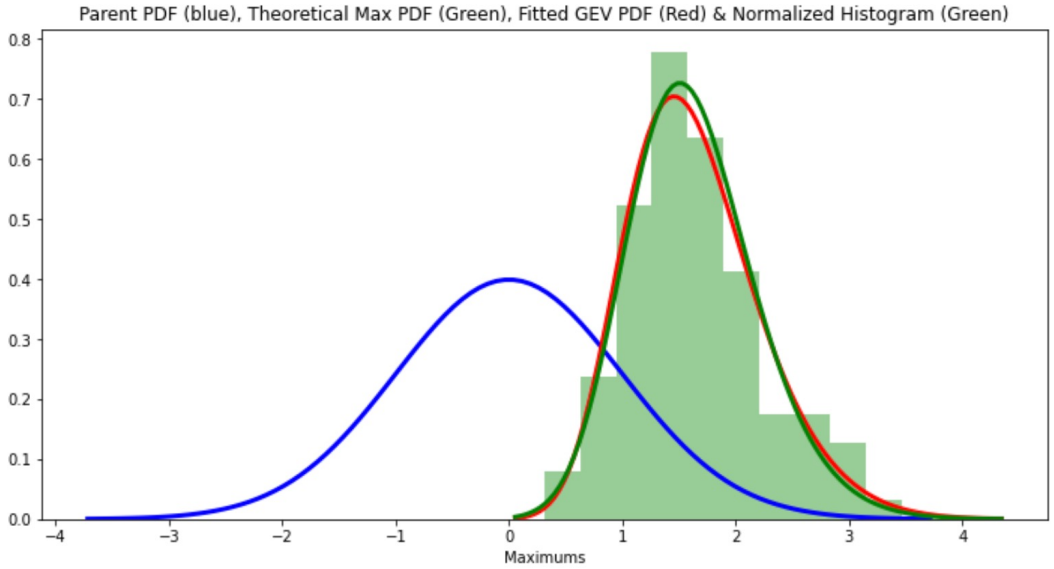
$$P(Y_n \leq w) \approx H\left(\frac{w-b_n^*}{a_n^*};\xi\right) = G^*(w)$$

using different location and scale values, a_n^* and b_n^*.

GEV MLE Fit to Maximums based on Std. Normal RVs

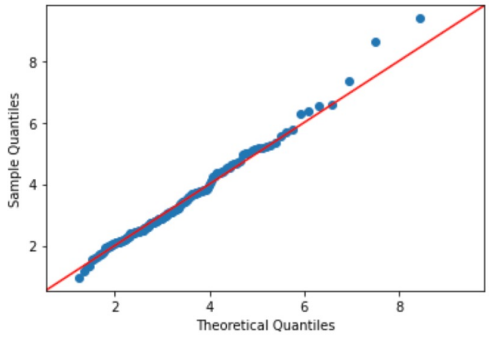
- Same as earlier normal plot, except that now the GEV fit (via MLE) is also shown (red)
- The QQ-plot shows the normal-based maximums (data) vs. the GEV fit

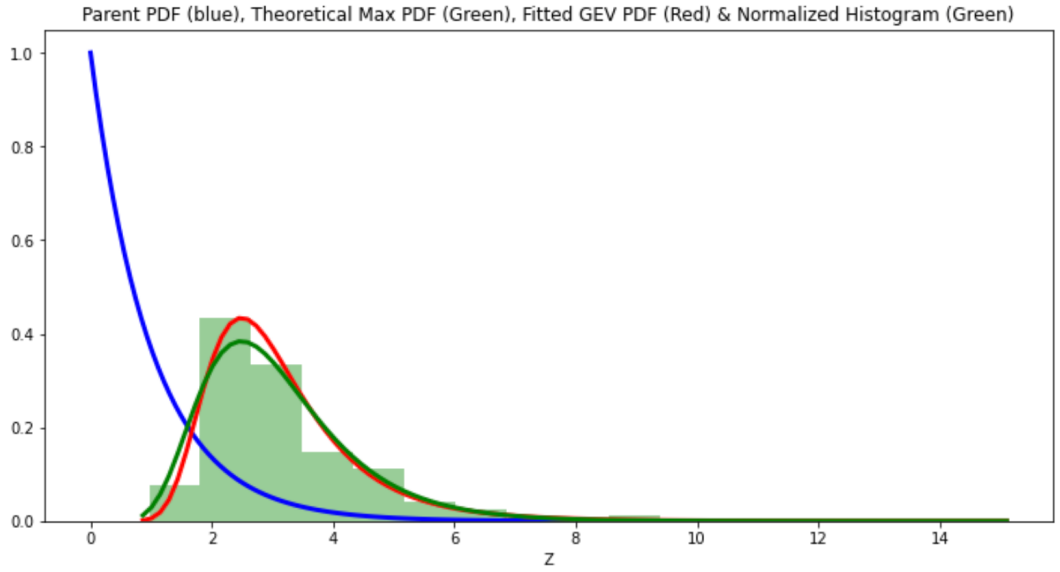




GEV MLE Fit to Maximums based on Exponential RVs

- Same as earlier exponential plot, except that now the GEV fit (via MLE) is also shown (red)
- The QQ-plot shows the exponential-based maximums (data) vs. the GEV fit





Return Levels & Return Periods

- The quantiles of the GEV can be interpreted as **return levels**
- A **return level** is the value expected to be exceeded on average once every 1/p periods, where 1 – p is the specific probability associated with the quantile $G(z_p) = 1 - p$

$$\Longrightarrow z_p = \mu - \frac{\sigma}{\xi}\left\{1 - [-\log(1-p)]^{-\xi}\right\}, \xi \neq 0$$

$$z_p = \mu - \sigma \log\{-\log(1-p)\}, \xi = 0$$

- z_p is the **return level** associated with the **return period** of $^1/_p$

EVA Software

- R:
  - **ismev**: https://cran.r-project.org/web/packages/ismev/index.html
  - **extRemes**: https://cran.r-project.org/web/packages/extRemes/index.html
  - and many, many more… see the following…
    - https://cran.r-project.org/web/views/ExtremeValue.html (Many links to other EVA packages in R)
    - *"A modeler's guide to extreme value software"*, Belzile, et al., arXiv:2205.07714v1, 16 May 2022
- Python:
  - **Pyextremes**: https://georgebv.github.io/pyextremes/
  - Scikit-extremes: https://kikocorreoso.github.io/scikit-extremes/
  - Wafo: https://pypi.org/project/wafo/
- Documentation
  - Both extRemes (R) and Pyextremes (Python) have excellent documentation
    - Not that other packages don't, but the docs for these two packages make good starting points for learning more about EVA.
  - W.r.t. a Good Book, I recommend Coles' book:
    - "An Introduction to Statistical Modeling of Extreme Values"
    - …that is, *ISMEV*

Example 1

GEV Fit using ISMEV (R)

Example 1: ismev (R)

```
> library(ismev)
Loading required package: mgcv
Loading required package: nlme
This is mgcv 1.8-40. For overview type 'help("mgcv-package")'.
> data(portpirie)
> plot(portpirie)
> ppfit <- gev.fit(portpirie[,2])
$conv
[1] 0

$nllh
[1] -4.339058

$mle
[1]  3.87474692  0.19804120 -0.05008773

$se
[1] 0.02793211 0.02024610 0.09825633

> |
```

Information, derived during the MLE fit, is stored here.

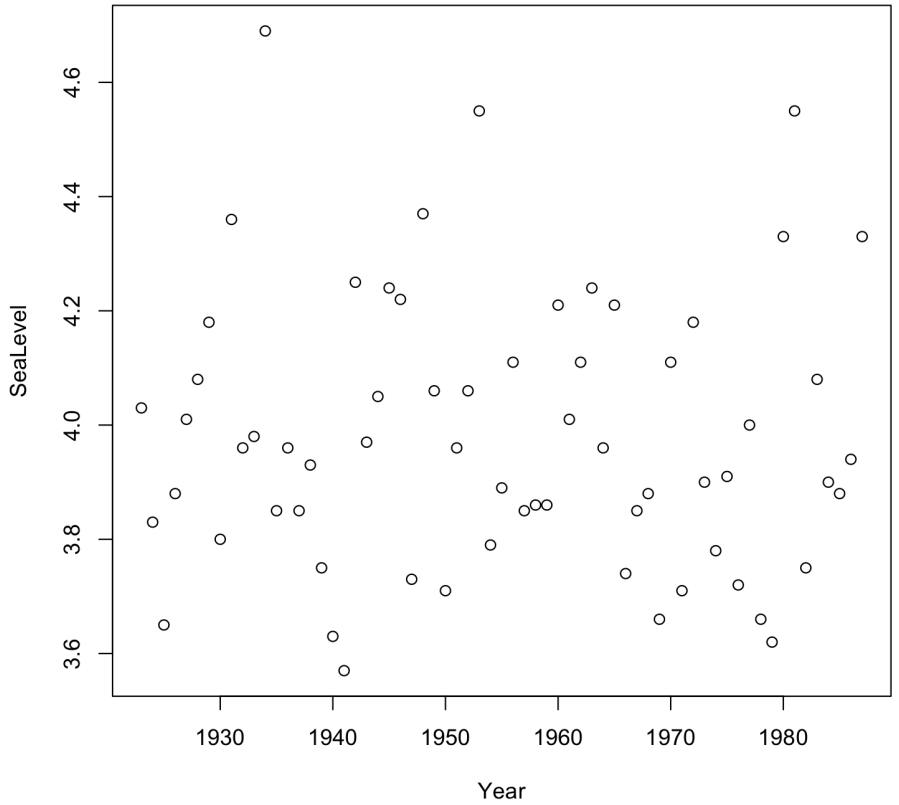
μ σ ξ

95% CI

-0.0500 ± 1.96 * 0.0983

$$-0.243 < \xi < 0.142$$

Annual Maximum Sea-Levels at Port Pirie, South Australia



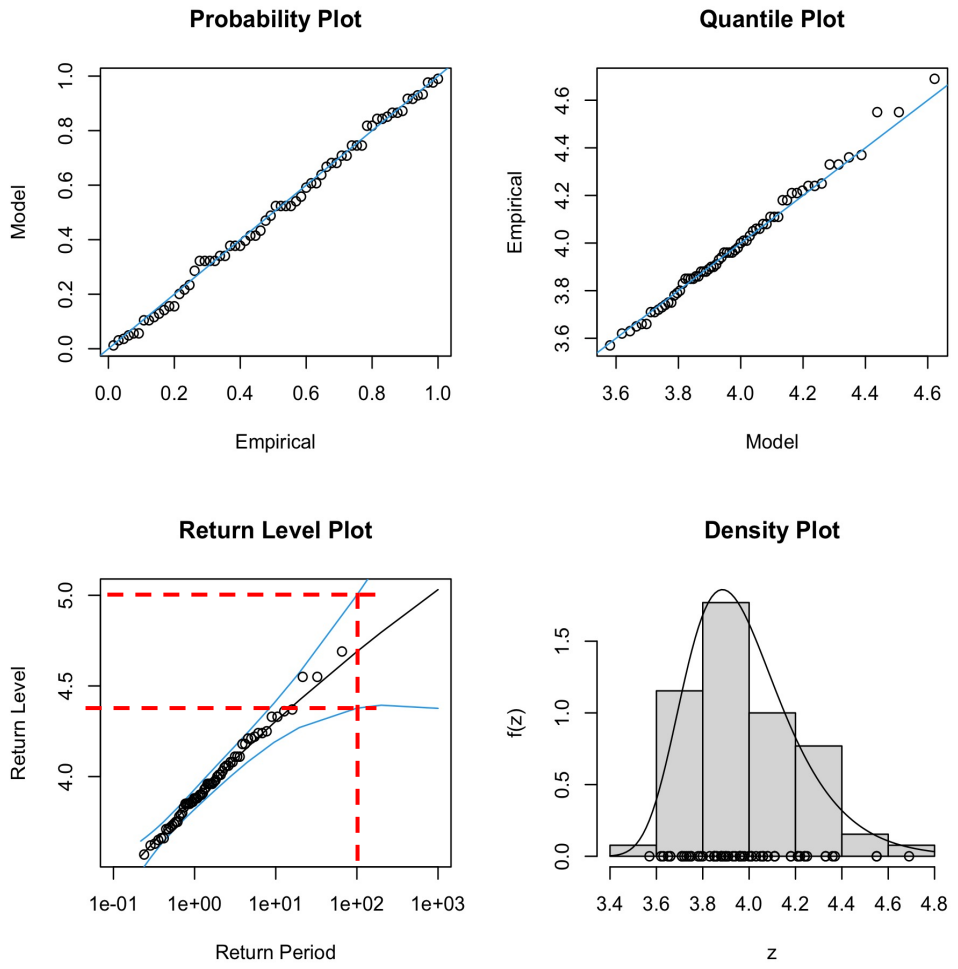
Example 1: Diagnostics

R function call:

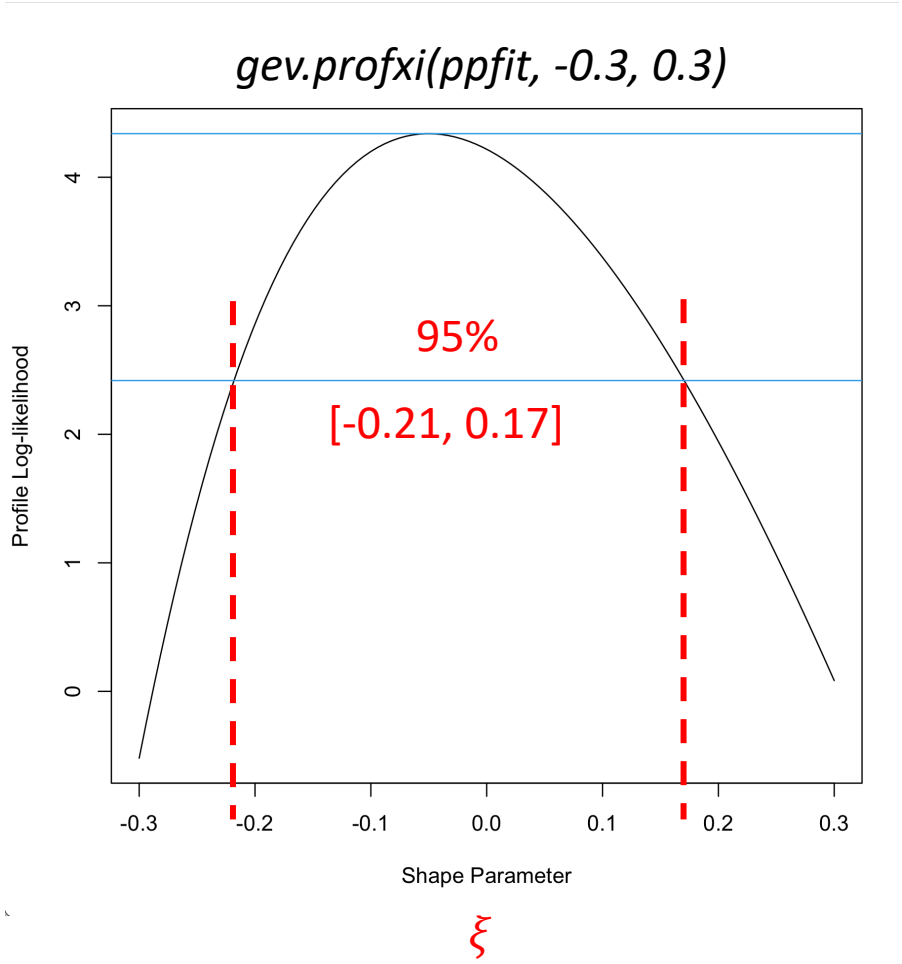
gev.diag(ppfit)

Return Levels (& Wald 95% CI)

- 10 year: $\hat{z}_{0.1} = 4.30$, [4.19, 4.41]
- 100 year: $\hat{z}_{0.01} = 4.69$, [4.38, 5.00]



Example 1: Profile Likelihood



- Confidence intervals produced using the profile likelihood method are derived from the asymptotic Chi-Square distribution of the likelihood ratio.
- They are "better" for asymmetric, sparse datasets, like those encountered in EVA.

Recall from the first slide of this example, we had:

$$-0.243 < \xi < 0.142$$

Peaks Over Threshold (POT)

and the Generalized Pareto Distribution (GPD)

Peaks Over Threshold (POT) & the Generalized Pareto Distribution (GPD)

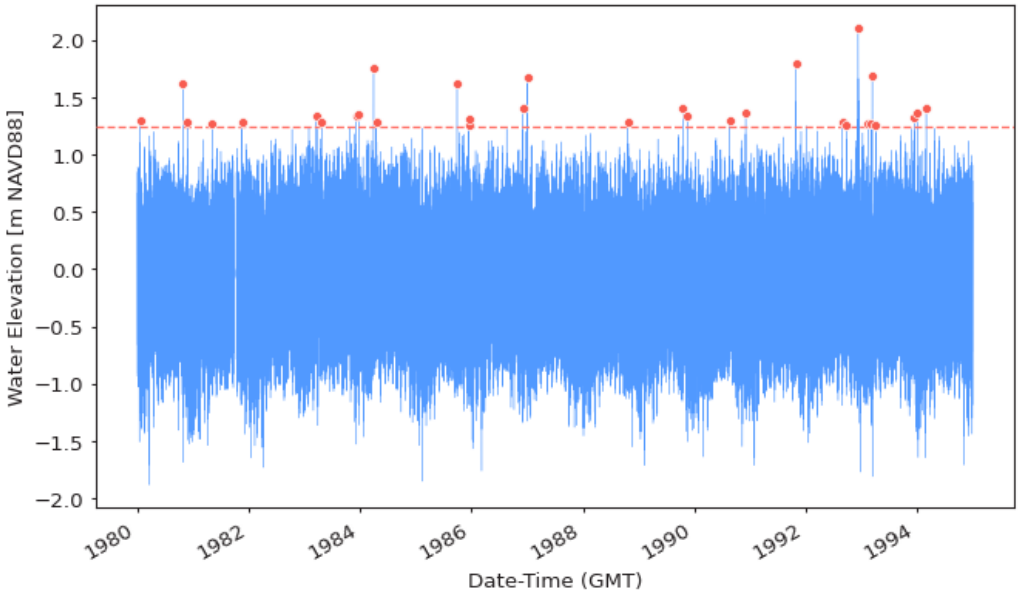
- Let $X_1, \ldots, X_n$ iid $\sim F$
- Define *extreme events* as those X_i's that exceed some high threshold, u.
- If F is known, then the distribution of threshold **exceedances** is:

$$P(X > u + y \mid X > u) = \frac{1 - F(u + y)}{1 - F(u)}$$

- Otherwise, if $Y_n = \max(X_1, \ldots, X_n)$ and $Y_n \dot{\sim} GEV(x; \mu, \sigma, \xi)$ then, for large enough u, the distribution of exceedances is approximately the **Generalized Pareto Distribution:**

$$H(y) = 1 - \left(1 + \frac{\xi y}{\tilde{\sigma}}\right)^{-1/\xi}$$

where $\{y: y > 0 \; and \; (1 + \xi y/\tilde{\sigma}) > 0\}$
and $\tilde{\sigma} = \sigma + \xi(u - \mu)$

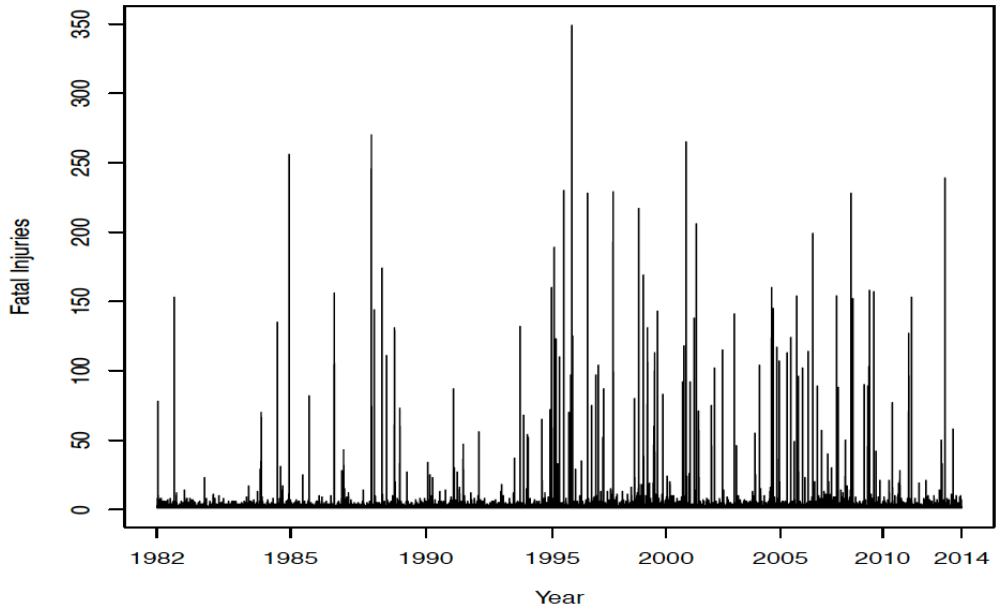


But now there's a problem…
How do we choose the threshold?

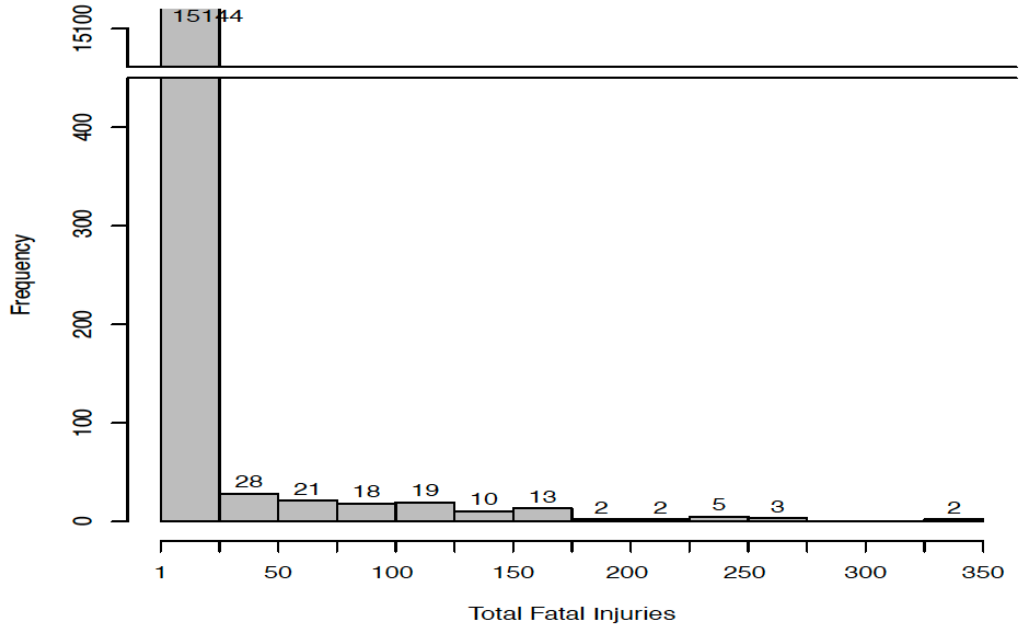
Choosing a Threshold for POT: Example

Fatal Injuries from Aviation Accidents

- "Quantification of the large accidents which have far reaching effect (fatality) would provide objective guidance in long-term planning and response for manufacturers, insurers and re-insurers." [Das 2016]



Numbers of fatal injuries from aviation accidents, 1982 - 2014



Histogram of fatal injuries from aviation accidents

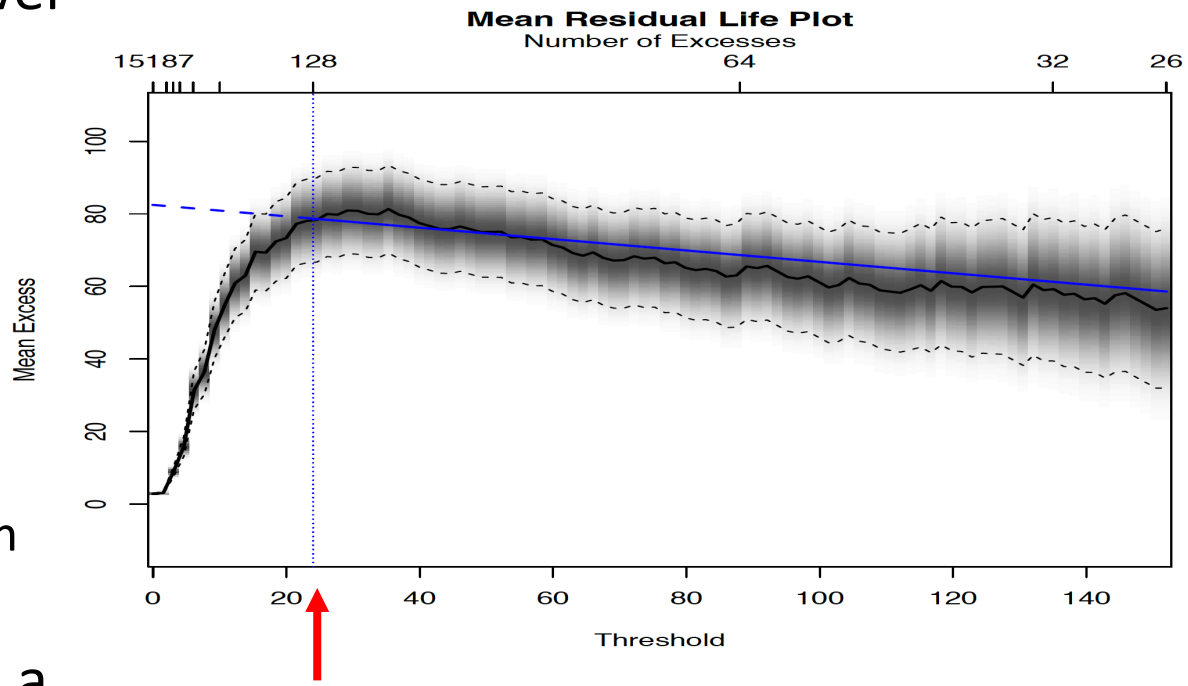
Source: [Das 2016]

Choosing a Threshold for POT: MRL

- If the tail data follow a GPD with lower bound of u, then the Mean Residual Life (MRL) plot should be approx. linear for values above u.

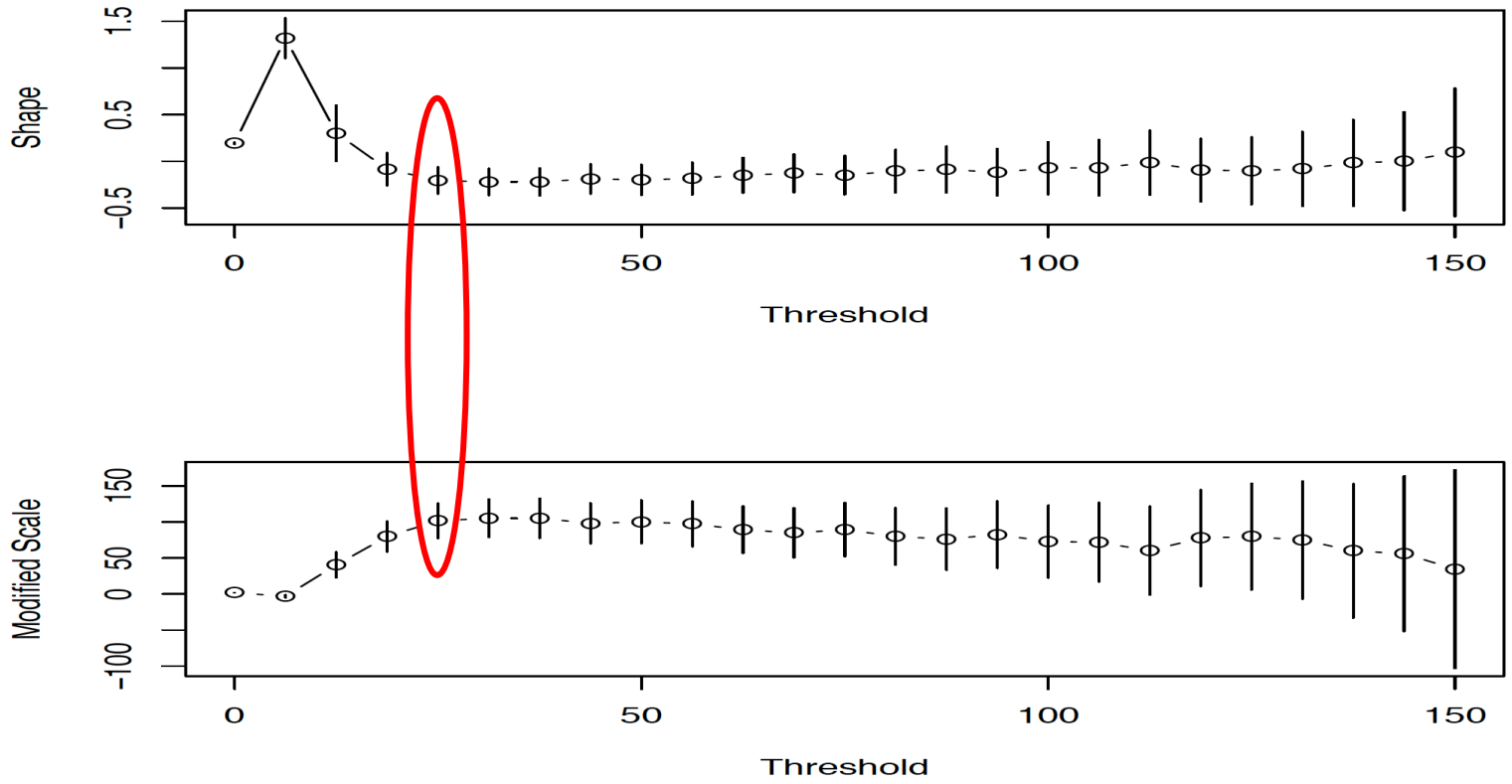
$$mrl(x) = E(X - x \mid X > x)$$

  - MRL is also sometimes called the Mean Excess Function
- So, select the smallest u which gives a linear MRL plot.



Source: [Das 2016]

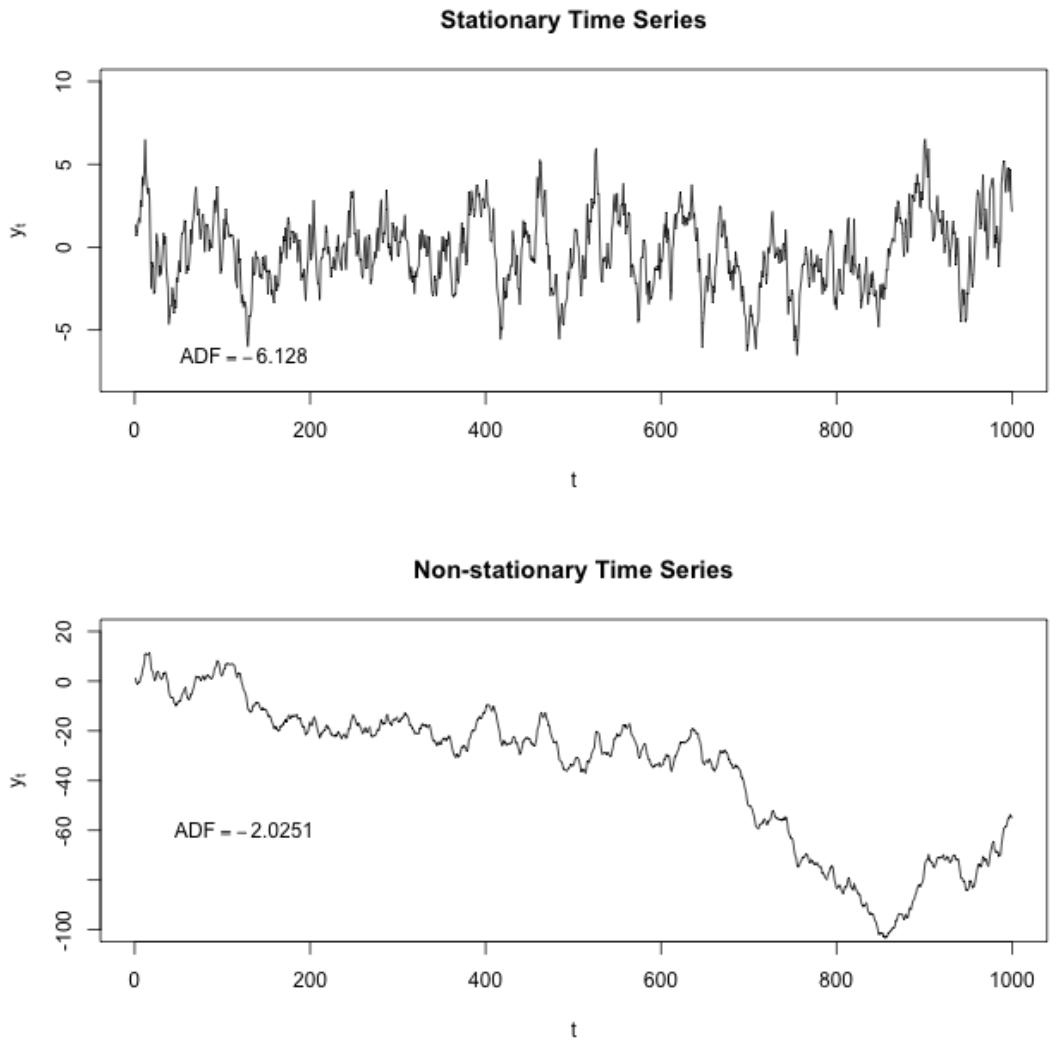
Choosing a Threshold for POT: Parameter Stability



Stationarity & Non-Stationarity

Stationarity

- $X_1, X_2, \ldots$ is a **stationary** random process if for any set of integers $\{i_1, \ldots, i_k\}$ and any integer m, the joint distributions of $(X_{i_1}, \ldots, X_{i_k})$ and $\left(X_{i_1+m}, \ldots, X_{i_k+m}\right)$ are identical.



https://en.wikipedia.org/wiki/Stationary_process

Dealing with Non-Stationarity

- Data often contains trends and seasonal cycles (financial, weather)
- Using BM with annual maximums can avoid seasonal cycles (weather)
- Trends and cycles can be removed via regression or time-series modeling.

$$\mu(t) = \mu_0 + \mu_1 t + \mu_2 t^2,$$

$$\sigma(t) = \sigma_0 + \sigma_1 t,$$

$$\xi(t) = \begin{cases} \xi_0, & t \leq t_0, \\ \xi_1, & t > t_0. \end{cases}$$

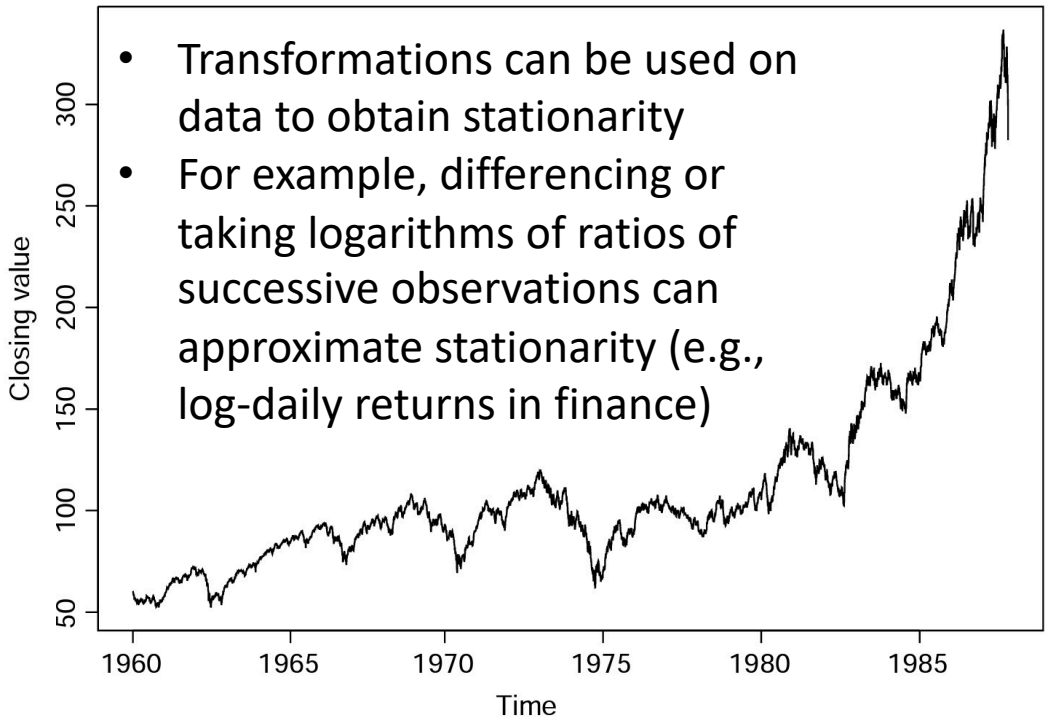
Example from *extRemes*
[Gilliland 2016]

Dealing with Non-Stationarity (cont.)

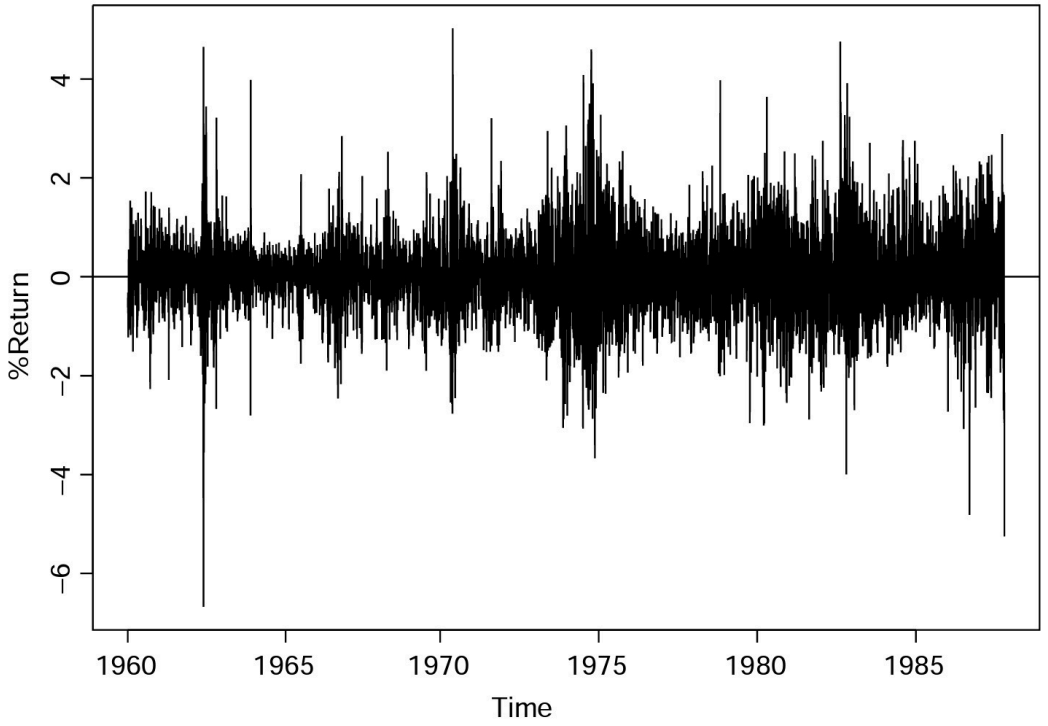
Financial Application: Estimate VaR (Value-at-Risk) for a given portfolio

S&P 500 Closing Values

- Transformations can be used on data to obtain stationarity
- For example, differencing or taking logarithms of ratios of successive observations can approximate stationarity (e.g., log-daily returns in finance)



Daily % Returns



Source: [Beirlant 2004]

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[Wikipedia GEV] “Generalized extreme value distribution”

Backup Slides

Central Limit Theorem†

- Let $X_1, X_2, X_3, \ldots, X_n$ be independent & identically distributed (iid) random variables (RVs)
- from a distribution that has mean μ and positive variance σ^2,
- and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

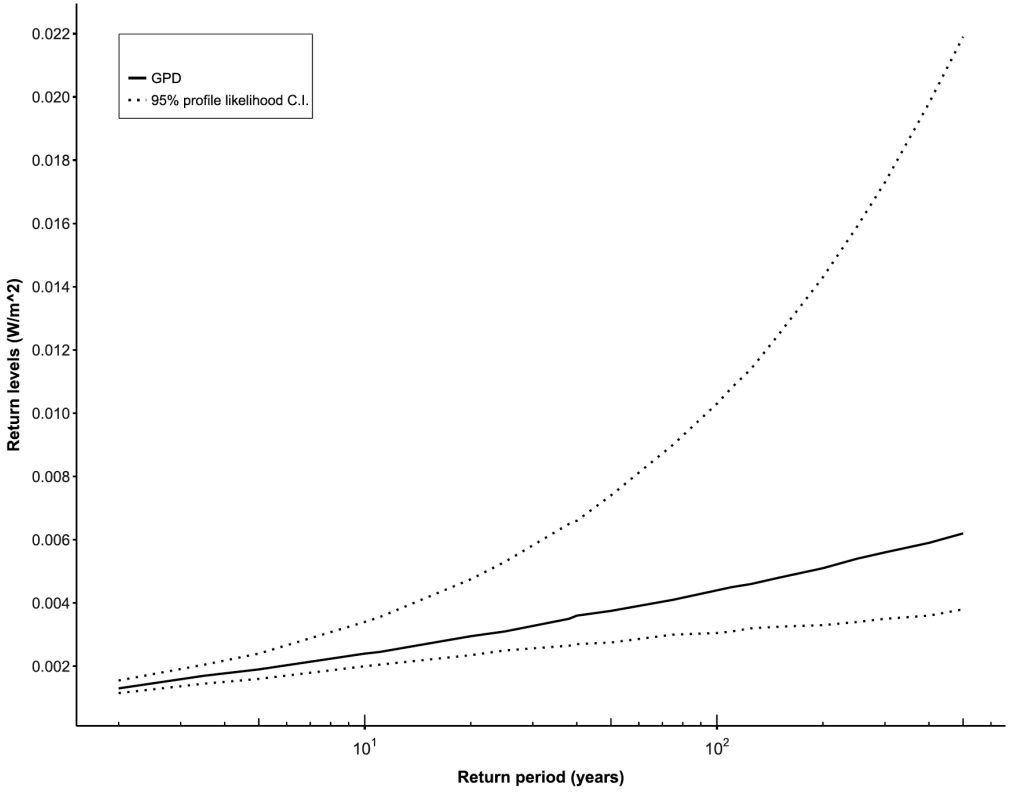
† This is a limited form of the CLT; other variants impose fewer conditions.

Case Studies (brief)

Solar Flares, California Droughts, and Human Life Span

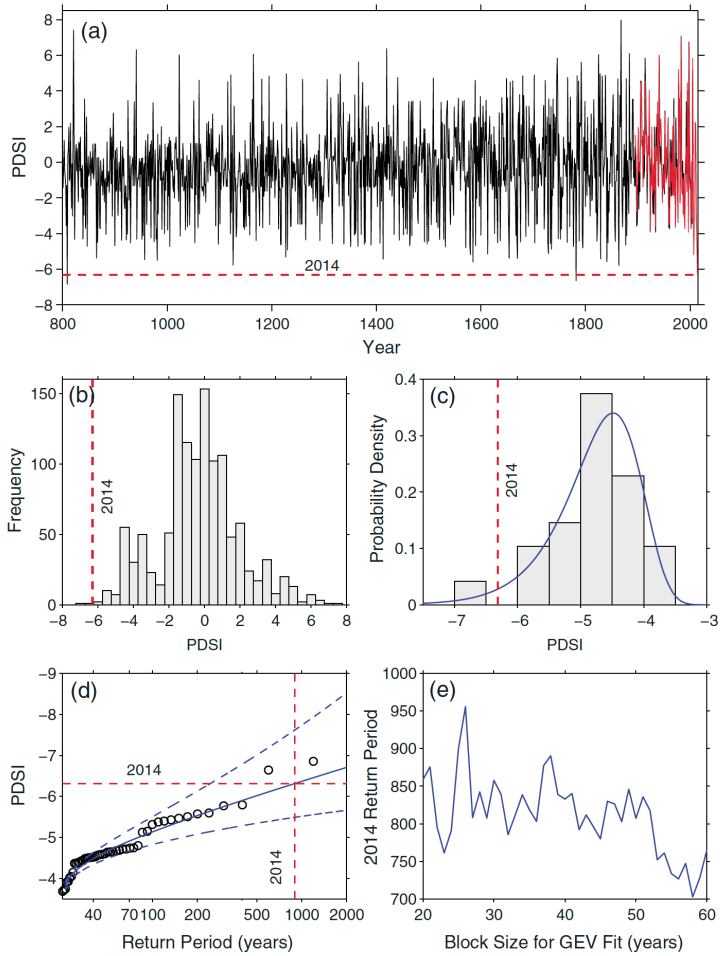
“EVA of Solar Flare Events” [Tsiftsi 2018]

- 1859 – the “Carrington Event” (X45), most intense geomagnetic storm in recorded history
  - Est. cost to U.S. of similar event: **$670 billion to $2.9 trillion (~ 3.6% – 15.5% annual GDP)**
    - https://en.wikipedia.org/wiki/Carrington_Event#Similar_events
  - **Return Period: 110 years**,
    - with profile likelihood CI ~ (20, 6500) years.
  - Probability of a Carrington-like event happening in the next decade is 9%
- 2003 – “Halloween solar storms” (X35) generated largest solar flare ever recorded by GOES
  - **Return Period: 38 years,**
    - with profile likelihood CI ~ (10, 300) years.
  - A Halloween-like event is expected in the next decade with probability 23.8%



California Droughts [Robeson 2015]

- The 1-year 2014 drought was most severe in the 1895–2014 record
  - Has a return period of 140–180 years,
  - however, *quantile mapping* produces return periods of 700–900 years
- Cumulative 3- and 4-year droughts are estimated to be much more severe
  - 2012–2014 drought is nearly a 10,000-year event
  - 2012–2015 drought has an almost incalculable return period and is completely without precedent



PDSI – Palmer Drought Severity Index

Limits to Human Life Span [Einmahl 2019]

- Used EVA to consider whether the human life span is bounded:
- 30 years of data from Dutch residents
- The estimated extreme value indices (ξ), exhibited in Figure 2, at right, are all negative, hinting at a finite upper endpoint, that is, a **finite maximum life span**.

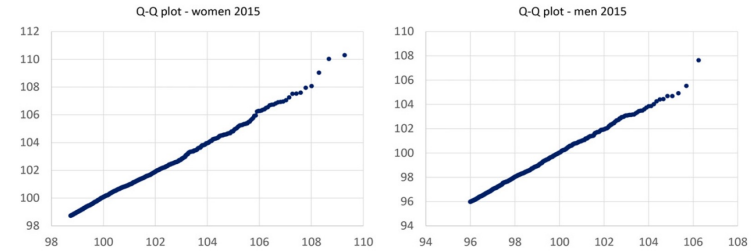


Figure 1. Generalized Pareto Q–Q plots for women and men for the year 2015.

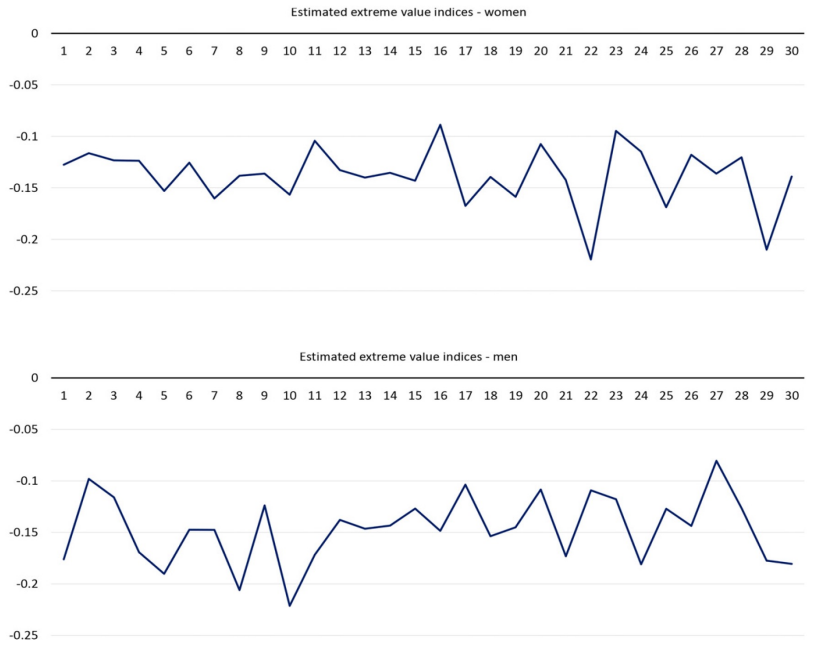


Figure 2. Estimated extreme values indices for the years of death $1985 + j, j = 1, \ldots, 30$.

Example 2

GEV Fit using PyExtremes (Python)

Example 2: pyextremes (Python)

- Raw Data (.csv) read & ”cleaned”
  - Sorted in ascending order
  - NaN entries removed
  - Converted to Pandas.Series
    - https://pandas.pydata.org/
  - Trend removed (+2.87 mm/yr)

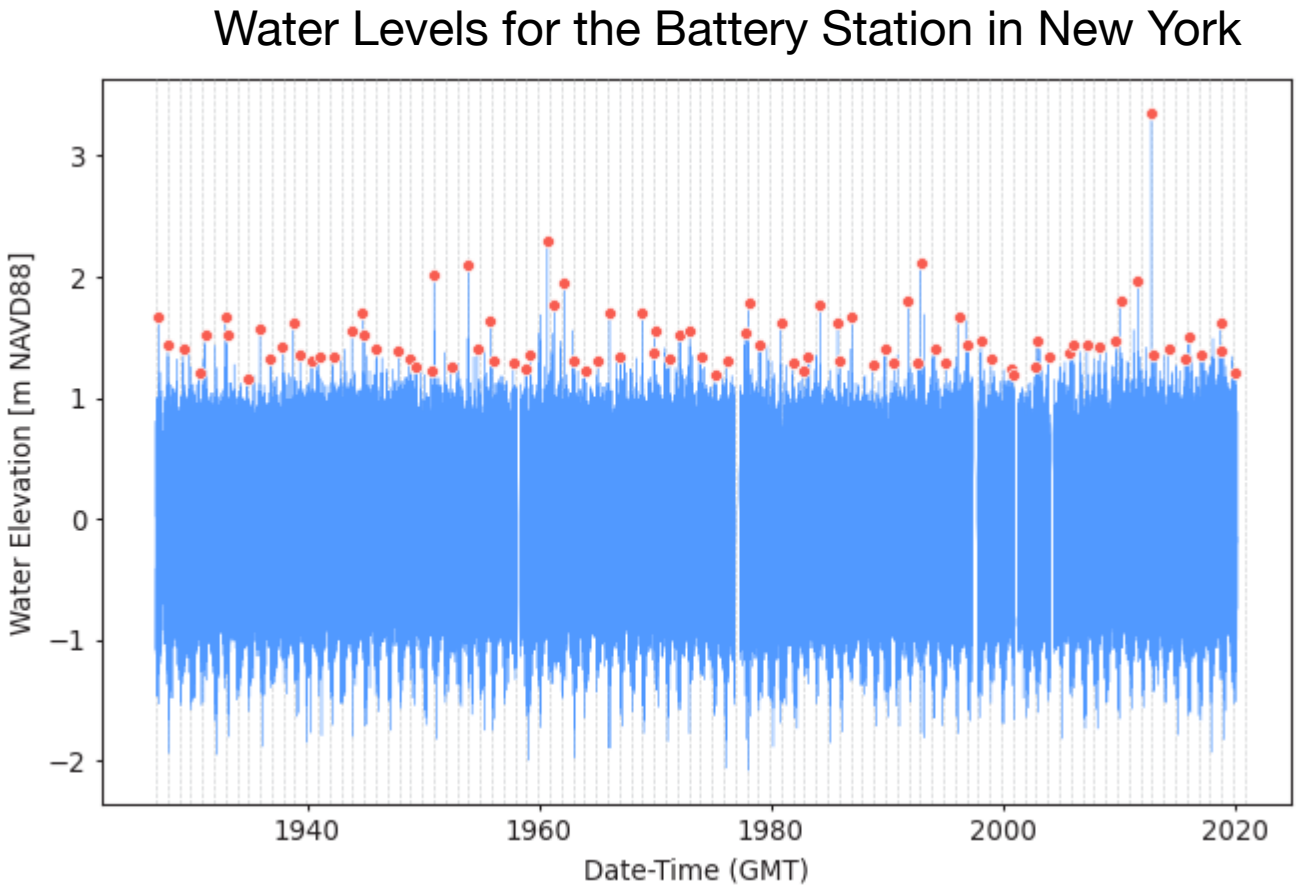
> from pyextremes import EVA

> model = EVA(data)

- EVA class provides interface to pyextremes library

> model.get_extremes(method="BM", block_size="365.2425D")

> model.plot_extremes()



See https://pypi.org/project/pyextremes/

Example 2: pyextremes (Model Fit)

> model.fit_model()

```
                          Univariate Extreme Value Analysis
================================================================================
                                    Source Data
--------------------------------------------------------------------------------
Data label:    Water Elevation [m NAVD88]      Size:                      796,751
Start:                        November 1926      End:                     March 2020
================================================================================
                                   Extreme Values
--------------------------------------------------------------------------------
Count:                                  94      Extraction method:             BM
Type:                                 high      Block size:      365 days 05:49:12
================================================================================
                                      Model
--------------------------------------------------------------------------------
Model:                                 MLE      Distribution:          genextreme
Log-likelihood:                     18.026      AIC:                      -29.786
--------------------------------------------------------------------------------
Free parameters:                  c=-0.266      Fixed parameters: All parameters are free
                                 loc=1.353
                               scale=0.146
================================================================================
```

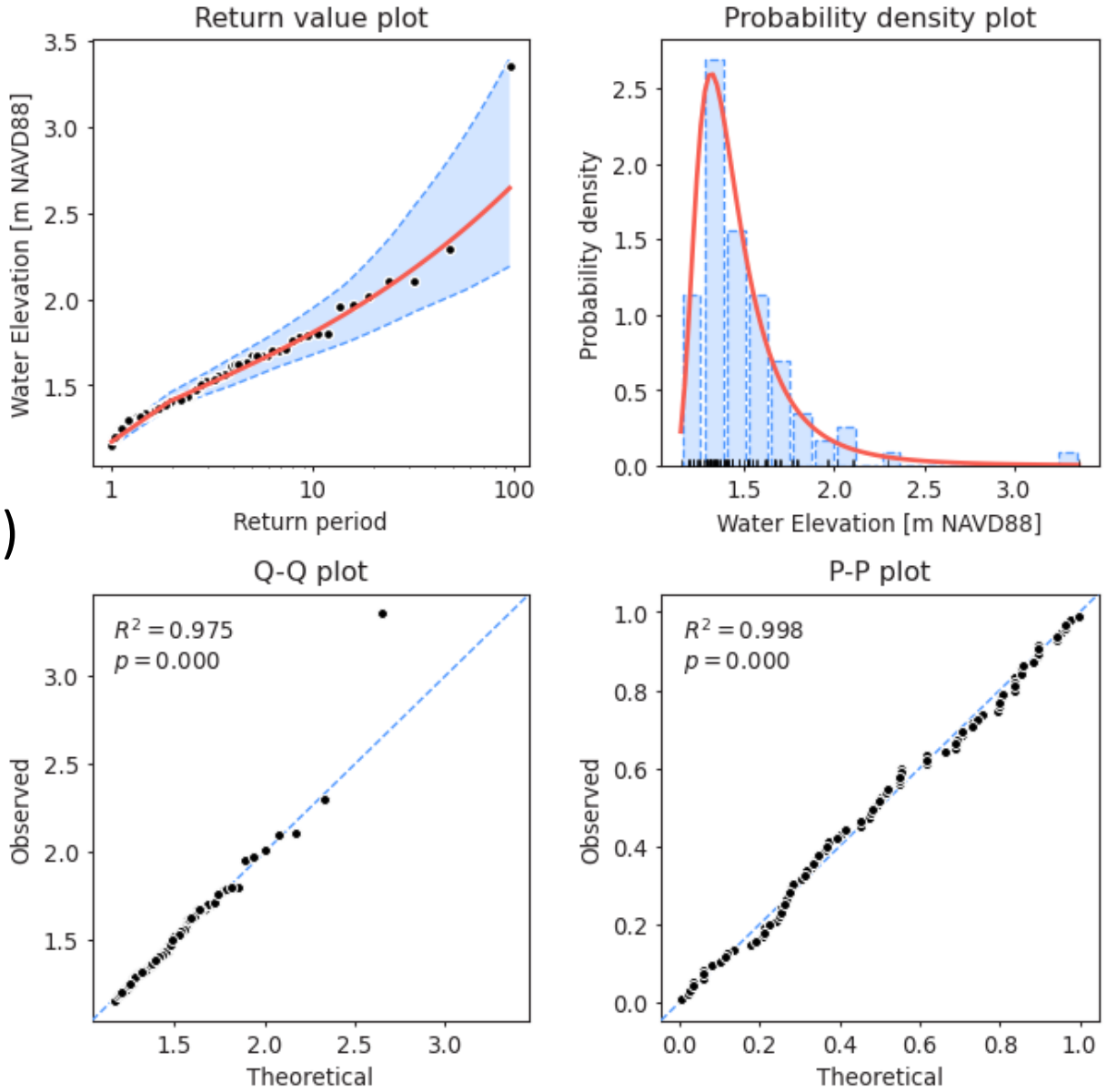
Example 2: pyextremes (Model Summary)

```
summary = model.get_summary(
    return_period=[1, 2, 5, 10, 25, 50, 100, 250, 500, 1000],
    alpha=0.95,
    n_samples=1000,
)
summary
```

return period	return value	lower ci	upper ci
1.0	0.802610	-0.313507	1.025702
2.0	1.409343	1.372263	1.453800
5.0	1.622565	1.547693	1.706435
10.0	1.803499	1.674898	1.951093
25.0	2.090267	1.854483	2.392612
50.0	2.354889	1.992968	2.875355
100.0	2.671313	2.139693	3.575801
250.0	3.188356	2.346309	4.843293
500.0	3.671580	2.522520	6.239443
1000.0	4.252220	2.704200	8.166698

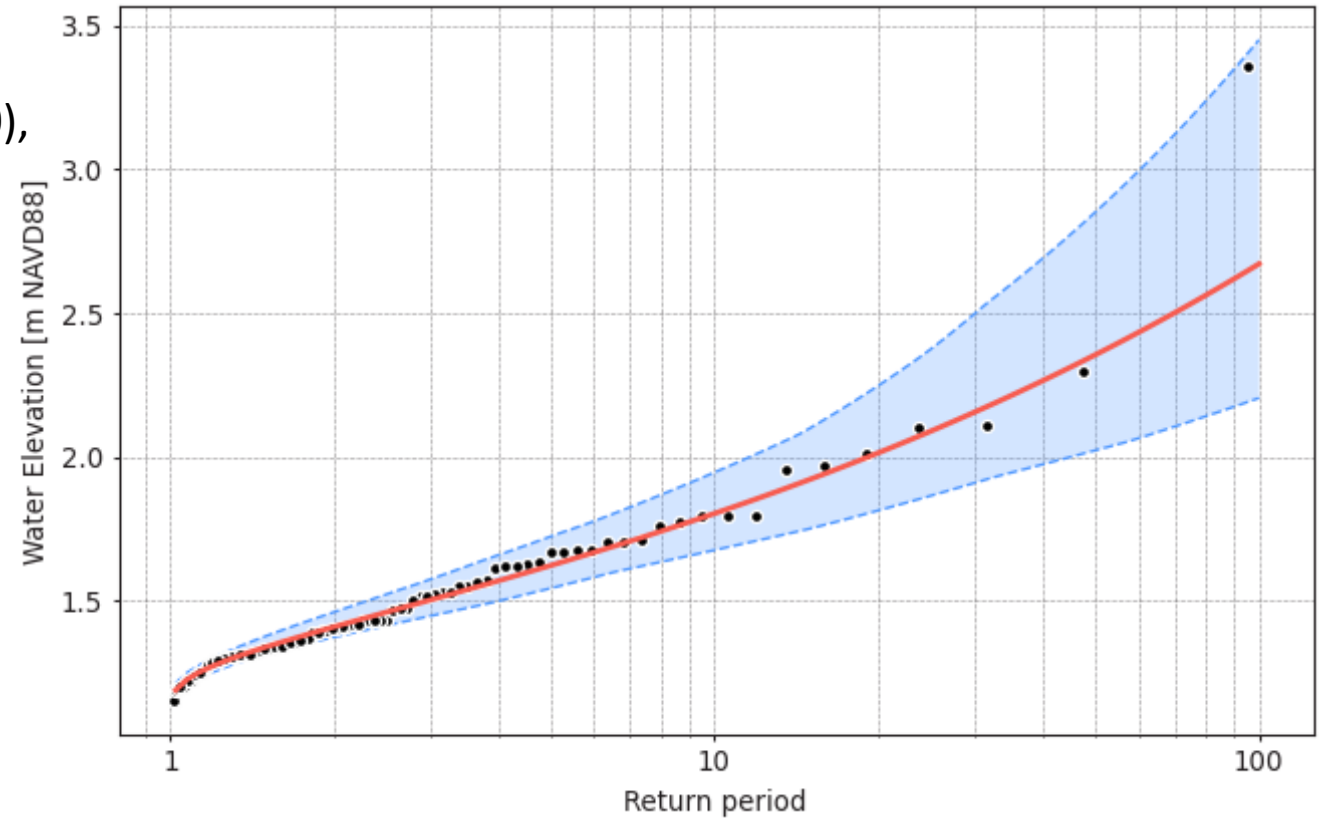
Example 2: pyextremes (Diagnostic Plots)

> model.plot_diagnostic(alpha=0.95)



Example 2: pyextremes (Return Values Plot)

```
> model.plot_return_values(
    return_period=np.logspace(0.01, 2, 100),
    return_period_size="365.2425D",
    alpha=0.95,
)
```



BM or POT?

- We cannot say that one method is better than another
- Different models and approaches (correctly applied) should converge to the same answer (within reasonable limits)
- So, investigate both
- BM is a simpler and more stable model
  - Requires very little input from the user
  - Use BM with a reasonable block size to avoid capturing seasonality
  - Get the initial estimates and see how the extremes behave
- Use POT with a reasonable threshold and declustering
  - To see how well the model behaves near the target return periods
  - and to gain more confidence in the results

Source: PyExtremes User Guide