Dynamic resource allocation problems in communication networks:

Weakly Coupled Markov decision processes

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- **Scenario:** N queues are processing jobs. The rate of each queue can be controlled. Moreover, the scheduler can also decide to which queue a job can be sent.
- **Challenge:** In order to minimize the total load on the system:
  - which queue should a job be *allocated* at each instant t?
  - which queue should see its *service rate* increased or decreased at each instant t ?

Arrival Process

We consider that at every time instant αN new jobs arrives in the system with probability $p \in (0, 1)$. Let $T_n \in \mathbb{N}_+$ be the arrival time of the n-th batch new jobs.

Note that:

$$\mathbb{P}(T_n - T_{n-1} = \tau) = (1 - p)^{\tau - 1} p, \ \forall \tau \geq 1, \ \forall n \in \mathbb{N}_+.$$

Dynamic of the queue

The length of the k-th queue, denoted by $S_k(T_{n+1})$ at instant T_{n+1} is given by:

$$S_k(T_{n+1}) = S_k(T_n) - D_k(T_{n+1} - T_n) + I\{S_k(T_n) < K\}A_k(T_n)$$

where:

- K is the finite buffer size of a queue;
- $D_k(T_n)$ the number of process jobs between T_n and T_{n+1}. We assume that the probability that a job is processed during one-time unit is equal to $B_k(t) \in \{\underline{b}, \bar{b}\}$. We assume that between two arrivals $B_k(t)$ is constant for all t and k;
- $A_k(T_n) \in \{0, 1\}$ is equal to one if one job from n-th batch is sent to the queue k.

Transition Probability

From that fact that the arrival are i.d.d. and the departure only depends on the inter-arrival time, we can rewrite the dynamic of the queue:

$$S_k(t+1) = S_k(t) - D_k(\tau) + I\{S_k(t) < K\}A_k(t).$$

We have the following lemma:

Lemma

For $s + a < k$, we have that the probability $\mathbb{P}(S_k(t+1) = s' | S_k(t+1) = s,\ A_k(t) = a,\ B_k(t) = b)$ is equal to

$$\sum_{\tau=1}^{+\infty} (1-p)^{\tau-1} p I_{s' < \min\{\tau, s+a\}} \binom{\tau}{s'} b^{\tau - s'} (1-b)^{s'}.$$

Cost functions and constraints

Costs: We will assume that there are two instantaneous costs:

- *Energy cost:* $\sum_k C_s(S_k(t)) + \sum_k C_q(B_k(t))$, where $C_s(\cdot)$ and $C_q(\cdot)$ are convex increasing.
- *Job rejection cost:* $-\gamma \sum_k A_k(t)$, with $\gamma > 0$. This cost implies that we prefer to send jobs.

Constraints: We will also assume that there are two instantaneous constraints:

$$\sum_k A_k(t) \leq \alpha N, \quad (1)$$

$$\sum_k B_k(t) \leq \beta N. \quad (2)$$

Motivation

Weakly Coupled Markov decision processes

Construction of LP-Admissible Policy

Multistage Convex Stochastic Optimization Problems

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- We assume that the decision-maker has to respect the following resource allocation constraints:

$$\sum_k D_l(S_k(t), A_k(t)) \leq N\alpha_l, \ \forall l = 1, \dots, L$$

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Discussion with respect to the constraints

We assume that all terms in $D_l(s, a)$ and α_l are non-negative numbers, and that $D(s, 0) = 0$.

This is a natural assumption under the resource allocation context in which $a = 0$ corresponds to a passive action that consumes no resources.

Implication: The later also implies that our resource constraint problem has at least a feasible solution by always choosing the passive action.

Mathematical Formulation

$$\min_{\pi} \quad \mathbb{E} \sum_{t=0}^{T-1} \sum_{s,a} r_s^a Y_{a,s}^{(N)}(t) := V_{opt}^{(N)}(m(0), T) \tag{3a}$$

s.t. Arms follow the Markovian evolution generated by $\Pi_n p^{a_n}_{s_n, s'_n}$, (3b)

$$\sum_a Y_{a,s}^{(N)}(t) = M_s^{(N)}(t),\ \forall t \in [[0, T-1]],\ \forall s \in \mathcal{S}, \tag{3c}$$

$$\sum_s D_l(s,a) Y_{s,a}^{(N)}(t) \leq \alpha_l\ \forall t \in [[0, T-1]],, \tag{3d}$$

$$M_s^{(N)}(0) = m_s(0),\ \forall s \in \mathcal{S}, \tag{3e}$$

where $m_s(0) = \frac{1}{N} \sum_{k=1}^{N} I\{S_k(0) = s\}$, for all $s \in \mathcal{S}$.

Difficulty

The key difficulty of Weakly Coupled Markov decision processes is coming from:

$$\sum_{s} D_l(s,a) Y_{s,a}^{(N)}(t) \leq \alpha_l \; \forall t \in [[0, T-1]],$$

which couples all the arms together.

Challenge of the day:
How to design an efficient heuristic to solve such problem?
A different one that the projection policy.

Outline of the approach

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2. **Interpolation:** Construct a sequence of decision rules $\pi_t : \Delta^d \to \Delta^{2d}$ which is optimal for the relaxed problem.

Relaxed problem

$$\min_{\pi} \quad \mathbb{E}[\sum_{t=0}^{T-1} \sum_{s,a} r_s^a Y_{a,s}^{(N)}(t)] =: V_{rel}^{(N)}(m(0), T) \tag{4a}$$

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$$M_s^{(N)}(0) = m_s(0), \ \forall s \in \mathcal{S}, \tag{4e}$$

LP formulation

Let us define the following LP problem:

$$\begin{aligned}
\min_{y \geq 0} \quad & \sum_{t=0}^{T-1} \sum_{s,a} r_s^a y_{s,a}(t) =: V_{LP}(m(0), T) \\
\text{s.t.} \quad & \sum_{a} y_{s,a}(t) = m_s(t),\ \forall t \in [[0, T-1]],\ \forall s \in \mathcal{S}, \\
& m_s(t) = \sum_{s'} \sum_{a} y_{s',a}(t-1) p_{s',s}^a \ \forall t \in [[1, T-1]],\ \forall s \in \mathcal{S}, \\
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We denote by $y^* := [[[y^*_{s,a}(t)]]]_{s,a,t}$ the optimal solution of (6) and we also define $m^* := [[m_s(t) := \sum_a y^*_{s,a}(t)]]_{s,t}$.

Feasible control

We define the set of feasible control at time t by:

$$\mathcal{Y}(M^{(N)}(t)) := \Big\{ y \in \mathbb{R}_+^{2S} | \sum_a y_{s,a} = M_s^{(N)}(t) \ \forall s \in \mathcal{S};$$
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Resolving policy

We redefine the following LP:

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\min_{y\geq 0} \quad & \sum_{t=0}^{T-t-1} \sum_{s,a} r_s^a y_{s,a}(t) =: V_{LP}(m(0), \textbf{T-t}) \\
\text{s.t.} \quad & \sum_a y_{s,a}(t) = m_s(t),\ \forall t \in [[0, \textbf{T-t-1}]],\ \forall s \in \mathcal{S}, \\
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& \sum_s y_{s,1}(t) \leq \alpha,\ \forall t \in [[0, \textbf{T-t-1}]],, \\
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The solution of this LP is denoted by

$$y^{Res}(m(0), T-t) = [y_{t'}^{Res}(m(0), T-t)]_{0 \leq t' \leq T-t-1}.$$

Algorithm to solve the LP

What could be a possible algorithm to solve this LP?

Solution 1: Simplex or Convex optimisation?

Solution 2: Dynamic programming. Observe that:

$$V_{LP}(m, T-t) = \min_{y \in \mathcal{Y}(m)} \sum_{s,a} r_s^a y_{s,a} + V_{LP}(\phi(m,y), T-t-1),$$

where $\phi_s(m,y) = \sum_{s'} \sum_a y_{s',a} p_{s',s}^a$ for all s.

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2. $\pi_t^{Res}(M^{(N)}) \in \mathcal{Y}(M^{(N)}(t))$;
3. $y^*(t) = pi_t^{Res}(m^*(t))$. (P-Admissible Policy)

Algorithm

Resolving Policy

- **Input:** Initial system configuration vector $m(0)$ and time horizon T.
- **Set** $\hat{M}$:= m(0);
- **For** $t = 0, 2, \ldots, T-1$ **do:**
  1. **Compute** $y^{Res}(\hat{M}, T-t)$; Set $\hat{y}(t) = y_0^{Res}(\hat{M}, T-t)$
  2. *Rounding step:* For all $s \in \mathcal{S}$, set:

$$\hat{Y}_{s,a}^{(N)}(t) = \begin{cases} N^{-1}\lfloor N\hat{y}_{s,1}(t) \rfloor & \text{if } a = 1, \\ \hat{M}_s - N^{-1}\lfloor N\hat{y}_{s,1}(t) \rfloor & \text{otherwise.} \end{cases}$$

  3. Use control $\hat{Y}^{(N)}$ to advance to the next time-step ;
  4. Set $\hat{M}$:= current empirical distribution;

Certainty equivalent control

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Principle of the CEC

Sub-optimal control that applies at each stage the control that would be optimal if some or all of the uncertain quantities were fixed at their expected values.

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3. How to handle continuous state?
4. Efficient algorithm to solve the LP when the parameters are unknown.

Example: Access control and Utility Maximization

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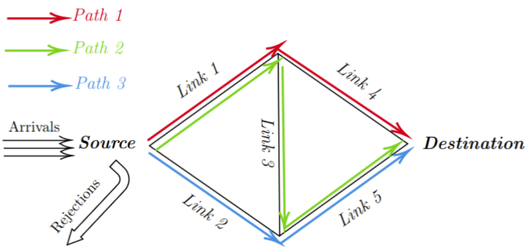
- **Challenge:** Maximize the total amount of Bandwidth sent into the network?



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$$0 \leq U_1(t) \leq W_1(t) \quad 0 \leq U_2(t) \leq W_2(t) \quad 0 \leq U_3(t) \leq W_3(t)$$

Bandwidth occupation

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- $(X_1(t), X_2(t), X_3(t))$ the bandwidth occupation of the three routing paths just arriving at time-step t.
- We assume that the evolution of $X_p(t)$ is given by:

$$X_p(t+1) = (X_p(t) + U_p(t)) \cdot q_p + \epsilon_p(t+1), \quad \text{for } 1 \le p \le 3 \text{ and } 1 \le t \le T,$$

where $\epsilon_p(t+1)$ is a r.v. with mean zero and support,

$$[-(X_p(t) + U_p(t)) \cdot q_p, (X_p(t) + U_p(t)) \cdot (1 - q_p)].$$

Capacity constraints

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- The constraints that each link should satisfy are given by for all $1 \leq t \leq T$:

$$\begin{aligned}
Y_1(t) &:= U_1(t) + X_1(t) + U_2(t) + X_2(t) \leq c_1 \\
Y_2(t) &:= U_3(t) + X_3(t) \leq c_2 \\
Y_3(t) &:= U_2(t) + X_2(t) \leq c_3 \\
Y_4(t) &:= U_1(t) + X_1(t) \leq c_4 \\
Y_5(t) &:= U_2(t) + X_2(t) + U_3(t) + X_3(t) \leq c_5.
\end{aligned}$$

Utility funcion

The decision-maker aims to maximize the following α-fairness utility (with $\alpha > 0$)

$$\mathbb{E}\sum_{t=1}^{T}\sum_{p=1}^{3}\frac{(X_p(t)+U_p(t))^{1-\alpha}}{1-\alpha}$$

gained by allocating and rejecting the bandwidth demands over a finite horizon T, while respecting the dynamics and constraints described in the previous slides.

Constrained Finite Horizon Stochastic Optimization Problem

For each time-step $t = 1, \ldots, T$:

1. The decision-maker gets the current system state $X(t)$;

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5. The system evolves to the next state $(t+1)$ such that $X(t+1) \sim \phi\,(X(t), W(t), U(t)) + \epsilon(X(t), W(t), U(t))$.

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Objective: Maximize the expected total sum of rewards over the T time-steps.

Mathematical model

$$\max_{[1,T]} \quad \mathbb{E}\left[\sum_{t=1}^{T} R_t\left(X(t), W(t), U(t)\right)\right] =: V_{\text{opt}}(x(1), T) \tag{8a}$$

$$\text{s.t.} \quad X(1) = x(1) \; a.s., \tag{8b}$$

$$g_{t,i}(X(t), W(t), U(t)) \leq 0, \; \forall t \text{ and } \forall j, \tag{8c}$$

$$h_{t,j}(X(t), W(t), U(t)) = 0, \; \forall t \text{ and } \forall j, \tag{8d}$$

$$X(t+1) = \phi\left(X(t), W(t), U(t)\right) + \epsilon(X(t), W(t), U(t)), \; \forall t \tag{8e}$$

The Certainty Equivalent Control (CEC)

Based on CEC, we apply the following relaxation to the original problem: define $\mathbb{E}X(t) := x(t)$ and $\mathbb{E}U(t) = u(t)$ where the expectation is taken with the whole trajectory. By Jensen's inequality, we have

$$\mathbb{E}[\sum_{t=1}^{T} R_t \left(X(t), W(t), U(t)\right)] \leq \sum_{t=1}^{T} R_t \left(x(t), \mathbb{E}[w], u(t)\right). \tag{9}$$

$$\mathbb{E}[g_{t,i}(X(t), W(t), U(t))] \geq g_{t,i}(x(t), \mathbb{E}[w], u(t)),\ \forall t, i \tag{10}$$

$$\mathbb{E}[h_{t,j}(X(t), W(t), U(t))] = h_{t,j}(x(t), \mathbb{E}[w], u(t))\ \ \forall t, j \tag{11}$$

Relaxed mathematical program

All this consideration leads to the following relaxed mathematical program with decision variables $u(t)$:

$$\begin{aligned}
\max_{u[1,T]} \quad & \sum_{t=1}^{T} R_t\left(x(t), \overline{w}, u(t)\right) && \text{(12a)}\\
\text{s.t.} \quad & x(1) = x, && \text{(12b)}\\
& g_{t,i}(x(t), \overline{w}, u(t)) \leq 0,\ \forall t, i, && \text{(12c)}\\
& h_{t,j}(x(t), \overline{w}, u(t)) = 0,\ \forall t, i, && \text{(12d)}\\
& x(t+1) = \phi\left(x(t), \overline{w}, u(t)\right),\ \forall t, i && \text{(12e)}
\end{aligned}$$

Algorithms

You can apply the resolving algorithm or the projection policy in this case. You can even have a combination of both.

But we don't have the symmetry property. So our error will be this time controlled by the **variances** of the different variables.

Bibliography

- The proof of the main theorem and more advance theorem can be found here: Gast, Nicolas, Bruno Gaujal, and Chen Yan. "The LP-update policy for weakly coupled Markov decision processes." arXiv preprint arXiv:2211.01961 (2022).
- If you want to have a quick introduction to dynamic programming, please have a look to the lecture note of Nahum Shimkin: `https://webee.technion.ac.il/shimkin/LCS11/LCS11index.html`