Homework 1

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PHYS 410
Summary of Answers are at the last page

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd #for tabling results
```

Question 1

Functions for Simple Bisection and 1-D Newton's Method:

In [2]:
```
"""
Code for a simple bisection
[a,b] signifies the starting interval.
givenFunction is the function of the sepcific problem we want
delta is a threshold value to function(x) for saying approximation
is good enough
"""
def bisection(a,b, givenFunction, delta = 0.01):
    if a>b:
        # Checking if a <= b
        return bisection(b,a, givenFunction, delta)
    mid = (a+b)/2
    if abs(givenFunction(mid)) < delta: # Test delta against value of root
        return mid # Return midpoint of interval if bisection is  satisfied
    else:
        if givenFunction(a)*givenFunction(mid) < 0:
            return bisection(a, mid, givenFunction, delta)
        else:
            return bisection(mid, b, givenFunction, delta)
```

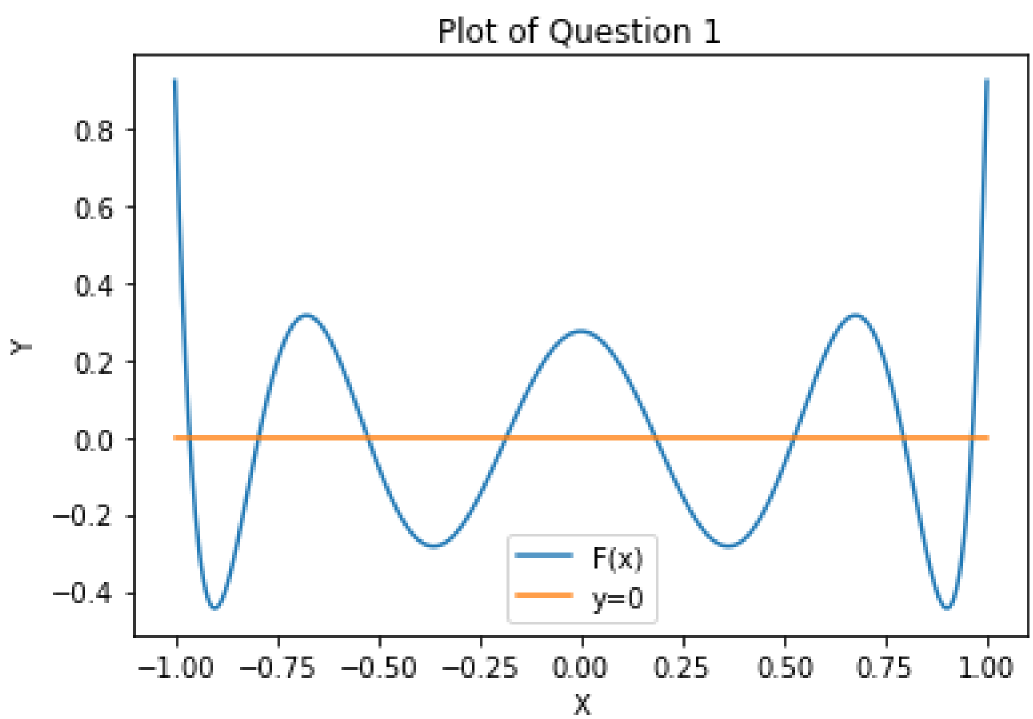
In [16]:
```
"""
Code for a 1-D Newton's method
x signifies the starting point.
givenFunction is the function of the sepcific problem we want
givenDerivative is the function of the sepcific problem we want
delta is a threshold value to function(x) for saying approximation
is good enough
a is the learning rate
"""
def one_Newton(x, givenFunction, givenDerivative, delta, a):
    value = givenFunction(x)
    slope = givenDerivative(x)

    if abs(value) < delta:
```

```
            return x
        else:
            xNew = x - (value/slope)
            if abs(xNew - x) < a:
                xNew = x - a*np.sign(value/slope)
            return one_Newton(xNew, givenFunction, givenDerivative, delta, a)
```

Function and Derivative for Question 1:

```
In [3]: def q1Function(x):
            return(6425*x**8 - 12012*x**6 + 6930*x**4 - 1260*x**2 + 35)/128

        def q1Derivative(x):
            return(8*6425*x**7 - 6*12012*x**5 + 4*6930*x**3 - 2*1260*x)/128
```

```
In [4]: x = np.linspace(-1,1,1000)
        y = np.apply_along_axis(q1Function,0, x)
        y2 = np.zeros(len(x))

        plt.figure(1)
        plt.plot(x,y, label = "F(x)")
        plt.plot(x,y2, label = "y=0")
        plt.xlabel("X")
        plt.ylabel("Y")
        plt.legend()
        plt.title("Plot of Question 1")
        plt.show()
```



By plotting, we now that there are 8 single roots in between $[-1, 1]$. We also now, by observation, that there are exactly one root in these intervals: $\{[-1, -0.90], [-0.90, -0.60], [-0.60, -0.30], [-0.30, 0.0], [0.0, 0.30], [0.30, 0.60], [0.60, 0.90], [0.90, 1.0]\}$

We want to use a hybrid method of Bisection and Newton. To do that, we first have to write a function that hybridizes them:

```
In [5]: """
        hybrid hybridizes the bisection and 1D Newtion method
        [a,b] signify the starting interval.
        givenFunction is the function of the sepcific problem we want
        givenDerivative is the function of the sepcific problem we want
        bis_delta is a threshold value to function(x) for saying bisection
        is good enough
        newton_delta is a threshold value to function(x) for saying bisection
        is good enough
        newton_a is the max learning rate for the 1D newton
        """
        def hybrid(a,b, bis_delta, newton_delta, newton_a, givenFunction,
                   givenDerivative):
            x0 = bisection(a,b,givenFunction, bis_delta)
            root = one_Newton(x0,givenFunction, givenDerivative, newton_delta,
                              newton_a)
            return root
```

Now, calculating approximates roots with threshold deltas:

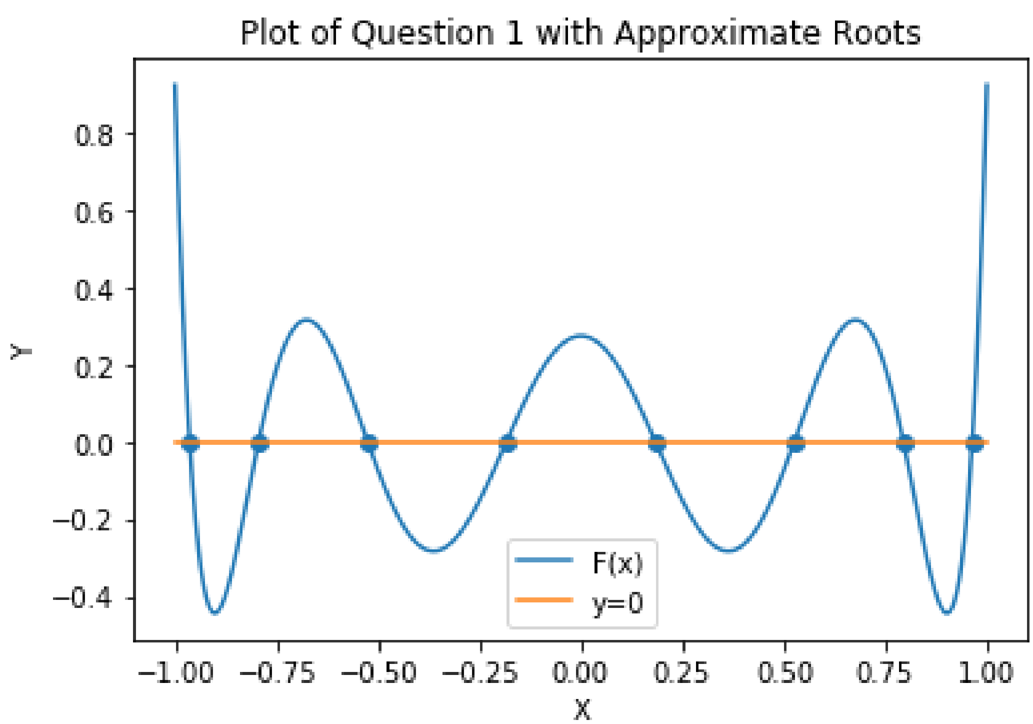
```
In [6]: cols = ["Interval Start", "Interval End", "Approximate Root", "Value of Approx"]
        data = []
        candidates = [[-1,-0.90],[-0.90,-0.60],[-0.60,-0.30],[-0.30,0.0],
                      [0.0,0.30],[0.30,0.60],[0.60,0.90],[0.90,1.0]]
        bisectionDelta = 0.001
        newtonDelta = 0.000001
        newtonA = 0.00000001

        for c in candidates:
            a = c[0]
            b = c[1]
            root = hybrid(a, b, bisectionDelta, newtonDelta, newtonA,
                          q1Function, q1Derivative)
            data.append([c[0],c[1],root, q1Function(root)])

        p = pd.DataFrame(data, columns=cols)
        p
```

Out[6]:

	Interval Start	Interval End	Approximate Root	Value of Approx
0	-1.0	-0.9	-0.963787	6.548784e-08
1	-0.9	-0.6	-0.794161	-7.321224e-08
2	-0.6	-0.3	-0.525686	2.095979e-08
3	-0.3	0.0	-0.183435	-5.958460e-10
4	0.0	0.3	0.183435	-5.958460e-10
5	0.3	0.6	0.525686	2.095979e-08
6	0.6	0.9	0.794161	-7.321224e-08
7	0.9	1.0	0.963787	6.548784e-08

Plotting the approximate roots on the graph:

In [7]:
```
x = np.linspace(-1,1,1000)
y = np.apply_along_axis(q1Function,0, x)
y2 = np.zeros(len(x))

plt.figure(2)
plt.plot(x,y, label = "F(x)")
plt.plot(x,y2, label = "y=0")
plt.scatter(p["Approximate Root"], p["Value of Approx"], label=None)
plt.xlabel("X")
plt.ylabel("Y")
plt.legend()
plt.title("Plot of Question 1 with Approximate Roots")
plt.show()
```



Question 2

Function for n-D Newton's Method:

```
In [8]: """
        Code for a n-D Newton's method
        X signifies the starting point. This is a 1xn array
        givenFunctions is an array of functions of the sepcific problem we want.
         Should be a 1xn array.
        givenDerivatives is a Jacobian matrix of partial derivatives of the
         sepcific problem we want. Should an nxn matrix
        delta is a threshold error for saying approximation is good enough.
         All functions must fall within threshold.
        """
        def n_Newton(X, givenFunctions, givenDerivatives, delta):
         # Values of each function at X
            b = np.array([g(X) for g in givenFunctions])
         # Jacobian Matrix evaluated at X
            J = np.array([np.array([g(X) for g in G]) for G in givenDerivatives])

            if all([abs(x) < delta for x in b]):
                return X
            else:
                dr = np.linalg.solve(J, -b) #Python's answer to MATLAB's linsolve
                xNew = X + dr
                return n_Newton(xNew, givenFunctions, givenDerivatives, delta)
```

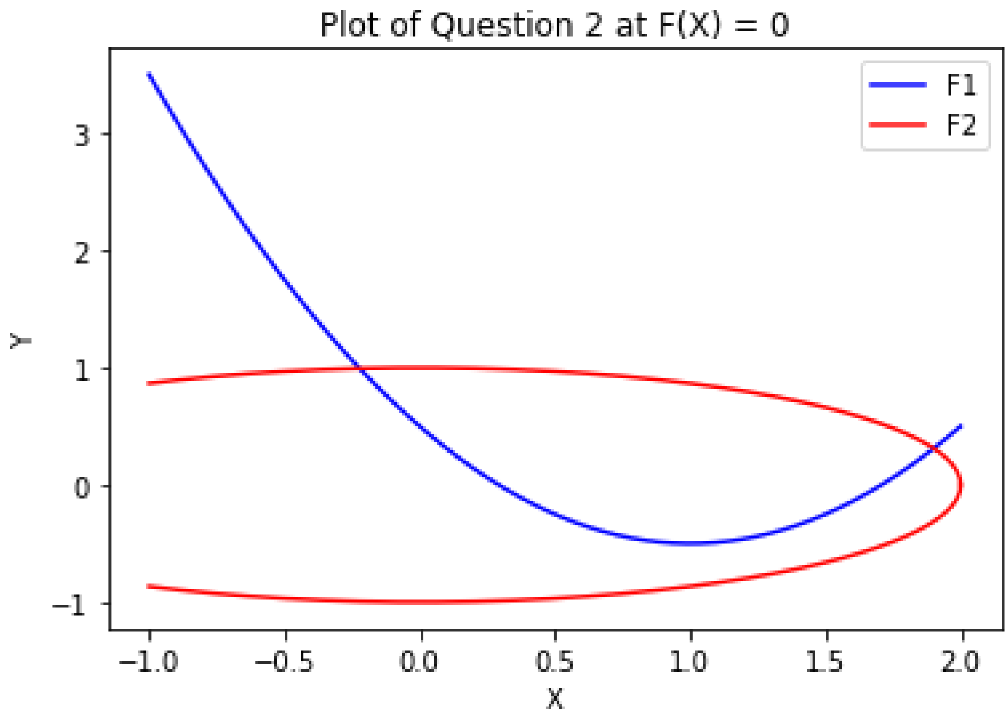
For this question, we have to deal with a 2-variable system, so we have two functions:

```
In [9]: def q2Function1(X): # Function for Function 1
            x1 = X[0]
            x2 = X[1]
            return(x1**2 - 2*x1 - x2 + 0.5)

        def q2Derivative11(X): #Partial Derivative of Function 1 on x1
            x1 = X[0]
            x2 = X[1]
            return(2*x1 - 2)

        def q2Derivative12(X): #Partial Derivative of Function 1 on x2
            x1 = X[0]
            x2 = X[1]
            return(-1)

        def q2Function2(X): # Function for Function 2
            x1 = X[0]
```

```
    x2 = X[1]
    return(x1**2 + 4*x2**2 - 4)

def q2Derivative21(X): #Partial Derivative of Function 2 on x1
    x1 = X[0]
    x2 = X[1]
    return(2*x1)

def q2Derivative22(X): #Partial Derivative of Function 2 on x2
    x1 = X[0]
    x2 = X[1]
    return(8*x2)
```

After declaring our functions and partial derivatives, it is now time to build the functional vector as well as the Jacobian Matrix:

```
In [10]: F = [q2Function1, q2Function2] #Functional vector
         #Jacobian Matrix
         J = [[q2Derivative11, q2Derivative12],[q2Derivative21, q2Derivative22]]

In [11]: x = np.linspace(-1,2,2000)
         y1 = np.apply_along_axis(lambda x: x**2 - 2*x + 0.5, 0, x)
         y21 = np.apply_along_axis(lambda x: -((-x**2 + 4)/4)**0.5, 0, x)
         y22 = np.apply_along_axis(lambda x: ((-x**2 + 4)/4)**0.5, 0, x)

         plt.figure(3)
         plt.plot(x,y1, color = 'b', label="F1")
         plt.plot(x,y21, color = 'r', label="F2") #Split F2 into 2 to plot the ellipse
         plt.plot(x,y22, color = 'r') #Split F2 into 2 to plot the ellipse
         plt.title("Plot of Question 2 at F(X) = 0")
         plt.legend()
         plt.xlabel("X")
         plt.ylabel("Y")
         plt.show()
```



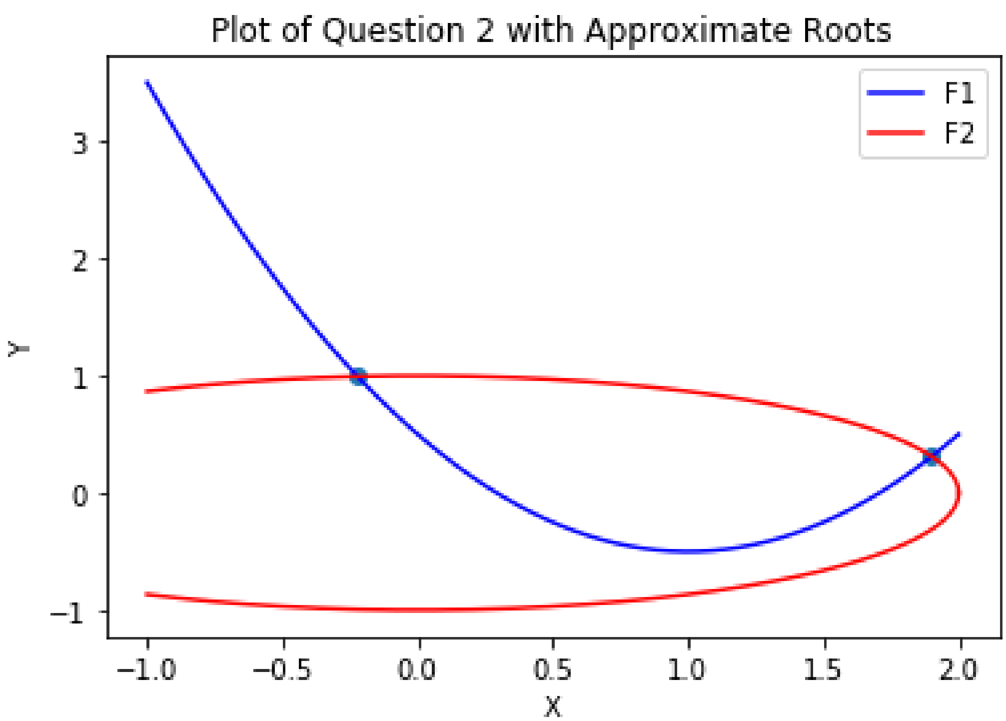
By plotting, we now that there are 2 simultaneous roots around (-0.2,0.9) and (1.9, 0.3). Now, calculating approximates roots with threshold deltas:

```
In [12]: cols = ["Starting Guess", "Approximate Roots (x)", "Approximate Roots (y)",
                 "Value of Approx for F1", "Value of Approx for F2"]
         data = []
         candidates = [[-0.2, 0.9],[1.9, 0.3]]
         delta = 0.000001

         for c in candidates:
             root = n_Newton(c, F, J, delta)
             data.append([c,root[0], root[1], q2Function1(root), q2Function2(root)])

         q = pd.DataFrame(data, columns=cols)
         q
```

```
Out[12]:   Starting Guess  Approximate Roots (x)  Approximate Roots (y)  \
         0    [-0.2, 0.9]              -0.222215               0.993808
         1     [1.9, 0.3]               1.900677               0.311219

            Value of Approx for F1  Value of Approx for F2
         0            4.738815e-11            6.167031e-10
         1            3.866634e-09            5.357242e-08
```

Plotting the approximate roots on the graph:

In [13]:
```
x = np.linspace(-1,2,2000)
y1 = np.apply_along_axis(lambda x: x**2 - 2*x + 0.5, 0, x)
y21 = np.apply_along_axis(lambda x: -((-x**2 + 4)/4)**0.5, 0, x)
y22 = np.apply_along_axis(lambda x: ((-x**2 + 4)/4)**0.5, 0, x)

plt.figure(4)
plt.plot(x,y1, color = 'b', label="F1")
plt.plot(x,y21, color = 'r', label="F2") #Split F2 into 2 to plot the ellipse
plt.plot(x,y22, color = 'r') #Split F2 into 2 to plot the ellipse
plt.scatter(q["Approximate Roots (x)"], q["Approximate Roots (y)"],
                    label = None)
plt.xlabel("X")
plt.ylabel("Y")
plt.title("Plot of Question 2 with Approximate Roots")
plt.legend()
plt.show()
```



Summary

Question 1 Answers

```
In [14]: p[["Approximate Root"]]
```

```
Out[14]:    Approximate Root
         0         -0.963787
         1         -0.794161
         2         -0.525686
         3         -0.183435
         4          0.183435
         5          0.525686
         6          0.794161
         7          0.963787
```

Question 2 Answers

```
In [15]: q[["Approximate Roots (x)", "Approximate Roots (y)"]]
```

```
Out[15]:    Approximate Roots (x)  Approximate Roots (y)
         0              -0.222215               0.993808
         1               1.900677               0.311219
```