Physics 410: Homework 2

Arnold Choa – 32038144

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Question 1

Please see `q1.py` in the `code` folder for the relevant code.

Part 1

Please see below for the values we got from Python. We note that, computationally, rational numbers that are not a factor of 2 are just hard for the computer to calculate. Case in point, even the 0.994 inputted, gets turned into a long string of floating points until a close enough factor of 2 is found. This is calculated until 15 significant digits

n	$Exact$	R	P	Q
0	0.5	0.993999999999999...	1	1
1	0.25	0.497000000000000	0.49699999999999...	0.49699999999999...
2	0.125	0.248500000000000	0.24550000000000	0.24250000000000
3	0.0625	0.124250000000000	0.119750000000000	0.109250000000000
4	0.03125	0.062125000000000	0.056875000000000	0.030625000000000
5	0.015625	0.031062500000000	0.025437500000000	−0.032687500000000
6	0.0078125	0.015531250000000	0.009718750000000	−0.112343750000000
7	0.00390625	0.007765625000000	0.001859375000000	−0.248171875000000
8	0.001953125	0.003882812500000	−0.002070312500000	−0.508085937500000
9	0.0009765625	0.001941406250000	−0.004035156250000	−1.02204296875000
10	0.00048828125	0.000970703125000	−0.005017578125000	−2.04702148437500
11	0.000244140625	0.000485351562500	−0.005508789062500	−4.09551074218750
12	0.0001220703125	0.000242675781250	−0.005754394531250	−8.19175537109380
13	0.00006103515625	0.000121337890625	−0.005877197265625	−16.3838776855470
14	0.000030517578125	0.0000606689453125	−0.0059385986328125	−32.7679388427737
15	0.0000152587890625	0.00003033447265625	−0.00596929931640625	−65.5359694213870
16	0.00000762939453125	0.000015167236328125	−0.00598464965820315	−131.071984710694
17	0.000003814697265625	0.0000075836181640625	−0.00599232482910158	−262.143992355348
18	0.0000019073486328125	0.00000379180908203125	−0.00599616241455077	−524.287996177676
19	$9.5367431640625E-7$	0.00000189590454101562	−0.00599808120727536	−1048.57599808884

Part 2

We can observe, even on the table above, that both the exact and r sequences converge to 0. As for the p sequence, though this dips below 0, it also converges, but to -0.006 (this was checked later up until n=199). Finally, for the q sequence, this does not only dip below 0, this sequence is unstable!

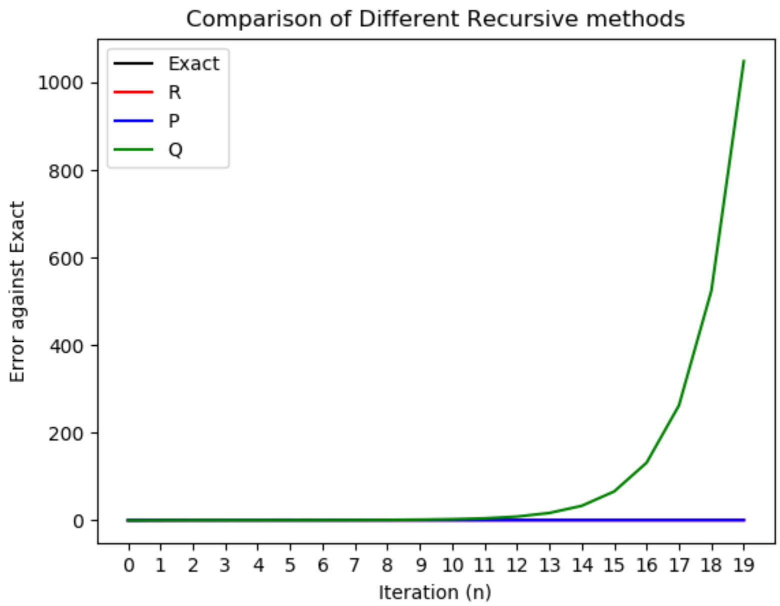

Figure 1: Plot of errors for the different pertured sequence. The sequences' values were baselined with the Exact value. We can see that the perturbed q sequence's errors grew exponentially with each iteration.

Part 3

First, we have to solve the general forms of the sequences:

For the r sequence, it is simply a geometric sequence:

$$r_n = \frac{1}{2}r_{n-1} = 2^{-n}r_0 \tag{1}$$

For the p sequence, we can assume a general solution of $p_n = As_1^n + Bs_2^n$. We can thus make a characteristic formula. Given reccurent sequence:

$$p_n = \frac{3}{2}p_{n-1} - \frac{1}{2}p_{n-2} \tag{2}$$

We can retrieve the characteristic polynomial:

$$s^2 = \frac{3}{2}s - \frac{1}{2} \tag{3}$$

with roots $s = 1, \frac{1}{2}$. We thus end up with the general solution:

$$p_n = A + B2^{-n} \tag{4}$$

with initial conditions $p_0 = p_0$ and $p_1 = p_1$. Solving for A and B:

$$p_n = (2p_1 - p_0) + (2\,(p_0 - p_1))\,2^{-n} \tag{5}$$

Finally, for the q sequence, we can again assume a general solution of $q_n = As_1^n + Bs_2^n$. We can thus make a characteristic formula. Given reccurent sequence:

$$q_n = \frac{5}{2}q_{n-1} - q_{n-2} \tag{6}$$

We can retrieve the characteristic polynomial:

$$s^2 = \frac{5}{2}s - 1 \tag{7}$$

with roots $s = 2, \frac{1}{2}$. We thus end up with the general solution:

$$q_n = 2^n A + B2^{-n} \tag{8}$$

with initial conditions $q_0 = q_0$ and $q_1 = q_1$. Solving for A and B:

$$q_n = \left(\frac{2q_1 - q_0}{3}\right)2^n + \left(\frac{2\,(2q_0 - q_1)}{3}\right)2^{-n} \tag{9}$$

Going back to our results, as $n \to \infty$:

- $r_n \to 0$
- $p_n \to 2p_1 - p_0$. In our case, $p_n \to 2(0.497) - 1 = -0.006$
- $q_n \to sign\left(\frac{2q_1 - q_0}{3}\right)\infty$ where the sign function gives 1 is positive, 0 if zero, and -1 if negative. In our case, $\left(\frac{2q_1 - q_0}{3}\right) = -0.002$ and so $q_n \to -\infty$. Note, we go by the convention that $0 \cdot \infty = 0$.

Question 2

Please see `q2.py` in the `code` folder for the relevant code.

Part 1

Solving by Gaussian elimination:

$$\left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{4} & -8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{36} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 105 \\ 0 & \frac{1}{36} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 210 \\ 0 & 1 & -312 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{3} & -\frac{1}{2} & -8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & -\frac{13}{18} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & 0 & -3 \\ 0 & -\frac{13}{18} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 12 \end{array}\right]$$

Thus, for our first system of equations: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 210 \\ -312 \end{bmatrix}$. As for our second system of equations: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$.

Part 2

We will change the top-right $\frac{1}{3}$ for both systems. Solving via Gaussian elimination:

$$\left[\begin{array}{cc|c} \frac{1}{2} & \frac{33}{100} & 1 \\ \frac{1}{3} & \frac{1}{4} & -8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & \frac{33}{100} & 1 \\ 0 & \frac{3}{100} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & 0 & \frac{289}{3} \\ 0 & \frac{3}{100} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{578}{3} \\ 0 & 1 & -\frac{2600}{9} \end{array}\right]$$

$$\left[\begin{array}{cc|c} \frac{1}{2} & \frac{33}{100} & 1 \\ \frac{1}{3} & -\frac{1}{2} & -8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & \frac{33}{100} & 1 \\ 0 & -\frac{18}{25} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \frac{1}{2} & 0 & -\frac{107}{36} \\ 0 & -\frac{18}{25} & -\frac{26}{3} \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{107}{18} \\ 0 & 1 & \frac{325}{27} \end{array}\right]$$

Thus, for our perturbed first system of equations: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{578}{3} \\ -\frac{2600}{9} \end{bmatrix}$. As for our perturbed second system of equations: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{107}{18} \\ \frac{325}{27} \end{bmatrix}$.

For relative error, we will use the formula:

$$\eta = \frac{|x - x_{approx}|}{|x|} \tag{10}$$

where x is a vector and $|x|$ denotes the length of said vector. Thus, for the relative error of the first system:

$$\eta_{first} = \frac{\left\| \begin{bmatrix} 210 \\ -312 \end{bmatrix} - \begin{bmatrix} \frac{578}{3} \\ -\frac{2600}{9} \end{bmatrix} \right\|}{\left\| \begin{bmatrix} 210 \\ -312 \end{bmatrix} \right\|} \approx \frac{\left\| \begin{bmatrix} \frac{52}{3} \\ -\frac{208}{9} \end{bmatrix} \right\|}{376.09} \approx \frac{28.89}{376.09} \approx 7.68\% \tag{11}$$

As for the relative error for the second system:

$$\eta_{second} = \frac{\left\| \begin{bmatrix} -6 \\ 12 \end{bmatrix} - \begin{bmatrix} -\frac{107}{18} \\ \frac{325}{27} \end{bmatrix} \right\|}{\left\| \begin{bmatrix} -6 \\ 12 \end{bmatrix} \right\|} \approx \frac{\left\| \begin{bmatrix} -\frac{1}{18} \\ -\frac{1}{27} \end{bmatrix} \right\|}{13.42} \approx \frac{0.067}{13.42} \approx 0.50\% \tag{12}$$

Part 3

No. The determinant is not an adequate measure of ill-conditioning. As a counter-example, let's take $10^{-1}\boldsymbol{I}$ where $\boldsymbol{I}$ is a $1000x1000$ identity matrix. Though its determinant is very small (10^{-1000} to be exact), any small pertubation to any entry will not cause a singularity.

We have to take what is called a *condition number* of the matrix. This is the ratio between the larget and smallest singular values (derived from singular value decomposition). The larger the condition number of the matrix, the more likely it is to be ill-conditioned.

Taking our systems, the first system has a condition number of $\approx$ 38 while the second system has a condition number of ≈ 1. This means that the first system is more ill-conditioned than the second.

Part 4

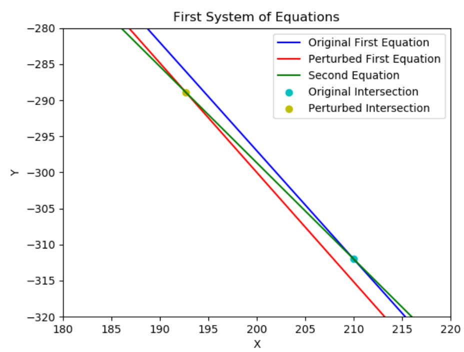

Figure 2: Geometrical Representation of the first system of equations. We can see that the two lines are close to being colinear, and thus, any slight pertubation results in a massive change in the intersection.

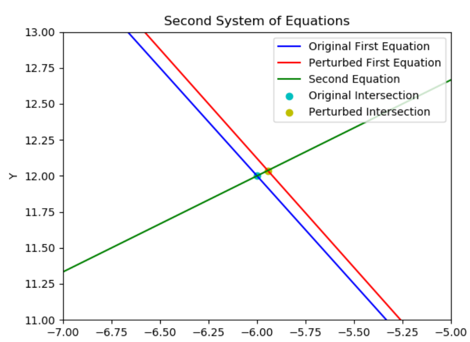

Figure 3: Geometrical Representation of the second system of equations. We can see that the two lines are close to being perpendicula, and thus, any slight pertubation results in a very small change in the intersection.

We can see more clearly, by using the geometrical representation, that the first system is very ill-conditioned as compared to the second. The major difference we can see from the two systems is that the ill-conditioned system is more colinear than the second (which is close to being orthogonal). Thus, given a slight pertubation to either equation, the first intersection of the first system will move wildly while the intersection of the second system will barely move at all, something that we have seen in the relative errors generated above.