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PHYS 410 - Homework 2

Question 1: Stable and unstable schemes

Consider the following three recurrence relations:

$$r_n = \frac{1}{2} r_{n-1}, \qquad \text{for } n = 1, 2, ..., \tag{1}$$

$$p_n = \frac{3}{2} p_{n-1} - \frac{1}{2} p_{n-2}, \qquad \text{for } n = 2, 3, ... \tag{2}$$

$$q_n = \frac{5}{2} q_{n-1} - q_{n-2}, \qquad \text{for } n = 2, 3, ... \tag{3}$$

For initial conditions $r_0 = p_0 = q_0 = 1$ and $p_1 = q_1 = \frac{1}{2}$, all three sequences above are exactly equivalent to the sequence $\{2^{-n}\}_{n=0}^{\infty}$.

1. Let's introduce a small error in the initial conditions and see how it propagates at each iteration. Write a short script in MATLAB that calculates the first 20 numerical approximations to the sequence $\{x_n\} = \{2^{-n}\}$ by using the initial conditions $r_0 = 0.994$, $p_0 = q_0 = 1$ and $p_1 = q_1 = 0.497$.
2. Plot the error sequences $\{x_n - r_n\}$, $\{x_n - p_n\}$ and $\{x_n - p_n\}$ for $n = 0, 1, ..., 19$. What do you observe? Which sequences are stable? Which are unstable?
3. Explain your results by solving the recurrence relations exactly. Hint: for the sequences $\{p_n\}$ and $\{q_n\}$, assume that the general solution is of the form $As_1^n + Bs_2^n$, with A and B determined by the initial conditions.

Question 2: Ill-conditioned matrices

A matrix M is said to be *ill-conditioned* if it is invertible but can become singular if one of its entries is slightly changed. Consider the following two systems of equations:

$$\frac{x}{2} + \frac{y}{3} = 1 \tag{4}$$

$$\frac{x}{3} + \frac{y}{4} = -8 \tag{5}$$

and

$$\frac{x}{2} + \frac{y}{3} = 1 \tag{6}$$

$$\frac{x}{3} - \frac{y}{2} = -8. \tag{7}$$

1. Find the exact solutions to these two systems of equations.
2. Replace one of the two off-diagonal coefficients from $1/3$ to 0.33 for both systems of equations (it doesn't matter which one). This corresponds to an error of 1% in that entry. What are the new solutions to these slightly perturbed equations? What is their relative error?

3. Is determinant an adequate mesure of ill-conditioning?
4. There is a geometrical explanation for this phenomenon. Remember that solving a system of equations corresponds to finding where two lines intersect in the xy-plane. Plot the two systems of equations near the location where they intersect. What is the difference between the ill-conditioned system and the good one?