Math 2001 Homework 6

Due 15 March 2019 (start of class)

1. (7.6.1) A *perfect number* is a number that is equal to the sum of its proper divisors, that is, the numbers that divide it, other than itself. For example, 6 is perfect, because $6 = 1 + 2 + 3$.

   Using a language with variables ranging over the natural numbers and suitable functions and predicates, write down first-order sentences asserting the following. Use a predicate **perfect** to express that a number is perfect.

   a. 28 is perfect.
   b. There are no perfect numbers between 100 and 200.
   c. There are (at least) two perfect numbers between 200 and 10,000. (Express this by saying that there are perfect numbers x and y between 200 and 10,000, with the property that $x \neq y$.)
   d. Every perfect number is even.
   e. For every number, there is a perfect number that is larger than it. (This is one way to express the statement that there are infinitely many perfect numbers.)

   Here, the phrase "between a and b" is meant to include a and b.

   By the way, we do not know whether the last two statements are true. They are open questions.

2. (7.6.2) Using a language with variables ranging over people, and predicates $\texttt{trusts}(x, y)$, $\texttt{politician}(x)$, $\texttt{crazy}(x)$, $\texttt{knows}(x, y)$, and $\texttt{related}(x, y)$, and $\texttt{rich}(x)$, write down first-order sentences asserting the following:

   a. Nobody trusts a politician.
   b. Anyone who trusts a politician is crazy.
   c. Everyone knows someone who is related to a politician.
   d. Everyone who is rich is either a politician or knows a politician. In each case, some interpretation may be involved. Notice that writing down a logical expression is one way of helping to clarify the meaning.

3. (8.6.2) Give a natural deduction proof of $\forall x\, B(x)$ from hypotheses $\forall x\, (A(x) \vee B(x))$ and $\forall y\, \neg A(y)$.

4. (8.6.3) From hypotheses

   $$\forall x\, (\texttt{even}(x) \vee \texttt{odd}(x)) \quad \text{and} \quad \forall x\, (\texttt{odd}(x) \to \texttt{even}(s(x)))$$

   give a natural deduction proof of $\forall x\, (\texttt{even}(x) \vee \texttt{even}(s(x)))$.
   (It might help to think of $s(x)$ as the function defined by $s(x) = x + 1$.)

5. (8.6.4) Give a natural deduction proof of

$$\exists x\, A(x) \vee \exists x\, B(x) \to \exists x\, (A(x) \vee B(x)).$$

6. (8.6.10) Give a natural deduction proof of $\forall x, y\, (x = y \to y = x)$ using only the following two hypotheses (and none of the new equality rules):

   - $\forall x\, (x = x)$
   - $\forall u, v, w\, (u = w \to (v = w \to u = v))$

   (Hint: Choose instantiations of u, v, and w carefully. You can instantiate all the universal quantifiers in one step, as on the last homework assignment.)

7. (8.6.11) Give a natural deduction proof of

$$\neg \exists x\, (A(x) \wedge B(x)) \leftrightarrow \forall x\, (A(x) \to \neg B(x)).$$