Math 2001, Homework 1

Due date: 25 January by 1pm

1. Decide whether or not the following are *statements*. In the case of a statement, and if so, say whether the statement is true or false, if possible.

   a. Some sets are finite.
   b. The sets $\mathbb{Z}$ and $\mathbb{N}$.
   c. $\mathbb{N} \notin \mathcal{P}(\mathbb{N})$
   d. $(\mathbb{R} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{R}) = \mathbb{N} \times \mathbb{N}$

2. Express each statement or open sentence in one of the forms $P \wedge Q$, $P \vee Q$, or $\neg P$. Be sure to also state exactly what statements P and Q stand for.

   a. At least one of the numbers x and y equals 0.
   b. $x \in A \cup B$

Exercises from *Logic and Proof* Chapter 11.

3. Let A and B be subsets of some domain Ω. Prove the following by showing that the two sides of the given equality have the same elements.

   a. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.
   b. $\overline{A \setminus B} = \overline{A} \cup B$.

4. (**LaP 11.5.3**) Two sets A and B are said to be **disjoint** if they have no element in common. Show that if A and B are disjoint, $C \subseteq A$, and $D \subseteq B$, then C and D are disjoint.

5. (**LaP 11.5.5**) Let A, B, and C be subsets of some domain Ω. Give a *calculational proof* of the identity $A \setminus (B \cup C) = (A \setminus B) \setminus C$, using the identities above. Also use the fact that, in general, $C \setminus D = C \cap \overline{D}$.

6. (**LaP 11.5.8**) Let I and J be sets and consider $\{A_{i,j} : i \in I, j \in J\}$ (that is, a family of sets indexed by I and J). Prove that

$$\bigcup_{i \in I} \bigcap_{j \in J} A_{i,j} \subseteq \bigcap_{j \in J} \bigcup_{i \in I} A_{i,j}.$$

   Also, find a family $\{A_{i,j} : i \in I, j \in J\}$ where the reverse inclusion does *not* hold.

7. (**LaP 11.5.11**) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

8. (**LaP 11.5.13**) Prove that that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Exercises from ***Logic and Proof*** Chapter 2.

9. (**LaP 2.5.2**) Here's a logic puzzle by George J. Summers, called "Murder in the Family." Murder occurred one evening in the home of a father and mother and their son and daughter. One member of the family murdered another member, the third member witnessed the crime, and the fourth member was an accessory after the fact.

   (a) The accessory and the witness were of opposite sex.
   (b) The oldest member and the witness were of opposite sex.
   (c) The youngest member and the victim were of opposite sex.
   (d) The accessory was older than the victim.
   (e) The father was the oldest member.
   (f) The murderer was not the youngest member.

   Using the mnemonics f (Father), m (Mother), etc., we can define propositional variables like

   f M : father is the Murderer,
   d V : daughter is the Victim,
   f O : father is Oldest,
   V Y : Victim is Youngest, etc.

   Notice that only the son or daughter can be the youngest, and only the mother or father can be the oldest. With these conventions, the first clue in the list above can be represented as follows:

   $$\text{(a)} \quad [(\mathrm{f\ A} \vee \mathrm{s\ A}) \to (\mathrm{m\ W} \vee \mathrm{d\ W})] \bigwedge [(\mathrm{m\ A} \vee \mathrm{d\ A}) \to (\mathrm{f\ W} \vee \mathrm{s\ W})]$$

   In other words, if the father or son was the accessory, then the mother or daughter was the witness, and vice-versa.

   Represent the other five clues in a similar manner.

10. (**LaP 2.5.3**) Consider the following three hypotheses:

    (a) Alan likes kangaroos, and either Betty likes frogs or Carl likes hamsters.

    (b) If Betty likes frogs, then Alan doesn't like kangaroos.

    (c) If Carl likes hamsters, then Betty likes frogs.

    Write a clear argument to show that these three hypotheses are contradictory.