**SUMMATION**

# 1 Introduction

The operation of *summation*—adding up aggregates of numbers—is of fundamental importance in the world of digital computing. While we humans are able to deal handily with abstractions such as "smoothness" and "continuity", we must employ sophisticated *discretizations* of these concepts in order to enlist the aid of digital computers in dealing with such abstractions. Summations provide a very useful discretization of "continuous" or "smooth" phenomena that are typically dealt with by humans with the aid of the calculus that was invented by Newton and Leibnitz for such dealings.

This chapter is dedicated to exploring how to employ summations as a computational tool. We deal throughout with *series*, i.e., (possibly infinite) sequences of numbers

$$a_1, a_2, \ldots$$

whose sum

$$a_1 + a_2 + \cdots$$

is of interest.

As a simple illustration, consider the following puzzle. You are presented with a traditional $8 \times 8$ chess board, plus an offer of a one-time gift of *one million euros*. In place of this one-time gift, you can opt for the money amassed in the following way.
I will go along the chessboard row by row. I will place 1 euro in the first square I encounter, 4 $(= 2^2)$ euros in the second, 9 $(= 3^2)$ euros in the third , ..., 4096 $(= 64^2)$ euros in the last square.
*WHICH CHOICE DO YOU MAKE?*

By the end of this chapter, you will be able to determine in minutes that the *alternative* offer (the one that uses the chessboard) would net you a sum of 8,390,656 euros—clearly the preferable choice!

# 2 Summing Structured Series

## 2.1 Arithmetic Sums and Series

We define arithmetic sequences and learn how to calculate their sums.

$$\begin{array}{l} \text{An } n\text{-term arithmetic sequence:} \\ \quad a,\ a+b,\ a+2b,\ a+3b,\ \ldots, a+(n-1)b \\ \\ \text{The corresponding arithmetic series:} \\ \quad a + (a+b) + (a+2b) + (a+3b) + \cdots + (a+(n-1)b) \\ \quad\quad = \ an + b \cdot (1 + 2 + \cdots + n - 1) \end{array} \tag{1}$$

We can, thus, sum the arithmetic series in (1) by determining the sum of the first $m$ positive integers; $m = n - 1$ in (1). We use this result as an opportubnity to introduce important notation.
**Proposition** For all $n \in \mathbb{N}$,

$$\begin{aligned} S_1(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} i \quad &\stackrel{\text{def}}{=} \quad 1 + 2 + \cdots + (n-1) + n \\ &= \quad \frac{1}{2} n(n+1) \\ &= \quad \binom{n+1}{2}. \end{aligned} \tag{2}$$

*Proof.* The *constructive* proof[1] of summation (2) that we present now employs a device known to the eminent German mathematician Karl Friedrich Gauss as a pre-teenager.

$$\begin{array}{lccccccccccc} \text{Write } S_1(n) \text{ "forwards":} & \sum_{i=1}^{n} & = & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ \text{Write } S_1(n) \text{ "in reverse":} & \sum_{i=1}^{n} & = & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \end{array} \tag{3}$$

Now add the two representations of $S_1(n)$ in (3) *columnwise*. Because each of the $n$ column-sums equals $n+1$, we find that $2S_1(n) = n(n+1)$, which we easily rewrite as in (2) (after multiplying both sides of the equation by 2). □ □

It follows that our original series in (1) sums as follows.

$$a + (a+b) + (a+2b) + (a+3b) + \cdots + (a+(n-1)b) \ = \ an + b \cdot \binom{n}{2}.$$

We can build on Proposition 2.1 to craft *two* "constructive" proofs—i.e., proof that explicitly calculate the summation—that each perfect square, say, $m^2$, is the sum of the first $m$ odd integers, $1, 3, 5, \ldots, 2m-1$. These proofs complement the "guess-and-verify" inductive proof of the same result in Proposition 3.3.

**Proposition** For all $n \in \mathbb{N}^+$,

$$\sum_{k=1}^{n} (2k-1) \ = \ 1+3+5+\cdots+(2n-1) \ = \ n^2. \tag{4}$$

That, is, the $n$th perfect square is the sum of the first $n$ odd integers.

Before presenting our two proofs of this result, we note that the notation in (4) is perfectly general: every positive odd integer $m$ can be written in the form $2n-1$ for some positive integer $n$.

*Proof.* (*Argument #1*.) By direct calculation, we have

$$\begin{array}{lcll} \displaystyle\sum_{k=1}^{n} (2k-1) & = & \displaystyle 2\sum_{k=1}^{n} k \ - \ n & \\ & = & \displaystyle 2\frac{n(n+1)}{2} \ - \ n & \text{by Proposition 2.1} \\ & = & (n^2+n)-n & \\ & = & n^2. \quad \square & \end{array}$$

□

*Proof.* (*Argument #2*.) Let us adapt Gauss's "trick" to this problem. Let us denote the target sum $\sum_{k=1}^{n} (2k-1)$ by $S(n)$.

$$\begin{array}{lcccccccccccc} S_n \text{ "forwards":} & S(n) & = & 1 & + & 3 & + & \cdots & + & (2n-3) & + & (2n-1) \\ S_n \text{ "in reverse":} & S(n) & = & (2n-1) & + & (2n-3) & + & \cdots & + & 3 & + & 1 \end{array} \tag{5}$$

Now add the two representations of $S(n)$ in (5) *columnwise*. Because each of the $n$ column-sums equals $2n$, we find that

$$2S(n) \ = \ 2\sum_{k=1}^{n} (2k-1) \ = \ 2n^2. \tag{6}$$

We thus derive the desired summation (4) when we divide both sides of equation (6) by 2. □ □

[1]The proof is *constructive* in that it actually derives an answer. This is in contrast to, say, the inductive proof of Proposition 3.3, which just verifies a "guessed" answer.

## 2.2 Geometric Sums and Series

We define geometric sequences and learn how to calculate their sums.

$$\begin{array}{l} \text{An } n\text{-term geometric sequence:} \\ \quad a,\ ab,\ ab^2,\ \ldots, ab^{n-1} \\ \\ \text{The corresponding geometric series:} \\ \quad a + ab + ab^2 + \cdots + ab^{n-1} \ = \ a(1 + b + b^2 + \cdots + b^{n-1}) \end{array} \tag{7}$$

Easily, we can sum the series in (7) by summing just the sub-series

$$S_b(n) \ \stackrel{\text{def}}{=} \ 1 + b + b^2 + \cdots + b^{n-1}. \tag{8}$$

We proceed as follows. Write $S_b(n)$ so that its terms are in *decreasing* order. We thereby isolate two cases.

1. Say first that $b > 1$. In this case, we write the series in the form

$$S_b^{b>1}(n) \ = \ b^{n-1} + b^{n-2} + \cdots + b^2 + b + 1,$$

   and we note that

$$S_b^{b>1}(n) \ = \ b^{n-1} \ + \ \frac{1}{b} \cdot S_b^{b>1}(n) \ - \ \frac{1}{b}.$$

   In other words, we have

$$\left(1 - \frac{1}{b}\right) S_b^{b>1}(n) \ = \ b^{n-1} - \frac{1}{b},$$

   or equivalently,

$$S_b^{b>1}(n) \ = \ \frac{b^n - 1}{b - 1}. \tag{9}$$

2. Alternatively, if $b < 1$, then we write the series in the form

$$S_b^{b<1}(n) \ = \ 1 + b + b^2 + b^3 + \cdots + b^{n-1}.$$

   and we note that

$$S_b^{b<1}(n) \ = \ 1 \ + \ b \cdot S_b^{b<1}(n) \ - \ b^n.$$

   In other words,

$$(1 - b) S_b^{b<1}(n) \ = \ 1 \ - \ b^n$$

   or equivalently,

$$S_b^{b<1}(n) \ = \ \frac{1 - b^n}{1 - b}. \tag{10}$$

Note that $S_b^{b>1}(n)$ and $S_b^{b<1}(n)$ actually have the same form. We have chosen to write them differently to stress their *approximate* values, which are useful in "back-of-the-envelope" calculations: For very large values of $n$, we have

$$S_b^{b>1}(n) \ \approx \ \frac{b^n}{b - 1} \quad \text{while} \quad S_b^{b<1}(n) \ \approx \ \frac{1}{1 - b}. \tag{11}$$

We now exploit our ability to sum geometric sums to illustrate a somewhat surprising, nontrivial fact about integers that are "encoded" in their positional numerals. We hope that this "fun" result will inspire the reader to seek kindred numeral-encoded properties of numbers.

**Proposition** *An integer $n$ is divisible by an integer $m$ if, and only if, $m$ divides the sum of the digits in the base-$(m+1)$ numeral for $n$.*

The most familiar instance of this result is phrased in terms of our traditional use of base-10 (decimal) numerals. *An integer $n$ is divisible by 9 if, and only if, the sum of the digits of $n$'s base-10 numeral is divisible by 9.*

*Proof.* (*Argument for general base b*). Of course, we lose no generality by focusing on numerals without leading 0's, for adding leading 0's does not alter a numeral's sum of digits.

To enhance legibility, let $b = m+1$, so that we are looking at the base-$b$ numeral for $n$. Say that

$$n \;=\; \delta_k \cdot b^k + \delta_{k-1} \cdot b_{k-1} + \cdots + \delta_1 \cdot b + \delta_0,$$

so that the sum of the digits of $n$'s base-$b$ numeral is

$$s_b(n) \;\stackrel{\text{def}}{=}\; \delta_k + \delta_{k-1} + \cdots + \delta_1 + \delta_0.$$

We next calculate the difference $n - s_b(n)$. We proceed as follows, digit by digit.

$$\begin{array}{ccccccccccc}
n & = & \delta_k \cdot b^k & + & \delta_{k-1} \cdot b^{k-1} & + & \cdots & + & \delta_1 \cdot b & + & \delta_0 \\
s_b(n) & = & \delta_k & + & \delta_{k-1} & + & \cdots & + & \delta_1 & + & \delta_0 \\
\hline
n - s_b(n) & = & \delta_k \cdot (b^k - 1) & + & \delta_{k-1} \cdot (b^{k-1} - 1) & + & \cdots & + & \delta_1 \cdot (b-1) & &
\end{array} \tag{12}$$

We now revisit summation (9). Because $b$ is a positive integer, so that $1 + b + \cdots + b^{a-2} + b^{a-1}$ is also a positive integer, we adduce from the summation that *the integer* $b^a - 1$ *is divisible by* $b-1$.

We are almost home. Look at the equation for $n - s_b(n)$ in the system (12). As we have just seen, every term on the righthand side of that equation is divisible by $b-1$. It follows therefore, that the lefthand expression, $n - s_b(n)$, is also divisible by $b-1$. An easy calculation, which we leave to the reader, now shows that this final fact means that $n$ is divisible by $b-1$ if, and only if, $s_b(n)$ is. □ □

## 2.3 Deriving and Solving Linear Recurrences

We have discussed already discussed the use of linear recurrences in Section We now derive the mathematics underlying this important topic.

By the time the reader has reached this paragraph, she has the mathematical tools necessary to prove and apply what is called *The Master Theorem for Linear Recurrences* [1]. This level of mathematical preparation should be adequate for most early-undergrad courses on data structures and algorithms, as well for for analyzing a large fraction of the algorithms that she is likely to encounter in daily activities.

**Theorem**[The Master Theorem for Linear Recurrences] *Let the function F be specified by the following linear recurrence.*

$$F(n) \;=\; \begin{cases} aF(n/b) + c & \text{for } n \geq b \\ c & \text{for } n < b \end{cases} \tag{13}$$

*Then the value of F on any argument n is given by*

$$\begin{aligned}
F(n) \;&=\; (1 + \log_b n)c && \text{if } a = 1 \\
&=\; \frac{1 - a^{\log_b n}}{1-a} \approx \frac{1}{1-a} && \text{if } a < 1 \\
&=\; \frac{a^{\log_b n} - 1}{a-1} && \text{if } a > 1
\end{aligned} \tag{14}$$

*Proof.* In order to discern the recurring pattern in (13), let us begin to "expand" the specified computation by replacing occurrences of $F(\bullet)$ as mandated in (13).

$$\begin{array}{rcccc}
F(n) & = & aF(n/b) + c & & \\
& = & a\left(aF(n/b^2) + c\right) + c & = & a^2F(n/b^2) + (1+a)c \\
& = & a^2\left(aF(n/b^3) + c\right) + (1+a)c & = & a^3F(n/b^3) + (1+a+a^2)c \\
& & \vdots & & \vdots \\
& = & \left(1 + a + a^2 + \cdots + a^{\log_b n}\right)c & &
\end{array} \tag{15}$$

The segment of (15) "hidden" by the vertical dots betokens an induction that is left to the reader. Equations (9) and (10) now enable us to demonstrate that (14) is the case-structured solution to (13). □ □

# 3 On Summing "Smooth" Series

We use the problem of summing the first $n$ integers as a running example.

## 3.1 Approximate Sums via Integration

This section illustrates a powerful strategy for obtaining nontrivial upper and lower bounds on summations, by finding continuous *envelopes* that bound the discrete summations both above and below. The areas under the enveloping continuous functions—which we can calculate via integration—provide the desired bounds on the summations.

The continuous functions that enable this method are obtained by the following stratagem, which we illustrate for

1. the *harmonic* summation $S_{(harmonic)}(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \ 1/i$ in Fig. 1.

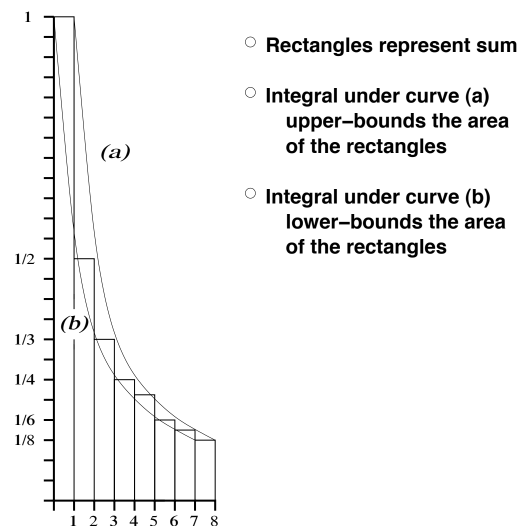


Figure 1: The summation $S_{(harmonic)}(n) = \sum_{i=1}^{n} \ 1/i$ represented by the aggregate area of a sequence of unit-width rectangles, and bounded above an below by and "enveloping" pair of integrals. The integral that provides the upper bound on the sum yields the area under the righthand continuous curve ($a$), namely, $\displaystyle\int_1^n \frac{1}{-}dx$. The integral that provides the lower bound yields the area under the lefthand continuous curve ($b$), namely, $\displaystyle\int_0^{n-1} \frac{1}{x+1}dx$.

2. the summation $S_2(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \ i^2$ in Fig. 2.

The strategem operates as follows.

1. Represent the summands seriatim as abutting unit-width rectangles. In Fig. 2, for instance, these rectangles have heights 1, 4, 9, 16, and 25. If we were to extend the figure, then next rectangle would have height 36.
2. Construct the continuous curve that will yield (via integration) the upper bound on the summation so that it passes through the upper lefthand corners of the unit-width rectangles specified by the summation. When constructed appropriately, the area subtended by the abutting rectangles lie completely under this curve.
3. Construct the continuous curve that will yield (via integration) the lower bound on the summation so that it passes through the upper righthand corners of the unit-width rectangles specified by the summation. When constructed appropriately, this curve lies completely within the area subtended by the abutting rectangles.

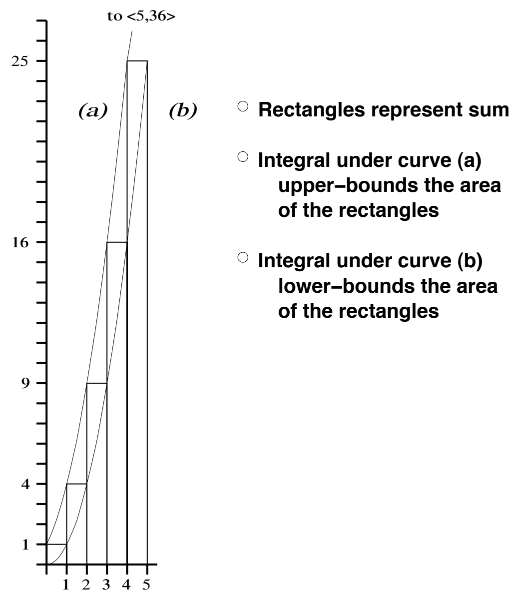


Figure 2: The summation $S_2(n) = \sum_{i=1}^{n} i^2$ represented by the aggregate area of a sequence of unit-width rectangles, and bounded above an below by and "enveloping" pair of integrals. The integral that provides the upper bound on the sum yields the area under the lefthand continuous curve $(a)$, namely, $\int_0^n (x+1)^2 dx$. The integral that provides the lower bound yields the area under the righthand continuous curve $(b)$, namely, $\int_1^n x^2 dx$.

We illustrate the technique of bounding summations via integrals by focusing on the summations

$$S_k(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} i^k,$$

for arbitrary positive integers $k$. The technique generalizes far beyond the summations $S_k(n)$, but this single illustration should give the reader a firm basis from which to extrapolate to more general summations.

We begin by describing the technique in detail for the summation $S_2(n) = \sum_{i=1}^{n} i^2$. We shall then indicate how to extrapolate from this case to general summations $S_k(n)$.

We obtain our upper bound on $S_2(n)$ by evaluating the integral that yields the area $\overline{A}_2(n)$ under the lefthand continuous curve $(a)$ in Fig. 2, namely,

$$\begin{aligned} \overline{A}_2(n) \ = \ \int_0^n (x+1)^2 dx \ &= \ \int_0^n x^2 dx \ + \ 2\int_0^n x dx \ + \ \int_0^n dx \\ &= \ \frac{1}{3}n^3 \ + n^2 \ + \ n \ + \ O(1). \end{aligned} \tag{16}$$

We obtain our lower bound on $S_2(n)$ by evaluating the integral that yields the area $\underline{A}_2(n)$ under the righthand continuous curve $(b)$ in the the figure, namely,

$$\underline{A}_2(n) \ = \ \int_1^n x^2 dx \ = \ \frac{1}{3}n^3 \ + \ O(1). \tag{17}$$

Combining these bounds, we finally have the following two-sided bound on $S_2(n)$.

$$\frac{1}{3}n^3 \ + \ O(1) \ \leq \ \sum_{i=1}^{n} i^2 \ \leq \ \frac{1}{3}n^3 \ + n^2 \ + \ n \ + \ O(1) \tag{18}$$

For summations $S_k(n)$ with $k > 2$, we follow the roadmap of the case $k = 2$, invoking as a technical tool the following special case of Newton's *Binomial Theorem*.[2]

**The Restricted Binomial Theorem.** *For all positive integers k,* $(x+1)^k \ = \ \sum_{i=0}^{k} \ \binom{k}{i} x^{k-i}$.[3]

The extrapolation from $S_2(n)$ to general $S_k(n)$ begins with the straightforward exercise of crafting analogues of Fig. 2 for arbitrary summations $S_k(n)$. Parallelling the reasoning that led us to the relations (16) and (17) we obtain:

- an upper bound on summation $S_k(n)$ by evaluating the integral that yields the area $\overline{A}_k(n)$ under the continuous curve that passes through the upper lefthand corners of the unit-width rectangles specified by summation $S_k(n)$.

$$\begin{aligned} \overline{A}_k(n) \ = \ \int_0^n (x+1)^k dx \ &= \ \int_0^n \left( \sum_{i=0}^{k} \ \binom{k}{i} x^{k-i} \right) dx \\ &= \ \sum_{i=0}^{k} \ \frac{1}{k-i+1} \binom{k}{i} n^{k-i+1} \ + \ O(1) \end{aligned} \tag{19}$$

  This is a proper upper bound because the region defined by this curve totally contains the region subtended by the rectangles.

- a lower bound on summation $S_k(n)$ by evaluating the integral that yields the area $\underline{A}_k(n)$ under the continuous curve that passes through the upper righthand corners of the unit-width rectangles specified by summation $S_k(n)$.

$$\underline{A}_k(n) \ = \ \int_1^n x^k dx \ = \ \frac{1}{k+1} n^{k+1} \ + \ O(1). \tag{20}$$

  This is a proper lower bound because the region subtended by the rectangles totally contains the below this curve

Using this strategy, one finds that for any positive integer $k$, summation $S_k(n)$ enjoys the following two-sided bound:

$$\frac{1}{k+1} n^{k+1} \ + \ O(1) \ \leq \ \sum_{i=1}^{n} i^k \ \leq \ \sum_{i=0}^{k} \ \frac{1}{k-i+1} \binom{k}{i} n^{k-i+1} \ + \ O(1) \tag{21}$$

It follows that $S_k(n)$ enjoys the following asymptotic behavior.

$$S_k(n) \ = \ \sum_{i=1}^{n} i^k \ \approx \ \frac{1}{k+1} \, n^{k+1}. \tag{22}$$

We now discuss how to "expand" bounds such as (21) into explicit expressions for the summations $S_k(n)$.

## 3.2 On Using *Undetermined Coefficients* to Obtain Explicit Sums

This section introduces the *Method of Undetermined Coefficients*, a tool that can sometimes refine approximations to summations to exact explicit expressions. We illustrate the tool on the summations $S_k(n)$ that we studied in Section 3.1; we use the tool to convert bounds on the $S_k(n)$, such as (22), to explicit expressions for the summations.

We illustrate the method by deriving explicit expressions for two rather simple summations: the sum $S_1(n)$ of the first $n$ positive integers and the sum $S_2(n)$ of the first $n$ perfect squares.

[2] The general form of the Binomial Theorem expands the polynomial $(x+y)^k$ rather than $(x+1)^k$.

[3] As usual, the *binomial coefficients* are defined and denoted as follows.

$$\binom{k}{i} \ = \ \frac{k!}{i!(k-i)!} \ = \ \frac{k(k-1)(k-2)\cdots(k-i+1)}{i(i-1)(i-2)\cdots 1}$$

*Expressing $S_1(n)$ explicitly.* Of course, we have already determined $S_1(n)$ in Section 2.1, by other means. We provide this alternative derivation as a gentle introduction to the method of undetermined coefficients.
**Proposition.** *For all $n \in \mathbb{N}$,*

$$S_1(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} i^2 \ = \ 1+2+\cdots+n \ = \ \frac{1}{2}n^2 \ + \ \frac{1}{2}n$$

*Proof.* We can reason from the case $k=1$ of (21) that

$$S_1(n) \ = \ \frac{1}{2}n^2 \ + \ cn$$

for some positive constant $c$, the eponymous *undetermined coefficient* for this case. We can discover the value of $c$ by evaluating $S_1(n)$ at the value $n=1$. Any value of $n$ will work; using the *smallest* one simplifies our calculation.

Because $S_1(1)=1$, we have

$$S_1(1) \ = \ 1 \ = \ \frac{1}{2} \ + \ c,$$

so that $c = 1/2$; in other words,

$$S_1(n) \ = \ \frac{1}{2}\left(n^2+n\right) \ = \ \frac{n(n+1)}{2}.$$

$\square$

We verify this expression by induction in Proposition 3.3.

*Expressing $S_2(n)$ explicitly.* A modestly more complicated undetermined-coefficient calculation allows us to evaluate the sum $S_2(n)$ of the first $n$ squares.
**Proposition.** *For all $n \in \mathbb{N}$,*

$$S_2(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} i^2 \stackrel{\text{def}}{=} 1+4+\cdots+(n-1)^2+n^2 \ = \ \frac{1}{3}n^3 \ + \ \frac{1}{2}n^2 \ + \ \frac{1}{6}n \tag{23}$$

$S_2(n)$ is often expressed in a more aesthetic form:

$$S_2(n) \ = \ \frac{1}{6}n(n+1)(2n+1) \ = \ \frac{2n+1}{3}\cdot\binom{n}{2}.$$

*Proof.* We can reason from the case $k=2$ of (21) that

$$\sum_{i=0}^{n} k^2 \ = \ \frac{1}{3}n^3+c_2n^2+c_1n. \tag{24}$$

for some positive numbers $c_2$ and $c_1$, the eponymous *undetermined coefficient* for this case.

We begin our determination of the constants $c_1$ and $c_2$ by instantiating the polynomial in (24) with the smallest two values of $n$, namely, $n=1,2$. Of course, any two values will work, but using the *smallest* ones simplifies calculations. These instantiations leave us with the following pair of linear equations.

$$\begin{array}{lllll} n=1: & \sum_{i=0}^{1} k^2 & = & 1 & = & 1/3 \ + \ c_2 \ + \ c_1 \\ n=2: & \sum_{i=0}^{2} k^2 & = & 5 & = & 8/3 \ + \ 4c_2 \ + \ 2c_1 \end{array}$$

By elementary arithmetic, these equations simplify to yield the pair

$$\begin{array}{rcl} c_2 \ + \ c_1 & = & 2/3 \\ 2c_2 \ + \ c_1 & = & 7/6 \end{array}$$

These equations reveal that

$$2/3 \ - \ c_2 \ = \ 7/6 \ - \ 2c_2$$

so that

$$c_2 \;=\; 1/2$$

which means that

$$c_1 \;=\; 1/6.$$

We have, thus, derived equation (23). □ □

With more (calculational) work, but no new (mathematical) ideas, one can derive explicit expressions for the sum of the first $n$ $k$th powers, i.e., the quantity $S_n^{(k)}$, for any positive integer $k$.

## 3.3 Validating Approximate Sums via Induction

We illustrate the proof technique of (Finite) Induction by proving the correctness of two familiar summation formulas: (1) the sum of the first $n$ positive integers and (2) the sum of the first $n$ odd positive integers.

For all $n \in \mathbb{N}$,

$$\begin{aligned}
S_n \;\stackrel{\text{def}}{=}\; \sum_{i=1}^{n} i \;\;&\stackrel{\text{def}}{=}\;\; 1+2+\cdots+(n-1)+n \\
&=\;\; \frac{1}{2}n(n+1) \qquad\qquad (25)\\
&=\;\; \binom{n+1}{2}.
\end{aligned}$$

*Proof.* For every positive integer $m$, let $\mathbf{P}(m)$ be the proposition

$$1+2+\cdot+m \;=\; \binom{m+1}{2}.$$

Let us proceed according to the standard format of an inductive argument.

**1.** Because $\binom{2}{2} = 1$, proposition $\mathbf{P}(1)$ is true.

**2.** Let us assume, for the sake of induction, that proposition $\mathbf{P}(m)$ is true for all positive integers strictly smaller than $n$. In particular, then, $\mathbf{P}(n-1)$ is true.

**3.** Consider now the summation

$$1+2+\cdots+(n-1)+n.$$

Because $\mathbf{P}(n-1)$ is true, we know that

$$1+2+\cdots+(n-1) \;=\; \binom{n}{2}.$$

By direct calculation, we see that

$$\begin{aligned}
\binom{n}{2}+n \;\;&=\;\; \frac{n(n-1)}{2} \;+\; n \\
&=\;\; \frac{n^2-n+2n}{2} \\
&=\;\; \frac{n^2+n}{2} \\
&=\;\; \binom{n+1}{2}
\end{aligned}$$

Because $n$ is an arbitrary positive integer, we conclude that $\mathbf{P}(n)$ is true whenever

- $\mathbf{P}(1)$ is true
- *and* $\mathbf{P}(m)$ is true for all $m < n$.

By the Principle of (Finite) Induction, then, we conclude that $\mathbf{P}(n)$ is true for all $n \in \mathbb{N}^+$. $\square$ $\square$

We turn now to our second summation, which asserts that each perfect square of a positive integer, say, $n^2$, is the sum of the first $n$ odd integers, $1, 3, 5, \ldots, 2n-1$. This proof complements the constructive proofs of the same result in Proposition 2.1.

For all $n \in \mathbb{N}^+$,

$$\sum_{k=1}^{n} (2k-1) \ = \ 1+3+5+\cdots+(2n-1) \ = \ n^2.$$

That, is, the $n$th perfect square is the sum of the first $n$ odd integers.

*Verification.* For every positive integer $m$, let $\mathbf{P}(m)$ be the proposition

$$m^2 \ = \ 1+3+5+\cdots+2m-1.$$

Let us proceed according to the standard format of an inductive argument.

**1.** Because $1 \cdot 1 = 1$, proposition $\mathbf{P}(1)$ is true.

**2.** Let us assume, for the sake of induction, that proposition $\mathbf{P}(m)$ is true for all positive integers strictly smaller than $n$. In particular, then, $\mathbf{P}(n-1)$ is true.

**3.** Consider now the summation

$$1+3+5+\cdots+2n-3+2n-1$$

Because $\mathbf{P}(n-1)$ is true, we know that

$$1+3+5+\cdots+2n-3+2n-1 \ = \ (n-1)^2+2n-1.$$

By direct calculation, we see that

$$(n-1)^2+2n-1 \ = \ (n^2-2n+1)+(2n-1) \ = \ n^2.$$

Because $n$ is an arbitrary positive integer, we conclude that $\mathbf{P}(n)$ is true whenever

- $\mathbf{P}(1)$ is true
- *and* $\mathbf{P}(m)$ is true for all $m < n$.

By the Principle of (Finite) Induction, then, we conclude that $\mathbf{P}(n)$ is true for all $n \in \mathbb{N}^+$. $\square$

# References

[1] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein (2001): *Introduction to Algorithms (2nd ed.)*. MIT Press, Cambridge, MA.