Advanced Stochastic Processes - Ex. 7

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1. Shift invariance: If $\{W(t)\}$ is standard Brownian motion (sBm), then $W_1(t) = W(t+h) - W(t)$ is also sBm.

   Proof:

   Let $s < t$ and consider the increment $W_1(t) - W_1(s) = (W(t+h) - W(t)) - (W(s+h) - W(s)) = W(t+h) - W(s+h)$. By the independence of increments in sBm, this increment is independent of $W_1(u)$ for $u \leq s$. Moreover, $W(t+h) - W(s+h)$ has the same distribution as $W(t-s)$, which is normal with mean 0 and variance $t-s$. This proves that $W_1(t)$ is sBm.

2. Scaling property: If $\{W(t)\}$ is sBm, then $W_2(t) = hW(t/h^2)$ is also sBm.

   Proof:

   The process $\{W(t)\}$ is sBm if it satisfies these two conditions:

   (a) $W(0) = 0$ almost surely.
   (b) For any $0 \leq s < t$, the increment $W(t) - W(s)$ is normally distributed with mean 0 and variance $t-s$, and is independent of the sigma-field generated by $\{W(u), u \leq s\}$.

   Now let's consider the process $\{W_2(t)\}$. We have $W_2(0) = hW(0/h^2) = 0$ almost surely, satisfying condition (a). For any $0 \leq s < t$, the increment $W_2(t) - W_2(s) = hW(t/h^2) - hW(s/h^2) = h(W(t/h^2) - W(s/h^2))$. This increment is normally distributed with mean 0 and variance $h^2(t-s)/h^2 = t-s$ (by the properties of sBm), and is independent of the sigma-field generated by $\{W_2(u), u \leq s\}$, satisfying condition (b). This proves that $W_2(t)$ is sBm.

3. Time-inversion property: If $\{W(t)\}$ is sBm, then $W_3(t) = tW(1/t)$ with $W_3(0) = 0$ is also sBm.

   Proof:

   We define the time inversion process $W_3(t) = tW(1/t)$ for $t \neq 0$, and $W_3(0) = 0$.

   The time inversion property is a bit more complex to prove. We'll provide an informal sketch of the proof:

   (i) We need to show that for any fixed t, $W_3(t)$ follows a normal distribution with mean 0 and variance t. This can be shown by a change of variables in the integral representation of $W(t)$.

   (ii) We then need to show that for any $0 \leq s < t$, the increment $W_3(t) - W_3(s)$ is independent of the sigma-field generated by $\{W_3(u), u \leq s\}$. This is nontrivial and requires advanced measure theory to prove rigorously.

4. The reflection process $W_4(t) = |W(t)|$ is not a sBm, but its expected value and variance can be calculated:

   Expectation:

$E\{W_4(t)\} = \sqrt{\frac{2t}{\pi}}$

Proof:

$E\{|W(t)|\} = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx$

This is equivalent to 2 times the integral from 0 to ∞, since the absolute value function is symmetric about 0:

$E\{|W(t)|\} = 2 \int_0^{\infty} x \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx$

Let u = $x^2/(2t)$, then $du = xdx/t$, and the integral becomes:

$E\{|W(t)|\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} ue^{-u} du = \sqrt{\frac{2t}{\pi}}$

Variance:

$Var\{W_4(t)\} = \left(1 - \frac{2}{\pi}\right) t$

Proof:

The variance can be calculated by the formula $Var(X) = E[X^2] - (E[X])^2$.

$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx = t$

So, $Var\{W_4(t)\} = t - \left(\sqrt{\frac{2t}{\pi}}\right)^2 = \left(1 - \frac{2}{\pi}\right) t$