Advanced Stochastic Processes - Ex. 9

Alireza Jamalie

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Problem

Let $N(t)$ denote the number of events up to time t in a Poisson process with rate λ. Each event i is associated with a random size Y_i, which are independent and identically distributed as Y with $E[Y] = E[Y_1]$ and $Var[Y] = \sigma^2$. Y_i are independent of $N(t)$. We define the sum of the process at time t, $S(t)$, as the sum of the event sizes:

$$S(t) = \sum_{i=1}^{N(t)} Y_i$$

Calculate the mean, variance and Laplace transform of $S(t)$.

Solution

Mean of $S(t)$

The expected value of $S(t)$ can be found using the law of total expectation:

$$E[S(t)] = E[E[S(t)|N(t)]]$$

For a fixed number of events n, $E[S(t)|N(t) = n] = nE[Y]$ because the Y_i are i.i.d. as Y. So we can write:

$$E[S(t)|N(t)] = N(t)E[Y]$$

Substituting this into the outer expectation and using independence between $N(t)$ and Y_i, we get:

$$E[S(t)] = E[N(t)]E[Y]$$

Since $E[N(t)] = \lambda t$ for a Poisson process and $E[Y] = E[Y_1]$, we find:

$$E[S(t)] = \lambda t E[Y_1]$$

Variance of $S(t)$

The variance of $S(t)$ can be found using the law of total variance:

$$Var[S(t)] = E[Var[S(t)|N(t)]] + Var[E[S(t)|N(t)]]$$

Given $N(t) = n$, $S(t)$ is the sum of n i.i.d. variables $Y_1, ..., Y_n$ with mean $E[Y]$ and variance σ^2. Therefore, $Var[S(t)|N(t) = n] = nE[Y^2]$. So, we can write $Var[S(t)|N(t)] = N(t)E[Y^2]$.

Using the law of total expectation and the fact that $E[N(t)] = \lambda t$, we have:

$$E[Var[S(t)|N(t)]] = E[N(t)E[Y^2]] = \lambda t E[Y^2]$$

The second term $Var[E[S(t)|N(t)]]$ is zero since $E[S(t)|N(t)] = N(t)E[Y]$ is a deterministic function of $N(t)$ (which means its variance is zero).

Therefore, the variance of $S(t)$ is:

$$Var[S(t)] = \lambda t E[Y^2]$$

Laplace Transform of $S(t)$

The Laplace transform of a random variable X is defined as:

$$\mathcal{L}\{X\}(s) = E[e^{-sX}]$$

So the Laplace transform of $S(t)$ is:

$$\mathcal{L}\{S(t)\}(s) = E[e^{-sS(t)}]$$

Conditioning on $N(t)$, we can write:

$$\mathcal{L}\{S(t)\}(s) = E[E[e^{-sS(t)}|N(t)]]$$

Given $N(t) = n$, $S(t)$ is the sum of n i.i.d. variables $Y_1, \ldots, Y_n$, and hence $e^{-sS(t)} = \prod_{i=1}^{n} e^{-sY_i}$. Because the Y_i's are i.i.d, $E[e^{-sY_i}]$ is the same for all i. Let this common value be $\phi(s) = E[e^{-sY}]$.

So we can write:

$$E[e^{-sS(t)}|N(t)] = \phi(s)^{N(t)}$$

Substituting this into the outer expectation, we get:

$$\mathcal{L}\{S(t)\}(s) = E[\phi(s)^{N(t)}]$$

$N(t)$ follows a Poisson distribution with parameter λt, hence its probability generating function is $E[z^{N(t)}] = e^{\lambda t(z-1)}$. Setting $z = \phi(s)$ in this expression gives the Laplace transform of $S(t)$:

$$\mathcal{L}\{S(t)\}(s) = e^{\lambda t(\phi(s)-1)}$$