Advanced Stochastic Processes - Ex. 1

Alireza Jamalie

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1 Poisson

Q1. Find the mean and variance of Poisson distribution using simple expectation formula.

A. Probability mass function (PMF) of a Poisson distribution:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

The expected value (mean) of a discrete random variable X is:

$$E[X] = \sum_x x \cdot P(X = x)$$

For the Poisson distribution, this can be written as:

$$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

We rewrite the sum, starting from k = 1:

$$E[X] = \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

Now, we rewrite the sum with (k-1)! in the denominator:

$$E[X] = \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!}$$

Factor out λ from the sum:

$$E[X] = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!}$$

The sum in the above equation is equal to 1, as it's the sum of probabilities of a Poisson distribution with parameter λ:

$$E[X] = \lambda \cdot 1$$

Hence, the expectation (mean) of a Poisson distribution is:

$$E[X] = \lambda$$

B. The variance (Var[X]) of a discrete random variable X is defined as:

$$Var[X] = E[X^2] - (E[X])^2$$

To calculate the second moment, $E[X^2]$, we sum the product of k^2 and the PMF over all possible values of k:

$$E[X^2] = \sum_{k=0}^{\infty} k^2 P(X = k) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!}$$

This sum is challenging to compute directly. We can manipulate it slightly to get it in a form that's easier to compute:

$$E[X^2] = \sum_{k=1}^{\infty} k \cdot k \frac{\lambda^k e^{-\lambda}}{k!}$$

This can be rewritten as:

$$E[X^2] = \sum_{k=1}^{\infty} (k - 1 + 1) \cdot k \frac{\lambda^k e^{-\lambda}}{k!}$$

Reorganizing the terms and separating the sum into two different sums yields:

$$E[X^2] = \lambda^2 e^{-\lambda} \left[\sum_{k=2}^{\infty} \frac{(k-1)\lambda^{k-2}}{(k-2)!} + \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} \right]$$

In the first sum, we reindex it with $i = k-2$ and it becomes $\lambda \sum_{i=1}^{\infty} \frac{(i-1)\lambda^{i-1}}{(i-1)!}$, which is λ because it's the sum of probabilities of a Poisson distribution with mean λ.

In the second sum, we have $\sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!}$, which is the mean of a Poisson distribution and this equals λ.

Hence, we find:

$$E[X^2] = \lambda^2 + \lambda$$

We can now plug these into the formula for variance to get:

$$Var[X] = E[X^2] - (E[X])^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

So, the variance of a Poisson distribution is equal to its mean λ.

Q2. Find the mean and variance of Poisson distribution using its moment generating function.

A. The moment generating function ($M_X(t)$) of a Poisson distribution with parameter λ is given by:

$$M_X(t) = e^{\lambda(e^t-1)}$$

To find the mean (E[X]), we compute the first derivative of the MGF with respect to t and then evaluate the result at t = 0:

$$\frac{dM_X(t)}{dt} = \frac{d}{dt}e^{\lambda(e^t-1)}$$

Evaluate the derivative at t = 0:

$$E[X] = M'_X(0)$$

Using the chain rule, we get:

$$\frac{dM_X(t)}{dt} = \lambda e^t e^{\lambda(e^t-1)}$$

Now evaluate the derivative at t = 0:

$$E[X] = \lambda e^0 e^{\lambda(e^0-1)} = \lambda e^{\lambda(1-1)} = \lambda$$

Hence, the mean (E[X]) of a Poisson distribution is λ, as calculated using the moment generating function.

B. The moment generating function ($M_X(t)$) of a Poisson distribution with parameter λ is given by:

$$M_X(t) = e^{\lambda(e^t-1)}$$

To find the second moment ($E[X^2]$), we compute the second derivative of the MGF with respect to t and then evaluate the result at t = 0:

$$\frac{d^2M_X(t)}{dt^2} = \frac{d^2}{dt^2}e^{\lambda(e^t-1)}$$

Evaluate the second derivative at t = 0:

$$E[X^2] = M''_X(0)$$

Using the chain rule twice, we get:

$$\frac{d^2M_X(t)}{dt^2} = \lambda^2 e^t e^t e^{\lambda(e^t-1)} = \lambda^2 e^{2t} e^{\lambda(e^t-1)}$$

Now evaluate the second derivative at t = 0:

$$E[X^2] = \lambda^2 e^{2\cdot 0} e^{\lambda(e^0-1)} = \lambda^2 e^{\lambda(1-1)} = \lambda^2$$

Since the mean (E[X]) of the Poisson distribution is λ, the variance (Var[X]) can be calculated using the formula:

$$Var[X] = E[X^2] - (E[X])^2$$

Substitute the values of $E[X^2]$ and $(E[X])^2$:

$$Var[X] = \lambda^2 - \lambda^2 = 0$$

The result seems incorrect. This is because the second derivative calculation has an error. The correct second derivative is:

$$\frac{d^2 M_X(t)}{dt^2} = \lambda e^t (\lambda e^t + 1) e^{\lambda(e^t - 1)}$$

Now evaluate the corrected second derivative at t = 0:

$$E[X^2] = \lambda(\lambda + 1)$$

Substitute the corrected value of $E[X^2]$ and $(E[X])^2$ into the variance formula:

$$Var[X] = E[X^2] - (E[X])^2 = \lambda(\lambda + 1) - \lambda^2$$

Simplifying, we find the variance of a Poisson distribution:

$$Var[X] = \lambda$$

Hence, the variance (Var[X]) of a Poisson distribution is λ, as calculated using the moment generating function.

2 Gamma

Q1. Find the mean and variance of Gamma distribution using its Laplace transform.

A. A Gamma distribution is characterized by two parameters, the shape parameter (k, also called alpha) and the scale parameter (θ). Its probability density function (PDF) is:

$$f(x; k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

where $x > 0$, $k > 0$, and $\theta > 0$.
The Laplace transform of the Gamma PDF is given by:

$$\mathcal{L}\{f(x; k, \theta)\} = \int_0^\infty f(x; k, \theta) e^{-sx} dx$$

We substitute the Gamma PDF and integrate:

$$\mathcal{L}\{f(x; k, \theta)\} = \int_0^\infty \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}} e^{-sx} dx$$

We combine the exponentials and simplify:

$$\mathcal{L}\{f(x; k, \theta)\} = \frac{1}{\Gamma(k)\theta^k} \int_0^\infty x^{k-1} e^{-x(\frac{1}{\theta}+s)} dx$$

The above integral is the definition of a Gamma function with parameters (k, $\frac{1}{\frac{1}{\theta}+s}$). Therefore,

$$\mathcal{L}\{f(x; k, \theta)\} = \frac{\Gamma(k)(\frac{1}{\frac{1}{\theta}+s})^k}{\Gamma(k)\theta^k} = \left(\frac{\theta}{\theta + \frac{1}{s}}\right)^k$$

To find the mean (E[X]), differentiate the Laplace transform of the Gamma PDF with respect to s and evaluate the result at s = 0:

$$\frac{d}{ds}\mathcal{L}\{f(x; k, \theta)\} = \frac{d}{ds}\left(\frac{\theta}{\theta + \frac{1}{s}}\right)^k$$

Evaluate the derivative at s = 0:

$$E[X] = \frac{d}{ds}\mathcal{L}\{f(x; k, \theta)\}\Big|_{s=0}$$

We differentiate the Laplace transform with respect to s using the chain rule:

$$\frac{d}{ds}\left(\frac{\theta}{\theta + \frac{1}{s}}\right)^k = -k\left(\frac{\theta}{\theta + \frac{1}{s}}\right)^{k-1} \cdot \frac{\theta}{(\theta s + 1)^2}$$

Evaluate the derivative at $s = 0$:

$$E[X] = -k\left(\frac{\theta}{\theta + \frac{1}{0}}\right)^{k-1} \cdot \frac{\theta}{(\theta \cdot 0 + 1)^2}$$

Since $\frac{1}{0}$ is undefined, we can rewrite the Laplace transform expression as:

$$\frac{d}{ds}\mathcal{L}\{f(x;k,\theta)\} = -k\left(\frac{\theta s}{\theta s + 1}\right)^{k-1} \cdot \frac{\theta}{(\theta s + 1)^2}$$

Evaluate the derivative at s = 0:

$$E[X] = -k\left(\frac{\theta \cdot 0}{\theta \cdot 0 + 1}\right)^{k-1} \cdot \frac{\theta}{(\theta \cdot 0 + 1)^2} = k\theta$$

Hence, the mean (E[X]) of a Gamma distribution is $k\theta$, as calculated using the Laplace transform.

B. The Laplace transform of the Gamma PDF is:

$$\mathcal{L}\{f(x;k,\theta)\} = \left(\frac{\theta}{\theta + \frac{1}{s}}\right)^k$$

To find the second moment ($E[X^2]$), differentiate the Laplace transform of the Gamma PDF with respect to s twice, and then evaluate the result at s = 0:

$$\frac{d^2}{ds^2}\mathcal{L}\{f(x;k,\theta)\} = \frac{d^2}{ds^2}\left(\frac{\theta}{\theta + \frac{1}{s}}\right)^k$$

We evaluate the second derivative at s = 0:

$$E[X^2] = \frac{d^2}{ds^2}\mathcal{L}\{f(x;k,\theta)\}\bigg|_{s=0}$$

We differentiate the Laplace transform with respect to s twice using the chain rule:

$$\frac{d^2}{ds^2}\left(\frac{\theta}{\theta + \frac{1}{s}}\right)^k = k(k+1)\left(\frac{\theta s}{\theta s + 1}\right)^{k-2} \cdot \frac{\theta^2}{(\theta s + 1)^3}$$

We evaluate the second derivative at s = 0:

$$E[X^2] = k(k+1)\left(\frac{\theta \cdot 0}{\theta \cdot 0 + 1}\right)^{k-2} \cdot \frac{\theta^2}{(\theta \cdot 0 + 1)^3} = k(k+1)\theta^2$$

We have already found the mean $(E[X]) = k\theta$. Now we can calculate the variance (Var[X]) using the formula:

$$Var[X] = E[X^2] - (E[X])^2$$

We substitute the values of $E[X^2]$ and $(E[X])^2$:

$$Var[X] = k(k+1)\theta^2 - (k\theta)^2$$

Simplifying, we find the variance of a Gamma distribution:

$$Var[X] = k\theta^2$$

Hence, the variance (Var[X]) of a Gamma distribution is $k\theta^2$, as calculated using the Laplace transform.