Advanced Stochastic Processes - Ex. 8

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Problem Statement

Given the density function of an Inverse Gaussian distribution:

$$f_{1G}(t) = \frac{\alpha}{\sqrt{2\pi\beta t^3}} \exp\left(-\frac{(\beta t - \alpha)^2}{2\beta t}\right) \quad for t > 0$$

We want to show that as $\alpha = |a|\sqrt{\beta}$ and $\beta \to 0$, the Inverse Gaussian distribution approaches a one-sided stable distribution of index 1/2.

Proof

The density function of a one-sided stable distribution of index 1/2 is given by:

$$f(x; c) = \sqrt{\frac{c}{2\pi x^3}} \exp\left(-\frac{c}{2x}\right) \quad for x > 0$$

where c is a scale parameter.

As $\beta \to 0$, the term $(\beta t - \alpha)^2/(2\beta t)$ in the exponent of the exponential function of the Inverse Gaussian distribution goes to 0, and the exponential function approaches 1.

If we let $\alpha = |a|\sqrt{\beta}$, then as $\beta \to 0$, $\alpha/\sqrt{\beta}$ approaches infinity. However, the $\sqrt{\beta}$ in the denominator of the fraction in front of the exponential function also approaches infinity, and the ratio $\alpha/\sqrt{\beta}$ remains finite.

Therefore, as $\beta \to 0$, the Inverse Gaussian distribution approaches:

$$f(t) = \frac{|a|}{\sqrt{2\pi t^3}} \quad for t > 0$$

which is the form of a one-sided stable distribution of index 1/2 with scale parameter $c = 2|a|^2$.

Therefore, the Inverse Gaussian distribution approaches a one-sided stable distribution of index 1/2 as $\beta \to 0$.