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College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

Part I: Introduction to Feedback Control

Lecture 7: The General Second Order System

Summer 2020

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Lecture Plan

- Objectives:
  - *Discuss the characteristics of the general second-order system*
  - *Discuss the effects of additional poles on the system response*
  - *Discuss the effects of zeros on the system response*
- Reading:
  - *Nise: 4.1-4.8*
- Practice Problems Included

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- To further analyze second-order systems, we treat the general form:

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \text{ from } G(s) = \frac{k}{ms^2+f_v s+k}$$

- Note: The numerator is set $= \omega_n^2$, normalizes the response:
  - The response's amplitude $= 1$, to a unit step input. Test by applying F.V.T.
- Natural Frequency: $\omega_n = \sqrt{\frac{k}{m}}$
- Damping Ratio: $\zeta = \frac{Exponential\ Decay\ Freq}{\omega_n}$
- The roots of the characteristic polynomial: $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- Damped Frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$, so $s = -\zeta\omega_n \pm \omega_d j$

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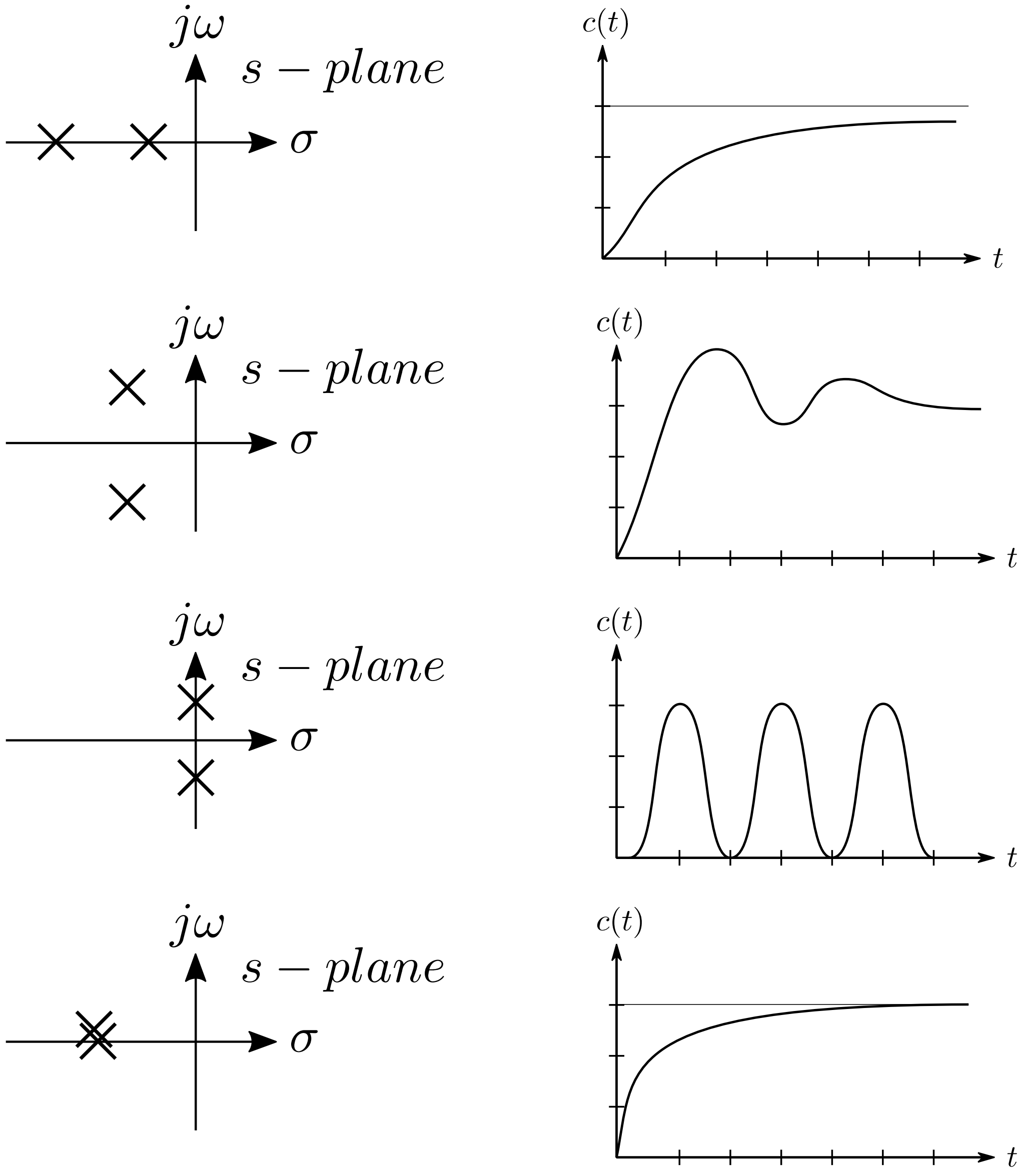
Second-Order System Response

$\zeta > 1$

$0 < \zeta < 1$

$\zeta = 0$

$\zeta = 1$



Underdamped Second-Order Systems

- Looking at the step response of the general second-order system

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- Assuming $0 < \zeta < 1$, doing the p.f.e and rearranging we get

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

- Taking the inverse Laplace Transform we get:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \phi\right)$$

$$\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

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Underdamped Second-Order Systems – Performance Specifications

- With respect to underdamped second-order systems, the following performance specifications are of concern
  - Rise Time, T_r: Time required for response to go from 10% to 90% of final value.
    - No precise analytical expression available
  - Peak Time, T_p: Time required to reach first, or maximum, peak.
    - $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$
  - Percent Overshoot, $\%OS$: Ratio of Peak – final value to final value
    - $\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$
    - $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$
  - Settling Time T_s: Time required to reach and stay within 2% of final value.
    - $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$
  - Where $s = \sigma_d \pm \omega_d j$
- Nise: 4.6, covers the derivation of the above expressions.

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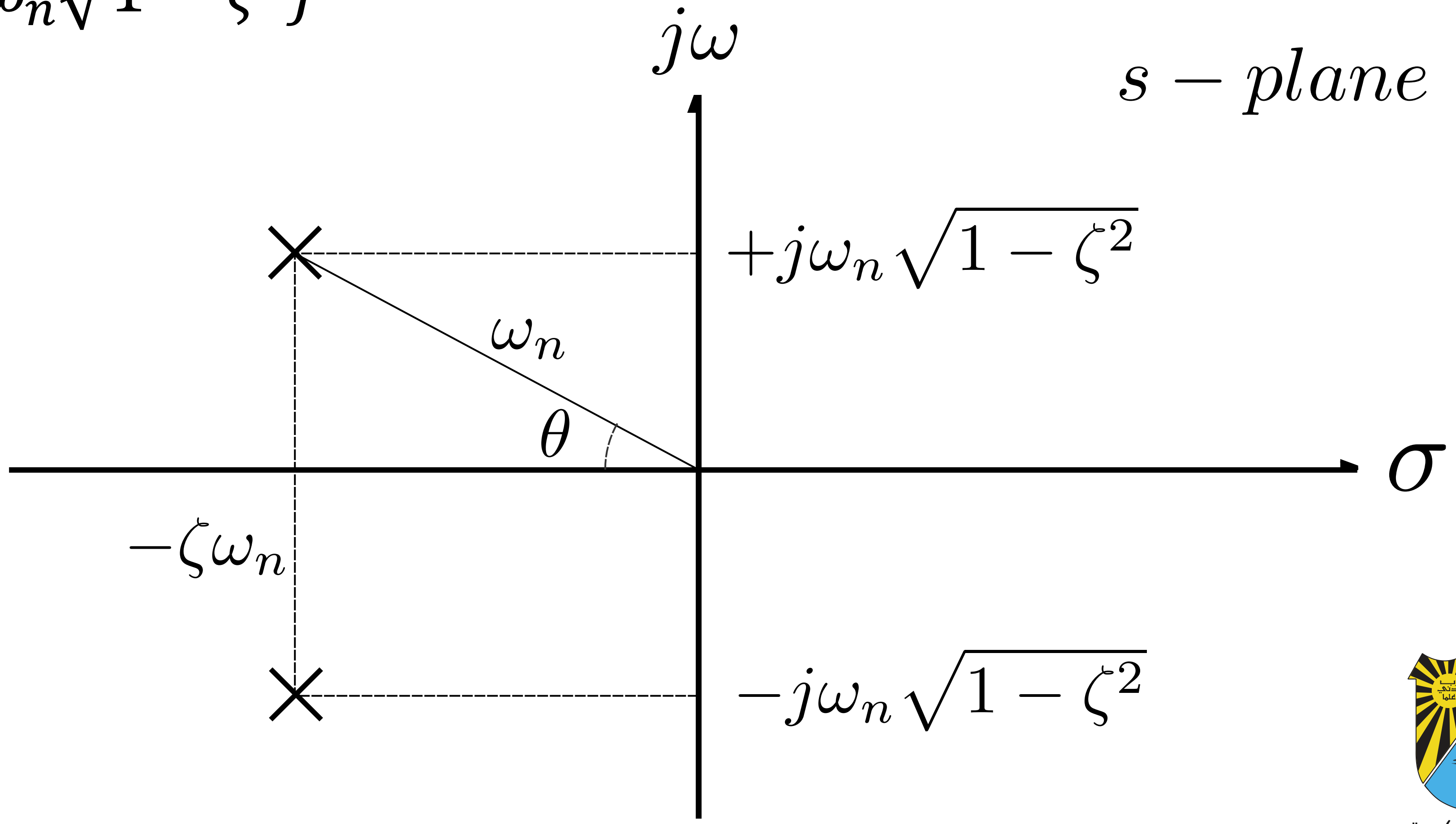
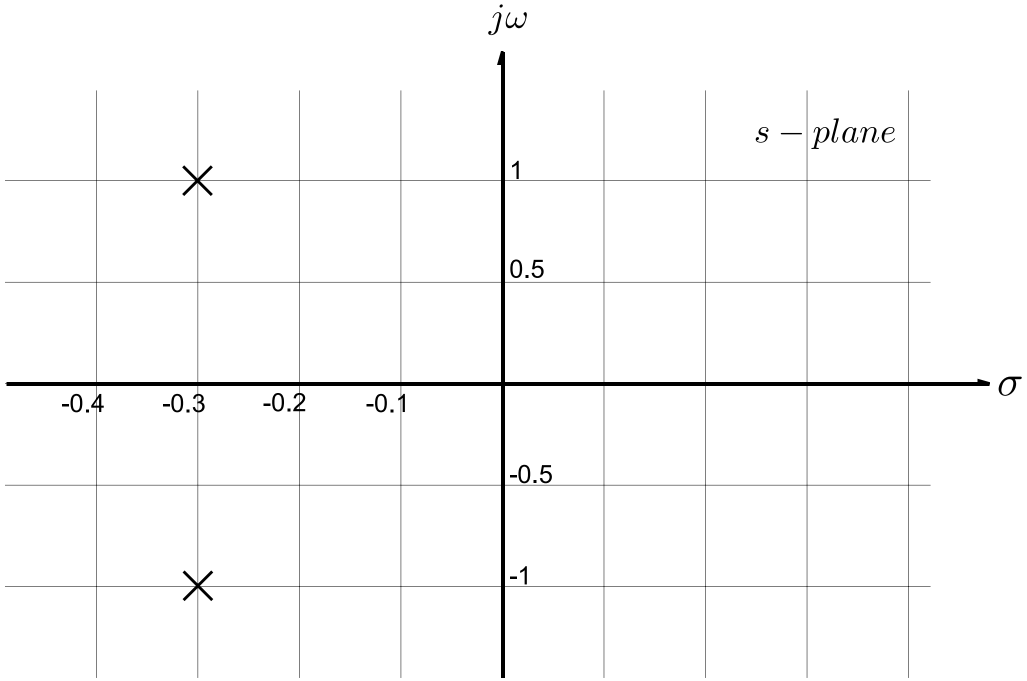
- Note that an underdamped general second-order system has two complex pole pairs on the left of the imaginary axis.
- $s = \sigma_d \pm \omega_d j = \zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}j$
- $\zeta = cos\theta$



Example 1

For the system represented by the poles shown on the s-plane, find out the values of Settling Time T_s, Peak Time T_p and $\%OS$ for a unit step input response. Assume a general second-order system "normalized response".



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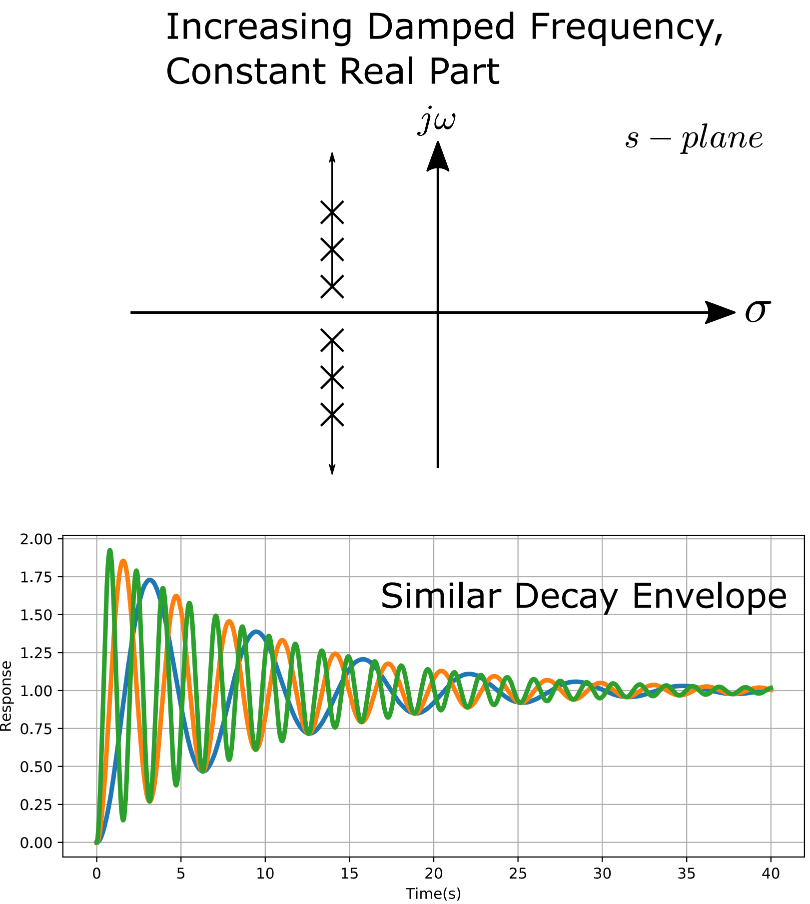
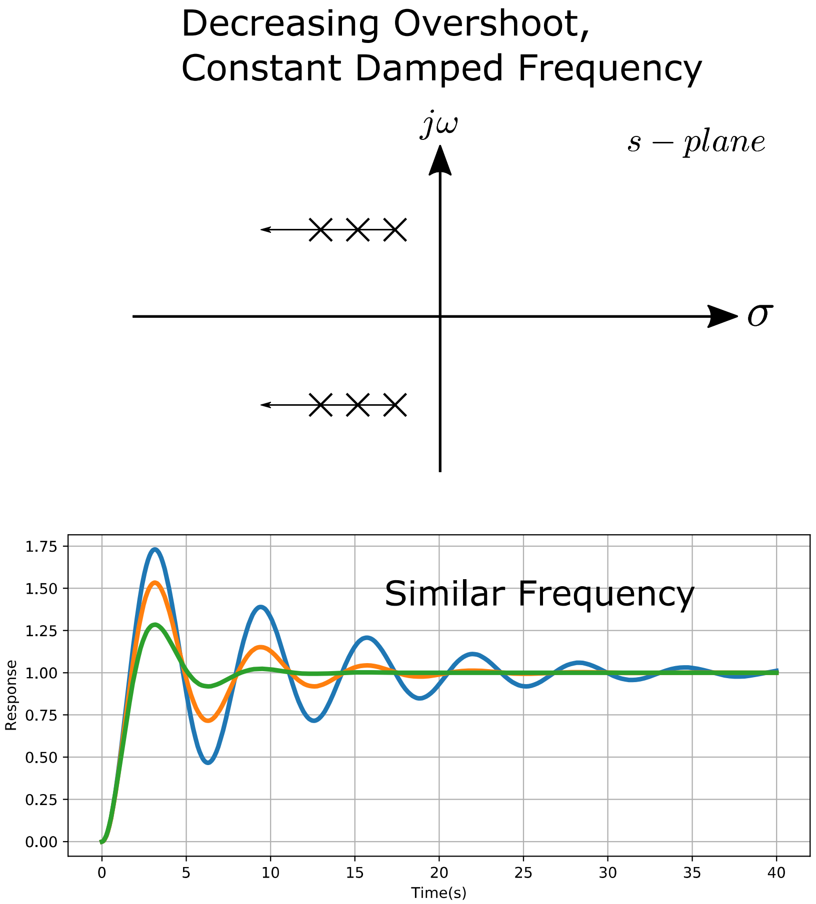
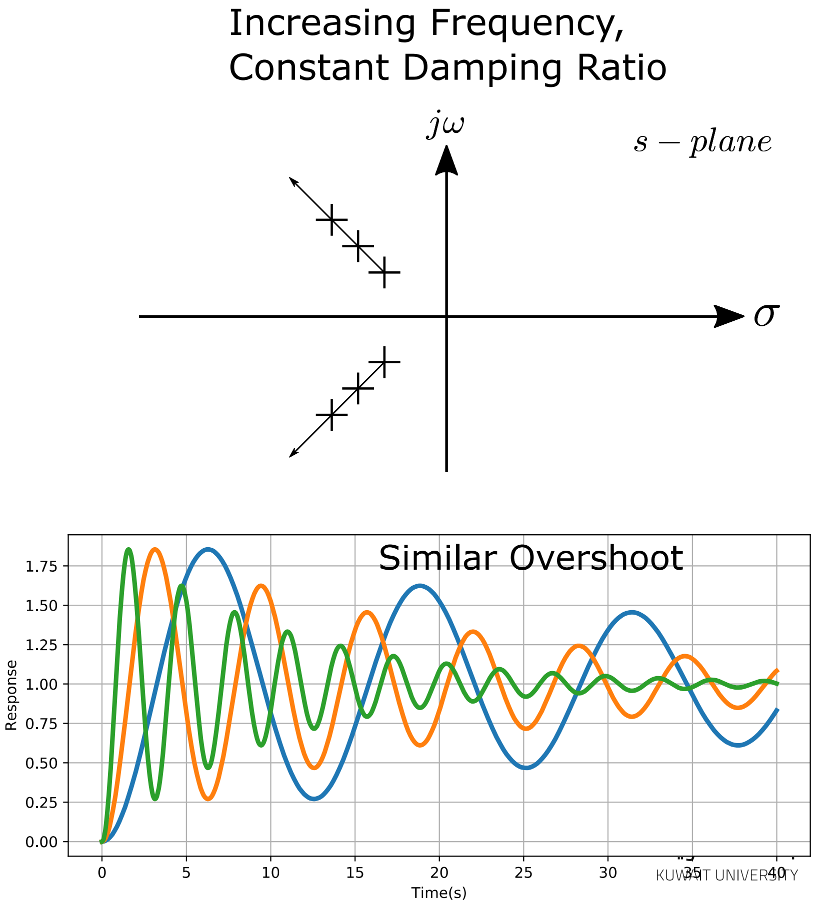
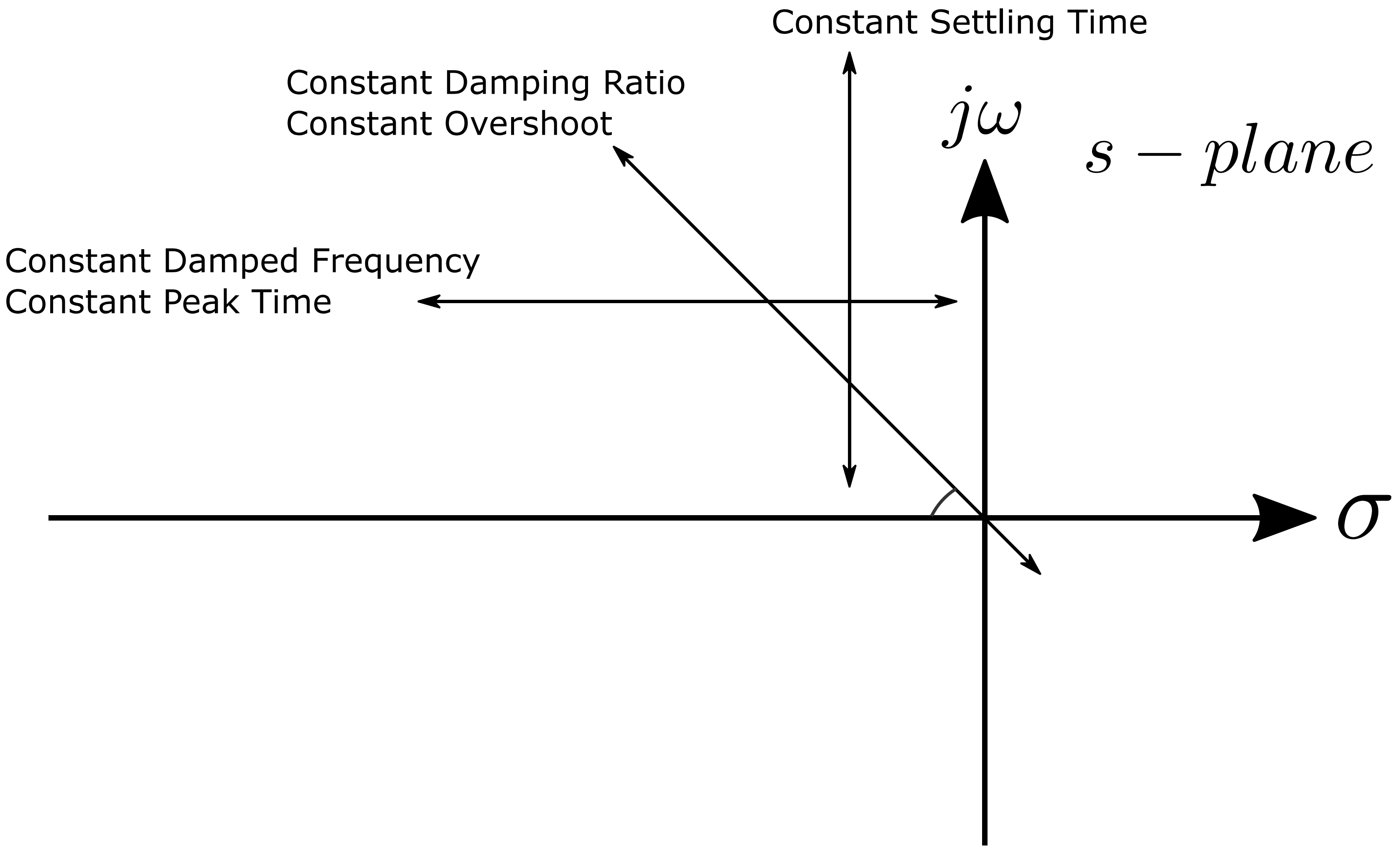
- Moving the poles on the s-plane, produces a defined qualitative change in the response.
- Moving the complex poles in three specific directions:



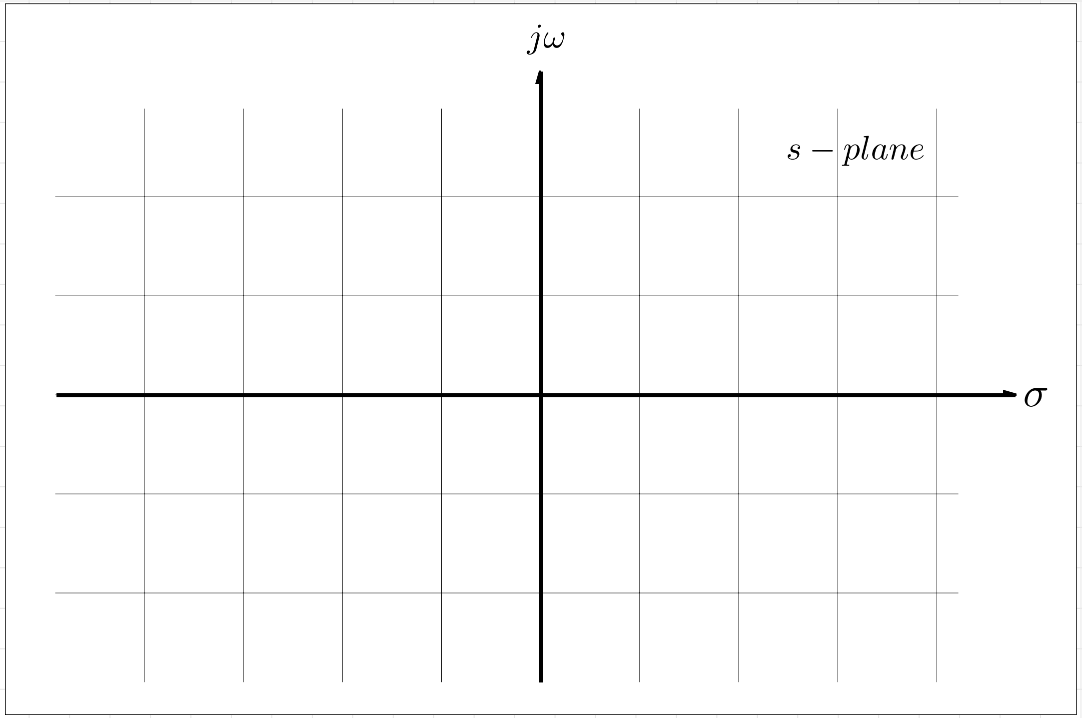


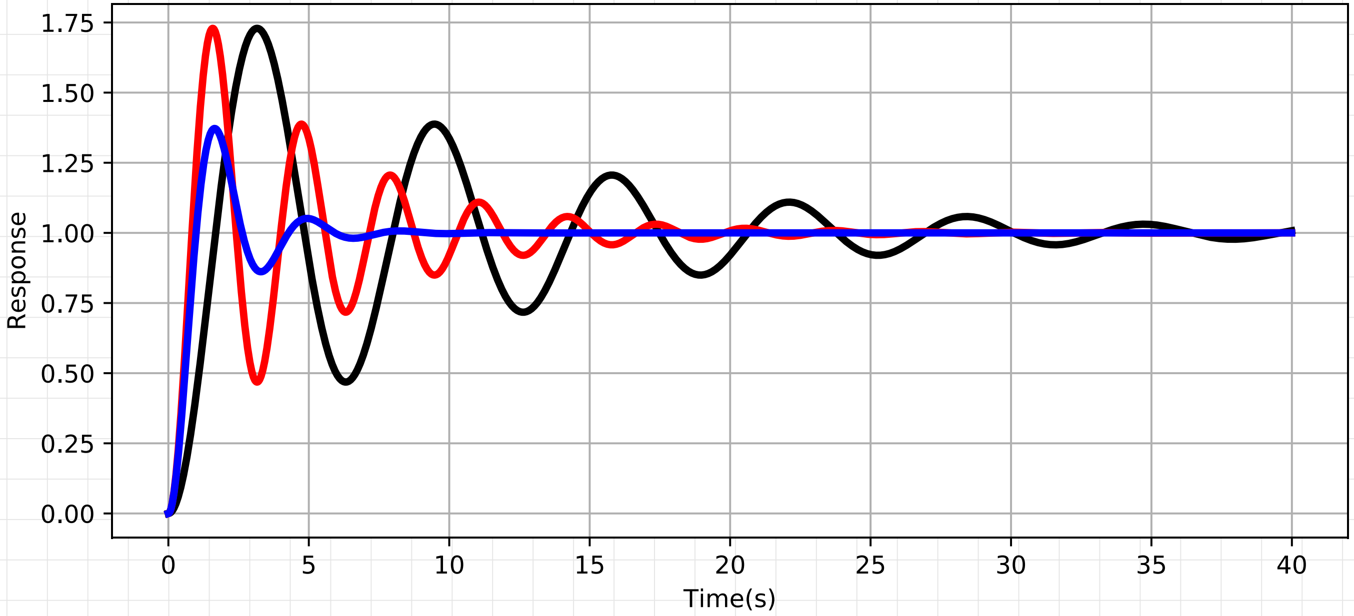


- Note the following when moving the poles along the following three directions on the s-plane.



Given the following three responses for a general second-order systems to a step input. Place, qualitatively, the poles of the systems on the s-plane, highlight which poles belong to which response.





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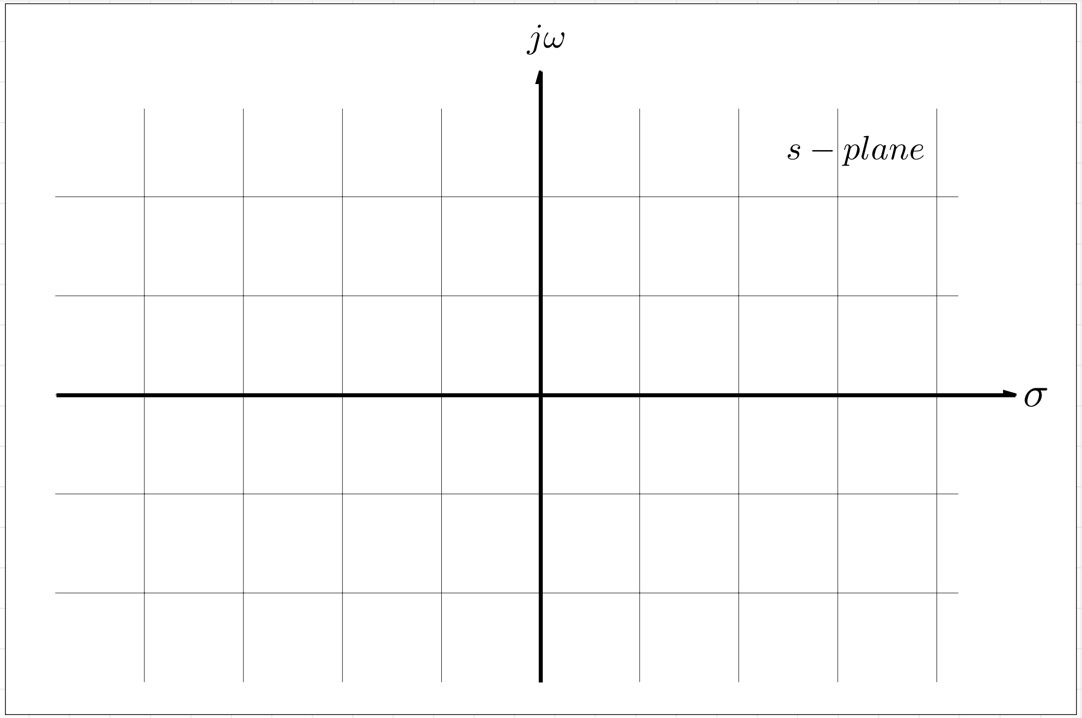

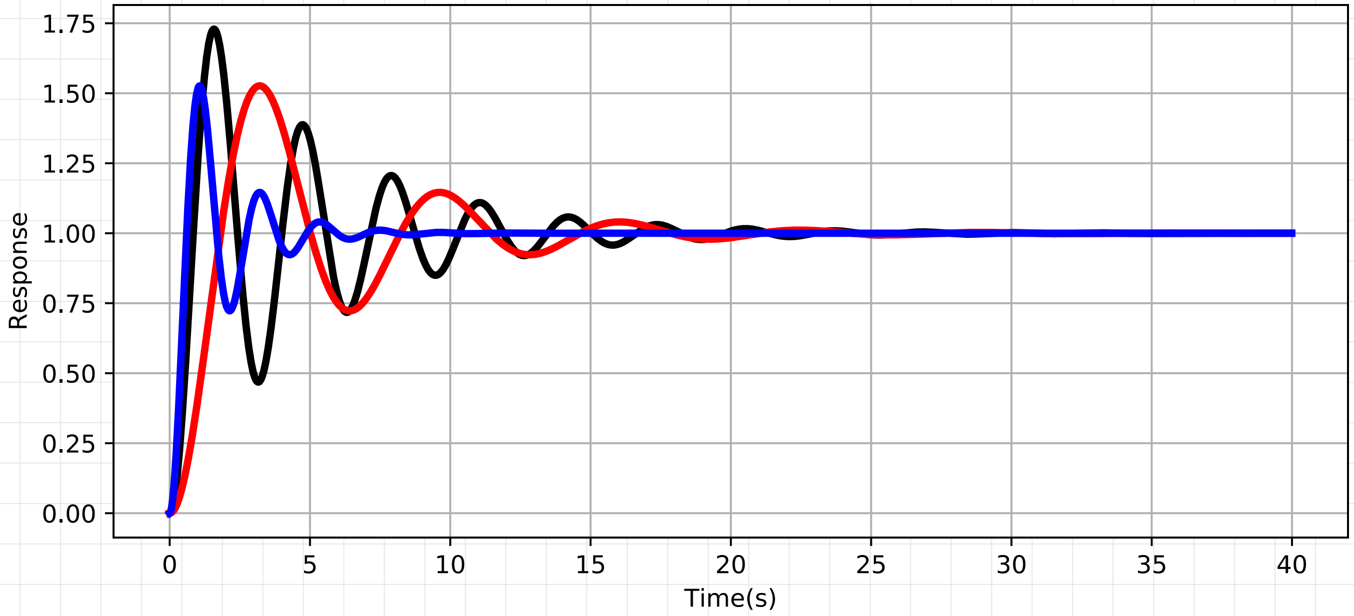

- Additional poles increase the order of the system
- But under certain conditions, we can approximate higher order system responses as second-order system responses.
- In these cases we evaluate the response from the **dominant** poles
- Adding a real pole to a general second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \times \frac{a}{\underbrace{s+a}_{Additional\ Pole}}$$

$$C(s) = R(s)G(s) = \frac{K_1}{s} + \frac{K_2(s+\zeta\omega_n) + K_3\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{K_4}{(s+a)}$$

$$c(t) = \underbrace{K_1 u(t)}_{Forced\ Resp.} + \underbrace{e^{-\zeta\omega_n t}(K_2 cos\omega_d t + K_3 sin\omega_d t)}_{Nat.Resp.: Complex\ Poles} + \underbrace{K_4 e^{-at}}_{Nat.Resp.: Additional\ Pole}$$

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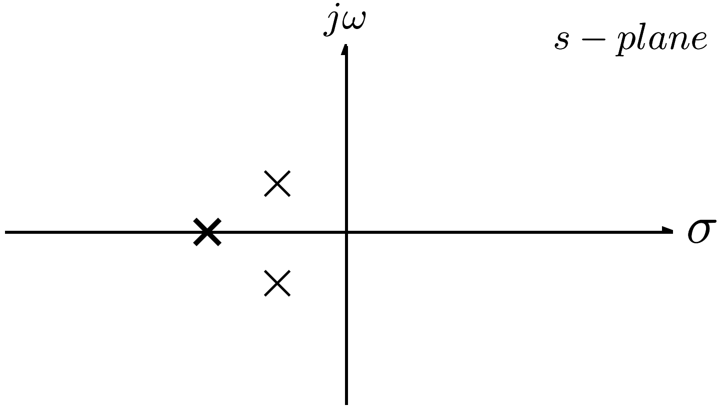
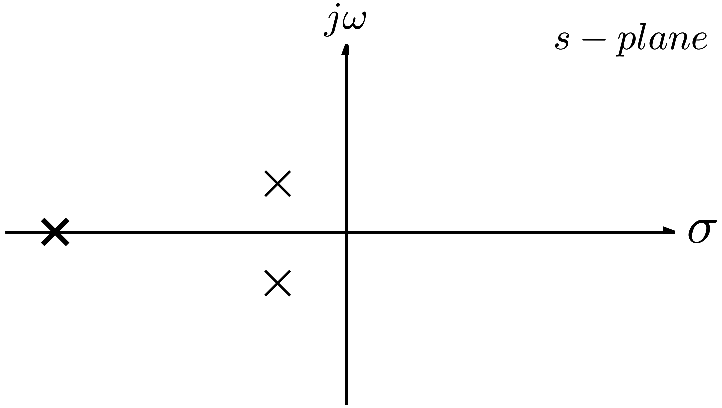
System Response with Additional Poles

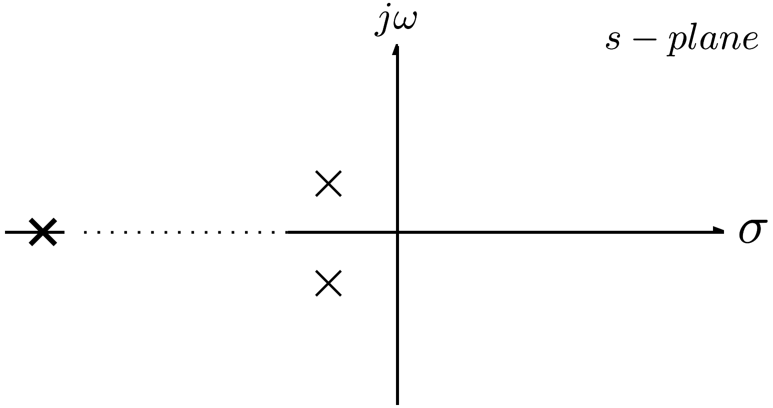
Case I: Additional Pole is close to dominant poles

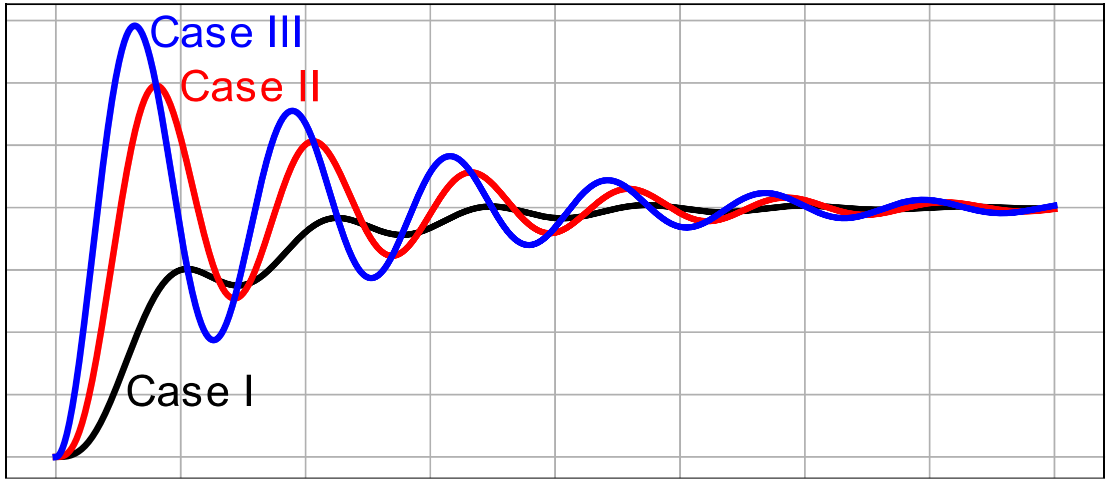


Case II: Additional Pole is far from dominant poles



Case III: Additional Pole is very far from dominant poles





Second-Order Approximation for Higher Order Systems

- Whenever possible, we would want to treat higher-order systems as second-order systems. *Why?*
- Deciding how to evaluate whether higher order systems is dependent on the designer's desired accuracy
- In general, the further away along the real axis, the additional poles are, from the dominant poles the more the second-order approximation is accurate.
- We can follow the "five times" rule of thumb. Which states:
  - If the real poles are at least five times farther to the left than the dominant poles we can treat the system as a second-order system:

$$If\ a \geq 5\zeta\omega_n\ then$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{a}{s+a} \approx \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

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Example - 4

Check whether a second-order approximation is valid for the following systems

$$G_1(s) = \frac{750}{(s+15)(s^2+4s+100)}$$

$$G_2(s) = \frac{360}{(s+4)(s^2+2s+90)}$$

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System Response with Zeros

- So far, we looked at first, second, and higher order systems assuming no zeros in the transfer function. What happens when we have a zero?
- If $C(s)$ is the response of a system without a zero, the added zero would result in a new response

$$C'(s) = \underbrace{(s+a)}_{zero}\ C(s) = \underbrace{sC(s)}_{Derivative\ of\ Original\ Response} + \underbrace{aC(s)}_{Scaled\ Original\ Response},\ a > 0$$

- The new response $C'(s)$ = the derivative of the original response $C(s)$ + a scaled version of the original response $C(s)$
- Effect from $\mathbf{a}\boldsymbol{C}(\boldsymbol{s})$**:** The larger the zero, the higher the amplitude of the response (acts like a gain factor)
- Effect from $\boldsymbol{sC}(\boldsymbol{s})$**:** The faster the response changes, the higher this component
  - At the beginning of a transient response, the rate of change is highest, as so, the effect of the derivative is highest.

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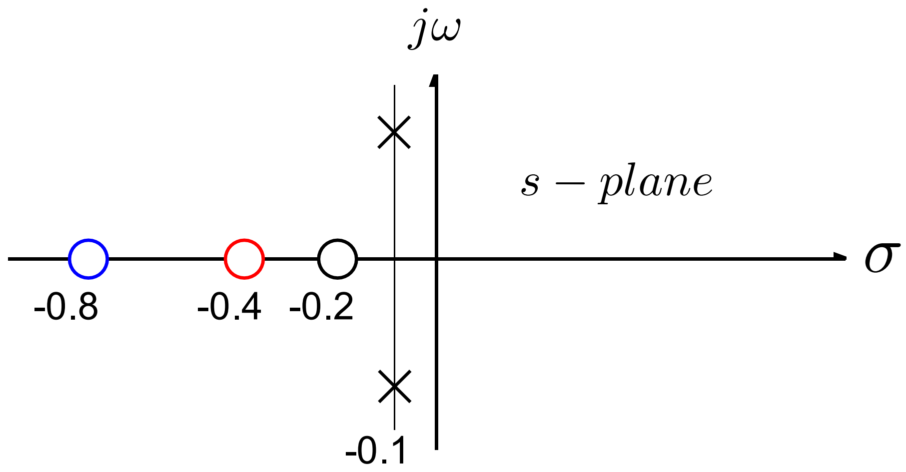
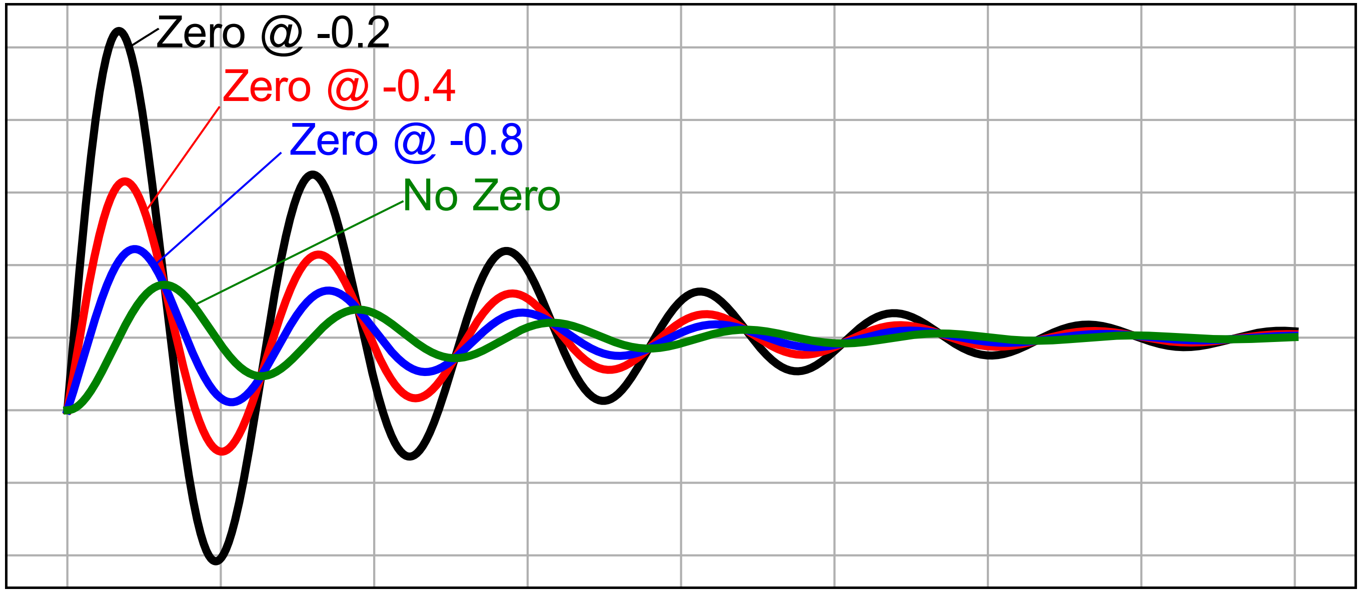
- The effect of adding a **negative real zero** to a general second-order system is shown
- Note that the derivative part contributes more to the new response since a is generally small, so the component $aC(s)$ remains small.
- Note how the zero can exacerbate the amplitude





- What if add a positive real zero to the system, $s = a,\ a > 0$?

- This would result in the following new response:

$$C'(s) = \underbrace{(s-a)}_{positive\ zero} C(s) = sC(s) - aC(s),\ a > 0$$

- The term derivative term "$sC(s)$" is of the opposite sign from the scaled response "$-aC(s)$"

- The derivative term will act in the opposite direction to the scaled response.

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- This results in the response initially going negative
  - Going in the “wrong” direction initially
  - Can you think of real-world examples of such systems?

- We call such systems: non-minimum phase systems
  - The term “*non-minimum phase*” can be understood in the context of the presence of phase delay between the response and desired output

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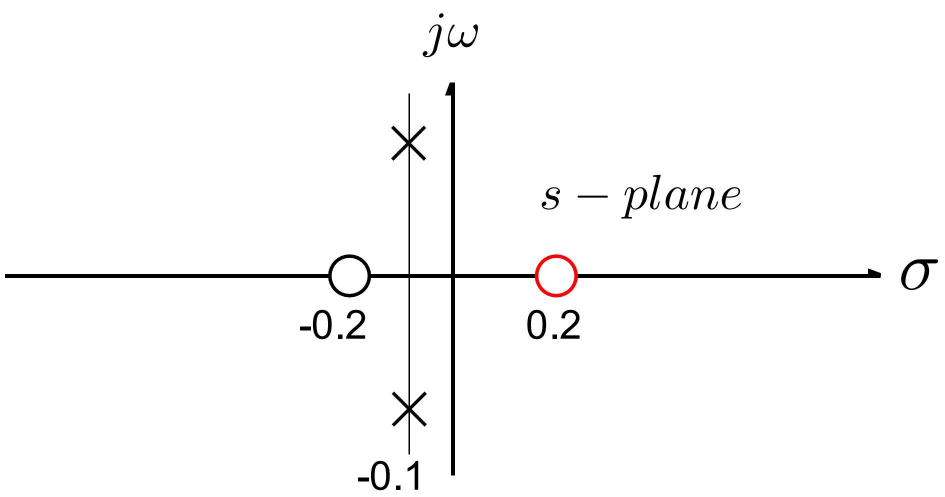
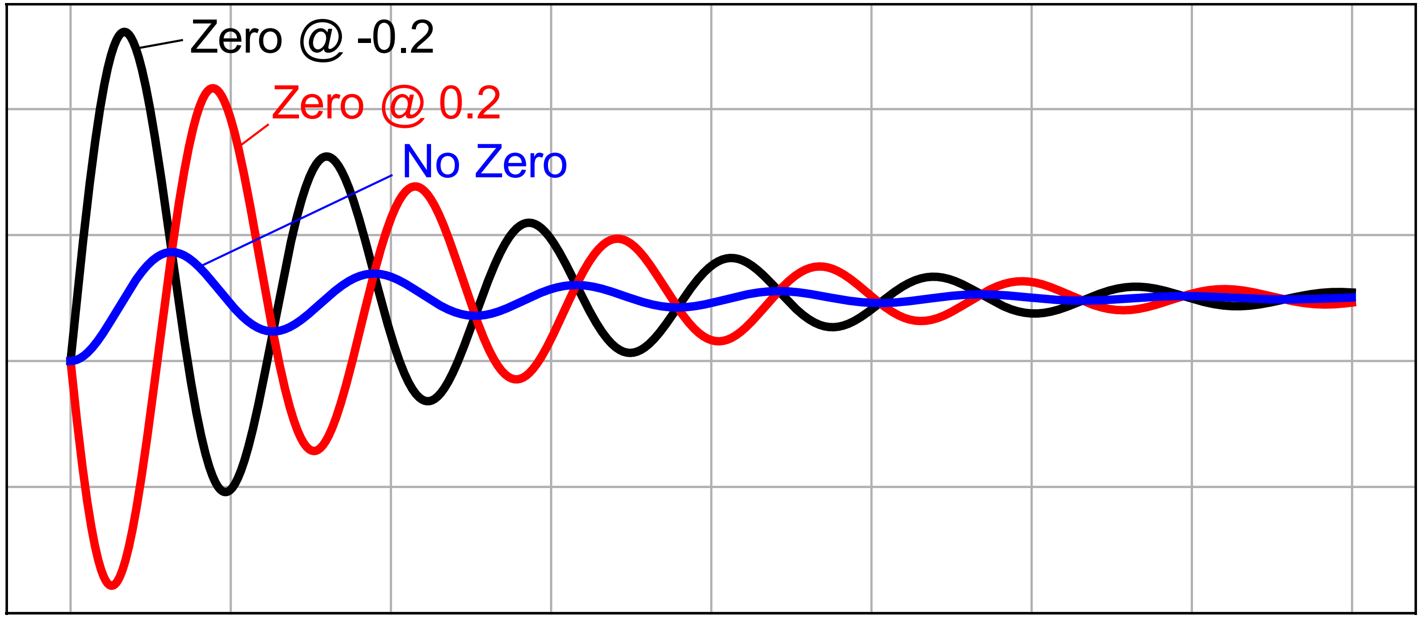
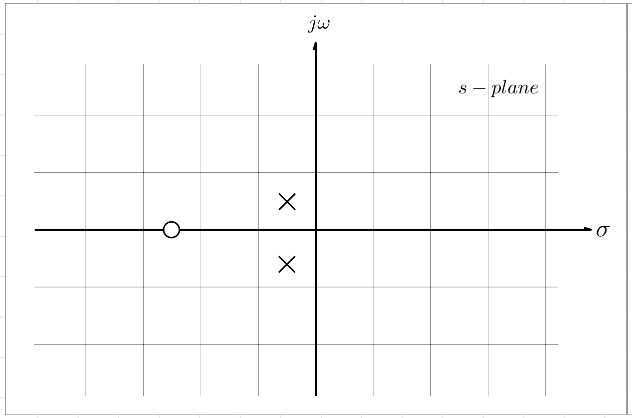
- Notice how the positive zero causes the response to initially go in the negative direction.
- Such systems pose a challenge when designing a controller for the purpose of tracking a changing input.

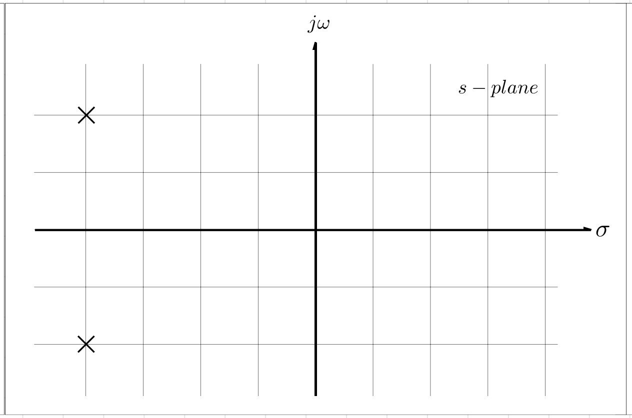


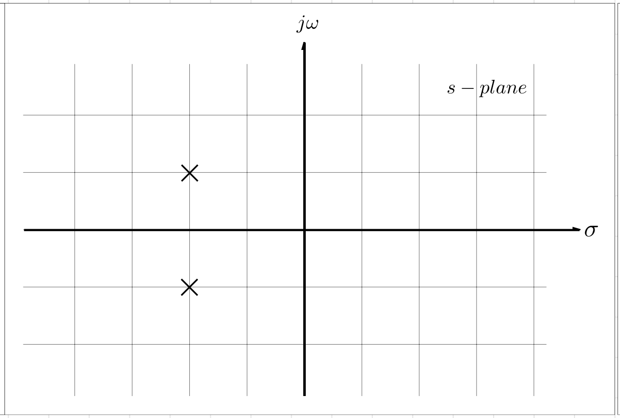


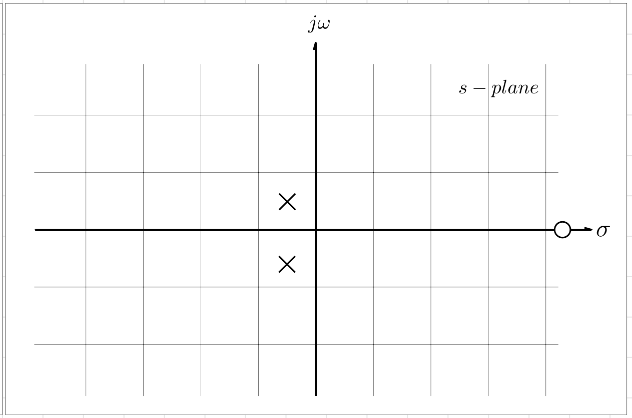
Example - 5

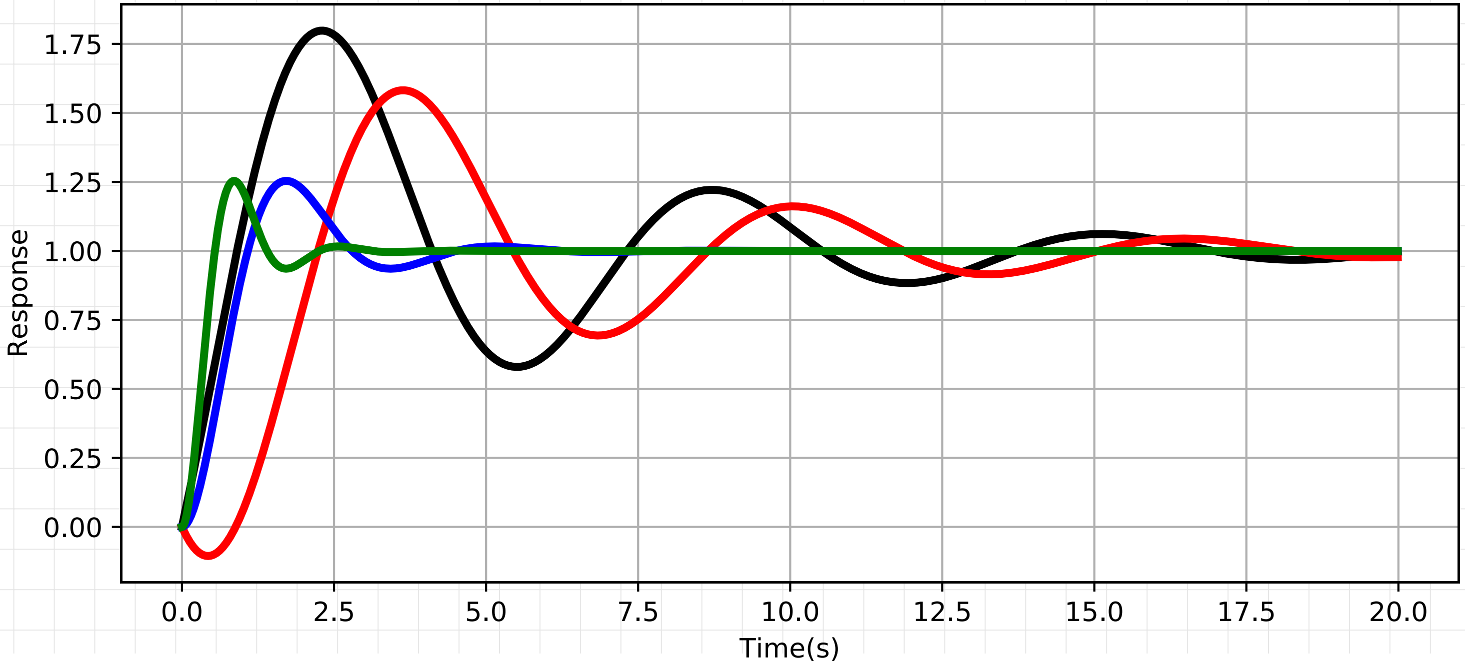
Given the following four responses for a general second-order systems to a step input. Match the pole pairs on the s-plane with the responses.











- Previously, we distinguished between the order of the system and the order of the **response** of the system, in the case

$$G(s) = \frac{(s+a)}{(s+b)(s+a)} = \frac{1}{(s+b)}$$

- That distinction is made to account for modeling uncertainties and errors which may make the above algebraic simplification not possible.
- However, for the above case and for cases where a pair of pole and zero are close enough to each other, we can simplify the system by making the pole-zero cancellation

$$G(s) = \frac{\cancel{(s+z_1)}}{(s+p_2)\cancel{(s+p_1)}} = \frac{1}{(s+p_2)}, \;\; if\, p_1 \approx z_1$$

- The "close enough" judgement is relative to the residues of the other poles of the response

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- Example: For following response function

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)} = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{\mathbf{1}}{s+3.5}$$

  - *The residue of the pole at 3.5 which is closest to the zero at 4, is equal to* ***1*** *and is not negligible compared to other residues.*

- Example: For following response function

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)} = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{\mathbf{0.033}}{s+4.01}$$

  - *The residue of the pole at 4.01 which is closest to the zero at 4, is equal to* ***0.033,*** *which is negligible in magnitude compared to other residues.*

- If the relative distance between the pole and zero pairs under investigation, is much smaller then the relative distance to other poles then the pole-zero cancellation is more likely to be valid.

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For each of the following systems, **justify** whether a second-order approximation is valid or not for a step response. State your assumptions.

a. $G(s) = \dfrac{1}{(s+4)(s^2+s+10)}$

b. $G(s) = \dfrac{100}{(s+4)(s^2+s+10)}$

c. $G(s) = \dfrac{300}{(s+6)(s^2+6s+2)}$

d. $G(s) = \dfrac{300}{(s+10)^2(s^2+s+4)}$

e. $G(s) = \dfrac{300}{(s^2+16s+84)(s^2+2s+10)}$

Ans. A. Valid B. Valid. C. Not Valid D. Valid C. Valid

For each pair of second-order system specifications that follow, find the location of the second-order pair of poles. Nise. 4-23

a. $\%OS = 12\%, T_s = 0.6s$

b. $\%OS = 10\%, T_s = 5s$

c. $T_s = 7s, T_p = 3s$

Practice Problem 3

For the following response functions, determine if pole-zero cancellation can be approximated. If it can, find percent overshoot, settling time, rise time, and peak time. Nise 4-32

a. $C(s) = \frac{(s+3)}{s(s+2)(s^2+3s+10)}$

b. $C(s) = \frac{(s+2.5)}{s(s+2)(s^2+4s+20)}$

c. $C(s) = \frac{(s+2.1)}{s(s+2)(s^2+4s+20)}$

d. $C(s) = \frac{(s+2.01)}{s(s+2)(s^2+5s+20)}$

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Find peak time, settling time, and percent overshoot for only those responses below that can be approximated as second-order responses. Nise 4

Practice Problem 4

a. $c(t) = 0.003500 - 0.001524e^{-4t}$
$-0.001976e^{-3t}\cos(22.16t)$
$-0.0005427e^{-3t}\sin(22.16t)$

b. $c(t) = 0.05100 - 0.007353e^{-8t}$
$-0.007647e^{-6t}\cos(8t)$
$-0.01309e^{-6t}\sin(8t)$

c. $c(t) = 0.009804 - 0.0001857e^{-5.1t}$
$-0.009990e^{-2t}\cos(9.796t)$
$-0.001942e^{-2t}\sin(9.796t)$

d. $c(t) = 0.007000 - 0.001667e^{-10t}$
$-0.008667e^{-2t}\cos(9.951t)$
$-0.0008040e^{-2t}\sin(9.951t)$

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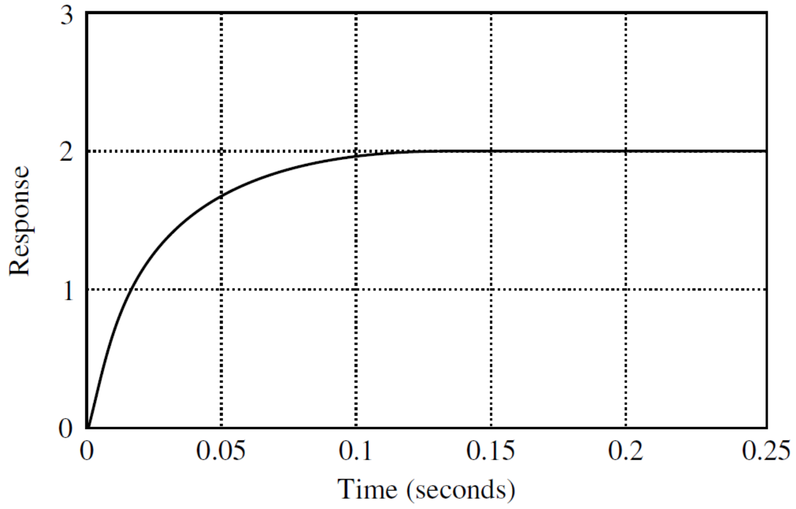
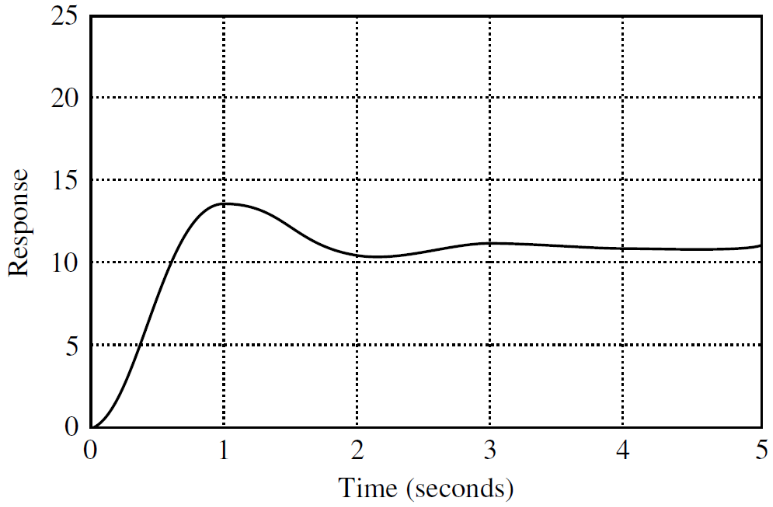
Practice Problem 5

For each of the unit step responses shown, find the transfer function of the system. Nise 4-29



(*a*)



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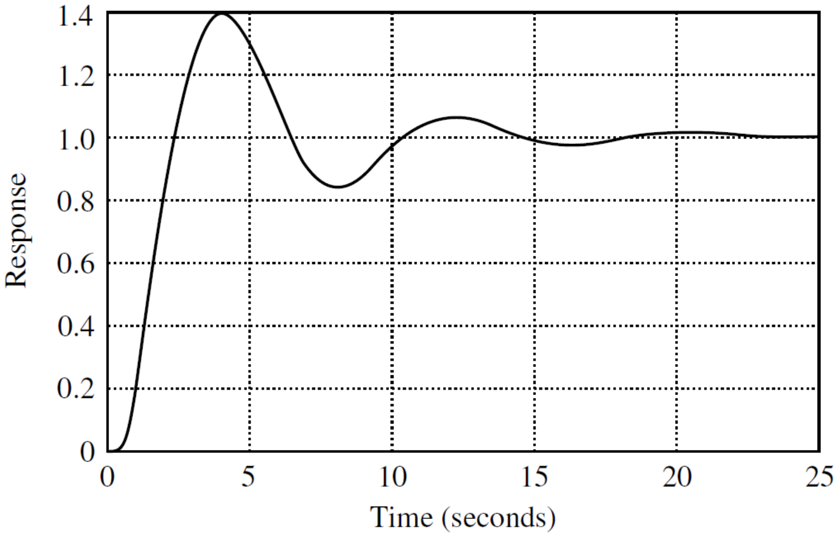

(*c*)

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Practice Problem 6

The following problems as well

Nise: 4-19, 4-20, 4-25, 4-28, 4-35

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