Kuwait University
College of Engineering and Petroleum

جامعة الكويت
KUWAIT UNIVERSITY

ME417 CONTROL OF MECHANICAL SYSTEMS

Part II: Controller Design Via State-Space

Lecture 2: Stability and Steady-State Error In State-Space

Summer 2020

Ali AlSaibie

Lecture Plan

- Objectives:
  - Define System Stability in State-Space
  - Derive Steady-State Error in for Systems in State-Space Form
- Reading:
  - *Nise: 6.5, 7.8*
- Practice Problems Included

جامعة الكويت
KUWAIT UNIVERSITY

- Given an **LTI** system in State-Space

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x} + \mathbf{D}\boldsymbol{u}$$

- The poles of the system are given by the eigenvalues of the system matrix $\mathbf{A}$

$$det(\lambda\mathbf{I} - \mathbf{A}) = det(s\mathbf{I} - \mathbf{A}) = \mathbf{0}$$

- Remember that stability is defined in the context of the natural response
  - $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$
- The term $det(s\mathbf{I} - \mathbf{A}) = 0$ is also the characteristic polynomial

جامعة الكويت
KUWAIT UNIVERSITY

The system is**:**

- **Stable:** If the real component of the roots are all strictly negative
  - $real(s_i) < 0, for\, all\, i = 1,2, \dots, n$
- **Marginally stable:** If there are no roots with positive real components, but there are roots with real components equal to zero, and with multiplicity no greater than 1
  - $real(s_i) = 0 \wedge imag(s_i) \neq imag(s_j)\, for\, any\, i \neq j,\ (i, j = 1,2, \dots, n)$
- **Unstable**: If there are roots with positive real components or roots with the real component equal 0 and multiplicity greater than 1.
  - $real(s_i) > 0 \vee imag(s_i) = imag(s_j)\, for\, any\, i \neq j, (i, j = 1,2, \dots, n)$

Determine the stability of the following system and derive the transfer function, then find the natural frequency of the system.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

جامعة الكويت
KUWAIT UNIVERSITY

- We can compute the steady-state error in two ways:

1. Final Value Theorem: $e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)(1 - G_{cL})$

2. Input Substitution: $e(\infty) = 1 - y(\infty)$
   - The input substitution method is better suited for numerical implementations.

- Remember that steady-state error is defined in the context of closed-loop systems only.

- Given a **closed-loop LTI** system in state-space form

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}r$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x}$$

- Note how we use r rather than u for closed-loop state-space representation.
- Convert the system into state space form $G_{CL} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$**,** then substitute in

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)(1 - G_{cL})$$

- The steady-state error for a closed-loop system represented in state-space:

$$e(\infty) = \lim_{s\to 0} sR(s)(1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\ )$$

- With input substitution, the idea is to express the steady state $x(\infty)$ and correspondingly, the output $y(\infty)$, assuming their final form, then solve for the steady state error.
- Given a stable closed-loop LTI system

$$\dot{\boldsymbol{x}} \in \mathcal{R}^n = \mathbf{A}\boldsymbol{x} + \mathbf{B}r$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x}$$

- The form of the steady state to a step input and ramp input are:

$$\boldsymbol{x}_{step}(\infty) = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \boldsymbol{V}, \quad \boldsymbol{x}_{ramp}(\infty) = \begin{bmatrix} V_1 t + W_1 \\ V_2 t + W_2 \\ \vdots \\ V_n t + W_n \end{bmatrix} = \boldsymbol{V}t + \boldsymbol{W}$$

- Where V_i and W_i are constants, $i = 1,2, \dots, n$
- Where $\boldsymbol{V}$ and $\boldsymbol{W}$ are constant vectors of size n

جامعة الكويت
KUWAIT UNIVERSITY

- For a step input, since $\boldsymbol{x}(\infty)$ is constant, then $\dot{\boldsymbol{x}}(\infty) = 0$, substituting into the state space equations, with $r = 1$ for unit step:

$$0 = \mathbf{A}\boldsymbol{V} + \mathbf{B} \Rightarrow \quad \boldsymbol{V} = -\mathbf{A}^{-1}\mathbf{B}$$

$$\boldsymbol{y}(\infty) = \mathbf{C}\boldsymbol{V} \Rightarrow \quad \boldsymbol{y}(\infty) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

- And since $\boldsymbol{e}(\infty) = 1 - \boldsymbol{y}(\infty)$ for a unit step, then the steady-state error:

$$\boldsymbol{e}(\infty) = \mathbf{1} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

جامعة الكويت
KUWAIT UNIVERSITY

- For a ramp input, $\dot{\boldsymbol{x}}(\infty) = \boldsymbol{V} = constant$, substituting into the state space equations, with $r = t$ for unit ramp:

$$\boldsymbol{V} = \mathbf{A}(\boldsymbol{V}t + \boldsymbol{W}) + \mathbf{B}\mathrm{t} \Rightarrow \quad \boldsymbol{V} = \mathbf{A}\boldsymbol{V}t + \mathbf{B}t + \mathbf{A}\boldsymbol{W}$$

$$\boldsymbol{y}(\infty) = \boldsymbol{C}(\boldsymbol{V}t + \boldsymbol{W})$$

- Equating the coefficients to balance the equation

$$\mathbf{A}\boldsymbol{V} = -\mathbf{B} \Rightarrow \boldsymbol{V} = -\mathbf{A}^{-1}\mathbf{B}$$

$$\boldsymbol{V} = \mathbf{A}\boldsymbol{W} \Rightarrow \boldsymbol{W} = \mathbf{A}^{-1}\boldsymbol{V} = -(\boldsymbol{A}^{-1})^{2}\mathbf{B}$$

- Substituting in $\boldsymbol{y}$

$$\boldsymbol{y}(\infty) = -\mathbf{C}\left(\mathbf{A}^{-1}\mathbf{B}t + \left(\mathbf{A}^{-1}\right)^{2}\mathbf{B}\right)$$

- And since $\boldsymbol{e}(\infty) = t - \boldsymbol{y}(\infty)$ for a unit ramp, then the steady-state error

$$\boldsymbol{e}(\infty) = \lim_{t\to\infty}\left[\mathrm{t} + \mathbf{C}\left(\mathbf{A}^{-1}\mathbf{B}t + \left(\boldsymbol{A}^{-1}\right)^{2}\mathbf{B}\right)\right]$$

$$\boldsymbol{e}(\infty) = \lim_{t\to\infty}\left[(1 + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})t + \mathbf{C}\left(\boldsymbol{A}^{-1}\right)^{2}\mathbf{B}\right]$$

For $A \in \mathcal{R}^{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C = adj(A)^T = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

•1966•
جامعة الكويت
KUWAIT UNIVERSITY

$$\text{For } A \in \mathcal{R}^{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$C^T = adj(A) = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

جامعة الكويت
KUWAIT UNIVERSITY

Example 2

Compute the steady-state error for the following stable closed-loop system, for a unit step input

a. Using the final value theorem

b. Using input substitution

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}$$

جامعة الكويت
KUWAIT UNIVERSITY

Nise 6th Global Edition:

6-51, 7-54

جامعة الكويت
KUWAIT UNIVERSITY