Kuwait University
College of Engineering and Petroleum

جامعة الكويت
KUWAIT UNIVERSITY

ME417 CONTROL OF MECHANICAL SYSTEMS

Part II: Controller Design Via State-Space

Lecture 3: State-Space Controller Representation and Design

Summer 2020

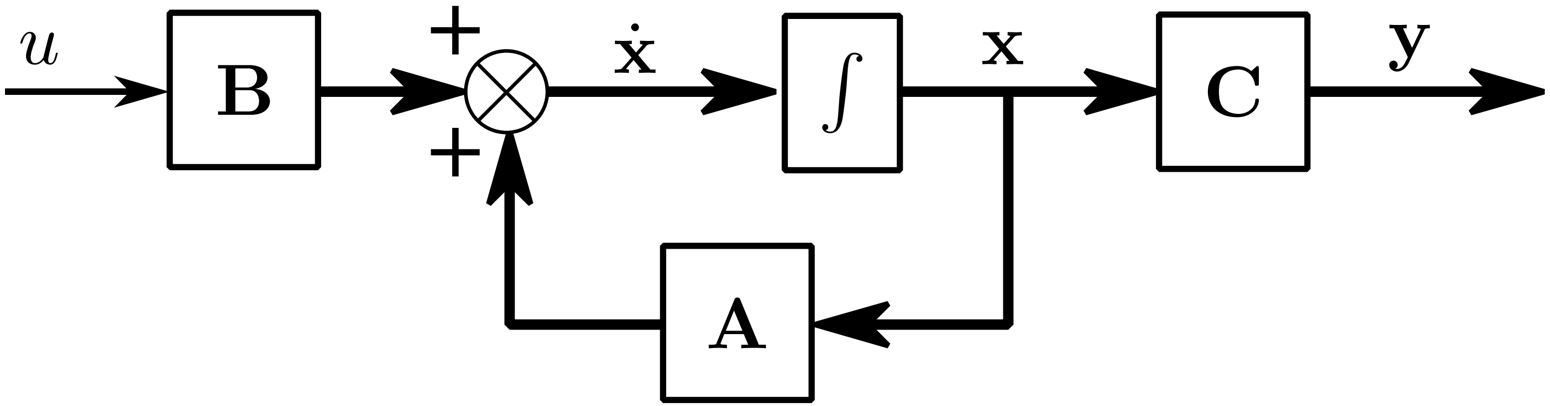
Ali AlSaibie

- Objectives:
  - Introduce the state-space feedback control form
  - Introduce the controller design by matching coefficients
  - Introduce the concept of controllability
- Reading:
  - *Nise: 12.1-12.3*
- Practice Problems Included

- The open-loop LTI system represented in state-space

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}u,\ u \in \mathcal{R}^1$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x}$$

- Can be represented in block diagram form
  - This is not a feedback loop, just a block diagram representation of the state-space equation

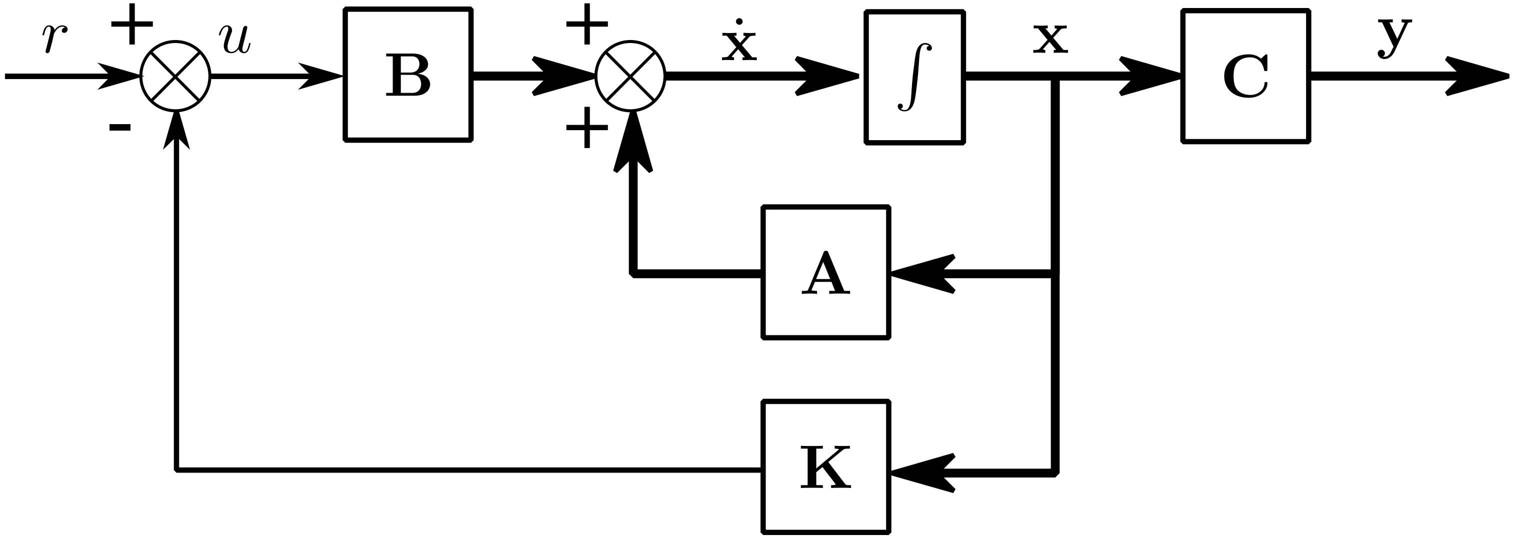


- There is no standard feedback form for systems in state-space. The form depends on the control method.
- A common feedback form for analyzing systems in state-space is the full state variable feedback control.
- With full-state feedback, it is assumed that all the system states $\boldsymbol{x}$ are available, the input becomes $u = r - \boldsymbol{K}\boldsymbol{x}, \quad \boldsymbol{K} \in \mathcal{R}^{1xn}, \boldsymbol{x} \in \mathcal{R}^{nx1}$

$$u = r - \begin{bmatrix} K_1 & K_2 & \dots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = r - (K_1x_1 + K_2x_2 + \cdots + K_nx_n)$$

- The gains in $\boldsymbol{K}$ are the controller gains in this architecture.
  - All the system states are *weighted* in computing the control action u

- The full state feedback, can be represented in block diagram as



$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}(r - \boldsymbol{K}\boldsymbol{x}) = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\boldsymbol{x} + \mathbf{B}r = \widetilde{\mathbf{A}}\boldsymbol{x} + \mathbf{B}r$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x}$$

- The eigenvalues of the new matrix $\widetilde{\mathbf{A}}$ give the ***closed-loop poles*** of the feedback system: $det\left(s\mathbf{I} - \widetilde{\mathbf{A}}\right) = \det(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\boldsymbol{K})) = 0$

- The output y and the output matrix $\mathbf{C}$ are normally set to reflect the sensors outputs.

- For example, if we have a motor model with $\boldsymbol{x} = \begin{bmatrix} \theta_m \\ \dot{\theta}_m \end{bmatrix}$ and a tachometer (rotational speed sensor) is the only sensor installed on the system.

- Then we only have one **output**, and the output $y = \dot{\theta}_m = [0 \quad 1] \begin{bmatrix} \theta_m \\ \dot{\theta}_m \end{bmatrix} = \mathbf{C}\boldsymbol{x}$

- With full state variable feedback, the design of the controller is done through pole placement (coefficient matching).

- Take the characteristic equation (the transfer function denominator) of the closed-loop system and equate it to a desired equivalent polynomial

$$det\left(s\mathbf{I}-\widetilde{\mathbf{A}}\right)=det(s\mathbf{I}-\mathbf{A}+\mathbf{B}\boldsymbol{K})=a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0$$

Where the constants $a_n, \ldots, a_0$ are **chosen** to result in a specific/desired closed loop pole locations, then the gains in $\boldsymbol{K}$ are computed by coefficient matching.

جامعة الكويت
KUWAIT UNIVERSITY

Example 1

Design a full state feedback controller to achieve a settling time of $T_s = 1s$ and damping ratio of $\zeta = 0.25$, for the following system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

Example 2

Design a full state feedback controller to achieve a settling time of $T_s = 1s$ and peak time of $T_p = 0.25s$, for the following system

$$G(s) = \frac{1}{s^2 + 4s + 8}$$

جامعة الكويت
KUWAIT UNIVERSITY

- Full state variable feedback is often unrealistic in real-world applications, and an observer is implemented instead.
    - Full state measurement means that we have sensors available to measure all the system states
    - An observer is a dynamic system, whose purpose is to ***estimate*** the states from the output y.
    - Observers are also called ***state estimators*** *(e.g.* Luenberger Observer, Recursive Least Squares Estimator, Kalman Filter*)*
    - Estimation Theory is a big subset of the world systems and control theory
        - It is also extremely beneficial in modern real-world applications
        - Artificial Intelligence has roots in Estimation Theory.

جامعة الكويت
KUWAIT UNIVERSITY

- For comparison, observe how a proportional feedback controller is represented in state-space
- For the following LTI system

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}u,\ u \in \mathcal{R}^1$$
$$\boldsymbol{y} \in \mathcal{R}^1 = \mathbf{C}\boldsymbol{x}$$

- The proportional controller is defined as

$$u = K_p e = K_p(r - y) = K_p(r - \mathbf{C}\boldsymbol{x})$$

- Substituting in the state-space form,

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \mathbf{B}K_p\mathbf{C})\boldsymbol{x} + \mathbf{B}K_p r$$
$$\boldsymbol{y} \in \mathcal{R}^1 = \mathbf{C}\boldsymbol{x}$$

- Comparing full state feedback to a proportional controller
  - Full State Feedback: $u = r - \boldsymbol{K}\boldsymbol{x} = r - (K_1 x_1 + \cdots + K_n x_n)$
  - Proportional Controller for $y = x_1$: $u = K_p(r - x_1)$

جامعة الكويت
KUWAIT UNIVERSITY

- What if you required the behavior of a second order system, but the system is higher than 2nd order, or if there is zero or more in the system?
- Several Options
  - Place the third, and higher poles, further into the LHP
  - If there is a **zero near the dominant poles**, place the third pole **on the zero** to cancel its effect.
- Remember that, with Full-State Feedback pole placement, if the system is **controllable**, you literally can place the poles where you desire.

Design a full state feedback controller to achieve a settling time of $T_s = 1s$ and damping ratio $\zeta = 0.5$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}$$

جامعة الكويت
KUWAIT UNIVERSITY

- Can we always apply full state feedback?
- Can we always control all the states of the system?
- Can we always place the n poles of the closed-loop system where we want?
- The answer to the above questions is, no, we can not always achieve those outcomes.
- For the above outcomes to be met, the system must be **Controllable**

"If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be ***controllable****; otherwise, it is* ***uncontrollable****"*

جامعة الكويت
KUWAIT UNIVERSITY

- Consider the following system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u$$

- Or

$$\begin{aligned} \dot{x}_1 &= -5x_1 \\ \dot{x}_2 &= -2x_2 + 2u \\ \dot{x}_3 &= -x_3 + u \end{aligned}$$

- Note that the states are decoupled, since one does not influence the other.
- The controller u only affects x_2 and x_3
- So, neither u affects x_1 nor the other states affect x_1 by proxy
- Thus, state x_1 is **uncontrollable**: The system is **uncontrollable**

جامعة الكويت
KUWAIT UNIVERSITY

- Consider the following system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

- Or

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -5x_1 - 3x_2 - x_3 + u$$

- Note that the states are coupled in $\dot{x}_3$
- The controller u affects $\dot{x}_3$
- And $\dot{x}_3$ is coupled with all states, so u affects all states by proxy.
- Thus, all states are **controllable**: The system is **controllable**

- The solution is generalized by considering the rank of the controllability matrix $\mathbf{C_M}$
- For a given dynamic system $\dot{\boldsymbol{x}} \in \mathcal{R}^n = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}$, the system is controllable if the rank of the controllability matrix C_M is n: The number of states of the system
- The controllability matrix for a system with n states is defined as

$$\mathbf{C_M} \in \mathcal{R}^{n\times n} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B} \quad \cdots \quad \mathbf{A^{n-1}B}]$$

- A sufficient condition for $rank(\mathbf{C_M}) = n$ is that

$$\det(\boldsymbol{C_M}) \neq 0$$

- In other words, if $\det(\boldsymbol{C_M}) \neq 0$, the system is **controllable**; otherwise, it is **uncontrollable**

Example 4

Can full state feedback be applied to the following system?

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & -5 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

جامعة الكويت
KUWAIT UNIVERSITY

Practice Problems

Nise 6th Global Edition:

12-4, 12-6, 12-7, 12-18

جامعة الكويت
KUWAIT UNIVERSITY