ME 417 - Homework #3

Control of Mechanical Systems - Summer 2020

Homework Due: Sun, 08 Nov 2020 23:59

Complete the following problems and submit a hard copy of your solutions. You are encouraged to work together to discuss the problems but submitted work **MUST** be your own. This is an **individually** submitted assignment.

Problem 1

Root Locus Sketching (20pts)

For each of the following transfer functions, sketch a general shape root-locus, and include, as applicable, asymptote intercepts and angles

a. $G(s) = \dfrac{s+5}{s^2+2s+6}$

b. $G(s) = \dfrac{(s-2)(s+10)}{s(s^2+4s+16)}$

c. $G(s) = \dfrac{s^2+4s+25}{s(s-8)(s-5)}$

d. $G(s) = \dfrac{(s-20)(s+4)}{s^2(s^2+4s+60)}$

Solution:

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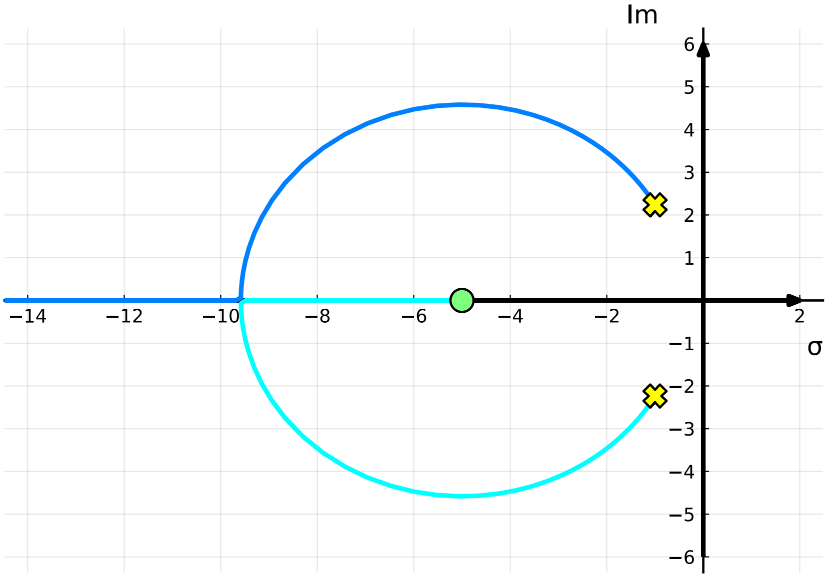
a.

The root-locus is shown below



We have $P_n - Z_n = 2 - 1 = 1$ asymptotes.

There is only one asymptote and it is on the real axis.

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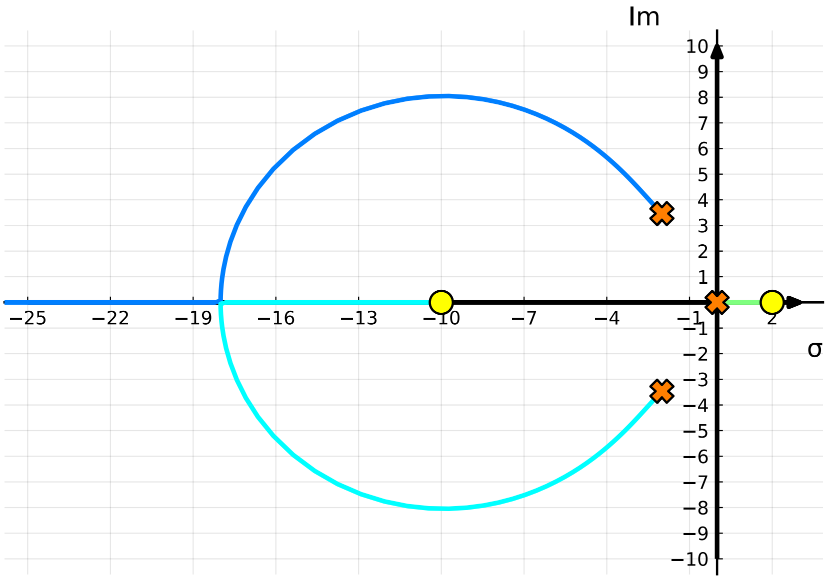
b.

The root-locus is shown below



We have $P_n - Z_n = 3 - 2 = 1$ asymptotes.

There is only one asymptote and it is on the real axis.

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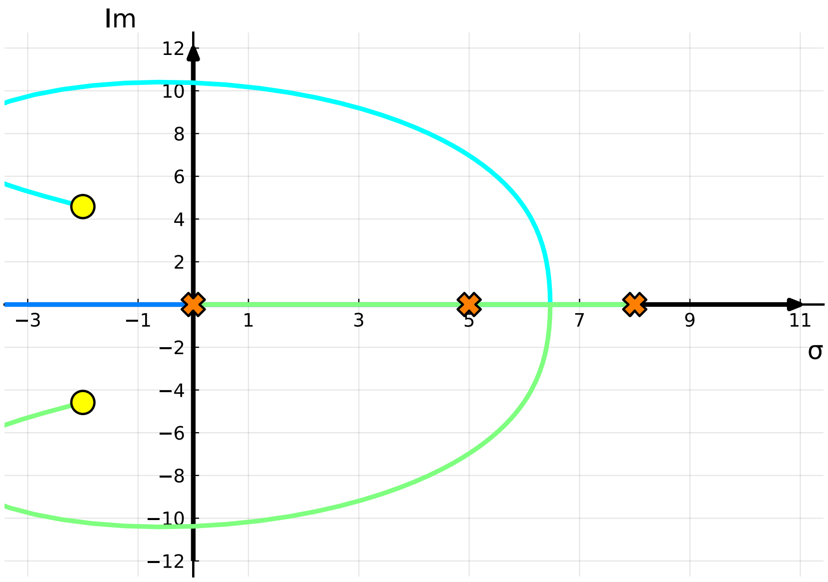
c.

The root-locus is shown below



We have $P_n - Z_n = 3 - 2 = 1$ asymptotes.

There is only one asymptote and it is on the real axis.

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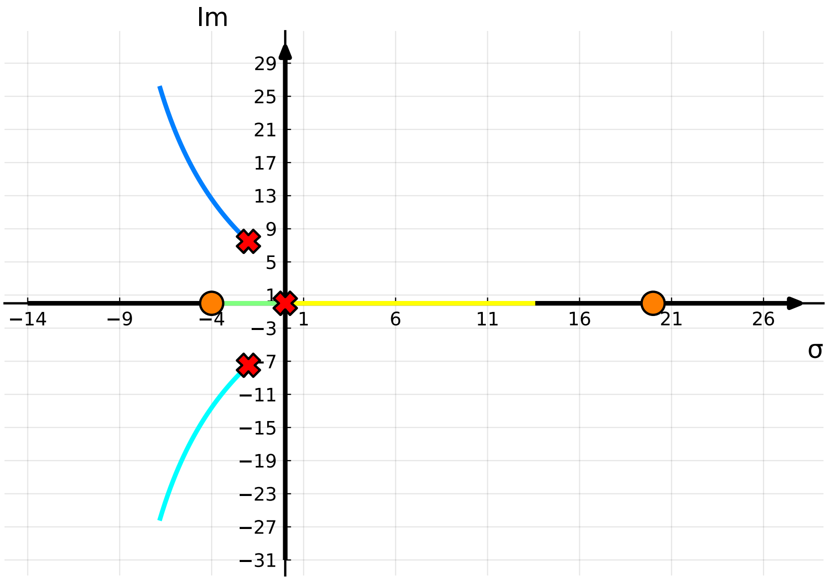
d.

The root-locus is shown below



We have $P_n - Z_n = 4 - 2 = 2$ asymptotes.

To find the the intercept of the asymptotes, we take the difference between the sum of the real locations of poles and real locations of zeros, divided by the number of asymptotes.

$$\frac{real(P) - real(Z)}{N_{asymptotes}} = \frac{-4.0 - 16.0}{2} = -10.0$$

And we have the asymptotes angles $\theta a = \frac{(2k+1)\pi}{2}, \forall k = 0:1 = \begin{bmatrix} 1.57 \\ 4.71 \end{bmatrix}$

Problem 2

Root Locus Sketching (20pts)

For the following open-loop transfer functions, sketch a refined root locus, compute any applicable break-away and break-in points as well as imaginary axis crossing. Highlight the range of K for which the system is stable.

a. $G(s) = \dfrac{s+5}{s^2+10s+61}$

b. $G(s) = \dfrac{(s-6)(s-1)}{(s+3)(s+8)}$

c. $G(s) = \dfrac{(s+10)(s+20)}{(s-8)(s-5)}$

d. $G(s) = \dfrac{(s-19)(s+20)}{s^2-4s+25}$

Solution:

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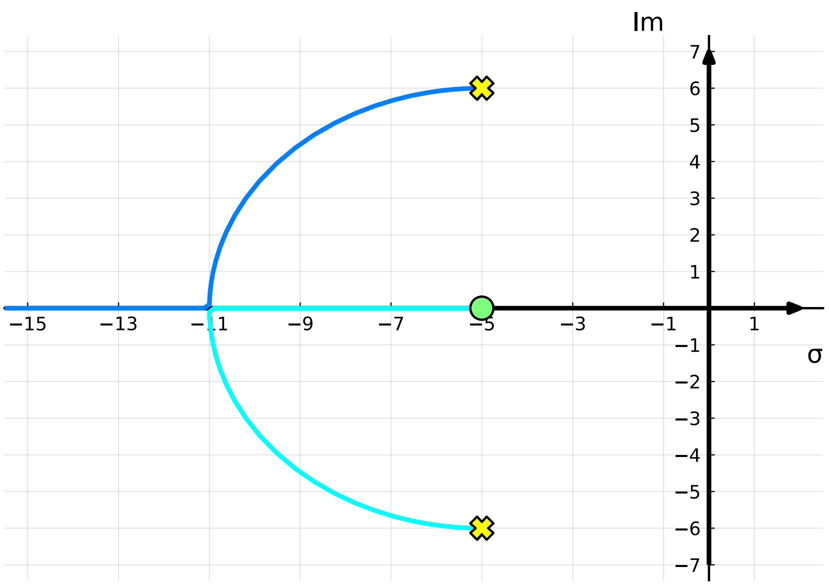
a.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\dfrac{\delta K(\sigma)}{\delta\sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 + 10.0\sigma + 61.0}{1.0\sigma + 5.0}$$

$$\frac{\delta K(\sigma)}{\delta \sigma} = \frac{-0.04\sigma^2 - 0.4\sigma + 0.44}{0.04\sigma^2 + 0.4\sigma + 1.0} = 0$$

Solving for σ we get $\left[-11.0\right]$, corresponding to $K = \left[12.0\right]$

There is no ωj crossing in this root-locus

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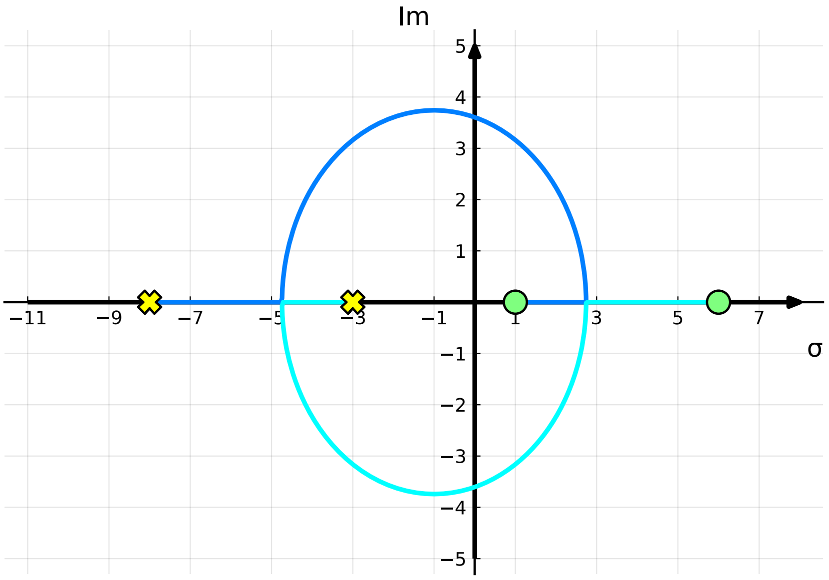
b.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\frac{\delta K(\sigma)}{\delta \sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 + 11.0\sigma + 24.0}{1.0\sigma^2 - 7.0\sigma + 6.0}$$

$$\frac{\delta K(\sigma)}{\delta\sigma} = \frac{1.0\left(0.367\sigma^2 + 0.735\sigma - 4.78\right)}{0.0204\sigma^4 - 0.286\sigma^3 + 1.24\sigma^2 - 1.71\sigma + 0.735} = 0$$

Solving for σ we get $\begin{bmatrix} -4.74 \\ 2.74 \end{bmatrix}$, corresponding to $K = \begin{bmatrix} 0.092 \\ 10.9 \end{bmatrix}$

To find the ωj crossing, we subsitute s in the characteristic polynomial $1 + KG(s) = 0$ with $\omega_d j$ and solve.

$1 + KG(s) = 0.0417Ks^2 - 0.292Ks + 0.25K + 0.0417s^2 + 0.458s + 1.0 = 0$, substituting for s, we get $1 + KG(\omega j) = -0.0417K\omega d^2 - 0.292iK\omega d + 0.25K - 0.0417\omega d^2 + 0.458i\omega d + 1.0 = 0.$ Solving for K and ω_d we get

$\omega_d = -3.61, K = 1.57$

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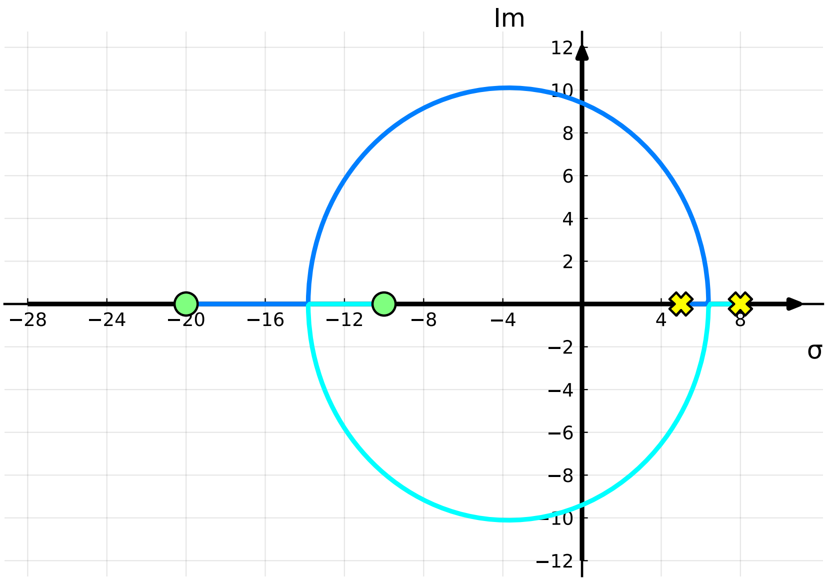
c.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\frac{\delta K(\sigma)}{\delta\sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 - 13.0\sigma + 40.0}{1.0\sigma^2 + 30.0\sigma + 200.0}$$

$$\frac{\delta K(\sigma)}{\delta\sigma} = \frac{-0.00108\sigma^2 - 0.008\sigma + 0.095}{2.5\cdot 10^{-5}\sigma^4 + 0.0015\sigma^3 + 0.0325\sigma^2 + 0.3\sigma + 1.0} = 0$$

Solving for σ we get $\begin{bmatrix} 6.39 \\ -13.8 \end{bmatrix}$, corresponding to $K = \begin{bmatrix} 0.00517 \\ 17.4 \end{bmatrix}$

To find the ωj crossing, we subsitute s in the characteristic polynomial $1 + KG(s) = 0$ with $\omega_d j$ and solve.

$1 + KG(s) = 0.025Ks^2 + 0.75Ks + 5.0K + 0.025s^2 - 0.325s + 1.0 = 0$, substituting for s, we get

$1 + KG(\omega j) = -0.025K\omega d^2 + 0.75iK\omega d + 5.0K - 0.025\omega d^2 - 0.325i\omega d + 1.0 = 0$. Solving for K and ω_d we get

$\omega_d = -9.4, K = 0.433$

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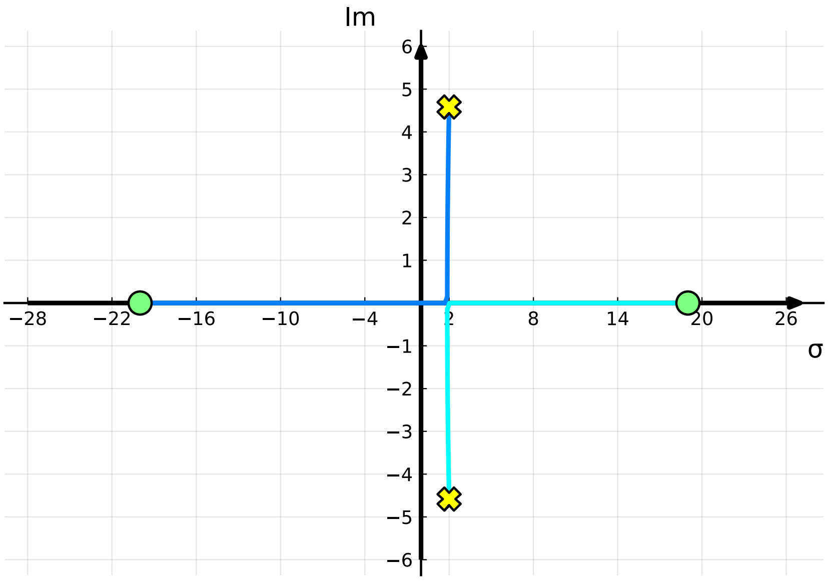
d.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\frac{\delta K(\sigma)}{\delta \sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 - 4.0\sigma + 25.0}{1.0\sigma^2 + 1.0\sigma - 380.0}$$

$$\frac{\delta K(\sigma)}{\delta \sigma} = \frac{-3.46 \cdot 10^{-5}\sigma^2 + 0.00561\sigma - 0.0104}{6.93 \cdot 10^{-6}\sigma^4 + 1.39 \cdot 10^{-5}\sigma^3 - 0.00526\sigma^2 - 0.00526\sigma + 1.0} = 0$$

Solving for σ we get $\left[1.87\right]$, corresponding to $K = \left[0.0561\right]$

To find the ωj crossing, we subsitute s in the characteristic polynomial $1 + KG(s) = 0$ with $\omega_d j$ and solve.

$1 + KG(s) = 0.04Ks^2 + 0.04Ks - 15.2K + 0.04s^2 - 0.16s + 1.0 = 0$, substituting for s, we get

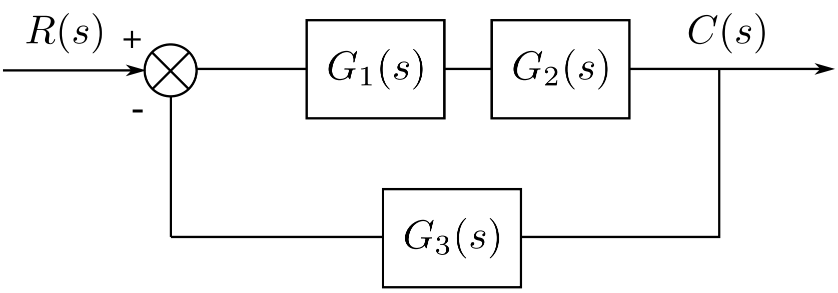
$1 + KG(\omega j) = -0.04K\omega d^2 + 0.04iK\omega d - 15.2K - 0.04\omega d^2 - 0.16i\omega d + 1.0 = 0$. Solving for K and ω_d we get

$\omega_d = 0.0, K = 0.0658$

Problem 3

Root Locus Sketching (20pts)

Given the following feedback system



With $G_1 = s + z, G_2 = \dfrac{10}{s^2 + 4s + 10}, G_3 = \dfrac{10}{s + 100}$

a. Derive the characteristic polynomial of the system in the form $1 + zG(s) = 0$

b. Sketch the root-locus of the system for varying values of the zero location z

c. Find the value of z that makes the closed-loop system's damped frequency $\omega_d = \pi rad/s$

Note that this is almost a PD controller design problem, but rather than try to find the zero location from the proportional gain root-locus, the characteristic polynomial is rearranged in a way to make the zero act like the system gain.

Solution:

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a.

From the feedback block diagram, the characteristic polynomial of the closed-loop system is $1 + \dfrac{100\,(s+z)}{(s+100)\,(s^2+4s+10)} = 0$, rearranging, we get

$100s + 100z + (s+100)\left(s^2+4s+10\right) = 0$, dividing by $100s + (s+100)\left(s^2+4s+10\right)$, we get

$$\frac{100z}{100s + (s+100)\,(s^2+4s+10)} + 1 = 0$$

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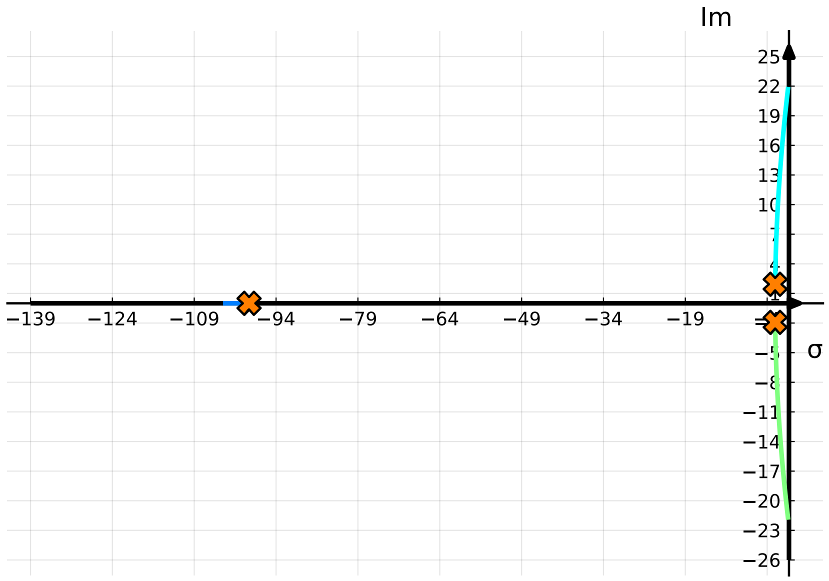
b. The open loop transfer function is then $G(s) = \dfrac{100}{100s + (s+100)\,(s^2+4s+10)}$, the root-locus given this open-loop transfer function is shown



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c.

The root-locus does interest the $\omega_d = \pi$ line. We can approximate the location to be at $s_d = -2.0 + 3.14im$

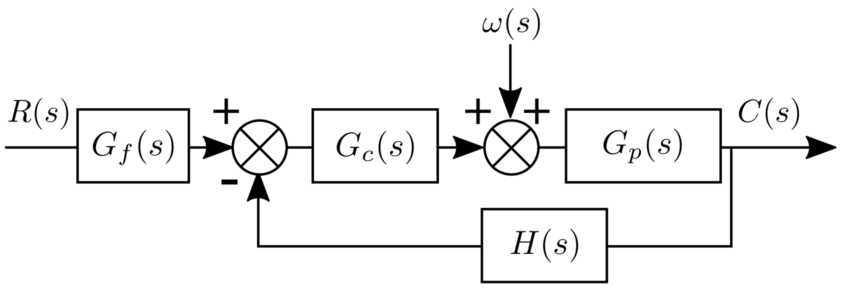
To find the gain, we can use the magnitude condition: $K = \dfrac{\prod L_p}{\prod L_z} = \dfrac{652.0}{1.0} = 652.0$

The gain of the controller is given by $K_c = K/K_{Gol} = 652.0/100.0 = 6.52$

Problem 4

Root Locus Sketching (20pts)

A closed-loop system with input disturbance is shown.



With $G_p = \dfrac{2s+2}{(s-2)\,(s+5)}, H = \dfrac{3}{s+1}, G_f = 5$

a. Design a controller that results in a stable response with

- $T_p = \dfrac{\pi}{2}s$
- Zero Steady-State error

Is a second-order approximation valid? Justify. Hint: Choose a convenient constraint for ζ or T_s to simplify your calculations.

b. Show that a steady-state error of the closed loop system, to a step input, is zero.

c. Given your designed controller, derive the transfer function that relates the input $r(t)$ to the controller output $u(t)$

Solution:

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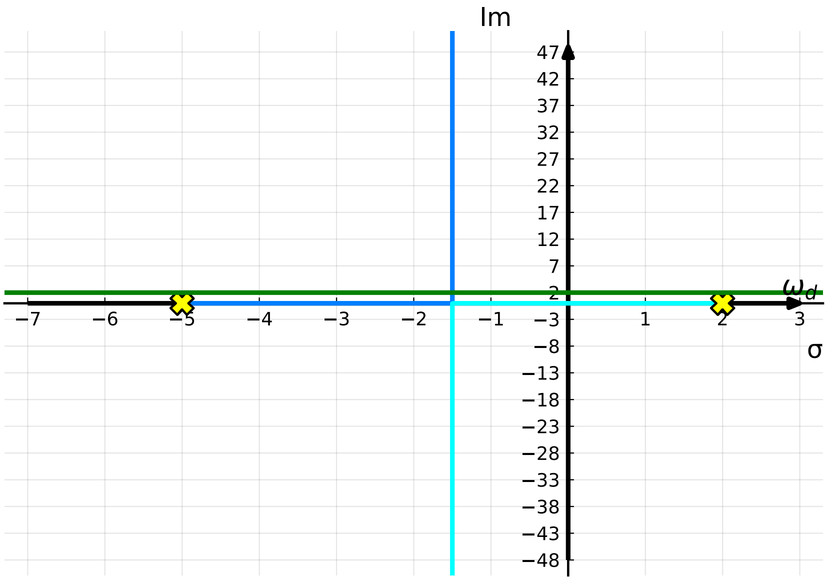
a.

The open-loop transfer function is $K\dfrac{6.0}{1.0s^2+3.0s-10.0}$

Let's first define the design point

The root-locus with the design criteria is shown



The root-locus does intersect the design point/line, the gain can be calculated geometrically or by substituting for the constraint in the characteristic equation and solving. The gain value at the intercept is $K = 2.71$

Factoring out the plant gain 6.0, the controller is then $G_c = 0.452$

The system is in fact second order.

To make the system have zero steady-state error, we can add a PI controller of the form $G_{PI} = \frac{s + 0.01}{s}$, but that would result in a closed-loop system that is unstable, since there will always be a segment in the RHP. $Gc = 0.452$

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b.

The steady-state error for the system is note zero, since the open-loop is type 0 and we can't add a PI controller.

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c.

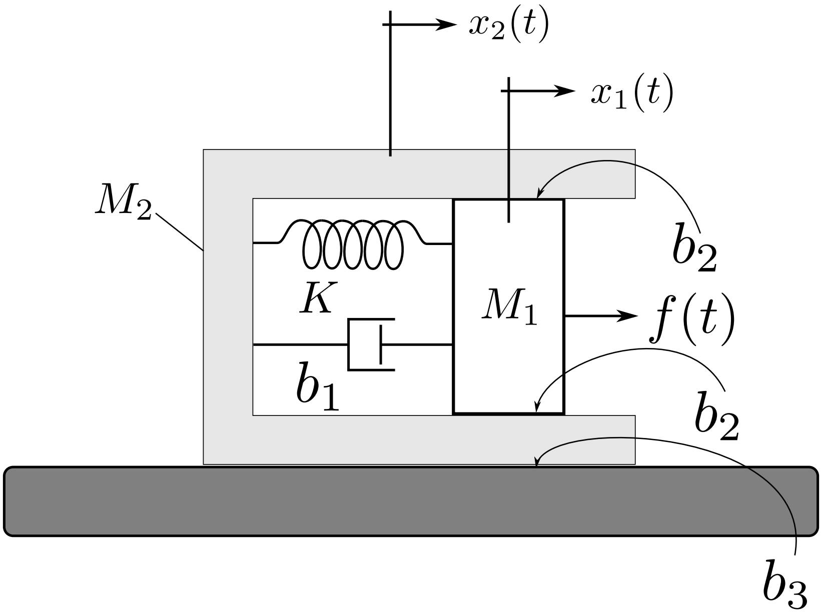
The closed-loop transfer function relating r to u is

$$G_f \frac{G_c}{1 + G_c G_p H} = \frac{2.26\,(s-2)\,(s+1)\,(s+5)}{2.712s + (s-2)\,(s+1)\,(s+5) + 2.712} = \frac{2.26s^2 + 6.78s - 22.6}{1.0s^2 + 3.0s - 7.29}$$

Problem 5

Root Locus Sketching (20pts)

Given the mechanical system shown on the figure. You can use MATLAB to aid in long calculations and verify your work.



With $M_1 = 5kg, M_2 = 3kg, b_1 = 5N \cdot s/m, b_2 = 5N \cdot s/m, b_3 = 5N \cdot s/m, K = 10N/m$

a. Derive the equations of motion for the system

b. Find the transfer function relating the input $f(t)$ to $x_2(t)$, $G_2(s) = \dfrac{X_2(s)}{f(s)}$

c. Analyze the stability of the system $G_2(s)$

d. Design a feedback controller, using root-locus technique, around G_2 to achieve

- Zero Steady-State error

- $T_s = 1s$

- $\zeta = 0.866$

Justify if the system can be approximated as second order.

e. Derive the transfer function relating the reference $r(t)$ to $x_2(t)$

f. Derive the transfer function relating the reference $r(t)$ to $x_1(t)$, with the feedback system derived above.

Solution:

a. The equations of motion for the system

$$K\left(x_1 - x_2\right) + M_1 \frac{d^2}{dt^2} x_1 - f + \left(b_1 + 2b_2\right)\left(\frac{d}{dt} x_1 - \frac{d}{dt} x_2\right) = 0$$

$$-K\left(x_1 - x_2\right) + M_2 \frac{d^2}{dt^2} x_2 + b_3 \frac{d}{dt} x_2 + \left(-b_1 - 2b_2\right)\left(\frac{d}{dt} x_1 - \frac{d}{dt} x_2\right) = 0$$

Taking the Laplace Transform, we get

$$K\left(X_1 - X_2\right) + M_1 X_1 s^2 - f + \left(b_1 + 2b_2\right)\left(X_1 s - X_2 s\right) = 0$$

$$-K\left(X_1 - X_2\right) + M_2 X_2 s^2 + X_2 b_3 s + \left(-b_1 - 2b_2\right)\left(X_1 s - X_2 s\right) = 0$$

b. Grouping the terms as $Ax = B$

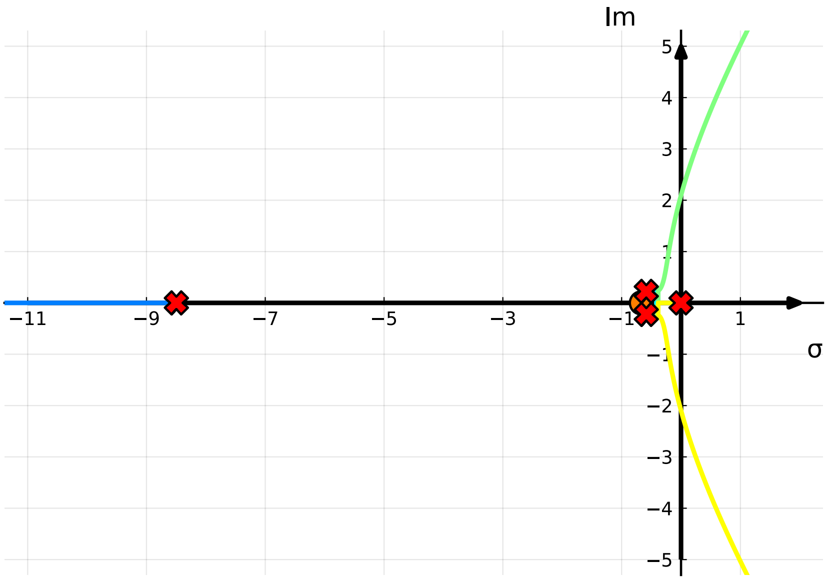
$$\begin{bmatrix} K + M_1 s^2 + b_1 s + 2b_2 s & -K - b_1 s - 2b_2 s \\ -K - b_1 s - 2b_2 s & K + M_2 s^2 + b_1 s + 2b_2 s + b_3 s \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for x, we find the two transfer functions relating the inputs to the two outputs of the system.

$$G_2(s) = \frac{K + b_1 s + 2b_2 s}{KM_1 s^2 + KM_2 s^2 + Kb_3 s + M_1 M_2 s^4 + M_1 b_1 s^3 + 2M_1 b_2 s^3 + M_1 b_3 s^3 + M_2 b_1 s^3 + 2M_2 b_2 s^3 + b_1 b_3 s^2 + 2b_2 b_3 s^2}$$

$$= \frac{3s + 2}{s\left(3s^3 + 29s^2 + 31s + 10\right)}$$

c. The root-locus of the system is shown



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d.

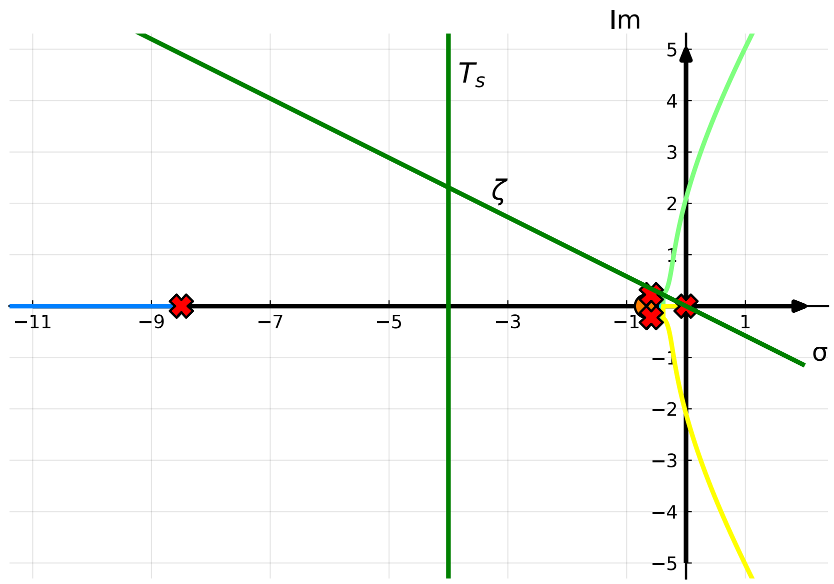
To get the desired closed-pool, a PD controller is required.

The open-loop transfer function is $G_c = \frac{1.0s + 0.667}{1.0s^4 + 9.67s^3 + 10.3s^2 + 3.33s}$.

Given the design requirements ζ and T_s, we get $\sigma = -\frac{4}{T_s} = -4.0$, $\omega_n = \frac{-\sigma}{\zeta} = 4.62$ and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.31$:

a design point at $s = \sigma \pm \omega_d i = -4.0 \pm 2.31i$

Sketching the root-locus, with the design requirements, we get



The controller $G_{PD} = K$ is not sufficient to place the closed-loop pole in the desired location. A PD, $G_{PD} = K(s+z)$, controller can be used to place the root-locus over the design point.

Find the location of the zero using the angle condition. Where θ_{zn} is the angle contribution of the added zero.

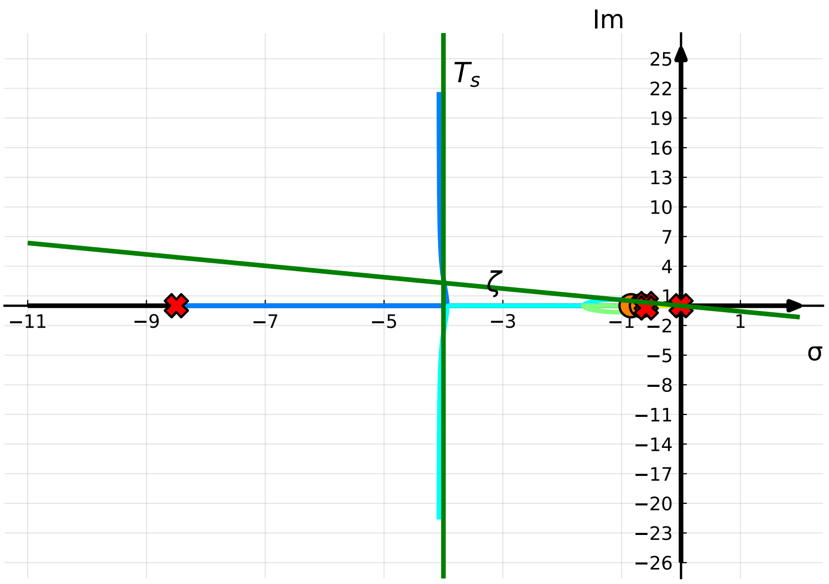
$\angle KG(s) = \sum \theta_p - \sum \theta_z = \pm(2k+1)180 = 8.19 - \theta_{addz} - 2.54 \longrightarrow \theta_{addz} = 5.65 - \pi = 2.51$

Solving for the zero location: $tan(\theta_{zn}) = \dfrac{im(s_d)}{re(s_d)+z} \longrightarrow z = \dfrac{im(s_d)}{tan(\theta_{zn})} - re(s_d) = 0.842$

The PD controller is now $G_{PD} = K(s+z_n) = K(s+0.842)$, and the gain can be computed by the magnitude condition.

To find the gain, we can use the magnitude condition: $K = \dfrac{\prod L_p}{\prod L_z} = \dfrac{397.0}{15.9} = 25.0$
The gain of the controller is given by $K_c = K/K_{Gol} = 25.0/1.0 = 25.0$

And the PD controller becomes $G_{PD} = 25.0(s+0.842)$



e.

This is the closed-loop transfer function.

$$G_{cl2} = \frac{G_cG_2}{1+G_cG_2} = \frac{1.0\left(75.0s^2+113.15s+42.1\right)}{3.0s^4+29.0s^3+106.0s^2+123.15s+42.1}$$

f.

$$G_{cl2} = \frac{G_c G_1}{1 + G_c G_2} = \frac{1.0\left(75.0s^3 + 563.15s^2 + 671.0s + 210.5\right)}{15.0s^4 + 145.0s^3 + 530.0s^2 + 615.75s + 210.5}$$