Kuwait University
College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

Part I: Introduction to Feedback Control

Lecture 10: Steady-State Error

Summer 2020

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- Objectives:
  - *Define Steady-State Error*
  - *Understand how to calculate the steady-state error for different test inputs*
  - *Introduce the concept of system types*
- Reading:
  - *Nise: 7.1-7.4*
- Practice Problems Included

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What is Steady-State Error?

- In control system analysis and design we focus on three performance aspects
  1. Stability:
     - A system must first be stable/stabilizable before we go any further.
  2. Transient Response
  3. Steady-State Error
- *Steady-State Error* for a system, is the difference between the input and the output of a system for a prescribed test input, as $t \to \infty$
  - Prescribed test inputs:
    - Step-Input: $R(s) = \frac{1}{s}, r(t) = 1$
    - Ramp-Input: $R(s) = \frac{1}{s^2}, r(t) = t$
    - Parabola-Input: $R(s) = \frac{1}{s^3}, r(t) = \frac{1}{2}t^2$
- In terms of error: Steady-State Error $e_{ss} = e(t = \infty) = c(\infty) - r(\infty)$

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- With LTI systems, calculating the steady-state error can be done using the final value theorem.
- The F.V.T is derived from the Laplace Transform of the derivative

$$\mathcal{L}[\dot{f}(t)] = \int_{0-}^{\infty} \dot{f}(t)e^{-st}dt = sF(s) - f(0-)$$

$$s \to 0$$

$$\int_{0-}^{\infty} \dot{f}(t)e^{-st}dt = f(\infty) - f(0\,-) = \lim_{s\to 0} sF(s) - f(0-)$$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

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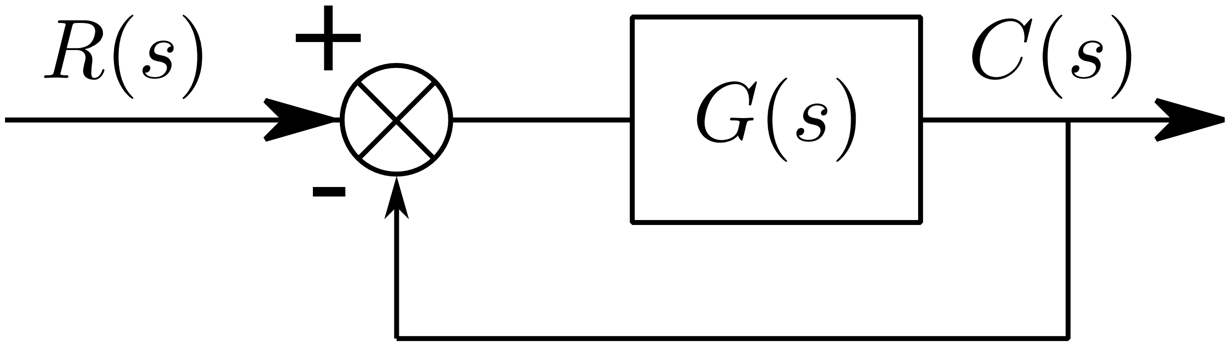
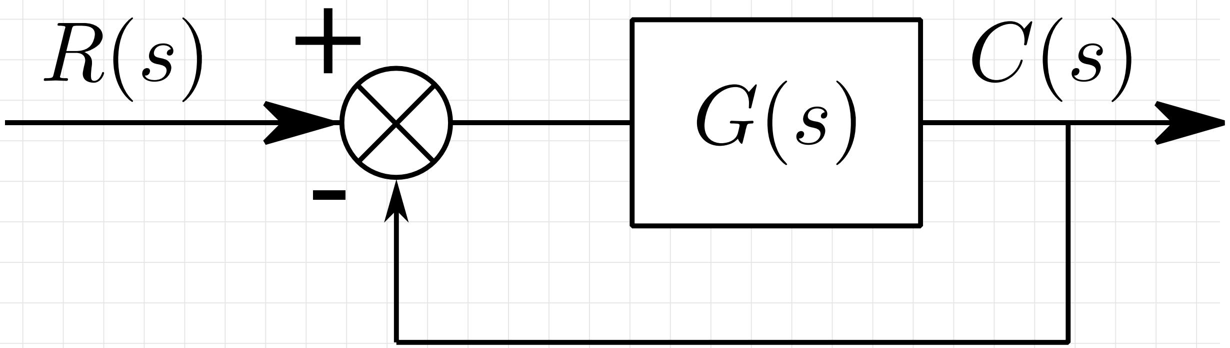
Steady-State Error in Terms of the Transfer Function

- The Error $E(s)$, is only defined in the context of a feedback system
- For a unity feedback control system with a unity proportional gain, the block diagram is shown below. Here, the error:
  - $E(s) = R(s) - C(s) = R(s) - E(s)G(s)$
  - $E(s) = \frac{R(s)}{1+G(s)}$
- From F.V.T:

$$e(\infty) = \lim_{s\to 0} sE(s) = lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$



Find the steady state error in a unity gain ($G_c(s) = 1$), unity feedback ($H(s) = 1$) control system, for each of the following open-loop transfer functions:



a. $G_1(s) = \frac{2}{s+4}$, to a step input

b. $G_2(s) = \frac{2}{s(s+4)}$, to a step and ramp inputs

c. $G_3(s) = \frac{2}{s^2(s+4)}$, to a step, ramp and parabola inputs

- In determining steady state error, LTI systems are also classified into "Types", by the order of the integrator.
- **Type 0** for systems with no integrator
  - They have a finite e_{ss} to step input, and $e_{ss} = \infty$ to both ramp and higher order inputs
- **Type 1** for first order integrator
  - They have zero e_{ss} for step input, finite for ramp and $e_{ss} = \infty$ for higher order inputs
- **Type 2** for second order integrator (a.k.a double integrator)
  - They have zero e_{ss} for step and ramp, finite for parabola, and $e_{ss} = \infty$ for higher order inputs

$$G(s) = \frac{K(s+z_1)\dots(s+z_i)}{s^n(s+p_1)\dots(s+p_m)}$$

Order of Integrator in transfer function determines system type

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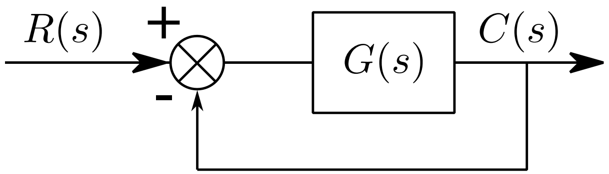
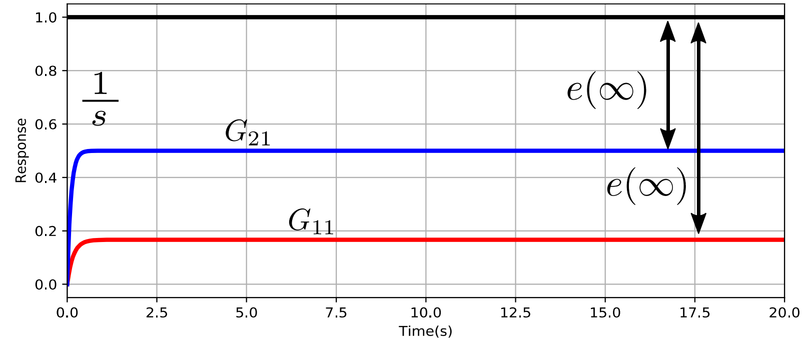
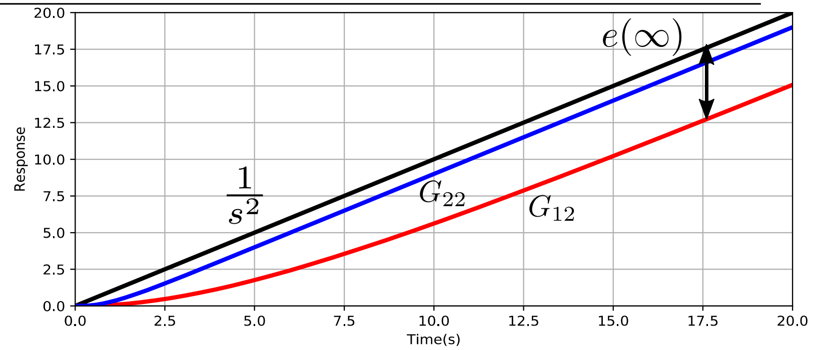
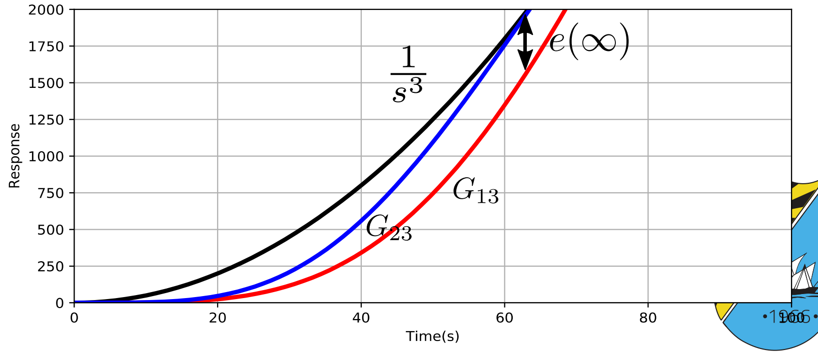
Unity Feedback Response Given the Open-Loop Transfer Function



	System Type	Transfer Function	Input
Step Input	Type 0	$G_{11} = \frac{1}{(s+5)}$	$R(s) = \frac{1}{s}$
	Type 0	$G_{21} = \frac{5}{(s+5)}$	$R(s) = \frac{1}{s}$
Ramp Input	Type 1	$G_{12} = \frac{1}{s(s+5)}$	$R(s) = \frac{1}{s^2}$
	Type 1	$G_{22} = \frac{5}{s(s+5)}$	$R(s) = \frac{1}{s^2}$
Parabola Input	Type 2	$G_{21} = \frac{1}{s^2(s+5)}$	$R(s) = \frac{1}{s^3}$
	Type 2	$G_{21} = \frac{5}{s^2(s+5)}$	$R(s) = \frac{1}{s^3}$







$$G(s) = \frac{K(s+z_1)\ldots(s+z_i)}{s^n(s+p_1)\ldots(s+p_m)}$$

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Static Error Constants

- From: $e(\infty) = \lim_{s\to 0} sE(s) = lim_{s\to 0} \frac{sR(s)}{1+G(s)}$
- For a Type 0 system with a Step Input:
  - $e(\infty) = lim_{s\to 0} \frac{s/s}{1+G(s)} = lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+lim_{s\to 0} G(s)}$
- For a Type 1 system with a ramp input:
  - $e(\infty) = lim_{s\to 0} \frac{s/s^2}{1+G(s)} = lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{lim_{s\to 0} sG(s)}$
- For a Type 2 system with a parabola input:
  - $e(\infty) = lim_{s\to 0} \frac{s/s^3}{1+G(s)} = lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{lim_{s\to 0} s^2G(s)}$
- The static error constant for each system type is the value of the respective denominator limit

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- The position constant, $K_p = lim_{s\to 0}G(s)$
  - For a Type 0 system, $K_p = constant$ and $e_{ss} = \frac{1}{1+K_p}$, to a step input
- The velocity constant, $K_v = lim_{s\to 0}sG(s)$
  - For a Type 1 system, $K_v = constant$ and $e_{ss} = \frac{1}{K_v}$, to a ramp input
- The acceleration constant, $K_a = lim_{s\to 0}s^2G(s)$
  - For a Type 2 system, $K_a = constant$ and $e_{ss} = \frac{1}{K_a}$, to a parabola input

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Steady-State Error

The following table summarizes the relationships between system type, steady-state-error, input, and static error constant.

		Type 0		Type 1		Type 2	
Input	**Steady-State Error**	**Static Error Constant**	**Error**	**Static Error Constant**	**Error**	**Static Error Constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = constant$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = constant$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = constant$	$\frac{1}{K_a}$

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The PID Gains

$$K_p, K_i, K_d$$

are **NOT** related* to the static error constants

$$K_p, K_v, K_a$$

*If a PID controller is used, the static error constants of the CL system will be influenced.

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Example

A unity feedback system has the following forward transfer function.

$$G(s) = \frac{10(s+4)}{s(s+3)(s+2)}$$

a. Evaluate system type, K_p, K_v, K_a

b. Use your answers to a. to find the steady-state errors for the standard step, ramp and parabolic inputs.

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- The static error constants are useful in inferring information regarding a closed loop system.
- Example: If we are told that for a system, the static error constant $K_v = 10$, we can infer the following
  - The system is stable
  - The system has a single integrator, and of Type 1. Since only Type 1 systems have a finite K_v value (not zero, nor ∞)
  - A ramp input is the test signal.
  - The steady-state error $e_{ss} = \frac{1}{K_v} = \frac{1}{10} = 0.1$, for a ramp input.
  - The steady-state error for a step input is zero.
  - The steady-state error for a parabola input is ∞.

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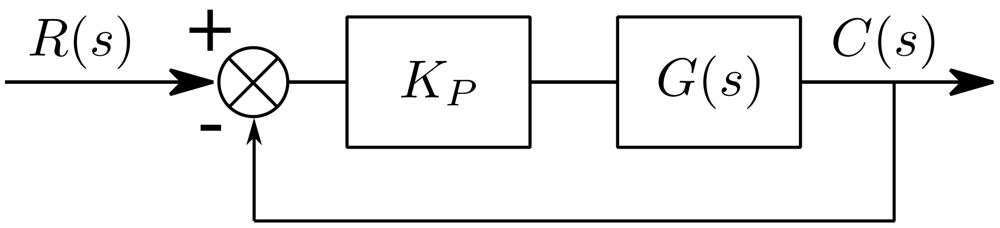
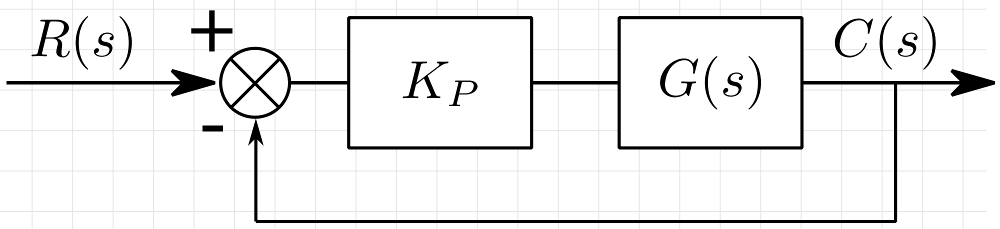
Steady-State Error Specification – Another perspective

- The order of system types 0, 1, 2 is analogous to the ability to track position, velocity and acceleration, respectively, <u>under the following conditions:</u>
  - Given the application of a proportional controller on a system, in a unity feedback system:
  - If the output $C(s)$ of a transfer function is <u>position</u>, and the input is a step-input:
- Then for said transfer function:
- If it's a Type 0 TF, we can tune the gain K to track position and reduce $e_{position\_ss}$, but with this controller we can't track a desired velocity nor acceleration with finite e_{ss}.
- If it's a Type 1 TF, we can track position with $e_{position\_ss} = 0$, and reduce $e_{velocity\_ss}$, but we can't track a desired acceleration
- If it's a Type 2 TF, we can track position and velocity with $e_{position\_ss} = e_{velocity\_ss} = 0$, and are able to reduce the acceleration steady-state error
- Note that the above logic applies if the transfer function's output is position: $G(s) = \frac{\Theta(s)}{T(s)}$ for instance, and input is a step $R(s) = \frac{k}{s}$



Example

Show that it is not possible to track velocity, given a step input, for the given feedback system, where the transfer function that relates torque to angular position is $G(s) = \frac{\Theta(s)}{T(s)} = \frac{5}{(s+10)}$



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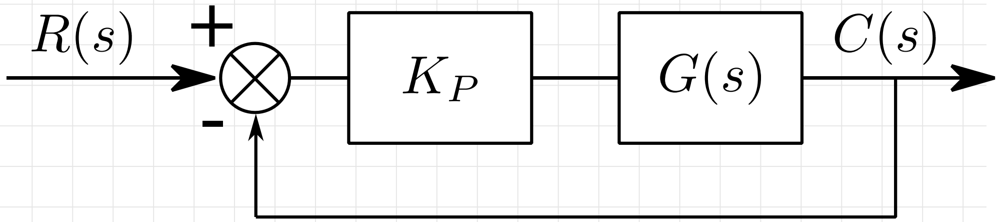
Example

The transfer function relating angular position to torque for a given mechanical system is $G(s) = \frac{\Theta(s)}{T(s)} = \frac{5}{s^2(s+5)(s+3)}$, if a unity gain, unity feedback system is applied to this system. Determine the angular velocity steady-state error to a ramp torque input with magnitude $3N \cdot m$



This is a type 2 system (w.r.t position as output and torque as input), due to the presence of a double integrator, and the transfer function output is position. The system is also part of a unity feedback proportional controller. Given all these conditions, we can expect the steady-state position error to be zero for a step input as well as a ramp input.

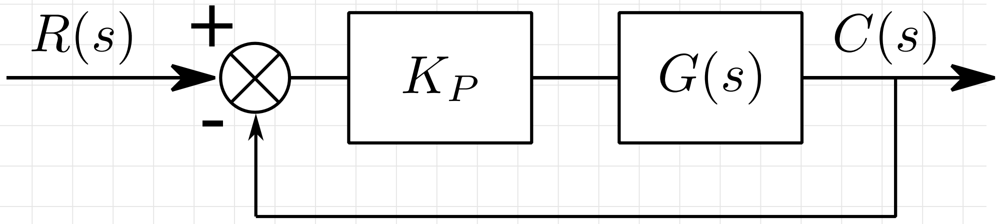
Unity gain: K=1

By definition the ss error in unity feedback: $e_{position}(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\cdot 3/s^2}{1+\frac{5}{s^2(s+5)(s+3)}} =$

$\lim_{s\to 0} \frac{3}{s+\frac{5}{s(s+5)(s+3)}} = \lim_{s\to 0} \frac{3}{0+\infty} = 0$, but this is the steady-state error for **position**, given a ramp **torque** input.

Example

The transfer function relating angular position to torque for a given mechanical system is $G(s) = \frac{\Theta(s)}{T(s)} = \frac{5}{s^2(s+5)(s+3)}$, if a unity gain, unity feedback system is applied to this system. Determine the angular velocity steady-state error to a ramp torque input with magnitude $3N \cdot m$



Continue:

To find the steady-state **velocity** error to a ramp input in torque, we need to express the transfer function with velocity as output $G_2(s) = \frac{\dot{\Theta}(s)}{T(s)} = \frac{s\Theta(s)}{T(s)} = \frac{5}{s(s+5)(s+3)}$, lowering the system to Type 1 rather than Type 2, then

$$e_{velocity}(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G_2(s)} = \lim_{s\to 0} \frac{s \cdot 3/s^2}{1+\frac{5}{s(s+5)(s+3)}} = \lim_{s\to 0} \frac{3}{s+\frac{5}{(s+5)(s+3)}} = \lim_{s\to 0} \frac{3}{0+5/(5\cdot 3)}$$

$$= 9$$

This example highlights the confusion that may be caused by the steady-state error terms, and it is important to remember that the steady state error is defined in terms of the transfer function output and input, and not by the static error constant name.

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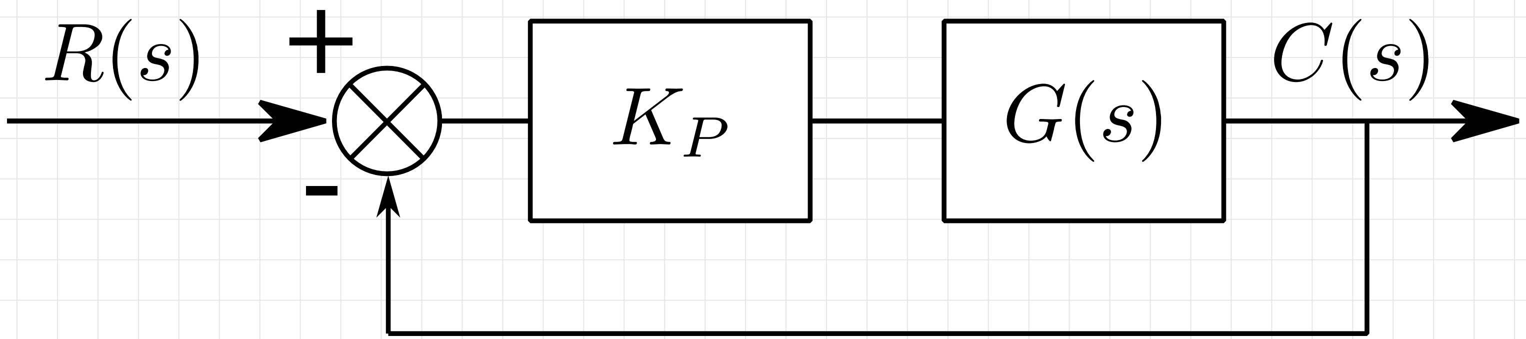
The transfer function relating **velocity** to force for a given mechanical system is $G(s) = \frac{\dot{X}(s)}{F(s)} = \frac{5}{s(s+5)(s+3)}$, if a unity gain, unity feedback system is applied to this system.

Determine the **position** steady-state error to a ramp force input with magnitude $3N$



Stability and System Type 2 Systems

- Note that Type 2 system have a double pole at the origin. As we will see in the root-locus design, we have to consider the proximity of the closed loop poles to the marginal stability line (imaginary axis).
- Algebraically, the steady-state error might be a finite value, but the realistically, and even numerically, the system may not behave in a stable way.

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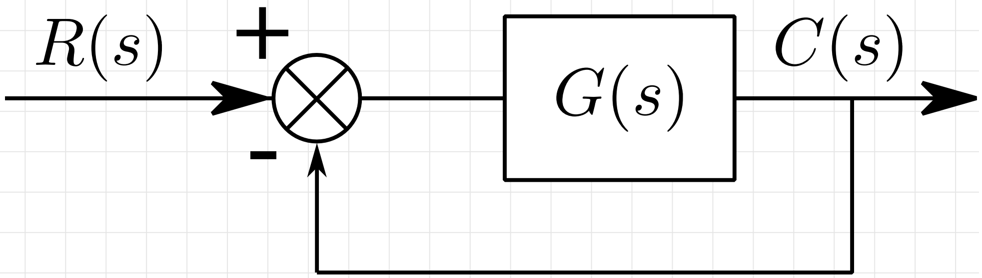
Example

The unity feedback system shown has a transfer function $G(s) = \frac{K(s+\alpha)}{(s^2+\beta)}$ is to be designed to meet the following requirements: The steady-state position error for a unit step input equals 0.2, the closed-loop poles will be located at $-2 \pm 1j$. Find K, α, β in order to meet the specifications.



Practice Problems

The following problems in chapter7:

International 6th ed:

7-1, 7-2,7-5, 7-18, 7-25, 7-27, 7-29

Equivalently in US 6th ed:

7-1, 7-2,7-5, 7-18, 7-25, 7-27, 7-29

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