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College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

Part I: Introduction to Feedback Control

Lecture 6: Poles, Zeros and System Response

Summer 2020

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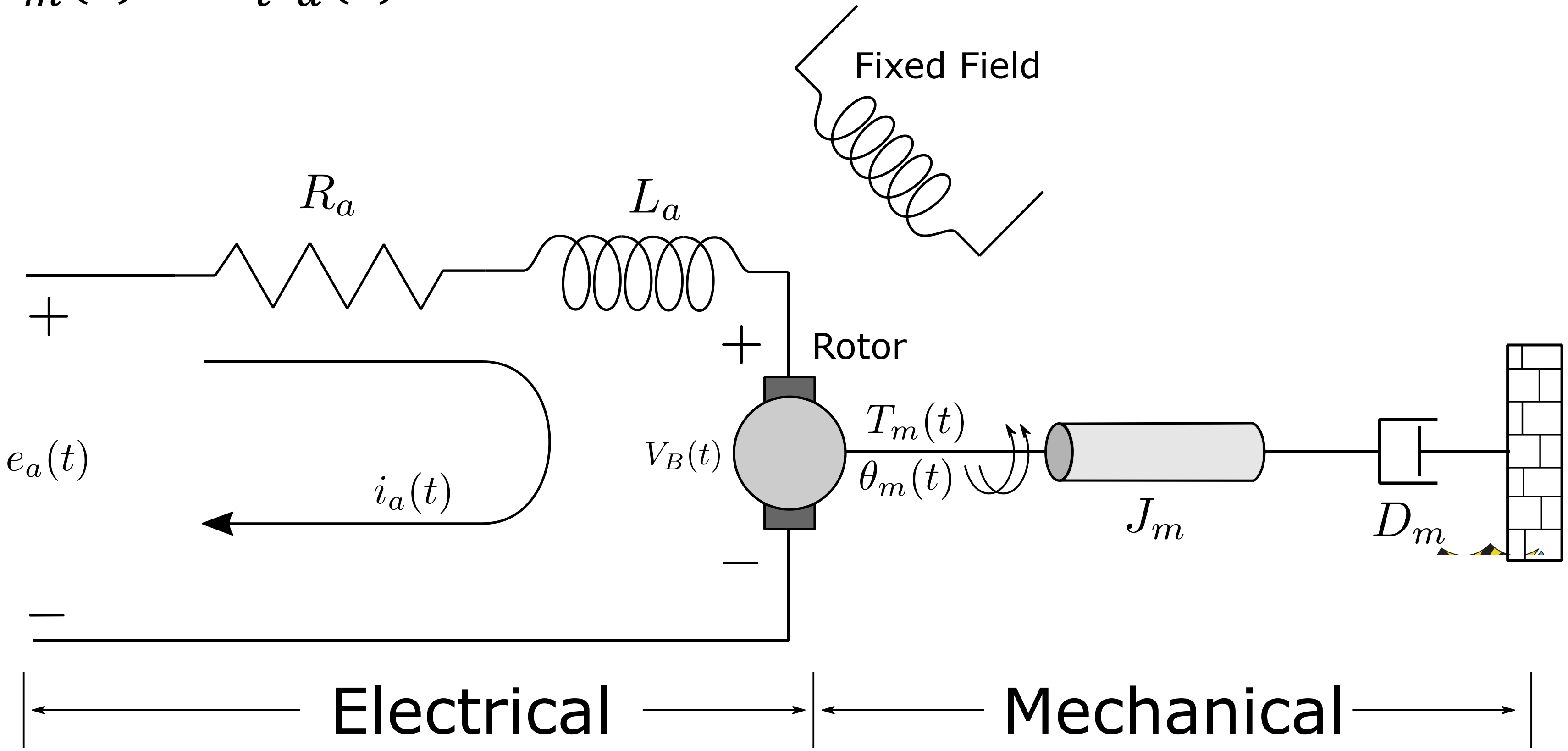
- Objectives:
    - *Introduce the concepts of Poles and Zeros of a Transfer Function*
    - *Review the response of a First-Order System*
    - *Introduce Second-Order Systems*
- Reading:
    - *Nise: 4.1-4.4*
- Practice Problems Included

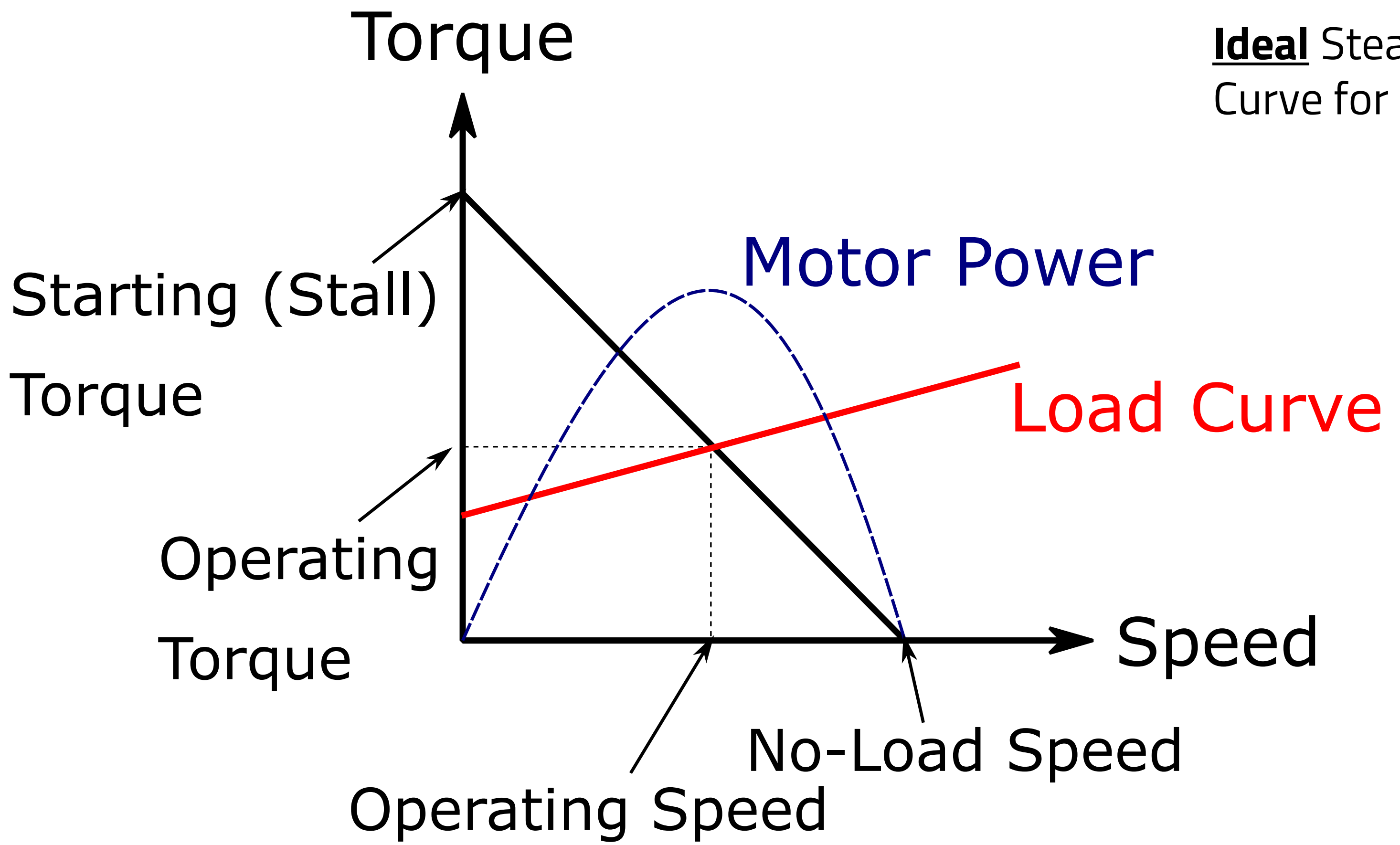
- DC Motor Model, Combining:
  - Electrical Part Gives: $R_a I_a(s) + L_a s I_a(s) + V_B(s) = E_a(s)$
  - Mechanical Part Gives: $T_m(s) = (J_m s^2 + D_m s)\Theta_m(s)$
  - Additional relationship 1: $V_B(s) = K_B s \Theta_m(s)$
  - Additional relationship 2: $T_m(s) = K_t I_a(s)$
- We get:

$$\frac{\Theta_m(s)}{E_m(s)} = \frac{K_t/(R_a J_m)}{s[s+\frac{1}{J_m}(D_m+\frac{K_t K_B}{R_a})]}$$



- Torque-Speed Curve

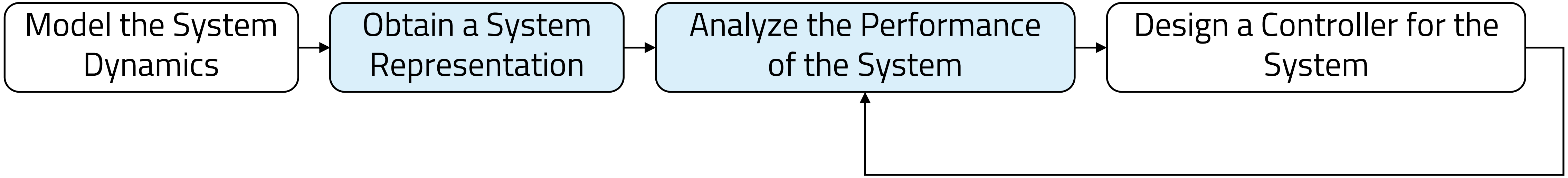
$$T_m(t) = -\frac{K_B K_t}{R_a}\omega_m(t) + \frac{K_t}{R_a}e_a(t)$$



Ideal Steady-State Torque-Speed Curve for a Brushed DC Motor

- In this lecture, we will start to classify systems based on their response characteristics
- So far, we introduced the transfer function as a way to represent a system model
- We will introduce a different way to represent a system (graphically)
- We will introduce standard performance specifications, that will help us in analyzing the performance of systems.

The Control System Design Process:



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- Output response of a system is the sum of: forced response and natural response
  - $c_t(t) = c_{forced}(t) + c_{natural}(t)$
- Given the equation of motion for a system, we can mathematically obtain its output response.
- But there are qualitative ways of studying the output response of the system
- One technique is to look at the **poles** and **zeros** of a system and their relationship to the output response.
- Poles × and zeros O can be derived from a system's transfer function.
  - They are generally defined in the context of a transfer function.

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Poles of a Transfer Function - Definition

- Poles (denoted by the symbol $\times$) of a transfer function $G(s)$ can either be:
  1. The values of s that cause the transfer function $G(s)$ to become infinite, or
  2. Any roots of the denominator of the transfer function $G(S)$ that are common to the numerator's.

- Example: The two poles of $G(s) = \frac{1}{s(s+4)}$, are $s = 0, s = -4$

- Example: The three poles of $G(s) = \frac{(s+3)}{(s^2+6s+18)(s+3)}$, are $s = -3, s = -3 \pm 3i$
  - Note that mathematically $G(s) = \frac{(s+3)}{(s^2+6s+18)(s+3)} = \frac{1}{(s^2+6s+18)}$ where the latter form has only two poles; however, $s = -3$ is still is a pole of the system, it is only that the **effect** of this pole is *cancelled* in this case.

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- Zeros (denoted by the symbol $\mathbf{O}$) of a transfer function $G(s)$ can either be
  1. The values of s that cause the transfer function $G(s)$ to become zero, or
  2. Any roots of the numerator of the transfer function $G(s)$ that are common with the denominator's
- Example: The zero of the transfer function $G(s) = \frac{(3s+1)}{s}$, is $s = -\frac{1}{3}$
- Example: The two zeros of the transfer function $G(s) = \frac{s(2s+1)}{(2s+1)}$, are $s = -\frac{1}{2}$, $s = 0$
- In later sections, we will learn why its important to keep in mind the pole-zero cancellation behavior.

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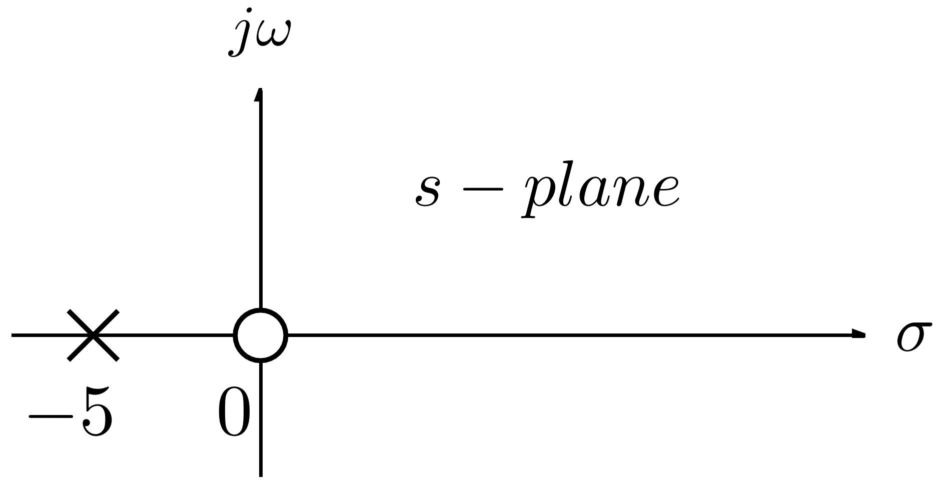
- In the Laplace domain, we defined $s = \sigma + j\omega$
- The $s - plane$ is where we plot the values of s
- Poles and Zeros are graphically placed on the $s - plane$
- The figure shows a graphical representation of the transfer function:

$$G(s) = 5\frac{s}{(s+5)}$$

Note that the gain 5 is not captured on the graph (we will deal with expressing the gain value under the Root-Locus section)



- With respect to the pole location on the s-plane:
  - A pole on the real negative axis produces an exponentially decaying response.

$$G(s) = \frac{1}{(s+3)}$$

  - A pole pair on the imaginary axis produce a sinusoidal response.

$$G(s) = \frac{1}{s^2+18}$$

  - A pole on the real positive axis produces an exponentially growing response. (unstable response)

$$G(s) = \frac{1}{(s-4)}$$

  - A pole at the origin produces a step response.

$$G(s) = \frac{1}{s}$$

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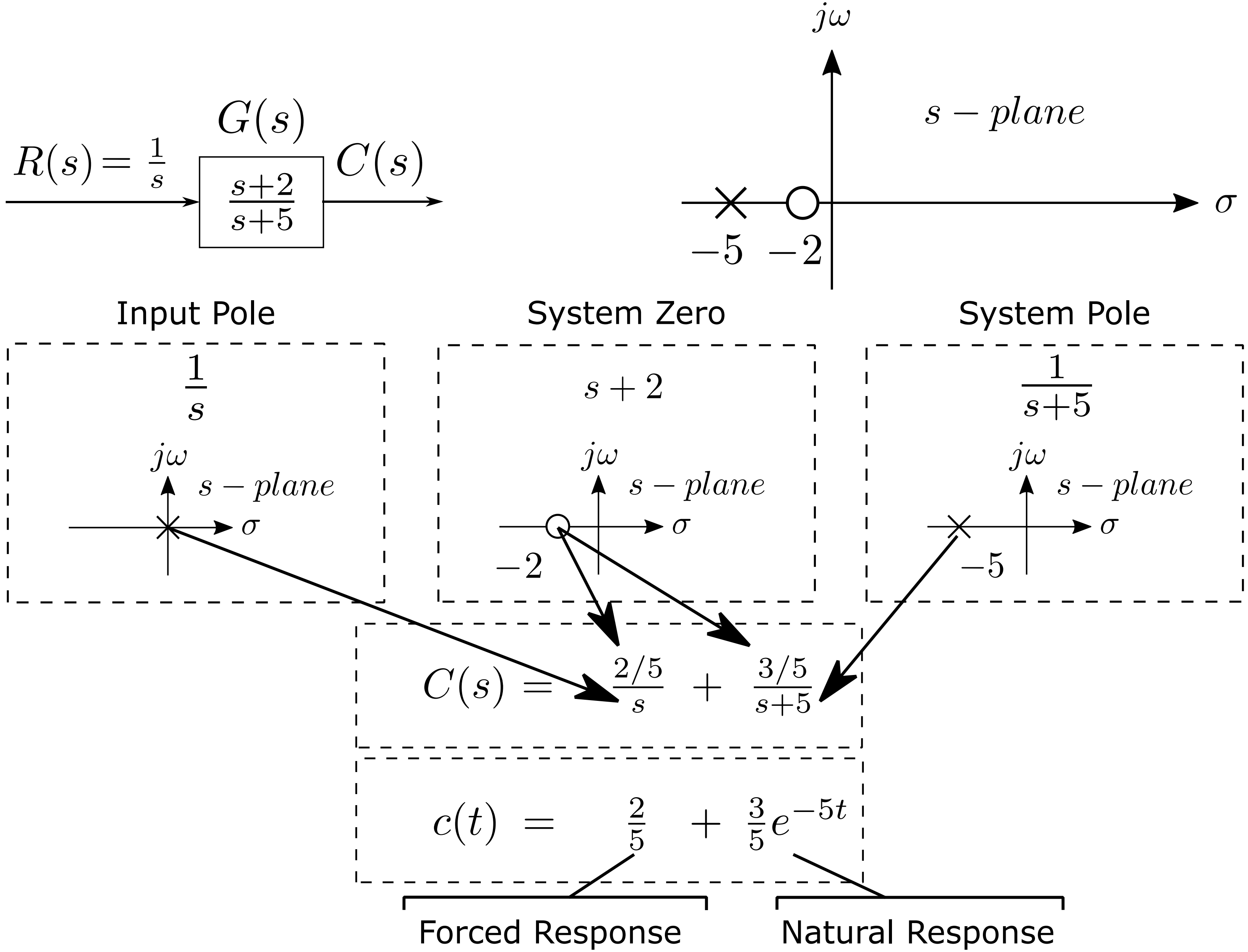
$G(s)$
$R(s) = \frac{1}{s}$
$\frac{s+2}{s+5}$
$C(s)$
$j\omega$
$s - plane$
σ
-5
-2
Input Pole
$\frac{1}{s}$
$j\omega$
$s - plane$
σ
System Zero
$s + 2$
$j\omega$
$s - plane$
σ
-2
System Pole
$\frac{1}{s+5}$
$j\omega$
$s - plane$
σ
-5
$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$
$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$
Forced Response
Natural Response

For the system with the transfer function $G(s) = \frac{C(s)}{R(s)} = 20\frac{(s+4)(s+3)(s-8)}{(s+1)(s+2)^2}$, write, by inspection, the output $c(t)$ in general terms, to a ramp input ($R(s) = \frac{1}{s^2}$) and specify the natural and forced response parts.

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- A first-order system is a system whose highest derivative order is 1, or who's characteristic equation is of degree 1.
  - Example: The system (internal system) with $G(s) = \frac{(s+2)}{(s+3)(s+2)} = \frac{1}{(s+3)}$, is not a first order system per se, but the system **response** is a first-order response, due to the pole-zero cancellation.
  - Example: The system with $G(s) = 10\,\frac{(s+3)}{(s+5)}$ is a first-order system with a zero
- Examples of First Order Systems:
  - Heat Transfer (Thermometer)
  - Interest Rate Growth
  - RC Circuit

- In the context of control system design, there are well defined performance specifications that we evaluate, such as:
  - Rise Time T_r, Settling Time T_s, Time Constant, Percentage Overshoot, Peak Time T_p
- These are common characteristics, but performance specifications are not limited to them in the design of real control systems.
- We will define some performance specifications for First-Order systems.

- Performance specifications for first-order systems are defined for $G(s) = \frac{a}{(s+a)}$, a first-order system with no zero.
- Moreover, the specifications are defined for a response to a unit-step input.
  - $R(s) = \frac{1}{s}$
  - $C(s) = \frac{a}{s(s+a)}$, giving:
  - $c(t) = c_{forced}(t) + c_{natural}(t) = 1 - e^{-at}$
- **Time Constant τ:** $\tau = \frac{1}{a}$, $c\left(t = \frac{1}{a}\right) = 1 - e^{-1} = 0.63$
  - Time constant is the time required for e^{-at} to decay to 37% of its initial value.
  - The reciprocal of the time constant, is called the exponential frequency

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- **Rise Time T_r:** The time for the response to go from 0.1 to 0.9 of its final value.
  - For the 1st Order System $G(s) = \frac{a}{(s+a)}$, $T_r = \frac{2.2}{a}$
- **Settling Time T_s:** The time required for the response to reach and stay within 2% of its final value.
  - For the 1st Order System $G(s) = \frac{a}{(s+a)}$, $T_s = \frac{4}{a}$
  - Other times, the settling time is defined with a different than 2% target, 5% is common as well. In this course we will use the 2% target when defining T_s
- Remember that the above defined equations are strictly for a system with the transfer function of the form $G(s) = \frac{a}{(s+a)}$, in response to a unit step input.

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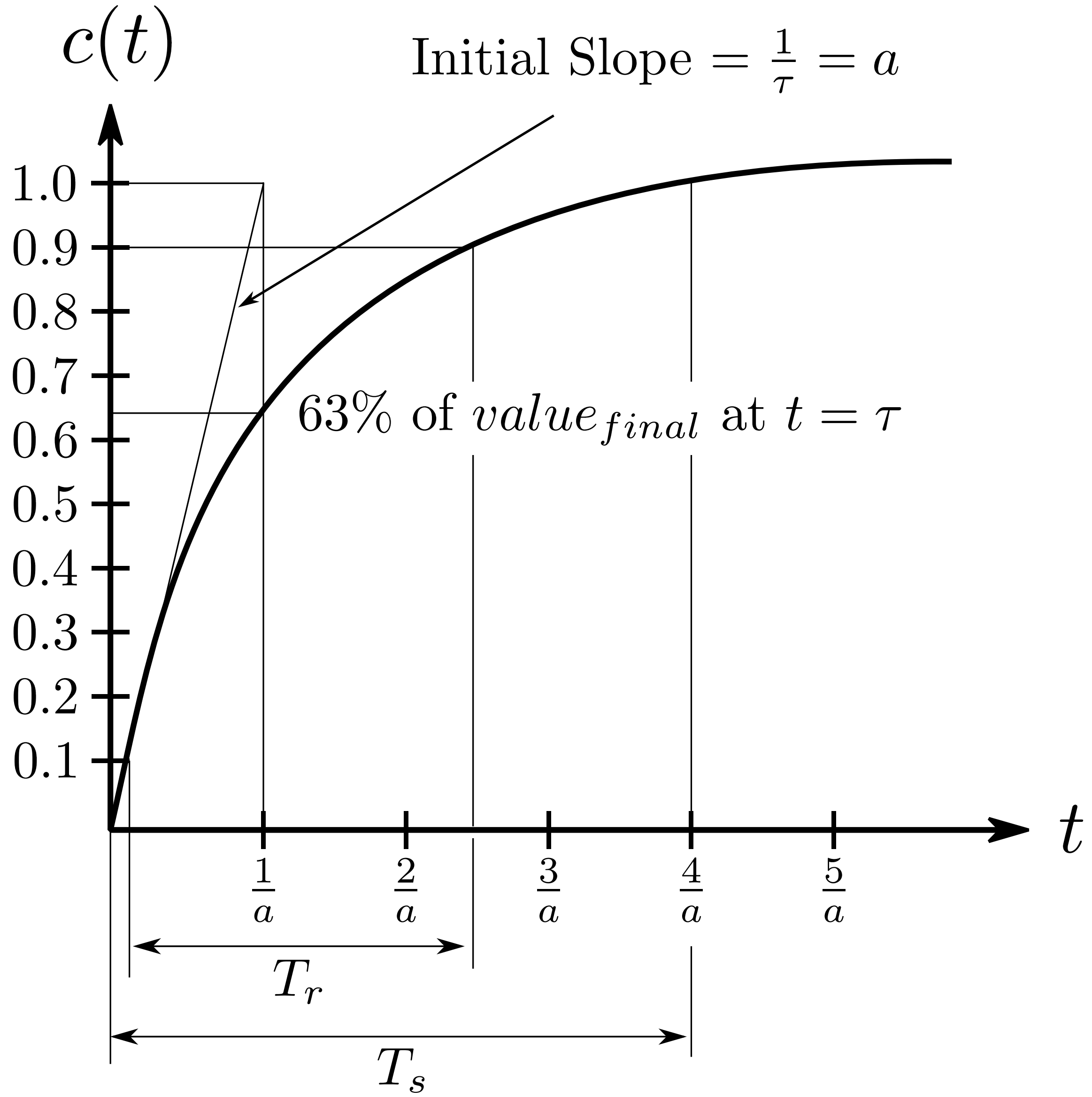
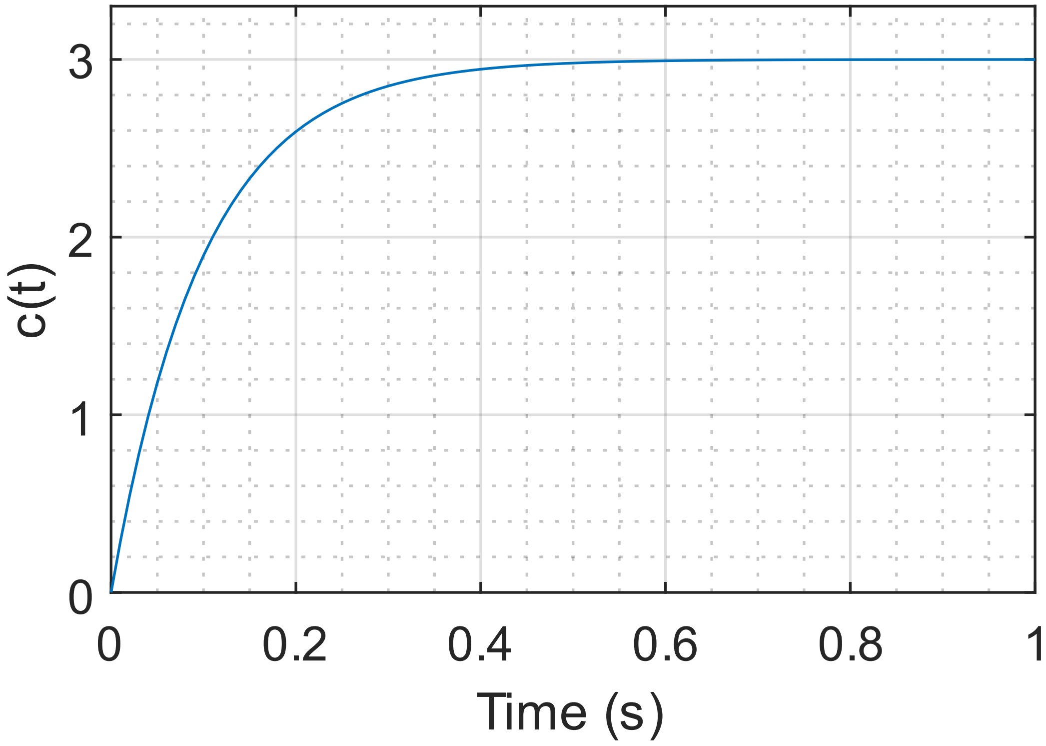
First-Order System Response to a Unit Step



Example 2

The figure shows the response of a system measured experimentally, identify the type of system response and find the time constant, rise time and settling time of the system.



- Second-Order Systems:
  - System’s differential equation highest derivative order is 2, or
  - The transfer function denominator’s degree is 2.
  - Example: A system with the transfer function $G(s) = \frac{(s+3)}{(s^2+6s+9)}$ is a second order system with a first-order **response** to a unit step input, but why?
- Second-order systems’ responses are of more interest in the study of control system design
- Examples of second-order systems:
  - All mechanical systems that have mass and undergo acceleration
  - First-order systems with unity feedback become 2^{nd} order systems.
  - Many more examples.
- Second-order systems are subclassified depending on their response

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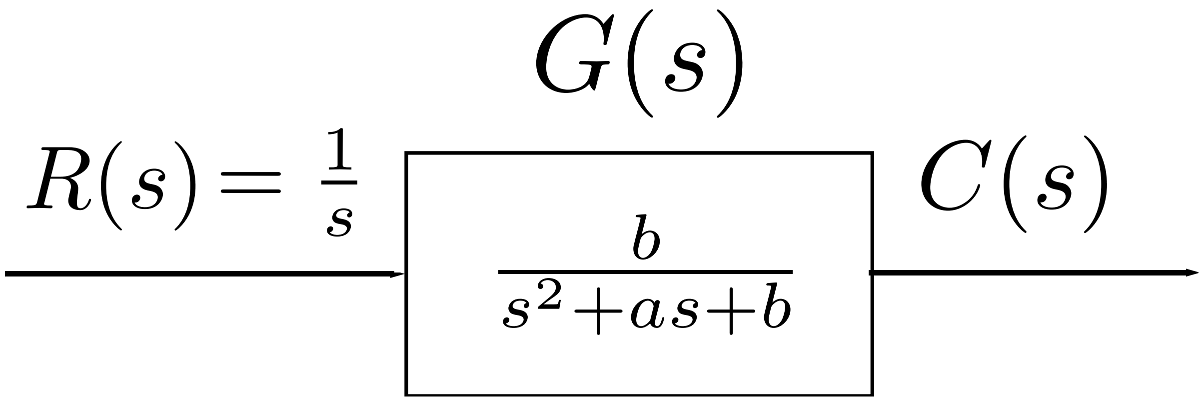
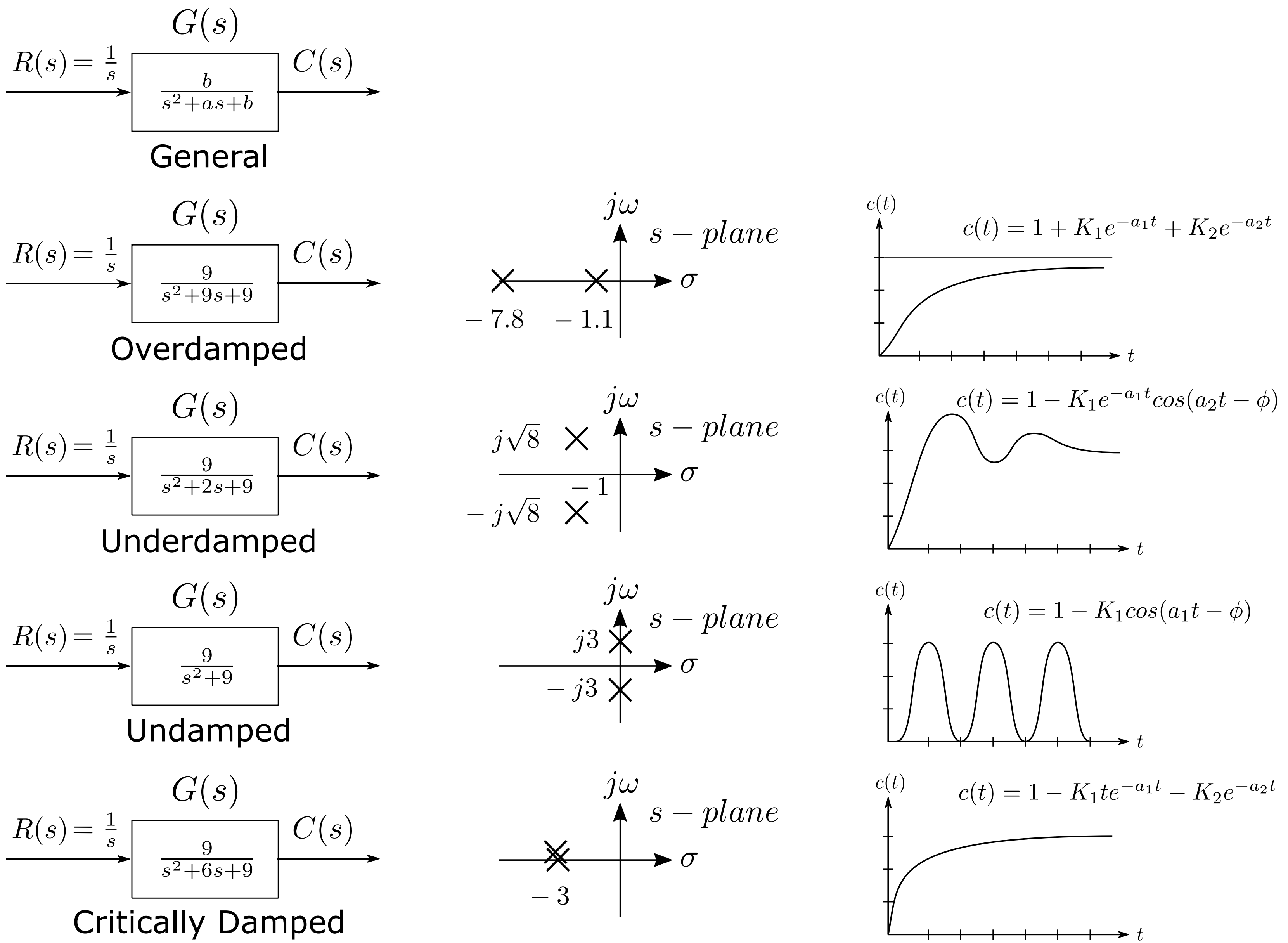
- Remember that we evaluate/subclassify second-order systems based on the system's response to a unit step input. That is, for $G(s) = \frac{C(s)}{R(s)}, R(s) = \frac{1}{s}$
- In our subclassification, we treat second-order systems with no zeros.
  - Note: The numerator is set $= b$ to produce a normalized response.





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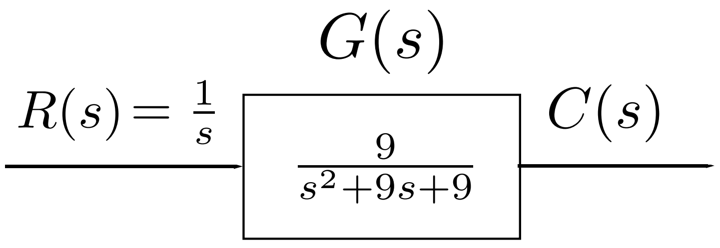
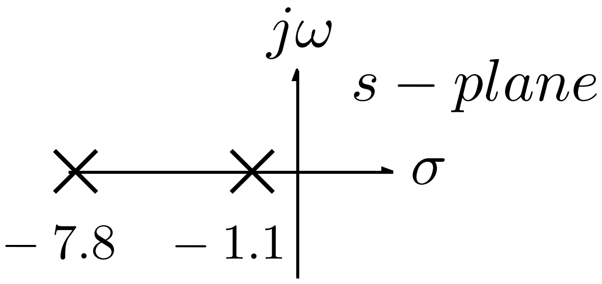
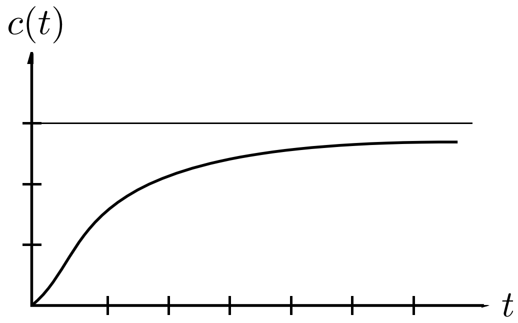
- For the response $C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)}$
- The pole at the origin comes from the step input, generating a constant forced response
- The two poles on the real negative axis produce two exponential decays, each at a different rate.
- The general form of the output is

$$c(t) = \underbrace{K_1}_{Forced\ Response} + \underbrace{K_2 e^{-7.854t} + K_3 e^{-1.146t}}_{Natural\ Response}$$



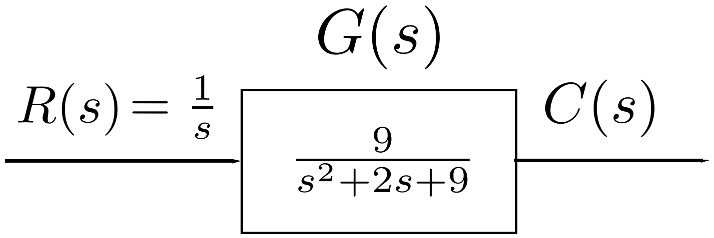
Overdamped



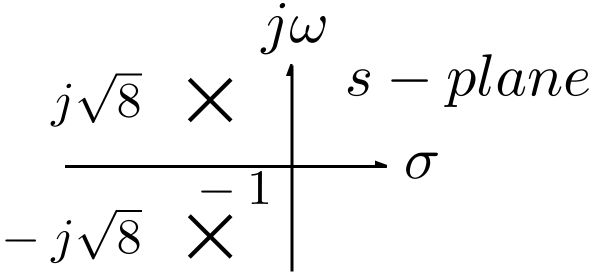


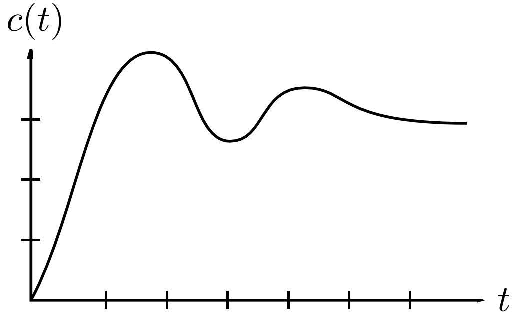
- For the response $C(s) = \frac{9}{s(s^2+2s+9)} = \frac{9}{s\left(s+1+j\sqrt{8}\right)(s+1-j\sqrt{8})}$
- The pole at the origin comes from the step input, generating a constant forced response
- The two complex pole pairs produce a decaying sinusoidal response
- The general form of the output is

$$c(t) = \underbrace{K_1}_{Forced\ Response} + \underbrace{K_2 e^{-t}\cos(\sqrt{8}t - \phi)}_{Natural\ Response}$$



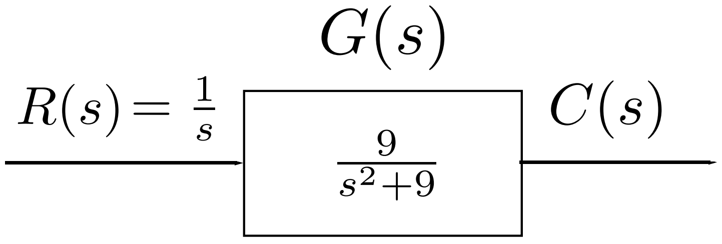
Underdamped



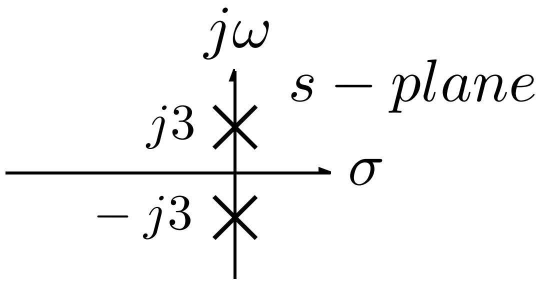


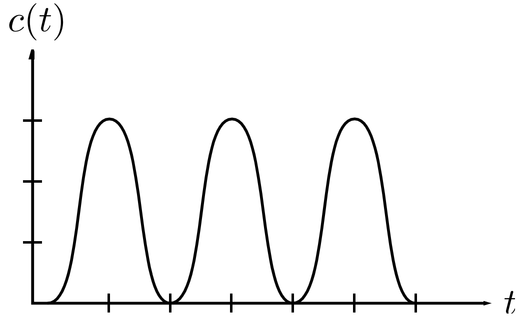
- For the response $C(s) = \frac{9}{s(s^2+9)} = \frac{9}{s(s+j3)(s-j3)}$
- The pole at the origin comes from the step input, generating a constant forced response
- The two imaginary pole pairs produce a pure sinusoidal response
- The general form of the output is

$$c(t) = \underbrace{K_1}_{Forced\ Response} + \underbrace{K_2 \cos(\sqrt{8}t - \phi)}_{Natural\ Response}$$



Undamped

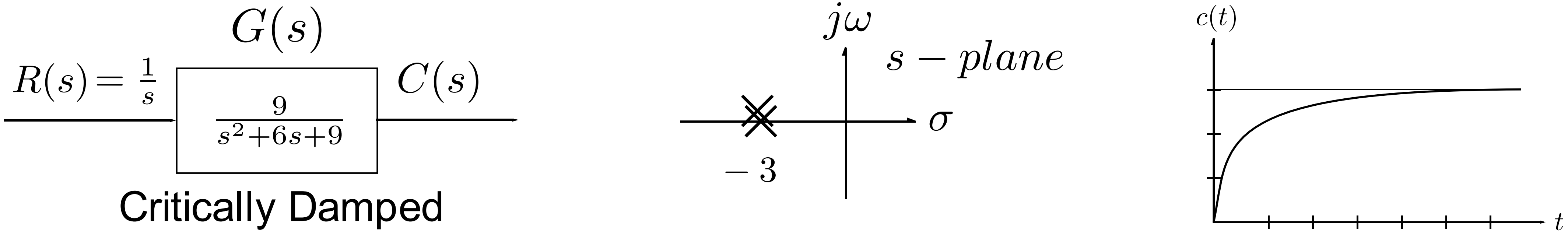




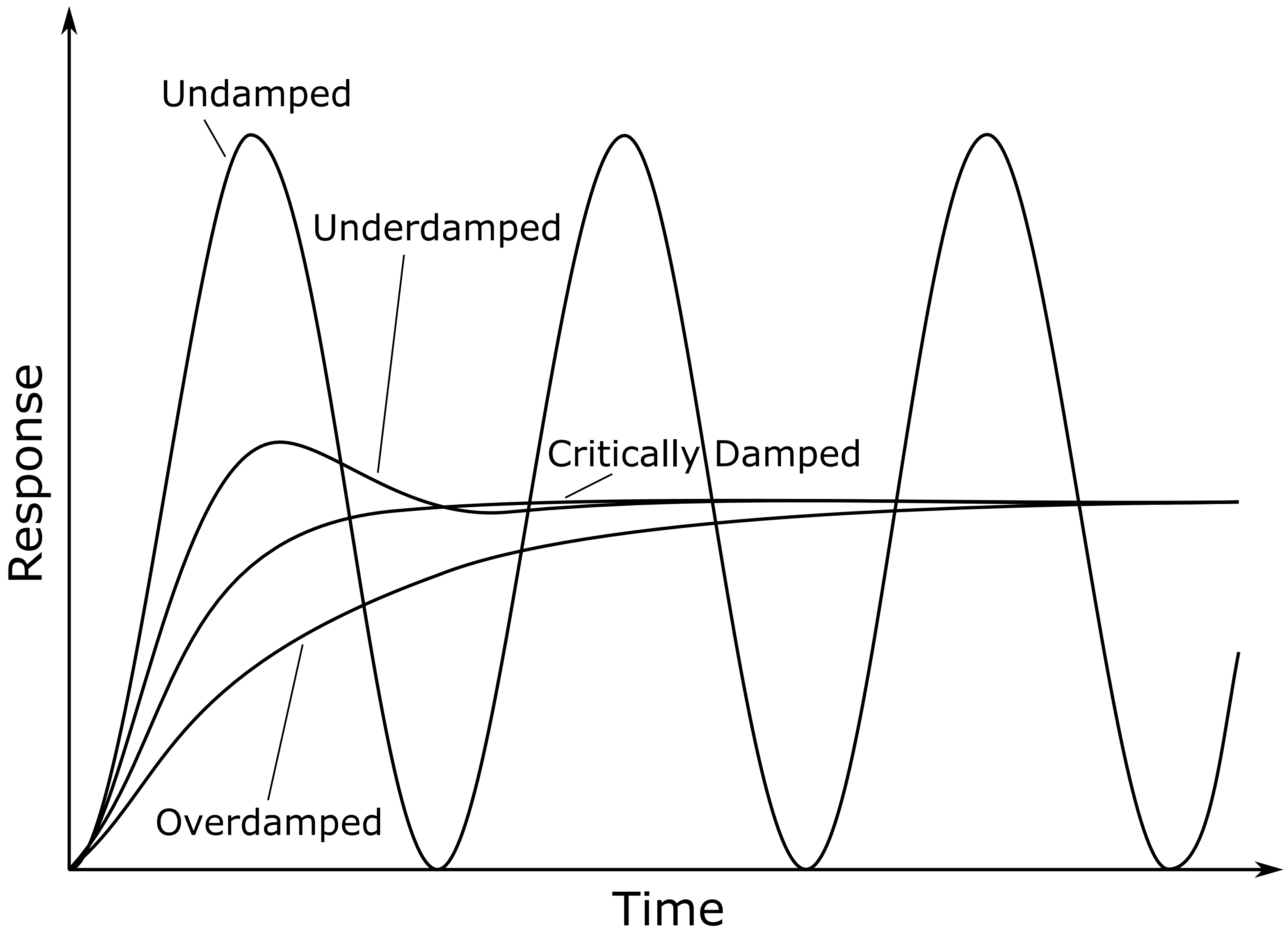
Second-Order System Subclassification – Critically Damped Response

- For the response $C(s) = \frac{9}{s(s^2+6s+9)} = \frac{9}{s(s+3)^2}$
- The pole at the origin comes from the step input, generating a constant forced response
- The double poles on the real negative axis produce an exponentially decaying response and an additional exponentially decaying response with a t term
- The general form of the output is

$$c(t) = \underbrace{K_1}_{Forced\ Response} + \underbrace{K_2te^{-3t} + K_3e^{-3t}}_{Natural\ Response}$$

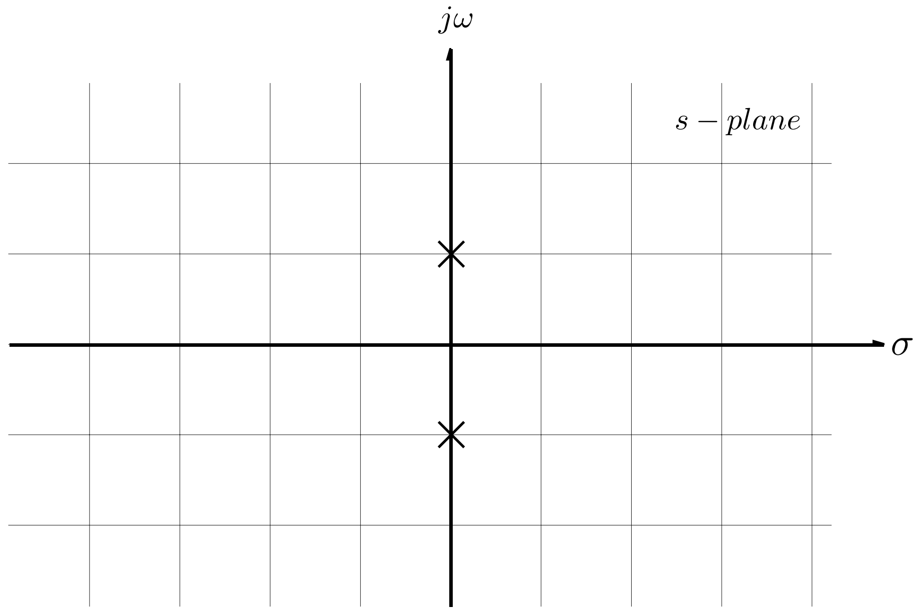


Critically Damped


Undamped
Underdamped
Critically Damped
Overdamped
Response
Time

Example 3

The figure shows poles and zeros for a system, placed on the s-plane, write the transfer function for the system as well as the output response equation to a step input, both in general terms. Sketch the system's time response to a step input



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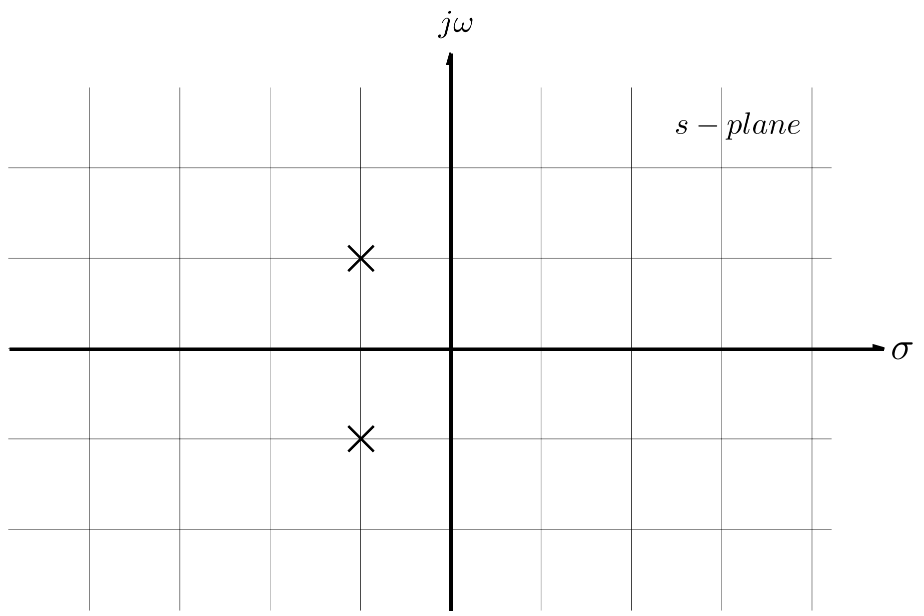
The figure shows poles and zeros for a system, placed on the s-plane, write the transfer function for the system as well as the output response equation to a step input, both in general terms. Sketch the system's time response to a step input

Example 4



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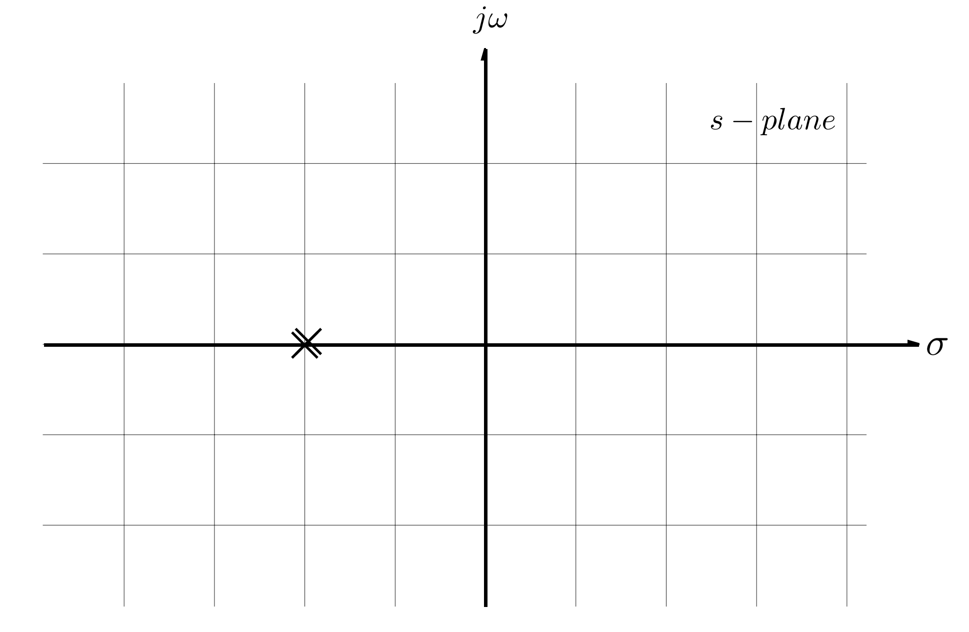
Example 5

The figure shows poles and zeros for a system, placed on the s-plane, write the transfer function for the system as well as the output response equation to a step input, both in general terms. Sketch the system's time response to a step input



A system has a transfer function, $G(s) = \frac{50}{(s+50)}$. Find the time constant, settling time and rise time.

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Answer the following questions.

1. The imaginary part of a pole generates what part of the response?
2. In a system with an input and an output, what poles generate the steady-state response and what poles generate the transient response?
3. The imaginary parts and the real parts of a pole generate what part of the response, respectively?
4. Are the following two systems equivalent? Explain
   - $G_{system\,1} = \frac{1}{(s+3)}$, $G_{system\,2} = \frac{(s+2)}{(s^2+5s+6)}$
5. Are the responses of the following two systems, to step input, equivalent? Explain
   - $G_{system\,1} = \frac{1}{(s+4)}$, $G_{system\,2} = \frac{(s+5)}{(s^2+9s+20)}$

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For each of the transfer functions shown below, find the locations of the poles and zeros, plot them on the s-plane, and then write an expression for the general form of the step response without solving for the inverse Laplace transform. State the nature of each response (overdamped, underdamped, and so on). Use MATLAB to verify your answers. Nise: 4-8

a. $G(s) = \frac{2}{(s+2)}$

b. $G(s) = \frac{5}{(s+3)(s+6)}$

c. $G(s) = \frac{10(s+7)}{(s+10)(s+20)}$

d. $G(s) = \frac{20}{(s^2+6s+144)}$

e. $G(s) = \frac{(s+2)}{(s^2+9)}$

f. $G(s) = \frac{(s+5)}{(s+10)^2}$

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