ME 417 - Homework #3

Control of Mechanical Systems - Fall 2020

Homework Due: Sun, 14 Feb 2021 23:59

Complete the following problems and submit a hard copy of your solutions. You are encouraged to work together to discuss the problems but submitted work **MUST** be your own. This is an **individually** submitted assignment.

Problem 1

Root Locus Sketching (20pts)

For each of the following transfer functions, sketch a general shape root-locus, and include, as applicable, asymptote intercepts and angles

a. $G(s) = \dfrac{s+10}{s^2+3s+15}$

b. $G(s) = \dfrac{s\,(s-5)}{(s+4)\,(2s^2+4s+4)}$

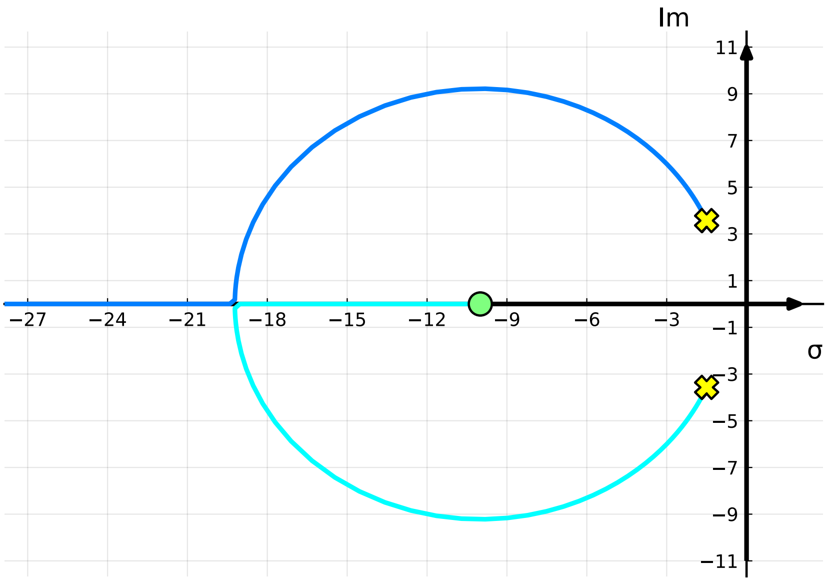
c. $G(s) = \dfrac{s^2+8s+40}{s\,(s-8)}$

d. $G(s) = \dfrac{(s-10)\,(s+8)}{s\,(s^2+2s+60)}$

Solution:

a.

The root-locus is shown below



We have $P_n - Z_n = 2 - 1 = 1$ asymptotes.

There is only one asymptote and it is on the real axis.

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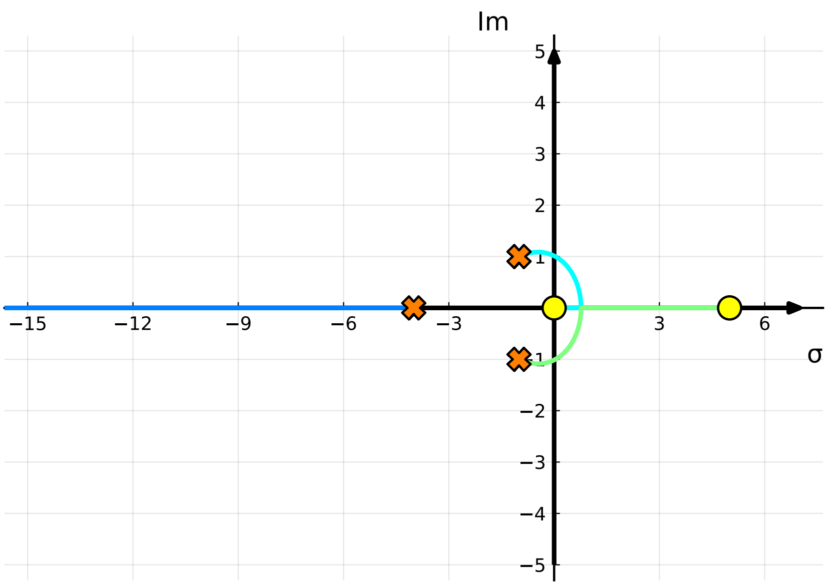
b.

The root-locus is shown below



We have $P_n - Z_n = 3 - 2 = 1$ asymptotes.

There is only one asymptote and it is on the real axis.

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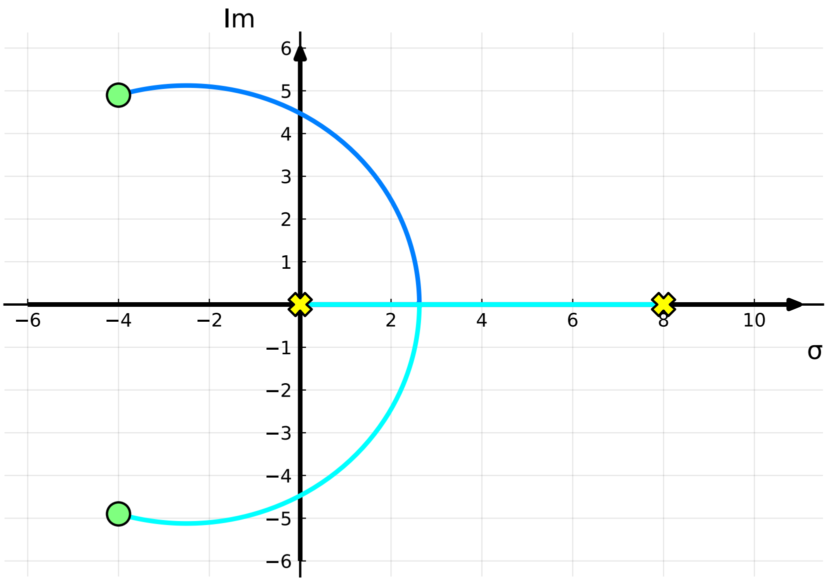
c.

The root-locus is shown below



There are no asymptotes, since $P_n - Z_n = 2 - 2 = 0$

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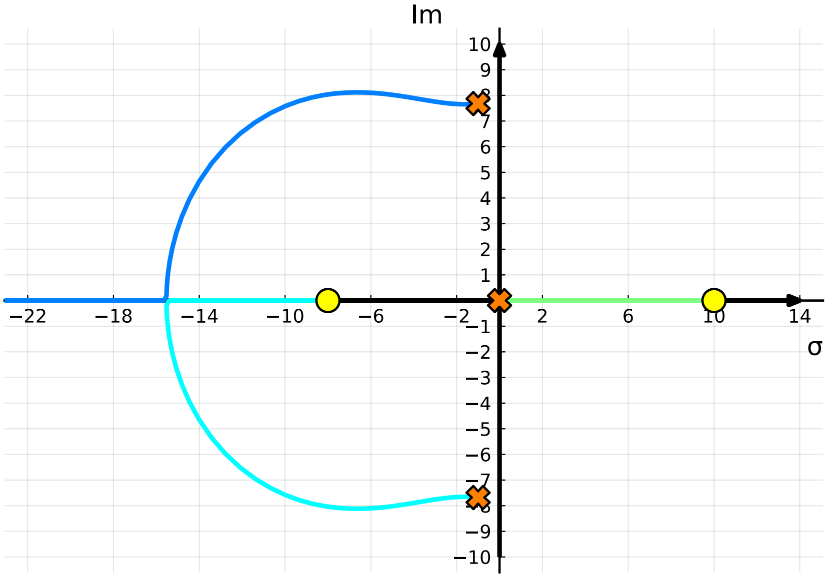
d.

The root-locus is shown below



We have $P_n - Z_n = 3 - 2 = 1$ asymptotes.

There is only one asymptote and it is on the real axis.

Problem 2

Root Locus Sketching (20pts)

For the following open-loop transfer functions, sketch a refined root locus, compute any applicable break-away and break-in points as well as imaginary axis crossing. Highlight the range of K for which the system is stable.

a. $G(s) = \dfrac{2s + 6}{s^2 + 10s + 61}$

b. $G(s) = \dfrac{s - 8}{s\,(s + 5)\,(s + 8)}$

c. $G(s) = \dfrac{(s + 20)\,(s + 40)}{(s - 20)\,(s - 10)}$

d. $G(s) = \dfrac{(s - 15)\,(s + 10)}{s^2 - 8s + 45}$

Solution:

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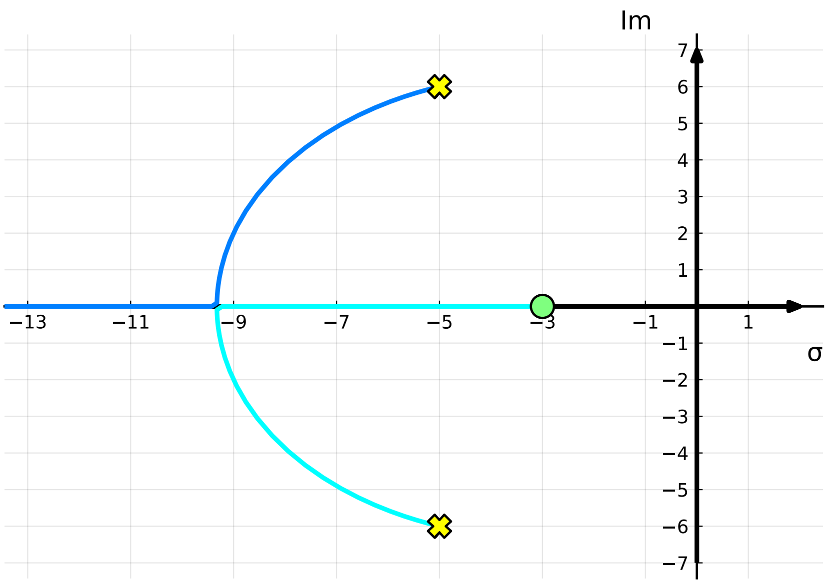
a.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\dfrac{\delta K(\sigma)}{\delta \sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 + 10.0\sigma + 61.0}{2.0\sigma + 6.0}$$

$$\frac{\delta K(\sigma)}{\delta\sigma} = \frac{-0.0556\sigma^2 - 0.333\sigma + 1.72}{0.111\sigma^2 + 0.667\sigma + 1.0} = 0$$

Solving for σ we get $\left[-9.32\right]$, corresponding to $K = \left[4.32\right]$

There is no ωj crossing in this root-locus

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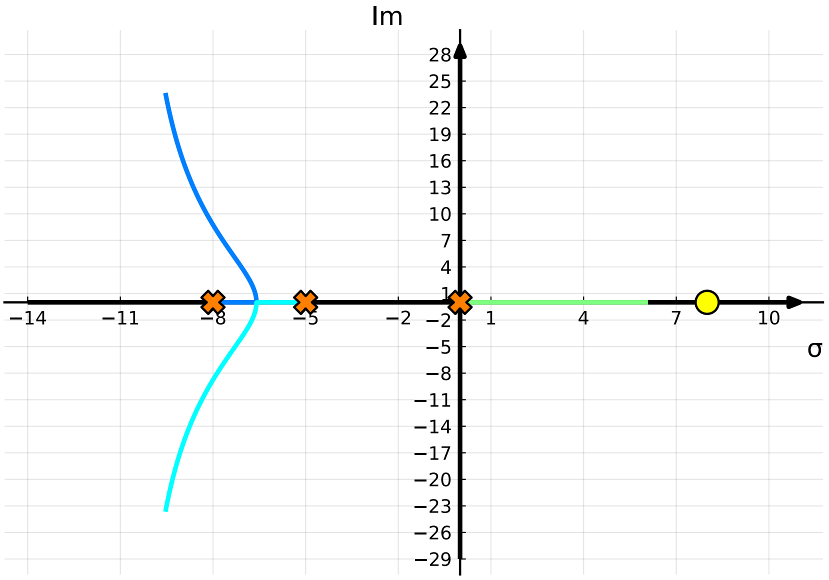
b.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\frac{\delta K(\sigma)}{\delta\sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^3 + 13.0\sigma^2 + 40.0\sigma}{1.0\sigma - 8.0}$$

$$\frac{\delta K(\sigma)}{\delta \sigma} = \frac{-0.0313\sigma^3 + 0.172\sigma^2 + 3.25\sigma + 5.0}{0.0156\sigma^2 - 0.25\sigma + 1.0} = 0$$

Solving for σ we get $\left[-6.59\right]$, corresponding to $K = \left[1.01\right]$

There is no ωj crossing in this root-locus

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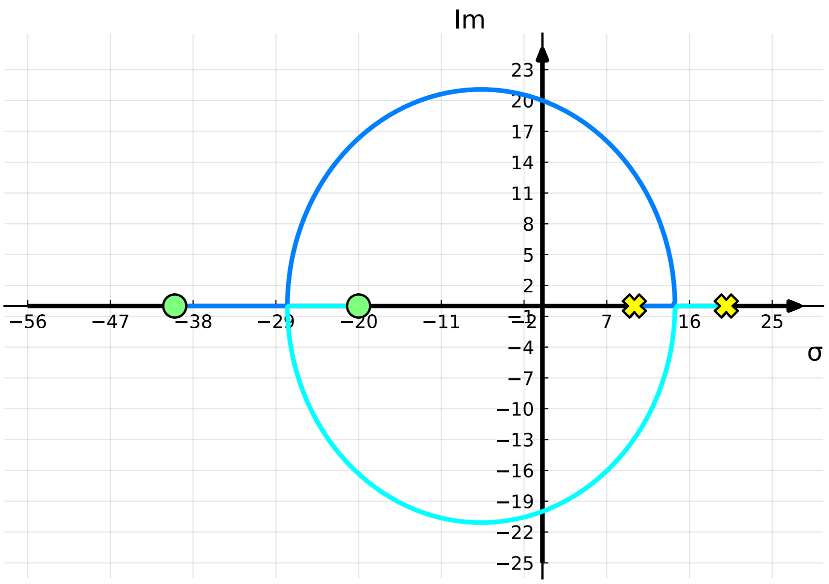
c.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\frac{\delta K(\sigma)}{\delta \sigma} = 0$

$$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 - 30.0\sigma + 200.0}{1.0\sigma^2 + 60.0\sigma + 800.0}$$

$$\frac{\delta K(\sigma)}{\delta\sigma} = \frac{-0.000141\sigma^2 - 0.00187\sigma + 0.0563}{1.56 \cdot 10^{-6}\sigma^4 + 0.000187\sigma^3 + 0.00813\sigma^2 + 0.15\sigma + 1.0} = 0$$

Solving for σ we get $\begin{bmatrix} 14.4 \\ -27.7 \end{bmatrix}$, corresponding to $K = \begin{bmatrix} 0.0132 \\ 19.0 \end{bmatrix}$

To find the ωj crossing, we subsitute s in the characteristic polynomial $1 + KG(s) = 0$ with $\omega_d j$ and solve.

$1 + KG(s) = 0.005Ks^2 + 0.3Ks + 4.0K + 0.005s^2 - 0.15s + 1.0 = 0$, substituting for s, we get

$1 + KG(\omega j) = -0.005K\omega d^2 + 0.3iK\omega d + 4.0K - 0.005\omega d^2 - 0.15i\omega d + 1.0 = 0$. Solving for K and ω_d we get $\omega_d = -20.0, K = 0.5$

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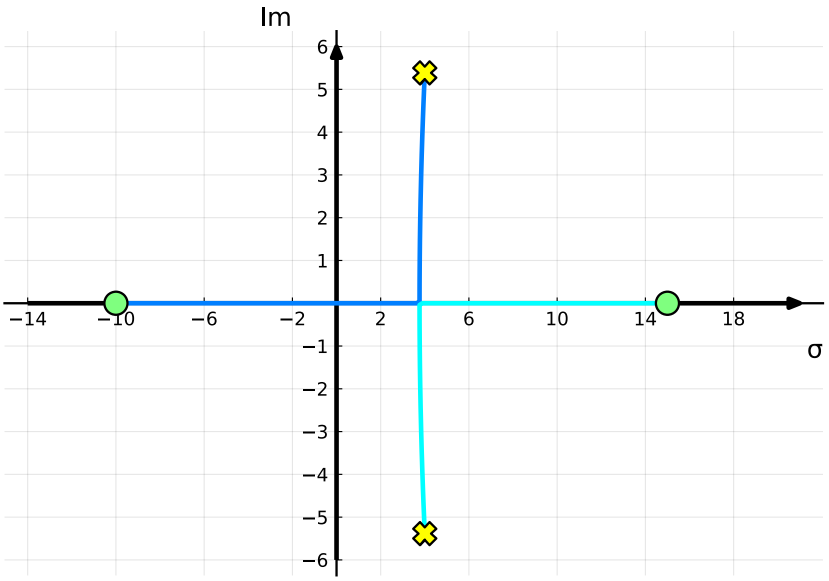
d.

The root-locus is shown below



To find the break points, we express the root-locus gain K as a function of s then solve for the maximum and minimum points of gain on the real-axis by first substituting s with σ in $K(s)$ and solving for σ in $\frac{\delta K(\sigma)}{\delta\sigma} = 0$

$K(\sigma) = -\frac{1}{G_{ol}(\sigma)} = -\frac{1.0\sigma^2 - 8.0\sigma + 45.0}{1.0\sigma^2 - 5.0\sigma - 150.0}$

$\frac{\delta K(\sigma)}{\delta\sigma} = \frac{-0.000133\sigma^2 + 0.0173\sigma - 0.0633}{4.44 \cdot 10^{-5}\sigma^4 - 0.000444\sigma^3 - 0.0122\sigma^2 + 0.0667\sigma + 1.0} = 0$

Solving for σ we get $\left[3.76\right]$, corresponding to $K = \left[0.188\right]$

To find the ωj crossing, we subsitute s in the characteristic polynomial $1 + KG(s) = 0$ with $\omega_d j$ and solve.

$1 + KG(s) = 0.0222Ks^2 - 0.111Ks - 3.33K + 0.0222s^2 - 0.178s + 1.0 = 0$, substituting for s, we get

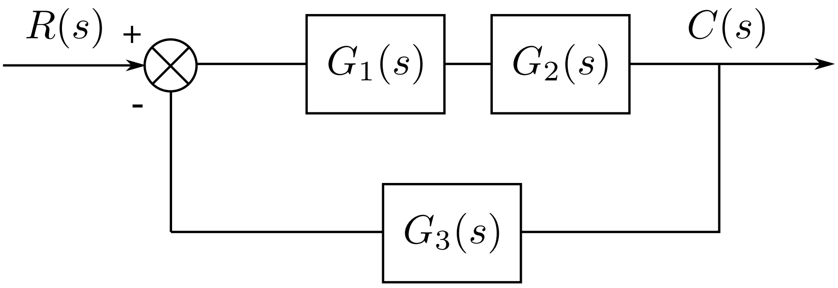
$1 + KG(\omega j) = -0.0222K\omega d^2 - 0.111iK\omega d - 3.33K - 0.0222\omega d^2 - 0.178i\omega d + 1.0 = 0$. Solving for K and ω_d we get

$\omega_d = 0.0, K = 0.3$

Problem 3

Root Locus Sketching (20pts)

Given the following feedback system



With $G_1 = s + z, G_2 = \frac{100}{s^2 + 4s + 20}, G_3 = \frac{20}{s + 10}$

a. Derive the characteristic polynomial of the system in the form $1 + zG(s) = 0$

b. Sketch the root-locus of the system for varying values of the zero location z

c. Find the value of z that makes the closed-loop system's damped frequency $\omega_d = 6.283185307179586 rad/s$

Solution:

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a.

From the feedback block diagram, the characteristic polynomial of the closed-loop system is $1 + \frac{2000\,(s+z)}{(s+10)\,(s^2+4s+20)} = 0$, rearranging, we get $2000s + 2000z + (s+10)\left(s^2+4s+20\right) = 0$, dividing by $2000s + (s+10)\left(s^2+4s+20\right)$, we get $\frac{2000z}{2000s + (s+10)\,(s^2+4s+20)} + 1 = 0$

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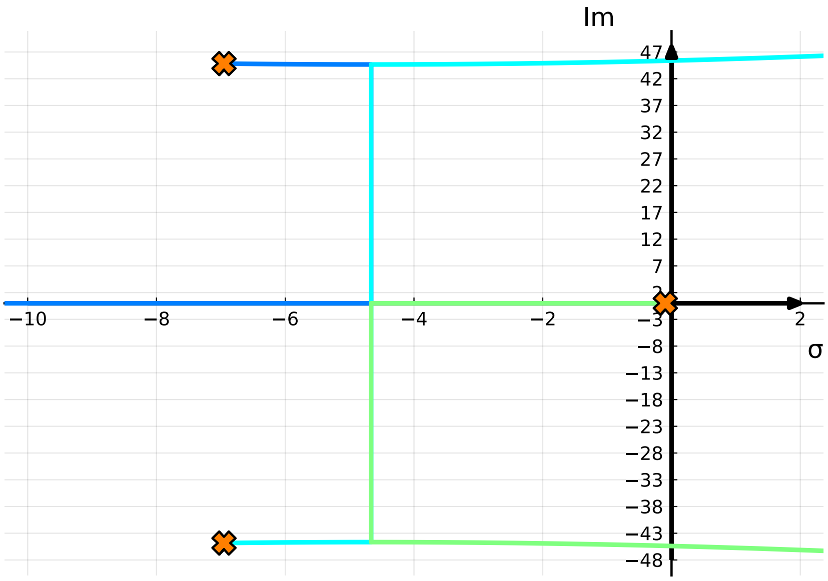
b. The open loop transfer function is then $G(s) = \frac{2000}{2000s + (s+10)\,(s^2+4s+20)}$, the root-locus given this open-loop transfer function is shown



c.

The root-locus does interest the $\omega_d = 6.283185307179586$ line. We can approximate the location to be at $s_d = -2.0 + 6.28im$

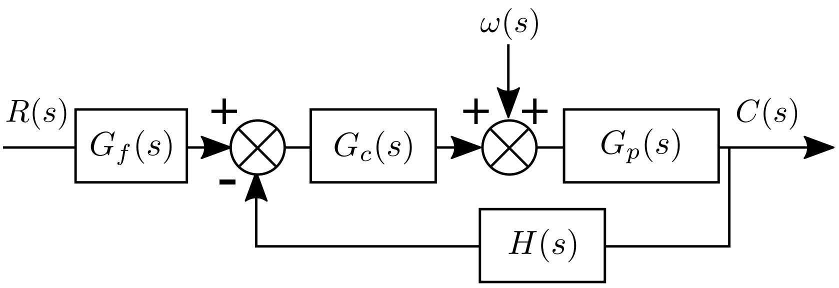
To find the gain, we can use the magnitude condition: $K = \frac{\prod L_p}{\prod L_z} = \frac{13100.0}{1.0} = 13100.0$

The gain of the controller is given by $K_c = K/K_{Gol} = 13100.0/2000.0 = 6.55$

Problem 4

Root Locus Sketching (20pts)

A closed-loop system with input disturbance is shown.



With $G_p = \dfrac{2s+4}{(s-3)(s+6)}, H = \dfrac{3}{s+1}, G_f = 5$

a. Design a controller that results in a stable response with

- $T_p = \dfrac{\pi}{4}s$
- Zero Steady-State error

Is a second-order approximation valid? Justify. Hint: Choose a convenient constraint for ζ or T_s to simplify your calculations.

b. Show that a steady-state error of the closed loop system, to a step input, is zero.

c. Given your designed controller, derive the transfer function that relates the input $r(t)$ to the controller output $u(t)$

Solution:

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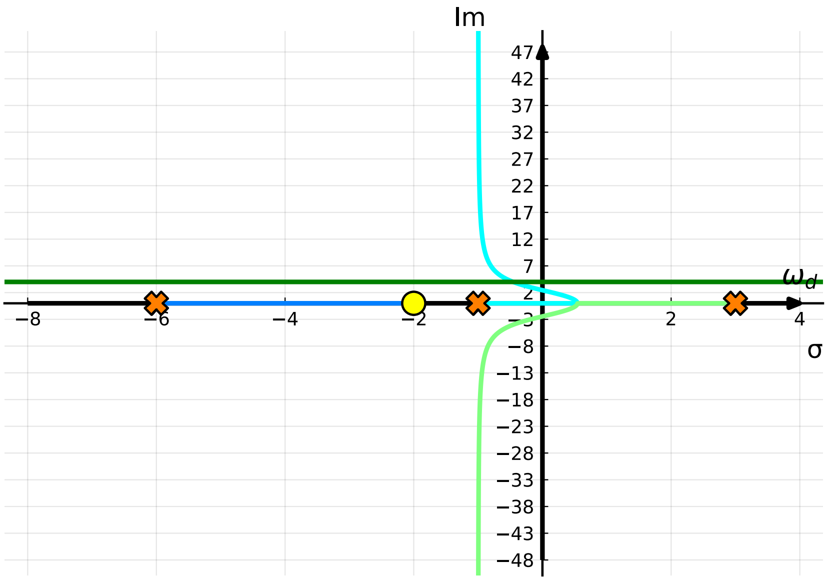
a.

The open-loop transfer function is $K\dfrac{6.0s+12.0}{1.0s^3+4.0s^2-15.0s-18.0}$

Let's first define the design point

The root-locus with the design criteria is shown



The root-locus does intersect the design point/line, the gain can be calculated geometrically or by substituting for the constraint in the characteristic equation and solving. The gain value at the intercept is $K = 5.67$

Factoring out the plant gain 6.0, the controller is then $G_c = 0.945$

The system is in fact second order.

To make the system have zero steady-state error, we can add a PI controller of the form $G_{PI} = \frac{s + 0.01}{s}$, but that would result in a closed-loop system that is unstable, since there will always be a segment in the RHP. $Gc = 0.945$

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b.

The steady-state error for the system is note zero, since the open-loop is type 0 and we can't add a PI controller.

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c.

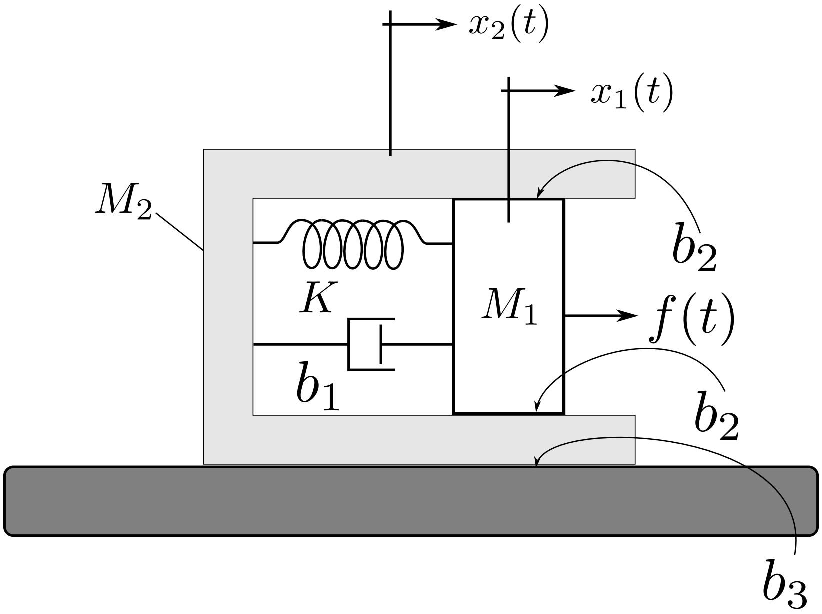
The closed-loop transfer function relating r to u is

$$G_f \frac{G_c}{1 + G_c G_p H} = \frac{4.725\,(s-3)\,(s+1)\,(s+6)}{5.67s + (s-3)\,(s+1)\,(s+6) + 11.34} = \frac{4.72s^3 + 18.9s^2 - 70.9s - 85.1}{1.0s^3 + 4.0s^2 - 9.33s - 6.66}$$

Problem 5

Root Locus Sketching (20pts)

Given the mechanical system shown on the figure. You can use MATLAB to aid in long calculations and verify your work.



With $M_1 = 2kg, M_2 = 2kg, b_1 = 7N \cdot s/m, b_2 = 4N \cdot s/m, b_3 = 2N \cdot s/m, K = 15N/m$

a. Derive the equations of motion for the system

b. Find the transfer function relating the input $f(t)$ to $x_2(t)$, $G_2(s) = \frac{X_2(s)}{f(s)}$

c. Analyze the stability of the system $G_2(s)$

d. Design a feedback controller, using root-locus technique, around G_2 to achieve

- Zero Steady-State error

- $T_s = 0.5s$

- $\zeta = 0.5$

Justify if the system can be approximated as second order.

e. Derive the transfer function relating the reference $r(t)$ to $x_2(t)$

f. Derive the transfer function relating the reference $r(t)$ to $x_1(t)$, with the feedback system derived above.

Solution:

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a. The equations of motion for the system

$K(x_1 - x_2) + M_1(0) - f + (b_1 + 2b_2)((0) - (0)) = 0$

$-K(x_1 - x_2) + M_2(0) + b_3(0) + (-b_1 - 2b_2)((0) - (0)) = 0$

Taking the Laplace Transform, we get

$K(X_1 - X_2) + M_1X_1s^2 - f + (b_1 + 2b_2)(X_1s - X_2s) = 0$

$-K(X_1 - X_2) + M_2X_2s^2 + X_2b_3s + (-b_1 - 2b_2)(X_1s - X_2s) = 0$

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b. Grouping the terms as $Ax = B$

$$\begin{bmatrix} K + M_1s^2 + b_1s + 2b_2s & -K - b_1s - 2b_2s \\ -K - b_1s - 2b_2s & K + M_2s^2 + b_1s + 2b_2s + b_3s \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for x, we find the two transfer functions relating the inputs to the two outputs of the system.

$$G_2(s) = \frac{K + b_1s + 2b_2s}{KM_1s^2 + KM_2s^2 + Kb_3s + M_1M_2s^4 + M_1b_1s^3 + 2M_1b_2s^3 + M_1b_3s^3 + M_2b_1s^3 + 2M_2b_2s^3 + b_1b_3s^2 + 2b_2b_3s^2}$$

$$= \frac{15}{2s(2s^2 + 30s + 15)}$$

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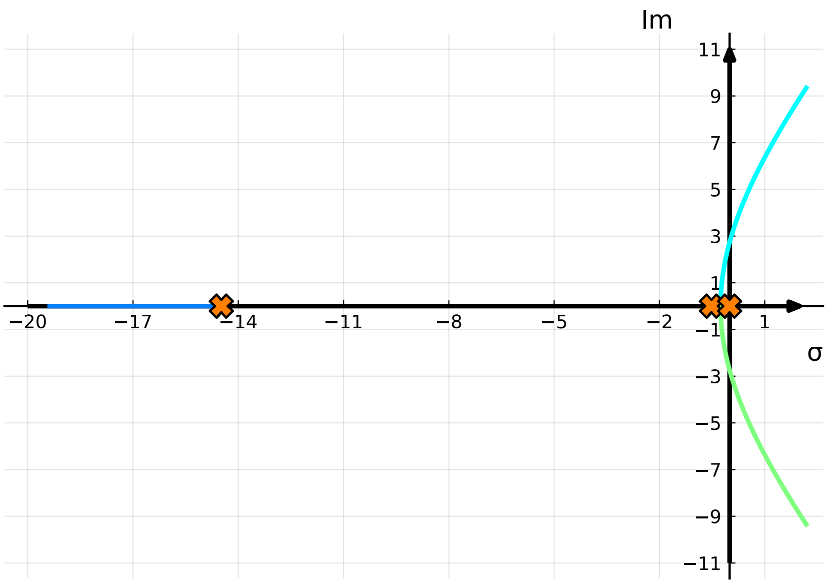
c. The root-locus of the system is shown



d.

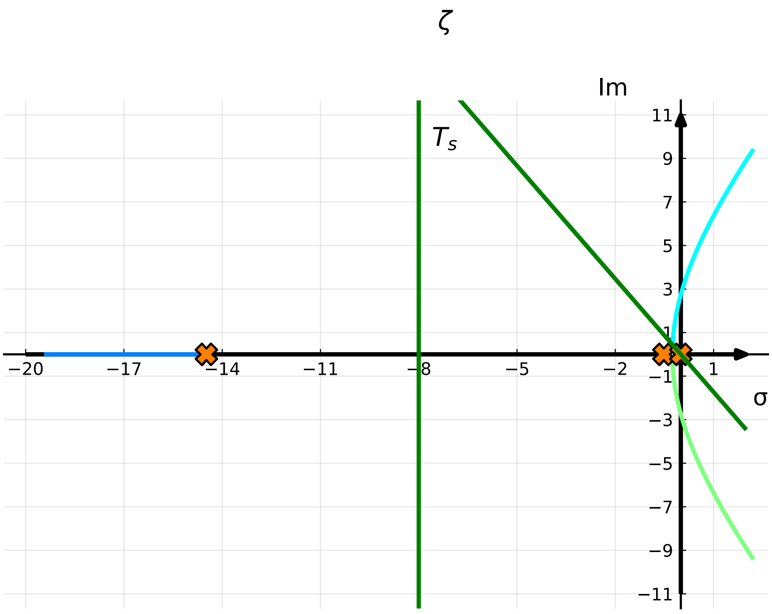
To get the desired closed-pool, a PD controller is required.

The open-loop transfer function is $G_c = \frac{3.75}{1.0s^3 + 15.0s^2 + 7.5s}$.

Given the design requirements ζ and T_s, we get $\sigma = -\frac{4}{T_s} = -8.0$, $\omega_n = \frac{-\sigma}{\zeta} = 16.0$ and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 13.9$:

a design point at $s = \sigma \pm \omega_d i = -8.0 \pm 13.9i$

Sketching the root-locus, with the design requirements, we get



The controller $G_{PD} = K$ is not sufficient to place the closed-loop pole in the desired location. A PD, $G_{PD} = K(s + z)$, controller can be used to place the root-locus over the design point.

Find the location of the zero using the angle condition. Where θ_{zn} is the angle contribution of the added zero.

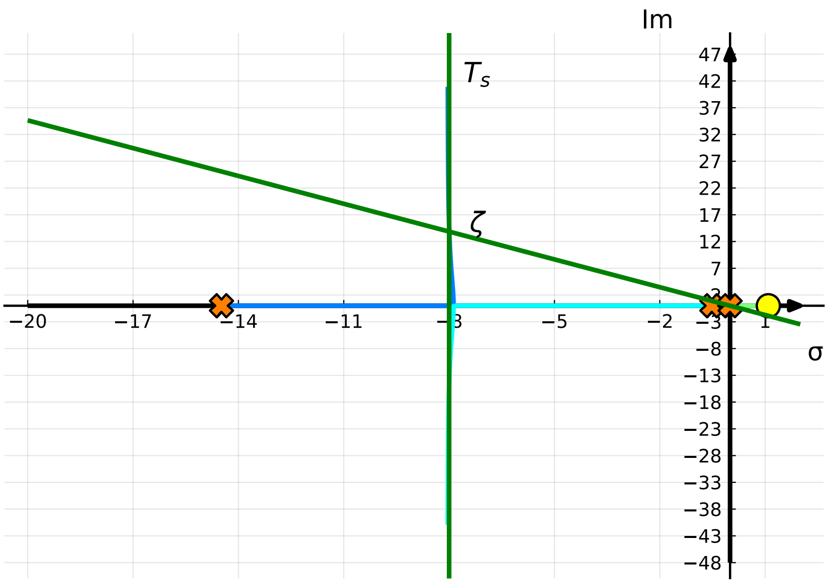
$\angle KG(s) = \sum \theta_p - \sum \theta_z = \pm(2k+1)180 = 5.29 - \theta_{addz} \rightarrow \theta_{addz} = 5.29 - \pi = 2.15$

Solving for the zero location: $tan(\theta_{zn}) = \dfrac{im(s_d)}{re(s_d) + z} \rightarrow z = \dfrac{im(s_d)}{tan(\theta_{zn})} - re(s_d) = -1.09$

The PD controller is now $G_{PD} = K(s + z_n) = K(s + -1.09)$, and the gain can be computed by the magnitude condition.

To find the gain, we can use the magnitude condition: $K = \dfrac{\prod L_p}{\prod L_z} = \dfrac{3880.0}{16.6} = 234.0$
The gain of the controller is given by $K_c = K/K_{Gol} = 234.0/3.75 = 62.4$

And the PD controller becomes $G_{PD} = 62.4(s + -1.09)$



e.

This is the closed-loop transfer function.
$G_{cl2} = \dfrac{G_c G_2}{1 + G_c G_2} = \dfrac{1.0\,(936.0s - 1020.24)}{4.0s^3 + 60.0s^2 + 966.0s - 1020.24}$

f.

$$G_{cl2} = \frac{G_c G_1}{1 + G_c G_2} = \frac{1.0\left(124.8s^2 + 799.968s - 1020.24\right)}{4.0s^3 + 60.0s^2 + 966.0s - 1020.24}$$