Kuwait University
College of Engineering and Petroleum

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ME417 CONTROL OF MECHANICAL SYSTEMS

Part II: Controller Design Via State-Space

Lecture 4: Introduction to Linear Optimal Control

Summer 2020

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- Objectives:
  - Introduce the concept of Optimal Control
  - Introduce the Linear Quadratic Regulator Problem LQR
- Reading:
  - *Modern Control Systems, Dorf & Bishop. Chapter 11 – Section on Optimal Control Systems*

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- Suppose you are working on a construction project
- You want to optimize the outcomes of the project:
  - You wish to minimize time and money, and maximize quality.
- Assume you can only control the effort you input into the system. Stated mathematically:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u, t) = \boldsymbol{A}(\boldsymbol{x}, t) + \boldsymbol{B}(u, t)$$

Where $\boldsymbol{x} = \begin{bmatrix} Time & Money & \frac{1}{Quality} \end{bmatrix}^T, u = Effort$

- Suppose for simplicity, that the system is **LTI** (absurd simplification)

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}u$$

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- Suppose you apply a feedback controller

$$u_{effort} = \boldsymbol{K}(\boldsymbol{x}_{desired} - \boldsymbol{x})$$
$$\dot{\boldsymbol{x}} = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\boldsymbol{x} + \mathbf{B}\boldsymbol{K}\boldsymbol{x}_{desired}$$

- $\boldsymbol{x}_{desired}$: Can be your desired target for money, time and quality for instance.
- You want to find the values of the gain matrix $\boldsymbol{K}$ that optimizes your project outcome:
  - So that you apply just the right amount of effort to achieve your goal.
- To achieve your optimization goal, you first must come up with a cost function in which you weight the relative importance of each state and input

$$f_{cost} = \int_{t=0}^{t=\infty} w_1 \cdot Time + w_2 \cdot Money + w_3 \frac{1}{Quality} + w_4 \cdot Effort$$

The Optimization Problem - Conceptually

$$f_{cost} = \int_{t=t_0}^{t=t_1} w_1 \cdot Time + w_2 \cdot Money + w_3 \frac{1}{Quality} + w_4 \cdot Effort$$

- The cost function computes the weighted sum of the four measures over a period of time.
- The coefficients $w_1, \ldots, w_4$ are weights we, the problem designers, place in the cost function based on the relative importance we give to each measure.
- So, suppose you want to penalize delays in the project heavily but can tolerate additional expenses, then you put more weight w_1 relative to w_2
- The weights have to be scaled to account for the units/scale of each variable.
    - Is delaying one extra day as bad as spending one extra KWD?

- The solution for the gain matrix $\boldsymbol{K}$, is computed by minimizing the cost function

$$\min f_{cost} \Rightarrow \partial f_{cost} = 0$$

Subject to: $\dot{\boldsymbol{x}} = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\boldsymbol{x} + \mathbf{B}\boldsymbol{K}\boldsymbol{x}_{desired}$

- Plus any other constraints we put on the system/project
  - Such as maximum effort, maximum delay, minimum quality, etc.

- In the optimization problem above, what you tune are the weights you give in the cost function, while the gains in $\boldsymbol{K}$ are to be *computed* by "mathematical" optimization.
- The cost function, is also referred to as the "performance index"

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- In applying the concept of optimization to our state-feedback control structure.
  - We don't "place" the poles of the system directly, but instead, solve the optimization problem, that would indirectly place the poles of the system for us, and consequently, compute the state feedback gain $\boldsymbol{K}$
- Given the following LTI mechanical system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

- We wish to apply a state feedback controller that regulates the system, and equally minimizes the states $x_1\ and\ x_2$ over time (position and velocity).
  - Regulates: returns the system to $\boldsymbol{x} = \boldsymbol{0}$, *with* $r = 0$
  - In the regulation problem the states correspond to the error $\boldsymbol{e} = 0 - \boldsymbol{x}$

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$$u = -\boldsymbol{K}\boldsymbol{x}$$
$$\dot{\boldsymbol{x}} = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\boldsymbol{x} = \widetilde{\boldsymbol{A}}\boldsymbol{x}$$

- We set the cost function (performance index) as

$$J = \int_0^\infty (x_1^2 + x_2^2)\, dt$$

- Note that $x_1^2 + x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{x}^T\boldsymbol{x}$, so

$$J = \int_0^\infty (\boldsymbol{x}^T\boldsymbol{x})\, dt$$

- To minimize the performance index J, we assume a form for J s.t.

$$\frac{d}{dt}(J) = \frac{d}{dt}(\boldsymbol{x}^T\mathbf{P}\boldsymbol{x}) = -\boldsymbol{x}^T\boldsymbol{x}$$

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- If we determine the constant matrix $\mathbf{P}$, then we solve for $\frac{d}{dt}(\boldsymbol{x}^T\mathbf{P}\boldsymbol{x}) = 0$, to minimize the performance index

$$\frac{d}{dt}(\boldsymbol{x}^T\mathbf{P}\boldsymbol{x}) = \dot{\boldsymbol{x}}^T\mathbf{P}\boldsymbol{x} + \boldsymbol{x}^T\mathbf{P}\dot{\boldsymbol{x}}$$

- Substitute $\dot{\boldsymbol{x}} = \widetilde{\mathbf{A}}\boldsymbol{x}$, in the above equation

$$\frac{d}{dt}(\boldsymbol{x}^T\mathbf{P}\boldsymbol{x}) = \left(\widetilde{\mathbf{A}}\boldsymbol{x}\right)^T\mathbf{P}\boldsymbol{x} + \boldsymbol{x}^T\mathbf{P}\left(\widetilde{\mathbf{A}}\boldsymbol{x}\right) = \boldsymbol{x}^T\widetilde{\mathbf{A}}\mathbf{P}\boldsymbol{x} + \boldsymbol{x}^T\mathbf{P}\widetilde{\mathbf{A}}\boldsymbol{x} = \boldsymbol{x}^T\left(\widetilde{\mathbf{A}}\mathbf{P} + \mathbf{P}\widetilde{\mathbf{A}}\right)\boldsymbol{x}$$

  - Using the matrix transpose property: $(\mathbf{AB})^{\boldsymbol{T}} = \mathbf{B}^{\boldsymbol{T}}\mathbf{A}^{\boldsymbol{T}}$

- Let's choose a $\mathbf{P}$ such that $\widetilde{\mathbf{A}}\mathbf{P} + \mathbf{P}\widetilde{\mathbf{A}} = -\mathbf{I}$, then

$\frac{d}{dt}(\boldsymbol{x}^T\mathbf{P}\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{T}}(-\mathbf{I})\boldsymbol{x} = -\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{x}$**,** the same equation we formed earlier

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- Now we substitute and integrate the performance index

$$J = \int_0^\infty \frac{d}{dt}(J)\,dt = \int_0^\infty \frac{d}{dt}(\boldsymbol{x}^T \mathbf{P}\boldsymbol{x})\,dt = -\boldsymbol{x}^T \mathbf{P}\boldsymbol{x}\Big|_0^\infty = \boldsymbol{x}^T(0)\mathbf{P}\boldsymbol{x}(\mathbf{0})$$

- Since we assume the system is stable and the states will return to $\mathbf{0}$, $as\ t \to \infty$

$$\boldsymbol{x}^T(\infty)\mathbf{P}\boldsymbol{x}(\infty) = 0$$

- The solution to our optimal controller given the performance index we formed is to find a P that satisfies the following equation

$$\tilde{\mathbf{A}}\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\boldsymbol{K}) = -\mathbf{I}$$

- The solution will be different if we choose a different performance index form.

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- In the previous example we postulated the performance index such that it equally minimizes the position and velocity errors
- But what if we want to penalize the states with relatively different weights?
  - Rather than equally penalize both position and velocity errors, we may want to penalize position error more than velocity error: $w_1 > w_2$

$$J = \int_0^\infty (w_1 x_1^2 + w_2 x_2^2)\, dt = \int_0^\infty \boldsymbol{x}^T \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \boldsymbol{x}\, dt$$

  - Perhaps we don't even care about velocity, we just want to get to position zero as fast as possible

$$J = \int_0^\infty x_1^2\, dt = \int_0^\infty \boldsymbol{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}\, dt$$

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- And what if we want to consider the input u?
  - Perhaps we have limited battery power and wasting a lot of energy to regulate the system is not desired.
  - We can include the input u as part of the performance index

$$J = \int_0^\infty (w_1 x_1^2 + w_2 x_2^2 + w_3 u^2)\, dt$$

- The general formulation for the above performance index is captured in what is called, the linear quadratic regulator, or LQR. Where the performance index is defined as

$$J = \int_0^\infty (q_{11} x_1^2 + \cdots + q_{nn} x_n^2) + (r_1 u_1^2 + \cdots + r_m u_m^2)\, dt$$

  - ignoring the coupling weights (e.g. $q_{12} x_1 x_2$)

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- Given a closed-loop LTI system with full state feedback: $u = r - \boldsymbol{K}\boldsymbol{x}$

$$\dot{\boldsymbol{x}} = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\boldsymbol{x} + \mathbf{B}r$$
$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x}$$

- But there is an optimal way to compute $\boldsymbol{K}$, the gain vector, via the Linear Quadratic Regulator (LQR)
- The LQR performance index

$$J = \int_0^\infty (\boldsymbol{x}^T \mathbf{Q}\boldsymbol{x} + u^T R u)dt$$

$\mathbf{Q}$ is a matrix we chose, to give weights to each state in $\boldsymbol{x}$

$\boldsymbol{R}$ is a weighting factor for the input to the system u

- The optimal $\boldsymbol{K}$ in $u = r - \boldsymbol{Kx}$ is found by minimizing J, subject to the closed-loop system dynamics:

$$min\ \ J = \int_0^{\infty} (\boldsymbol{x}^T \mathbf{Q} \boldsymbol{x} + u^T R u) dt$$

$$\text{s.t. } \dot{\boldsymbol{x}} = (\mathbf{A} - \mathbf{B}\boldsymbol{K})\boldsymbol{x} + \mathbf{B}r$$

- After some more math….
- The solution for $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}$$

- Where $\boldsymbol{P}$ is found from solving the Riccati Equation:

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{PA} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0}$$

- This is not meant to be solved by hand, and we can use MATLAB’s *lqr()* function to solve for $\boldsymbol{K}$, directly.
  - The function *lqr()* also returns the optimal pole locations of the Closed-Loop system.
- $\mathbf{R}$ and $\mathbf{Q}$ must be positive definite

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- As a starting point, choose $\mathbf{Q} = \mathbf{I}$ and $R = 1$
- To improve $\mathbf{Q}$, place more/less weights on the states you care more/less to stabilize (regulate: bring to zero).
- If you care to minimize the input to the system (the effort/energy you spend on the system), then you might want to increase R and vice versa.
- You can generally achieve a good outcome by keeping the **Q** matrix diagonal
  - Unless you want to weight a coupling effect between states.

- Given the two degree of freedom system shown, whose state-space representation is given by

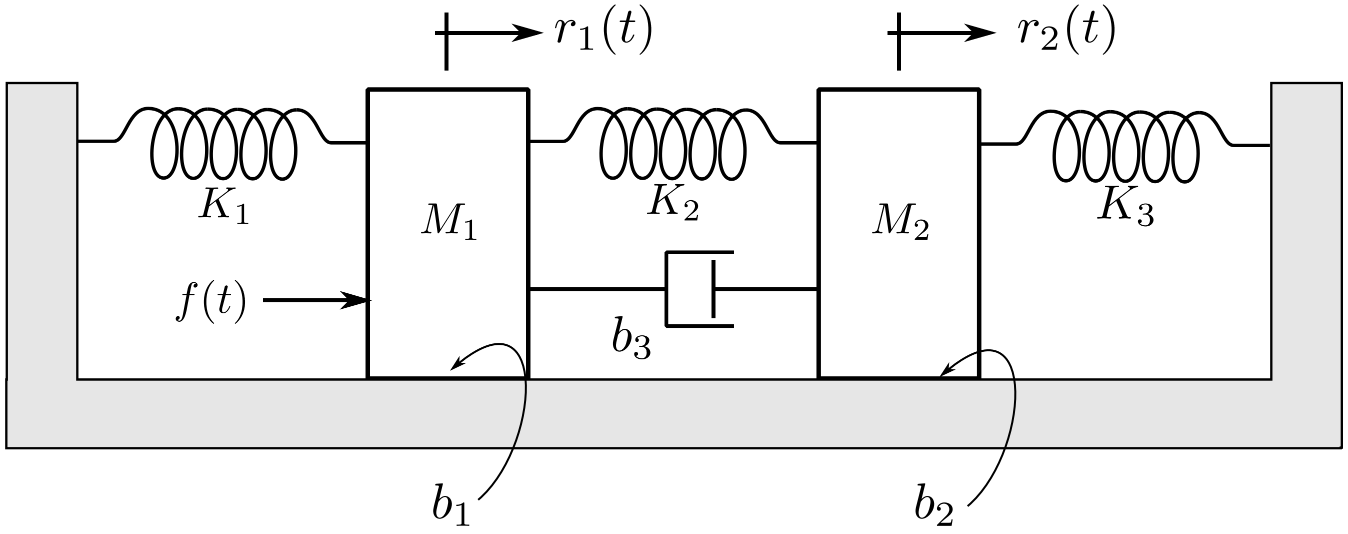
$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-(K_1+K_2)}{m_1} & \dfrac{-(b_1+b_3)}{m_1} & \dfrac{K_2}{m_1} & \dfrac{b_3}{m_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{K_2}{m_2} & \dfrac{b_3}{m_2} & \dfrac{-(K_2+K_3)}{m_2} & \dfrac{-(b_2+b_3)}{m_2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}$$

Let's design a feedback controller

$u = r - \boldsymbol{K}\boldsymbol{x}$, to stabilize the system

From a nonzero initial condition.



- See "*PIII_ L4_ Examples.m*"