ME 417 - Homework #1

Control of Mechanical Systems - Spring 2021

Homework Due: Thu, 22 Apr 2021 23:59

Complete the following problems and submit a hard copy of your solutions. You are encouraged to work together to discuss the problems but submitted work **MUST** be your own. This is an **individually** submitted assignment.

Problem 1

System Modeling (25pts)

An inverted pendulum on a rotating disk is shown. Where θ is the pendulum angle measured from the vertical and ψ is the disk angle.

The equations of motion for the system are given as:

$$l^2 m\ddot{\theta} + lmr\cos(\theta)\ddot{\psi} = b_1\dot{\theta} + glm\sin(\theta)$$
$$lmr\cos(\theta)\ddot{\theta} + \left(J + mr^2\right)\ddot{\psi} = b_2\dot{\psi} + lmr\sin(\theta)\dot{\theta}^2 + \tau$$

a. Linearize the equations of motion (small angle approximation)

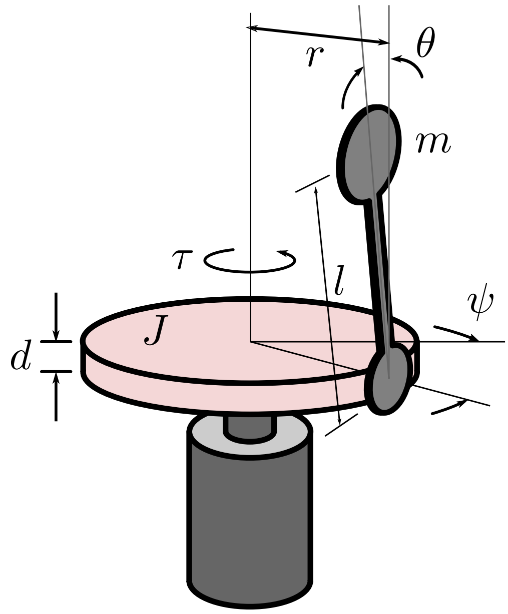
b. Find the transfer function that relates τ to θ and τ to $\dot{\theta}$

c. Draw the block diagram if feedback control is applied to control the pendulum angle θ

c. Find the pole locations of the transfer function derived in part (b)

Given: $r = 12.0cm$, $m_r = 0.3kg$, $J = 15kg \cdot m^2$, $l = 20.0cm$

Neglect friction in the system.



Solution:

a.

The linearized equations of motion are (neglecting friction)

$l^2 m\ddot{\theta} + lmr\ddot{\psi} = glm\theta$

$lmr\ddot{\theta} + \left(J + mr^2\right)\ddot{\psi} = \tau$

b.

The transfer function that relates τ to θ is $\frac{r}{Jg - Jls^2 + gmr^2} = \frac{0.12}{147.1923792 - 3.0s^2}$

The transfer function that relates τ to $\dot{\theta}$ is $\frac{rs}{Jg - Jls^2 + gmr^2} = \frac{0.12s}{147.1923792 - 3.0s^2}$

c.

d.

The poles of the transfer function in b are $8.573214099741124 + 0.0im, -8.573214099741122 + 0.0im$

Problem 2

System Modeling (25pts)

Given the following system

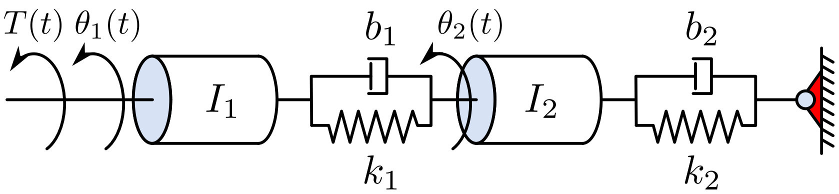
a. Derive the equations of motion for the system

b. Find the transfer function that relates T to θ_2

c. Find the steady state value of θ_2 given a step-input $T(t) = 20$

d. Draw a feedback block diagram if you wanted to control θ_2, show how the output signal θ_1 will be represented.

Given: $I_1 = 0.3kg \cdot m^2$, $I_2 = 0.25kg \cdot m^2$, $k_1 = 280N/m$, $k_2 = 380N/m$, $b_1 = 45N \cdot s/m$, $b_2 = 35N \cdot s/m$



Solution:

a.

By using the impedance method

$[I_1 s^2 + b_1 s + k_1]\Theta_1(s) - [b_1 s + k_1]\Theta_2(s) = T(s)$

$-[b_1 s + k_1]\Theta_1(s) + [I_2 s^2 + (b_1 + b_2)s + (k_1 + k_2)]\Theta_2(s) = 0$

b.

Decoupling the EOM using Cramer's rule, we can find $G_2(s) = \dfrac{\theta_2(s)}{T(s)}$

$$G_2(s) = \frac{-a_2 y_1}{\Delta} = \frac{b_1 s + k_1}{I_1 I_2 s^4 + I_1 b_1 s^3 + I_1 b_2 s^3 + I_1 k_1 s^2 + I_1 k_2 s^2 + I_2 b_1 s^3 + I_2 k_1 s^2 + b_1 b_2 s^2 + b_1 k_2 s + b_2 k_1 s + k_1 k_2}$$

$$G_2(s) = \frac{45.0s + 280.0}{0.075s^4 + 35.3s^3 + 1.84 \cdot 10^3 s^2 + 2.69 \cdot 10^4 s + 1.06 \cdot 10^5}$$

c.

To find the steady state value, we apply the final value theorem

$$c_{ss}(t) = \lim_{s \to 0} s\Theta_2(s) = \lim_{s \to 0} sG(s)R(s) = \lim_{s \to 0} \frac{20\,(45.0s + 280.0)}{0.075s^4 + 35.3s^3 + 1.84 \cdot 10^3 s^2 + 2.69 \cdot 10^4 s + 1.06 \cdot 10^5} =$$

0.0526

Problem 3

Time Response (25pts)

Given the following transfer function relating force to position

$$\frac{X}{F} = \frac{40}{s\,(s+5)}$$

Derive the partial fraction expansion form for the output, sketch (by hand) the time response for position and velocity on the same figure, and find the steady-state output value for position for each of the following inputs.

a. $u_a(t) = 2$

b. $u_b(t) = 6t + 3$

c. $u_c(t) = 0.2e^{-2t}$

d. $u_d(t) = 2te^{-4t}$

Solution:

a.

Converting the input into the Laplace dominant

$$u(t) \Rightarrow U(s) = \mathcal{L}_t\,[2]\,(s) = \frac{2}{s}$$

The output in the Laplace domain is then:

$$C(s) = \frac{80}{s^2\,(s+5)}$$

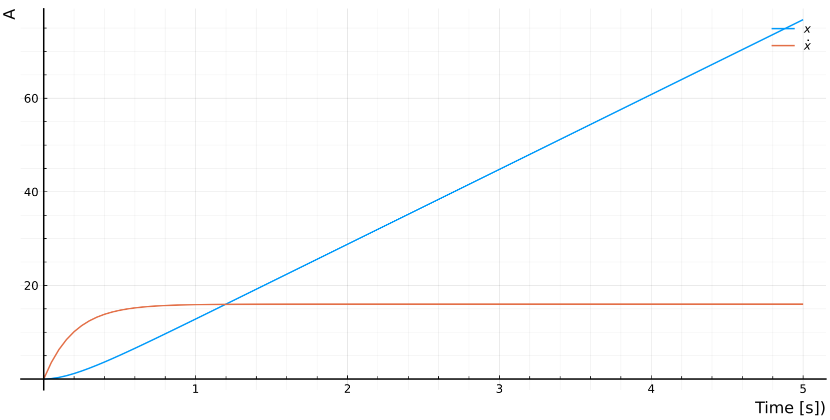
After partial fraction expansion, we get:

$$C(s) = \frac{3.2}{s+5.0} - \frac{3.2}{s} + \frac{16.0}{s^2}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[\frac{3.2}{s+5.0} - \frac{3.2}{s} + \frac{16.0}{s^2}\right](t) = 16t - \frac{16}{5} + \frac{16e^{-5t}}{5}$$

The following is the time response for position and velocity.



b.

Converting the input into the Laplace dominant

$u(t) \Rightarrow U(s) = \mathcal{L}_t\left[6t + 3\right](s) = \frac{3\,(s+2)}{s^2}$

The output in the Laplace domain is then:

$$C(s) = \frac{120\,(s+2)}{s^3\,(s+5)}$$

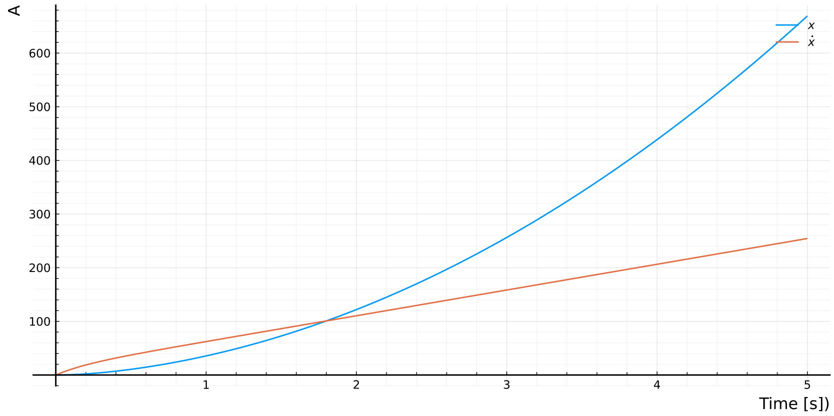
After partial fraction expansion, we get:

$$C(s) = \frac{2.88}{s+5.0} - \frac{2.88}{s} + \frac{14.4}{s^2} + \frac{48.0}{s^3}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[\frac{2.88}{s+5.0} - \frac{2.88}{s} + \frac{14.4}{s^2} + \frac{48.0}{s^3}\right](t) = 24t^2 + \frac{72t}{5} - \frac{72}{25} + \frac{72e^{-5t}}{25}$$

The following is the time response for position and velocity.



c.

Converting the input into the Laplace dominant

$$u(t) \Rightarrow U(s) = \mathcal{L}_t\left[0.2e^{-2t}\right](s) = \frac{0.2}{s+2}$$

The output in the Laplace domain is then:

$$C(s) = \frac{8.0}{s(s+2)(s+5)}$$

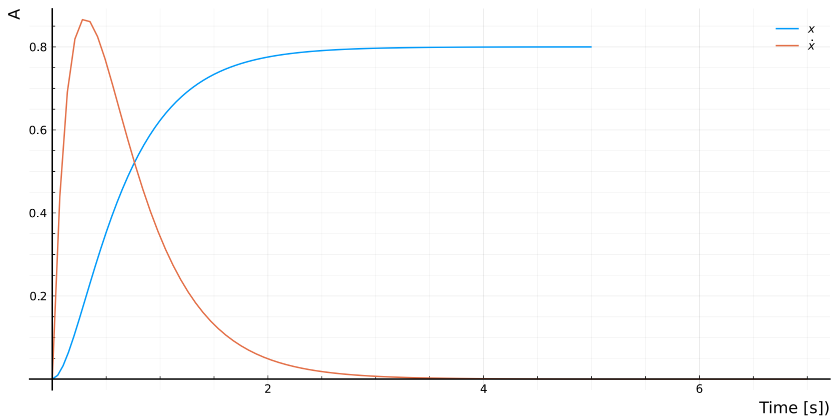
After partial fraction expansion, we get:

$$C(s) = -\frac{0.667}{0.5s+1.0} + \frac{0.107}{0.2s+1.0} + \frac{0.8}{s}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[-\frac{0.667}{0.5s+1.0} + \frac{0.107}{0.2s+1.0} + \frac{0.8}{s}\right](t) = 0.8 - 1.33333333333333e^{-2t} + 0.533333333333333e^{-5t}$$

The following is the time response for position and velocity.



d.

Converting the input into the Laplace dominant

$$u(t) \Rightarrow U(s) = \mathcal{L}_t\left[2te^{-4t}\right](s) = \frac{2}{(s+4)^2}$$

The output in the Laplace domain is then:

$$C(s) = \frac{80}{s\,(s+4)^2\,(s+5)}$$

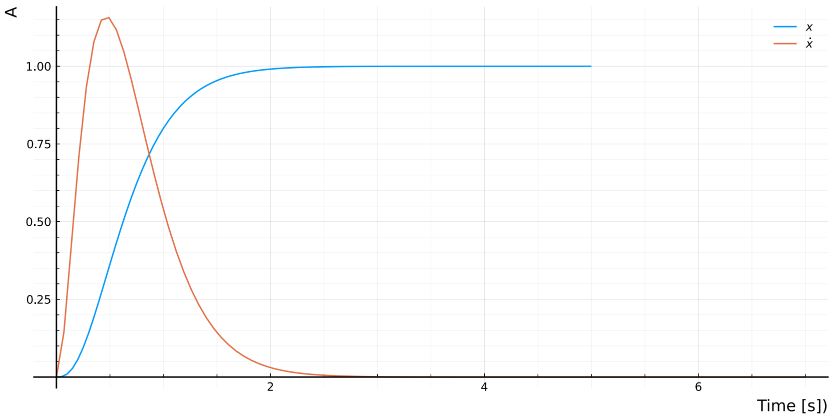
After partial fraction expansion, we get:

$$C(s) = -\frac{16.0}{s+5.0} + \frac{15.0}{s+4.0} - \frac{1.25}{(0.25s+1)^2} + \frac{1}{s}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[-\frac{16.0}{s+5.0} + \frac{15.0}{s+4.0} - \frac{1.25}{(0.25s+1)^2} + \frac{1}{s}\right](t) = -20te^{-4t} + 1 + 15e^{-4t} - 16e^{-5t}$$

The following is the time response for position and velocity.


A
1.00
0.75
0.50
0.25
2
4
6
Time [s])
x
ẋ

Problem 4

Transfer Function Components (25pts)

For each of the following 3rd order systems, perform a partial fraction expansion, then cancel the third pole term if it is real magnitude is five times or higher than the real magnitude of the other two poles

a. $G(s) = \dfrac{10}{(s+10)(s^2+2s+20)}$

b. $G(s) = \dfrac{4}{(s+4)(s+5)(s+30)^2}$

c. $G(s) = \dfrac{10}{(s+5)(s^2+2s+8)}$

d. $G(s) = \dfrac{1}{(s+18)(s^2+6s+100)}$

e. $G(s) = \dfrac{5}{(s+5)(s^2+4s+20)}$

Solution:

a.

Partial fraction expansion: $G(s) = -\dfrac{s-8}{10(s^2+2s+20)} + \dfrac{1}{10(s+10)}$

The third pole @ -10.0 is more than five times further away on the real-axis relative to the dominant poles @ -1.0 , and the pole term is cancelled.

b.

Partial fraction expansion: $G(s) = \dfrac{51}{105625(s+30)} + \dfrac{2}{325(s+30)^2} - \dfrac{4}{625(s+5)} + \dfrac{1}{169(s+4)}$

The third pole @ -30 is more than five times further away on the real-axis relative to the dominant poles @ -4 , and the pole term is cancelled.

c.

Partial fraction expansion: $G(s) = -\dfrac{10(s-3)}{23(s^2+2s+8)} + \dfrac{10}{23(s+5)}$

The third pole @ -5.0 is more than five times further away on the real-axis relative to the dominant poles @ -1.0 , and the pole term is cancelled.

d.

Partial fraction expansion: $G(s) = -\dfrac{s-12}{316(s^2+6s+100)} + \dfrac{1}{316(s+18)}$

The third pole @ -18.0 is more than five times further away on the real-axis relative to the dominant poles @ -3.0 , and the pole term is cancelled.

e.

Partial fraction expansion: $G(s) = -\dfrac{s-1}{5\left(s^2+4s+20\right)} + \dfrac{1}{5\left(s+5\right)}$

The third pole @ -5 is not more than five times further away on the real-axis relative to the dominant poles @ -2, and the pole term is not cancelled.