ME 417 - Homework #1

Control of Mechanical Systems - Fall 2020

Homework Due: Thu, 24 Dec 2020 23:59

Complete the following problems and submit a hard copy of your solutions. You are encouraged to work together to discuss the problems but submitted work **MUST** be your own. This is an **individually** submitted assignment.

Problem 1

System Modeling (25pts)

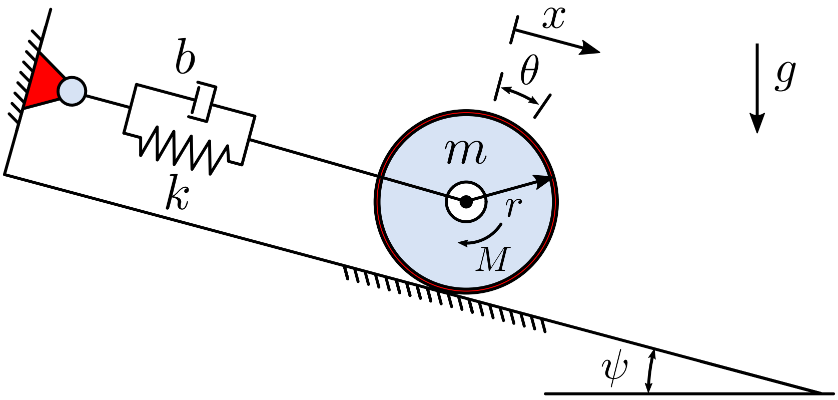
A disk of uniform mass rolls without slipping on an inclined surface as shown.

a. Derive the equations of motion for the system.

b. Find the transfer function that relates M to $\dot{\theta}$.

c. Find the pole locations of the transfer function derived in part (b)

Given: $r = 0.2m$, $m_r = 2.5kg$, $k = 150N/m$, $b = 60N \cdot s/m$, $\psi = 20^o$



Solution:

a.

We can sum the forces in the x-y direction and the moment in the z-direction to get:

$\sum F_x = -b\dot{x} - f_t + gm\sin(\psi) - kx = m\ddot{x}$

$\sum M = M + f_t r = I\ddot{\theta}$

At equilibruim, we can ignore gravity:

$\sum F_x = -b\dot{x} - f_t - kx = m\ddot{x}$

b.

Given no slip condition, $x = r\theta$, subtituting and taking the laplace transform, we get:

$$\frac{-I\ddot{\theta} + M - r\left(br\dot{\theta} + kr\theta + mr\ddot{\theta}\right)}{r} = 0$$

$$-\frac{Is^2 + r^2\left(bs + k + ms^2\right)}{r}\Theta(s) = \frac{1}{r}M(s)$$

And the transfer function becomes:

$$\frac{\dot{\Theta}(s)}{M(s)} = -\frac{s}{Is^2 + r^2\left(bs + k + ms^2\right)} = -\frac{s}{0.15s^2 + 2.4s + 6.0}$$

c.

The poles of the system are $\begin{bmatrix} -12.9 \\ \\ -3.1 \end{bmatrix}$

Problem 2

System Modeling (25pts)

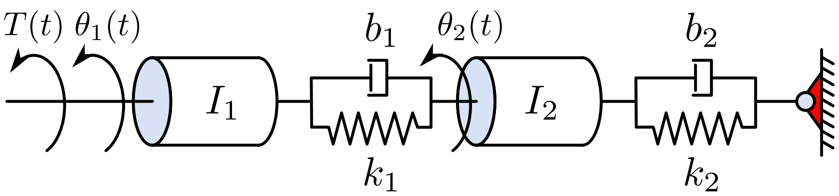
Given the following system

a. Derive the equations of motion for the system

b. Find the transfer function that relates T to θ_1

c. Find the steady state value of θ_1 given a step-input $T(t) = 10t$

Given: $I_1 = 0.2kg \cdot m^2$, $I_2 = 0.15kg \cdot m^2$, $k_1 = 280N/m$, $k_2 = 180N/m$, $b_1 = 35N \cdot s/m$, $b_2 = 25N \cdot s/m$



Solution:

a.

By using the impedance method

$[I_1 s^2 + b_1 s + k_1]\Theta_1(s) - [b_1 s + k_1]\Theta_2(s) = T(s)$

$-[b_1 s + k_1]\Theta_1(s) + [I_2 s^2 + (b_1 + b_2)s + (k_1 + k_2)]\Theta_2(s) = 0$

b.

Decoupling the EOM using Cramer's rule, we can find $G_1(s) = \dfrac{\theta_1(s)}{T(s)}$

$$G_1(s) = \frac{b_2 y_1}{\Delta} = \frac{I_2 s^2 + b_1 s + b_2 s + k_1 + k_2}{I_1 I_2 s^4 + I_1 b_1 s^3 + I_1 b_2 s^3 + I_1 k_1 s^2 + I_1 k_2 s^2 + I_2 b_1 s^3 + I_2 k_1 s^2 + b_1 b_2 s^2 + 4 b_1 k_1 s + b_1 k_2 s + b_2 k_1 s + k_1 k_2}$$

$$G_1(s) = \frac{0.15s^2 + 60.0s + 460.0}{0.03s^4 + 17.3s^3 + 1.01 \cdot 10^3 s^2 + 5.25 \cdot 10^4 s + 5.04 \cdot 10^4}$$

c.

To find the steady state value, we apply the final value theorem

$c_{ss}(t) = \lim_{s \to 0} s\Theta_1(s) = \lim_{s \to 0} sG(s)R(s) = \lim_{s \to 0} \dfrac{10\left(0.15s^2 + 60.0s + 460.0\right)}{s\left(0.03s^4 + 17.3s^3 + 1.01 \cdot 10^3 s^2 + 5.25 \cdot 10^4 s + 5.04 \cdot 10^4\right)} =$
$\tilde{\infty}$

Problem 3

Time Response (25pts)

Given the following transfer function relating force to position

$$\frac{X}{F} = \frac{50.0}{s\,(s + 6.0)}$$

Derive the partial fraction expansion form for the output, sketch (by hand) the time response for position and velocity on the same figure, and find the steady-state output value for position for each of the following inputs.

a. $u_a(t) = 5$

b. $u_b(t) = 12.0t + 6.0$

c. $u_c(t) = 0.5e^{-4t}$

d. $u_d(t) = 3.0te^{-2t}$

Solution:

a.

Converting the input into the Laplace dominant

$u(t) \Rightarrow U(s) = \mathcal{L}_t\,[5]\,(s) = \dfrac{5.0}{s}$

The output in the Laplace domain is then:

$$C(s) = \frac{250.0}{s^2\,(s + 6.0)}$$

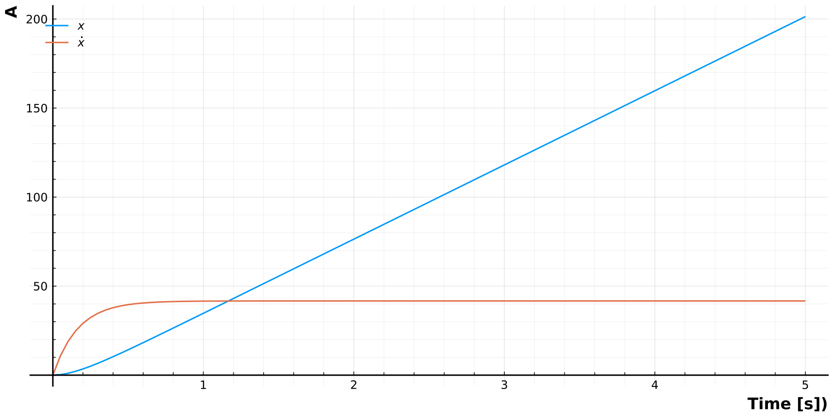
After partial fraction expansion, we get:

$$C(s) = \frac{6.94}{s + 6.0} - \frac{6.94}{s} + \frac{41.7}{s^2}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[\frac{6.94}{s + 6.0} - \frac{6.94}{s} + \frac{41.7}{s^2}\right](t) = 41.7t - 6.94 + 6.94e^{-6t}$$

The following is the time response for position and velocity.



b.

Converting the input into the Laplace dominant

$u(t) \Rightarrow U(s) = \mathcal{L}_t\left[12t + 6\right](s) = \frac{6.0\,(s + 2.0)}{s^2}$

The output in the Laplace domain is then:

$$C(s) = \frac{300.0\,(s + 2.0)}{s^3\,(s + 6.0)}$$

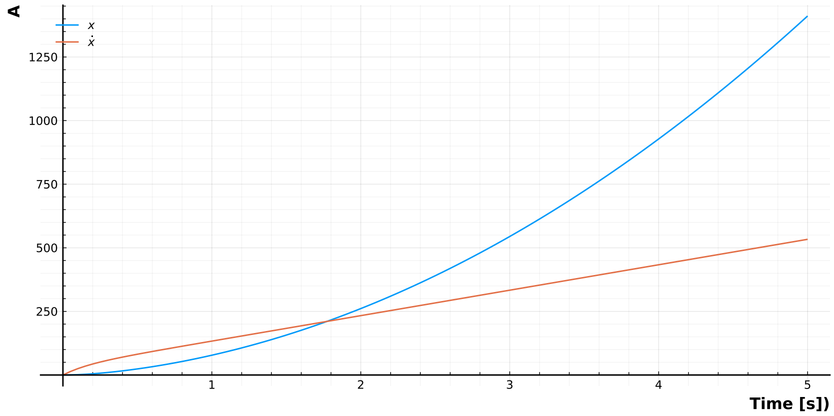
After partial fraction expansion, we get:

$$C(s) = \frac{5.56}{s + 6.0} - \frac{5.56}{s} + \frac{33.3}{s^2} + \frac{100.0}{s^3}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[\frac{5.56}{s + 6.0} - \frac{5.56}{s} + \frac{33.3}{s^2} + \frac{100.0}{s^3}\right](t) = 50.0t^2 + 33.3t - 5.56 + 5.56e^{-6t}$$

The following is the time response for position and velocity.



c.

Converting the input into the Laplace dominant

$$u(t) \Rightarrow U(s) = \mathcal{L}_t\left[0.5e^{-4t}\right](s) = \frac{0.5}{s + 4.0}$$

The output in the Laplace domain is then:

$$C(s) = \frac{25.0}{s\,(s + 4.0)\,(s + 6.0)}$$

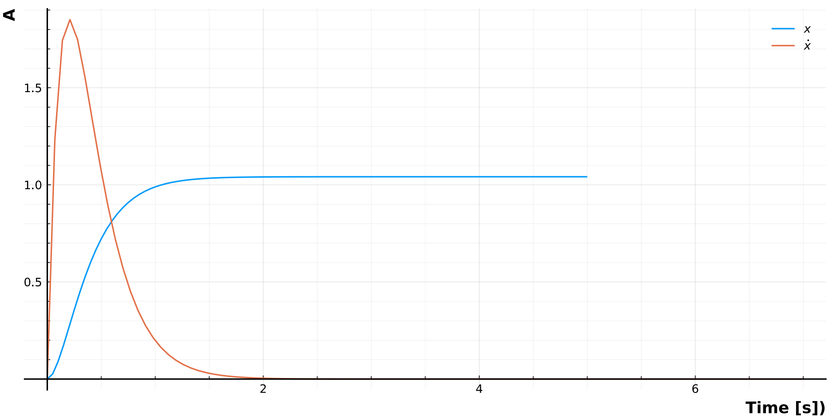
After partial fraction expansion, we get:

$$C(s) = -\frac{0.781}{0.25s + 1.0} + \frac{0.347}{0.167s + 1.0} + \frac{1.04}{s}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[-\frac{0.781}{0.25s + 1.0} + \frac{0.347}{0.167s + 1.0} + \frac{1.04}{s}\right](t) = 1.04 - 3.13e^{-4t} + 2.08e^{-6t}$$

The following is the time response for position and velocity.



d.

Converting the input into the Laplace dominant

$$u(t) \Rightarrow U(s) = \mathcal{L}_t\left[3te^{-2t}\right](s) = \frac{0.75}{(0.5s+1)^2}$$

The output in the Laplace domain is then:

$$C(s) = \frac{37.5}{s\,(0.5s+1)^2\,(s+6.0)}$$

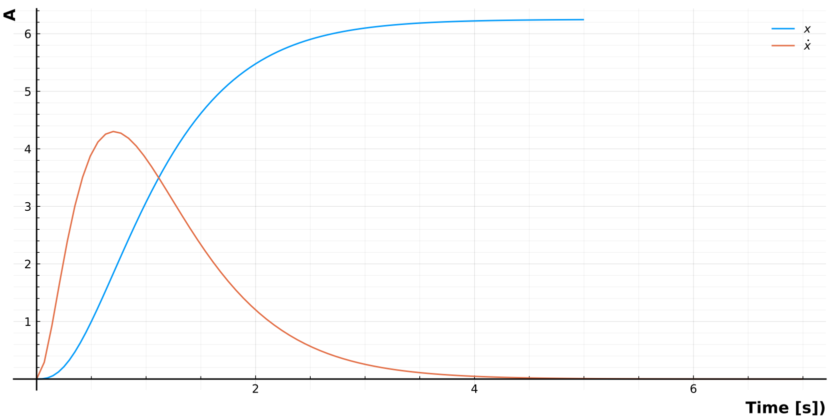
After partial fraction expansion, we get:

$$C(s) = -\frac{1.56}{s+6.0} - \frac{4.69}{s+2.0} - \frac{4.69}{(0.5s+1.0)^2} + \frac{6.25}{s}$$

Taking the Laplace inverse to get the output in the time-domain, we get:

$$c(t) = \mathcal{L}_s^{-1}\left[-\frac{1.56}{s+6.0} - \frac{4.69}{s+2.0} - \frac{4.69}{(0.5s+1)^2} + \frac{6.25}{s}\right](t) = -18.8te^{-2t} + 6.25 - 4.69e^{-2t} - 1.56e^{-6t}$$

The following is the time response for position and velocity.



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Time [s])
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Problem 4

Transfer Function Components (25pts)

For each of the following 3rd order systems, perform a partial fraction expansion, then cancel the third pole term if it is real magnitude is five times or higher than the real magnitude of the other two poles

a. $G(s) = \dfrac{20}{(s+4)(s^2+3s+20)}$

b. $G(s) = \dfrac{4}{(s+1)(s+2)(s+20)^2}$

c. $G(s) = \dfrac{2}{(s+10)(s^2+2s+8)}$

d. $G(s) = \dfrac{1}{(s+10)(s^2+5s+100)}$

e. $G(s) = \dfrac{5}{(s+1)(s^2+3s+20)}$

Solution:

a.

Partial fraction expansion: $G(s) = -\dfrac{5(s-1)}{6(s^2+3s+20)} + \dfrac{5}{6(s+4)}$

The third pole @ -4.0 is not more than five times further away on the real-axis relative to the dominant poles @ -1.5, and the pole term is not cancelled.

b.

Partial fraction expansion: $G(s) = \dfrac{37}{29241(s+20)} + \dfrac{2}{171(s+20)^2} - \dfrac{1}{81(s+2)} + \dfrac{4}{361(s+1)}$

The third pole @ -20 is more than five times further away on the real-axis relative to the dominant poles @ -1 , and the pole term is cancelled.

c.

Partial fraction expansion: $G(s) = -\dfrac{s-8}{44(s^2+2s+8)} + \dfrac{1}{44(s+10)}$

The third pole @ -10.0 is more than five times further away on the real-axis relative to the dominant poles @ -1.0 , and the pole term is cancelled.

d.

Partial fraction expansion: $G(s) = -\dfrac{s-5}{150(s^2+5s+100)} + \dfrac{1}{150(s+10)}$

The third pole @ -10.0 is not more than five times further away on the real-axis relative to the dominant poles @ -2.5, and the pole term is not cancelled.

e.

Partial fraction expansion: $G(s) = -\frac{5\,(s+2)}{18\,(s^2+3s+20)} + \frac{5}{18\,(s+1)}$

The third pole @ -1.5 is not more than five times further away on the real-axis relative to the dominant poles @ -1.0, and the pole term is not cancelled.