`laplace/body_ax` - an excerpt from the book C.Pozrikidis 'A Practical Guide to Boundary-Element Methods'

with additional notes and commentaries

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Introduction

- This is a code that computes potential flow past an axisymmetric, compact (singly connected) or toroidal (doubly connected) body with arbitrary geometry, as illustrated in Figure
- original *Fortran77* code is here CFDLAB code for `laplace/body_ax`

Mathematical Formulation

The velocity u is decomposed into three parts as

$$u = u^{\infty} + v + u^{D} \tag{1}$$

where:

- u^{∞} is the far-field component prevailing far from the body, expressing uniform (streaming) flow along the x axis of symmetry.
- v is the velocity due to a line vortex ring with specified strength situated in the interior of the body, generating circulation around the toroidal body. In the case of flow past a compact body, this component is inconsequential, and serves only to modify the disturbance velocity.
- u^{D} is a disturbance velocity expressed by the gradient of the single-valued harmonic potential Φ^{D},

$$u^{D} = \nabla \Phi^{D} \tag{2}$$

Requiring the no-penetration boundary condition $u \cdot n = 0$ around the contour of the body in a meridional plane of constant angle φ, where n is the unit vector normal to the body, we derive a boundary condition for the normal derivative of the disturbance potential,

$$n \cdot \nabla \phi^{D} \equiv \frac{\partial \phi^{D}}{\partial n} = -\left(u^{\infty} + v\right) \cdot n \tag{3}$$

Using the standard boundary-integral formulation, we find that the disturbance potential satisfies the integral equation of the second kind

$$\phi^{D}(x_0) = -2 \int_C G(x, x_0)[n(x) \cdot \nabla \phi^{D}(x)] dl(x) + 2 \int_C^{PV} \phi^{D}(x)[n(x) \cdot \nabla G(x, x_0)] dl(x) \tag{4}$$

where $G(x, x_0)$ is the free-space Green's function of Laplace's equation in an axisymmetric domain, and the point x_0 lies on the contour of the body C. Inserting the boundary condition (3) into (4) and rearranging, we obtain

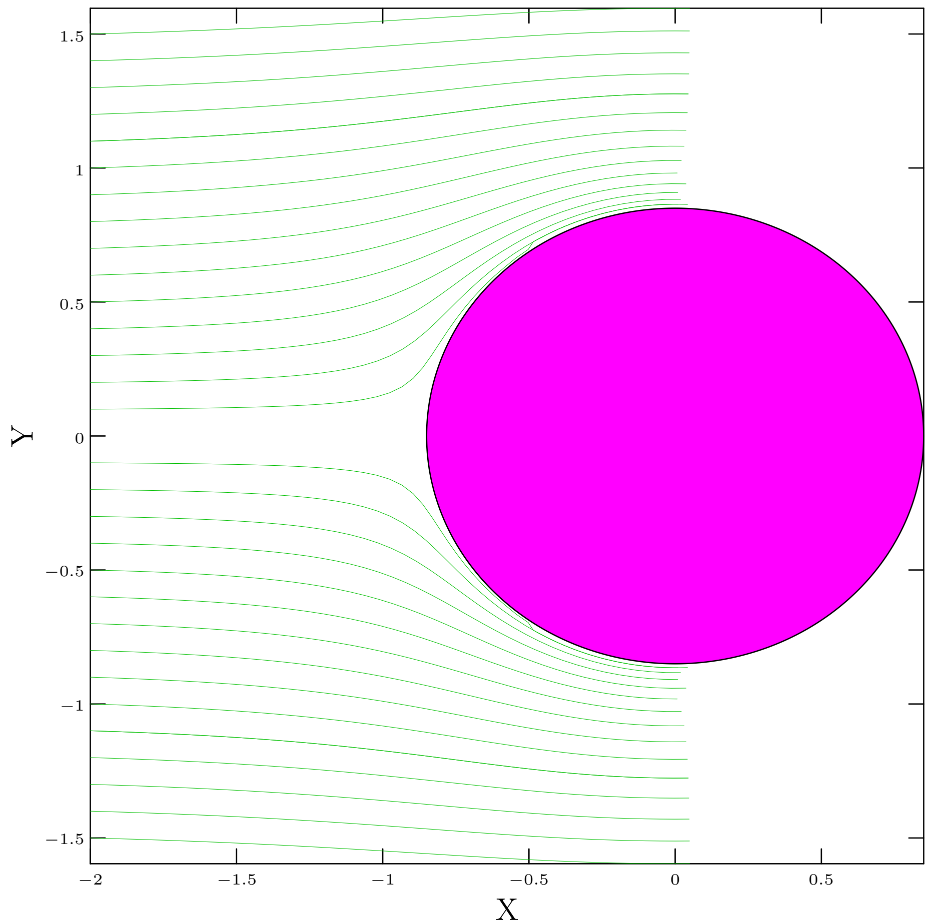

Figure 1: Streamlines of flow past a sphere

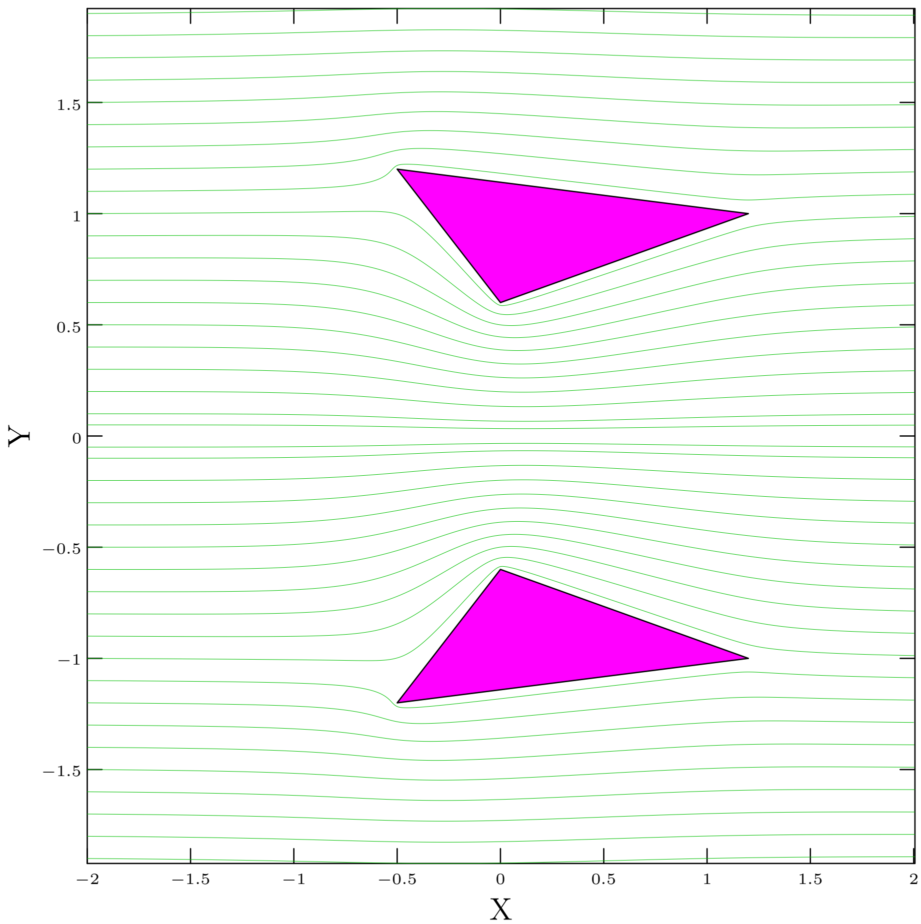

Figure 2: Streamlines of flow past a triangular thorus

$$\phi^D(x_0) - 2\int_C^{PV} \phi^D(x)[n(x)\cdot\nabla G(x,x_0)]dl(x) = 2\int_C (u^\infty + v)\cdot G(x,x_0)n(x)dl(x) \tag{5}$$

Numerical Method

The contour of the body in a meridional plane is discretized into a collection of N boundary elements denoted by $E_i, i = 1, \ldots, N$. The boundary elements can be straight segments or circular arcs. The disturbance potential and its normal derivative are approximated with constant functions the ith element denoted respectively by φ_i^D and

$$\left(\frac{\partial \phi^D}{\partial n}\right)_i = (u^\infty + v^{(i)}) \cdot n^{(i)} \tag{6}$$

where $n^{(i)}$ and $v^{(i)}$ are the normal vector and velocity induced by the line vortex ring evaluated at the mid-point of the i-th element. Subject to these approximations, the integral equation (5) assumes the discretized form

$$\phi_j^D - 2\sum_{i=1}^{N} \phi_i^D \int_{E_i}^{PV} n(x)\nabla G\left(x, x_j^M\right) dl(x) = 2\sum_{i=1}^{N} \left(u_\infty + v^{(i)}\right)\cdot n^{(i)} \int_{E_i} G\left(x, x_j^M\right) dl(x) \tag{7}$$