`laplace/body_ax` - an excerpt from the book C.Pozrikidis ‘A Practical Guide to Boundary-Element Methods’

with additional notes and commentaries

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Introduction

- This is a code that computes potential flow past an axisymmetric, compact (singly connected) or toroidal (doubly connected) body with arbitrary geometry, as illustrated in Figure
- original *Fortran77* code is here CFDLAB code for `laplace/body_ax`

Mathematical Formulation

The velocity u is decomposed into three parts as

$$u = u^{\infty} + v + u^{D} \tag{1}$$

where:

- u^{∞} is the far-field component prevailing far from the body, expressing uniform (streaming) flow along the x axis of symmetry.
- v is the velocity due to a line vortex ring with specified strength situated in the interior of the body, generating circulation around the toroidal body. In the case of flow past a compact body, this component is inconsequential, and serves only to modify the disturbance velocity.
- u^{D} is a disturbance velocity expressed by the gradient of the single-valued harmonic potential Φ^{D},

$$u^{D} = \nabla \Phi^{D} \tag{2}$$

Requiring the no-penetration boundary condition $u \cdot n = 0$ around the contour of the body in a meridional plane of constant angle φ, where n is the unit vector normal to the body, we derive a boundary condition for the normal derivative of the disturbance potential,

$$n \cdot \nabla \phi^{D} \equiv \frac{\partial \phi^{D}}{\partial n} = -\left(u^{\infty} + v\right) \cdot n \tag{3}$$

Using the standard boundary-integral formulation, we find that the disturbance potential satisfies the integral equation of the second kind

$$\begin{aligned} \phi^{D}(x_0) = &-2 \int_C G(x, x_0)[n(x) \cdot \nabla \phi^{D}(x)] dl(x) \\ &+ 2 \int_C^{PV} \phi^{D}(x)[n(x) \cdot \nabla G(x, x_0)] dl(x) \end{aligned} \tag{4}$$

where $G(x, x_0)$ is the free-space Green's function of Laplace's equation in an axisymmetric domain, and the point x_0 lies on the contour of the body C. Inserting the boundary condition (3) into (4) and rearranging, we obtain

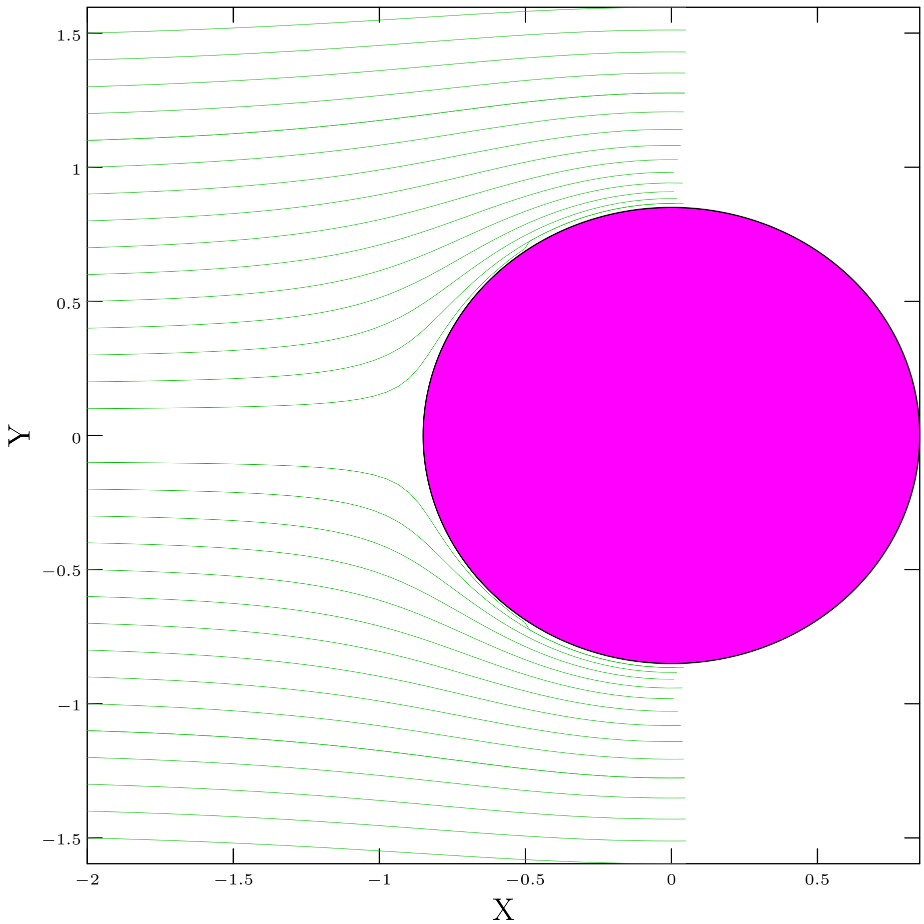

Figure 1: Streamlines of flow past a sphere

$$\begin{aligned} \phi^D(x_0) - 2\int_C^{PV} \phi^D(x)[n(x)\cdot\nabla G(x, x_0)]dl(x) \\ = 2\int_C (u^\infty + v)\cdot G(x, x_0)n(x)dl(x) \end{aligned} \tag{5}$$

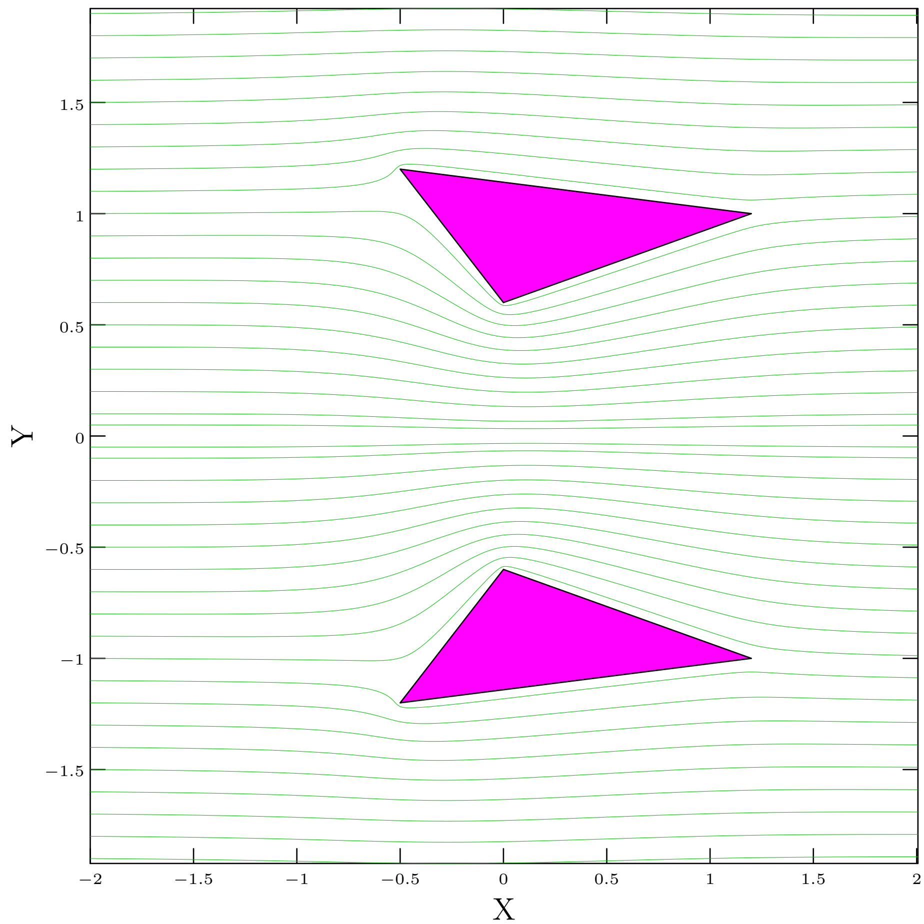

Figure 2: Streamlines of flow past a triangular thorus

Numerical Method

The contour of the body in a meridional plane is discretized into a collection of N boundary elements denoted by $E_i, i = 1, \ldots, N$. The boundary elements can be straight segments or circular arcs. The disturbance potential and its normal derivative are approximated with constant functions the ith element denoted respectively by φ_i^D and

$$\left(\frac{\partial \phi^D}{\partial n}\right)_i = (u^\infty + v^{(i)}) \cdot n^{(i)} \tag{6}$$

where $n^{(i)}$ and $v^{(i)}$ are the normal vector and velocity induced by the line vortex ring evaluated at the mid-point of the i-th element. Subject to these approximations, the integral equation (5) assumes the discretized form

$$\begin{aligned} \phi^D(x_0) - 2\sum_{i=1}^{N} \phi_i^D \int_{E_i}^{PV} n(x)\nabla G\left(x, x_0\right) dl(x) \\ = 2\sum_{i=1}^{N} \left(u^\infty + v^{(i)}\right) \cdot n^{(i)} \int_{E_i} G\left(x, x_0\right) dl(x) \end{aligned} \tag{7}$$

Identifying the point x_0 with the mid-point of the j-th element denoted by x_j^M, where $j = 1, \ldots, N$, we obtain a system of linear equations for the unknown values ϕ_i^D,

$$\begin{aligned} \phi_j^D - 2\sum_{i=1}^{N} \phi_i^D \int_{E_i}^{PV} n(x)\nabla G\left(x, x_j^M\right) dl(x) \\ = 2\sum_{i=1}^{N} \left(u^\infty + v^{(i)}\right) \cdot n^{(i)} \int_{E_i} G\left(x, x_j^M\right) dl(x) \end{aligned} \tag{8}$$

where $j = 1, \ldots, N$.

To illustrate the structure of the linear system more clearly, we collect the two terms on the left-hand side of (8) and rearrange to obtain

$$\begin{aligned} \sum_{i=1}^{N} \phi_i^D \left[\frac{1}{2}\delta_{ij} - \int_{E_i}^{PV} n(x)\nabla G\left(x, x_j^M\right) dl(x)\right] \\ = \sum_{i=1}^{N} \left(u^\infty + v^{(i)}\right) \cdot n^{(i)} \int_{E_i} G\left(x, x_j^M\right) dl(x) \end{aligned} \tag{9}$$

The integrals in equation (9) are computed by numerical methods and the linear system is solved by Gauss elimination.

Tangential Velocity

The normal component of the disturbance velocity along the boundary contour is computed by solving an integral equation. The tangential component is computed numerically by differentiating the disturbance potential with respect to arc length along the contour. The tangential component is then added to the normal compo- nent to yield the total disturbance velocity. The total velocity is computed from the decomposition expressed by equation (1).

Computation of the Velocity at a Point in the Flow

To obtain the velocity at a point in the flow, we compute the gradient of the disturbance potential on the right-hand side of equation (2) using centered differences. The disturbance potential is evaluated using the discretized integral representation

$$\begin{aligned}\phi^D(x_0) &= \sum_{i=1}^{N} \left(u^\infty + v^{(i)}\right) \cdot n^{(i)} \int_{E_i} G\left(x, x_0\right) dl(x) \\ &\quad + \sum_{i=1}^{N} \phi_i^D \int_{E_i} n(x) \cdot \nabla G\left(x, x_0\right) dl(x)\end{aligned} \tag{10}$$

where the point x_0 lies in the domain of the flow.

Program Depiction

- Main program: *body_ax*
  The main program solves an integral equation of the second kind for the boundary distribution of the disturbance potential, computes the boundary distribution of the pressure coefficient and the force exerted on the body, and generates streamlines originating from specified points in the flow.
- Files to be linked

1. *elm_arc*
   Discretization of a circular segment into arc elements.
2. *elm_line*
   Discretization of a straight segment into straight (linear) elements.

3. *body_ ax_ geo*
   Discretization of the boundary geometry.
4. *body_ ax_ sdlp*
   Evaluation of the single- and double-layer harmonic potential over boundary elements.
5. *body_ ax_ vel*
   Evaluation of the velocity at a specified point in the flow.
6. *ell_ int*
   Evaluation of complete elliptic integrals of the first and second kind.
7. *gauss_ leg*
   Base points and weights for the Gauss-Legendre quadrature.
8. *gel*
   Solution of a linear algebraic system by Gauss elimination.
9. *lgf_ ax_ fs*
   Free-space Green's function of Laplace's equation in an axisymmetric domain.
10. *lvr_ ax_ fs*
    Potential and velocity due to a line vortex ring.

- Input files:

1. *sphere.dat*
   Parameters for flow past a sphere.
2. *torus_ trgl.dat*
   Parameters for flow past a triangular torus.

- Output files:

1. *body ax.str*
   Streamlines.
2. *body 2d.out*
   Boundary distribution of the disturbance potential and tangential velocity.