CS 374 HW 0 Problem 1

quddus2, Hieu Huynh, Aldo Sanjoto

TOTAL POINTS

100 / 100

QUESTION 1

1 A **40 / 40**

✓ **- 0 pts** **Correct**

- 20 pts Pigeonhole Principle - but not proving n/m

- 10 pts Correct floor/ceiling

- 15 pts Proof by contradiction

- 5 pts $M = \beta - \alpha + 1$ possible values

- 10 pts Correct conclusion & proper formal language

- 10 pts IDK

QUESTION 2

2 B **40 / 40**

✓ **- 0 pts** **Correct - uses part (a)**

- 15 pts Differentiating between 0-degree and no 0-degree cases

- 20 pts There are at most n-1 degree values

- 5 pts Using part (a.)

- 10 pts Bad arithmetic with ceiling

- 10 pts Correct, but not using part a.

- 5 pts Argument by contradiction

- 10 pts All degree values appear once

- 15 pts Contradiction in deg = 0 and deg = n-1

- 10 pts Not using part a.

- 10 pts IDK

QUESTION 3

3 C **20 / 20**

✓ **- 0 pts** **Correct - multiple components**

- 5 pts Noticing every component has at least 2 vertices

- 10 pts The size of a component bounds the degree of the vertices (define the max)

- 5 pts applying the same argument as in part (b)

- 0 pts Correct - can just restrict to a connected component

- 5 pts Every component has at least 2 vertices

- 5 pts Noting that the degrees in the component equals the degrees in G.

- 10 pts Arguing correctly that (b) applies

- 5 pts IDK

(Q1)(A)

- Prove that: $\left\lceil \frac{a+1}{b} \right\rceil = \begin{cases} \left\lceil \frac{a}{b} \right\rceil & \text{, if } a \bmod b \neq 0 \\ \left\lceil \frac{a}{b} \right\rceil + 1 & \text{, if } a \bmod b = 0 \end{cases}$ (a, b are positive integers)

- <u>Case 1</u>: $a \bmod b = 0$

$\Rightarrow a = b \cdot q$ (with q is an interger)

$\Rightarrow \frac{a+1}{b} = q + \frac{1}{b} \Rightarrow q < \frac{a+1}{b} \leq q+1$ (because $b>0$)

$\Rightarrow \left\lceil \frac{a+1}{b} \right\rceil = q+1$

- We also have that: ~~a mod b = 0~~ $\Rightarrow \frac{a}{b} = q$: an integer

$\Rightarrow \left\lceil \frac{a}{b} \right\rceil = q$

Therefore, $\left\lceil \frac{a+1}{b} \right\rceil = \left\lceil \frac{a}{b} \right\rceil + 1$

- <u>Case 2</u>: $a \bmod b \neq 0 \Rightarrow a = b \cdot q + r$ (with q and r be integers and $0<r<b$)

$\Rightarrow \frac{a+1}{b} = q + \frac{r+1}{b}$ (because $0<r<b \Rightarrow r+1 \leq b \Rightarrow \frac{r+1}{b} \leq 1$ and $\frac{r+1}{b} > 0$)

$\Rightarrow q < \frac{a+1}{b} \leq q+1$

$\Rightarrow \left\lceil \frac{a+1}{b} \right\rceil = q+1$

- We also have that: $a = b \cdot q + r \Rightarrow \frac{a}{b} = q + \frac{r}{b}$

$\Rightarrow q < \frac{a}{b} < q+1$ (because $0<r<b \Rightarrow 0 < \frac{r}{b} < 1$)

$\Rightarrow \left\lceil \frac{a}{b} \right\rceil = q+1$

Therefore $\left\lceil \frac{a+1}{b} \right\rceil = \left\lceil \frac{a}{b} \right\rceil$ (because both $= (q+1)$)

Based on those 2 cases, we can conclude that:

$$\left\lceil \frac{a+1}{b} \right\rceil = \begin{cases} \left\lceil \frac{a}{b} \right\rceil & \text{, if } a \bmod b \neq 0 \\ \left\lceil \frac{a}{b} \right\rceil + 1 & \text{, if } a \bmod b = 0 \end{cases}$$

Induction on n

Base case: n = 0:

$$\lceil \frac{0}{\beta - \alpha + 1} \rceil = 0$$

In the sequence of 0 number, there's always 0 numbers that are all equal. Therefore, base case is correct

Hypothesis: Assume for any non-negative integer $n \leq k$, in the sequence $x_0, x_1, .. x_k$, such that $\alpha \leq x_i \leq \beta$, there are at least $\lceil n/(\beta - \alpha + 1) \rceil$ numbers that are all equals.

For n = k+1: Need to prove that the sequence $x_0, x_1, .. x_k\ x_{k+1}$, such that $\alpha \leq x_i \leq \beta$, has at least $\lceil (k+1)/(\beta - \alpha + 1) \rceil$ numbers that are all equals

Based on the proof above, we have that for any non-negative integers a, b, we have the fact that:

$$\lceil \frac{a+1}{b} \rceil = \begin{cases} \lceil \frac{a}{b} \rceil & a \mod b \neq 0 \\ \lceil \frac{a}{b} \rceil + 1 & a \mod b = 0 \end{cases}$$

Case 1: k mod $(\beta - \alpha + 1) \neq 0$

Because k mod $(\beta - \alpha + 1) \neq 0$, we have $\lceil k/(\beta - \alpha + 1) \rceil = \lceil (k+1)/(\beta - \alpha + 1) \rceil$. Therefore, we only need to prove that the sequence $x_0, x_1, .. x_k, x_{k+1}$, such that $\alpha \leq x_i \leq \beta$, has at least $\lceil k/(\beta - \alpha + 1) \rceil$ numbers that are all equals.

Based on the hypothesis, the sequence $x_0, x_1, .. x_k$, such that $\alpha \leq x_i \leq \beta$, there are at least $\lceil k/(\beta - \alpha + 1) \rceil$ numbers that are all equals. Therefore, adding any integer to the sequence will give us the sequence $x_0, x_1, .. x_k, x_{k+1}$ that has at least $\lceil k/(\beta - \alpha + 1) \rceil$ numbers that are equal. Therefore, the statement is true for this case.

Case 2: k mod $(\beta - \alpha + 1) = 0$, we have $\lceil (k+1)/(\beta - \alpha + 1) \rceil = 1 + \lceil k/(\beta - \alpha + 1) \rceil$.

Let m be the maximum number of numbers that are all equal in the sequence $x_0, x_1, .. x_k$.

Based on the hypothesis, we have that $m \geq \lceil k/(\beta - \alpha + 1) \rceil$

<u>Case a:</u> $m > \lceil k/(\beta - \alpha + 1) \rceil \Leftrightarrow m \geq \lceil k/(\beta - \alpha + 1) \rceil + 1$. Therefore, the sequence $x_0, x_1, .. x_k$ has at least $\lceil k/(\beta - \alpha + 1) \rceil + 1$ numbers that are all equal. Therefore, adding any integer to the sequence will give us the sequence $x_0, x_1, .. x_k, x_{k+1}$ that has at least $\lceil k/(\beta - \alpha + 1) \rceil$ numbers that are equal. Therefore, the statement is true for this case.

<u>Case b:</u> $m = \lceil k/(\beta - \alpha + 1) \rceil = k/(\beta - \alpha + 1)$ because k mod $(\beta - \alpha + 1) = 0$. The sequence $x_0, x_1, .. x_k$ has k elements in the sequence with each element in the range $[\beta, \alpha]$, and $m = k/(\beta - \alpha + 1)$. Therefore, each number in the sequence must be repeated exactly m times. By adding any number in range $[\beta, \alpha]$to that sequence, we have the sequence $x_0, x_1, .. x_k, x_{k+1}$ that has (m+1) numbers that are all equal. Therefore, the sequence $x_0, x_1, .. x_k, x_{k+1}$ has at least $(\lceil k/(\beta - \alpha + 1) \rceil + 1) = \lceil (k+1)/(\beta - \alpha + 1) \rceil$ numbers that are equal.

In all cases, the claim is true.

Therefore, the statement is true for all value of n.

1 A 40 / 40

✓ - **0 pts** Correct

- **20 pts** Pigeonhole Principle - but not proving n/m
- **10 pts** Correct floor/ceiling
- **15 pts** Proof by contradiction
- **5 pts** $M = \beta - \alpha + 1$ possible values
- **10 pts** Correct conclusion & proper formal language
- **10 pts** IDK

(Q1)

(B) Because $|V| \geq 2$, there are 3 cases:

Case 1: There are at least 2 distinct nodes in G that have degree 0

There are 2 distinct nodes u and v with $deg(u) = deg(v) = 0$

$\Rightarrow$ There are 2 distinct nodes u and v that degree of u is equal

$\Rightarrow$ There are 2 distinct nodes u and v that degree of u is equal to degree of v

Case 2: There ~~are~~ is exactly 1 node that has degree 0.

- For this case, $|V|$ cannot be 2 because if there are only 2 nodes in the graph, and 1 of them has degree 0, the other node must also have degree 0

$\Rightarrow |V| > 2$.

- Because there's exactly 1 node that has degree 0, there are $(|V|-1)$ nodes are connected with the degree of each node is at least 1.
- Because there's no loop, each vertex in the $(|V|-1)$ nodes can be adjacent to at most $(|V|-2)$ vertices

$\Rightarrow$ The degree of those connected $(|V|-1)$ is in range $[1, |V|-2]$

- Let $n = |V| - 1$; $\alpha = 1$; $\beta = |V| - 2$ (we have $\beta \geq \alpha$ because $|V| > 2$)
- Let $x_1, x_2, .. x_n$ be the sequence of integers numbers which represent the degree of $(|V|-1)$ connected nodes in G with $\alpha \leq x_i \leq \beta$.

- Based on the result from part (A), we have that there are at least $\left\lceil \frac{n}{\beta - \alpha + 1} \right\rceil$ numbers in the sequence that're all equal

- $\left\lceil \frac{n}{\beta - \alpha + 1} \right\rceil = \left\lceil \frac{|V| - 1}{(|V|-2) - 1 + 1} \right\rceil = \left\lceil \frac{|V|-1}{|V|-2} \right\rceil = 2$

Therefore, there're at least 2 numbers in the sequence $x_1, x_2, .., x_n$ that are all equal

$\Rightarrow$ There're at least 2 distinct nodes that have the same degree

Case 3: G is a connected ~~not~~ graph. This means that all vertices in G have degree that's greater or equal 1

- Because there's no loop, and there are $|V|$ nodes $(|V| > 2)$ in G, each vertex can be adjacent to at most $(|V| - 1)$ nodes

$\Longrightarrow$ All nodes in G have degree in range $[1, |V|-1]$

• Repeat the same argument for case 2 above, we have that there are at least $\left\lceil \frac{|V|}{(|V|-1)-1+1} \right\rceil = \left\lceil \frac{|V|}{|V|-1} \right\rceil = 2$ nodes that have the same degree.

For all cases, the claim is true.

(1 C): Because all vertices in G are of degree at least one, there are at least 2 vertices in G. $\Rightarrow |V| \geq 2$.

Also, there exist a subgraph G' of G, such that all vertices in G' are connected

Let $|V'|$ be the number of vertices in G'. Because all nodes in G' are connect and has degree of at least 1
$\Rightarrow |V'| \geq 2$.

• Based on part (B), the graph G' has $|V'| \geq 2$, therefore there are at least 2 distinct nodes u an v in G' such that ~~d~~egree of u is equal to degree of v. And we have that G' is a ~~non~~ connected graph. Therefore, there is a simple path between u an v

Therefore, there is a simple path between 2 distinct nodes that have same degree in G'.

Because G' is a sub-graph of G, there's also a simple path between 2 distinct nodes that have the same degree in G.

2 B **40 / 40**

✓ - **0 pts** **Correct - uses part (a)**

- **15 pts** Differentiating between 0-degree and no 0-degree cases

- **20 pts** There are at most n-1 degree values

- **5 pts** Using part (a.)

- **10 pts** Bad arithmetic with ceiling

- **10 pts** Correct, but not using part a.

- **5 pts** Argument by contradiction

- **10 pts** All degree values appear once

- **15 pts** Contradiction in deg = 0 and deg = n-1

- **10 pts** Not using part a.

- **10 pts** IDK

• Repeat the same argument for case 2 above, we have that there are at least $\left\lceil \frac{|V|}{(|V|-1)-1+1} \right\rceil = \left\lceil \frac{|V|}{|V|-1} \right\rceil = 2$ nodes that have the same degree.

For all cases, the claim is true.

(1 C): Because all vertices in G are of degree at least one, there are at least 2 vertices in G. $\Rightarrow |V| \geq 2$.

Also, there exist a subgraph G' of G, such that all vertices in G' are connected

Let $|V'|$ be the number of vertices in G'. Because all nodes in G' are connect and has degree of at least 1

$\Rightarrow |V'| \geq 2$.

• Based on part (B), the graph G' has $|V'| \geq 2$, therefore there are at least 2 distinct nodes u an v in G' such that ~~d~~degree of u is equal to degree of v. And we have that G' is a ~~non~~ connected graph. Therefore, there is a simple path between u an v

Therefore, there is a simple path between 2 distinct nodes that have same degree in G'.

Because G' is a sub-graph of G, there's also a simple path between 2 distinct nodes that have the same degree in G.

3 C **20 / 20**

- ✓ **- 0 pts** **Correct - multiple components**
- **- 5 pts** Noticing every component has at least 2 vertices
- **- 10 pts** The size of a component bounds the degree of the vertices (define the max)
- **- 5 pts** applying the same argument as in part (b)
- **- 0 pts** Correct - can just restrict to a connected component
- **- 5 pts** Every component has at least 2 vertices
- **- 5 pts** Noting that the degrees in the component equals the degrees in G.
- **- 10 pts** Arguing correctly that (b) applies
- **- 5 pts** IDK