CS 374 HW 0 Problem 3

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TOTAL POINTS

100 / 100

QUESTION 1

1 3A **50 / 50**

- + **12.5 pts** IDK
- + **5 pts** Correct Induction Var
- + **5 pts** Base Case
- + **10 pts** Correct IH
- + **5 pts** Case x=y in inductive step
- + **25 pts** Case x>y (or x<y) in inductive step
- + **5 pts** Partial Correct IH
- + **0 pts** No IH
- + **0 pts** Induction on WRONG var
- + **0 pts** Wrong induction
- + **0 pts** No answer
- ✓ + **50 pts Correct**

QUESTION 2

2 3B **50 / 50**

- ✓ + **50 pts Correct**
- + **12.5 pts** IDK
- + **0 pts** wrong
- + **10 pts** Argue log bound
- + **0 pts** No answer
- + **10 pts** Explanation rather than formal proof
- + **40 pts** Argue the max/sum reduced by a factor
- + **30 pts** Assumptions used without sufficiently justified.

(3A) Proof: That starting with any point p on the positive integer grid, the sequence $W(p)$ is finite.

Base Case: Since X_0 & y_0 are positive integers; $P_0 = (X_0, Y_0)$. We consider the case where the $\max(X_0, Y_0) = 1$. Since X_0 & y_0 are positive integers, then X_0 & y_0 need to be equal to 1.
Since according to the def'n, if $x = y$ then it is an good point, meaning $W(P_0) = p_0 \Rightarrow W(p_0)$ is finite

Inductive Hypothesis:
let x and y be any positive integers then assume that for any non-negative integer k, if $\max(x, y) \le K$ then $W(P)$ is finite where $P = (x, y)$

Inductive Step: For $P_0 = (x_0, y_0)$, such that $\max(x_0, y_0) = k+1$
1st case: $x_0 > y_0$

$$\Rightarrow P_1 = d(P_0) = (\underbrace{X_0 - Y_0 - 1}_{x_1}, \underbrace{Y_0}_{y_1}) \quad // \text{ definition of } d(P)$$

Since we assumed that $x_0 > Y_0$ & assumed the $\max(X_0, Y_0) = k+1$ then we know that $X_0 = k+1$ // definition of $\max()$

$$\max(x_1, y_1) = \max(X_0 - Y_0 - 1,\ Y_0) = \max((k+1) - Y_0 - 1,\ Y_0) \quad // \text{ Substitution}$$
$$= \max(k - Y_0,\ Y_0)$$

We have: $k - y_0 < K$ (because Y_0 is a positive integer.)
and $y_0 < x_0 \Leftrightarrow y_0 < k+1$ (because $x_0 = k+1$)
$\Leftrightarrow y_0 \le K$

Since $X_1 = k - y_0$ is $< K$ and $y_1 = y_0 \le K$
So the $\max(X_1, y_1) \le k$ and by the inductive hypothesis $W(P_1)$ is finite and so if $W(P_1)$ is finite and $W(P_0) = P_0, P_1 P_2 \ldots = P_0 W(P_1) \Rightarrow W(P_0)$ is finite

2^{nd} case: $y_0 > x_0$

$P_1 = d(P_0) = (\underbrace{X_0}_{x_1}, \underbrace{y_0 - X_0 - 1}_{y_1})$ // definiton of $d(P)$

Since we assumed that $y_0 < X_0$ & $\max(X_0, y_0) = k+1$, then we know ~~that~~ that $y_0 = k+1$ // definition of max()

$\max(X_1, y_1) = \max(X_0, (k+1) - X_0 - 1)$ // substitution
$= \max(X_0, k - X_0)$

Since $k - X_0$ is less than k because X_0 is a positive integer.
Also, since $x_0 < y_0 \iff x_0 < k+1$ (because $y_0 = k+1$)
$\iff x_0 \leq k$

$\Rightarrow \max(x_1, y_1) \leq k$

Because $P_1 = (x_1, y_1)$ and $\max(x_1, y_1) \leq k$
$\Rightarrow$ By the inductive hypothesis $W(P_1)$ is finite and so if $W(P_1)$ is finite and $W(P_0) = P_0 P_1 P_2 \ldots = P_0 W(P_1)$

$\Rightarrow W(P_0)$ is finite

3^{rd} case: $x_0 = y_0$

with $P_0 = (x_0, y_0)$ then P_0 is already a good point due to the definition of a good point $(x = y)$.

So $W(P_0) = P_0 \Rightarrow W(P_0)$ is finite

For all cases, the claim is true.

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- ✓ + **50 pts** **Correct**

(Q3)(B)

- Let $P_i = (x_i, y_i)$ and $P_n = (x_n, y_n)$ be the starting point and ending point of a horizontal run.

- We have:

$$x_n < y_n \quad \text{(Because next move is vertical)}$$

$$\Rightarrow x_{n-1} - y_{n-1} - 1 < y_n \quad (x_n = x_{n-1} - y_{n-1} - 1)$$

$$\Rightarrow x_{n-1} < 2y_{n-1} + 1 \quad \text{(Because } y_n = y_{n-1} = y_i \text{ since it's horizontal run)}$$

$$\Leftrightarrow y_{n-1} > \frac{x_{n-1} - 1}{2} \quad (1)$$

- We have $x_n = x_{n-1} - y_{n-1} - 1$

$$\Rightarrow x_n < x_{n-1} - \frac{x_{n-1} - 1}{2} - 1 \quad \text{(Because of (1))}$$

$$\Rightarrow x_n < \frac{x_{n-1}}{2} - \frac{1}{2} < \frac{x_{n-1}}{2} < \frac{x_i}{2} \quad \text{(because } x_{n-1} < x_i\text{)}$$

$$\Rightarrow x_n < \frac{x_i}{2}$$

$$\Rightarrow x_n \cdot y_n < \frac{x_i}{2} \cdot y_n = \frac{x_i \cdot y_i}{2} \quad \text{(because } y_i = y_n\text{)}$$

Therefore, the value of $(x_i \cdot y_i)$ decreases at least by a factor of 2 after a horizontal run.

* Repeat the same argument for a vertical run, we also have that if $P_i = (x_i, y_i)$ is the starting point of a vertical run, after the vertical run, the value of $(x_i \cdot y_i)$ decreases at least by a factor of 2

Therefore, with $P = (x, y)$ be the starting point of the sequence $W(p)$, the value of $(x \cdot y)$ decreases at least by a factor of 2 after every run until they reach 0 or $x_i = y_i$

$\Rightarrow$ There are at most $O(\log(xy)) = O(\log x + \log y)$ runs in the sequence $W(p)$

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