CS 374 HW 0 Problem 2

quddus2, Hieu Huynh, Aldo Sanjoto

TOTAL POINTS

99 / 100

QUESTION 1

1 2B **25 / 25**

- **3 pts** Minor issue
- ✓ **0 pts** **Good Job!**
- **12.5 pts** A lot of flaws
- **18.75 pts** IDK

QUESTION 2

2 2C **25 / 25**

- ✓ **0 pts** **Correct**
- **5 pts** base case
- **5 pts** inductive hypothesis
- **10 pts** prove for (i) x=aw, y; where lwl+lyl=n, a is some symbol
- **5 pts** Similarly, prove for (ii) x, y=bu; where lxl+lul=n, b is some symbol
- **18.75 pts** IDK

QUESTION 3

3 2D **24 / 25**

- **0 pts** Correct
- **3 pts** Incorrect/Missing Induction Proof Case x = eps or y = eps
- **4 pts** Incorrect/Missing Induction Proof Case x = 0x'
- **4 pts** Incorrect/Missing Induction Proof Case y = 0y'
- **4 pts** Incorrect/Missing Induction Proof Case x = 1x'
- ✓ **1 pts** **Warning/Minor error**
- **5 pts** Incorrect induction hypothesis
- **5 pts** Incorrect/Missing base case
- **25 pts** Submission missing
- **18.75 pts** IDK
- **15 pts** Incorrect/Missing Induction Proof
- **3 pts** Equate string with set of strings

Use induction instead of ...

QUESTION 4

4 2E **25 / 25**

- ✓ **0 pts** **Correct**
- **5 pts** Incorrect/Missing proof for (x,y)^s, x^s, y^s are of form 0*1*
- **5 pts** Incorrect/Missing proof for m(x^s, y^s) is of the form 0*1*
- **5 pts** Incorrect/Missing proof for #0(m(x^s, y^s) = #0(x^s) + #0(y^s)
- **5 pts** Incorrect/Missing proof for #0(x^s) + #0(y^s) = #0(x) + #0(y)
- **5 pts** Incorrect/Missing proof for #0(x) + #0(y) = #0(x.y)^s
- **25 pts** Missing solution
- **1 pts** Warning/Minor error
- **18.75 pts** IDK
- **15 pts** Incorrect/Missing induction proof
- **5 pts** Incorrect/Missing Base case
- **5 pts** Incorrect/Missing Induction Hypothesis
- **5 pts** Incorrect/Missing induction proof on case x=0x'
- **5 pts** Incorrect/Missing induction proof on case y=0y'
- **5 pts** Incorrect/Missing induction proof on case x=1x'

gradescope

Q2.) (B)

Hypothesis:

Let w be an arbitrary string $w \in \{0,1\}^*$ and assume for every String X such that $|X| < |w|$ that $\#_0(X) = \#_0(X^s)$

Base case:

let w be the empty string where $w = \varepsilon$ then

$\#_0(w) = \#_0(\varepsilon)$ // substitution $w = \varepsilon$

$\#_0(w) = 0$ // definition of $\#_0$ function since there are no 0's in ε

$\#_0(w) = \#_0(\varepsilon) = \#_0(\varepsilon^s)$ // definition of $w^s \Rightarrow \varepsilon^s \Rightarrow \varepsilon$

$\#_0(w) = \#_0(\varepsilon^s)$ // substitution $\Rightarrow \varepsilon = \varepsilon^s$

$\#_0(w) = \#_0(w^s)$ // substitution $\Rightarrow w = \varepsilon$ $\Rightarrow$ Base case is true

Inductive step:

Suppose that w is an arbitrary string $w \in \{0,1\}^*$ that is not empty and $w = a \cdot x$ where $|x| < |w|$ and $\underline{a = 0}$

$\#_0(w) = \#_0(a \cdot x)$ // substitution $w = a \cdot x$

$= \#_0(0 \cdot x)$ // substitution $a = 0$

$= 1 + \#_0(x)$ // definition of the function $\#_0$

$= 1 + \#_0(x^s)$ // Inductive hypothesis since $|x| < |w|$ then $\#_0(x) = \#_0(x^s)$

$= \#_0(0 \cdot x^s)$ // definition of $\#_0$ function

$= \#_0(w^s)$ // definition of w^s

Suppose that w is an arbitrary string which is not empty and $w \in \{0,1\}^*$

$w = a \cdot x$ where $|x| < |w|$ and $\underline{a = 1}$ then

$$\begin{aligned}
\#_0(w) &= \#_0(1 \cdot x) && \text{// substitution } w = a \cdot x,\ a = 1 \\
&= \#_0(x) && \text{// definition of } \#_0 \\
&= \#_0(x^s) && \text{// inductive hypothesis where } |x| < |w| \text{ then } \#_0(x) = \#_0(x^s) \\
&= \#_0(x^s \cdot 1) && \text{// definition of } \#_0 \\
&= \#_0(w^s) && \text{// definition of } w^s
\end{aligned}$$

In all cases we conclude that $\#_0(w) = \#_0(w^s)$

By the same argument as above we can conclude that $\#_1(w) = \#_1(w^s)$ and from these two ~~facts~~ facts we can conclude that $|w| = |w^s| = \#_0(w) + \#_1(w)$

Since w & w^s are $\in \{0,1\}^*$

1 2B **25** / **25**

- **3 pts** Minor issue
- ✓ **0 pts** **Good Job!**
- **12.5 pts** A lot of flaws
- **18.75 pts** IDK

gradescope

(2c)

Base case: $|x| + |y| = 0 \Rightarrow |x| = |y| = 0 \Rightarrow x = y = \varepsilon$

$\Rightarrow \#_0(m(x,y)) = \#_0(m(\varepsilon,\varepsilon))$ // substitution

$= \#_0(\varepsilon)$ // definition of m

$= \#_0(\varepsilon) + \#_0(\varepsilon)$

$= \#_0(x) + \#_0(y)$ // $x = y = \varepsilon$

$\Rightarrow \#_0(m(x,y)) = \#_0(x) + \#_0(y)$

$\Rightarrow$ Base case is true

Hypothesis:

For any string x & y ($x, y \in \{0,1\}^*$) that $|x| + |y| = n$ where $n \leq k$ (both n & k are non-negative integers) that the following claim holds: $\#_0(m(x,y)) = \#_0(x) + \#_0(y)$

Inductive Step:

Now we prove that $\#_0(m(x,y)) = \#_0(x) + \#_0(y)$ for any $x, y \in \{0,1\}^*$ that have length $n = k+1$

Suppose that $|x| = 0$ & $|y| \neq 0$ then $x = \varepsilon$ by definition 1)

then

$\#_0(m(x,y)) = \#_0(m(\varepsilon, y))$ // substitution $x = \varepsilon$

$= \#_0(y)$ // definition of m function

$= \#_0(\varepsilon) + \#_0(y)$ // definition of $\#_0$ since $\#_0(\varepsilon) = 0$

$= \#_0(x) + \#_0(y)$ // substitution $x = \varepsilon$

Suppose that $|y| = 0$ & $|x| \neq 0$ then $y = \varepsilon$ by definition 1)

then

$\#_0(m(x,y)) = \#_0(m(x, \varepsilon))$ // substitution $y = \varepsilon$

$\#_0(m(x,y)) = \#_0(x)$ // definition of m function

$\#_0(m(x,y)) = \#_0(x) + \#_0(\varepsilon)$ // definition of $\#_0$ since $\#_0(\varepsilon) = 0$

$\#_0(m(x,y)) = \#_0(x) + \#_0(y)$ // substitution $y = \varepsilon$

Suppose that $|x|$ & $|y| \neq 0$ then we have the following cases

let $x = a \cdot x'$ where a & $b \in \{0,1\}$, $x, y \in \{0,1\}^*$
$y = b \cdot y'$ & $|x| + |y| = k+1$ $|x'| < |x|$
$|y'| < |y|$

There are 3 cases

Case 1:
Suppose that $a=0$ then

$$
\begin{aligned}
\#_0(m(x,y)) &= \#_0(m(a \cdot x', y)) && \text{// Substitution } x = a \cdot x' \\
&= \#_0(m(0 \cdot x', y)) && \text{// } a=0 \\
&= \#_0(0 \cdot m(x', y)) && \text{// definition of } m \\
&= 1 + \#_0(m(x', y)) && \text{// definition of } \#_0 \\
&= 1 + \#_0(x') + \#_0(y) && \text{// Inductive hypothesis since } x = a \cdot x', |x| + |y| = k+1, |x'| + |y| = k \\
&= \#_0(0 \cdot x') + \#_0(y) && \text{// definition of } \#_0 \\
&= \#_0(a \cdot x') + \#_0(y) && \text{// Substitution } a=0 \\
&= \#_0(x) + \#_0(y) && \text{// Substitution } x = a \cdot x'
\end{aligned}
$$

~~Suppose~~
Case 2:
Suppose that $a \neq 0$ & $b = 0$

$$\begin{aligned}
\#_0(m(x,y)) &= \#_0(m(x, b\cdot y')) && \text{// substitution } y = b\cdot y' \\
&= \#_0(m(x, 0\cdot y')) && \text{// substitution } b = 0 \\
&= \#_0(0\cdot m(x, y')) && \text{// definition of } m \\
&= 1 + \#_0(m(x, y')) && \text{// definition of } \#_0 \\
&= 1 + \#_0(x) + \#_0(y') && \text{// inductive hypothesis } y = b\cdot y' \\
&= \#_0(x) + \#_0(0\, y') && \text{// definition of } \#_0 \\
&= \#_0(x) + \#_0(b y') && \text{// substitution } b = 0 \\
&= \#_0(x) + \#_0(y) && \text{// substitution } y = b\cdot y'
\end{aligned}$$

$|x| + |y| = k+1$
$|x| + |y'| = k$

Case 3:
Suppose that $a = 1$ & $b \neq 0$

$$\begin{aligned}
\#_0(m(x,y)) &= \#_0(m(a\cdot x', y)) && \text{// substitution } x = a\cdot x' \\
&= \#_0(m(1\cdot x', y)) && \text{// substitution } a = 1 \\
&= \#_0(1\, m(x', y)) && \text{// definition of } m \\
&= \#_0(m(x', y)) && \text{// definition } \#_0 \\
&= \#_0(x') + \#_0(y) && \text{// inductive hypothesis } x = a\cdot x' \\
&= \#_0(1x') + \#_0(y) && \text{// definition } \#_0 \\
&= \#_0(a\cdot x') + \#_0(y) && \text{// substitution } a = 1 \\
&= \#_0(x) + \#_0(y) && \text{// substitution } x = a\cdot x'
\end{aligned}$$

$|x| + |y| = k+1$
$|x'| + |y| = k$

In all cases we conclude that $\#_0(m(x,y)) = \#_0(x) + \#_0(y)$

By the same argument as above we can conclude that $\#_1(m(x,y)) = \#_1(x) + \#_1(y)$ and from these two facts we can conclude that

$$|m(x,y)| = |x| + |y|$$ because

$$|x| = \#_0(x) + \#_1(x)$$

$$|y| = \#_0(y) + \#_1(y)$$

$$|m(x,y)| = \#_0(m(x,y)) + \#_1(m(x,y))$$

$$|m(x,y)| = \#_0(x) + \#_1(x) + \#_0(y) + \#_1(y)$$

Therefore, $\boxed{|m(x,y)| = |x| + |y|}$

2 2C 25 / 25

✓ - 0 pts Correct

- 5 pts base case
- 5 pts inductive hypothesis
- 10 pts prove for (i) x=aw, y; where |w|+|y|=n, a is some symbol
- 5 pts Similarly, prove for (ii) x, y=bu; where |x|+|u|=n, b is some symbol
- 18.75 pts IDK

gradescope

Q2) D) Prove that $\forall x, y \in 0^*1^*$, $m(x,y) \in 0^*1^*$

- Induction on $\#_0(x)$.
- <u>Base case</u>: $\#_0(x) = 0 \Rightarrow x = 1^{|x|}$ (because $x \in 0^*1^*$)
  - Let t be an integer, such that $t = \#_0(y)$

And we have that: $\#_1(y) = |y| - \#_0(y) = |y| - t$

Also, $y \in 0^*1^* \Rightarrow y = 0^t 1^{|y|-t}$

$$\Rightarrow m(x,y) = m(1^{|x|}, 0^t 1^{|y|-t})$$
$$= 0^t m(1^{|x|}, 1^{|y|-t}) \quad // \text{ definition of } m.$$
$$= 0^t 1^{|x|} m(\varepsilon, 1^{|y|-t}) \quad //$$

i.e
$$m(1^{|x|}, 0^t 1^{|y|-t})$$
$$= 0\, m(1^{|x|}, 0^{t-1} 1^{|y|-t})$$
$$= 0^2 m(1^{|x|}, 0^{t-2} 1^{|y|-t})$$
$$= 0^t m(1^{|x|}, 1^{|y|-t})$$

because of the definition of m, and there are no 0 in x and y.

$$\Rightarrow m(1^{|x|}, 1^{|y|-t}) = 1\, m(1^{|x|-1}, 1^{|y|-t})$$
$$= 1^2 m(1^{|x|-2}, 1^{|y|-t})$$
$$= \dots$$
$$= 1^{|x|} m(\varepsilon, 1^{|y|-t})$$

$$\Rightarrow m(x,y) = 0^t 1^{|x|} m(\varepsilon, 1^{|y|-t})$$
$$= 0^t 1^{|x|} 1^{|y|-t} \quad (\text{because } m(\varepsilon, y) = y.)$$
$$= 0^t 1^{|x|+|y|-t}$$

$\Rightarrow m(x,y) \in 0^*1^* \Rightarrow$ The base case is true.

- <u>Hypothesis</u>: Assume for any non-negative integer k, if $x, y \in 0^*1^*$ and $\#_0(x) \le k$, then we have $m(x,y) \in 0^*1^*$
- <u>Inductive Step</u>: Let x and y be string of form 0^*1^*, and $\#_0(x) = k+1 > 0$

$\Rightarrow x = 0 \cdot x'$ (with x' be string of form 0^*1^* and $\#_0(x') = \#_0(x) - 1 = k+1-1 = k$)

$$\Rightarrow m(x,y) = m(0 \cdot x', y)$$
$$= 0 \cdot m(x', y) \quad // \text{ definition of } m$$

Based on the hypothesis, we have that $x', y \in 0^*1^*$ and $\#_0(x') = k$

$$\Rightarrow m(x', y) \in 0^*1^* \Rightarrow 0 \cdot m(x', y) \in 0^*1^*$$
$$\Rightarrow m(x,y) \in 0^*1^*$$

Therefore, the claim is correct.

3 2D 24 / 25

- 0 pts Correct
- 3 pts Incorrect/Missing Induction Proof Case x = eps or y = eps
- 4 pts Incorrect/Missing Induction Proof Case x = 0x'
- 4 pts Incorrect/Missing Induction Proof Case y = 0y'
- 4 pts Incorrect/Missing Induction Proof Case x = 1x'
- ✓ 1 pts **Warning/Minor error**
- 5 pts Incorrect induction hypothesis
- 5 pts Incorrect/Missing base case
- 25 pts Submission missing
- 18.75 pts IDK
- 15 pts Incorrect/Missing Induction Proof
- 3 pts Equate string with set of strings

Use induction instead of ...

gradescope

(2)

(E) Prove $(x \cdot y)^s = m(x^s, y^s)$ with $\forall x, y \in \{0,1\}^*$

- Based on the result part (A), $(x \cdot y) \in \{0,1\}^* \Rightarrow (x \cdot y)^s \in 0^*1^*$
- Based on the result of part (A), $x, y \in \{0,1\}^* \Rightarrow x^s \in 0^*1^*$; $y^s \in 0^*1^*$
- Based on the result of part (D), $x^s \in 0^*1^*$ and $y^s \in 0^*1^*$, therefore $m(x^s, y^s) \in 0^*1^*$

Therefore, both $(x \cdot y)^s$ and $m(x^s y^s)$ are of the form 0^*1^*. (1)

- Based on the result of part (B) and (C), we have that:

$\#_0(x^s) = \#_0(x)$; $\#_0(y^s) = \#_0(y)$ (from part (C))

$\Rightarrow \#_0(m(x^s, y^s)) = \#_0(x^s) + \#_0(y^s)$ (from part (B))

$= \#_0(x) + \#_0(y)$ (from part (B))

- We also have that: $\#_0((xy)^s) = \#_0(xy)$ (from part (B))

$= \#_0(x) + \#_0(y)$ (because of definition of $\#_0$)

Therefore, $\#_0(m(x^s, y^s)) = \#_0((xy)^s)$ (because both $= \#_0(x) + \#_0(y)$) (2)

- Based on the result of part (B) and (C) we have that:

$\#_1(m(x^s, y^s)) = \#_1(x^s) + \#_1(y^s)$, (from part (B))

$= \#_1(x) + \#_1(y)$ (from part (B))

- We also have that $\#_1((xy)^s) = \#_1(xy)$ (from part (B))

$= \#_1(x) + \#_1(y)$ (definition of $\#_1$)

Therefore, $\#_1(m(x^s, y^s)) = \#_1((xy)^s)$ (because both $= \#_1(x) + \#_1(y)$) (3)

From (1), (2), (3), we have that:

$$\begin{cases} m(x^s, y^s) \text{ and } (xy)^s \text{ ~~has~~ are of the form } 0^*1^* \\ \#_0(m(x^s, y^s)) = \#_0((xy)^s) \\ \#_1(m(x^s, y^s)) = \#_1((xy)^s) \end{cases}$$

Therefore, $m(x^s, y^s) = (xy)^s$.

4 2E 25 / 25

✓ - 0 pts Correct

- 5 pts Incorrect/Missing proof for (x,y)^s, x^s, y^s are of form 0*1*
- 5 pts Incorrect/Missing proof for m(x^s, y^s) is of the form 0*1*
- 5 pts Incorrect/Missing proof for #0(m(x^s, y^s) = #0(x^s) + #0(y^s)
- 5 pts Incorrect/Missing proof for #0(x^s) + #0(y^s) = #0(x) + #0(y)
- 5 pts Incorrect/Missing proof for #0(x) + #0(y) = #0(x.y)^s
- 25 pts Missing solution
- 1 pts Warning/Minor error
- 18.75 pts IDK
- 15 pts Incorrect/Missing induction proof
- 5 pts Incorrect/Missing Base case
- 5 pts Incorrect/Missing Induction Hypothesis
- 5 pts Incorrect/Missing induction proof on case x=0x'
- 5 pts Incorrect/Missing induction proof on case y=0y'
- 5 pts Incorrect/Missing induction proof on case x=1x'

gradescope