Generative Classifiers

LDQ, QDA, Naive Bayes

DS 6030 | Fall 2021

gen-classifiers.pdf

Contents

1 Classification and Pattern Recognition

- The outcome variable is categorical and denoted $G \in \mathcal{G}$
  - Default Credit Card Example: $\mathcal{G} = \{$"Yes", "No"$\}$
  - Medical Diagnosis Example: $\mathcal{G} = \{$"stroke", "heart attack", "drug overdose", "vertigo"$\}$
- The training data is $D = \{(X_1, G_1), (X_2, G_2), \ldots, (X_n, G_n)\}$
- The optimal decision/classification is often based on the posterior probability $\Pr(G = g \mid \mathbf{X} = \mathbf{x})$

1.1 Binary Classification

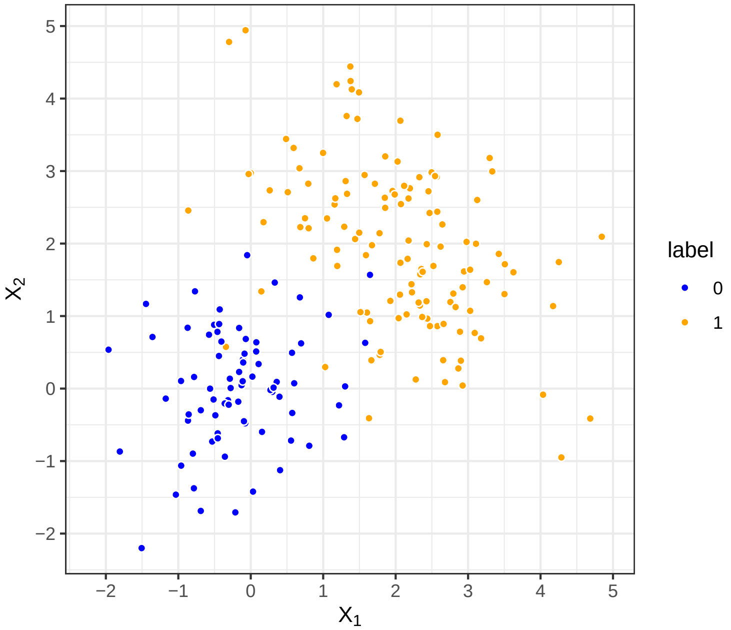
- Classification is simplified when there are only 2 classes.
  - Many multi-class problems can be addressed by solving a set of binary classification problems (e.g., one-vs-rest).
- It is often convenient to transform the outcome variable to a binary $\{0, 1\}$ variable:

$$Y_i = \begin{cases} 1 & G_i = \mathcal{G}_1 \quad \text{(outcome of interest)} \\ 0 & G_i = \mathcal{G}_2 \end{cases}$$

- Or, like with SVM, as a $\{-1, +1\}$ variable:

$$Y_i = \begin{cases} +1 & G_i = \mathcal{G}_1 \quad \text{(outcome of interest)} \\ -1 & G_i = \mathcal{G}_2 \end{cases}$$

1.2 Two-Class Example



1.3 Discriminative Models

- The models we have covered so far (Linear Regression, Logistic Regression, SVM, and KNN) can be considered *discriminative* models.
- Their goal is to directly estimate $\Pr(Y = 1 \mid X = x)$ **conditional on** $X = x$.

$$p(x) = \Pr(Y = 1 \mid X = x)$$

a. **Linear Regression (for binary outcomes)**

$$\hat{p}(x) = \hat{\beta}^{\mathsf{T}} x$$

b. **Logistic Regression**

$$\log\left(\frac{\hat{p}(x)}{1 - \hat{p}(x)}\right) = \hat{\beta}^{\mathsf{T}} x$$

and thus,

$$\begin{aligned}\hat{p}(x) &= \frac{e^{\hat{\beta}^{\mathsf{T}} x}}{1 + e^{\hat{\beta}^{\mathsf{T}} x}} \\ &= \left(1 + e^{-\hat{\beta}^{\mathsf{T}} x}\right)^{-1}\end{aligned}$$

c. **kNN (for binary outcomes)**

$$\begin{aligned}\hat{p}(x; k) &= \frac{1}{k} \sum_{i: x_i \in N_k(x)} y_i \\ &= \text{Avg}(y_i \mid x_i \in N_k(x))\end{aligned}$$

- $N_k(x)$ are the set of k nearest neighbors

d. **Support Vector Machines (SVM)**

$$\hat{g}(x) = \hat{\beta}_0 + \sum_{i=1}^{n} \hat{\alpha}_i \, y_i \, K(x, x_i)$$

- Decide $\hat{Y} = 1$ if $\hat{g}(x) > 0$
- Or calibrated probability: $\log \frac{\hat{p}(x)}{1 - \hat{p}(x)} = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{g}(x)$
  - I.e., using logistic regression with $\hat{g}(x)$ as the predictor.

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#> Warning: package 'e1071' was built under R version 4.0.3
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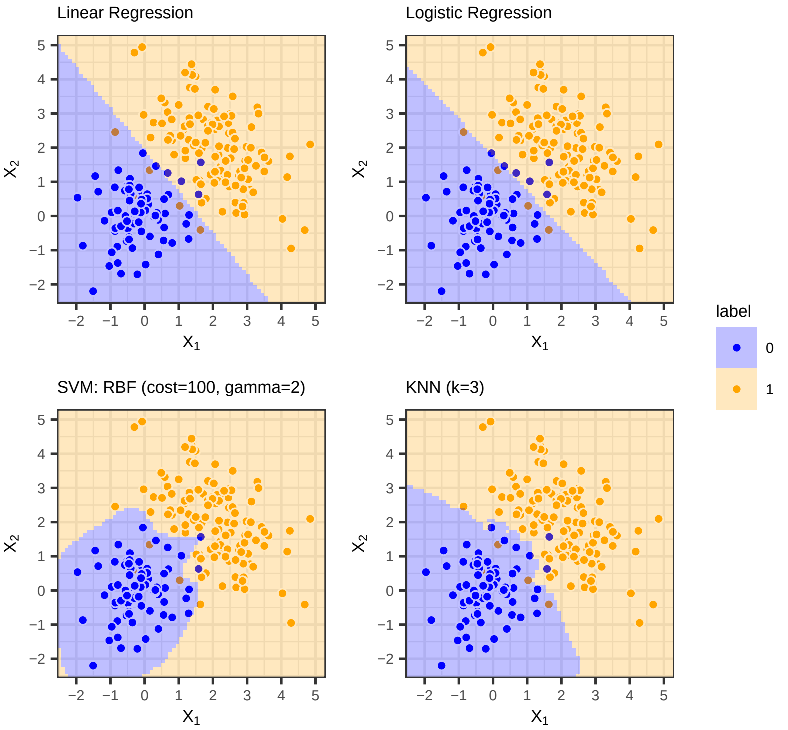

Linear Regression
Logistic Regression
SVM: RBF (cost=100, gamma=2)
KNN (k=3)
X_1
X_2
−2
−1
0
1
2
3
4
5
label
0
1

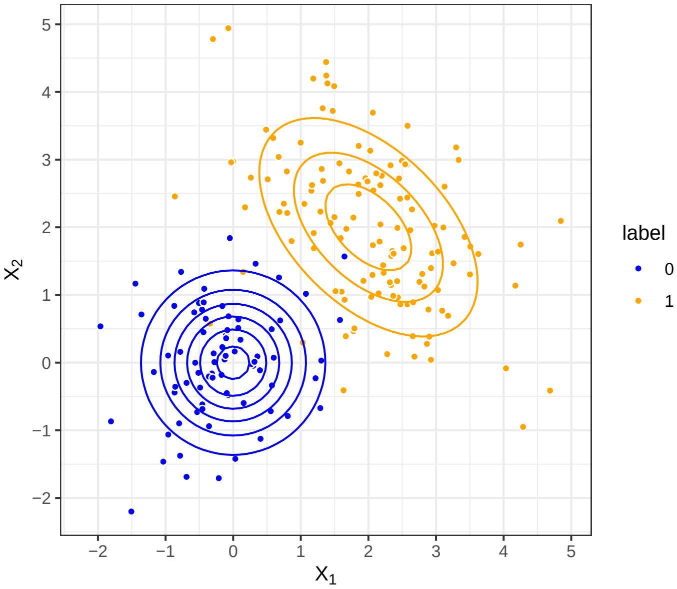
2 Generative Classification Models

Consider how the data $D = \{(X_1, G_1), (X_2, G_2), \ldots, (X_n, G_n)\}$ could be generated.

1. First, the class label is selected according to the *prior probabilities* $\pi = [\pi_1, \ldots, \pi_K]$.
   - That is, $\Pr(G_i = k) = \pi_k$
2. Given the class is k, the X value is generated $X \mid G = k \sim f_k$
   - Let $f_k(\mathbf{x})$ be the (pdf/pmf/mixed) of the predictors from class k.
3. Repeat n times

Example

- Two classes, $k \in \{0, 1\}$
  - $\pi_1 = 0.6$, $\pi_0 = 0.4$
  - I expect 60% of the observations to be from class 1.
- If $G_i = 1$, then $X \sim N\left(\mu_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\right)$
- If $G_i = 0$, then $X \sim N\left(\mu_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\right)$



2.1 From Discriminative to Generative, and Back Again

- The models we have discussed so far are considered *discriminative* and focused on estimating the **conditional** probability $\Pr(Y = k \mid X = x)$
- But there is another class of models termed *generative* which try to directly estimate the **joint** probability $\Pr(Y = k, X = x) \propto \Pr(X = x \mid Y = k) \Pr(Y = k)$

2.1.1 The Bayes Breakdown (Binary Classification)

Bayes Theorem

$$\begin{aligned} p(x) = \Pr(Y = 1 \mid X = x) &= \frac{\Pr(X = x \mid Y = 1) \Pr(Y = 1)}{\Pr(X = x)} \\ &= \frac{f_1(x)\pi}{f_1(x)\pi + f_0(x)(1-\pi)} \end{aligned}$$

- $f_k(x)$ is the *class conditional density*
- $0 \le \pi_k \le 1$ are the *prior class probabilities*
- $\pi_0 + \pi_1 = 1$

Recall our notation for the log-odds:

- $\gamma(x) = \log \frac{p(x)}{1-p(x)}$

The log-odds reduces to a combination of prior odds and density (likelihood) ratios

$$\begin{aligned} \gamma(x) &= \log\left(\frac{p(x)}{1-p(x)}\right) \\ &= \log\left(\frac{f_1(x)\pi}{f_0(x)(1-\pi)}\right) \\ &= \underbrace{\log\left(\frac{\pi}{1-\pi}\right)}_{\text{log prior odds}} + \underbrace{\log\left(\frac{f_1(x)}{f_0(x)}\right)}_{\text{log density ratio}} \end{aligned}$$

Likelihood Ratio and Bayes Factor

$\frac{f_1(x)}{f_0(x)}$ is usually called a *likelihood ratio* when estimated via MLE and *Bayes Factor* when integrating over the model parameters

2.1.2 Decision-Making (Hard Classification)

- We can see that the optimal decision can be based on the density ratios

$$
\begin{aligned}
&\text{Choose } \hat{G}(x) = 1 \text{ if:}\\
&\hat{\gamma}(x) > \log\left(\frac{C_{\text{FP}}}{C_{\text{FN}}}\right)\\
&\log\left(\frac{1-\hat{\pi}}{\hat{\pi}}\right) + \log\widehat{\left(\frac{f_1(x)}{f_0(x)}\right)} > \log\left(\frac{C_{\text{FP}}}{C_{\text{FN}}}\right)\\
&\log\widehat{\left(\frac{f_1(x)}{f_0(x)}\right)} > \log\left(\frac{1-\hat{\pi}}{\hat{\pi}}\right) + \log\left(\frac{C_{\text{FP}}}{C_{\text{FN}}}\right)
\end{aligned}
$$

2.1.3 Estimation

- $\hat{\pi}_k = n_k/n$ is a natural estimate for the class priors if we think the testing data will have the same proportions as the training data
- The other term to estimate is the log density ratio: $\log\widehat{\left(\frac{f_1(x)}{f_0(x)}\right)}$
- Generative Models estimate this term by

$$
\log\widehat{\left(\frac{f_1(x)}{f_0(x)}\right)} = \log\left(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\right)
$$

- That is, generative models estimate the class conditional densities $\{f_k(\cdot)\}$
- The different generative models take different approaches to estimate these component densities

Generative Models

Generative Classification Models use *density estimation* to make predictions!

2.1.3.1 Linear/Quadratic Discriminant Analysis (LDA/QDA)

- Both LDA and QDA model the class conditional densities $f_k(x)$ with a *Gaussian* density
  - Thus, they model the observations as coming from a *Gaussian mixture model*
  - Each class has its own mean vector μ_k
  - The difference between LDA and QDA is what they use for their covariance matrix
- **LDA**

$$
f_k(x) = (2\pi)^{-p/2}|\Sigma|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mu_k)^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\mu_k)\right\}
$$

  - $\Sigma_k = \Sigma \quad \forall k$ (*uses the same variance-covariance for all classes*)

- **QDA**

$$
f_k(x) = (2\pi)^{-p/2}|\Sigma_k|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mu_k)^{\mathsf{T}}\Sigma_k{}^{-1}(\mathbf{x}-\mu_k)\right\}
$$

  - Σ_k is *different* for each classes

2.1.3.2 Kernel Discriminant Analysis (KDA)

- Model the class conditional densities $f_k(x)$ with a multivariate *kernel density estimate (KDE)*

$$\hat{f}_k(x) = \frac{1}{n_k} \sum_{i:g_i=k} K(x - x_i; H)$$

where H is the $p \times p$ bandwidth matrix.

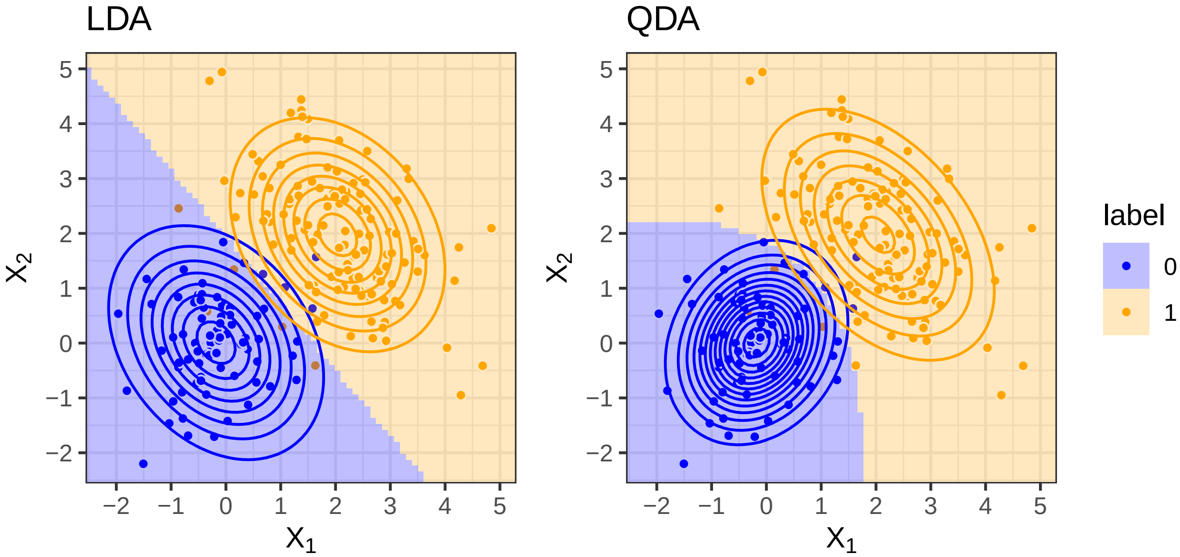
2.1.3.3 Naive Bayes

- **Naive Bayes** ignores potential associations between predictors and estimates the density of each predictor variable independently.

$$\hat{f}_k(x) = \sum_{j=1}^{p} \hat{f}_{jk}(x_j)$$

  - This greatly simplifies the estimation
  - You will often find $\hat{f}_{jk}(u) = \mathcal{N}(u; \hat{\mu}_{jk}, \hat{\sigma}_{jk})$
  - But KDE is a great approach $\hat{f}_{jk}(u) = \frac{1}{n_k} \sum_{\{i:G_i=k\}} K_h(u - x_{ij})$
  - And mix continuous and discrete variables is very easily

3 Linear/Quadratic Discriminant Analysis (LDA/QDA)



- *Linear Discriminant Analysis (LDA)* finds *linear* boundaries between classes
- *Quadratic Discriminant Analysis (QDA)* finds *quadratic* boundaries between classes

- Setup: $K = |\mathcal{G}|$ classes in the training data, $D = \{(\mathbf{X}_i, G_i)\}_{i=1}^n$
  - where $\mathbf{X}_i \in \mathbf{R}^p$, $G_i \in \mathcal{G}$
- The posterior probability of class g, given $X = x$,

$$\begin{aligned}\Pr(G = g \mid \mathbf{X} = \mathbf{x}) &= \frac{f(x \mid G = g)\Pr(G = g)}{f(x)} \\ &= \frac{f_g(x)\pi_g}{\sum_{k=1}^K f_k(x)\pi_k}\end{aligned}$$

  - $f_k(x)$ is the *class conditional density*
  - $0 \le \pi_k \le 1$ are the *prior class probabilities*; $\sum_{k=1}^K \pi_k = 1$

3.1 Estimation

- Both LDA and QDA model the class conditional densities $f_k(x)$ with *Gaussians*
  - Thus, they model the observations as coming from a K component *Gaussian mixture model*
  - Each class has its own mean vector μ_k
  - The difference between LDA and QDA is what they use for their covariance matrix

$$f_k(x) = \mathcal{N}(x; \mu_k, \Sigma_k)$$

- LDA: $\hat{\Sigma}_1 = \hat{\Sigma}_2 = \ldots = \hat{\Sigma}_K = \hat{\Sigma}$ Common covariance
- QDA: $\hat{\Sigma}_1 \neq \hat{\Sigma}_2 \neq \ldots \neq \hat{\Sigma}_K$ Different covariances

- **LDA**

$$f_k(x) = (2\pi)^{-p/2} |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \mu_k)^\mathsf{T} \Sigma^{-1} (\mathbf{x} - \mu_k) \right\}$$

  - $\Sigma_k = \Sigma \quad \forall k$ (*uses the same variance-covariance for all classes*)

- **QDA**

$$f_k(x) = (2\pi)^{-p/2} |\Sigma_k|^{-1/2} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \mu_k)^\mathsf{T} {\Sigma_k}^{-1} (\mathbf{x} - \mu_k) \right\}$$

  - Σ_k is *different* for each classes

Your Turn #1 : Model Complexity

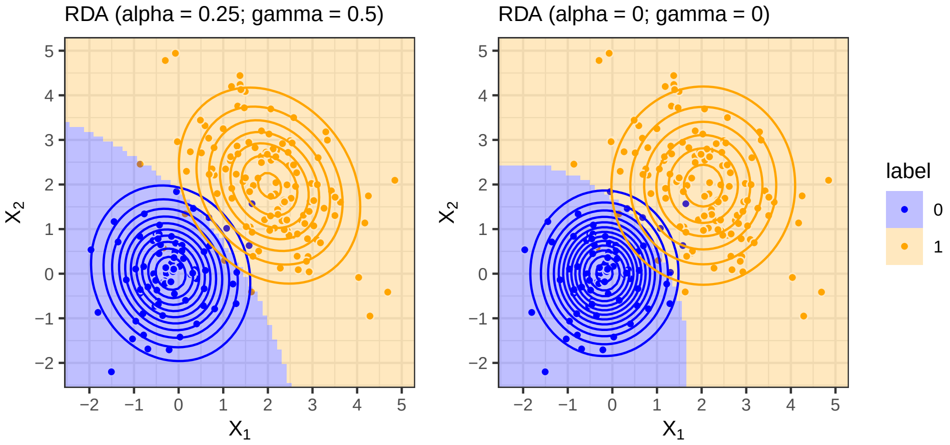
The LDA model uses a common covariance matrix while QDA allows each class to have a different covariance (which permits quadratic boundaries). But this flexibility comes at a cost.

1. How many parameters have to be estimated in an LDA model with K classes and p dimensions?

2. How many parameters have to be estimated in an QDA model with K classes and p dimensions?

- There are a few methods to maintain some flexibility, yet protect the model from high variance
- One is to use a *regularlized covariance matrix* (see ESL 4.3.1). Called Regularlized Discriminant Analysis (RDA)

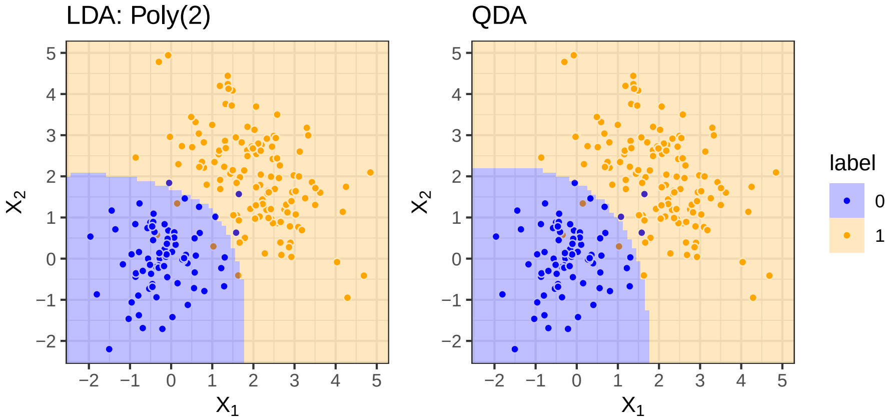
$$\hat{\Sigma}_k(\alpha, \gamma) = \alpha \hat{\Sigma}_k + (1 - \alpha)\{\gamma \hat{\Sigma} + (1 - \gamma)\hat{\sigma}^2 I_p\}$$



- A special case of above using diagonal covariance matrices only ($\hat{\Sigma}_k(\alpha = 0, \gamma = 0)$). This covariance matrix has all off-diagonal terms set to 0.

$$\begin{aligned}\hat{\Sigma}_k &= \hat{\sigma}^2 I_p \\ &= \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p^2 \end{bmatrix}\end{aligned}$$

  - This treats predictors/features as uncorrelated/independent.
  - It is a special case of *Naive Bayes*!
- In some settings (large K, small p), edf could be reduced by fitting an LDA model in an *enlarged feature space*
  - E.g., for $p = 2$ dimensions, use $X_1, X_2, X_1 \cdot X_2, X_1^2, X_2^2$ instead of QDA in X_1, X_2.
  - Think basis expansion like what we did with polynomial regression or B-splines
  - Or kernels with SVM



Mahalanobis Distance

Notice that a multivariate normal density is a function of the squared *Mahalanobis* distance from x to the mean.

$$\begin{aligned} f(\mathbf{x}) &= (2\pi)^{-p/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\mu)\right\} \\ &= (2\pi)^{-p/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}D_m^2\right\} \end{aligned}$$

where

$$D_m = \sqrt{(\mathbf{x}-\mu)^{\mathsf{T}}\Sigma^{-1}(\mathbf{x}-\mu)}$$

is the Mahalanobis distance.

3.2 LDA/QDA in Action

- In **R**, LDA and QDA can be implemented with the `lda()` and `qda()` functions from the `MASS` package.
- See ISLR 4.7 for details

3.3 Connections: LDA, QDA, and Logistic Regression

ISL 4.5 and ESL 4.4.5 show more details about the parametric form LDA and QDA take.

Recall the notation for generative models:

$$\begin{aligned} \hat{\gamma}(x) &= \log\left(\frac{\hat{p}(x)}{1-\hat{p}(x)}\right) \\ &= \log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \log\left(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\right) \end{aligned}$$

Logistic Regression

$$\hat{\gamma}(x) = \hat{\beta}_0 + \sum_{j=1}^{p} \hat{\beta}_j x_j$$

LDA

$$\hat{\gamma}(x) = \hat{\alpha}_0 + \sum_{j=1}^{p} \hat{\alpha}_j x_j$$

$$\begin{aligned}
\hat{a}_0 &= \log \frac{\hat{\pi}}{1-\hat{\pi}} - \frac{1}{2}(\hat{\mu}_1 - \hat{\mu}_0)^{\mathsf{T}} \hat{\Sigma}^{-1} (\hat{\mu}_1 - \hat{\mu}_0) \\
\hat{a}_j &= \text{the } j\text{th element of } \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_0)
\end{aligned}$$

QDA

$$\hat{\gamma}(x) = \hat{\alpha}_0 + \sum_{j=1}^{p} \hat{\alpha}_j x_j + \sum_{j=1}^{p} \sum_{k=1}^{p} \hat{a}_{jk} x_j x_k$$

$$\begin{aligned}
\hat{a}_0 &= \log \frac{\hat{\pi}}{1-\hat{\pi}} - \frac{1}{2} \log \frac{|\hat{\Sigma}_1|}{|\hat{\Sigma}_0|} - \frac{1}{2} \left(\hat{\mu}_1^{\mathsf{T}} \Sigma_1^{-1} - \hat{\mu}_0^{\mathsf{T}} \Sigma_0^{-1} \right) \\
\hat{a}_j &= \text{the } j\text{th element of } \hat{\Sigma}_1^{-1} \hat{\mu}_1 - \hat{\Sigma}_0^{-1} \hat{\mu}_0 \\
\hat{a}_{jk} &= \text{the } (j,k)\text{th element of } (\hat{\Sigma}_0^{-1} - \hat{\Sigma}_1^{-1})/2
\end{aligned}$$

3.3.1 Estimation

LDA and QDA estimates model parameters by maximizing the *joint* likelihood:

$$\begin{aligned}
\hat{\alpha} &= \arg\max_{\alpha} \ \Pr(X, Y) \\
&= \arg\max_{\alpha} \ \Pr(X \mid Y) \Pr(Y) \\
&= \arg\max_{\alpha} \ \Pr(Y \mid X) \Pr(X)
\end{aligned}$$

Logistic Regression estimates model parameters by maximizing the *conditional* likelihood

$$\hat{\beta} = \arg\max_{\beta} \ \Pr(Y \mid X)$$

4 Kernel Discriminant Analysis (KDA)

- Model the class conditional densities $f_k(x)$ with a multivariate *kernel density estimate (KDE)*

$$f_k(x) = \frac{1}{n_k} \sum_{i:g_i=k} K(x - x_i; H)$$

where H is the $p \times p$ bandwidth matrix.

There are three primary approaches to multivariate (p dimensional) KDE:

1. Multivariate kernels
   - e.g., $K(u) = N(\mathbf{0}, H)$:

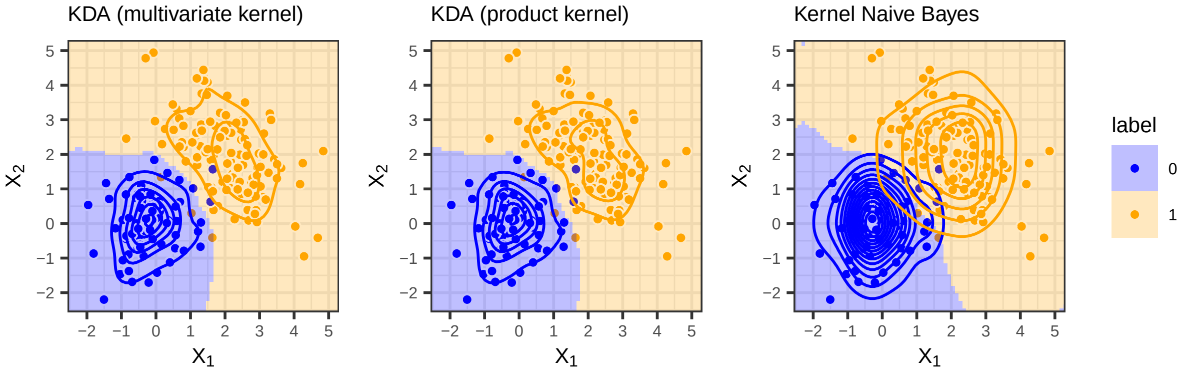
$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}|H|^{1/2}n} \sum_{i=1}^{n} \exp\left(-\frac{1}{2}(x - x_i)^\mathsf{T} H^{-1}(x - x_i)\right)$$

2. Product Kernels
   - $H = diag(h_1, h_2, \ldots, h_p)$

$$\hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} \left(\prod_{j=1}^{p} K(x_j - x_{ij}; h_j) \right)$$

3. Independence
   - This is a special case of *Naive Bayes* (Kernel Naive Bayes)!

$$\begin{aligned}\hat{f}(x) &= \prod_{j=1}^{p} \hat{f}_j(x) \\ &= \prod_{j=1}^{p} \left(\frac{1}{n} \sum_{i=1}^{n} K(x_j - x_{ij}; h_j) \right)\end{aligned}$$



4.1 KDA with R

- In **R**, the `ks::kda()` function (`ks` package) implements Kernel Discriminant Analysis.

5 Naive Bayes

Naive Bayes is a generative model that ignores potential associations between predictors and estimates the density of each predictor variable independently.

$$\hat{f}_k(x) = \prod_{j=1}^{p} \hat{f}_{kj}(x_j)$$

- This greatly simplifies the estimation
- The densities do *not* have to be Gaussian (e.g., KDE is a good option)
- Categorical densities (i.e., pmfs) can be thrown in the mix without a problem
- Because of the independence, this is easy to implement in parallel (and thus can be fast)
- The decision function becomes:

$$\begin{aligned}
\hat{\gamma}(x) &= \log\left(\frac{\hat{p}(x)}{1-\hat{p}(x)}\right) \\
&= \log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \log\left(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\right) \\
&= \log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \log\left(\frac{\prod_{j=1}^{p} \hat{f}_{1j}(x_j)}{\prod_{j=1}^{p} \hat{f}_{0j}(x_j)}\right) \\
&= \log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \log\left(\prod_{j=1}^{p} \frac{\hat{f}_{1j}(x_j)}{\hat{f}_{0j}(x_j)}\right) \\
&= \log\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) + \sum_{j=1}^{p} \log\left(\frac{\hat{f}_{1j}(x_j)}{\hat{f}_{0j}(x_j)}\right)
\end{aligned}$$

5.1 Gaussian Naive Bayes

- Recall in LDA/QDA, the class conditional densities were estimated as Gaussians:

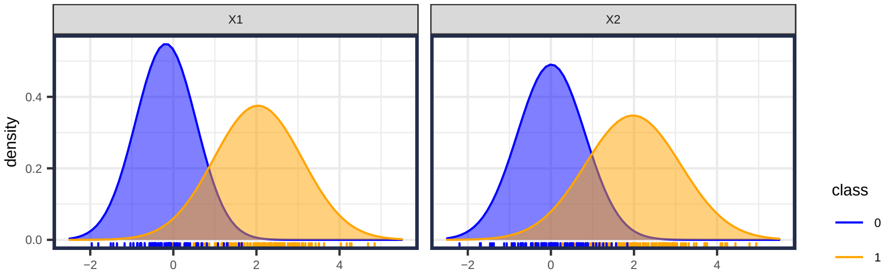
$$\hat{f}_k(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \hat{\mu}_k, \hat{\Sigma}_k)$$

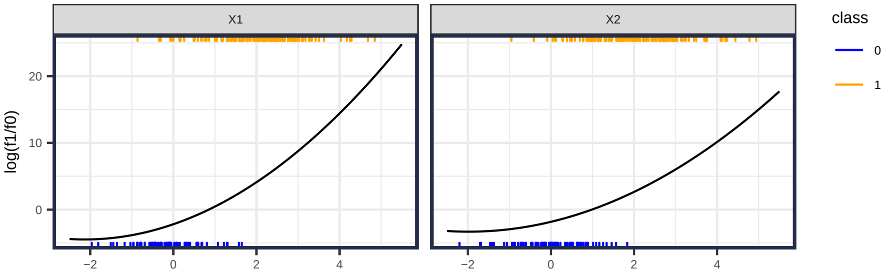
  - But when the dimensionality of $\mathbf{x}$ gets large or there is high correlation, estimation of $\hat{\Sigma}_k$ can be poor
- If we force $\hat{\Sigma}_k$ to be *diagonal* then the densities are product of univariate Gaussians (called Gaussian Naive Bayes)

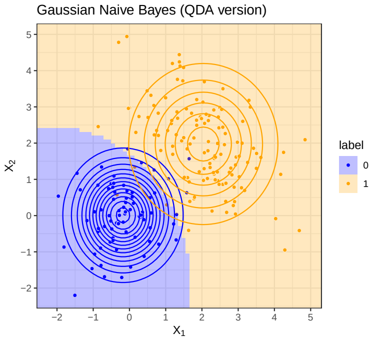
$$\hat{f}_k(\mathbf{x}) = \prod_{j=1}^{p} \mathcal{N}(x_j; \mu_{kj}, \sigma_{kj})$$

  - Even if the data are not independent, this may give better estimates by reducing the variance (at the expense of a bit of bias)
  - This is a special case of QDA, where we restrict the off-diagonal terms in the variance-covariance to be 0.

class	predictor	mu	sd
0	X1	-0.18	0.73
0	X2	0.01	0.81
1	X1	2.04	1.06
1	X2	1.97	1.15



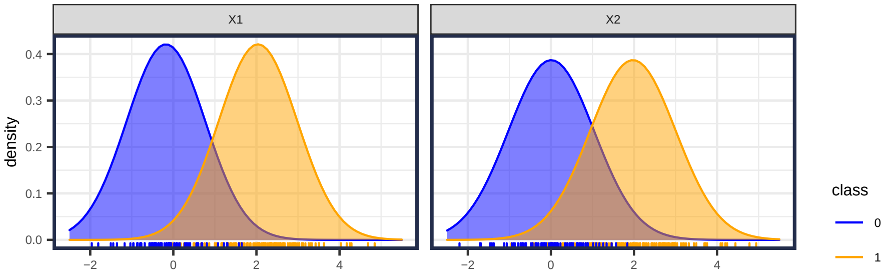


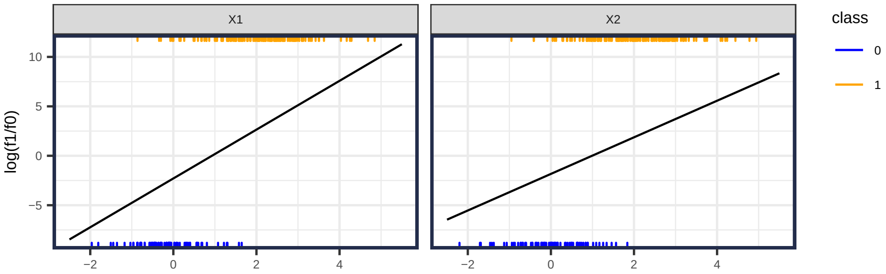


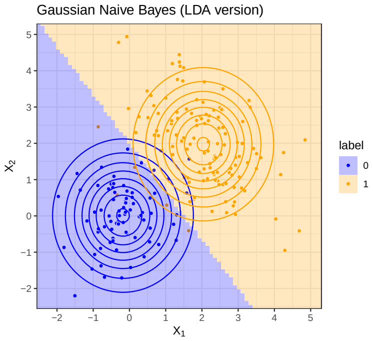
- A simpler model (less complexity/edf) forces a common standard deviation for all class (special case of LDA)

$$\hat{f}_k(\mathbf{x}) = \prod_{j=1}^{p} \mathcal{N}(x_j; \mu_{kj}, \sigma_j)$$

class	predictor	mu	sd
0	X1	-0.18	0.95
0	X2	0.01	1.03
1	X1	2.04	0.95
1	X2	1.97	1.03







5.2 Kernel Naive Bayes

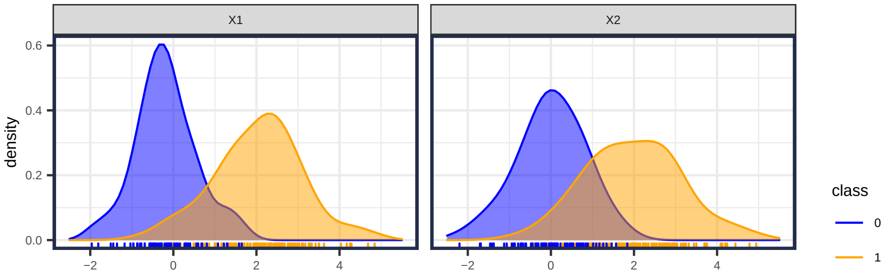
In *kernel density* Naive Bayes, use Kernel Density Estimation (KDE) to estimate each component density:

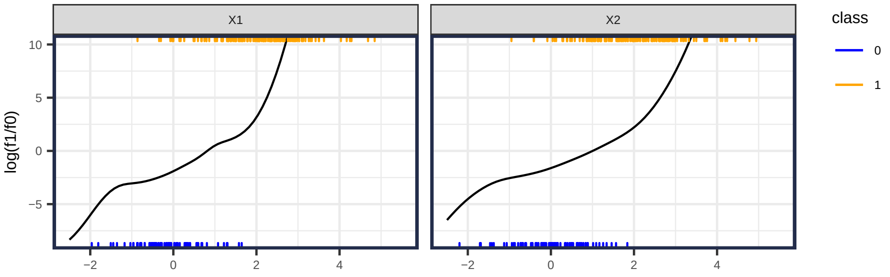
$$\hat{f}_{kj}(x_j) = \frac{1}{n_k} \sum_{i:g_i=k} K(x_j - x_{ij}; h_{kj})$$

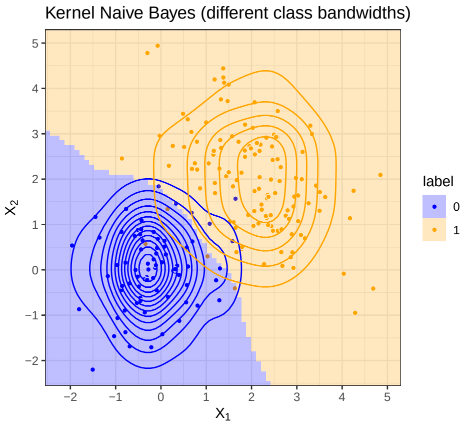
with bandwidth parameter h_{kj}.

The density ratio becomes

$$\frac{\hat{f}_{1j}(x_j)}{\hat{f}_{0j}(x_j)} = \frac{\frac{1}{n_1}\sum_{i:g_i=1} K(x_j - x_{ij}; h_{1j})}{\frac{1}{n_0}\sum_{i:g_i=0} K(x_j - x_{ij}; h_{0j})}$$







- for less complex models, use same bandwidth parameter for each class.

Note: this gives a different solution than using KDE with a *product kernel*! (which is not a naive bayes model)

$$\hat{f}_k(\mathbf{x}) = \frac{1}{n_k} \sum_{i:g_i=k} \prod_{j=1}^{p} K(x_j - x_{ij}; h_{kj})$$

6 Connections: Generalized Additive Models (GAM)

It turns out that there is a close connection between Logistic Regression, Naive Bayes, and LDA. To help see this, notice that all three methods can be written:

$$\begin{aligned}\gamma(x) &= \log\left(\frac{\pi}{1-\pi}\right) + \log\left(\frac{f_1(x)}{f_0(x)}\right)\\ &= \alpha_0 + \sum_{j=1}^{p} \alpha_j S_j\end{aligned}$$

- **Logistic Regression**

$$\begin{aligned}\hat{\alpha}_0 &= \hat{\beta}_0\\ \hat{\alpha}_j &= \hat{\beta}_j\\ \hat{S}_j &= x_j\end{aligned}$$

- **LDA**

$$\begin{aligned}\hat{\alpha}_0 &= \log\frac{\hat{\pi}}{1-\hat{\pi}} - \frac{1}{2}(\hat{\mu}_1 + \hat{\mu}_0)^{\mathsf{T}}\hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_0)\\ \hat{\alpha}_j &= \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_0)\\ \hat{S}_j &= x_j\end{aligned}$$

- **Naive Bayes**

$$\begin{aligned}\hat{\alpha}_0 &= \log\frac{\hat{\pi}}{1-\hat{\pi}}\\ \hat{\alpha}_j &= 1\\ \hat{S}_j &= \log\frac{\hat{f}_{1j}(x_j)}{\hat{f}_{0j}(x_j)}\end{aligned}$$

- **Generalized Additive Models (GAM)**
  - GAM models are made to directly estimate models of this form.

$$\hat{\gamma}(x) = \hat{\alpha} + \sum_{j=1}^{p} \hat{g}_j(x_j)$$

  - In **R**, the `mgcv` package is worth becoming familiar with to implement GAM.
  - See ESL 9.1 for more details