**Chapter 1 – Systems of Linear Equations**

**A system of linear equations has either;**

1. no solution, or

2. exactly one solution, or

3. infinitely many solutions.

Consistent 🡪 One or many solutions.

Incosistent 🡪 No solutions.

**Elementary Row operations**

1. Replacement, replace one row by the sum of itself and a multiple of another row.

2. Interchange, interchange two rows.

3. Scaling, multiplying all entries in a row by a **nonzero** constant.

**Echelon Form**

1. All nonzero rows are above any rows of all zeroes.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

**Reduced Echelon form**

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

\*\*Each matrix is equivalent to one and only one RREF

**Pivot position and pivot column** – easy peezy

**Span**{**v1**,…,**v**n} = {c1**v1** + … + cn**vn** | **v**i **V** , ci }

Note that the zero vector is always in the span!

A system is consistent IFF the right most column in an augmented matrix is NOT a pivot column.

If the system is consistent with no free variables, then there is one solution.

If the system is consistent with one or more free variables, then there is infinitely many solutions.

**Product of a matrix and a vector.**

A**x** = [**a**1 **a**2 ... **a**n] = x1 **a**1 + … + xn**a**n

This produces a vector, if A is MxN, then x must be Nx1 and the produced vector will be Mx1.

If A is an mxn matrix, with columns **a**1, ….. , **a**n, and if **b** is in m, the matrix equation

A**x** = **b**

Has the same solution set as the vector equation

x1**a**1 + … + xn**an = b**

which is turn has the same solution set as the augmented matrix

[**a**1 … **a**n **b**]

The equation A**x** = **b** has a solution IFF **b** is a linear combination of the columns of A.

Let A be an m x n matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each **b** in m , the equation A**x** = **b** has a solution

b. Each **b** in m is a linear combination of the columns of A.

c. The columns of A span m

d. A has a pivot position in every row.

If A is an m x n matrix, **u** and **v** are vectors in m, and c is a scalar, then:

a. A(**u + v**) = A**u** + A**v**

b. A(c**u**) = c(A**u**)

proof page 45

The homogeneous equation A**x** = **0** has a nontrivial solution IFF the equation has at least one free variable.

**Parametric vector form**

x = x2 + x3

**Linearly independent**, x1**v1** + … + xn**v**n = **0** has only the trivial solution.

**Linearly dependant**, …. , has a non trivial solution.

An indexed set S = {**v**1,…,**v**p} of two or more vectors is linearly dependant IFF at least one of the vectors in S is a linear combination of the others. In fact if S is linearly dependent and **v**1 != **0**, then some **v**j(with j > 1) is a linear combination of the preceding vectors, **v**1, … , **v**j-1

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set {**v**1,…,**v**p} in m is linearly dependent if p > n.

Proof

Let A = [**v**1 … **v**p]. Then A is n x p, and the equation A**x**=**0** corresponds to a system of n equations in p unknowns. If p > n, there are more variables than equations, so there must be a free variable. Hence A**x** = **0** has a nontrivial solution, and the columns of A are linearly dependent.

If a set contains the zero vector, than it is linearly dependent.

**Transformations**

When we have T(**x**) = A**x** , A is called the standard matrix for T

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| Let T : n 🡪 m be a linear transformation. Then there exists a unique matrix A such that  T(**x**) = A**x** for all **x** in n  In fact, A is the m x n matrix whose jth column is the tect T(**e**j), where **e**j is the jth column of the identity matrix in n.  A = [T(**e**1) … T(**e**n)]  PROOF  Write **x** = In**x** = [**e1** … **e**n]**x** = x1**e**1 + … + xn**e**n, and use the linearity of T to compute  T(**x**) = T(x1**e**1 + … + xn**e**n) = x1T(**e**1) + … + xnT(**e**n)  =[T(**e**1) … T(**e**n)]= A**x** |

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| Let T : n 🡪 m be a linear transformation. Then T is one-to-one IFF the equation T(**x**) = **0** has only the trivial solution.  PROOF  Since T is linear, T(**0**) = **0**. If T is on-to-one, then the equation T(**x**) = **0** has at most one solution and hence only the trivial solution. If T is not one-to-one, then there is a **b** that is the image of at least two different vectors in n –say, **u** and **v**. That is, T(**u**) = **b**, T(**v**) = **0**. But then since T is linear,  T(**u** – **v**) = T(**u**) – T(**v**) = **b** – **b** = **0**  The vector **u** – **v** is not zero, since **u**!=**v**. Hence the equation T(**x**) = **0** has more than one solution. So, either the two conditions in the theorem are both true or they are both false. |

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| Let T : n 🡪 m be a linear transformation and let A be the standard matrix for T. Then:  a. T maps n onto m IFF the columns of A span m.  b. T is one-to-one IFF the columns of A are linearly independent. |

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| Let T : n 🡪 m be a linear transformation. Then there exists a unique matrix A such that T(**x**) = A**x** for every **x** n  PROOF  Proof Idea:  Observe what T does to the columns of the identity matrix In  Let A be the matrix [T(**e**1)… T(**e**n)]  Let **x** =  A**x** = x1T(**e**1) + … + xnT(**e**n)  =T(x1**e**1+ … +xn**e**n) = T(**x**)  So T(**x**) = A**x** for all **x**  But we are not done: we need to prove that A is unique.  If T(**x**) = B**x** for all **x** , then in particular B(**e**i) = T(**e**i) = A(**e**i) for all I = 1 …… n  But B(**e**i) is the ithcolumn of B, and similarily for A. So A and B have the same columns and thus they are equal! |

**Chapter 2 - Matrix Algebra**

**Transpose**

AT is the matrix whose columns are forms from the rows of A.

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| Property. Let A and B be two m x n matrices , and c be a scalar  a. (AT)T = A  b. (A+B)T = AT + B T  c. (cA)T = cAT  d. (AB)T = BTAT |

**Matrix Multiplication**

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| Let A be an m x p matrix and B be an p x n matrix.  Then the product AB is the m x n matrix whos columns are  [A**b1 …** A**bn**] |

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| Show:  Let **x** = , what are B**x** and A(B**x**) ?  B**x** = **x1b1**+ … + **xnbn**  Now apply A, A(B**x**) = A(**x1b1**+ … + **xnbn**) = **x1**(A**b1**) + … + **xn**(A**bn**)  So, the vector A(B**x**) is a linear combination of the vectors A**b1**, … , A**bn**  Using the entries of **x** as weights |

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| Properties that matrix multiplication **DOES** **NOT** satisfy!  1. AB = BA  2. AB = AC implies B=C  3. AB = 0 implies A = 0 or B = 0 |

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| **Properties of matrix multiplication**  Property. Let A,B,C be matrices and c be a scalar. When the the following products are defined we have.  1. A(BC) = (AB)C (associative law of multi)  2. A(B + C) = AB + AC (Left distributive law)  3. (B + C)A = BA + CA (Right distributive law)  4. c(AB) = (cA)B = A(cB)  5. ImA = A = AIn (Identity for matrix multi)  6. (AB)T = BTAT |

**Matrix Inverse**

AA-1 = I and A-1A=I

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| **Proposition:** If A has an inverse then it is unique.  Proof by contradiction.  Suppose that A is invertible and it has two inverses, namely C and B  B = IB  = (CA)B since C is an inverse of A, AC = I  = C(AB) by associativity of matrix multiplication  = C(I) since B is an inverse of A, AB = I  B = C  Which is a contradiction! And thus A-1 is unique! |

**Chapter 5 – Eigenvecotors and Eigenvalues**

An **eigenvector** of a nxn matrix A is a nonzero vector **x** such that A**x** = λ**x** for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of A**x**; such an **x** is called an eigenvector corresponding to λ.

Determining if some vector **v** is an eigenvector or if some λ is an eigen value is easy.

A**u**, **u** is an eigenvector IFF A**u** = c**u** has a non trivial solution, say n (an eigenvalue). Then **u** is said to be an eigen vector corresponding to the eigenvalue n.

A**x** = 7**x** , 7 is an eigenvalue IFF A**x** = 7**x**  has a non trivial solution.

A**x** = λ**x** 🡪 (A -λI)**x** = **0**

If A –λI has linearly independent columns, then it clearly has non trivial solutions, which shows that λ is an eigenvalue for A.

The set of all solutions (eigenvectors) is just the null space of A –λI . Called the **eigenspace**.

This is solved by taking A –λI into RREF and solving, solve for all x’s in a general equation in terms of the fee variables.

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof pg 306.

**Triangular Matrices**

Upper Triangular: All entries below the main diagonal are zero.

Lower Triangular: All entries above the main diagonal are zero.

If both upper and lower, it’s called a diagonal matrix.